03. Divide-and-Conquer

CPSC 535

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Divide-and-Conquer

One of the *big ideas* of computer science problem solving

1. **Divide** a problem into smaller parts
2. **Conquer** the smaller problems recursively
3. **Combine** the smaller solutions into one solution for the original problem

(The term carries some baggage from the age of imperialism.)

Divide-and-conquer, outside of algorithm design

- Software design; breaking features into classes, functions
- Networking; OSI seven layer model
- Parallel processing; MapReduce
- Software process; agile methods; sprints

Divide-and-conquer at a high level

function DIVIDE-AND-CONQUER(INPUT)
    if INPUT is base case **then**
        return trivial base case solution
    else
        $x_1, x_2, \ldots, x_k$ = divide INPUT into k pieces (often 2)
        s_1 = DIVIDE-AND-CONQUER(x_1)
        $\ldots$
        s_k = DIVIDE-AND-CONQUER(x_k)
        S = combine $s_1, \ldots, s_k$ into one solution
        return S
    end if
end function

Time complexity recurrences

Recur<u>sive</u> pseudocode leads to recu<u>rrences</u> in run-time functions

Suppose base case is $n = 1$ and takes $\Theta(1)$ time; in the recursive case we divide evenly into k pieces of size $\approx n/k$, recurse once on each, and spend $f(n)$ time in the *divide* and *conquer* phases:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ kT(n/k) + f(n) & \text{if } n > 1. \end{cases}$$

Recall merge sort divides into $k = 2$ pieces, merge takes $\Theta(n)$ time:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

Taking liberties with recurrences

General math: bound recurrences precisely including constant factors

Algorithm analysis: ordinarily bounding asymptotically; Θ notation will hide constant factors anyway; drop math details that can only impact constants and add clutter

- drop ceilings/floors, so write e.g. $n/2$ in lieu of $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$ is more precise
- when the base case is $\Theta(1)$ time for $n < c$ for some $c \in \Theta(1)$, don't bother writing it explicitly; so

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

is abbreviated as

$$T(n) = 2T(n/2) + \Theta(n)$$

Maximum subarray problem

Maximum subarray problem
input: an array $\langle p_1, p_2, \ldots, p_n \rangle$ where each $p_i \in \mathbb{R}$ is a *profit* (or loss) on day i
output: indices s, e with $s \leq e$, maximizing the total profit

$$\sum_{i=s}^{e} p_i$$

Applications

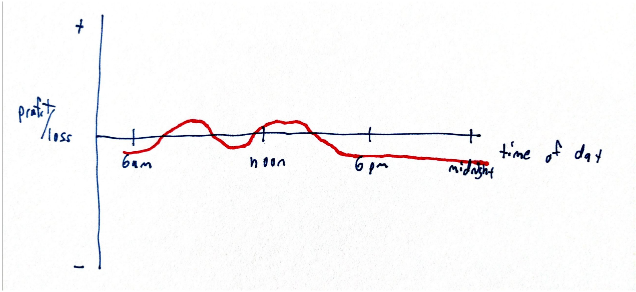
- buy then sell a stock/security
- pick opening/closing time of a retail store with slow periods
- computer vision, data mining: identify region most consistent with a pattern e.g. street striping

Examples

The optimal subarray may involve negative elements:

$$\langle 100, -1, -1, -1, 5, 3 \rangle$$

Application: when to open/close a cafe:



Greedy fails

Straightforward greedy algorithm would be:

- buy at the lowest price or sell at the highest price
- incorrect; best "run" could be elsewhere
- example: $\langle 0, 1, 10, 4, 4, 4, 4 \rangle$
  - $\langle 1, 10 \rangle$ is the biggest trough-to-peak; sum 11
  - but slow-and-steady $\langle 4, 4, 4, 4 \rangle$ has sum 12
- not always correct $\implies$ not actually an algorithm

Brute force

Exhaustive search: try every legal start/end

```
function BRUTE-FORCE-MAX-SUBARRAY(P)
    s = e = 1
    for i from 1 to n do
        for j from i to n do
            if (∑ p_i ... p_j) > (∑ p_s ... p_e) then
                s = i, e = j
            end if
        end for
    end for
    return (s, e)
end function
```

$\Theta(n^3)$ time as written; can cache sums to achieve $\Theta(n^2)$

Divide-and-conquer brainstorm

Divide: chop array in half into two smaller arrays L, R
Conquer: recursively compute maximum subarray in L and in R
Combine: maximum subarray of entire P could be

1. subarray entirely in L;
2. subarray entirely in R; or
3. *crossing* subarray that starts in L and ends in R

(exhaustive case analysis)
Theme with **combine**: choose best among small solutions (easy) or a distinct solution that crosses boundaries (trickier)

