06. Maximum Flow
CPSC 535

Kevin A. Wortman

CALIFORNIA STATE UNIVERSITY
FULLERTON

Big Idea: Algorithm Frameworks

Algorithm framework: an algorithm with modular parts that can be swapped in for different performance properties; or to solve different but related problems

Example: hash tables are a framework, can swap in
- different collision resolution strategy (chaining, probing)
- different hash function (universal hash, linear congruential hash, etc.)

A framework generalizes several algorithm ideas into one pattern; "chunking"

Big Idea: Iterative Pattern

Recall greedy pattern:

1. initialize base-case result
2. for each piece of input, update result

Iterative pattern (a.k.a. *fixed-point algorithm*):

1. initialize base-case result
2. while result is not optimal:
   2.1 improve result one step

The *fixed point* is the moment when the result becomes optimal.

Both use a *greedy heuristic*; iterative pattern makes a problem-wide decision.

Defining Maximum Flow 1/2: Flow Networks

flow network: graph representing resource flows

- ▶ directed graph $G = (V, E)$
- ▶ designated *source vertex* $s \in V$ and *sink vertex* $t \in V$
- ▶ **no self-loop**: $\forall v \in V, (v, v) \notin E$
- ▶ **no antiparallel edges**: for any $\forall (u, v) \in E, (v, u) \notin E$
- ▶ flow is possible through every vertex: $\forall v \in V$, there exists some path $s \rightsquigarrow v \rightsquigarrow t$
- ▶ *capacity:* $\forall (u, v) \in E$, there is a defined, non-negative real capacity $c(u, v)$
- ▶ implies: G is connected and $|E| \geq |V| - 1$

Flow Network Sketches
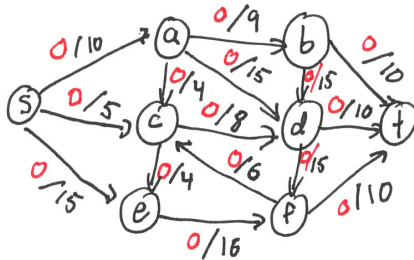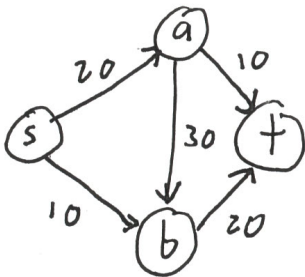
Defining Maximum Flow 2/2: Flows

*flow:* settings for how much capacity to use on each edge

- ▶ candidate for maximum flow: follows the "rules," but not necessarily optimal
- ▶ modeled as function $f(u, v)$ over vertices u, v
- ▶ **nonexistent edges**: if $(u, v) \notin E$ then $f(u, v) = 0$
- ▶ **capacity constraint**: $0 \le f(u, v) \le c(u, v)$
- ▶ **flow conservation**: (flow-in) = (flow-out), except for source and sink; formally, $\forall u \in V - \{s, t\}$,

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

- ▶ *value* $|f|$ = net flow into sink

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

Flow Sketch
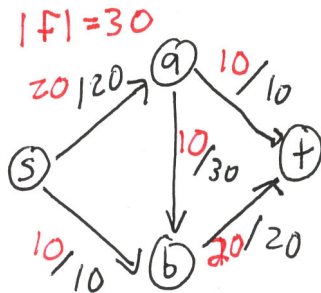
Maximum Flow Problem Definition

maximum flow problem
input: a flow network G
output: a flow f of maximum value $|f|$

Ford-Fulkerson Method

"method" because this is a framework for specific max-flow algorithms

- ▶ not a complete, clear alg. yet
- ▶ based on iterative improvement pattern

```
1: function ITERATIVE-IMPROVEMENT(input)
2:     result = base-case result
3:     while result is not optimal do
4:         improve result
5:     end while
6:     return result
7: end function
```

Ford-Fulkerson Method

```
1: function FORD-FULKERSON-METHOD(G, s, t)
2:     f = flow with every edge set to zero
3:     initialize residual network G_f
4:     while there exists an augmenting path p in G_f do
5:         augment flow f along path p
6:     end while
7:     return f
8: end function
```

