2D FDTD with PML based on Matlab

Junda Feng

***Abstract*—In this literature, a FDTD program with a PML absorbing boundary condition is developed to solve the time-domain Maxwell's Equation numerically and calculate the radiation of an infinitely long current. After the code is validated, a case in which the infinitely large conducting sheet with one or two slots is considered. In addition, one program version with GPU acceleration is presented to compare with CPU version.**

***Index Terms*—Finite difference time domain (FDTD), Perfect matched layer (PML), Matlab, Graphics processing unit (GPU).**

I. INTRODUCTION

TRANSIENT electromagnetic (EM) analysis is of critical importance in computational physics society. Time-domain Maxwell's Equations (ME) can be expressed as a group of partial differential equations (PDEs) that are widely used in modeling all types of continuous phenomena in nature.

We have proved the correctness of the ME for countless times through empirical observations since James Clerk Maxwell proposed it, and use it as one of the foundations of modern world of electronics. However, we actually can use any kind of differential equations to describe the oscillatory nature of EM wave in time and space. Thus the ME, in its best-known form, can be seen, from the perspective of Ockham Razor, as the simplest possible way to connect time and space together.

Everybody loves FDTD[1][2], not only because of its simplicity, but FDTD can also gives solutions to the problem at many frequencies simultaneously by doing inverse Fourier transform of the time domain data, whereas finite difference method in frequency domain only gives us solution at one single frequency. What's more, from a perspective of high performance computing (HPC), FDTD can be considered as an ideal algorithm, because $\mathcal{O}(N^\alpha)$ values are produced from $\mathcal{O}(N^\alpha)$ input values in $\mathcal{O}(N^\alpha)$ operations. FDTD is also suitable for implementations based on the SIMD parallel programming strategy because of the data reuse patterns (time locality and spatial locality) of the stencil operations contained in its calculations [3].

In reality, many EM problems are open-region ones. For example, when simulating dipole antennas that transmit electromagnetic waves into infinitely large space, we do not expect any artificial reflection when the radiated wave passing through the truncation air box we create for the numerical simulation. Although there is no perfect solution for now, mathematical boundary conditions such as absorbing boundary conditions (ABCs) [4][5] and fictitious absorbing material boundary such as [6] and perfect matched layer (PML) [7] are proposed to tackle this problem and minimize reflections. The latter category that utilize theoretically designed conventional absorptive materials as absorbers is suitable for SIMD parallel computing and can ease the programming skill requirements for writing the FDTD source code in that the FDTD grids at the edge of computation domain can be implemented with the same FDTD stencil operation as that of the interior grids by simply modifying the material conductivity parameter of those grids. Specifically, the PML proposed by Berenger is in such a more general way that the modified Maxwell's equations in stretched coordinates can be reduced to the Maxwell equations in general form by simply removing the additional degrees of freedom.

The organization of this course project report is as follows. In Section II, we first give the traditional FDTD time stepping formula based on Yee grid as well as modified FDTD time stepping formula based on stretched coordinate method used for deriving PML. Then the information about current source $\mathcal{J}_z$ and object (conductor sheet with one slot) are presented. In Section III, results are shown with respect to time domain simulation snapshots, time and frequency analysis at the observing point and GPU acceleration provided by the library of Matlab. Finally, in Section IV, conclusions are drawn.

II. FORMULATION

IN this section, first, we are going to derive the time stepping formula in a leapfrog fashion of two-dimensional FDTD. Then, the line source that is perpendicular to the simulation domain is formulated. In the third part of this section, for mimicking boundless space in the finite simulation domain, we give a brief derivation of the uni-axial absorptive material known as PML to absorb without reflection the electromagnetic waves. In the end, we illustrate the geometry and material properties of the whole computation domain of our numerical simulation.

A. FDTD Formulation

Suppose the source is electric current flowing in the z-direction, it will generate an electric field with only a z-component (TM polarization). As we can see from Equation (1) (2) and (3), the magnetic field is transverse to the z direction with only x and y component. For the numerical computation of FDTD, the time axis can be uniformly discretized into N time steps as $t = n\Delta t$ for electric field and $t = (n + \frac{1}{2})\Delta t$ for magnetic field and electric current with $\Delta t = T/N$ and $n \subset \mathbb{Z}^*$ and the spatial solution domain enclosed in a rectangular box can be discretized into an equi-spaced grid of mesh points with each lattice having a size of $\Delta x \times \Delta y$. Again, a half cell offset exists between the sampling grid of electric field and magnetic field due

