

13. Dynamic Programming for Longest Common Subsequence and Optimal Binary Search Trees

CPSC 535

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Big Idea: Alternative Kinds of Solutions

- ▶ So far
 - ▶ Step 2. Derive a **recurrence** for an optimal value.
 - ▶ Recall rod cutting:

$$r_i = \max_{1 \leq i \leq n} (p_i + r_{n-i})$$

- ▶ Recall matrix chain multiplication:

$$r_{i,j} = \min_{i \leq k \leq j} r_{i,k} + r_{k+1,j} + p_{i-1}p_kp_j$$

- ▶ Now: longest common subsequence (LCS)
 - ▶ not simply minimizing/maximizing one expression
 - ▶ instead, choose between **three alternatives**
 - ▶ **2D table**, like matrix chain

Subsequences

- ▶ Let $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ be two sequences
- ▶ Define **prefix** notation: $X_k = \langle x_1, \dots, x_k \rangle$; $X_0 = \langle \rangle$
 - ▶ if $X = \langle 2, 7, 8, 1, 7, 1, 2 \rangle$ then $X_3 = \langle 2, 7, 8 \rangle$
- ▶ Informally: a **subsequence** of Y is a copy of Y with some elements removed
- ▶ Formally: X is a **subsequence** of Y if there exists an increasing sequence of indices $\langle i_1, i_2, \dots, i_k \rangle$ such that, for all $j \in [1, k]$, $x_j = y_{i_j}$
- ▶ Example: for $X = \langle B, C, D, B \rangle$ and $Y = \langle A, B, C, B, D, A, B \rangle$, X is a subsequence of Y with index sequence $\langle 2, 3, 5, 7 \rangle$

Common Subsequence

- ▶ Z is a **common subsequence** of X and Y , if Z is a subsequence of X and Z is a subsequence of Y
- ▶ a **longest common subsequence** is a common subsequence of maximum length
- ▶ Example: let $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$
- ▶ $Z = \langle B, C, A \rangle$ is a common subsequence
- ▶ $Z = \langle B, C, B, A \rangle$ is a longest common subsequence

Longest Common Subsequence

Longest Common Subsequence (LCS) solution problem

input: sequences $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$

output: a longest common subsequence of X and Y

Longest Common Subsequence (LCS) value problem

input: (same)

output: the length of a longest common subsequence of X and Y

Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
 - 4.1 **top-down** alternative: add table base case (**memoization**)
 - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

Longest Common Subsequence Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
 - ▶ **solution**: a sequence e.g. $\langle B, C, B, A \rangle$
 - ▶ **value**: integer length of a sequence e.g. 4
 - ▶ eventual goal is solution
 - ▶ start with value

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

- ▶ Recall input: $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$
- ▶ Recall *prefix*: X_i is first i elements of X
- ▶ Define $LCS(X, Y) \equiv$ length of longest common subsequence of X and Y
- ▶ We need to define LCS recursively

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

- ▶ **Idea:** If last symbols $x_m = y_n$ match, then extend a shorter common subsequence: $LCS(X, Y) = LCS(X_{m-1}, Y_{n-1}) + 1$
- ▶ Else ($x_m \neq y_n$), have to omit x_m or y_n
 - ▶ Omit x_m : $LCS(X, Y) = LCS(X_{m-1}, Y)$
 - ▶ Omit y_n : $LCS(X, Y) = LCS(X, Y_{n-1})$
 - ▶ Want **longest** so

$$LCS(X, Y) = \max(LCS(X_{m-1}, Y), LCS(X, Y_{n-1}))$$

Example

- ▶ Suppose $X = \langle A, B, A, D \rangle$ and $Y = \langle B, B, A, C, D \rangle$
- ▶ Last symbols match, $x_4 = y_5 = D$, so

$$\begin{aligned}LCS(X, Y) &= LCS(X_{m-1}, Y_{n-1}) + 1 \\ &= LCS(\langle A, B, A \rangle, \langle B, B, A, C \rangle) + 1\end{aligned}$$

