

# 12. Dynamic Programming for Matrix Chain Multiplication

## CPSC 535

Kevin A. Wortman



CALIFORNIA STATE UNIVERSITY  
**FULLERTON**



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## Big Idea: 2D Table

- ▶ *Recall:* dynamic programming
  - ▶ problem has recursive structure
  - ▶ overlapping subproblems
  - ▶ use table to store solutions, avoid duplicated effort
  - ▶ top-down or bottom-up
- ▶ so far: **1D table** – has one index
- ▶ now: **2D table** – has *two* indices

## Matrix Multiplication

for matrices  $A_1, A_2$  :

$$A_1 A_2$$

Recall:

$$\begin{bmatrix} 5 & 12 & 5 \\ 16 & 9 & 4 \end{bmatrix} \times \begin{bmatrix} 19 & 2 \\ 9 & 5 \\ 8 & 11 \end{bmatrix} = \begin{bmatrix} 5 \times 19 + 12 \times 9 + 5 \times 8 & 125 \\ & 417 \\ & 121 \end{bmatrix}$$

## Matrix Multiplication Algorithms

Recall:

- ▶ Naïve algorithm: three nested loops,  $O(n^3)$
- ▶ Strassen's algorithm: divide-and-conquer,  $\approx O(n^{2.8074})$
- ▶ Those analyses assumed  $A_1, A_2$  are both square  $n \times n$  matrices
- ▶ Now: matrix sizes may differ
- ▶ **Compatible:**  $A_1$  and  $A_2$  are compatible when  $A_1.columns = A_2.rows$

## Naïve Matrix Multiplication Algorithm

```
1: function MATRIX-MULTIPLY(A, B)
2:   C = new A.rows × B.columns matrix
3:   for i from 1 to A.rows do
4:     for j from 1 to B.columns do
5:       cij = 0
6:       for k from 1 to A.columns do
7:         cij = cij + aik · bkj
8:       end for
9:     end for
10:  end for
11:  return C
12: end function
```

**Analysis:**  $\Theta(A.rows \cdot A.columns \cdot B.columns)$

## Matrix Chain Multiplication

Given  $n$  compatible matrices  $A_1, A_2, \dots, A_n$ , compute

$$A_1 A_2 \dots A_n$$

- ▶ Recall: matrix multiplication is **associative**
- ▶ May parenthesize  $A_1 A_2 \dots A_n$  in any order
- ▶ Q: which order is most efficient?

## Equivalent Parenthesizations

$$\begin{aligned}A_1 A_2 A_3 A_4 &= A_1 (A_2 (A_3 A_4)) \\&= A_1 ((A_2 A_3) A_4) \\&= (A_1 A_2) (A_3 A_4) \\&= (A_1 (A_2 A_3)) A_4 \\&= ((A_1 A_2) A_3) A_4\end{aligned}$$

Total runtime depends on the dimensions of  $A_1 \dots A_4$ .

## Example: Different Runtimes

Given three matrices  $A_1, A_2, A_3$  with dimensions

matrix	rows	columns
$A_1$	10	100
$A_2$	100	5
$A_3$	5	50

- ▶  $((A_1 A_2) A_3)$  costs  
 $10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 5,000 + 2,500 = 7,500$  scalar multiplies
- ▶  $(A_1 (A_2 A_3))$  costs  
 $100 \cdot 5 \cdot 50 + 10 \cdot 100 \cdot 50 = 25,000 + 50,000 = 75,000$  scalar multiplies
- ▶ first is order of magnitude faster



## Matrix Chain Multiplication Problem

*matrix chain multiplication problem*

**input:** a sequence  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n > 0$  compatible matrices, and sequence  $p = \langle p_0, p_1, \dots, p_n \rangle$  of integers, where matrix  $A_i$  has  $p_{i-1}$  rows and  $p_i$  columns

**output:** a parenthesization of  $A_1 A_2 \dots A_n$  that minimizes scalar multiplications

*matrix chain multiplication value problem*

**input:** a sequence  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n > 0$  compatible matrices, and sequence  $p = \langle p_0, p_1, \dots, p_n \rangle$  of integers, where matrix  $A_i$  has  $p_{i-1}$  rows and  $p_i$  columns

**output:** the minimum number of scalar multiplies necessary to multiply  $A_1 A_2 \dots A_n$

## Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
  - 4.1 **top-down** alternative: add table base case (**memoization**)
  - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

