11. Linear Programming

CPSC 535

Kevin A. Wortman

CALIFORNIA STATE UNIVERSITY FULLERTON

Big Ideas

- duality — same problem from different perspectives
- formulations, reductions
- visualizing high geometric dimensions

Overview

- *programming* in math involves finding some kind of optimal solution subject to mathematically-codified constraints
  - (not coding e.g. C++ programming)
- *linear programming (LP):* optimize a linear *objective function* subject to inequalities
- very general framework
- pioneered by Soviet economist Leonid Kantorovich circa 1930s; goal was to optimize supply/demand in a communist economy in lieu of prices
- now used in business *(operations research)*
  - scheduling UPS deliveries, optimizing farm production, allocating investment portfolios, etc.

Computational Complexity

- many tough problems in P, including max-flow, reduce to LP
- on the border of P
- simplex algorithm technically takes $O(2^n)$ worst-case time, but is fast polynomial on most practical inputs
- we have pseudopolynomial algorithms with e.g. $O(n^{2.5}W)$ runtime and expensive constant factors
- open question whether there is a strongly polynomial LP algorithm with runtime e.g. $O(n^3)$, not a function of W

Standard Form

- *standard form:* restricted/simplified LP, easier for algorithms to solve
- later: *general form* which is more convenient for end-user formulations
- general reduces to standard with constant overhead
- similar situation to max-flow and robust max-flow
- actual solver algorithm sees a simplified standard form; reduction algorithm "frontend" accepts a generalized problem that is more convenient for end-users

Standard Form

standard form with n variables and m constraints:

maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_{1,n} & \leq & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_{2,n} & \leq & b_2 \\ \vdots & & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_{m,n} & \leq & b_m \\ x_1, x_2, \ldots, x_n & \geq & 0 \end{array}$$

variables: $x_1, \ldots, x_n \in \mathbb{R}$
objective function defined by coefficients $c_1, \ldots, c_n \in \mathbb{R}$
constraints defined by coefficients $a_{i,j}, b_i \in \mathbb{R}$

Standard Form Example

maximize $2x_1 + x_2 - \frac{1}{3}x_3$
subject to

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ -x_3 &\leq -2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Standard Form Matrix Notation

- more compact math notation
- collect:
  - variables into vector $x = \langle x_1, \ldots, x_n \rangle$
  - objective coefficients into vector $c = \langle c_1, \ldots, c_n \rangle$
  - r.h.s. of inequalities into vector $b = \langle b_1, \ldots, b_m \rangle$
  - $a_{i,j}$ coefficients into matrix A
- LP can be written in terms of dot-product and matrix-vector multiplication as (and note the transpose c^T):

maximize $c^T x$
subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Possible Outcomes

LPs are not always solvable!

there are three outcomes:

1. **solution**: concrete values for $x_1, \ldots, x_n$ that maximize $c^T x$ (good, usually the goal)
2. **unbounded**: objective can be made arbitrarily large i.e. $+\infty$ (bad, usually means there is a bug in your LP that makes it nonsensical)
3. **infeasible**: impossible to satisfy all constraints simultaneously (bad, usually means that either your LP is nonsensical; or your LP makes sense but meeting all your goals is impossible)

Standard-Form LP Problem

standard-form linear programming problem
input: vector $c \in \mathbb{R}^n$, vector $b \in \mathbb{R}^m$, and $m \times n$ matrix A of real numbers
output: one of

1. "unbounded";
2. "infeasible"; or
3. "solution" with a vector $x \in \mathbb{R}^n$ maximizing the objective function

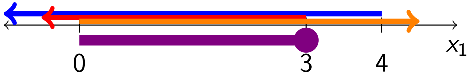
Exploring the Three Outcomes

- we will explore unbounded/infeasible/solution in 1D, then 2D
- *dimension* of an LP: #variables n
- *feasible region:* space of x vectors that satisfy all constraints
- *halfspace:* half of all geometric space,
  - 1D: one side of a point on the number line e.g. $x = 3$
  - 2D: one side of a line e.g. $y = 3x + 2$
  - 3D: one side of a plane e.g. $2x + 3y - z = 5$
- each new constraint limits the feasible region to a halfspace
- as we go, make note of
  - the shape of the feasible region
  - optimal solutions are found at extreme points ("corners") of halfspaces
  - unbounded $\Leftrightarrow$ feasible region extends out infinitely
  - infeasible $\Leftrightarrow$ empty feasible region