Need to explain

- ▶ *residual network*
- ▶ *augmenting path*
- ▶ why this terminates and is correct
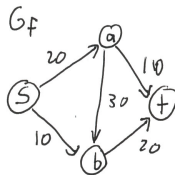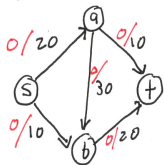
Residual Networks

- residual network G_f has same vertices as flow network $G = (V, E)$
- edges reflect how much capacity is still available
- G_f only contains edges with positive available capacity
- also add "backwards" edges to allow us to take-back some positive flow
- define *residual capacity* between vertices $v, w \in V$ as

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$
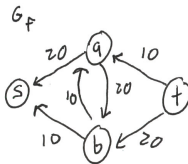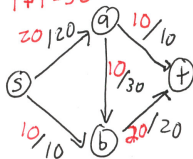
- (recall that in a flow network either $(u, v) \in E$ or $(v, u) \in E$ but not both)

Residual Network Example

Augmenting Paths

- *augmenting path:* simple path from source s to sink t in residual network c_f (*simple* $\equiv$ no repeated vertices)
- recall: residual network G_f only contains edges with leftover capacity
- $\implies$ if path p exists in G_f, then every edge along p has positive weight in G_f
- $\implies$ we can legally increase net $s \rightsquigarrow t$ flow by increasing weights in G_f
- i.e. increasing flow across the forwards edges in G_f, sometimes decreasing flow acress the backwards edges
- $c_f(p) =$ *residual capacity* of p = minimum weight $c_f(u, v)$ of an edge (u, v) in p

Ford-Fulkerson Method Recap

Recall the Ford-Fulkerson method/pattern:

```
1: function FORD-FULKERSON-METHOD(G, s, t)
2:     f = flow with every edge set to zero
3:     initialize residual network G_f
4:     while there exists an augmenting path p in G_f do
5:         augment flow f along path p
6:     end while
7:     return f
8: end function
```

still need to

▶ clarify how to pick p: modular choice leading to specific algorithms

▶ prove correctness and termination: *max-flow min-cut theorem*

Max-Flow Min-Cut Theorem

Lemma: Augmenting a flow f with path p increases $s \rightsquigarrow t$ flow by $c_f(p)$.

Max-Flow Min-Cut Theorem: flow f is maximum iff G_f contains no augmenting path.

If true, any Ford-Fulkerson algorithm computes a correct maximum flow.
But,

- ▶ does not imply that the algorithm terminates
- ▶ does not imply that the # loop iterations is small
- ▶ need to decide how to pick paths carefully
- ▶ we'll come back to this later

Cuts

- *cut:* partition $V = S \cap T$, where $s \in S$ and t $in T$
- *net flow* across f is

$$f(S, T) = (\text{total flow from } S \text{ to } T) - (\text{flow from } T \text{ to } S)$$

- *minimum cut* = a cut whose net flow is minimum

Lemma: for any cut (S, T), net flow $f(S, T) = |f|$.
Proof sketch: since $s \in S$ and $t \in T$, total flow $|f|$ must cross the S–T boundary.

Max-Flow Min-Cut Proof Sketch

Show all these are equivalent conditions:

1. f is a maximum flow
2. G_f contains no augmenting path
3. $|f| = c(S, T)$ for some cut (S, T)

$(1) \implies (2)$: by definitions of residual network and augmenting path, a maximum flow has no capacity leftover so no paths in G_f

$(2) \implies (3)$: consider a cut where all vertices reachable from s in G_f are in S and the unreachables are in T; since there is no $s \rightsquigarrow t$ path in G_f, all edges across the S–T boundary must already be at full capacity

$(3) \implies (1)$: trivially $|f| \leq c(S, T)$, and if $|f| = c(S, T)$ then this (S, T) is maximum

Ford-Fulkerson Detailed Pseudocode

```
1: function FORD-FULKERSON-METHOD(G = (V, E), s, t)
2:     for each edge (u, v) in E do
3:         (u, v).f = 0
4:     end for
5:     while there exists an augmenting path p in G_f do
6:         c_f(p) = min{c_f(u, v) : (u, v) ∈ p}
7:         for each edge (u, v) ∈ p do
8:             if (u, v) ∈ E then
9:                 (u, v).f = (u, v).f + c_f(p)
10:            else
11:                (u, v).f = (v, u).f - c_f(p)
12:            end if
13:        end for
14:    end while
15:    return flow on .f fields
16: end function
```

Still abstract — need to clarify how we choose path p.

