Local Class Field Theory

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I'll then talk about how these isomorphisms change as we vary K and L.

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Note first that G is cyclic with a canonical generator F_K (arithmetic Frobenius, sending $x \in \mathbb{F}_L$ to $x^{|\mathbb{F}_K|}$) (recall that G acts on L/K but also on $\mathbb{F}_L/\mathbb{F}_K$).

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Our formula for the fundamental class in this case: it's the cohomology class in $H^2(G, L^\times)$ associated to the 2-cocycle $G^2 \to L^\times$ sending (F^i, F^j) to 1 if $i + j < d$ and ϖ_K if $i + j \geq d$.

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In fact ϖ_K can be replaced by any unformiser of L (as $H^2(G, \mathcal{O}_L^\times) \cong 0$).

We have explicit isomorphisms $H^2(G, L^\times) \cong H^2(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$ and can just compute that $fc_{L/K}$ maps to 1.

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We constructed $fc_{L'/K} \in H^2(L'/K, L'^\times)$ as above, inflated this element to $H^2(M/K, M^\times)$ and then showed that the resulting element is in the image of $H^2(L/K, L^\times)$ and defined $fc_{L/K}$ to be the resulting element.

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As far as I can see, we don't really have a formula for $fc_{L/K}$, or even for a 2-cocycle representing it.

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We do have a formula for the 2-cocycle in $H^2(M/K, M^\times)$ (given $(a, b) \in Gal(M/K)^2$ just restrict them to $Gal(L'/K)$ and now they're powers F_K^i and F_K^j of F_K and now just ask if $i + j < [L : K]$ etc).

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So we have to be careful.

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Cyclicity of $H^2(L/K, L^\times)$ is a powerful new result.

For example, the previously slightly mysterious higher inf-res short exact sequence for $M/L/K$ all finite Galois is

$$0 \to H^2(L/K, L^\times) \to H^2(M/K, M^\times) \to H^2(M/L, M^\times)$$

and now we see that this is just isomorphic to

$$0 \to \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/ab\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$$

and in particular the last map is surjective (we didn't know this before).

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In fact we can do a bit better: if $M/L/K$ and M/K is finite Galois (but L/K might not be) then $res : H^2(M/K, M^\times) \to H^2(M/L, M^\times)$ is surjective.

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This is because $H^2(M/K, M^\times)$ is cyclic of order $[M : K]$ and $H^2(M/L, M^\times)$ is cyclic of order $[M : L]$, so the kernel of the map must have size at least $[L : K]$ by the first isomorphism theorem.

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So the original map has kernel of size $[L : K]$ and is hence surjective.

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Then the well-understood $H^2(L'/K, L'^\times)$ and the mysterious $H^2(L/K, L^\times)$ are both *the same cyclic subgroup* of order d of the big cyclic group $H^2(N/K, N^\times) \cong \mathbb{Z}/nd\mathbb{Z}$.

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Why? Because our construction of $fc_{L/K}$ using $M = L'L$ can all be regarded as going on within N, and $H^2(M/K, M^\times) \subseteq H^2(N/K, N^\times)$ by higher Inf-res.

Continuous cohomology interpretation

The inclusion $H^2(K^{nr}/K, K^{nr\times}) \subseteq H^2(K^{sep}/K, K^{sep\times})$ is actually an isomorphism, and our unramified fundamental classes give an explicit isomorphism $H^2(K^{nr}/K, K^{nr\times}) \cong \mathbb{Q}/\mathbb{Z}$.

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The embedding $H^2(L/K, L^\times) \subseteq H^2(K^{sep}/K, K^{sep\times})$ identifies $H^2(L/K, L^\times)$ with $\frac{1}{d}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$ and the fundamental class is $1/d$.

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We have a formula for both directions of the isomorphism $H^2(K^{nr}/K, K^{nr\times}) \cong \mathbb{Q}/\mathbb{Z}$ and also a formula for the isomorphism $H^2(K^{nr}/K, K^{nr\times}) \to H^2(K^{sep}/K, K^{sep\times})$.

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But we do not have a formula for the inverse $H^2(K^{sep}/K, K^{sep\times}) \to H^2(K^{nr}/K, K^{nr\times})$; the reason we know the map in the other direction is surjective is because of a nonconstructive argument (counting).

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Note: it was learning something about constructive mathematics that taught me that this was what was going on.

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Note: the actual "Artin maps" are the maps the other way, $K^\times \to G^{ab}$, but these are harder to work with.

