Probability & Statistics

SE30

Portfolio

Table of Content

1. Sets ~~→~~ notation

A set is a collection of elements

- explicit: • coin $\{heads, tails\}$
  • bits $\{0, 1\}$
  • die $\{1, 2, 3, 4, 5, 6\}$
- implicit: • digits $\{0, 1, \dots, 9\}$
  • letters $\{a, b, \dots, z\}$
  • days $\{Monday, \dots, Sunday\}$
- descriptive $\{$four-letter words$\}$

more compact
ambiguous ⚠

▲ Integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ $\mathbb{Z}$
▲ Naturals $\{0, 1, 2, \dots\}$ $\mathbb{N}$
▲ Positives $\{1, 2, 3, \dots\}$ $\mathbb{P}$
▲ Rationals $\{$integer ratios m/n, $n \neq 0\}$ $\mathbb{Q}$
▲ Reals $\{$"everything from above"$\}$ $\mathbb{R}$

Sets in UPPERCASE, elements in lowercase

$x \in A \Rightarrow x$ is a member of A e.g: $0 \in \{0, 1\}$

$A \ni x \Rightarrow A$ contains x e.g: $\mathbb{R} \ni \pi$

$x \notin A \Rightarrow x$ does not belong to A e.g: $2 \notin \{0, 1\}$

$A \not\ni x \Rightarrow A$ does not contain x e.g: $\{0, 1\} \not\ni x$

▸ order in sets doesn't matter

Special Sets

- empty set - contains no elements - $\emptyset$ or $\{\}$ - $\forall x, x \notin \emptyset$
- universal set - contains all possible elements - Ω - $\forall x, x \in \Omega$
  → changes depending on the application

Sets within sets

e.g. $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$ $\mathbb{P} = \{x \in \mathbb{N} \mid x > 0\}$

$\{x \in \mathbb{R} \mid x^2 \geq 0\} = \mathbb{R}$ $\{x \in \mathbb{R} \mid x^2 = 0\} = \{0\}$ → "Singleton"

Intervals

$[n] = \{1, \dots, n\}$

$[a,b] = \{x \in \overset{\mathbb{Z}}{\cancel{\mathbb{R}}} \mid a \le x \le b\}$ $\quad [3,5] = \{3,4,5\}$

$(a,b) = \{x \in \overset{\mathbb{Z}}{\cancel{\mathbb{R}}} \mid a < x < b\}$ $\quad (3,5) = \{4\}$

$[a,b) = \{x \in \overset{\mathbb{Z}}{\cancel{\mathbb{R}}} \mid a \le x < b\}$ $\quad [3,5) = \{3,4\}$

$(a,b] = \{x \in \overset{\mathbb{Z}}{\cancel{\mathbb{R}}} \mid a < x \le b\}$ $\quad (3,5] = \{4,5\}$

e.g. $[3,2] = [3,3) = (3,3] = \emptyset$

Set of Multiples

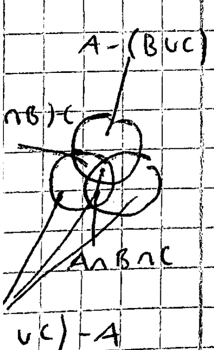
- $m \in \mathbb{Z}$, ${}_m\mathbb{Z} \overset{def}{=} \{i \in \mathbb{Z} : m \mid i\}$ → integer multiples of m
  - e.g.: ${}_2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\} \overset{def}{=} \mathbb{E}$ → even numbers
- $m \in \mathbb{Z}, n \in \mathbb{P}$ ${}_m[n] \overset{def}{=} \{i \in [n] : m \mid i\}$ → multiples of m in $\{1, \dots, n\}$
  - e.g: ${}_3[13] = \{3, 6, 9, 12\}$

Identities - One Set

- Identity $A \cap \Omega = A$ $\quad A \cup \Omega = \Omega$
- Universal bound $A \cap \emptyset = \emptyset$ $\quad A \cup \emptyset = A$
- Idempotent $A \cap A = A$ $\quad A \cup A = A$
- Complement $A \cap A^c = \emptyset$ $\quad A \cup A^c = \Omega$

Two and Three Sets - laws

- commutative $A \cap B = B \cap A$ $\quad A \cup B = B \cup A$
- associative $(A \cap B) \cap C = A \cap (B \cap C)$
- distributive $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- de morgan $(A \cap B)^c = A^c \cup B^c$



Set Difference

A - B is the set of elements in A but not in B

! Also $A \setminus B$

Symmetric difference

Cartesian Products

The cartesian product of A and B is the set $A \times B$ of ordered pairs (a,b) where $a \in A$ and $b \in B$

$A \times B = \{(a,b) : a \in A, \ b \in B\}$

$A \times A$ denoted A^2

$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ Cartesian Plane

$A, B \subseteq \mathbb{R}$ $\quad A \times B \subseteq \mathbb{R}^2$ Rectangle

e.g. $A = [0,2]$ $\quad B = [1,4]$

$A \times B = \{(x,y) : x \in [0,2], \ y \in [1,4]\}$

Discrete Sets

$$\{a,b\} \times \{1,2,3\} = \{(x,y) : x \in \{a,b\}, y \in \{1,2,3\}\}$$
$$= \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$$

	1	2	3
a	(a,1)	(a,2)	(a,3)
b	(b,1)	(b,2)	(b,3)

$A \times \emptyset = \emptyset \times A = \emptyset$

$A \times (B \cup C) = A \times B \cup A \times C$

$A \times (B \cap C) = A \times B \cap A \times C$

$A \times (B - C) = A \times B - A \times C$

Russell's Paradox - sets in sets

"This sentence is a lie"

Can a set belong to itself? $S \in S$?

eg. $N = \{$anything that is not Trump$\}$

$= \{0, \{1,2\}, \ldots, NT\}$ $\rightarrow NT \in NT$

$\rightarrow$ ⚠ infinite recursion

Define a set that cannot exist

$R = \{$sets that don't belong to themselves$\} = \{S : S \notin S\}$

$R \in R \rightarrow R \notin R$ ↯

$R \notin R \rightarrow R \in R$ ↯

2. Counting

Set size – number of elements in a set
denoted $|S|$

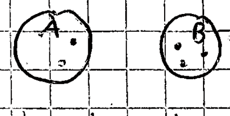
e.g. $|\{0,1\}| = 2$

$|\emptyset| = 0$

$|\mathbb{Z}| = \infty$

Disjoint Unions:

$|A| = 2$ $\quad |B| = 3$



$|A \cup B| = |A| + |B| = 2 + 3 = 5$

$|\Omega| = |A| + |A^c|$

$|A^c| = |\Omega| - |A|$

e.g. $D = \{i \in [6] : 3 \mid i\} = \{3, 6\}$ $\quad |D| = 2$

$D^c = \{i \in [6] : 3 \nmid i\} = \{1, 2, 4, 5\}$ $\quad |D^c| = 4$

$|D^c| = 4 = 6 - 2 = |\Omega| - |D|$

General Unions:

$|A \cup B| \neq |A| + |B|$ $\quad |\{a\} \cup \{a\}| = |\{a\}| = 1 \neq 2$!