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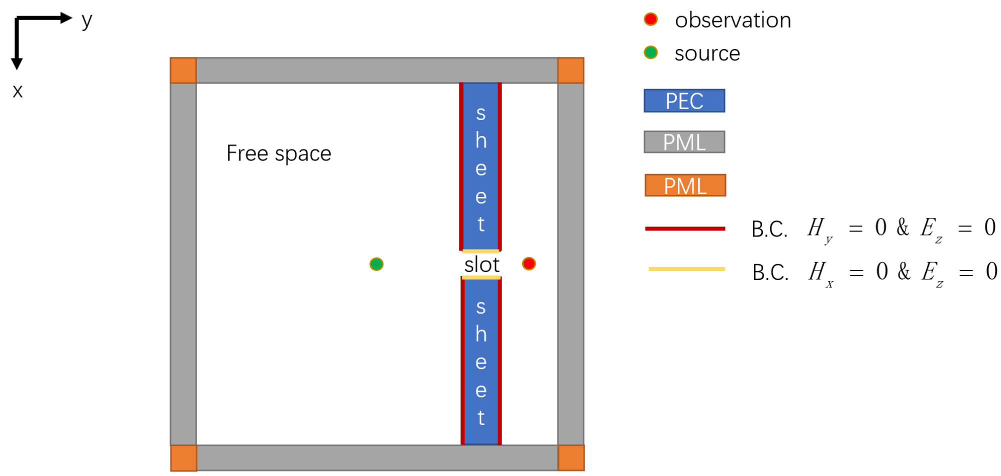


Fig. 1: Structure of FDTD problem: source, object, free space region and PML

to the staircased nature of Yee grid. To achieve a second-order accurate, we apply the centered differencing to the discretization of derivatives in both time and space.

$$\frac{\partial \mathcal{E}_z}{\partial y} = -\mu \frac{\partial \mathcal{H}_x}{\partial t} \tag{1}$$

$$\frac{\partial \mathcal{E}_z}{\partial x} = \mu \frac{\partial \mathcal{H}_y}{\partial t} \tag{2}$$

$$\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = \epsilon \frac{\partial \mathcal{E}_z}{\partial t} + \sigma \mathcal{E}_z + \mathcal{J}_z \tag{3}$$

$$\mathcal{H}_x^{n+\frac{1}{2}}(i, j+\frac{1}{2}) = \mathcal{H}_x^{n-\frac{1}{2}}(i, j+\frac{1}{2}) - \frac{\Delta t}{\mu \Delta y}\left[\mathcal{E}_z^n(i, j+1) - \mathcal{E}_z^n(i, j)\right] \tag{4}$$

$$\mathcal{H}_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j) = \mathcal{H}_y^{n-\frac{1}{2}}(i+\frac{1}{2}, j) + \frac{\Delta t}{\mu \Delta x}\left[\mathcal{E}_z^n(i+1, j) - \mathcal{E}_z^n(i, j)\right] \tag{5}$$

$$\begin{aligned}\mathcal{E}_z^{n+1}(i,j) = \left[\frac{1}{\frac{\epsilon_{ij}}{\Delta t} + \frac{\sigma}{2}}\right] \Bigg\{ \left[\frac{\epsilon_{ij}}{\Delta t} - \frac{\sigma}{2}\right] \mathcal{E}_z^n(i,j) + \frac{1}{\Delta x}\left[\mathcal{H}_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j) - \mathcal{H}_y^{n+\frac{1}{2}}(i-\frac{1}{2}, j)\right] - \\ \frac{1}{\Delta y}\left[\mathcal{H}_x^{n+\frac{1}{2}}(i, j+\frac{1}{2}) - \mathcal{H}_x^{n+\frac{1}{2}}(i, j-\frac{1}{2})\right] - \mathcal{J}_z^{n+\frac{1}{2}}(i,j) \Bigg\}\end{aligned} \tag{6}$$

B. *Current Source formulation*

In FDTD simulation, we usually would like to have the information of entire frequency range with one single time-stepping calculation. Thus when choosing the time domain profile of the current source, an shifted Gaussian pulse with a start value of zero (as depicted in Equation 7 and Fig. 2) would be a good candidate. Time constant τ_p is used for determine the maximum frequency (or frequency range in passband scenario) you want to simulate.

