

05. Linear-Time Sorting and Selection

CPSC 535

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Big Idea in Computer Science

“What is old is new again”: CS principles are solved science; society’s needs, economic factors, and fads dictate which are prominent and which are in the background

- ▶ thin clients dominated the mainframe era; thick clients dominated the PC era; thin clients dominate the web app era
- ▶ memory conservation was critical prior to the 90s; programmer labor was more important, until mobile phones
- ▶ Unix rose (70s), fell (80s-90s), rose again (MacOS, iOS, Android, ChromeOS, Linux, PlayStation, embedded)
- ▶ algorithms were considered ivory-tower theory until recently

Protip: the CS material that seems irrelevant now, will probably become extremely marketable later in your career

Big Idea in Algorithm Design

Parameterized complexity: algorithm complexity measured both in terms of input size n , **and** some parameter describing the values in the input

- ▶ machine word size W (e.g. $W = 64$ on modern PCs)
- ▶ # distinct values k

Pseudopolynomial: polynomial over both n , and also parameters

- ▶ radix sort takes $\Theta(nW)$ time
- ▶ strictly speaking W could be as large as n , so $\Theta(nW) = \Theta(n^2)$, unimpressive
- ▶ in practice all real-world computers have $W \in \Theta(1)$ so $\Theta(nW) = \Theta(n)$, faster than $\Theta(n \log n)$
- ▶ arguably defying the spirit of the Random Access Model

Tool to circumvent lower bounds, NP -hardness

The Lower Bound for the Sorting Problem

Recall the precise phrasing of the theorem:

Any comparison sort algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

Bad news: $O(n \log n)$ “speed limit” for this important problem

Good news:

- ▶ optimal $\Theta(n \log n)$ -time algorithms: mergesort, heapsort, quicksort
- ▶ loophole: theorem only applies to “comparison sorts”
- ▶ loophole: theorem applies to the general sorting problem, but we could make the problem more specific

Counting Sort Problem

Recall the classical *sorting problem*:

input: a sequence of n numbers $A = \langle a_1, a_2, \dots, a_n \rangle$

output: a permutation (reordering) $\langle a'_1, a'_2, \dots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \dots, a'_n$.

What if the inputs a_i are all bounded integers?

counting sort problem: (changes are underlined)

input: an integer $k \geq 0$, and a sequence of n integers
 $A = \langle a_1, a_2, \dots, a_n \rangle$, where each $a_i \in [0, k]$

output: same

Turns out this change admits a $\Theta(n)$ -time algorithm.

Counting Sort Idea

- ▶ each a_i can work as an array index
- ▶ count the number of occurrences of each value; create array C where

$C[x] =$ the number of times that x appears in A

- ▶ let B be the output array
- ▶ use the counts in C to plan out which indices of B will hold each a_i
- ▶ fill in B using this information

Counting Sort Pseudocode

```

1: function COUNTING-SORT( $A, B, k$ )
2:   allocate new array  $C[0, \dots, k]$ , initialized to all zeroes
3:   for  $j$  from 1 to  $A.length$  do
4:      $C[A[j]] + +$   $\triangleright C[i]$  is the number of elements  $= i$ 
5:   end for
6:   for  $i$  from 1 to  $k$  do
7:      $C[i] + = C[i - 1]$   $\triangleright C[i]$  is the number of elements  $\leq i$ 
8:   end for
9:   for  $j$  from  $A.length$  down to 1 do
10:     $B[C[A[j]]] = A[j]$ 
11:     $C[A[j]] - -$ 
12:  end for
13: end function

```

Counting Sort Analysis

- ▶ allocate C : $\Theta(k)$ time
- ▶ first **for** loop: $\Theta(n)$ time
- ▶ second **for** loop: $\Theta(k)$ time
- ▶ third **for** loop: $\Theta(n)$ time
- ▶ total = $\Theta(k + n + k + n) = \Theta(2k + 2n) = \Theta(k + n)$ time

When Counting Sort Wins

If $k \in O(n)$, then counting sort takes $\Theta(k + n) = \Theta((n) + n) = \Theta(n)$ time.