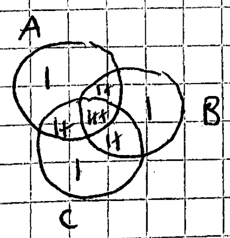
↳ (intersection is counted twice)

Can we determine $|A \cup B|$ in general?

$\rightarrow |A \cup B| = |A| + |B| - |A \cap B|$

Multiple Sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



n-sets

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \dots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

The size of a Cartesian Product is the product of the set sizes

$|A \times B| = |A| \times |B|$

Analogies between number and set operations

numbers	sets
addition	disjoint union
subtraction	complement
multiplication	cartesian product
exponents	? → cartesian power

Cartesian Powers

$A^2 = A \times A$ Cartesian square

$A^n = \underbrace{A \times A \times \dots \times A}_{n}$ n'th Cartesian power

$|A^n| = |A \times A \times \dots \times A| = |A| \times |A| \times \dots \times |A| = |A|^n$

application: subsets

The power set of S, denoted $\mathbb{P}(S)$, is the collection of all subsets of S

$\mathbb{P}(\{a,b\}) = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$

$|\mathbb{P}(S)| = ?$

binary strings of length $|S|$

→ 1-1 correspondence between $\mathbb{P}(S)$ and $\{0,1\}^{|S|}$

$\mathbb{P}(\{a,b\})$	a	b	$\{0,1\}^2$
$\{\}$	×	×	00
$\{b\}$	×	✓	01
$\{a\}$	✓	×	10
$\{a,b\}$	✓	✓	11

$\implies |\mathbb{P}(S)| = |\{0,1\}^{|S|}| = 2^{|S|}$

(The size of the power set is 2 to the power of size set)

Counting Variations

e.g. PIN (Personal Identification Number)

~~wha~~ how many 4-digit PINs are there?

$D: \{0, \dots, 9\}$

Power set → $\{\text{4-digit PINs}\} = D^4$

$|D^4| = |D|^4 = 10^4 = 10\,000$

Variable Length

3-5 digit PINs 314 2246 79380

$D = \{0, \dots, 9\}$ $\{PINs\} = D^3 \cup D^4 \cup D^5$

$\# PINs = |D^3 \cup D^4 \cup D^5| = |D^3| + |D^4| + |D^5|$

$= 10^3 + 10^4 + 10^5$

$= 1000 + 10\,000 + 100\,000$

$= 111\,000$

eg: PINs containing zero

$D = \{0, 1, \dots, 9\}$

$Z = \{0\}$ $\overline{Z} = \{1, 2, \dots, 9\}$ (complement of Z)

$x^n \triangleq x_1, \dots, x_n$ → n-digit sequence

$\exists Z = \{x^n \in D^n : \exists i \ x_i \in Z\}$ → {n-digit PINs containing 0}

$|\exists Z| = ?$

- 2 - Digits: Inclusion - Exclusion

$\exists Z = \{x_1 x_2 : \exists i \ x_i = 0\}$ 00 03 50 73

$Z_1 = \{x_1 x_2 : x_1 = 0\}$ 00 03 ~~50~~ ~~73~~ $|Z_1| = 10$

$Z_2 = \{x_1 x_2 : x_2 = 0\}$ 00 ~~03~~ 50 ~~73~~ $|Z_2| = 10$

$\exists Z = Z_1 \cup Z_2$ $Z_1 \cap Z_2 = \{00\}$

$|\exists Z| = |Z_1| + |Z_2| - |Z_1 \cap Z_2|$

$= 10 + 10 - 1$

$= \underline{\underline{19}}$

- 2 - Digits: Complement Rule

$\Omega = D^2$ → all 2-digit PINs

$\overline{\exists Z} = \{x_1 x_2 : \forall i \ x_i \neq 0\} = \overline{Z} \times \overline{Z}$ $\{1..9\} \times \{1, \dots, 9\}$

both digits nonzero

$|\overline{\exists Z}| = |\overline{Z} \times \overline{Z}| = |\overline{Z}|^2 = 9^2 = 81$

$|\exists Z| = |D^2| - |\overline{\exists Z}| = 100 - 81 = \underline{\underline{19}}$

exercise 1: How many 5-digit ternary strings are there without 4 consecutive 0s, 1s or 2s?

For example 01210 and 11211 are counted but 20000, 11112, and 22222 are excluded

$D = \{0, 1, 2\}$

$X = \{$ 5-digit ternary strings without 4 consecutives $\}$

$\overline{X_1} = \{ x_1 - x_4$ $(x_1 = x_2 = x_3 = x_4)$ are either 0s, 1s or 2s $\}$ 11111 22220 ~~02111~~ 0222

$\overline{X_2} = \{ x_2 - x_5$ are either 0s, 1s or 2s $\}$ 11111 22220 ~~02111~~ 0222

$|\overline{X_1}| = 9 \quad |\overline{X_2}| = 9$

$$\overline{X} = \overline{X_1} \cup \overline{X_2} \longrightarrow |\overline{X}| = |\overline{X_1}| \cup |\overline{X_2}|$$

$$= 9 + 9 - 3$$

$$= 15$$

$\Omega = D^5$

$$|X| = |D^5| - |\overline{X}| = 3^5 - 15 = 243 - 15 = \underline{\underline{228}}$$

exercise 2:

In the US, telephone numbers are 7-digit long. While real numbers have some restrictions, here we assume that all 7-digit sequences are possible, even 0000000.

a) "How many phone numbers start or end with two identical digits, e.g. 0012345, 1234511, or 2222222

$10 \cdot 10^5 = 10^6$ phone numbers start with two identical digits, and the same number end with two identical digits, furthermore $10^2 \cdot 10^3 = 10^5$ numbers start and end with two identical digits

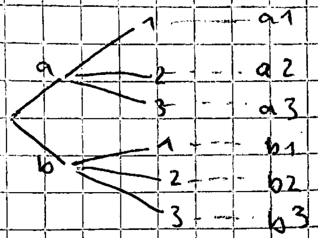
By inclusion-exclusion, the answer is $2 \cdot 10^6 - 10^5 = 1\,900\,000$

b) " " contain a substring of 5 consecutive digits
For example, 0045678, 2567892, or 0123456.