$$f(t) = e^{-\frac{1}{2}(\frac{t}{\tau_p})^2} \tag{7}$$

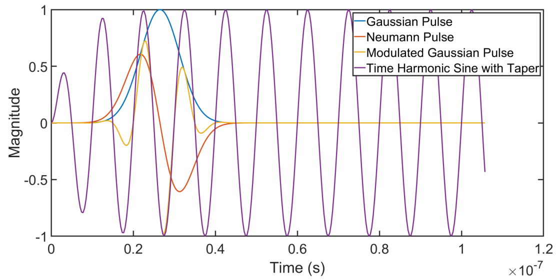
The large DC component in the Gaussian pulse may make the time domain simulation unstable. We can use Neumann pulse (as depicted in Equation 8 and Fig. 2), whose time domain profile is the time derivative of the Gaussian pulse, to get rid of the DC component.

$$f(t) = -\frac{t}{\tau_p} e^{-\frac{1}{2}(\frac{t}{\tau_p})^2} \tag{8}$$

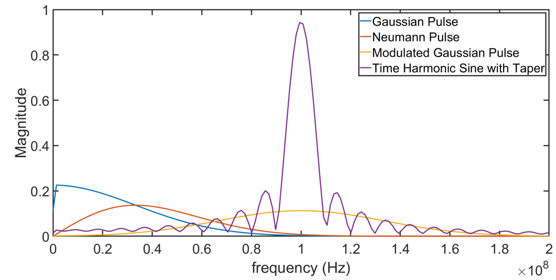
For some kind of applications such as radio frequency (RF) devices, we do not care about the low frequency components of spectrum. We can modulate the Gaussian pulse to shift the spectrum to higher frequency range, as depicted in Equation 9 and Fig. 2, where ω_0 is the carrier frequency.

$$f(t) = e^{-\frac{1}{2}(\frac{t}{\tau_p})^2} \sin \omega_0 t \tag{9}$$

Sometimes, people are also interested in solving the EM problem at one frequency. In this case, we can use the finite difference method frequency domain (FDFD) method and solve large equation system. We can also still use the FDTD



(a) Time Domain



(b) Frequency Domain

Fig. 2: Time domain (a) and frequency domain (b) profile of the four pulses introduced during the lectures. (for sine function, if we may ignore the side lopes caused by time domain aperiodic cutoff)

with a time-harmonic source to achieve this goal. However, a direct use of $\sin \omega_0 t$ will introduce numerical error because of the quick jump along the time. Thus, a taper function is multiplied with the time harmonic sine function as depicted in Equation 10 and Fig. 2.

$$f(t) = \left[1 - e^{-\frac{t}{\tau_p}}\right] \sin \omega_0 t \tag{10}$$

In this project, we set $f_0 = 100$ MHz as the main frequency of the passband simulation and single frequency simulation and the maximum frequency of the baseband simulation (set $\tau_p = \frac{3}{\omega_0}$).

C. PML formulation

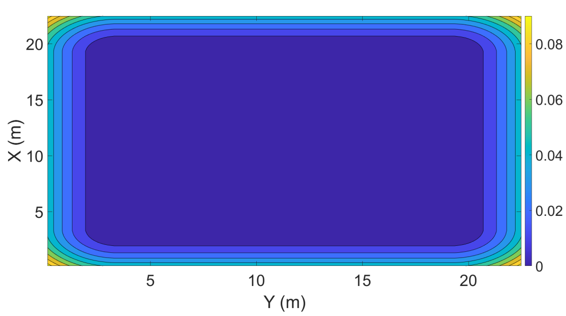


Fig. 3: A rough visualization of PML with specific width.

For purely simulation purposes, we can use PML to truncate the simulation domain in Cartesian coordinate. The perfectly matched layer (PML) as a material absorbing boundary condition is equivalent to a uniaxial, anisotropic, dispersive, fictitious medium that can absorb both electric and magnetic field. For a two dimensional domain, four perfectly matched interfaces (PMIs) are needed to build a is an interface between two half spaces, one of which is lossy, but the interface does not reflect a plane wave for all frequencies, all types of polarizations and all angles of incidence. The loss of the wave in the lossy half space is introduced only in the direction normal to the interface. As such, PML allows us to simulate open-region problems effectively without much error introduced. Together with other benefits mentioned in Section I, PML has been considered as the most important development of FDTD after the initial proposition of FDTD itself and recognized as an optimal absorbing boundary condition for numerical simulation of partial differential equations.