Identify crossing subarray — try 1

Suppose the two pieces of P are $P[low \ldots mid]$ and $P[mid+1 \ldots high]$

Tempting to try all pairs of $s \in \{low, \ldots, mid\}$ and $e \in \{mid+1, \ldots, high\}$

Would work, but

- time becomes $T(n) = 2T(n/2) + \Theta(n^2)$ which is $\Theta(n^2)$ by master theorem
- same time as brute force, but more complicated $\implies$ not a win

Identify crossing subarray — insight

Theme in algorithm design: in general, a more specific problem admits a faster and/or simpler algorithm

First try is not using the fact that a *crossing* subarray must cross *mid*

- substantially simplifies the search
- s is how far before *mid*; separately, e is how far after *mid*?
- two separate 1D searches $\implies$ two linear loops
- $\Theta(n) + \Theta(n) = \Theta(2n) = \Theta(n)$ time
- versus: s is where, and e is how much later?
- 2D search $\implies$ two nested loops $\implies$ $\Theta(n^2)$ time
- location of the "2" is profound; $\Theta(2n) \ll \Theta(n^2)$

Identify crossing subarray — try 2

```
function MAX-CROSSING-SUBARRAY(P, low, mid, high)
    leftsum = rightsum = −∞
    sum = 0
    for i from mid down to low do
        sum = sum + P[i]
        if sum > leftsum then
            leftsum = sum
            maxleft = i
        end if
    end for
    sum = 0
    for i from mid + 1 to high do
        sum = sum + P[i]
        if sum > rightsum then
            rightsum = sum
            maxright = i
        end if
    end for
    return (maxleft, maxright, leftsum + rightsum)
end function
```

$\Theta(n)$ time

(Note scoping of *maxleft*, *maxright*, and that they are inevitably initialized.)

Maximum subarray algorithm

```
function MAX-SUBARRAY(P, low, high)
    if low == high then
        return (low, high, P[low])
    else
        mid = ⌈(low + high)/2⌉
        (leftlow, lefthigh, leftsum) = MAX − SUBARRAY(P, low, mid)
        (rightlow, righthigh, rightsum) = MAX − SUBARRAY(P, mid + 1, high)
        (crosslow, crosshigh, crosssum) = MAX − CROSSING − SUBARRAY(A, low, mid, high)
        if leftsum ≥ rightsum and leftsum ≥ crosssum then
            return (leftlow, lefthigh, leftsum)                ▷ entirely-left subarray
        else if rightsum ≥ leftsum and rightsum ≥ crosssum then
            return (rightlow, righthigh, rightsum)             ▷ entirely-right subarray
        else
            return (crosslow, crosshigh, crosssum)             ▷ mid-crossing subarray
        end if
    end if
end function
```

Maximum subarray analysis

D&C runtime is

$$T(n) = 2T(n/2) + \Theta(n)$$

Solves to $\Theta(n \log n)$, by master theorem, same as merge sort.

Brute force was $\Theta(n^2)$

- D&C is much faster
- perhaps counterintuitive due to recursion's reputation for sloth
- D&C benefits from observation that subarrays are contiguous, so extend in two directions from a middle
- brute force is oblivious to this
- human mathematical insight eliminates wasted effort

Matrix multiplication

Matrix multiplication problem
input: A, B each an $n \times n$ matrix
output: matrix product $C = AB$

Recall notation: element at row i and column j of matrix A is denoted a_{ij}

Definition of matrix multiplication:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}.$$

Naïve matrix multiplication

```
function MATRIX-MULTIPLY(A, B)
    C = new n × n matrix
    for i from 1 to n do
        for j from 1 to n do
            c_ij = 0
            for k from 1 to n do
                c_ij = c_ij + a_ik · b_kj
            end for
        end for
    end for
    return C
end function
```

$\Theta(n^3)$ time

Is naïve optimal?

The definition of matrix multiplication involves a sum that is iterated n times, for each of the $n \times n$ elements of C, which might seem to require exactly n^3 scalar multiply instructions, and imply an $\Omega(n^3)$ lower bound for matrix multiplication.

Surprise! Strassen's algorithm (1969) takes $O(n^{\lg 7}) = O(n^{2.81})$ time; more complicated Williams-Le Gall algorithm (2014) takes $O(n^{2.37})$ time

Insight: per the definition of matrix multiplication, some elements of A and B are multiplied together more than once; avoid duplicating these efforts.

Moving to divide-and-conquer

Suppose n is an even power of 2, i.e. $n = 2^k$ for $k \geq 0$
(Can preprocess A, B by adding padding zeroes, then trim the zeroes out of C.)
Divide A into four equal-size submatrices, and same for B, C.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

so we can compute C as

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Moving to divide-and-conquer (continued)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

can be broken down into four separate computations

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

each of which can be performed recursively.