Let L, M and R be the sets of phone numbers whose 5 left, middle and right digits, respectively, are consecutive. The set of phone numbers with five consecutive digits is $L \cup M \cup R$. $|L| = |M| = |R| = 6 \cdot 10^2 = 600$, while $|L \cap M| = |M \cap R| = 5 \cdot 10 = 50$, and $|L \cap R| = |L \cap M \cap R| = 4$
By inclusion-exclusion, $|L \cup M \cup R| = 3 \cdot 600 - 2 \cdot 50 - 4 + 4 = \underline{\underline{1700}}$

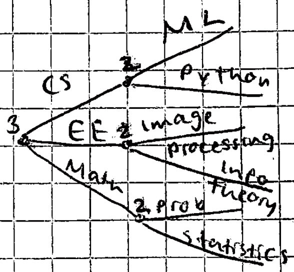
Counting Trees

Cartesian products as trees



$\{a, b\} \times \{1, 2, 3\}$

→ trees are more general

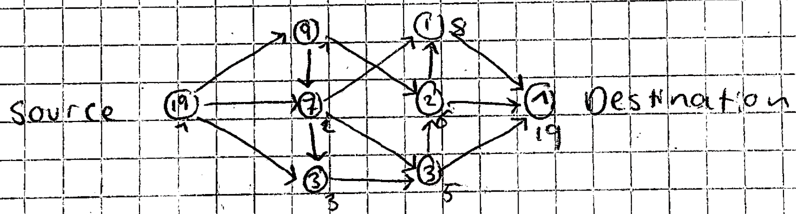


→ departments offer <u>different</u> courses
→ NOT cartesian product

> still each level, all degrees equal
→ # courses = $3 \times 2 = 6$

A tree can represent any set of sequence.

Paths from Source to Destination



3. Combinatorics

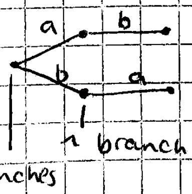
Permutations

A permutation is an ordering of a set of objects

<u>2 objects</u>

2 choices, 1 choice

a b
b a

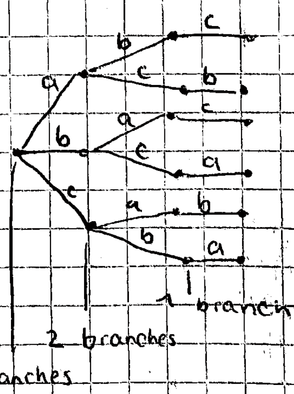


$2 \times 1 = \underline{2}$

<u>3 objects</u>

3 choices, 2 choices, 1 choice

a b c
a c b
b a c
b c a
c a b
c b a



$3 \times 2 \times 1 = \underline{6}$

permutations of n objects $= n \times (n-1) \times \ldots \times 2 \times 1$
$\stackrel{!}{=} n!$

n! can be defined recursively

$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

$\quad = n \cdot [(n-1) \cdot \ldots \cdot 2 \cdot 1]$

$\quad = n \cdot (n-1)! \qquad \forall n \geq 1 \qquad (\to 0! = 1$

e.g.: # anagrams of 5 distinct letters PEARS

SPEAR
...
EAPRS } $5! = 5 \times 4 \times 3 \times 2 \times 1 = \underline{\underline{120}}$

- constrained anagrams of PEARS: A, R stay adjacent in order

PARSE
...
SEPAR } permutations of P, E, AR, S $= 4! = \underline{\underline{24}}$

- A, R are adjacent in any order

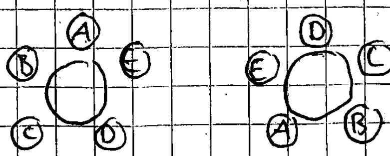
SPARE
...
RAESP } 2 orders 24 anagrams each $\xrightarrow{\text{Product rule}} 2 \times 24 = \underline{\underline{48}}$

- A, R are not adjacent

AESPR
...
SRPAE } $5! - 48 = 120 - 48 = 72$

- # ways 3 distinct boys & 2 distinct girls can stand in a raw
  - alternating boys and girls: $3! \times 2! = 6 \times 2 = 12$
  - unconstrained: $(3+2)! = 5! = 120$
  - boys together & girls together: $2 \times 3! \times 2! = 24$
  - unconstrained, but orientation doesn't matter: $5!/2 = 60$
- circular arrangements: # ways 5 people can sit at a round table

rotations matter $\neq$ $\to 5! = 120$



rotations don't matter $=$ $\to 5!/5 = 4! = 24$

Partial Permutations

orders of some of the n objects = ?

2-digit PINs
- any digits: 11 45 — 10×10
- distinct digits: 05 32 ~~33~~ — 10×9

3-letter words
- any letters: mom xyz — $26 \times 26 \times 26$
- distinct letters: abc ~~dad~~ — $26 \times 25 \times 24$ (we don't want to arrange all letters!)

→ # permutations of k out of n objects

$$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!} = n^{\underline{k}} \text{ or } P(n,k)$$

e.g. How many 2-permutations do we have for set $\{1,2,3,4\}$?

$$\rightarrow \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 12$$

permutations - arranging
combinations - selecting

Combinations

A subset of size k is a k-subset

$\binom{[n]}{k}$ - collection of k-subsets of $[n] = \{1, 2, \ldots, n\}$

k-subsets of an n-set — n-bit sequences with k 1's

e.g. $\binom{[3]}{1} = \{\{1\}, \{2\}, \{3\}\}$ — 100 010 001

$\binom{[3]}{2} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ — 110 101 011

$\binom{[4]}{2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

Number of n-Bit sequences with k 1's

$\binom{n}{k} = \left|\binom{[n]}{k}\right|$ - binomial coefficient

$\binom{3}{2} = \left|\binom{[3]}{2}\right| = |\{110, 101, 011\}| = 3$

$= \frac{3^{\underline{2}}}{2} = \frac{6}{2} = 3$

permutations		
1 2	110	110
1 3	101	101
2 1	110	
2 3	011	011
3 1	101	
3 2	011	

Calculating the Binomial Coefficients

$$\binom{n}{k} \stackrel{?}{=} \left| \binom{[n]}{k} \right| = ?$$

specify locations of the k 1's (in the n-bit sequence) in order

$\rightarrow$ 123, 531, 213

each location $\in [n]$

$\rightarrow$ # ordered locations $= n^k$

Every binary sequence with k 1's corresponds to $k!$ ordered locations

$\rightarrow$ 10101 $\leftrightarrow$ 1,3,5, 1,5,3, 3,1,5, 3,5,1, 5,1,3, 5,3,1

$$k! \binom{n}{k} = n^k \Rightarrow \binom{n}{k} = \frac{n^k}{k!} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{[3]}{1} = \begin{Bmatrix} 001 \\ 010 \\ 100 \end{Bmatrix} \qquad \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