As we have mentioned in Section I, the FDTD with PML can be alternatively seen as the FDTD of a modified form of Maxwell's equations based on stretched coordinates. If we define the stretched del operator as

$$\nabla_s = \hat{x}\frac{1}{s_x}\frac{\partial}{\partial x} + \hat{y}\frac{1}{s_y}\frac{\partial}{\partial y} + \hat{z}\frac{1}{s_z}\frac{\partial}{\partial z} \tag{11}$$

where the stretch factors in s_x, s_y and s_z are complex numbers. To make reflection coefficient independent of frequency and achieve the good performance of PML in time domain simulation, we can choose the imaginary part of them, which is related to attenuation, to be frequency dependent in the first place, as shown in Equation (12), where σ and ϵ are respectively the artificial conductivity and permittivity.

$$\begin{aligned} s_x &= 1 - j\frac{\sigma_x}{\omega\epsilon} \\ s_y &= 1 - j\frac{\sigma_y}{\omega\epsilon} \\ s_z &= 1 - j\frac{\sigma_z}{\omega\epsilon} \end{aligned} \tag{12}$$

To derive time-stepping formula, we would like to write the Maxwell's equations in time domain by using the relationship $\frac{\partial}{\partial t} \leftrightarrow j\omega$ which is a common technique used for linking time derivative in time domain and to angular frequency in frequency domain when deriving formulas in finite difference methods. Let us only consider TM polarization in two-dimensional case, where $\boldsymbol{\mathcal{E}} = \hat{z}\mathcal{E}_z$ and $\boldsymbol{\mathcal{H}} = \hat{x}\mathcal{H}_x + \hat{y}\mathcal{H}_y$. The modified time domain Maxwell equations become

$$\frac{\partial \mathcal{E}_z}{\partial x} = \mu\frac{\partial \mathcal{H}_y}{\partial t} + \frac{\sigma_x \mu}{\epsilon}\mathcal{H}_y \tag{13}$$

$$\frac{\partial \mathcal{E}_z}{\partial y} = -\mu\frac{\partial \mathcal{H}_x}{\partial t} - \frac{\sigma_y \mu}{\epsilon}\mathcal{H}_x \tag{14}$$

$$\frac{\partial \mathcal{H}_y}{\partial x} = \epsilon\frac{\partial \mathcal{E}_{sx,z}}{\partial t} + \sigma_x \mathcal{E}_{sx,z} \tag{15}$$

$$\frac{\partial \mathcal{H}_x}{\partial y} = -\epsilon\frac{\partial \mathcal{E}_{sy,z}}{\partial t} - \sigma_y \mathcal{E}_{sy,z} \tag{16}$$

where $\mathcal{E}_{sx,z} + \mathcal{E}_{sy,z} = \mathcal{E}_z$. Applying finite difference discretization using Yee grid to the Equation (13), (14),

$$\mathcal{H}_x^{n+\frac{1}{2}}(i,j+\frac{1}{2}) = \frac{1}{\frac{\epsilon_{i,j+\frac{1}{2}}}{\Delta t}+\frac{\sigma_{y,i,j+\frac{1}{2}}}{2}}\left\{\left[\frac{\epsilon_{i,j+\frac{1}{2}}}{\Delta t}-\frac{\sigma_{y,i,j+\frac{1}{2}}}{2}\right]\mathcal{H}_x^{n-\frac{1}{2}}(i,j+\frac{1}{2})-\frac{\Delta t}{\mu\Delta y}\left[\mathcal{E}_z^n(i,j+1)-\mathcal{E}_z^n(i,j)\right]\right\} \tag{17}$$

$$\mathcal{H}_y^{n+\frac{1}{2}}(i+\frac{1}{2},j) = \frac{1}{\frac{\epsilon_{i+\frac{1}{2},j}}{\Delta t}+\frac{\sigma_{x,i+\frac{1}{2},j}}{2}}\left\{\left[\frac{\epsilon_{i+\frac{1}{2},j}}{\Delta t}-\frac{\sigma_{x,i+\frac{1}{2},j}}{2}\right]\mathcal{H}_y^{n-\frac{1}{2}}(i+\frac{1}{2},j)+\frac{\Delta t}{\mu\Delta x}\left[\mathcal{E}_z^n(i+1,j)-\mathcal{E}_z^n(i,j)\right]\right\} \tag{18}$$

$$\mathcal{E}_{sx,z}^{n+1}(i,j) = \frac{1}{\frac{\epsilon_{i,j}}{\Delta t}+\frac{\sigma_{x,i,j}}{2}}\left\{\left[\frac{\epsilon_{i,j}}{\Delta t}-\frac{\sigma_{x,i,j}}{2}\right]\mathcal{E}_{sx,z}^n(i,j)+\frac{1}{\Delta x}\left[\mathcal{H}_y^{n+\frac{1}{2}}(i+\frac{1}{2},j)-\mathcal{H}_y^{n+\frac{1}{2}}(i-\frac{1}{2},j)\right]\right\} \tag{19}$$

$$\mathcal{E}_{sy,z}^{n+1}(i,j) = \frac{1}{\frac{\epsilon_{i,j}}{\Delta t}+\frac{\sigma_{y,i,j}}{2}}\left\{\left[\frac{\epsilon_{i,j}}{\Delta t}-\frac{\sigma_{y,i,j}}{2}\right]\mathcal{E}_{sy,z}^n(i,j)-\frac{1}{\Delta y}\left[\mathcal{H}_x^{n+\frac{1}{2}}(i,j+\frac{1}{2})-\mathcal{H}_x^{n+\frac{1}{2}}(i,j-\frac{1}{2})\right]\right\} \tag{20}$$