1D Solution

maximize $2x_1$
subject to

$$
\begin{aligned}
x_1 &\leq 4 \\
x_1 &\leq 3 \\
x_1 &\geq 0
\end{aligned}
$$

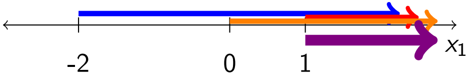


- feasible region = intersection of all arrows = is line segment $[0, 3]$
- solution ● is $x_1 = 3$
- optimal objective function value is $2x_1 = 2(3) = 6$

1D Unbounded

maximize $2x_1$
subject to

$$\begin{array}{rcl} -x_1 & \leq & 2 \\ -x_1 & \leq & -1 \\ x_1 & \geq & 0 \end{array}$$



- feasible region = intersection of all arrows = open interval $[1, +\infty)$
- solution is undefined
- optimal objective function value is $2x_1 = 2(\infty) = \infty$

1D Infeasible

maximize $2x_1$
subject to

$$
\begin{aligned}
x_1 &\leq 1 \\
-x_1 &\leq -3 \\
x_1 &\geq 0
\end{aligned}
$$

- feasible region $=$ intersection of all arrows $= \emptyset$
- solution is undefined
- cannot evaluate objective function

2D Solution

maximize x_2
subject to

$$\begin{aligned}
\frac{1}{4}x_1 + x_2 &\leq 2 \\
-\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\
x_1, x_2 &\geq 0
\end{aligned}$$

Sidebar: Math Definition of a Line

- recall
  - slope-intercept form $y = mx + b$
  - 2D LP constraint is $c_1x_1 + c_2x_2 \leq b$
- substitute $x_1 = x, x_2 = y$, rearrange to slope-intercept:

$$\begin{aligned} c_1x_1 + c_2x_2 &\leq b \\ c_1(x) + c_2(y) &\leq b \\ -(c_1x) & \quad -(c_1x) \\ c_2y &\leq -c_1x + b \end{aligned}$$

if $c_2 > 0$ then

$$y \leq -\frac{c_1}{c_2}x + \frac{b}{c_2}$$

else, $c_2 < 0$, dividing by c_2 flips $\leq$ to $\geq$, and

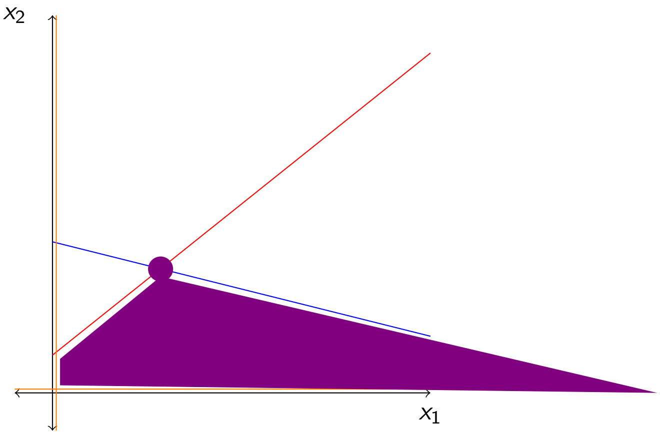
$$y \geq -\frac{c_1}{c_2}x + \frac{b}{c_2}$$

2D Solution

maximize x_2
subject to

$$\begin{aligned} \frac{1}{4}x_1 + x_2 &\leq 2 \\ -\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

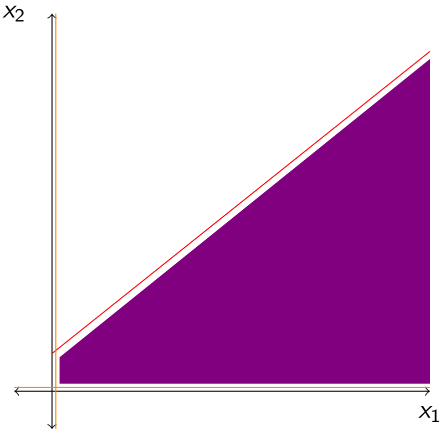
- feasible region is intersection of halfspaces $\Leftrightarrow$ polygon
- optimal solution is intersection of lines at $x_1 \approx 1.43, x_2 \approx 1.64$