- ▶ Now suppose $X = \langle A, B, A, D \rangle$ and $Y = \langle B, B, A, C, C \rangle$
- ▶ Last symbols differ ($x_4 = D$ but $y_5 = C$), so

$$\begin{aligned}LCS(X, Y) &= \max(LCS(X_{m-1}, Y_n), LCS(X_m, Y_{n-1})) \\ &= \max(\langle A, B, A \rangle, \langle B, B, A, C, C \rangle), LCS(\langle A, B, A, D \rangle, \langle B, B, A, C \rangle))\end{aligned}$$

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

$$LCS(X_m, Y_n) = \begin{cases} 0 & m = 0 \\ 0 & n = 0 \\ LCS(X_{m-1}, Y_{n-1}) + 1 & x_m = x_n \\ \max(LCS(X_{m-1}, Y_n), LCS(X_m, Y_{n-1})) & \text{otherwise} \end{cases}$$

Longest Common Subsequence Step 3

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
1: function LCS-DC( $X[1..m]$ ,  $Y[1..n]$ )
2:   if  $m == 0$  or  $n == 0$  then
3:     return 0
4:   else if  $X[m] == Y[n]$  then
5:     return LCS-DC( $X[1..m-1]$ ,  $Y[1..n-1]$ ) + 1
6:   else
7:     return max(LCS-DC( $X[1..m-1]$ ,  $Y[1..n]$ ), LCS-DC( $X[1..m]$ ,  $Y[1..n-1]$ ))
8:   end if
9: end function
```

Matrix Chain Multiplication Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.

4.1 **top-down** alternative: add table base case (**memoization**)

- ▶ Recall **memoization**: use a hash dictionary to make a “memo” of pre-calculated solutions
- ▶ create hash table T
- ▶ use pair (m, n) as key in table T , storing $LCS(X_m, Y_n)$

Matrix Chain Multiplication Step 4.a

```
1: function LCS-MEMOIZED( $X[1..m]$ ,  $Y[1..n]$ )
2:   HASH-TABLE-CREATE( $T$ )
3:   return LCS-M( $T$ ,  $X$ ,  $Y$ )
4: end function
5: function LCS-M( $T$ ,  $X[1..m]$ ,  $Y[1..n]$ )
6:    $q$  = HASH-TABLE-SEARCH( $T$ ,  $(m, n)$ )
7:   if  $q \neq \text{NIL}$  then
8:     return  $q$ 
9:   end if
10:  if  $m == 0$  or  $n == 0$  then
11:     $q = 0$ 
12:  else if  $X[m] == Y[n]$  then
13:     $q$  = LCS-M( $T$ ,  $X[1..m-1]$ ,  $Y[1..n-1]$ ) + 1
14:  else
15:     $q$  = max(LCS-M( $X[1..m-1]$ ,  $Y[1..n]$ ), LCS-M( $X[1..m]$ ,  $Y[1..n-1]$ ))
16:  end if
17:   $q.\text{key} = (m, n)$ 
18:  HASH-TABLE-INSERT( $q$ )
19:  return  $q$ 
20: end function
```

Memoized Algorithm Analysis

- ▶ T contains $\Theta(n^2)$ pairs (m, n)
- ▶ each entry is inserted exactly once
- ▶ in the general case, LCS-M takes $\Theta(1)$ expected time
- ▶ \Rightarrow LCS-MEMOIZED takes $\Theta(n^2)$ expected time

Longest Common Subsequence Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
 - 4.1 **top-down** alternative: add table base case (**memoization**)
 - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
- ▶ create 2D array c where $c[i][j] = LCS(X_i, Y_j)$
 - ▶ **bottom-up**: write an explicit **for** loop that computes and stores every general case
 - ▶ need to order loops so we never use an uninitialized element
 - ▶ \therefore initialize all base cases before any general case

Longest Common Subsequence Step 4.b