D. Position of the Source and Construction of the Object

In this problem, we choose to place the current source which can be viewed as an dipole antenna at the middle grid point of the simulation domain, which keeps some distance away from the four PMIs to avoid nonphysical reflection caused by near grazing incident angle.

As for the object, we choose to use a perfect electric conducting sheet with one slot. A perfect conductor or perfect electric conductor (PEC) is an idealized material exhibiting infinite electrical conductivity [8]. One method for modeling a PEC in a TM grid, if an Ez node falls within the PEC, it is set to zero. Here, according to the definition, we set the conductivity of the conductor sheet to a very large number ($\sigma_{sheet} = 10^{12}$) for simplicity of implementation, which is also beneficial to SIMD stencil operation. In addition, PEC boundary condition ($\mathcal{E}_{\parallel} = 0$ and $\mathcal{H}_{\perp} = 0$) [9] is manually enforced at the boundary of the object as shown in Fig. 1.

III. Results

In this section, snapshots of simulation result are presented, including the snapshots of electric field ($\mathcal{E}_z$) and magnetic field ($\mathcal{H}_x$ and $\mathcal{H}_y$) at different time steps and the electric field in both time and frequency domain at the observation point.

A. FDTD simulation result

In this project, we use the Neumann Pulse to simulate the time domain electromagnetic propagation problem with a frequency range of 0 to 0.1 GHz. The size of Yee grid is chosen to be $\Delta x = \Delta y = \frac{\lambda_{min}}{30}$ and the size of time step is chosen to be $\Delta t = \frac{1}{c\sqrt{\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}}}$ according to the empirical value and stability condition described in [1]. The width of PML is set to be 64 around the boundary and total size of Yee grid is set to be 2048 (physically 204.8 m in both height and width). 3000 time steps in total are simulated, which have an approximate time span of 707.2 ns. To have a better visual effect of the fields, the respective fields are renormalized according to the maximum (l_∞ norm) field value of the current simulation time step. The results are shown in Fig. (4), (5) and (6).

B. Field at the Observing Point

To show how we can get frequency information from time domain calculation, we record the all the value of electric field $\mathcal{E}_z$ at a specific point at the right of the slot. The time domain profile is recorded as shown in Fig. 7 (a). Frequency domain information can be obtained after doing fast Fourier transform (FFT) to the time domain data. We can tell from the frequency information that the energy at the right side of the sheet is more focused on higher frequency range than that of the source, as shown in Fig. 2 (b). This can be explained by inspecting the physical size of the aperture on the sheet, which is about 8.09 meter in diameter (width). If we see this size as the wavelength of a time harmonic wave, it corresponds to a frequency of 0.037 GHz. The harmonic components of the radiating Neumann pulse whose frequency is lower than this is less likely to pass through the slot (aperture).

C. Time Consumption Comparison between CPU and GPU computation.

Compare with MPI cluster parallel programming that needs programmer to partition the input and output data into cluster nodes by domain decomposition, CUDA GPU provides shared memory for parallel execution in GPU can achieve significant speedup over sequential execution on a single CPU core if the problem is suitable for parallel execution. For a time domain radiating problem of a size of 3000 time steps and 2048*2048 Yee grid. GPU calculation interface provided by Matlab 2020b can save about 27.3% of time that is needed by CPU implementation on a personal laptop with an Intel i7-8650U CPU and a Nvidia GeForce GTX 1050 GPU. This is because GPU contains large number of threads and is good at single program multiple data (SPMD) tasks.