2D Unbounded

maximize x_2
subject to

$$\begin{aligned} -\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$



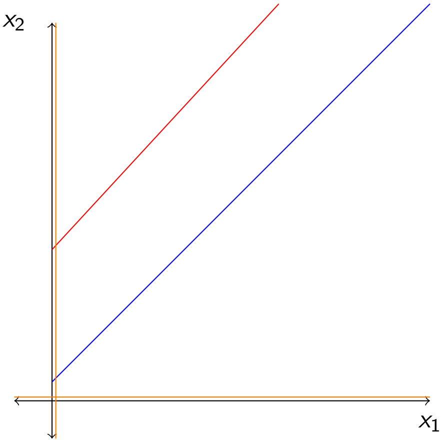
- feasible region is intersection of halfspaces $\Leftrightarrow$ some polygon sides, one infinite side
- optimal solution undefined

2D Infeasible

maximize x_2
subject to

$$\begin{aligned} -x_1 + x_2 &\leq .25 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- feasible region is intersection of halfspaces $\Leftrightarrow$ empty set
- optimal solution undefined



Recall: Standard Form

standard form with n variables and m constraints:

maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to

$$
\begin{array}{rcl}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_{1,n} & \leq & b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_{2,n} & \leq & b_2 \\
\vdots & & \vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_{m,n} & \leq & b_m \\
x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

variables: $x_1, \ldots, x_n \in \mathbb{R}$
objective function defined by coefficients $c_1, \ldots, c_n \in \mathbb{R}$
constraints defined by coefficients $a_{i,j}, b_i \in \mathbb{R}$

Recall: Standard Form Matrix Notation

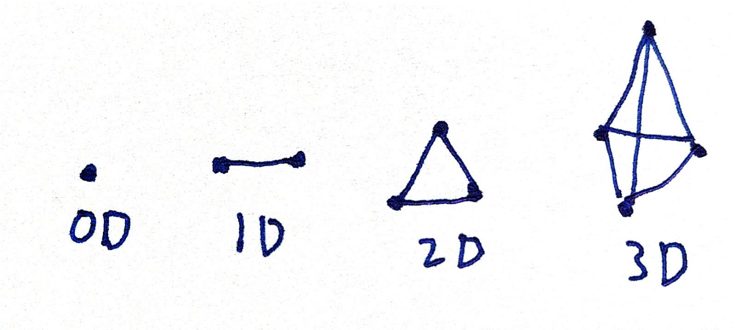
- more compact math notation
- collect:
  - variables into vector $x = \langle x_1, \ldots, x_n \rangle$
  - objective coefficients into vector $c = \langle c_1, \ldots, c_n \rangle$
  - r.h.s. of inequalities into vector $b = \langle b_1, \ldots, b_m \rangle$
  - $a_{i,j}$ coefficients into matrix A
- LP can be written in terms of dot-product and matrix-vector multiplication as (and note the transpose c^T):

maximize $c^T x$
subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

What is a Simplex?

simplex: generalization of a triangle to arbitrary dimensions



Slack Form

duality: the simplex algorithm views one LP in two ways,

1. standard form
2. *slack form*

- standard form: constraint says l.h.s $\leq$ r.h.s.
- $\Rightarrow$ the difference or "slack" between l.h.s. and r.h.s. is ≥ 0
- *slack form:* constraint says l.h.s. $+$ **slack** $=$ r.h.s.
- increasing objective $=$ decreasing slack
- introduce one new *basic variable* to represent slack in each constraint
- (pre-existing variables are *nonbasic*)
- $z =$ value of objective function
- don't bother writing "maximize" or "subject to"

Standard versus Slack Form

maximize $x_1 + 2x_2 - \frac{1}{2}x_3$
subject to

$$\begin{aligned} \frac{1}{3}x_1 + x_3 &\leq 5 \\ x_1 + x_2 + x_3 &\leq 100 \\ x_1 - x_2 &\leq -3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} z &= x_1 + 2x_2 - \frac{1}{2}x_3 \\ x_4 &= 5 - \frac{1}{3}x_1 - x_3 \\ x_5 &= 100 - x_1 - x_2 - x_3 \\ x_6 &= -3 - x_1 + x_2 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

