# 15. Approximate Set Cover and Bin Packing

CPSC 535

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## Set Cover: Intuition

- list of **needs**
- list of **services**
  - each service meets some of the needs
- **puzzle:** shortest list of products that meets all the needs?

## Set Cover: Formal Definition

*set cover problem*
**input**: a universe set $X$, and family $\mathfrak{F}$ of subsets of $X$, such that $X = \bigcup_{S \in \mathfrak{F}} S$
**output**: a minimum size subfamily $\mathfrak{C} \subseteq \mathfrak{F}$ whose members cover all of $X$, so $X = \bigcup_{S \in \mathfrak{C}} S$

## Application: Streaming Services

- **needs:** stream TV shows $A, B, C, D, E, F$
- $X = \{A, B, C, D, E, F\}$
- **services:** alternative streaming services; each offer only some shows
- $\mathfrak{F} = \{\{A, F\}, \{A, C, E\}, \{B, E\}, \dots\}$
- **puzzle:** subscribe to smallest number of services that provide all desired shows

## Application: Menu Design

- **needs:** menu has a food option available for various dietary needs
- $X = \{carnivore, vegan, kosher, halal, glutenfree, \ldots\}$
- **services:** alternative entrees
- $\mathfrak{F} = \{\{carnivore, halal\}, \{vegan\}, \{kosher, carnivore\}, \ldots\}$
- **puzzle:** design a menu with the fewest number of food options so that everyone can eat something

## Set Cover Hardness

- set cover is *NP*-complete
- baseline algorithm: for each subset $\mathfrak{C} \subseteq \mathfrak{F}$, check if the sets in $\mathfrak{C}$ contain all elements, keep track of the smallest such $\mathfrak{C}$
- $\Theta(2^n \cdot n)$ time, slow

## Set Cover Approximation Algorithm

```
function APPROX-SET-COVER($X, \mathfrak{F}$)
    $U_0 = X$                                          ▷ still-uncovered elements
    $\mathfrak{C} = \emptyset$
    $i = 0$
    while $U_i \neq \emptyset$ do
        // choose set with most currently-uncovered elements
        Find $S \in \mathfrak{F}$ that maximizes $|S \cap U_i|$
        $U_{i+1} = U_i - S$
        $\mathfrak{C} = \mathfrak{C} \cup \{S\}$
        $i = i + 1$
    end while
    return $\mathfrak{C}$
end function
```

## Efficiency Analysis

- while loop: $\Theta(n)$ iterations
  - Find: $\Theta(n)$ time (assuming fast data structure to look up $U_i$)
  - $U_{i+1} = U_i - S$: $\Theta(n)$ time
- other steps: $\Theta(1)$ time each
- total time $\Theta(n^2)$
- can be sped up to $\Theta(n)$ (CLRS Exercise 35.3-3)

## Approximation Ratio

**Theorem:** APPROX-SET-COVER is a $O(\lg n)$-approximation algorithm
Proof sketch:

- Let $\mathfrak{C}^\star$ be the optimal cover and $k^\star = |\mathfrak{C}^\star|$
- $\mathfrak{C}^\star$ covers all of $X$, and each $U_i \subseteq X$, so $\mathfrak{C}^\star$ covers each $U_i$
- each $U_i$ can be covered with $\leq k^\star$ sets from $\mathfrak{F}$
- on average, $\mathfrak{C}^\star$ covers $n/k^\star$ elements/set
- so at least one set in $\mathfrak{F}$ covers $\geq n/k^\star$ elements
- APPROX-SET-COVER picks the set that covers the most elements, so each $S$ covers at least $n/k^\star$ additional elements, and

$$|U_{i+1}| \leq |U_i| - |U_i|/k^\star = |U_i|(1 - 1/k^\star)$$

## Approximation Ratio (continued)

$$|U_{i+1}| \leq |U_i|(1 - 1/k^\star)$$

- algorithm stops when some $|U_i| = 0$
- as a recurrence,
$$T(n) = (1 - 1/k^\star)n$$
- algebra and log rules show $T(n) \in O(k^\star \lg n)$
- each iteration adds one set to $\mathfrak{C}$, so APPROX-SET-COVER picks $O(k^\star \lg n)$ sets
- $k^\star$ is the optimal number of sets, so
- $\therefore$ APPROX-SET-COVER is a $O(\lg n)$-approximation algorithm $\square$