IV. Conclusion

In the first project of ECE 540, we implement an modified FDTD algorithm based on the stretched coordinate derivation of PML. A PEC object is constructed through modifying the material properties of the discretized grid and enforcing the boundary conditions. In addition, the comparison between Matlab implementation with GPU acceleration and the corresponding CPU version shows the potential that GPU

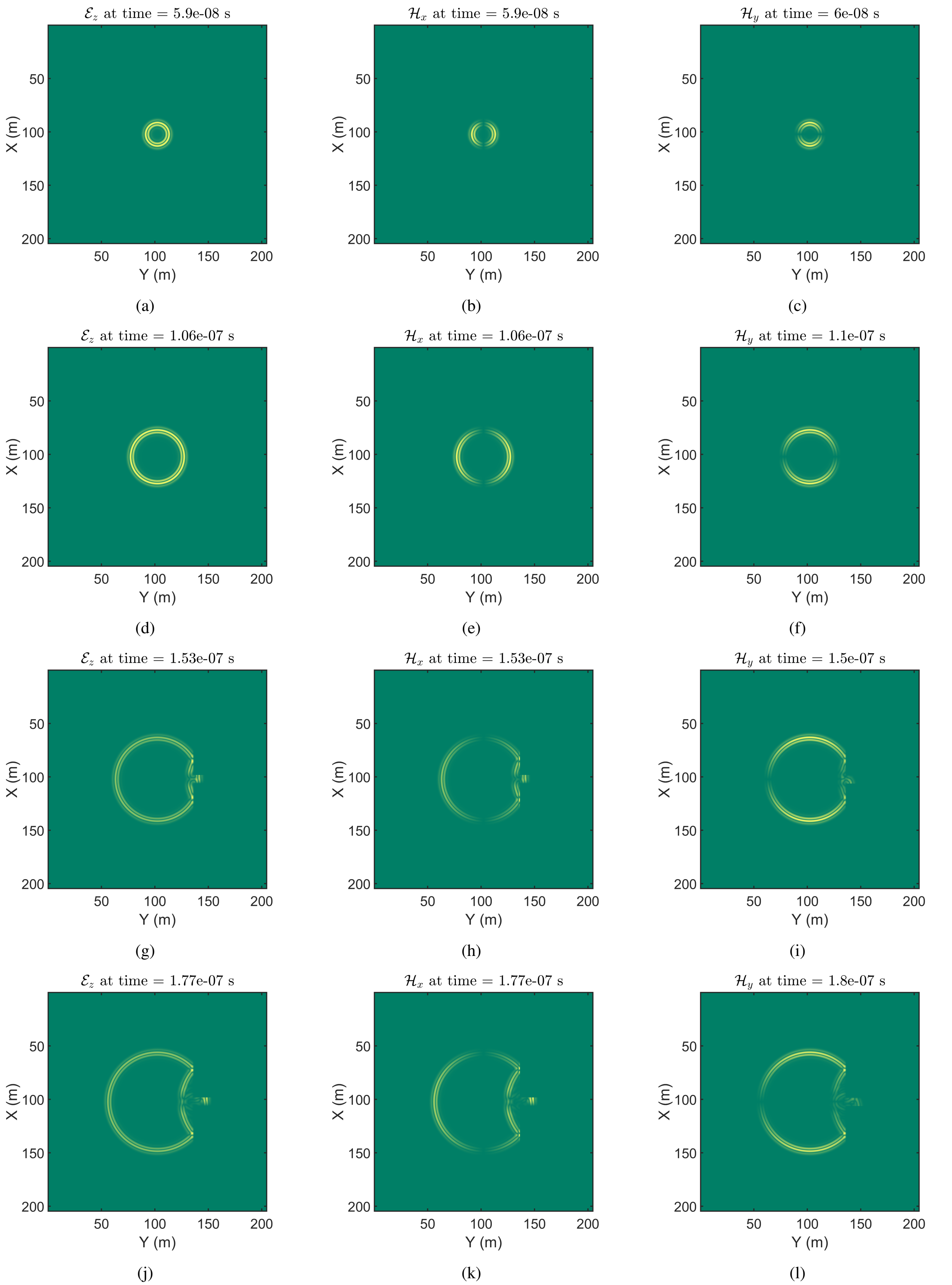


Fig. 4: Snapshots of the electric field and magnetic field intensity (absolute value) at different time steps. (Continue)