Edmonds-Karp Algorithm

Edmonds-Karp Algorithm is

- ▶ Ford-Fulkerson method from previous page, and...
- ▶ use breadth-first search (BFS) to find the shortest augmenting path
- ▶ (shortest $\equiv$ fewest vertices, irrespective of weights)
- ▶ now a concrete, runnable, implementable algorithm
- ▶ performs $O(|V| \cdot |E|)$ augmentations
- ▶ takes $O(|V| \cdot |E|^2)$ time
- ▶ for $n = |V|$, this is $O(n^3)$ in a sparse graph and $O(n^5)$ in a dense graph
- ▶ more complicated **relabel-to-front** algorithm takes $O(|V|^3) = O(n^3)$ time

Edmonds-Karp Pseudocode for Worked Examples

```
 1: function EDMONDS-KARP(G = (V, E), s, t)
 2:     initialize each edge's flow to 0
 3:     repeat
 4:         for k = 2, 3, . . . , |V| do
 5:             if ∃ augmenting path p of length k   then
 6:                 c_f(p) = minimum excess capacity of any edge in p
 7:                 for edge e in p do
 8:                     if p follows e forwards   then
 9:                         increase e's flow by c_f(p)
10:                     else
11:                         decrease e's flow by c_f(p)
12:                     end if
13:                 end for
14:                 break loop
15:             end if
16:         end for
17:     until no path can be found
18:     return flow based on current capacities
19: end function
```

Identifying Edge Capacity in G

When running this algorithm by hand,

- you *could* sketch the residual network each time, but this is tedious
- instead, when looking at edge e with flow x/c
- if $x < c$, you *may* follow e forwards and add up to $(c - x)$ flow
- if $x > 0$, you *may* follow e backwards and subtract up to x flow
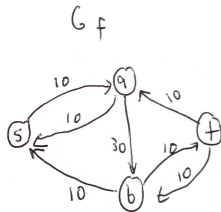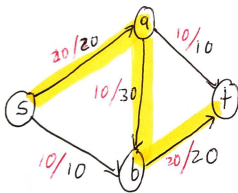
Edmonds-Carp Example 1/2
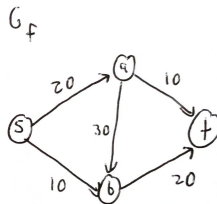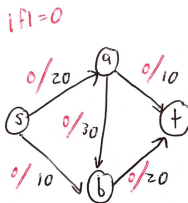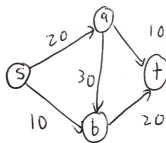
Edmonds-Carp Example 1/2
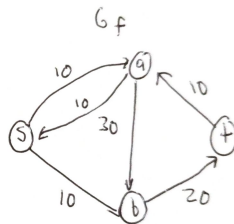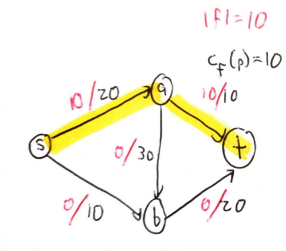
Edmonds-Carp Example 1/2
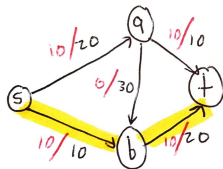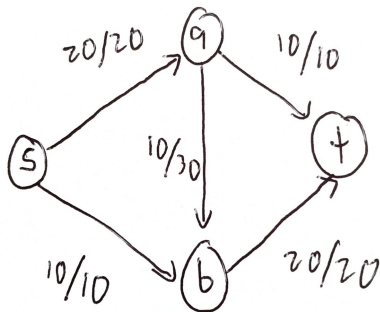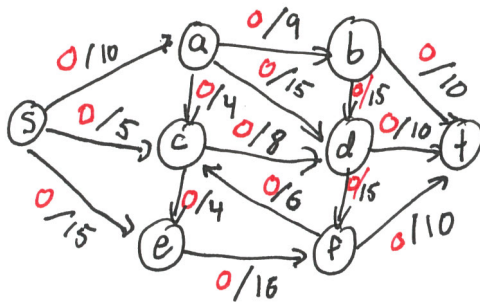

