

07. Maximum Flow

CPSC 535

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Big Idea: Algorithm Frameworks

Algorithm framework: an algorithm with modular parts that can be swapped in for different performance properties; or to solve different but related problems

Example: hash tables are a framework, can swap in

- ▶ different collision resolution strategy (chaining, probing)
- ▶ different hash function (universal hash, linear congruential hash, etc.)

A framework generalizes several algorithm ideas into one pattern; “chunking”

Big Idea: Iterative Pattern

Recall greedy pattern:

1. initialize base-case result
2. for each piece of input,
update result

Iterative pattern (a.k.a.
fixed-point algorithm):

1. initialize base-case result
2. while result is not optimal:
 - 2.1 improve result one step

The *fixed point* is the moment when the result becomes optimal.

Both use a *greedy heuristic*; iterative pattern makes a problem-wide decision.

Big Idea: Problem Duality

problem duality: when the input/output mathematical definition of a problem can be interpreted by humans in two (or more) very different ways

- ▶ one algorithm can solve multiple problems with different “stories”
- ▶ algorithms, computers, don’t actually care what data values mean
- ▶ turns out max-flow and min-cut are two different stories for the same problem
- ▶ max-flow and min-cut are the *dual of each other*

Duality Example

maximum y coordinate

input: a set of (x, y) points $S = \{(x, y) \mid x, y \in \mathbb{R}\}$

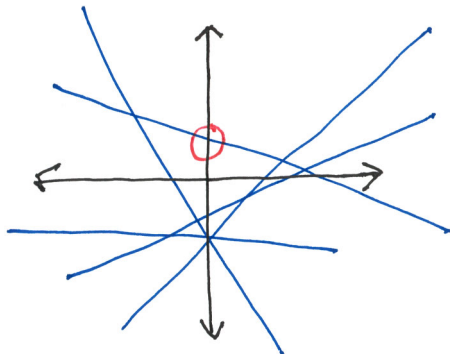
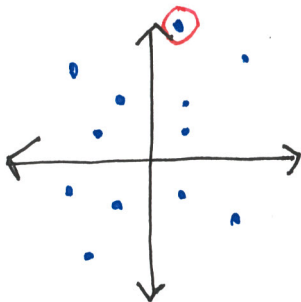
output: the greatest y -coordinate in S

highest y -intercept point problem

input: a set of $y = mx + b$ lines $L = \{(m, b) \mid m, b \in \mathbb{R}\}$

output: the greatest y -intercept b in L

Geometry Sketch



C++ functions for these would be declared like:

```
double maximum_y_coord(vector<pair<double, double>>& points);  
double highest_y_intercept (vector<pair<double, double>>& lines);
```

As far as the computer is concerned, these are interchangeable!

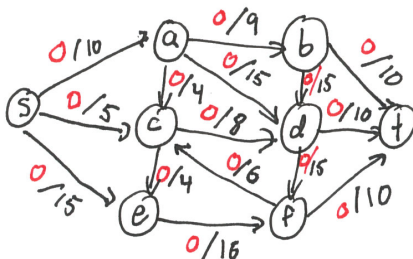
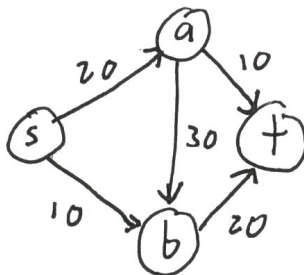
Only the human story differs. The **maximum y coordinate** and **highest y-intercept point problem** problems are the dual of each other.

Defining Maximum Flow 1/2: Flow Networks

flow network: graph representing resource flows

- ▶ directed graph $G = (V, E)$
- ▶ designated *source vertex* $s \in V$ and *sink vertex* $t \in V$
- ▶ **no self-loop**: $\forall v \in V, (v, v) \notin E$
- ▶ **no antiparallel edges**: for any $\forall (u, v) \in E, (v, u) \notin E$
- ▶ flow is possible through every vertex: $\forall v \in V$, there exists some path $s \rightsquigarrow v \rightsquigarrow t$
- ▶ *capacity*: $\forall (u, v) \in E$, there is a defined, non-negative real capacity $c(u, v)$
- ▶ implies: G is connected and $|E| \geq |V| - 1$

Flow Network Sketches



Defining Maximum Flow 2/2: Flows

flow: settings for how much capacity to use on each edge

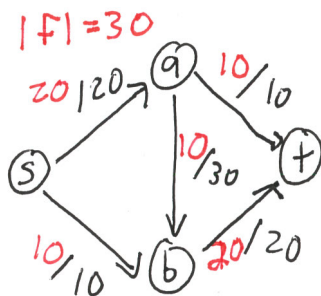
- ▶ candidate for maximum flow: follows the “rules,” but not necessarily optimal
- ▶ modeled as function $f(u, v)$ over vertices u, v
- ▶ **nonexistent edges**: if $(u, v) \notin E$ then $f(u, v) = 0$
- ▶ **capacity constraint**: $0 \leq f(u, v) \leq c(u, v)$
- ▶ **flow conservation**: (flow-in) = (flow-out), except for source and sink; formally, $\forall u \in V - \{s, t\}$,