Splitting modules

If G is a finite group and M is a (additive) G-module and $\sigma : G^2 \to M$ is a 2-cocycle, then we can make an extension $\mathit{split}(\sigma)$ of $\mathit{aug}(G) := \{f : G \to \mathbb{Z} \mid \sum_{x \in G} f(x) = 0\}$ by M which as a set is $M \times \mathit{aug}(G)$ and satisfies $g \bullet (m, f) = (g \bullet m + \sum_{x \in G} f(x)\sigma(g, x), g \bullet f)$.

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This extension has been explicitly made so that the image of the 2-cocycle $\sigma \in H^2(G, M)$ is a 2-coboundary in $H^2(G,$ *split*$(\sigma))$ coming from the 1-cochain sending x to $(x \bullet \sigma(1, 1), [x] - [1])$.

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This is a general construction which works for all 2-cocycles.

Splitting modules

If G is a finite group and M is a (additive) G-module and $\sigma : G^2 \to M$ is a 2-cocycle, then we can make an extension $split(\sigma)$ of $aug(G) := \{f : G \to \mathbb{Z} \mid \sum_{x \in G} f(x) = 0\}$ by M which as a set is $M \times aug(G)$ and satisfies $g \bullet (m, f) = (g \bullet m + \sum_{x \in G} f(x)\sigma(g, x), g \bullet f)$.

There's a short exact sequence $0 \to M \to split(\sigma) \to aug(G) \to 0$.

This extension has been explicitly made so that the image of the 2-cocycle $\sigma \in H^2(G, M)$ is a 2-coboundary in $H^2(G, split(\sigma))$ coming from the 1-cochain sending x to $(x \bullet \sigma(1, 1), [x] - [1])$.

This is a general construction which works for all 2-cocycles.

Note that $0 \to aug(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ is another short exact sequence, and the resulting "2-extension" $0 \to M \to split(\sigma) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ is the 2-extension corresponding to the 2-cocycle.

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Recall that this means: for all subgroups $S \subseteq G$ we have $H^i(S, \mathit{split}(\sigma)) \cong 0$ for all $i \geq 1$ (and $H^i_{Tate}(S, \mathit{split}(\sigma)) \cong 0$ for all $i \in \mathbb{Z}$).

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I'll just check that these vanish for $S = G$ but the general case is the same (and the proof is in the blueprint).

Splitting modules for fundamental classes

We have an exact sequence $H^1(G, L^\times) \to H^1(G, split(\sigma)) \to H^1(G, aug(G)) \to H^2(G, L^\times) \to H^2(G, split(\sigma)) \to H^2(G, aug(G))$.

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Finally $H^1(G, aug(G)) = H^0_{Tate}(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$ has size $|G|$ and $H^2(G, L^\times)$ has size $|G|$ (a profound result) so the surjection is an isomorphism and $H^1(G, split(\sigma)) = 0$ as well.

The Artin reciprocity isomorphism

The short exact sequence of G-modules $0 \to aug(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ shows that the boundary map $\delta : H^{-2}_{Tate}(G, \mathbb{Z}) \to H^{-1}_{Tate}(G, aug(G))$ an isomorphism.

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So we could expect a formula for $G^{ab} \to K^\times/N(L^\times)$ but it will depend on the cocycle (which we don't really know).

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I should say: it's not obvious (to me) that $\prod_{x \in G} \sigma(x, g) \in K^\times$ rather than $L^\times$ (but it might be a fun exercise).

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Note: some authors will take the inverse of our Artin maps everywhere, and send a uniformiser to the reciprocal of an arithmetic Frobenius (a so-called geometric Frobenius).

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Let's start with the first question.

Doing the calculation

Set-up: $M/L/K$ all finite Galois, and we want to compare $rec_{L/K} : Gal(L/K)^{ab} \to K^\times/N(L^\times)$ and $rec_{M/K} : Gal(M/K)^{ab} \to K^\times/N(M^\times)$.

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And this we can do.

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Then $H^2(X/K, X^\times)$ for $X = L, M, L', M'$ are all subgroups of the big cyclic group $H^2(N/K, N^\times)$.

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We know that $fc_{M/K}$ and $fc_{M'/K}$ are equal, as are $fc_{L/K}$ and $fc_{L'/K}$, when inflated to $H^2(N/K, N^\times)$.

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Let $M/L/K$ all be Galois, and choose some gigantic extension N/K containing M and L and also the unramified extensions M' and L' of K of the same degrees as M and L.