```
1: function LCS-BU( $X[1..m]$ ,  $Y[1..n]$ )
2:   Create array  $c[0..m][0..n]$                                 ▷ unusual index range
3:   for  $i$  from 0 to  $m$  do
4:      $c[i][0] = 0$ 
5:   end for
6:   for  $j$  from 1 to  $n$  do                                    ▷ only initialize  $c[0][0]$  once
7:      $c[0][j] = 0$ 
8:   end for
9:   for  $i$  from 1 to  $m$  do
10:    for  $j$  from 1 to  $n$  do
11:      if  $X[i] == Y[j]$  then
12:         $c[i][j] = c[i-1][j-1] + 1$ 
13:      else
14:         $c[i][j] = \max(c[i-1][j], c[i][j-1])$ 
15:      end if
16:    end for
17:  end for
18:  return  $c[m][n]$ 
19: end function
```

Bottom-Up Analysis

- ▶ LCS-BU is clearly $\Theta(n^2)$ time
- ▶ (easy analysis)

Longest Common Subsequence Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

- ▶ **idea:** for each (i, j) , record which alternative sub-solution defines $c[i][j]$:

- ▶ $\nwarrow \equiv c[i-1][j-1]$
- ▶ $\uparrow \equiv c[i-1][j]$
- ▶ $\leftarrow \equiv c[i][j-1]$

- ▶ define

$$b[i][j] \in \{\nwarrow, \uparrow, \leftarrow\}$$

- ▶ rewrite $\max(c[i-1][j], c[i][j-1])$ as **if/else** so we can update $b[i][j]$

Longest Common Subsequence Step 5

```

1: function LCS-SOLUTION( $X[1..m]$ ,  $Y[1..n]$ )
2:   Create arrays  $c[0..m][0..n]$  and  $b[1..m][1..n]$ 
3:   for  $i$  from 0 to  $m$  do
4:      $c[i][0] = 0$ 
5:   end for
6:   for  $j$  from 1 to  $n$  do
7:      $c[0][j] = 0$ 
8:   end for
9:   for  $i$  from 1 to  $m$  do
10:    for  $j$  from 1 to  $n$  do
11:      if  $X[i] == Y[j]$  then
12:         $c[i][j] = c[i-1][j-1] + 1$ 
13:         $b[i][j] = \nwarrow$ 
14:      else if  $c[i-1][j] \geq c[i][j-1]$  then
15:         $c[i][j] = c[i-1][j]$ 
16:         $b[i][j] = \uparrow$ 
17:      else
18:         $c[i][j] = c[i][j-1]$ 
19:         $b[i][j] = \leftarrow$ 
20:      end if
21:    end for
22:  end for
23:  return LCS-BTRACK( $b, X, m, n$ )
24: end function

```

▷ different index ranges

▷ only initialize $c[0][0]$ once

Longest Common Subsequence Step 5