Choose location of 1

$$\binom{[3]}{2} = \begin{Bmatrix} 011 \\ 101 \\ 110 \end{Bmatrix} \qquad \binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Choose location of 0

$$\binom{[4]}{2} = \begin{Bmatrix} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{Bmatrix} \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Simple $\binom{n}{k}$

$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$ All-zero n-bit sequence

$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$ All-one n-bit sequence

$\binom{n}{1} = \frac{n!}{(n-1)!} = n$ Choose location of single 1

$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$

1st location: n ways
2nd location: $n-1$ ways
each sequence chosen twice

! $\binom{n}{k} = \binom{n}{n-k}$ if $k > \frac{n}{2}$ calculate $\binom{n}{n-k}$

$$\binom{7}{5} = \binom{7}{2} = 21$$

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

n \ k	0	1	2	3	4	5
0	1	0				
1	1	1	0			
2	1	2	1	0		
3	1	3	3	1	0	
4	1	4	6	4	1	0
5	1	5	10	10	5	1

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0} = 2$$

$$\binom{3}{2} = \binom{2}{2} + \binom{2}{1} = 3$$

$$\binom{4}{3} = \binom{3}{3} + \binom{3}{2} = 4$$

$$\binom{5}{4} = \binom{4}{4} + \binom{4}{3} = 5$$

$$\binom{2}{2} - \binom{\cancel{2}1}{\cancel{2}1} = 0$$

$$\binom{2}{3} = \binom{1}{3} + \binom{1}{2}$$

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\boxed{(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \forall a,b \quad \forall n \geq 0}$$

↳ **Binomial Theorem** (gives binomial coefficients their name)

e.g.: Polynomial Coefficient

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

- coefficient of x^2 in $(1+x)^7$

$$(1+x)^7 = \sum_{i=0}^{7} \binom{7}{i} x^i \qquad \binom{7}{2} = 21$$

- coefficient of x^3 in $(3+2x)^5$

$$(3+2x)^5 = \sum_{i=0}^{5} \binom{5}{i} 3^{5-i} (2x)^i \qquad \binom{5}{3} 3^2 \cdot 2^3 = 720$$

~~Multinomialhomial~~ Multinomial Coefficients

$k_1 + k_2 + k_3 = n$

$\# \{1,2,3\}$ sequences with $\begin{cases} k_1 \text{ } 1\text{'s} \\ k_2 \text{ } 2\text{'s} \\ k_3 \text{ } 3\text{'s} \end{cases}$

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1! \cdot \cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2! \cdot \underbrace{(n-k_1-k_2)!}_{k_3}}$$

! note: remaining spots are 3's

$\binom{n}{k_1}$: n-bit sequences with k_1 1's

$\binom{n-k_1}{k_2}$: $n-1$-bit sequences with k_2 2's

$$= \frac{n!}{k_1! \cdot k_2! \cdot k_3!}$$

$$\hat{=} \binom{n}{k_1, k_2, k_3}$$

Binomial: $\binom{n}{k, n-k} = \binom{n}{k}$

e.g.: # sequences over $\{1,2,3,4\}$

digit	1	2	3	4
# times	1	4	4	2

$$\binom{11}{1,4,4,2} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$$

Multinomial Theorem

$$(a_1 + a_2 + \ldots + a_m)^n = \sum_{\substack{k_1 + k_2 + \ldots + k_m = n \\ k_1, k_2, \ldots k_m \geq 0}} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} a_t^{k_t}$$

e.g. $(a+b+c)^2$

$$(a+b+c)^2 = \sum_{\substack{i+j+k=2 \\ i,j,k \geq 0}} \binom{2}{i,j,k} a^i b^j c^k$$

$$= \binom{2}{2,0,0} a^2 + \binom{2}{0,2,0} b^2 + \binom{2}{0,0,2} c^2$$

$$+ \binom{2}{1,1,0} ab + \binom{2}{1,0,1} ac + \binom{2}{0,1,1} bc$$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Sum of Multinomials

Recall binomial identity $\quad 2^n = \sum_{i=0}^{n} \binom{n}{i}$

similarly for multinomials

$$m^n = (1+1+\dots+1)^n = \sum_{\substack{k_1+\dots+k_m = n \\ k_1,\dots,k_m \geq 0}} \binom{n}{k_1,k_2,\dots,k_m}$$

$$3^2 = 9 = \underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2}$$

exercise: What is the coefficient of xy in the expansion of $(x+y+2)^4$?

$$\longrightarrow \binom{4}{1,1,2} x^1 \cdot y^1 \cdot 2^2$$

$$= \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 1\cdot 2\cdot 1} \cdot 4 \cdot xy = 12 \cdot 4\ xy = \underline{\underline{48\,xy}}$$

another example: bookkeeper rule

\# permutations of the word bookkeeper ?

- \# perms $b\, o_1 o_2 k_1 k_2 e_1 e_2 p\, e_3 r = 10!$

  map perm $o_1 b e_1 o_2 k_1 r k_2 e_2 p e_3$ to

  $o\ b\, e\ o\ k\, r\, k\, e\, p\, e$

  2 o's, 2 k's, 3 e's $\hat{=}\ 2! \cdot 2! \cdot 3! \Rightarrow \boxed{\frac{10!}{2!\,2!\,3!}}$

$\Rightarrow$ \# permutations of length-n word with n_1 a's, n_2 b's, ..., n_k z's

$$\Rightarrow \frac{n!}{n_1!\,n_2!\dots n_k!} = \binom{n}{n_1,n_2,\dots,n_k}$$

What is the coefficient of BA^3N^2 in the expansion of $(B+A+N)^6$?

$$\rightarrow \binom{6}{1,3,2} = \frac{6!}{3!\cdot 2!} = 60$$

Stars and Bars - application

– counting sums!

ways to write 5 as a sum of 3 positive integers, when order matters
– to partition 5 items into 3 groups, when order matters