## Set Cover Summary

- set cover is *NP*-complete, exact algorithm takes exponential time
- fast $O(\lg n)$-approximate algorithm
- showed $\Theta(n^2)$ time
- $\Theta(n)$ time is possible

## Big Idea: Linear Programming Relaxation

Recall:

- linear programming with real-valued variables is fast (polynomial time)
- integer linear programming (MIP) is *NP*-complete and slow (exponential time)

Idea:

- formulate our problem as a MIP
- "cheat" and solve it as a LP
- round off each solution variable to the nearest integer

## Big Idea: Linear Programming Relaxation

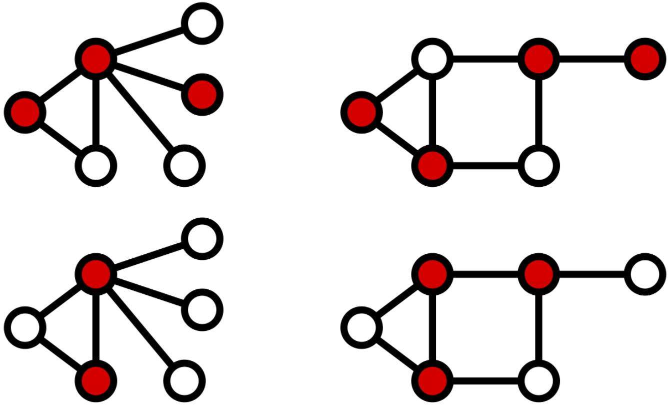
- **LP relaxation:** MIP formulation with integrality constraints removed
- not correct in general
- **but,** sometimes we can prove an approximate performance ratio
- next: algorithm that uses LP relaxation to solve vertex cover

## Review: Vertex Cover

*vertex cover problem*
**input**: undirected graph $G = (V, E)$
**output**: set of vertices $C \subseteq V$, of minimal size $|C|$, such that every edge in $E$ is incident on at least one vertex in $C$

- $NP$-complete
- previous deck: greedy algorithm, 2-approximate, $\Theta(m + n)$ time

## Vertex Cover Example



Images credit: Wikipedia user Miym, CC BY-SA 3.0, `https://commons.wikimedia.org/wiki/File:Vertex-cover.svg`, `https://commons.wikimedia.org/wiki/File:Minimum-vertex-cover.svg`

## Review: Formulating Vertex Cover

**Variables:** for each $v \in V$, create an integer variable $x_v$ such that

$$x_v = 1 \Leftrightarrow v \in C$$

**Objective:** minimize

$$\sum_{v \in V} x_v$$

**Constraints:**

$$\begin{array}{lll} 0 \le x_v \le 1 & \forall v \in V & \text{(0 or 1 indicator)} \\ x_u + x_v \ge 1 & \forall (u,v) \in E & \text{(each edge is covered)} \end{array}$$

## Review: Vertex Cover Outcomes

- **Infeasible:**
  - never happens
  - $\exists$ a solution: setting all $x_v = 1$ satisfies all constraints
- **Unbounded:**
  - never happens
  - objective is bounded: the objective function is to minimize

    $$\sum_{v \in V} x_v;$$

    since every $x_v \geq 0$, the minimum objective value is zero, which is finite, so the program is never unbounded
- **Solution:** Construct $C$ as

  $$C = \{v \mid v \in V \text{ and } x_v = 1\}$$

## Vertex Cover LP Relaxation

**Variables:** for each $v \in V$, create a real-valued variable $x_v$ such that

$$x_v = 1 \Leftrightarrow v \in C$$

**Objective:** minimize

$$\sum_{v \in V} x_v$$

**Constraints:**

$$\begin{array}{lll} 0 \le x_v \le 1 & \forall v \in V & \text{(fuzzy 0 or 1 indicator)} \\ x_u + x_v \ge 1 & \forall (u, v) \in E & \text{(each edge is covered)} \end{array}$$

## LP Relaxation Vertex Cover Algorithm

```
1:  function APX-VC-RELAX($G = (V, E)$)
2:      $C = \emptyset$
3:      $LP$ = the linear program from the previous slide
4:      $\bar{x} = SOLVE - LP(LP)$                    ▷ assume $LP$ has solution
5:      for each vertex $v \in V$ do
6:          if $x_v \geq \frac{1}{2}$ then             ▷ using $x_v \in \bar{x}$
7:              $C = C \cup \{v\}$
8:          end if
9:      end for
10:     return $C$
11: end function
```