```

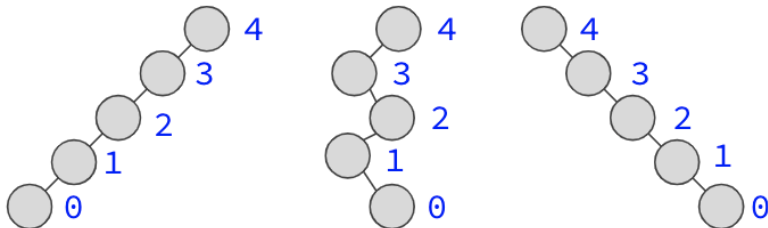
1: function LCS-BTRACK( $b[1..m][1..n], X[1..m], i, j$ )
2:   if  $i == 0$  or  $j == 0$  then
3:     return  $\langle \rangle$                                 ▷ empty sequence
4:   end if
5:   if  $b[i][j] == \nwarrow$  then
6:     return  $\text{LCS-BTRACK}(b, X, i - 1, j - 1) + \langle x_i \rangle$     ▷ append
7:   else if  $b[i][j] == \uparrow$  then
8:     return  $\text{LCS-BTRACK}(b, X, i - 1, j)$ 
9:   else
10:    return  $\text{LCS-BTRACK}(b, X, i, j - 1)$ 
11:  end if
12: end function

```

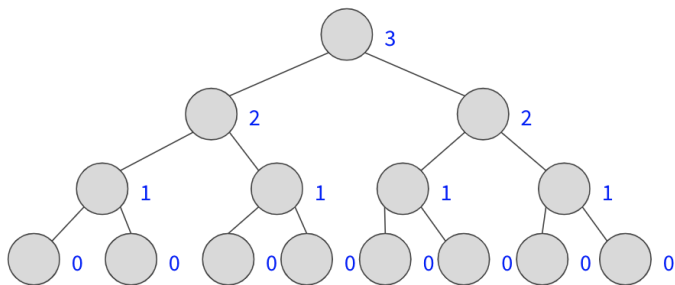
Review: Binary Search Trees

- ▶ Recall **Binary Search Tree (BST)**: fundamental data structure
- ▶ **Depth** of node x = length of path from root to x
- ▶ **Height** of tree = maximum depth of any node
- ▶ Time of a search = **depth** of search path
- ▶ Height
 - ▶ worst case = $\Theta(n)$
 - ▶ best case = $\Theta(\log n)$
- ▶ self-balancing BST maintains $\Theta(\log n)$ height

Worst-Case BSTs



Best-Case BST



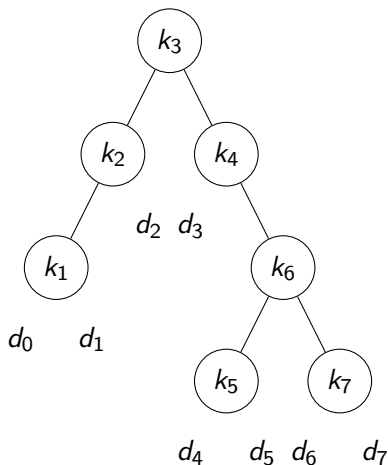
Optimal BSTs

- ▶ Fix the sequence of search operations
- ▶ **Optimal** BST: minimizes total search time
 - ▶ including constant factors
- ▶ Total time for n elements and k searches:
 - ▶ any self-balancing BST: $O(k \log n)$
 - ▶ optimal BST: $O(k \log n)$ with lowest possible constant factor
- ▶ Goal
 - ▶ frequently-visited elements near root
 - ▶ rarely-visited elements near leaves
 - ▶ tricky because a path visits multiple nodes; all count

Problem Setup

- ▶ Given:
 - ▶ ordered keys $K = \langle k_1, k_2, \dots, k_n \rangle$
 - ▶ “dummy” values d_0, d_1, \dots, d_n represent values of failed searches, between keys
- ▶ for a given search and index i ,
 - ▶ p_i = probability that this is a successful search for k_i
 - ▶ q_i = probability that this is a failed search for value d_i

Problem Setup



Expected Search Cost

Every search ends in a key k_i or dummy d_i , so

$$\sum_{i=1}^n p_i + \sum_{i=0}^n q_i = 1.$$

For tree T ,

$$\begin{aligned} E[\text{search in } T] &= \sum_{i=1}^n (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i)) \cdot q_i \\ &= 1 + \sum_{i=1}^n (\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i)) \cdot q_i \end{aligned}$$

Have: probabilities p_1, \dots, p_n and q_0, \dots, q_n

Need: shape T to minimize sum

Optimal BST Problem

Optimal Binary Search Tree (BST) solution problem

input: keys $K = \langle k_1, k_2, \dots, k_n \rangle$; successfull-search probabilities p_1, p_2, \dots, p_n ; and failed-search probabilities q_0, q_1, \dots, q_n

output: a BST T that contains K with minimum expected search cost

Optimal Binary Search Tree (BST) value problem

input: successfull-search probabilities p_1, p_2, \dots, p_n ; and failed-search probabilities q_0, q_1, \dots, q_n

output: the minimum expected search cost of a tree that contains K

(Note: keys K unneeded for value problem.)

Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
 - 4.1 **top-down** alternative: add table base case (**memoization**)
 - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

Optimal BST Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.

- ▶ **solution:** a BST T

- ▶ **value:**

$$E[\text{search in } T] = 1 + \sum_{i=1}^n (\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i)) \cdot q_i$$

- ▶ goal is value

Optimal BST Step 2

2. Derive a **recurrence** for an optimal value.
 - ▶ Make one decision and recurse for the rest
 - ▶ Decision: **choose some key to be root**
 - ▶ Define $e[i, j] = E[\text{search in optimal tree containing } k_i, \dots, k_j]$
 - ▶ Denote empty tree with $j = i - 1$
 - ▶ Base case: empty tree; cost is q_{i-1}
 - ▶ General case:
 - ▶ choose a split index r
 - ▶ recursively compute left subtree $e[i, r - 1]$
 - ▶ recursively compute right subtree $e[r + 1, j]$
 - ▶ add root on top; increases depths of subtrees

Optimal BST Step 2

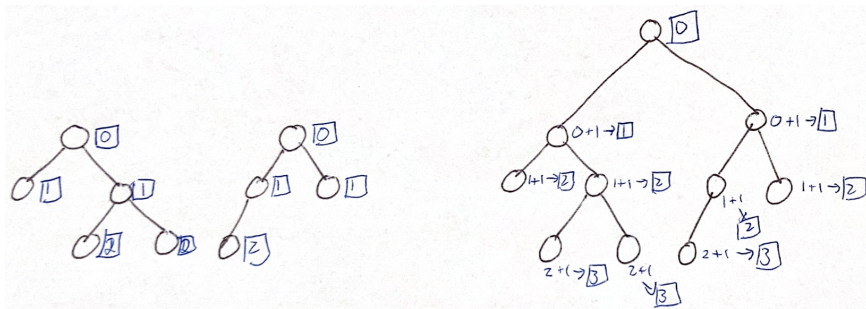
- ▶ Place root atop two subtrees
- ▶ +1 to path length of every descendant
- ▶ Recall

$$\begin{aligned} E[\text{search in } T] &= \sum_{i=1}^n (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i)) \cdot q_i \\ &= 1 + \sum_{i=1}^n (\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i)) \cdot q_i \end{aligned}$$

- ▶ +1 to each path increases $E[\text{search in } T]$ by $\sum_{i=1}^n p_i + \sum_{i=0}^n q_i$
- ▶ Define

$$w(i, j) = \sum_{k=1}^i p_k + \sum_{k=0}^j q_k$$

Adding a Root Increments Path Lengths



Optimal BST Step 2

For a chosen root index r ,

$$e[i, j] = e[i, r - 1] + e[r + 1, j] + w(i, j)$$

Optimize by choosing whichever root has minimal total cost:

$$e[i, j] = \begin{cases} q_{i-1} & \text{if } j = i - 1 \\ \min_{r \in [i:j]} (e[i, r - 1] + e[r + 1, j] + w(i, j)) & \text{if } i \leq j \end{cases}$$

Optimal BST Step 3 – core function

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
1: function OBST-REC( $p[1..n], q[0..n], i, j$ )
2:   if  $j == (i - 1)$  then
3:     return  $q[i - 1]$ 
4:   end if
5:    $e = \infty$ 
6:   for  $r$  from  $i$  to  $j$  do
7:      $t = \text{OBST-REC}(p, q, i, r - 1) + \text{OBST-REC}(p, q, r + 1, j) + W(p, q, i, j)$ 
8:     if  $t < e$  then
9:        $e = t$ 
10:    end if
11:  end for
12:  return  $e$ 
13: end function
```

