

Backtracking, also called depth-first search, will be the focus of the present paper. Backtracking can be used in programs that enumerate all solutions to a given set of constraints. Backtrack is the technique of dancing links in computer science. Backtracking programs often know the value of simplicity as a byproduct of their normal operation. The program logic that traverses lists can often be run in reverse. Scott's pentomino problem is a special case of the exact cover problem. Each row contains a 1 in the column identifying the piece, and there are exactly 1568 such rows. We can name the twelve columns F I L P N T U V W X Y Z. Backtracking is the process of traversing the tree in preorder, "depth first" Backtracking involves choosing a row, r , such that $A[r, c] = 1$ (non-deterministically) and deleting a row from matrix A . The nondeterministic choice of r means that the algorithm essentially clones itself into independent sub-algorithms. The search trees for Scott's pentomino problem then have only 10,421 nodes (X at 23) and 12,900 nodes (X at 24) The algorithm X is to represent each 1 in the A as a data object x with a list header and column object. The size is the number of 1s in the column, and the name is a symbolic identifier for printing the answers. The operation of covering column c is more interesting: It removes c from the headerlist and removes all rows in c 's own list from the other column lists they are in. Operation (1), which I mentioned at the outset of this paper, is used here to remove objects in both the horizontal and vertical directions. In Figure 2 we show how search (0) affects the data of (3) Column A is covered by removing both of its rows from their other columns. The structure now takes the form of Figure 3. The outer level routine, search (0), will proceed to transform Figure 4. The running time of algorithm DLX is essentially proportional to the number of times it applies operation (1) to remove an object from a list. A total of 28 updates are performed during the solution of (3) if we repeatedly choose the shortest column. A backtrack program usually spends most of its time on only a few levels of the searchtree. Figure 5 shows the search tree for the case $X = 23$ of Dana Scott's pentomino problem. The heuristic multiplies the total number of memory accesses by a factor of approximately $(14 \times 3,617,723)/(8 \times 17,818,752) \times 36\%$ in this example. Extra work on the lower levels has reduced the need for hard work at the higher levels. Algorithm is faster than Scott's $8 \times 8 \times 2 \times 2$ problem, because the backtrack tree is larger and there are 2339 essentially different solutions [21] The linked-list approach of algorithm DLX performs a total of 13,691,897 updates, or about 192 million memory accesses. This solves the problem in less than a minute on an IBM 7094. Algorithm DLX begins to outperform other pentomino-placing procedures in problems where the search tree has many levels. It knows no difference, because pieces and cells are simply columns of the given input matrix. The solutions fall into four classes, depending on the behavior at the four corners. In the late 1950s, T. H. O'Beirne introduced a pleasant variation on polyominoes by substituting triangles for squares. The twelve hexiamonds were

independently discovered by J. E. Reeve and J. A. Tyrell. They found more than forty ways to arrange them into a 6×6 rhombus. O'Beirne was fascinated by the fact that seven of the twelve hexiamonds have different shapes when they are flipped over. In November of 1959, after three months of trials, he found a solution; and two years later he challenged the readers of New Scientist to match this feat. A solution to the hexiamond problem is maximally symmetric if it has the highest horizontal or vertical symmetry score. Figure 8 shows solutions to O'Beirne's hexiamond hexagon problem, with the small hexagon at various distances from the center of the large one. There are 46 ways to pack one-sided pentominoes into a 3 × 30 rectangle. The Monte Carlo estimation procedure of [24] suggests that about 19 quadrillion updates will be needed. If that estimate is correct, I could have the result in a few months. The cells of the hexiamond-hexagon problem can be colored black and white in a similar fashion. Instead of making pieces by joining squares or triangles together, Brian Barwell considered making them from line segments or sticks. The tetrasticks are especially interesting from a recreational standpoint. Filling a 5×5 grid with 15 of the 16 tetrasticks, we must leave out either the H, the J, the L, the N, or the Y. We also need secondary columns I_{xy} to represent interior junction points (x, y). Each row represents a possible placing of a piece. There are ten one-sided welded tetrasticks if we add the mirror images of the unsymmetrical pieces. Only three solutions are possible, including two perfectly symmetric solutions shown. I've decided not to show the third solution, which has the X piece in the middle. The Nqueens problem is actually a special case of the generalized cover problem in the previous section. For example, the 4 queens problem is just the task of covering eight primary columns (R0, R1, R2, R3, F0, F1, F2, F3) given the sixteen rows. Consider the eight queens problem. There are 8 possible ways to occupy each rank and each file. It is better to place queens near the middle of the board because central positions rule out more possibilities. The order in which header nodes are linked together at the start of algorithm DLX can have a significant effect on the running time. Algorithm DLX, which uses dancing links to implement the "nongenicural" algorithm for exact cover problems, can run even faster than other algorithms. In this paper I have used the exact cover problem to illustrate the versatility of dancing links. The approach works nicely with the Waltz filtering algorithm. I believe that a terpsichorean technique is significantly better than the current state at every level. The implementation of algorithm DLX that I used when preparing this paper is on webpage <http://www-cs-faculty.stanford.edu/~knuth/grotesqueprograms.html>. 845 Combinations Puzzles: 845 Interestingly Combinations (Taiwan: R.O.C. Patent 66009). This puzzle, which is available from www.puzzletts.com, actually has only 83 solutions. It carries a Chinese title, "Dr. Dragon's Intelligence Pro*t

System. *Richard K. Guy, *O*Beirne*s Hexiamond,* in The Mathematician and Pied Puzzler. *Packing a square with Y-pentominoes,*
Journal of Recreational Mathematics 7(1974), 229*229. Alfred Wassermann of Universität Bayreuth
covered the Aztec diamond of Figure 15 with one-sided tetrasticks. He subsequently enumerated the 10,440,433
solutions to the 9×10 one-sided pentomino problem.