Then $H^2(X/K, X^\times)$ for $X = L, M, L', M'$ are all subgroups of the big cyclic group $H^2(N/K, N^\times)$.

We know that $fc_{M/K}$ and $fc_{M'/K}$ are equal, as are $fc_{L/K}$ and $fc_{L'/K}$, when inflated to $H^2(N/K, N^\times)$.

To to figure out the number n such that $n \times fc_{M/K} = fc_{L/K}$ we just need to do the same calculation for M'/K and L'/K, where we know all the cocycles.

The calculation

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We take the carry cocycle for L'/K, inflate to M'/K, and apply the explicit map sending a cocycle σ to $\sum_{x\in G} v(\sigma(x, F_K))$ which gives an isomorphism $H^2(X/K, X^\times) = \mathbb{Z}/[X:K]\mathbb{Z}$ for X any finite unramified extension.

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Upshot: $[M : L]fc_{M/K} = fc_{L/K}$.

Sanity check: $fc_{M/K}$ has order $[M : K]$ and multiplying by $[M : L]$ gives an element of order $[L : K]$.

Consequence for reciprocity map

It is now an easy exercise (actually I don't know how to do it) to check that the obvious map $Gal(M/K)^{ab} \to Gal(L/K)^{ab}$ and the obvious projection $K^\times/N(M^\times) \to K^\times/N(L^\times)$ and the two reciprocity maps $rec_{L/K} : Gal(L/K)^{ab} \to K^\times/N(L^\times)$ and $rec_{M/K} : Gal(M/K)^{ab} \to K^\times/N(M^\times)$ make a commutative square.

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This one I can do.

Restriction

This time the key question is: we have $M/L/K$ and this time we don't need to assume L/K is Galois.

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I claim that the restriction of $fc_{M/K}$ is *equal* to $fc_{M/L}$.

Granted this, it now really is a straightforward diagram chase to verify that the natural maps $Gal(M/L)^{ab} \to Gal(M/K)^{ab}$ and the norm map $N_{L/K} : L^\times/N_{M/L}(M^\times) \to K^\times/N_{M/K}(M^\times)$ make the diagram commute (this is lemma 73 in the blueprint).

What remains

What remains then, is to check that if $M/L/K$ with M/K Galois then $res : H^2(M/K, M^\times) \to H^2(M/L, M^\times)$ sends $fc_{M/K}$ to $fc_{M/L}$.

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Given $M/L/K$ as above, set $a = [L : K]$ and $b = [M : L]$, so $[M : K] = ab$. let N be the unramified extension of M of degree $[M : K] = ab$. Then N contains the unramified extensions of K and L of degree ab.

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Instead of everything going on in one large H^2, this time everything will be happening in two large H^2, namely $H^2(N/K, N^\times)$ and $H^2(N/L, N^\times)$. The restriction map from the bigger group to the smaller is a surjection (because all the groups are cyclic and after composing with corestriction the kernel has the right size).

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The diagram chase we will do has all horizontal maps as res.

Diagram chases

The first diagram which I claim commutes is

$$\begin{tikzcd}
H^2(N/K, N^\times) \arrow[r, "res"] & H^2(N/L, N^\times) \\
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This diagram commutes because all of the maps can be defined at cocycle level and the claim boils down to the assertion that the two maps $Gal(N/L) \to Gal(N/K) \to Gal(M/K)$ and $Gal(N/L) \to Gal(M/L) \to Gal(M/K)$ are equal.

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This means that to compute the image of $fc_{M/K}$ it suffices to compute the restriction to $H^2(N/L, N^\times)$ of its inflation in $H^2(N/K, N^\times)$.

Diagram chases

The second diagram which I claim commutes involves some new fields. Let M' be the unramified subextension of N/K of degree $[M : K]$ and let M'' the unramified subextension of N/L of degree $[M : K]$ (note: not degree $[M : L]$).

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I claim that even though M'/K might not be Galois, there is still a "restriction" map $Gal(M''/L) \to Gal(M'/K)$ which factors through $Aut(M''/K)$, the key point being that any K-algebra isomorphism $M'' \to M''$ will send M' to M' as M' is the unique unramified subextension of M'' of degree $[M:K]$.