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

- ▶ *value* $|f|$ = net flow into sink

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

Flow Sketch



Maximum Flow Problem Definition

maximum flow problem

input: a flow network G

output: a flow f of maximum value $|f|$

Ford-Fulkerson Method

“method” because this is a framework for specific max-flow algorithms

- ▶ not a complete, clear alg. yet
- ▶ based on iterative improvement pattern

```
1: function ITERATIVE-IMPROVEMENT(input)
2:   result = base-case result
3:   while result is not optimal do
4:     improve result
5:   end while
6:   return result
7: end function
```

Ford-Fulkerson Method

```
1: function FORD-FULKERSON-METHOD( $G, s, t$ )
2:    $f$  = flow with every edge set to zero
3:   initialize residual network  $G_f$ 
4:   while there exists an augmenting path  $p$  in  $G_f$  do
5:     augment flow  $f$  along path  $p$ 
6:   end while
7:   return  $f$ 
8: end function
```

Need to explain

- ▶ *residual network*
- ▶ *augmenting path*
- ▶ why this terminates and is correct

Residual Networks

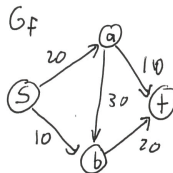
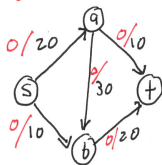
- ▶ residual network G_f has same vertices as flow network $G = (V, E)$
- ▶ edges reflect how much capacity is still available
- ▶ G_f only contains edges with positive available capacity
- ▶ also add “backwards” edges to allow us to take-back some positive flow
- ▶ define *residual capacity* between vertices $v, w \in V$ as

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$

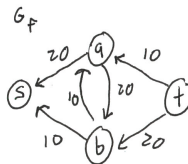
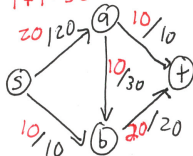
- ▶ (recall that in a flow network either $(u, v) \in E$ or $(v, u) \in E$ but not both)

Residual Network Example

$$|f| = 0$$



$$|f| = 30$$



Augmenting Paths

- ▶ *augmenting path*: simple path from source s to sink t in residual network c_f (*simple* \equiv no repeated vertices)
- ▶ recall: residual network G_f only contains edges with leftover capacity
- ▶ \implies if path p exists in G_f , then every edge along p has positive weight in G_f
- ▶ \implies we can legally increase net $s \rightsquigarrow t$ flow by increasing weights in G_f
- ▶ i.e. increasing flow across the forwards edges in G_f , sometimes decreasing flow across the backwards edges
- ▶ $c_f(p) = \text{residual capacity of } p = \text{minimum weight } c_f(u, v) \text{ of an edge } (u, v) \text{ in } p$

Ford-Fulkerson Method Recap

Recall the Ford-Fulkerson method/pattern:

```
1: function FORD-FULKERSON-METHOD( $G, s, t$ )  
2:    $f$  = flow with every edge set to zero  
3:   initialize residual network  $G_f$   
4:   while there exists an augmenting path  $p$  in  $G_f$  do  
5:     augment flow  $f$  along path  $p$   
6:   end while  
7:   return  $f$   
8: end function
```

still need to

- ▶ clarify how to pick p : modular choice leading to specific algorithms
- ▶ prove correctness and termination: *max-flow min-cut theorem*

Max-Flow Min-Cut Theorem

Lemma: Augmenting a flow f with path p increases $s \rightsquigarrow t$ flow by $c_f(p)$.

Max-Flow Min-Cut Theorem: flow f is maximum iff G_f contains no augmenting path.

If true, any Ford-Fulkerson algorithm computes a correct maximum flow.

But,

- ▶ does not imply that the algorithm terminates
- ▶ does not imply that the $\#$ loop iterations is small
- ▶ need to decide how to pick paths carefully
- ▶ we'll come back to this later

Cuts

- ▶ *cut*: partition $V = S \cup T$, where $s \in S$ and $t \in T$
- ▶ *net flow* across f is

$$f(S, T) = (\text{total flow from } S \text{ to } T) - (\text{flow from } T \text{ to } S)$$

- ▶ *minimum cut* = a cut whose net flow is minimum

Lemma: for any cut (S, T) , net flow $f(S, T) = |f|$.