Divide-and-conquer matrix multiplication — try 1

```
function MMR(A, B)
    C = new n × n matrix
    if n == 1 then
        c_11 = a_11 · b_11
    else
        quadrisect A, B, C
        C_11 = MMR(A_11, B_11) + MMR(A_12, B_21)
        C_12 = MMR(A_11, B_12) + MMR(A_12, B_22)
        C_21 = MMR(A_21, B_11) + MMR(A_22, B_21)
        C_22 = MMR(A_21, B_12) + MMR(A_22, B_22)
    end if
    return C
end function
```

Analysis

- each of the submatrices A_{11}, etc. has size $n/2$
- quadrisecting A, B is $\Theta(n^2)$ time; same for assembling C
- each matrix $+$ takes $\Theta((\frac{n}{2})^2) = \Theta(\frac{n^2}{4}) = \Theta(n^2)$ time
- 8 recursive calls

$$T(n) = 8T(n/2) + \Theta(n^2)$$

Solves to $T(n) \in \Theta(n^3)$ by master theorem; same as naïve

Observe: the 8 factor is meaningful, but the $\frac{1}{4}$ isn't
$\implies$ it's a win to have fewer recursive calls, but more work (by a constant factor) in the **combine** step

Strassen's insight

Use algebra to refactor into 7 recursive multiplies instead of 8

1. quadrisect A, B, C as before
2. create 10 $(n/2) \times (n/2)$ submatrices $S_1, \ldots, S_{10}$ using matrix $+$ and -
3. recursively compute 7 submatrix products $P_1, \ldots, P_7$ in terms of the matrices from steps 1, 2
4. compute $C_{11}, C_{12}, C_{21}, C_{22}$ using matrix $+$ and -

$$
\begin{aligned}
T(n) &= \Theta(n^2) + \Theta(10\frac{n}{4}) + 7T(n/2) + T(4\frac{n}{4}) \\
&= 7T(n/2) + \Theta(n^2) \\
&\in \Theta(n^{\lg 7})
\end{aligned}
$$

by master theorem

Divide-and-conquer matrix multiplication — try 2

```
function MMS(A, B)
    C = new n × n matrix
    if  n == 1  then
        c11 = a11 · b11
    else
        quadrisect A, B, C
        form S1, . . . , S10 as shown on next slide
        P1 = MMS(A11, S1)
        P2 = MMS(S2, B22)
        P3 = MMS(S3, B11)
        P4 = MMS(A22, S4)
        P5 = MMS(S5, S6)
        P6 = MMS(S7, S8)
        P7 = MMS(S9, S10)
        C11 = P5 + P4 − P2 + P6
        C12 = P1 + P2
        C21 = P3 + P4
        C22 = P5 + P1 − P3 − P7
    end if
    return C
end function
```

Details of Strassen's algorithm

$$
\begin{aligned}
S_1 &= B_{12} - B_{22} \\
S_2 &= A_{11} + A_{12} \\
S_3 &= A_{21} + A_{22} \\
S_4 &= B_{21} - B_{11} \\
S_5 &= A_{11} + A_{22} \\
S_6 &= B_{11} + B_{22} \\
S_7 &= A_{12} - A_{22} \\
S_8 &= B_{21} + B_{22} \\
S_9 &= A_{11} - A_{21} \\
S_{10} &= B_{11} + B_{12}
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 \\
P_2 &= S_2 \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 \\
P_5 &= S_5 \cdot S_6 \\
P_6 &= S_7 \cdot S_8 \\
P_7 &= S_9 \cdot S_{10} \\
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Editorial Commentary

- proof that 7 recursive multiplies suffice, instead of 8, is surprising and therefore interesting
- equations on previous slide are relatively uninteresting (though not unimportant) **technical detail**
- $o(n^3)$ matrix multiply is of great theoretical interest (because surprise)
- but the naïve alg. has substantially better constant factors, and the gap between $\Theta(n^3)$ and $\Theta(n^{2.81})$ is narrow
- Strassen (and descendants) are only practical for very large n
- in practice: naïve alg. for base case $n < 128$ (say)

Takeaways

Recall

- insertion sort is $\Theta(n^2)$; D&C merge sort is $\Theta(n \log n)$
- brute force maximum subarray is $\Theta(n^2)$; D&C alg. is $\Theta(n \log n)$
- naïve matrix multiply is $\Theta(n^3)$; Strassen's alg. is $\Theta(n^{2.81})$

In each case study,

- first try was no faster; just using D&C isn't an automatic improvement
- master method analyses hinted at the bottleneck
- shift work around to decrease asymptotic time complexity (but increase constant factors); beneficial trade-off
- optimization comes from human insight into the problem
- unclear how to make these insights w/o the D&C framing

Master Method

- **Master method:** plug-and-chug process for solving some recurrences
  - doesn't work for *all*
  - but works for *typical* D&C recurrences
- **Master theorem:** proof that the method is sound

Master Theorem

Let $a \geq 1, b > 1$ be constants, $f(n)$ be a function, and $T(n)$ be the recurrence

$$T(n) = aT(n/b) + f(n).$$

Then

1. If $\exists \epsilon > 0$ such that $f(n) = O(n^{log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = O(n^{log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{log_b a + \epsilon})$, and $af(n/b) \leq cf(n)$ for some $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$.