Applications where $k \ll n$

- ▶ DNA sequences have only $k = 4$ bases A, C, G, T; human genome has $n \approx 3$ billion bases
- ▶ n -character ASCII string has only $k = 127$ character values
- ▶ log of website analytics has n hits but only k distinct web page URLs; each page is visited many times so $n \gg k$

When Counting Sort Loses

In the general sorting problem, each a_i is unbounded, so the maximum element could use $\Theta(n)$ bits, have value

$$a_i = 2^n = k$$

and force counting sort to take

$$\Theta(k + n) = \Theta((2^n) + n) = \Theta(2^n)$$

time, which is **exponential** in n and much more expensive than $\Theta(n \log n)$

\implies counting sort is optimal when the software designer knows that the input is always a set of k integers with $k \in O(n)$

\implies **but** if that is not guaranteed, comparison sorts are still optimal

Stable Sorting

stable sorting algorithm: does not swap order of ties

if $a_i = a_j$ and $i < j$ then $i' < j'$

Ex.: suppose we sort

$[(47, a), (13, c), (28, b), (13, d)]$

by the first element of each pair; a *stable* sort guarantees $(13, c)$ comes before $(13, d)$

Stability is a convenient, desirable property

Stable: insertion sort, mergesort, counting sort

Not stable: heapsort, quicksort

“Radix”

Vocabulary quiz:

- ▶ what does **radix** mean?
- ▶ where else do we use the word “radix”?

Radix Sort Overview

- ▶ make counting sort more robust to large elements
- ▶ sort one digit at a time
- ▶ i.e. sort by least-significant-digit, then by second-least-significant-digit, ..., sort by most-significant digit
- ▶ e.g. to sort names in a spreadsheet: sort by first name, then by last name
- ▶ originally **used by pre-digital punchcard sorting machines** (what's old...)
- ▶ now used for parallel sort in GPU (...is new again)

Radix Sort Worked Example

(Sorting one base-10 digit at a time.)

7	7	5	7	9	1	5	2	1	0	5	3
0	5	3	5	2	1	2	2	5	2	2	5
7	6	2	7	6	2	3	3	4	3	3	4
7	9	1	0	5	3	0	5	3	5	2	1
6	7	4	6	7	4	7	6	2	6	7	4
5	2	1	3	3	4	6	7	4	7	6	2
3	3	4	7	7	5	7	7	5	7	7	5
2	2	5	2	2	5	7	9	1	7	9	1

Radix Sort, 1 bit at a time

```
1: function RADIX-SORT-1( $A, W$ )  
2:   for  $i$  from 1 to  $W$  do  
3:     use a stable sort to sort  $A$  based only on bit position  $i$   
4:   end for  
5: end function
```

Using counting sort as the stable sort, we have $k = 2$ (bit values 0 or 1) so each loop iteration takes $\Theta(k + n) = \Theta(2 + n) = \Theta(n)$ time

Clearly W iterations $\implies \Theta(nW)$ total time

Radix Sort, 8 bits at a time

```
1: function RADIX-SORT-8(A, W)
2:   for  $i$  from 1 to  $\lceil W/8 \rceil$  do
3:     stably sort  $A$  on bits  $8i - 7$  through  $8i$ 
4:   end for
5: end function
```

$$k = 2^8 = 256 \in \Theta(1)$$

number of iterations is $\lceil nW/8 \rceil \in \Theta(n)$

\implies still $\Theta(nW)$ time, but with different constant factors

Let $r = \#$ bits per pass; optimal choice of r minimizes

$$\lceil W/r \rceil \cdot (2^r + n)$$

Minimum and Maximum

```
1: function MINIMUM(A)
2:   min = A[1]
3:   for i from 2 to A.length do
4:     if min > A[i] then
5:       min = A[i]
6:     end if
7:   end for
8:   return min
9: end function
```

$\Theta(n)$ time

can also find maximum in $\Theta(n)$ time, or both in $\Theta(n)$ time

Selection Problem and Baseline Algorithm

input: array of n numbers $A = \langle a_1, a_2, \dots, a_n \rangle$; index $i \in \{1, 2, \dots, n\}$

output: the i th smallest element of A

```
1: function SELECTION-BY-SORTING( $A, i$ )  
2:   return  $MERGE - SORT(A)[i]$   
3: end function
```

Clearly $\Theta(n \log n)$ time

Surprise: selection can be solved in only $\Theta(n)$ time

Randomized Quicksort Review

```

1: function RQSORT(A, p, r)
2:   if  $p < r$  then
3:      $q = \text{RPART}(A, p, r)$ 
4:      $\text{RQSORT}(A, p, q - 1)$ 
5:      $\text{RQSORT}(A, q + 1, r)$ 
6:   end if
7: end function

```

Non-stable sort in $\Theta(n \log n)$
 expected time but $\Theta(n^2)$
 worst-case time

```

1: function RPART(A, p, r)
2:    $i = \text{random in } [p, r]$ 
3:    $\text{swap}(A[i], A[r])$ 
4:    $\text{pivot} = A[r]$ 
5:    $i = p$ 
6:   for  $j$  from  $p$  to  $r - 1$  do
7:     if  $A[j] < \text{pivot}$  then
8:        $\text{swap}(A[i], A[j])$ 
9:        $i++$ 
10:    end if
11:  end for
12:   $\text{swap}(A[i], A[r])$ 
13:  return  $i$ 
14: end function