## Correctness

- $LP$ is never unbounded or infeasible
- must prove that $C$ is a valid vertex cover
- need, for each edge $\{u, v\} \in E$, that $u \in C$ or $v \in C$ (or both)
- LP relaxation has constraints $\quad x_u + x_v \geq 1 \quad \forall (u, v) \in E \quad$ (each edge is covered)
- solution $\bar{x}$ satisfies all constraints, so

  $$x_u + x_v \geq 1$$

  is true and

  $$x_u \geq \frac{1}{2} \text{ or } x_v \geq \frac{1}{2} \text{ (or both)}$$
- so the **for** loop adds at least one of $u, v$ to $C$

## Efficiency Analysis

- create $LP : \Theta(n+m)$
- solve $LP$ : polynomial
- post-processing **for** loop: $\Theta(n)$
- total time

$$\Theta(n+m) + \Theta(\text{solve LP}) + \Theta(n) = \Theta(\text{solve LP})$$

- polynomial time

## Approximation Ratio

**Theorem:** APX-VC-RELAX is a 2-approximation algorithm

Proof sketch:

- let $C^\star$ be an optimal vertex cover for $G$
- need to prove $|C| \le 2|C^\star|$
- use a “common ground” comparison between $|C|$ and $|C^\star|$
- let

$$\begin{aligned} z^\star &= \text{objective function value of LP} \\ &= \sum_{v \in V} x_v \text{ using each } x_v \in \bar{x} \end{aligned}$$

- we use $z^\star$ to relate $|C|$ to $|C^\star|$

## Relating $z^\star$ to $|C^\star|$

- $z^\star$ is objective value $f(C)$ for our relaxed LP
- $C^\star$ is solution to MIP, with more constraints (integer $x_v$)
- so

$$\begin{aligned} z^\star &\leq f(C^\star) \\ &= |C^\star| \end{aligned}$$

## Relating $z^\star$ to $|C|$

- now relate $z^\star$ to $|C|$ :

$$\begin{aligned} z^\star &= \sum_{v \in V} x_v \\ &\geq \sum_{v \in V, x_v \geq 1/2} x_v \\ &\geq \sum_{v \in V, x_v \geq 1/2} \left(\frac{1}{2}\right) \\ &= \sum_{v \in C} \frac{1}{2} \\ &= \frac{1}{2}|C| \end{aligned}$$

## Completing the Proof of Approximation Ratio

Combine

$$z^\star \leq |C^\star|$$

with

$$z^\star \geq \frac{1}{2}|C|$$

to obtain

$$\frac{1}{2}|C| \leq z^\star \leq |C^\star|$$

or

$$|C| \leq 2 \cdot |C^\star|.$$

QED.

## Vertex Cover LP Relaxation Summary

- LP relaxation approach:
  - formulate vertex cover as MIP
  - remove integer constraints, solve as LP
  - round each solution variable to nearest integer
- same polynomial runtime as linear programming
- 2-approximation
- compared to greedy algorithm in previous slides, this algo. is
  - simpler
  - slower
- generalizes to **weighted** case (see textbook section 35.5)

## Bin Packing: Intuition

- have a collection of **items**
- want to **pack** them tightly into containers
- **puzzle**: which items go together in each container?

## Bin Packing: Formal Definition

*bin packing problem*
**input**: a multiset $U = \{u \in \mathbb{Q}, 0 < u \leq 1\}$ of item sizes
**output**: a partition $B_1, B_2, \ldots, B_k$ of $U$ into $k$ multisets, such that the sum of each $B_i$ is at most 1

- bin capacity is 1
- each size $u \in U$ is a fraction between 0 and 1
- ex. $U = \{\frac{2}{3}, \frac{1}{2}, \frac{1}{9}, \frac{1}{2}, \ldots\}$
- $k$ = number of bins used

## Example Applications

- Given a sink full of dirty dishes, how to load the dishwasher to clean all the dishes in the fewest loads?
- Given an Amazon order for items of varying weights, how to pack the items into the fewest shipping boxes?
- Given a set of virtual machines (VMs) of varying memory sizes, how to host them on the fewest physical servers?