Proof sketch: since $s \in S$ and $t \in T$, total flow $|f|$ must cross the S - T boundary.

Max-Flow Min-Cut Proof Sketch

Show all these are equivalent conditions:

1. f is a maximum flow
2. G_f contains no augmenting path
3. $|f| = c(S, T)$ for some cut (S, T)

(1) \implies (2) : by definitions of residual network and augmenting path, a maximum flow has no capacity leftover so no paths in G_f

(2) \implies (3) : consider a cut where all vertices reachable from s in G_f are in S and the unreachable are in T ; since there is no $s \rightsquigarrow t$ path in G_f , all edges across the S - T boundary must already be at full capacity

(3) \implies (1): trivially $|f| \leq c(S, T)$, and if $|f| = c(S, T)$ then this (S, T) is maximum

Ford-Fulkerson Detailed Pseudocode

```
1: function FORD-FULKERSON-METHOD( $G = (V, E), s, t$ )
2:   for each edge  $(u, v)$  in  $E$  do
3:      $(u, v).f = 0$ 
4:   end for
5:   while there exists an augmenting path  $p$  in  $G_f$  do
6:      $c_f(p) = \min\{c_f(u, v) : (u, v) \in p\}$ 
7:     for each edge  $(u, v) \in p$  do
8:       if  $(u, v) \in E$  then
9:          $(u, v).f = (u, v).f + c_f(p)$ 
10:      else
11:         $(u, v).f = (v, u).f - c_f(p)$ 
12:      end if
13:     end for
14:   end while
15:   return flow on  $.f$  fields
16: end function
```

Still abstract — need to clarify how we choose path p .

Edmonds-Karp Algorithm

Edmonds-Karp Algorithm is

- ▶ Ford-Fulkerson method from previous page, and...
- ▶ use breadth-first search (BFS) to find the shortest augmenting path
- ▶ (shortest \equiv fewest vertices, irrespective of weights)
- ▶ now a concrete, runnable, implementable algorithm
- ▶ performs $O(|V| \cdot |E|)$ augmentations
- ▶ takes $O(|V| \cdot |E|^2)$ time
- ▶ for $n = |V|$, this is $O(n^3)$ in a sparse graph and $O(n^5)$ in a dense graph
- ▶ more complicated **relabel-to-front** algorithm takes $O(|V|^3) = O(n^3)$ time

Edmonds-Karp Pseudocode for Worked Examples

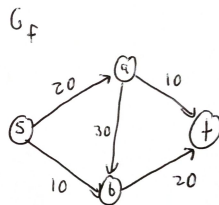
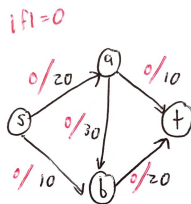
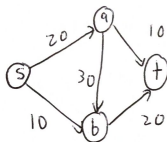
```
1: function EDMONDS-KARP( $G = (V, E), s, t$ )
2:   initialize each edge's flow to 0
3:   repeat
4:     for  $k = 2, 3, \dots, |V|$  do
5:       if  $\exists$  augmenting path  $p$  of length  $k$  then
6:          $c_f(p)$  = minimum excess capacity of any edge in  $p$ 
7:         for edge  $e$  in  $p$  do
8:           if  $p$  follows  $e$  forwards then
9:             increase  $e$ 's flow by  $c_f(p)$ 
10:          else
11:            decrease  $e$ 's flow by  $c_f(p)$ 
12:          end if
13:        end for
14:        break loop
15:      end if
16:    end for
17:  until no path can be found
18:  return flow based on current capacities
19: end function
```


Identifying Edge Capacity in G

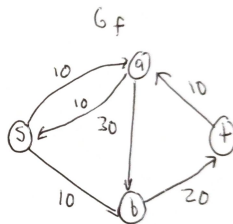
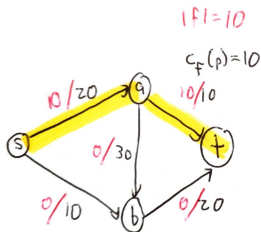
When running this algorithm by hand,

- ▶ you *could* sketch the residual network each time, but this is tedious
- ▶ instead, when looking at edge e with flow x/c
- ▶ if $x < c$, you *may* follow e forwards and add up to $(c - x)$ flow
- ▶ if $x > 0$, you *may* follow e backwards and subtract up to x flow