$3+1+1$	★ ★ ★ | ★ | ★	Addition	Partition
$2+2+1$	★ ★ | ★ ★ | ★	sum of 5	5 stars
$2+1+2$	★ ★ | ★ | ★ ★	3 pos. terms	3 consecutive star intervals
$1+3+1$	★ | ★ ★ ★ | ★	2 +'s	2 bars
$1+2+2$	★ | ★ ★ | ★ ★		4 inter-star gap
$1+1+3$	★ | ★ | ★ ★ ★		choose 2 of 4 gaps

= 6

$\rightarrow \binom{4}{2} = 6$

k Terms Adding to n

ways to write n as a sum of k positive integers, when order matters

n stars

n-1 gaps

★ ★ | ★ ★ | ★ | ★ ★

k-1 bars

$\rightarrow \boxed{\#\text{ sums} = \binom{n-1}{k-1}}$

[redacted]

	3	★ ★ ★
	$1+2$	★ | ★ ★
$n=3$	$2+1$	★ ★ | ★
	$1+1+1$	★ | ★ | ★

$\# = 4 = 2^2$ (putting bars)

general n

n

★ ★ ★ ★

bar or no bar

$\# = 2^{n-1}$

n as sum of $k \in [n]$: $\binom{n-1}{k-1}$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$$

Nonnegative Terms

ways to write n as sum of k nonnegative integers

2 as a sum of 3 nonnegatives

$2+0+0$	★ ★ | |
$0+2+0$	| ★ ★ |
$0+0+2$	| | ★ ★
$1+1+0$	★ | ★ |
$1+0+1$	★ | | ★
$0+1+1$	| ★ | ★

= 6

n = ★ k-1 |

ways to order 2 ★ and 2 |

$\rightarrow \binom{4}{2} = 6$

k nonnegatives adding to n

$\Longrightarrow \binom{n+k-1}{k-1}$

e.g.: 4-Letter Words combinations

\# 4-letter words when order matters $26^4 = 456\,976$

\# 4-letter words when order does not matter

evil = vile = live = ...

→ determind by composition: #a, #b, #c, ... #z

$\#a + \#b + \#c + \ldots + \#z = 4$

26 nonnegative terms $k = 26$ sum to $4 \rightarrow n = 4$

$$\binom{4+26-1}{26-1} = \binom{29}{25} = \binom{29}{4} = \underline{\underline{23{,}751}}$$

~~exercise 1~~: In how many different ways can you write 11 as a sum of 3 positive integers if order matters?

$\star \ \star \ \star \ \star \mid \star \ \star \ \star \mid \star \ \star \ \star \ \star$

11 stars with 3 partitions and therefore 2 bars
There are $11 - 1 = 10$ spaces to place the bars

$$\rightarrow \binom{10}{2} = \frac{10!}{2!\,(10-2)!} = \frac{10!}{2!\ 8!} = \frac{90}{2} = \underline{\underline{45}}$$

~~exercise 2~~: How many terms are there in the expansion of $(x+y+z)^{10}$?

Every coefficient corresponds to a term of the form $x^{a_1} y^{a_2} z^{a_3}$ with $a_1 + a_2 + a_3 = 10$, such that $a_i \geq 0$ (= nonneg

$\rightarrow n = 10, \quad k = 3$

$$\rightarrow \binom{n+k-1}{k-1} = \binom{10+2}{2} = \binom{12}{2} = \frac{12!}{2!\cdot 10!} = 6 \cdot 11 = \underline{\underline{66}}$$

exercise 3: 3 girls and 5 boys are to be assigned to eight seats in a row, with the stipulation that a girl sits in the second seat.
How many arrangements are there?

$$\binom{7}{2,5} = \frac{7!}{2!\,5!} = \frac{7 \cdot \cancel{6}^{3}}{\cancel{2}} = \underline{\underline{21}}$$

5. Probability.

Why? + background

The theory stems from gambling and games of chance, where the first attempt at some mathematical rigor is credited to Laplace.
It is of comparatively recent origin. It was Kolmogorov who axiomatized probability in his fundamental work in 1933.

Probability allows you to analyze chance events in a logically sound manner

Use cases:
- in medicine: to test drugs, to work out the chance that patients' develop side effects
- in finance: to assess the risk of investing in a company
  (the insurance idustry is based on the idea of risk the chance of your house burning down is quite small, but if it does happen you lose everything)
  → risk of fire vs cost of fire

...

Experiment & Sample Space

Possible experiment results are called outcomes
Set of possible outcomes is the sample space, denoted Ω

eg:

experiment	Ω
coin	$\{h, t\}$
die	$\{1, 2, \dots, 6\}$
gender	$\{m, f, o\}$
age	$\{\mathbb{N}$
temperature	$\mathbb{R}$

2 types of sample spaces

1. A finite or countably infinite sample space is discrete
   eg: $\{h, t\}$ $\mathbb{N}$ $\{\text{words}\}$

2. An uncountably infinite sample space is continuous
   eg: $\mathbb{R}$ $\{\text{temperatures}\}$

Random value of outcome is denoted by X ($x \in \Omega$)
The probability of an outcome $x \in \Omega$ denoted $P(x)$ or $P(X = x)$, is the fraction of times x will occur when the experiment is repeated many times

eg fair coin: As # experiments $\to \infty$, $P(X = h) = 1/2$

Probability Distribution Function

When viewing at the probabilities of the whole sample space, a pattern emerges

Coin: $P(h) = 1/2$ $P(t) = 1/2$
Die: $P(1) = 1/6$ ... $P(6) = 1/6$
Rain: $P(\text{rain}) = 10\%$ $P(\text{no rain}) = 90\%$

P maps outcomes in Ω to nonnegative values that sum to 1

$$P: \Omega \to \mathbb{R} \quad P(x) \geq 0 \quad \sum_{x \in \Omega} P(x) = 1$$

$\downarrow$

Probability distribution function

→ Sample space Ω and distribution P define the whole <u>probability space</u>.

Distribution Types:

Uniform probability spaces
> all outcomes are equally likely

$$\forall x \in \Omega \quad P(x) = p$$
$$1 = \sum_{x \in \Omega} P(x) = \sum_{x \in \Omega} p = |\Omega| \cdot p \quad \rightarrow \quad p = \frac{1}{|\Omega|}$$

> e.g: coin flip $\Omega = \{h, t\}$ $|\Omega| = 2$

$$P(h) = P(t) = \frac{1}{|\Omega|} = \frac{1}{2}$$

! In nature, nonuniform spaces abound (rain, illness, grades, ...)