Optimal BST Step 3 – helper functions

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
1: function OBST-DC( $p[1..n]$ ,  $q[0..n]$ )
2:   return OBST-DC-REC( $p$ ,  $q$ , 1,  $n$ )
3: end function
4: function W( $p[1..n]$ ,  $q[0..n]$ ,  $i$ ,  $j$ )
5:    $w = 0$ 
6:   for  $k$  from  $i$  to  $j$  do
7:      $w = w + p[k]$ 
8:   end for
9:   for  $k$  from  $i - 1$  to  $j$  do
10:     $w = w + q[k]$ 
11:   end for
12:   return  $w$ 
13: end function
```

Optimal BST Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.

4.1 **top-down** alternative: add table base case (**memoization**)

- ▶ create hash table T
- ▶ use pair (i, j) as key in table T , storing $\text{OBST-REC}(p, q, i, j)$

Optimal BST Step 4.a – helper functions

```
1: function OBST-MEMOIZED( $p[1..n], q[0..n]$ )
2:   HASH-TABLE-CREATE( $T$ )
3:   return OBST-DC-REC( $p, q, T, 1, n$ )
4: end function
5: function W( $p[1..n], q[0..n], i, j$ )
6:    $w = 0$ 
7:   for  $k$  from  $i$  to  $j$  do
8:      $w = w + p[k]$ 
9:   end for
10:  for  $k$  from  $i - 1$  to  $j$  do
11:     $w = w + q[k]$ 
12:  end for
13:  return  $w$ 
14: end function
```

Optimal BST Step 4.a – core function