```

Randomized Selection Overview

- ▶ combining ideas from binary search and quicksort
- ▶ recursively search for i th smallest element
- ▶ do randomized partition; then
- ▶ three cases
 - ▶ pivot happens to be i th smallest
 - ▶ need to keep searching before pivot
 - ▶ need to keep searching after pivot
- ▶ expected runtime is $T(n) \approx T(n/2) + \Theta(n)$
- ▶ counterintuitively, that solves to $\Theta(n)$

Randomized Selection Pseudocode

```
1: function RSELECT( $A, p, r, i$ )
2:   if  $p == r$  then
3:     return  $A[p]$                                 ▷ base case, done
4:   end if
5:    $q = \text{RPART}(A, p, r)$                           ▷ partition,  $q$  is pivot index
6:    $k = q - p + 1$                                 ▷  $k$  = number of elements before pivot
7:   if  $i == k$  then
8:     return  $A[q]$                                 ▷ pivot is answer
9:   else if  $i < k$  then
10:    return  $\text{RSELECT}(A, p, q - 1, i)$ 
11:  else
12:    return  $\text{RSELECT}(A, q + 1, r, i - k)$  ▷  $i$  decreases by  $k$ 
13:  end if
14: end function
```

Randomized Selection Analysis

- ▶ at most one recursive call, on $n/2$ elements on average
- ▶ partitioning takes $\Theta(n)$ time
- ▶ rest of algorithm takes $\Theta(1)$ time
- ▶ expected running time

$$T(n) = T(n/2) + \Theta(n)$$

which is only $\Theta(n)$ by master theorem

- ▶ worst case is the same for quicksort, extreme pivot at each step, $\Theta(n^2)$ time
- ▶ **takeaway:** randomized selection takes $\Theta(n)$ expected time and $\Theta(n^2)$ worst-case time

Deterministic Selection Overview

- ▶ **deterministic:** perfectly predictable; not randomized
- ▶ recall that $T(n) = T(fn) + \Theta(n)$ is $\Theta(n)$ for any fraction $0 < f < 1$, not just $f = 1/2$
- ▶ need: deterministic process to find a not-terrible pivot
- ▶ i.e. need at least fn elements on each side of the pivot, so that the worst-case recursive call is $T((1 - f)n)$
- ▶ e.g. need at least $\frac{1}{3}n$ elements on each side of the pivot, so that there is a $T(\frac{1}{3}n)$ or $T(\frac{2}{3}n)$ call; worst-case is $T(\frac{2}{3}n)$; so

$$T(n) = T\left(\frac{2}{3}n\right) + \Theta(n)$$

which is still $\Theta(n)$ (though with worse constants)

Deterministic Selection Process

1. divide n elements into $\approx n/5$ groups of 5 elements each
2. find the median of each group with
SELECTION – BY – SORTING; $\Theta(n(5 \log 5)) = \Theta(n)$ time
3. form a new array of the medians, and recursively select the median of this array = “median-of-medians”; $T(n/5)$ time
4. partition as usual, using median-of-medians as the pivot; $\Theta(n)$ time
5. same three cases: either pivot is answer, or recurse before pivot, or recurse after pivot;
 $T(\text{max. \# elements on either side of pivot})$

Deterministic Selection Analysis

- ▶ let x be the median-of-medians; count elements $\geq x$
- ▶ suppose W.L.O.G. that input elements are distinct
- ▶ \therefore at least half of the group-medians are $\geq x$
- ▶ \therefore at least half of the groups contain at least 3 elements $\geq x$ each; except for the group containing x , and possibly one group with < 5 elements
- ▶ \therefore #elements $\geq x$ is at least

$$3\left(\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3}{10}n - 6$$

- ▶ symmetrically there are at least $\frac{3}{10}n - 6$ elements $\leq x$
- ▶ \therefore recursively select at most $n - (\frac{3}{10}n - 6) = \frac{7}{10}n + 6$ elements

Deterministic Selection Analysis (continued)

For some $t \in \Theta(1)$,

$$T(n) \leq \begin{cases} O(1) & n < t \\ T(\lceil n/5 \rceil) + T(\frac{7}{10}n + 6) + O(n) & n \geq t. \end{cases}$$

The master theorem does not apply, but the substitution method can be used to show $T(n) \in O(n)$.

Takeaway: Deterministic selection takes $O(n)$ worst-case time.

Surprise: selection can be **derandomized** from $O(n)$ expected time to $O(n)$ worst-case time with no asymptotic overhead.

Impractical; much worse constant factors, not usually worth it.