Events:

Often we are interested in a set of possible outcomes: e.g:

- temperatur > 32 degrees
- stock will close higher
- pass a course

→ set of outcomes is a subset of Ω → event $\subseteq \Omega$

event	name
$\{1, \ldots, 6\}$	Ω
$\{2, 4, 6\}$	even
$\{1, 4\}$	square
$\{5, 6\}$	$>4, \geq 5$
$\{1, 2, 5\}$	$\{1, 2, 5\}$

complement →

event	name
$\{\}$	$\emptyset$
$\{1, 3, 5\}$	odd
$\{2, 3, 5, 6\}$	none square
$\{1, 2, 3, 4\}$	$\leq 4, < 5$
$\{3, 4, 6\}$	$\{3, 4, 6\}$

An event occurs if it contains the observed outcome

The probability of ~~o~~ the event occuring is $P(E)$

Fraction of experiments where E occurs, as # experiments grows

> ~~Relation of the probability of event to the probability of its elements~~

~~e.g.~~ → # times even occurs = sum of # times 2 and 4 occur

$$P(\text{Even}) = P(2) + P(4)$$

→ ~~P(E) = sum of fraction of times its elements occur~~

in uniform spaces:

$$P(E) = \sum_{x \in E} P(x) = \sum_{x \in E} \frac{1}{|\Omega|} = \frac{\sum_{x \in E} 1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

e.g. die $\Omega = \{1, 2, 3, 4, 5, 6\}$ $|\Omega| = 6$

event	set	|Event|	$P(\text{Event}) = \frac{\|\text{Event}\|}{6}$
even	$\{2, 4, 6\}$	3	$\frac{3}{6} = \frac{1}{2}$
squar	$\{1, 4\}$	2	$\frac{2}{6} = \frac{1}{3}$

If events A and B are disjoint, they are <u>mutually exclusive</u>

Repeated Experiments

Composite experiments

- Sample space is a cartesian product

independent repitition means all experiments are of same type but where each experiment's outcome is independent of previous or future outcomes

e.g.: two coins (fair, independent flips)

	coin 1: h	t
coin 2: h	1/4	1/4
t	1/4	1/4

$\Omega = \{hh, ht, th, tt\} = \{h, t\}^2$ Cartesian power

$|\Omega| = 2^2 = 4$

<u>events</u> 2 coins

- $P(\text{different outcomes}) = P(\{ht, th\}) = \frac{2}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$
- $P(\text{at least one } h) = P(\{ht, th, hh\}) = \frac{3}{|\Omega|} = \frac{3}{4}$

<u>3 coins</u> → $|\Omega| = 2^3 = 8$

- $P(\text{Alternating}) = P(\{hth, tht\}) = \frac{2}{8} = \frac{1}{4}$
- $P(\text{odd \# h's}) = P(\{htt, tht, tth, hhh\}) = \frac{4}{8} = \frac{1}{2}$

Sampling - two types: with or without replacement

- with replacement: - reuse selected element
  - outcomes can repeat
  - (like coin / dice) - experiments often independent
- without replacement: - do not reuse selected element
  - outcomes cannot repeat
  - (selecting sb. for a committee) - experiments dependent

difference largest for small +

e.g.: balls in a jar

> with replacement: yellow and blue ball in a jar

1. pick a ball → blue or yellow
2. replace it
3. pick again → blue or yellow

→ second selection from original set

1st ball \ 2nd ball	1/2 y	1/2 b
1/2 y	1/4	1/4
1/2 b	1/4	1/4

↳ uniform

$\Omega = \{yy, yb, by, bb\}$

$|\Omega| = 4$

$|\Omega| = |\Omega_1|^2$

> without replacement:

1. pick a ball
2. do not return it
3. pick second (i.e. remaining) ball

→ second selection from a subset

1st ball \ 2nd ball	y	b
$\frac{1}{2}$ y	0	1/2
$\frac{1}{2}$ b	1/2	0

$\Omega = \{yb, by\}$

$|\Omega| = 2$

$|\Omega| = |\Omega_1| \cdot (|\Omega_1| - 1)$

→ so far order mattered

When order does not matter

tuple of outcomes → set of outcomes

$(2,5)\ (5,2) \to \{2,5\}$ Event $\{(2,5), (5,2)\}$

e.g.: 2 dice

$P(\{1,2\}) = P(\{(1,2), (2,1)\}) = P(1,2) + P(2,1) = \frac{2}{36}$

$P(\{1,1\}) = P(1,1) = \frac{1}{36}$

↳ not uniform!

check $\sum$ Probabilities $= 1$

$\binom{6}{2} \cdot \frac{2}{36} + \binom{6}{1} \cdot \frac{1}{36} = \frac{5}{6} + \frac{1}{6} = 1$

e.g. 2: drawing 2 cards $\in \{1, \dots, 6\}$ without replacement

- order matters: $i \neq j$ $P(i,j) = 1/6 \cdot 1/5 = 1/30$
- doesn't matter: $P(\{1,2\}) = P(\{(1,2),(2,1)\}) = P(1,2) + P(2,1) = \frac{2}{30}$

$\{1,1\}$ cannot happen

$\hookrightarrow$ Uniform!

check $\binom{6}{2} \cdot \frac{2}{30} = 1$

exercise 1

What is the probability that two cards drawn from a standard deck without replacement have the same rank?

The number of ways to draw two cards from a specific rank is $\binom{4}{2} = 6 = |E|$ and $|\Omega| = \binom{52}{2}$

Hence the probability is $13 \cdot \frac{6}{\binom{52}{2}} = \frac{\cancel{13} \cdot \cancel{6}^{3}}{\cancel{26}_{2} \cdot 51} = \frac{3}{51} = \underline{\underline{\frac{1}{17}}}$

exercise 2 Find the probability that a five-card hand contains:

- ace of diamonds

The number of hands containing the ace of diamonds is $\binom{51}{4}$, corresponding to the choice of the remaining 4 cards from the other 51. Hence the probability is

$$\frac{\binom{51}{4}}{\binom{52}{5}} = \frac{\frac{51 \cdot 50 \cdot 49 \cdot 48}{24}}{\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2}}$$

$$= \frac{\cancel{51} \cdot \cancel{50}^{5} \cdot \cancel{49} \cdot \cancel{48}^{2}}{\cancel{24} \cdot 52 \cdot \cancel{51} \cdot \cancel{10} \cdot \cancel{49} \cdot \cancel{2}} = \underline{\frac{5}{52}}$$

- at least an ace

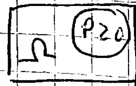
The number of ways to draw 5 cards without any ace is $\binom{48}{5}$. By the complement rule, the answer is $\frac{1 - \binom{48}{5}}{\binom{52}{5}} = 0{,}3412$

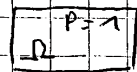
- at least a diamond

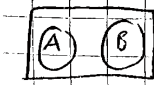
The number of ways to draw 5 cards without any diamond is $\binom{39}{5}$