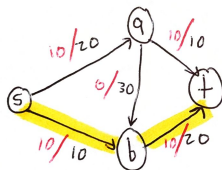
Edmonds-Carp Example 1/2



Edmonds-Carp Example 1/2

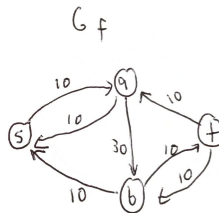


Edmonds-Carp Example 1/2

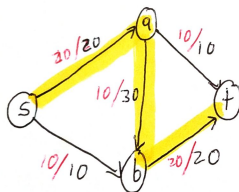


$$c_f(p) = 10$$

$$|f| = 20$$

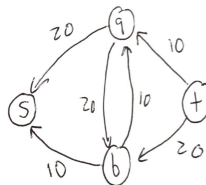


Edmonds-Carp Example 1/2



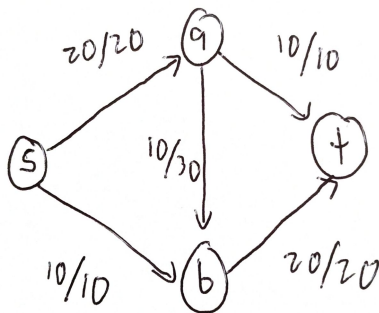
$$C_f(p) = 10$$

$$|f| = 30$$

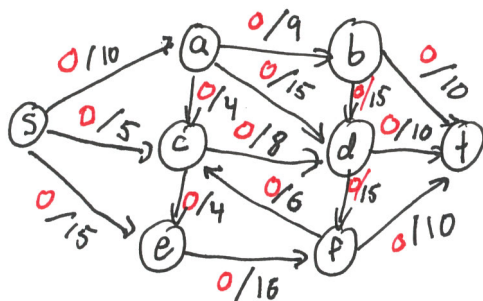


Edmonds-Carp Example 1/2

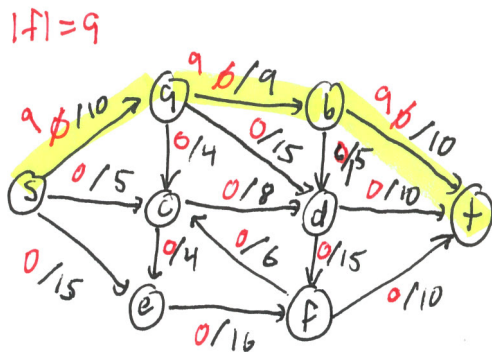
$$|f| = 30$$



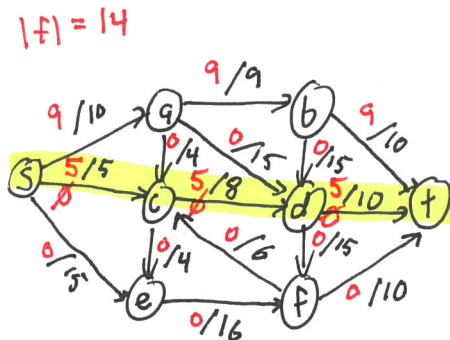
Edmonds-Carp Example 2/2



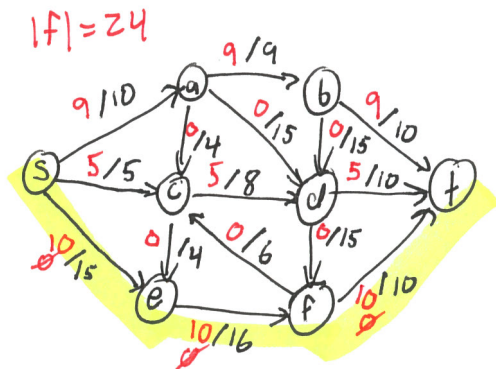
Edmonds-Carp Example 2/2



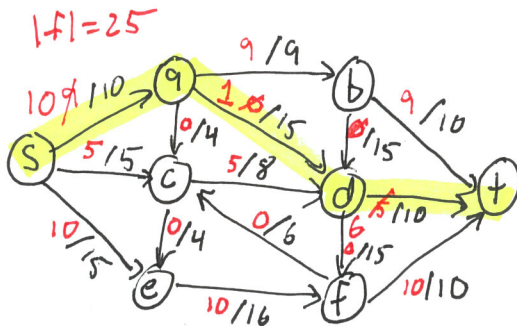
Edmonds-Carp Example 2/2



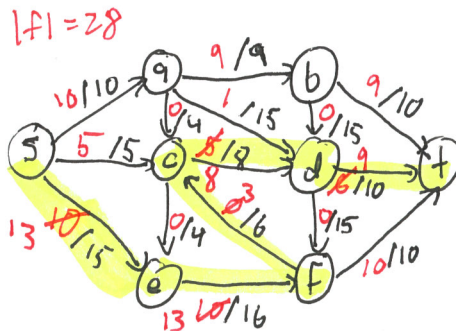
Edmonds-Carp Example 2/2



Edmonds-Carp Example 2/2



Edmonds-Carp Example 2/2



Edmonds-Carp Example 2/2

