

15. Convex Hulls

CPSC 535

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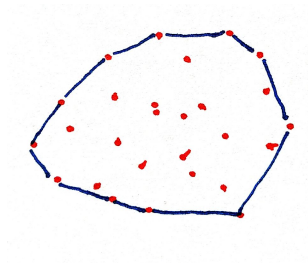
Convex Hulls

convex hull problem

input: set of $n \geq 3$ points Q

output: $CH(Q)$, the subset of Q that is the set of vertices on the convex hull of Q

convex hull \equiv boundary of convex polygon enclosing all of Q



Convex Hull Applications

- ▶ object intersection in raytracing, video games, GUIs
- ▶ drawing implicit regions in GIS
- ▶ finding farthest points (they're always *CH* vertices)
- ▶ component of other algorithms

Approaches to Convex Hulls

Like the sorting problem, many algorithm patterns work for convex hulls, and there is a rich literature of competitive algorithms.

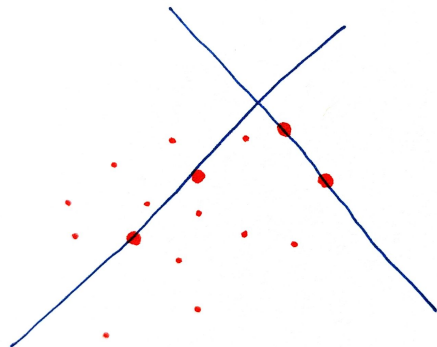
- ▶ Greedy pattern: line-sweep, update hull as we go
- ▶ Divide-and-conquer: divide Q in half, compute convex hulls for each half, merge two convex hulls into one
- ▶ Iterative improvement: start with a superset of $CH(Q)$; refine by repeatedly eliminating a constant fraction of the points until only $CH(Q)$ remains

Baseline Algorithm

Observe

- ▶ any two input points define a line ℓ
- ▶ when those points are both in $CH(Q)$, remaining $n - 2$ points are all on the same side of ℓ (**a geometric property**)
- ▶ \implies for each pair of input points p, q , see whether all other points are on the same side of ℓ
- ▶ if so include p, q in $CH(Q)$

Baseline Algorithm



Baseline Pseudocode

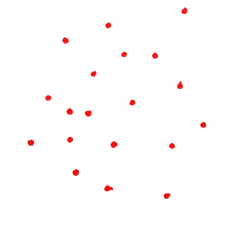
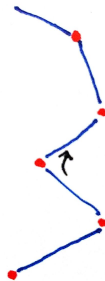
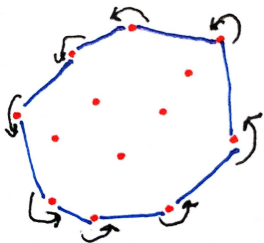
```
1: function NAIVE-CONVEX-HULL( $Q$ )
2:    $H = \emptyset$ 
3:   for distinct points  $p, q \in Q$  do
4:     form line  $\ell$  intersecting  $p$  and  $q$ 
5:      $k = \#$  points above  $\ell$ 
6:     if  $k = (n - 2)$  or  $k = 0$  then
7:        $H = H \cup \{p, q\}$ 
8:     end if
9:   end for
10:  return  $H$ 
11: end function
12:
```

Analysis: $\Theta(n^2)$ iterations, counting $\#$ points is $\Theta(n)$
 $\implies \Theta(n^3)$ time

Graham Scan Idea

- ▶ greedy pattern, reduction-to-sorting
- ▶ **Geometric property:** when touring a CH in counter-clockwise order, we **only make left turns**
- ▶ right turn = exiting a concavity, middle point not in hull
- ▶ \therefore sweep counter-clockwise, keep points that participate in left turns, drop points in the middle of right turns
- ▶ alternative kind of line sweep: rotating the line (not left-to-right)

Graham Scan Idea



Graham Scan Greedy Heuristic

- ▶ $p_1, \dots, p_m = Q$ sorted into counter-clockwise order, eliminating ties
- ▶ stack S of points; contains hull of points visited *already*
- ▶ base case: push first 3 points onto S
 - ▶ for any three points p, q, r forming a non-degenerate triangle, $CH(\{p, q, r\}) = \{p, q, r\}$
- ▶ inductive case:
 - ▶ examine next input point p_i , top of stack t , next-lowest stack point r
 - ▶ if $\angle rtp_i$ is not a left turn $\implies t$ not on hull
- ▶ Note: need stack data structure w/ accessor to top **two** elements