## Matrix Chain Multiplication Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.

*matrix chain multiplication value problem*

**input:** a sequence  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n > 0$  compatible matrices, and sequence  $p = \langle p_0, p_1, \dots, p_n \rangle$  of integers, where matrix  $A_i$  has  $p_{i-1}$  rows and  $p_i$  columns

**output:** the minimum number of scalar multiplies necessary to multiply  $A_1 A_2 \dots A_n$

- ▶ **solution:** parenthesized expression e.g.  $(A_1(A_2 A_3))(A_4 A_5)$
- ▶ **value:** number of multiplications e.g. 75,000
- ▶ goal: **value**

## Matrix Chain Multiplication Step 2

2. Derive a **recurrence** for an optimal value.

- ▶ define  $r_{i,j}$  = minimum number of multiplies for  $A_i A_{i+1} \dots A_j$
- ▶ (note: **two** indices)
- ▶ solution to whole problem is  $r_{1,n}$
- ▶ base case:  $A_i$  by itself; so when  $i = j$ ,  $r_{i,j} = 0$
- ▶ general case:
  - ▶ **think** divide-and-conquer; define  $r_{i,j}$  in terms of  $r_{<i,<j}$
  - ▶ make the problem **one piece** smaller
  - ▶ given  $A_i A_{i+1} \dots A_j$ , split w/ parenthesis at index  $k$  :

$$A_i A_{i+1} \dots A_j = (A_i A_{i+1} \dots A_k)(A_{k+1} A_{k+2} \dots A_j)$$

- ▶ try every option and keep the optimal one

$$r_{i,j} = \min_{i \leq k \leq j} r_{i,k} + r_{k+1,j} + p_{i-1} p_k p_j$$

## Matrix Chain Multiplication Step 3

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
1: function MATRIX-CHAIN-VALUE-DC( $p[0..n]$ )
2:   return MC-DC( $p, 0, n$ )
3: end function
4: function MC-DC( $p[0..n], i, j$ )
5:   if  $i == j$  then
6:     return 0
7:   end if
8:    $q = \infty$ 
9:   for  $k$  from  $i$  to  $j - 1$  do
10:     $q = \min(q, \text{MC-DC}(p, i, k) + \text{MC-DC}(p, k + 1, j) + p[i - 1] \times p[k] \times p[j])$ 
11:  end for
12:  return  $q$ 
13: end function
```

## Sidebar: Analysis of MATRIX-CHAIN-VALUE-DC

- ▶ MC-DC-REC calls itself  $O(n)$  times in general case
- ▶ like CUT-ROD-DC
- ▶ exponential time
- ▶ again, dynamic programming will circumvent all this recursion

## Matrix Chain Multiplication Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.

4.1 **top-down** alternative: add table base case (**memoization**)

- ▶ Recall **memoization**: use a hash dictionary to make a “memo” of pre-calculated solutions
- ▶ create hash table  $T$
- ▶ use pair  $(i, j)$  as key in table  $T$ , storing  $r_{i,j}$

## Matrix Chain Multiplication Step 4.a

```
1: function MATRIX-CHAIN-VALUE-MEMOIZED( $p[0..n]$ )
2:   HASH-TABLE-CREATE( $T$ )
3:   return MC-M( $T, p, 1, n$ )
4: end function
5: function MC-M( $T, p[0..n], i, j$ )
6:    $q = \text{HASH-TABLE-SEARCH}(T, (i, j))$ 
7:   if  $q \neq \text{NIL}$  then
8:     return  $q$ 
9:   end if
10:  if  $i == j$  then
11:     $q = 0$ 
12:  else
13:     $q = \infty$ 
14:    for  $k$  from  $i$  to  $j - 1$  do
15:       $q = \min(q, \text{MC-M}(p, i, k) + \text{MC-M}(p, k + 1, j) + p[i - 1] \times p[k] \times p[j])$ 
16:    end for
17:  end if
18:   $q.\text{key} = (i, j)$ 
19:  HASH-TABLE-INSERT( $q$ )
20:  return  $q$ 
21: end function
```



## Memoized Algorithm Analysis

- ▶  $T$  contains  $\Theta(n^2)$  pairs  $(i, j)$
- ▶ each entry is inserted exactly once
- ▶ in the general case, MC-M takes  $\Theta(n)$  expected time
- ▶  $\Rightarrow$  MATRIX-CHAIN-VALUE-MEMOIZED takes  $\Theta(n^3)$  expected time

## Matrix Chain Multiplication Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
    - 4.1 **top-down** alternative: add table base case (**memoization**)
    - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
- ▶ create 2D array  $m$  where  $m[i][j] = r_{i,j}$
  - ▶ **bottom-up**: write an explicit **for** loop that computes and stores every general case
  - ▶ need to order loops so we never use an uninitialized element
  - ▶  $\therefore$  initialize chain length 1(base case), 2,  $\dots$ ,  $n$

## Matrix Chain Multiplication Step 4.b