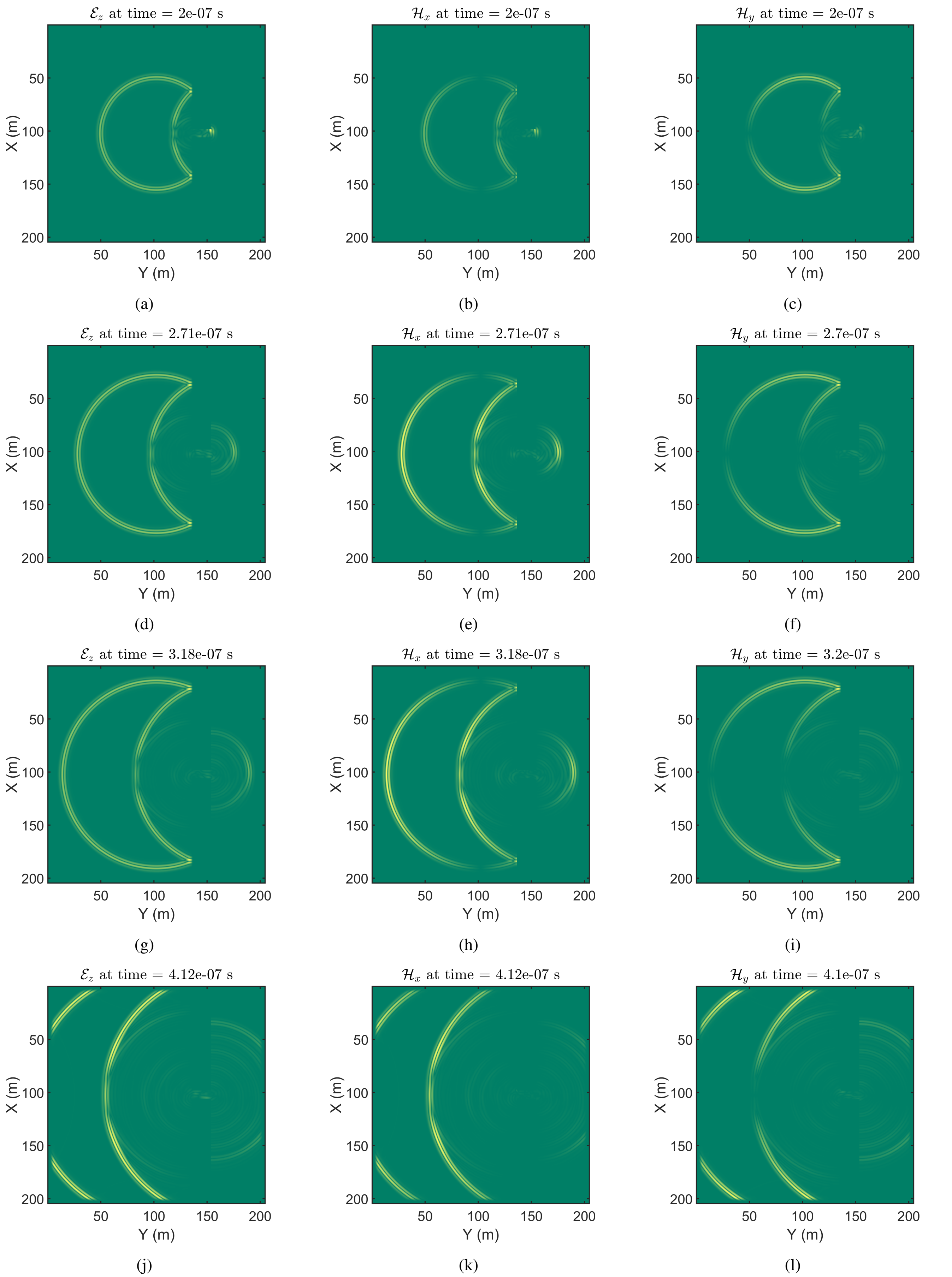
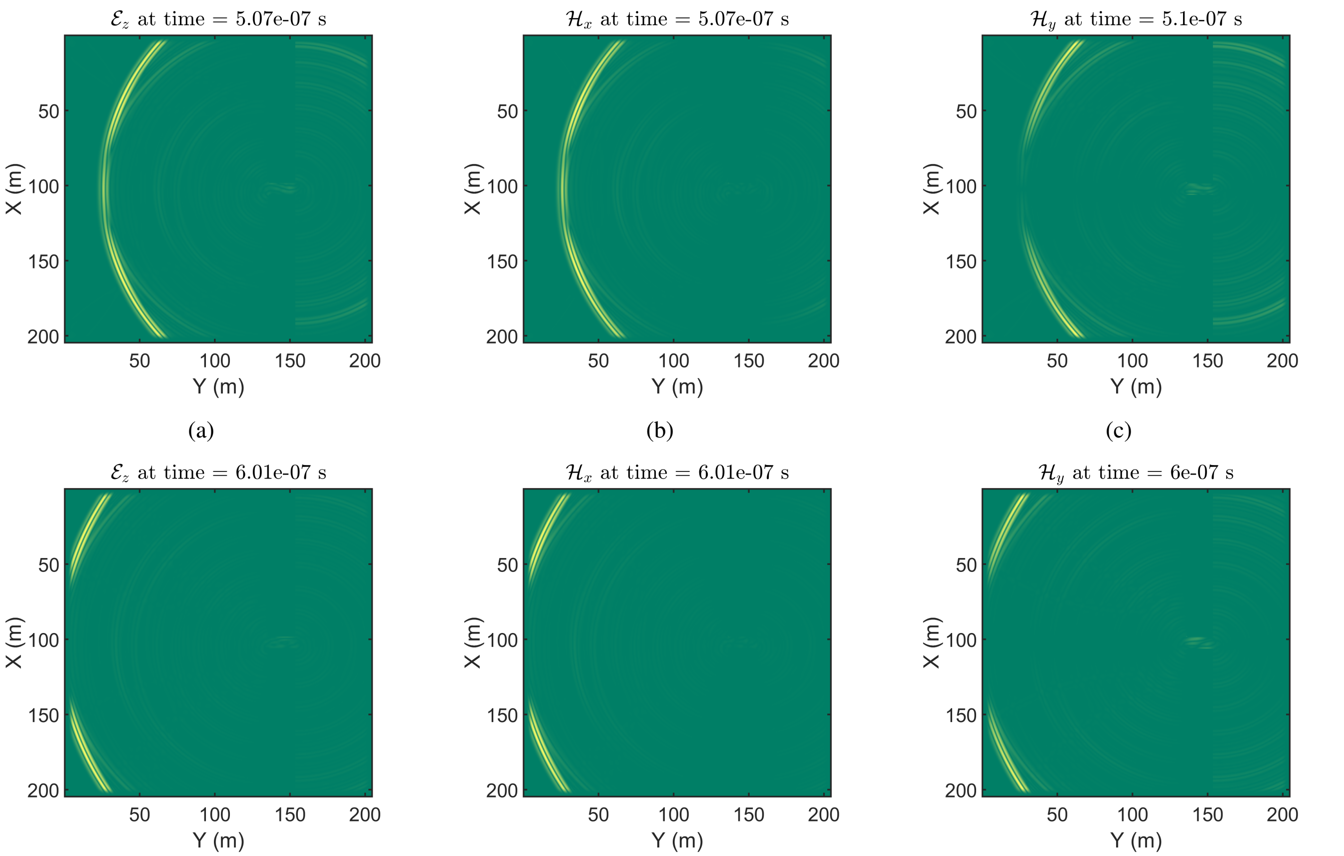