$c_f(p) = 10$

$|f| = 20$

Edmonds-Carp Example 1/2

$c_f(\rho) > 10$
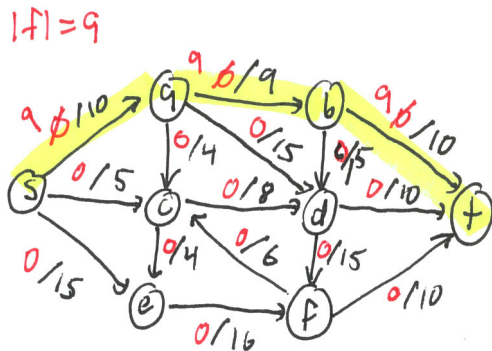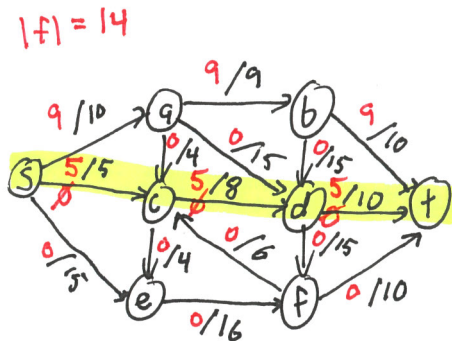
$|f| = 30$
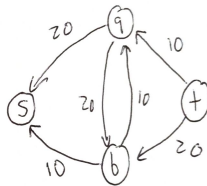
Edmonds-Carp Example 1/2
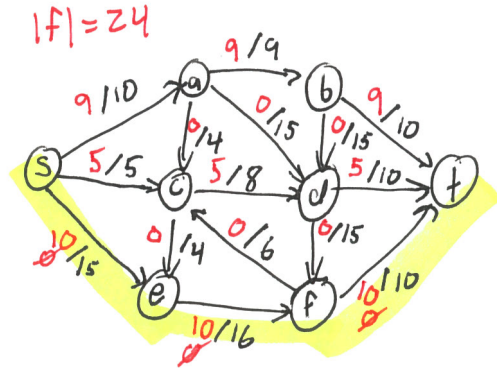

$|f| = 30$
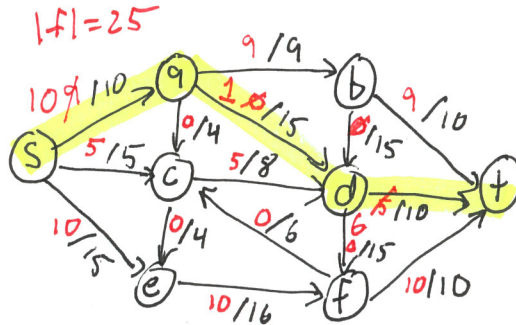
Edmonds-Carp Example 2/2
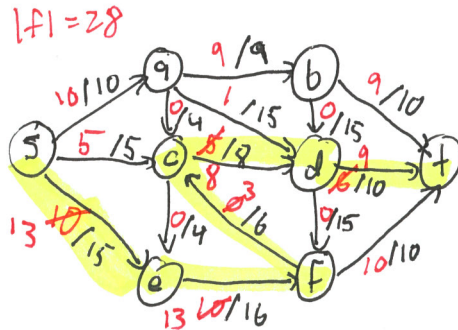
Edmonds-Carp Example 2/2

Edmonds-Carp Example 2/2

Edmonds-Carp Example 2/2

Edmonds-Carp Example 2/2

Edmonds-Carp Example 2/2

Edmonds-Carp Example 2/2