```
1: function MATRIX-CHAIN-BU( $p[0..n]$ )
2:   Create array  $m[1..n][1..n]$ 
3:   for  $i$  from 1 to  $n$  do
4:      $m[i][i] = 0$  ▷ base case, length=1
5:   end for
6:   for  $\ell$  from 2 to  $n$  do ▷  $\ell$  = general-case length
7:     for  $i$  from 1 to  $(n - \ell + 1)$  do
8:        $j = i + \ell - 1$ 
9:        $q = \infty$ 
10:      for  $k$  from  $i$  to  $j - 1$  do
11:         $q = \min(q, m[i][k] + m[k + 1][j] + p[i - 1] \times p[k] \times p[j])$ 
12:      end for
13:       $m[i][j] = q$ 
14:    end for
15:  end for
16:  return  $m[1, n]$ 
17: end function
```

## Matrix Chain Multiplication Analysis

- ▶ MATRIX-CHAIN-BU is clearly  $\Theta(n^3)$  time
- ▶ top-down memoized algorithm:  $\Theta(n^3)$  expected time
- ▶ bottom-up algorithm:  $\Theta(n^3)$  time with faster constant factors

## Matrix Chain Multiplication Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

*matrix chain multiplication value problem*

**input:** a sequence  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n > 0$  compatible matrices, and sequence  $p = \langle p_0, p_1, \dots, p_n \rangle$  of integers, where matrix  $A_i$  has  $p_{i-1}$  rows and  $p_i$  columns

**output:** the minimum number of scalar multiplies necessary to multiply  $A_1 A_2 \dots A_n$

*matrix chain multiplication problem*

**input:** (same)

**output:** a parenthesization of  $A_1 A_2 \dots A_n$  that minimizes scalar multiplications

## Matrix Chain Multiplication Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

- ▶ **idea:** for each  $(i, j)$ , record which  $k$  defines the minimum  $m[i][j]$
- ▶ happens inside the inner-most  $k$  loop
- ▶ define

$s[i][j] =$  the index  $k$  that minimizes  $r_{i,k} + r_{k+1,j} + p_{i-1}p_kp_j$

- ▶ rewrite  $\min(q, \dots)$  statement as an **if** so we can update  $s[i][j]$

## Matrix Chain Multiplication Step 5

```

1: function MATRIX-CHAIN-SOLUTION( $p[0..n]$ )
2:   Create arrays  $m[1..n][1..n]$  and  $s[1..n][1..n]$ 
3:   for  $i$  from 1 to  $n$  do
4:      $m[i][i] = 0$  ▷ base case, length=1
5:   end for
6:   for  $\ell$  from 2 to  $n$  do ▷  $\ell$  = general-case length
7:     for  $i$  from 1 to  $(n - \ell + 1)$  do
8:        $j = i + \ell - 1$ 
9:        $q = \infty$ 
10:      for  $k$  from  $i$  to  $j - 1$  do
11:         $q' = m[i][k] + m[k + 1][j] + p[i - 1] \times p[k] \times p[j]$ 
12:        if  $q' < q$  then
13:           $q = q'$ 
14:           $s[i][j] = k$ 
15:        end if
16:      end for
17:       $m[i][j] = q$ 
18:    end for
19:  end for
20:  return MC-BTRACK( $s, 1, n$ )
21: end function

```

## Matrix Chain Multiplication Step 5

```
1: function MC-PARENS( $s[1..n][1..n]$ ,  $i, j$ )  
2:   if  $i == j$  then                                     ▷ single matrix  
3:     return " $A_i$ "  
4:   end if  
5:    $k = s[i][j]$   
6:   return "(" + MC-BTRACK( $s, i, k$ ) + ")(" + MC-BTRACK( $s, k, j$ ) + ")"  
7: end function
```