Graham Scan Pseudocode

```
1: function GRAHAM-SCAN( $Q$ )                                ▷ guaranteed  $|Q| \geq 3$ 
2:    $p_0$  = lowest point in  $Q$  (break ties by choosing leftmost point)
3:    $p_1 \dots p_m$  = sort  $Q - \{p_0\}$  into counter-clockwise order, by polar
   angle with  $p_0$ ; break ties by keeping only the point farthest from  $p_0$ 
4:    $S$  = new stack
5:    $S.PUSH(p_0)$ 
6:    $S.PUSH(p_1)$ 
7:    $S.PUSH(p_2)$ 
8:   for  $i$  from 3 through  $m$  do
9:     while  $\angle p_i, S.TOP, S.BELOWTOP$  is non-left turn do
10:       $S.POP()$ 
11:     end while
12:      $S.PUSH(p_i)$ 
13:   end for
14:   return set of point still in  $S$ 
15: end function
```

Graham Scan Analysis

- ▶ find p_0 : $\Theta(n)$
- ▶ sort: $\Theta(n \log n)$
- ▶ eliminate tied points: $\Theta(n)$
- ▶ each stack operation is $\Theta(1)$
- ▶ **for** loop repeats $m < n$ times
- ▶ turn angle test, stack operations are $\Theta(1)$
- ▶ $\Rightarrow \Theta(n \log n)$ time
- ▶ dominating term is sort (*reduction to sorting*)
- ▶ organizing data structure is arrayed stack
- ▶ \Rightarrow good constant factors

Jarvis March

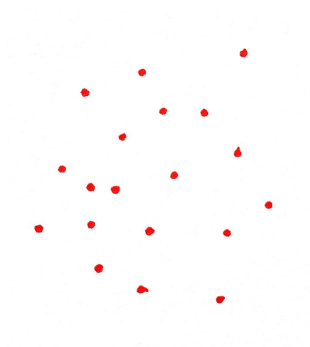
Alternative greedy heuristic: moving around the hull counter-clockwise, each step from one vertex to the next is *the input point whose angle is shallowest*.

⇒ we can start from a *CH* point, then incrementally find one more *CH* point until we're done.

Called “gift wrapping” b/c this resembles carefully wrapping up an irregular object in paper or foil.

(Jarvis march is sometimes called the *gift-wrapping algorithm*.)

Jarvis March



Jarvis March Pseudocode

Jarvis march (Q)

1. $H = \emptyset$
2. Let ℓ = lowest point in Q (min. y -coord.)
3. Let h = highest point in Q
4. (right chain) Starting from ℓ and until we reach h :
 - 4.1 Linear search Q for the next point p_i , minimizing the angle between p_i and the previous point
 - 4.2 Include p_i in H and continue the loop at p_i .
5. (left chain) Repeat the previous process but starting from h and ending at ℓ .
6. Return H

Jarvis March Analysis

Preprocessing to find $h, \ell : \Theta(n)$

Each iteration of the left/right-chain loops identifies one hull point
 \implies in total they iterate h times, where $h \equiv$ number of points on the hull.

linear search inside the loops takes $\Theta(n)$ time.

$\therefore \Theta(nh)$ total time.

Comparison of Convex Hull Algorithms

Algorithm	Time	Main Idea
Graham Scan	$\Theta(n \log n)$	sort, skip right turns
Jarvis March	$\Theta(nh)$	gift-wrapping

What is the relationship between n and h ?

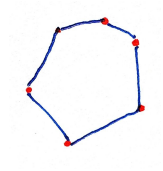
n vs h

Recall

- ▶ $n \equiv \#$ input points = $|Q|$
- ▶ $h \equiv \#$ output points = $\#$ vertices of convex hull = $|CH(Q)|$

For fixed n ,

- ▶ minimum $h = 3$ when all input points are enclosed in a triangle
- ▶ maximum $h = n$ when all input points happen to be convex hull vertices
- ▶ $3 \leq h \leq n$



Summary of Convex Hull Algorithms

FYI

- ▶ **Chan's algorithm** is an optimal output-sensitive algorithm
- ▶ (not covered in book or class)
- ▶ combines both algorithms, divides input points using Graham's heuristic, merges hulls using Jarvis' heuristic
- ▶ $\Theta(n \log h)$ time

Algorithm	Time	$h \in O(1)$	$h \in \Theta(n)$
Graham Scan	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$
Jarvis March	$\Theta(nh)$	$\Theta(n)$	$\Theta(n^2)$
Chan's algorithm	$\Theta(n \log h)$	$\Theta(n)$	$\Theta(n \log n)$