```
1: function OBST-M( $p[1..n], q[0..n], T, i, j$ )
2:    $q = \text{HASH-TABLE-SEARCH}(T, (i, j))$ 
3:   if  $q \neq \text{NIL}$  then
4:     return  $q$ 
5:   end if
6:   if  $j == (i - 1)$  then
7:     return  $q[i - 1]$ 
8:   end if
9:    $e = \infty$ 
10:  for  $r$  from  $i$  to  $j$  do
11:     $t = \text{OBST-M}(p, q, T, i, r - 1) + \text{OBST-M}(p, q, T, r + 1, j) + W(p, q, i, j)$ 
12:    if  $t < e$  then
13:       $e = t$ 
14:    end if
15:  end for
16:   $e.\text{key} = (i, j)$ 
17:   $\text{HASH-TABLE-INSERT}(e)$ 
18:  return  $e$ 
19: end function
```


Optimal BST Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
 - 4.1 **top-down** alternative: add table base case (**memoization**)
 - 4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
- ▶ create 2D array e where $e[i][j] = \text{OBST-REC}(p, q, i, j)$
- ▶ **bottom-up**: write an explicit **for** loop that computes and stores every general case

Optimal BST Step 4.b

```

1: function OBST-BU( $p[1..n]$ ,  $q[0..n]$ )
2:   Create array  $e[1..n+1][1..n+1]$ 
3:   for  $i$  from 1 to  $n+1$  do
4:      $e[i][i-1] = q[i-1]$ 
5:   end for
6:   for  $\ell$  from 1 to  $n$  do
7:     for  $i$  from 1 to  $n-\ell+1$  do
8:        $j = i + \ell - 1$ 
9:        $e[i][j] = \infty$ 
10:      for  $r = i$  to  $j$  do
11:         $t = e[i][r-1] + e[r+1][j] + W(p, q, i, j)$ 
12:        if  $t < e[i][j]$  then
13:           $e[i][j] = t$ 
14:        end if
15:      end for
16:    end for
17:  end for
18:  return  $e[1][n]$ 
19: end function

```

▷ unusual index range
 ▷ base cases

Optimal BST Bottom-Up Analysis

- ▶ Create array e : $\Theta(n^2)$
- ▶ Base cases: $\Theta(n)$
- ▶ General cases:
 - ▶ **for** loop over ℓ : $\Theta(n)$ iterations
 - ▶ nested **for** loop over i : $\Theta(n)$ iterations
 - ▶ nested **for** loop over r : $\Theta(n)$ iterations
 - ▶ call $W(p, q, i, j)$: $\Theta(n)$ time
- ▶ total $\Theta(n^4)$ time
- ▶ bottleneck is calls to W
- ▶ can precompute and cache W values in their own table

Optimal BST Final Draft

```

1: function OBST-BU( $p[1..n]$ ,  $q[0..n]$ )
2:   Create array  $e[1..n+1][1..n+1]$ 
3:   Create array  $w[1..n+1][0..n]$ 
4:   for  $i$  from 1 to  $n+1$  do
5:      $e[i][i-1] = q[i-1]$ 
6:      $w[i][i-1] = q[i-1]$ 
7:   end for
8:   for  $\ell$  from 1 to  $n$  do
9:     for  $i$  from 1 to  $n-\ell+1$  do
10:       $j = i + \ell - 1$ 
11:       $e[i][j] = \infty$ 
12:       $w[i][j] = w[i][j-1] + p[j] + q[j]$ 
13:      for  $r = i$  to  $j$  do
14:         $t = e[i][r-1] + e[r+1][j] + W(p, q, i, j)$ 
15:        if  $t < e[i][j]$  then
16:           $e[i][j] = t$ 
17:        end if
18:      end for
19:    end for
20:  end for
21:  return  $e[1][n]$ 
22: end function

```

▷ unusual index range
 ▷ $w[i][j] = W(p, q, i, j)$

▷ base cases

Optimal BST Final Draft Analysis

- ▶ three nested loop
- ▶ body of innermost loop is now only $\Theta(1)$
- ▶ OBST-BU takes $\Theta(n^3)$ time