$\rightarrow P(\text{at least diamond}) = \frac{1 - \binom{39}{5}}{\binom{52}{5}} = 0{,}7785$

Probability Axioms

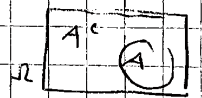
(≥) • non-negativity $P(A) \geq 0$ $\quad \forall A \; P(A) \geq 0$ 

(1) • unilarity $P(\Omega) = 1$ 

(+) • addition rule A, B disjoint $\rightarrow P(A \cup B) = P(A) + P(B)$ 

$A_1, A_2, \dots$ disjoint $\rightarrow P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$

Complement rule

$A \cup A^c = \Omega \qquad |A^c| = |\Omega| - |A|$ 

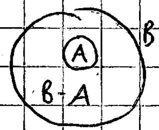
$P(A \cup A^c) = P(\Omega) = 1 = P(A) + P(A^c)$

$P(A^c) = 1 - P(A)$

Substraction rule - Nested sets

$A \subseteq \Omega \qquad P(A^c) = 1 - P(A) \qquad P(\Omega - A) = P(\Omega) - P(A)$

$\rightarrow A \subseteq B \qquad \longrightarrow \qquad P(B - A) = P(B) - P(A)$



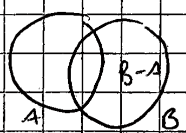
$B = A \cup (B - A)$

$P(B) = P(A \cup (B - A))$

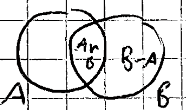
$= P(A) + P(B - A)$

$\Rightarrow \boxed{P(B - A) = P(B) - P(A)}$

Substraction rule - general sets



$B - A = B - (A \cap B)$



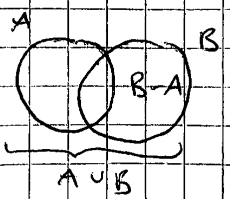
$A \cap B \subseteq B \rightarrow$ nested sets, can apply rule

$P(B - A) = P(B - (A \cap B)) = P(B) - P(A \cap B)$

Inclusion Exclusion

$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$A \cup B = A \cup B - A$



$P(A \cup B) = P(A \cup B - A)$

General substraction

$\overset{\text{addition rule}}{=} P(A) + P(B - A) \qquad P(B - A) = P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

Probability of Null Event

$\Omega = \emptyset \cup \Omega$

$P(\cancel{\Omega}) = P(\emptyset \cup \Omega) \overset{+}{=} P(\emptyset) + P(\cancel{\Omega})$

$\boxed{P(\emptyset) = 0}$

Probability inequalities

$0 \leq P(A) \leq 1$

$\geq$ $P(A) \geq 0$ show: $P(A) \leq 1$

$A \cup A^c = \Omega$

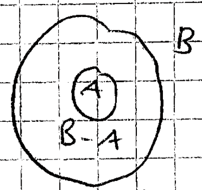
$P(A) \overset{\geq}{\leq} P(A) + P(A^c) \overset{+}{=} P(A \cup A^c) = P(\Omega) \overset{1}{=} 1$

$\rightarrow P(A) \leq 1$ Probability always between 0 & 1

Subset

$A \subseteq B$ $P(A) \leq P(B)$

$B = A \cup (B - A)$

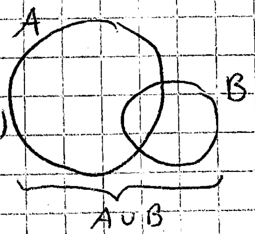


$P(B) = P(A \cup (B-A)) \overset{+}{=} P(A) + P(B-A) \geq P(A)$

$\rightarrow P(A) \geq P(B)$ ✓

Union

$\max(P(A), P(B)) \leq P(A \cup B) \leq P(A) + P(B)$



left $\leq$ $A, B \subset A \cup B$

$\rightarrow P(A), P(B) \leq P(A \cup B)$

right $\leq$ $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\geq 0}$

$\rightarrow P(A \cup B) \leq P(A) + P(B)$

e.g. The Linda Problem

As a student Linda was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which is more probable

A Active in feminist movement B - Bank teller

$P(B)$ vs. $P(B \cap A)$ $B \supseteq B \cap A$ $\boxed{P(B) \geq P(B \cap A)}$

5 Conditional Probability

E, F - events

$P(F|E)$ = Prob that F happens given that E happened

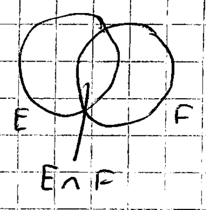
= Fraction of E occurences that F also occurs

e.g. fair die

- $P(2 | \text{Even}) = \frac{2}{5}$ $\quad \boxed{2}$ 1 3 6 4 $\boxed{2}$ 5 4 3 ...
- $P(4 | \geq 3) = P(4 | \{3, 4, 5, 6\}) = \frac{1}{4}$
- $P(4 | \leq 3) = P(4 | \{1, 2, 3\}) = 0$
- $P(\leq 2 | \leq 4) = P(\{1, 2\} | \{1, 2, 3, 4\}) = \frac{1}{2}$
- $P(\leq 2 | \geq 2) = P(\{1, 2\} | \{2, 3, 4, 5, 6\}) = \frac{1}{5}$

General Events - Uniform Spaces

$$P(F|E) = P(X \in F | X \in E)$$
$$= P(X \in E \cap X \in F | X \in E)$$
$$= P(X \in E \cap F | X \in E)$$
$$= \frac{|E \cap F|}{|E|}$$



e.g.: fair die

$$P(\text{Prime} | \text{Odd}) = P(\{2, 3, 5\} | \{1, 3, 5\})$$
$$= \frac{|\{2,3,5\} \cap \{1,3,5\}|}{|\{1,3,5\}|} = \frac{|\{3,5\}|}{|\{1,3,5\}|} = \underline{\underline{\frac{2}{3}}}$$

" " - General Spaces

↯ cannot use cardinality of sets since the elements have different probabilities

$$P(F|E) = \frac{n \cdot P(E \cap F)}{n \cdot P(E)}$$ ~~num~~ # elements in $E \cap F$

$$= \frac{P(E \cap F)}{P(E)}$$

e.g.: 4-Sided Die

side	1	2	3	4
Prob	0.1	0.2	0.3	0.4

$$P(\geq 2 | \leq 3) = \frac{P(\geq 2 \cap \leq 3)}{P(\leq 3)}$$
$$= \frac{P(\{2,3,4\} \cap \{1,2,3\})}{P(\{1,2,3\})}$$
$$= \frac{P(\{2,3\})}{P(\{1,2,3\})} = \frac{0.5}{0.6} = \frac{5}{6}$$

Product Rule

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

$\rightarrow$ $P(E \cap F) = P(E) \cdot P(F|E)$

↳ helps calculate regular probabilities

e.g.: ○ ○ ○ Probability of both red?
blue red red

R_1 - first ball red R_2 - second ball red

$$P(\text{both red}) = P(R_1 \cap R_2)$$
$$= P(R_1) \cdot P(R_2 | R_1)$$
$$= \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

General

$$P(E \cap F \cap G) = P((E \cap F) \cap G)$$
$$= P(E \cap F) \cdot P(G | E \cap F)$$
$$= P(E) \cdot P(F|E) \cdot P(G | E \cap F)$$

exercise:

~~Does~~ Let A and B be two positive-probability events.
Does $P(A|B) > P(A)$ imply $P(B|A) > P(B)$?