Step 1: Identify Relevant Case

3 cases: $f(n)$ is asymptotically

1. less than,
2. equal,
3. greater than

the benchmark $n^{\log_b a}$.

So identify a and b, substitute into $n^{\log_b a}$, simplify, and decide among the cases.

If unsure, take the limit

$$\lim_{n \to \infty} \frac{f(n)}{n^{\log_b a}}.$$

Step 1: Identify Relevant Case

Example:

$$T(n) = 2T(n/2) + 7$$

Identify: $a = 2$, $b = 2$
Plug and chug:

$$n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$$

Is 7 less than, equal, or greater than n?

Intuition: less than

Check:

$$\lim_{n \to \infty} \frac{7}{n} = 0$$

Step 2 alternative 1: Justify Case 1

Need to Prove: If $\exists\epsilon > 0$ such that $f(n) = O(n^{log_b a-\epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.

Prove by showing an example of a ϵ that makes $f(n) = O(n^{log_b a-\epsilon})$.

Continuing example: show ϵ s.t. $7 = O(n^{1-\epsilon})$ Choose $\epsilon = 1$, so we have $7 = O(n^{1-(1)}) = O(n^0) = O(1)$

Justification: "We have $f(n) = 7$ and $n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$. Let $\epsilon = 1$; then $f(n) = O(n^{\log_b a-\epsilon}) = O(n^{(1)-(1)}) = O(n^{(0)}) = O(1)$, and by case 1 of the master theorem, $T(n) = \Theta(n^{\log_b a}) = \Theta(n)$."

Step 2 alternative 2: Justify Case 2

Need to Prove: If $f(n) = O(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.

Case 2 is true without qualification; don't need to show anything else.

Example: $T(n) = 2T(n/2) + 5n$, so $a = 2$, $b = 2$, and $f(n) = 5n$.

$n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$ is asymptotically equal to $f(n) = 5n$ so case 2 applies and $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n \log n)$.

Justification: "We have $f(n) = 5n$ and $n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$. Case 2 of the master theorem applies, so $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^{(1)} \log n) = \Theta(n \log n)$. "

Step 2 alternative 3: Justify Case 3

Need to Prove: If $\exists\epsilon > 0$ such that $f(n) = \Omega(n^{log_b a+\epsilon})$, and $af(n/b) \leq cf(n)$ for some $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$.

Prove by showing examples of ϵ, c that makes $f(n) = O(n^{log_b a+\epsilon})$ and $af(n/b) \leq cf(n)$ for large n.

Example: $T(n) = 2T(n/2) + n^2$, so $a = 2$, $b = 2$, and $f(n) = n^2$.

$n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$. Can choose $\epsilon = 1$ to have $\Omega(n^{1+(1)})$.

Step 2 alternative 3: Justify Case 3 (cont'd)

Need

$$af(n/b) \leq cf(n)$$
$$(2)f(\frac{n}{2}) \leq c(n^2)$$
$$2 \cdot (\frac{n}{2})^2 \leq cn^2$$
$$2 \cdot \frac{n^2}{4} \leq cn^2$$
$$\frac{1}{2} \leq c$$

Any c satisfying $\frac{1}{2} \leq c < 1$ can work; arbitrarily choose $c = \frac{3}{4}$.

Step 2 alternative 3: Justify Case 3 (cont'd)

Justification: "We have $f(n) = n^2$ and $n^{\log_b a} = n^{\log_{(2)}(2)} = n^{(1)} = n$. Let $\epsilon = 1$; then $f(n) = \Omega(n^{\log_b a + \epsilon}) = \Omega(n^{(1)+(1)}) = \Omega(n^2)$. Let $c = \frac{3}{4}$; then $af(n/b) = (2)f(n/(2)) = 2(n/2)^2 = 2(n^2/4) = n^2/2 \leq cf(n) = (\frac{3}{4})n^2$ for sufficiently large n. Case 3 of the master theorem applies, so $T(n) = \Theta(f(n)) = \Theta(n^2)$. "

Limitations Of The Master Method

Master Theorem is a big "if/then"

Theorem does not apply when:

- $T(n)$ not in the necessary form, or
- none of cases 1, 2, 3 apply

There are gaps between the cases.

However, when an algo. is designed according to the D&C pattern, the master theorem almost always applies.