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So we have

$$\begin{array}{ccc} H^2(N/K, N^\times) & \xrightarrow{res} & H^2(N/L, N^\times) \\ {\scriptstyle infl}\uparrow & & \uparrow{\scriptstyle infl} \\ H^2(M'/K, M'^\times) & \xrightarrow{''res''} & H^2(M''/L, M''^\times) \end{array}$$

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Note: Serre also uses this "res" trick in his proof of the infinite Galois group of this result.

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We are trying to understand what $res : H^2(M/K, M^\times) \to H^2(M/L, M^\times)$ does to $fc_{M/K}$, and we're hoping it sends it to $fc_{M/L}$.

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What about $fc_{M/L}$?

The diagram chase

Well $fc_{M/L} \in H^2(N/L, N^\times)$ is equal to $fc_{M'''/L}$ where M''' is the subfield of M''/L of degree equal to $[M : L]$ (recall that M''/L has degree $[M : K]$).

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Because "res" is a map into $H^2(M''/L, M''^\times)$, to locate $fc_{M/L}$ we need to figure out the image of $fc_{M''/L}$ in $H^2(M''/L, M''^\times)$.

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Summary of what we're about to do: both $res(fc_{M/K}) = \text{"}res\text{"}(fc_{M'/K})$ and $fc_{M/L} = fc_{M'''/L}$ can be regarded as elements of $H^2(M''/L, M''^\times)$ where M''/L is the unramified extension of degree $[M : K]$, and hence they're both multiples of $fc_{M''/L}$ (because this group is cyclic with $fc_{M''/L}$ as a generator)

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We have explicit formulae for cocycles representing these two cohomology classes in $H^2(M''/L, M''^\times)$ so all we need to do is evaluate the map $\sigma \mapsto \sum_{i=0}^{[M:K]-1} v_L(\sigma(F_L^i, F_L))$ (which gives an isomorphism of $H^2(M''/L, M''^\times)$ with $\mathbb{Z}/[M : K]\mathbb{Z}$) on these two cocycles.

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This is represented by the 2-cocycle on $Gal(M'''/L)$ sending (F_L^i, F_L^j) (here $0 \leq i, j < [M:L]$) to 1 if $i + j < [M:L]$ and ϖ_L otherwise.

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So its inflation is the 2-cocycle on $Gal(M''/L)$ which sends (F_L^i, F_L^j) (here $0 \leq i, j < [M : K]$) to 1 if $(i \ mod \ [M : L]) + (j \ mod \ [M : L]) < [M : L]$ and ϖ_L otherwise.

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The invariant of this cocycle σ in $\mathbb{Z}/[M:K]\mathbb{Z}$ is $\sum_{i=0}^{[M:K]-1} v_L(\sigma(F_L^i, F_L))$ which is a sum of 0s and 1s, with 1's occurring whenever i mod $[M:L]$ is $[M:L]-1$, and this occurs $[L:K]$ times.

Conclusion: $fc_{M/L} = [L:K]fc_{M''/L}$.

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Conclusion: $fc_{M/L} = [L : K] fc_{M''/L}$.

This is a general result about unramified cocycles and we should probably do all of these first.

The final calculation

Now let's let σ be the restriction to M''/L of the standard cocycle $\sigma_{M'/K}$ representing $fc_{M'/K}$, i.e., the restriction of the cocycle sending (F_K^i, F_K^j) to 1 or ϖ_K depending on whether $i + j < [M : K]$ or not.

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If the answer is $[L : K]$ then we have proved $res(fc_{M/K}) = fc_{M/L}$.

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Now let's let σ be the restriction to M''/L of the standard cocycle $\sigma_{M'/K}$ representing $fc_{M'/K}$, i.e., the restriction of the cocycle sending (F_K^i, F_K^j) to 1 or ϖ_K depending on whether $i+j < [M:K]$ or not.

What remains is to compute $\sum_{i=0}^{[M:K]-1} v_L(\sigma(F_L^i, F_L))$ for this cocycle.

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So now we can do the calculation.

The final calculation

The sum $\sum_{i=0}^{[M:K]-1} v_L(\sigma(F_L^i, F_L))$ is going to be e times the number of i's such that there's a carry when we add if modulo $[M:K]$ and f.

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What's left: (1) how to check $Art_{M/K}$ and $Art_{L/K}$ are compatible (we know the relationship between the fundamental classes but I don't know what to do next), and (2) uniqueness of Artin maps subject to those diagrams all commuting. There is also an option (3) prove that a subgroup of $K^\times$ is open of finite index iff it's $N(L^\times)$ for L/K a finite abelian extension.