Yes $\rightarrow$ $P(A|B) = \frac{P(B, A)}{P(B)}$

$$P(B|A) = \frac{P(B, A)}{P(A)}$$

~~Hence~~ if $P(A|B) > P(A) \Leftrightarrow \frac{P(B,A)}{P(B)} > P(A) \Leftrightarrow P(B,A) > P(A) \cdot P$

$= \frac{P(B,A)}{P(A)} > P(B)$

$= P(B|A) > P(B)$

Conditional probabilities allow us to account for information we have about our system of interest. For instance, we might expect the prob. that it will rain tomorrow to be smaller than the prob. it will rain tomorrow given it is cloudy today. This latter probability is a conditional probability. Mathematically speaking, we are shrinking our sample space to a particular event.
So in the rain example, instead of looking at how often it rains on any day in general, our sample space consists of only those days for which the previous day was cloudy. We then determine how many of those days were rainy.

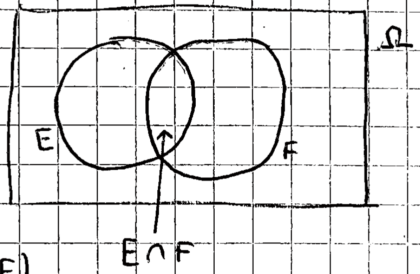
Independence:

Events E and F are independent ($E \perp F$) if occurrences of one does not change the probability that the other occurs

$$\rightarrow P(F|E) = P(F)$$

visually:

$P(F)$ - fraction of Ω that is F



$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

↳ (fraction of E that is $E \cap F$)

$$\rightarrow P(E \cap F) = P(E) \cdot P(F)$$

e.g. Die

Event	Set	Probability	intersection	set	Prob	=?	Product	Independ
Prime	$\{2,3,5\}$	$1/2$	Prime ∩ odd	$\{3,5\}$	$1/3$	$\neq$	$1/2 \cdot 1/2 = 1/4$	depend
odd	$\{1,3,5\}$	$1/2$	Prime ∩ square	$\{\ \}$	0	$\neq$	$1/6$	depend
square	$\{1,4\}$	$1/3$	Odd ∩ square	$\{1\}$	$1/6$	$=$	$1/6$	indepen

Coins

H_1	$\{h**\}$	$1/2$	$H_1 \cap H_2$	$\{hh*\}$	$1/4$	$=$	$1/2 \cdot 1/2 = 1/4$	independ
H_2	$\{*h*\}$	$1/2$	$H_2 \cap HH$	$\{hht, thh\}$	$1/4$	$\neq$	$1/2 \cdot 1/4 = \frac{1}{8}$	depend
HH	$\{hht, thh\}$	$1/4$	$H_1 \cap HH$	$\{hht\}$	$1/8$	$=$	$1/2 \cdot 1/4 = \frac{1}{8}$	indepen

Can two disjoint events be independent?
- Yes, but only if at least one of the two events has zero probability.

Sequential Probability

Product Rule: $P(E \cap F) = P(E) \cdot P(F|E)$

Sequential Selection ⓑ ⓡ ⓡ

draw 2 balls without replacement $\quad R_i$ - i'th ball is red

$P(\text{both red}) = P(R_1) \cdot P(R_2 | R_1) \qquad P(R_1) = 2/3$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(R_2 | R_1) = 1/2$

$= 2/3 \cdot 1/2 = \underline{\underline{1/3}}$

For 3 events: $P(E \cap F \cap G) = P((E \cap F) \cap G)$

$= P(E \cap F) \cdot P(G | E \cap F)$

$= P(E) \cdot P(F|E) \cdot P(G | E \cap F)$

e.g.: Odd Ball

$n-1$ red balls and one blue ball, pick n balls without replacement

$P(\text{last ball is blue}) = ?$ R_i - ith ball is red $R^i = R_1, R_2, \dots, R_i$

$$P(\text{last ball blue}) = P(R_1)\,P(R_2 \mid R_1)\,P(R_3 \mid R^2) \dots P(R_{n-1} \mid R^{n-2})$$

$$= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{n}$$

e.g.: The Birthday Paradox

How many people does it take so that 2 will likely share a birthday?

Choose n random integers each $\in \{1, \dots, 365\}$, with replacement

$B(n)$ - probability that two (or more) are the same

For which n does $B(n)$ exceed, say, $1/2$?

Set of all possible birthdays sequences $\Omega = \{1, 2, \dots, 365\}^n$ $|\Omega| = 365^n$

individual birthday prob uniform $\longrightarrow$ Ω uniform

B_n {sequences with repetition} - n people have birthday repetition

$$P(\text{repetition}) = \frac{|B_n|}{|\Omega|}$$

B_n^c n people, no two share a birthday

$$P(B_n^c) = \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365-n+1}{365}$$

$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$1 - x \leq e^{-x}$

$$\leq \prod_{i=1}^{n-1} e^{-\frac{i}{365}} = e^{\left(-\frac{1}{365} \cdot \sum_{i=1}^{n-1} i\right)} = e^{\left(-\frac{n(n-1)}{2 \cdot 365}\right)}$$

$$\approx e^{\left(-\frac{n^2}{2 \cdot 365}\right)}$$

$$e^{\left(-\frac{n^2}{2 \cdot 365}\right)} = 0{,}5 \quad \Big| \ln()$$

$$-\frac{n^2}{2 \cdot 365} = \ln(0{,}5) = -\ln(2)$$

$$n \approx \sqrt{-2 \cdot 365 \cdot \ln(0{,}5)} = 22{,}494$$ $\rightarrow$ Complement prob

$\rightarrow$ with at least 23 people it is 50% likely that at least two people share a birthday