CT-216

Introduction To Communication System

POLAR CODES

Lab Group: 6 Project Group: 3

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Polar Transform

The polar transformation matrix for $N = 2$ is defined as:

$$\mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The transformation matrix for N is defined as the Kronecker product of $\mathbf{G}_2$ with itself n times, where $n = \log_2 N$:

$$\mathbf{G}_N = \mathbf{G}_2^{\otimes n}$$

Theoritical Analysis

Definition: Binary Discrete Memoryless Channel

A B-DMS W is defined as a channel with input alphabets X, output alphabets Y, and a probability distribution $w(y|x)$ that describes the probability of receiving y given that x was transmitted.

The entropy $H(W)$ and the mutual information $I(W)$ of a channel W are related by the equation:

$$I(W) = 1 - H(W)$$

Assuming X follows $Bernoulli(q = 0.5)$ The capacity of a BSC(p) is given by:

$$I(W) = 1 - H_2(p)$$

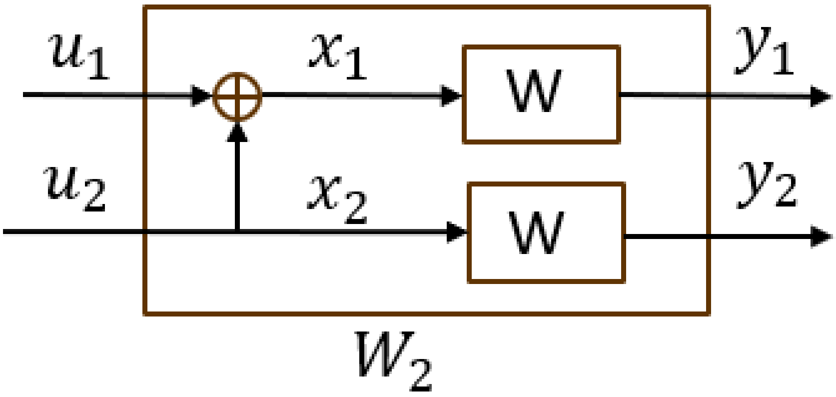
The capacity of a BEC(p) is given by:

$$I(W) = 1 - p$$

Channel Polarization

Now let's define two input-two output channel W_2 as below

$$W_2(y_1, y_2|u_1, u_2) = W(y_1|u_1 \oplus u_2) \cdot W(y_2|u_2)$$



This is basically a G_2 Polar Transform where,

$$u = [u_1 u_2]$$

$$Y = [y_1 y_2] = uG_2$$

Two virtual channels W^+ and W^- are defined as:

$$W^- : U_1 \to (Y_1, Y_2)$$

$$W^+ : U_2 \to (Y_1, Y_2, U_1)$$

If we consider BEC channel with erasure probability p then, the erasure probabilities for W^+ and W^- are given by:

$$\text{Erasure Probability for } W^+ = p^2$$
$$\text{Erasure Probability for } W^- = p(2 - p)$$

Assuming X follows $Bernoulli(q)$, then the information BEC is given by

$$I(W) = H_2(q)(1 - ErasureProbability)$$

So for W^+ and W^-, the information will be as follows:-

$$I(W^+) = H_2(q)(1 - p^2)$$

$$I(W^-) = H_2(q)(1 - 2p + p^2)$$

We can clearly see that the virtual channel W+ is performing better than the original channel W and the virtual channel W- is performing worse than the original channel W.

$$I(W^-) \le I(W) \le I(W^+)$$

Similarly for W4 we can define 4-virtual channels, where some channels will perform well while others will get worse :-

$$W_4(y_1, y_2, y_3,\ y_4 | u_1,\ u_2, u_3,\ u_4) = W_2(y_1,\ y_2 | u_1 \oplus\ u_2, u_3 \oplus\ u_4)\ W_2(y_3,\ y_4 | u_2, u_4)$$

For larger N, Recursively doing this Half of the channel becomes highly informative and half of the channel becomes information-less.

This relates with the Matthew Effect which is based on the concept that the rich gets richer and poor gets poorer. In our scenario, by performing polar coding, the channel is divided into two halves, which are "very good channels" and "very bad channels".
For block length N we have N bit-channels and all Ui are i.i.d so the conservation of entropy/information property holds for information.

So the information for W_1, W_2, $\ldots$, W_N is same

$$I(W_1) = I(W_2) = \ldots = I(W_N) = I(W)$$

$$\sum_{i=1}^{N} I(W_i) = N \cdot I(W)$$

For larger N fraction of bit-channels that are noise-less will be I(W) and the fraction of bit-channels that are highly-noisy will be $1 - I(W)$

So the Number of bit-channels that are noiseless is NI(W) and the number of bit-channels that are highly noisy are $N(1–I(W))$

As We are setting message positions which are highly informative so the number of message bits $K = NI(W)$

Rate is defined as Number of Message bits divided by Total Number bits

$$Rate = K/N$$

For large N,

$$\text{Rate} = \lim_{N\to\infty} \frac{\text{number of good channels}}{\text{Total Number of channels}}$$

$$= \lim_{N\to\infty} \frac{K}{N}$$

$$= \lim_{N\to\infty} \frac{N \cdot I(W)}{N}$$

$$= I(W)$$

So for larger N, polar codes arbitrarily achieve Shannon's Channel Capacity.

Conclusion

In conclusion as N grows large, polar codes arbitarily approach Shannon's channel capacity. This notable property positions them as leading contenders for achieving near-optimal communication performance, holding immense promise for advancing modern communication systems.

Bibliography

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