basic var's: x_4, x_5, x_6
nonbasic var's: x_1, x_2, x_3

High-Level Simplex Algorithm

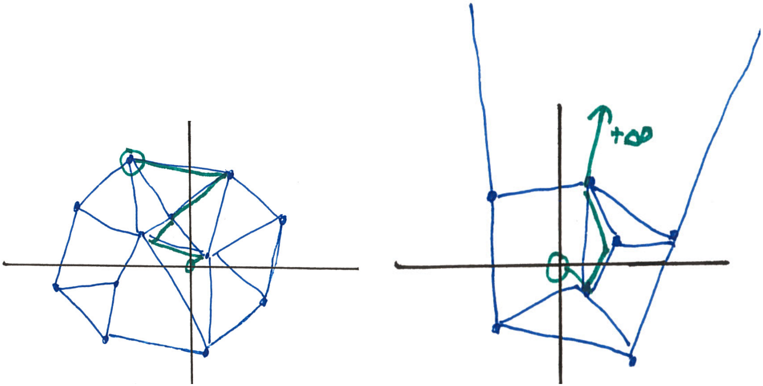
- convert standard form LP to slack form
- find a feasible (probably non-optimal) initial solution
  - intuitively: each $x_i = 0$
  - if this does not exist, return "infeasible"
- repeat:
  - choose a nonbasic variable x_i with positive coefficient in objective function (increasing x_i increases z)
    - if no such x_i exists, return solution (it's optimal)
  - increase x_i until some basic variable x_j is decreased to zero ("tighten" the slack until we're up against a constraint)
    - if none exists, return "unbounded"
  - swap roles: rewrite slack form with x_i as basic variable and x_j as nonbasic variable

(for further details, see CLRS section 29.3)

Geometric Intuition

- a solution is a point in *n*-dimensional space
- intuitively, initial solution is at the origin where $x_1, \ldots, x_n = 0$
- (for further details, see CLRS section 29.5)
- each iteration "reels in" the solution to hug the intersection between two constraints
- continues until we either
  1. go "off the map" and know the LP is infeasible; or
  2. cannot improve any further $\Rightarrow$ found optimal solution
- each step moves us along the border of a *simplex*

Geometric Intuition

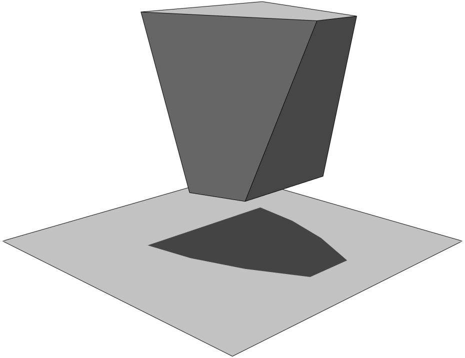


Analysis

- in LP's formulated to solve practical problems, usually
  - each of the m halfspaces intersects $O(m)$ other halfspaces
  - $\Rightarrow O(m^2)$ intersection points in the feasible region
  - $\Rightarrow$ simplex iterates $O(m^2)$ times
  - each iteration involves evaluating n-dimension obj. function
  - $\Rightarrow O(m^2 n)$ worst-case time
  - order-3 polynomial, same as max-flow
  - often faster b/c each step can "jump" pretty far
- **however,** $\exists$ feasible LP's that force simplex to take $\Omega(2^m)$ time
- *Klee-Minty cube:* $\forall d$, has $n = d$ variables, $n = d$ constraints, 2^d vertices, simplex is "tricked" into visiting all vertices
- this is a rare example of worst-case asymptotic analysis being misleading

Klee-Minty Cube

Klee-Minty Cube in 3D:



(image credit: Sophie Huiberts, CC-BY 4.0, https://commons.wikimedia.org/wiki/File:Klee-Minty-cube-for-shadow-vertex-pivot-rule.png)

Summary

- for a standard-form LP with n variables and m constraints...
- simplex algorithm is fast in practice, technically takes $O(2^m)$ worst-case time
- Khachiyan's *ellipsoid algorithm* takes $O(n^4 W)$ time
  - seminal result, proved that sub-exponential algorithms are possible
- now have faster pseudopolynomial algorithms, e.g Vaidya's alg. takes $O((n+m)^{1.5} nW)$ time
- open questions:
  - Is there a strongly-polynomial algorithm, or is LP NP-complete?
  - Is there an algorithm that has **both** simplex' practical speed **and** provable pseudonomial runtime?