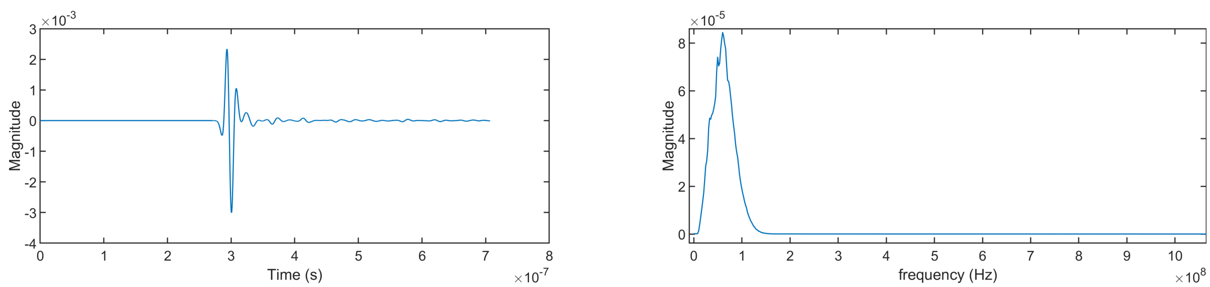
Fig. 5: Snapshots of the electric field and magnetic field intensity (absolute value) at different time steps. (Continue)



(a) (b) (c)

(d) (e) (f)

Fig. 6: Snapshots of the electric field and magnetic field intensity (absolute value) at different time steps. (Continue)



(a) (b)

Fig. 7: $\mathcal{E}_z$ in time and frequency domain (magnitude only) at observation point

can speed up FDTD calculation when the domain size is large even without special optimization.

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TABLE I: Time consumption of 2D-FDTD problem of size (number of time steps, grid number along x direction, grid number along y direction) implemented on different devices

	(3000, 256, 256)	(3000, 1024, 1024)	(3000, 2048, 2048)	(3000, 4096, 4096)
GPU(seconds)	46.24	518.78	2029.43	8226.00
CPU(seconds)	13.40	679.03	2791.29	10050.33

magnetic waves," *Journal of computational physics*, vol. 114, no. 2, pp. 185–200, 1994.
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