## Generalizations of Bin Packing

Problem statement can be generalized to be more realistic:

- items are 2D shapes instead of numbers (physical object shipping)
- items can partially overlap (VM shared memory can overlap)
- one bin, different values: *knapsack problem*
- minimize bins, and also waste: *cutting stock problem*

## Bin Packing Hardness

- bin packing is *NP*-complete
- generalizations (ex. 2D shapes) are even harder
- baseline algorithm:
  - loop through each possible number of bins $k = 1, \dots, n$
  - for each item $u \in U$ : try placing $u$ in all $k$ bins, then recursively place the remaining items
- $\Theta(n \cdot n!)$ time, extremely slow
- (exponential time is possible too)

## Greedy Algorithm Idea

- keep a list of bins
- for each item $u$ : find any bin with enough room for $u$, and put it there
- if no bin has enough room: start a new bin holding just $u$
- **“first fit”** algorithm

## First-Fit Algorithm

```
function FIRST-FIT-BIN-PACK($U$)
    $B = \varnothing$                                   ▷ the bins
    for $u \in U$ do
        $packed = false$
        for $B_i \in B$ do
            if $(u + \sum B_i) \leq 1$ then             ▷ does $u$ fit in $B_i$?
                $B_i = B_i \cup \{u\}$
                $packed = true$
                break loop
            end if
        end for
        if $packed == false$ then
            $B_k = \{u\}$                               ▷ $u$ in its own new bin
            $B = B \cup \{B_k\}$
        end if
    end for
    return $B$
end function
```

## Efficiency Analysis

- outer **for** loop: $\Theta(n)$ iterations
- $|B| \leq n$, so
- inner **for** loop: $\Theta(n)$ iterations
- $\sum B_i : \Theta(n)$ time
- other statements are $\Theta(1)$ time
- $\therefore\ \Theta(n^3)$ total time
- can speed up to $\Theta(n \log n)$ by caching totals and storing bins in a BST

## Approximation Ratio

**Theorem**: FIRST-FIT-BIN-PACK is a 2-approximation algorithm.
Proof sketch:

- Recall $k = |B|$
- Let $B^\star$ be an optimal multiset of bins, and $k^\star = |B^\star|$
- (again) use "common ground" comparison between $k$ and $k^\star$
- Let $t = \sum U$ be the sum of all items; use $t$ as common ground
- Best possible packing fills every single bin with no leftover space, so

$$k^\star \geq \frac{\text{total size of items}}{\text{size of each bin}} = \frac{(t)}{(1)} = t$$

## There Is At Most One Light Bin

- Call a bin $B_i$ **"light"** when $\sum B_i \leq \frac{1}{2}$, otherwise **"heavy"**
- Invariant: there is at most one light bin
- Induction on number of items in bins
- Base case: zero items $\implies$ zero bins $\implies$ no light bin
- Inductive case: given new item $u$, four cases:

| | $u \leq \frac{1}{2}$ | $u > \frac{1}{2}$ |
|---|---|---|
| no light bin | $u$ starts only light bin | $u$ starts new heavy bin |
| $\exists$ light bin $B_i$ | $u$ joins $B_i$ | $u$ either joins $B_i$ making it heavy; or starts a new heavy bin |

## Approximation Ratio (continued)

- worst case is one light bin and $k - 1$ barely-heavy bins
- $k$ is at most

$$k \leq \frac{\text{total size of items}}{\text{size of each bin}} + 1 < \frac{(t)}{(1/2)} + 1 = 2t + 1$$

- we showed $k^\star \geq t$, so

$$k < 2t + 1 \leq 2(k^\star) + 1$$

- for large $n$,

$$k \propto 2k^\star$$

- QED

## Tight Approximation Ratio

- (Johnson, Demers, Ullman, Garey, Graham 1974)
- FIRST-FIT-BIN-PACK is a 1.7-approximation algorithm
  - More elaborate case analysis
- There is a $U$ for which FIRST-FIT-BIN-PACK achieves only a 1.7 performance ratio
  - $U = \{\frac{6}{101} \times 7, \frac{10}{101} \times 7, \frac{16}{101} \times 3, \frac{34}{101} \times 10, \frac{51}{101} \times 10\}$
  - $k^\star = 10$
  - $k = 17$
- $\therefore$ the $1.7k^\star$ bound is tight

## Bin Packing Summary

- bin packing is *NP*-complete, exact algorithm takes exponential time
- $\Theta(n \log n)$ time, 1.7-approximation algorithm
- we only proved $\Theta(n^3)$ time, 2-approximation