08. Dynamic Programming for Longest Common Subsequence and Optimal Binary Search Trees

CPSC 535

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Big Idea: Alternative Kinds of Solutions

- So far
  - Step 2. Derive a **recurrence** for an optimal value.
  - Recall rod cutting:

$$r_i = \max_{1 \le i \le n} (p_i + r_{n-i})$$

  - Recall matrix chain multiplication:

$$r_{i,j} = \min_{i \le k \le j} r_{i,k} + r_{k+1,j} + p_{i-1} p_k p_j$$

- Now: longest common subsequence (LCS)
  - not simply minimizing/maximizing one expression
  - instead, choose between **three alternatives**
  - **2D table**, like matrix chain

Subsequences

- Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be two sequences
- Define **prefix** notation: $X_k = \langle x_1, \ldots, x_k \rangle; X_0 = \langle \rangle$
  - if $X = \langle 2, 7, 8, 1, 7, 1, 2 \rangle$ then $X_3 = \langle 2, 7, 8 \rangle$
- Informally: a **subsequence** of Y is a copy of Y with some elements removed
- Formally: X is a **subsequence** of Y if there exists an increasing sequence of indices $\langle i_1, i_2, \ldots, i_k \rangle$ such that, for all $j \in [1, k], x_j = y_{i_j}$
- Example: for $X = \langle B, C, D, B \rangle$ and $Y = \langle A, B, C, B, D, A, B \rangle$, X is a subsequence of Y with index sequence $\langle 2, 3, 5, 7 \rangle$

Common Subsequence

- Z is a **common subsequence** of X and Y, if Z is a subsequence X and Z is a subsequence of Y
- a **longest common subsequence** is a common subsequence of maximum length
- Example: let $X = \langle A, B, C, B, D, A, B\rangle$ and $Y = \langle B, D, C, A, B, A\rangle$
- $Z = \langle B, C, A\rangle$ is a common subsequence
- $Z = \langle B, C, B, A\rangle$ is a longest common subsequence

Longest Common Subsequence

Longest Common Subsequence (LCS) solution problem
input: sequences $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$
output: a longest common subsequence of X and Y

Longest Common Subsequence (LCS) value problem
input: (same)
output: the length of a longest common subsequence of X and Y

Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)
   4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

Longest Common Subsequence Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.

- **solution:** a sequence e.g. $\langle B, C, B, A \rangle$
- **value:** integer length of a sequence e.g. 4
- eventual goal is solution
- start with value

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

- Recall input: $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$
- Recall *prefix:* X_i is first i elements of X
- Define $LCS(X, Y) \equiv$ length of longest common subsequence of X and Y
- We need to define LCS recursively

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

- **Idea:** If last symbols $x_m = y_n$ match, then extend a shorter common subsequence: $LCS(X, Y) = LCS(X_{m-1}, Y_{n-1}) + 1$
- Else $(x_m \neq y_n)$, have to omit x_m or y_n
  - Omit x_m: $LCS(X, Y) = LCS(X_{m-1}, Y)$
  - Omit y_n: $LCS(X, Y) = LCS(X, Y_{n-1})$
  - Want **longest** so
$$LCS(X, Y) = \max(LCS(X_{m-1}, Y), LCS(X, Y_{n-1}))$$

Example

- Suppose $X = \langle A, B, A, D \rangle$ and $Y = \langle B, B, A, C, D \rangle$
- Last symbols match, $x_4 = y_5 = D$, so

$$\begin{aligned} LCS(X, Y) &= LCS(X_{m-1}, Y_{n-1}) + 1 \\ &= LCS(\langle A, B, A \rangle, \langle B, B, A, C \rangle) + 1 \end{aligned}$$

- Now suppose $X = \langle A, B, A, D \rangle$ and $Y = \langle B, B, A, C, C \rangle$
- Last symbols differ ($x_4 = D$ but $y_5 = C$), so

$$\begin{aligned} LCS(X, Y) &= \max(LCS(X_{m-1}, Y_n), LCS(X_m, Y_{n-1})) \\ &= \max(\langle A, B, A \rangle, \langle B, B, A, C, C \rangle), LCS(\langle A, B, A, D \rangle, \langle B, B, A, C \rangle)) \end{aligned}$$

Longest Common Subsequence Step 2

2. Derive a **recurrence** for an optimal value.

$$LCS(X_m, Y_n) = \begin{cases} 0 & m = 0 \\ 0 & n = 0 \\ LCS(X_{m-1}, Y_{n-1}) + 1 & x_m = x_n \\ \max(LCS(X_{m-1}, Y_n), LCS(X_m, Y_{n-1})) & \text{otherwise} \end{cases}$$

Longest Common Subsequence Step 3

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
function LCS-DC(X[1..m], Y[1..n])
    if m == 0 or n == 0 then
        return 0
    else if X[m] == Y[n] then
        return LCS-DC(X[1..m-1], Y[1..n-1]) + 1
    else
        return max(LCS-DC(X[1..m-1], Y[1..n]), LCS-DC(X[1..m], Y[1..n-1]))
    end if
end function
```

Matrix Chain Multiplication Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)

- Recall **memoization:** use a hash dictionary to make a "memo" of pre-calculated solutions
- create hash table T
- use pair (m, n) as key in table T, storing $LCS(X_m, Y_n)$

Matrix Chain Multiplication Step 4.a

```
function LCS-MEMOIZED(X[1..m], Y[1..n])
    HASH-TABLE-CREATE(T)
    return LCS-M(T, X, Y)
end function
function LCS-M(T, X[1..m], Y[1..n])
    q = HASH-TABLE-SEARCH(T, (m, n))
    if  q ≠ NIL  then
        return  q
    end if
    if  m == 0 or n == 0  then
        q = 0
    else if  X[m] == Y[n]  then
        q = LCS-M(T, X[1..m − 1], Y[1..m − 1]) + 1
    else
        q = max(LCS-M(X[1..m − 1], Y[1..n]), LCS-M(X[1..m], Y[1..n − 1])
    end if
    q.key = (m, n)
    HASH-TABLE-INSERT(q)
    return q
end function
```

Memoized Algorithm Analysis

- T contains $\Theta(n^2)$ pairs (m, n)
- each entry is inserted exactly once
- in the general case, LCS-M takes $\Theta(1)$ expected time
- $\Rightarrow$ LCS-MEMOIZED takes $\Theta(n^2)$ expected time

Longest Common Subsequence Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)
   4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion

- create 2D array c where $c[i][j] = LCS(X_i, Y_j)$
- **bottom-up:** write an explicit **for** loop that computes and stores every general case
- need to order loops so we never use an uninitialized element
- $\therefore$ initialize all base cases before any general case

Longest Common Subsequence Step 4.b

```
function LCS-BU(X[1..m], Y[1..n])
    Create array c[0..m][0..n]                    ▷ unusual index range
    for i from 0 to m do
        c[i][0] = 0
    end for
    for j from 1 to n do                          ▷ only initialize c[0][0] once
        c[0][j] = 0
    end for
    for i from 1 to m do
        for j from 1 to n do
            if X[i] == Y[j] then
                c[i][j] = c[i − 1][j − 1] + 1
            else
                c[i][j] = max(c[i − 1][j], c[i][j − 1])
            end if
        end for
    end for
    return c[m][n]
end function
```

Bottom-Up Analysis

- LCS-BU is clearly $\Theta(n^2)$ time
- (easy analysis)

Longest Common Subsequence Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

- **idea:** for each (i, j), record which alternative sub-solution defines $c[i][j]$:
  - $\nwarrow \equiv c[i-1][j-1]$
  - $\uparrow \equiv c[i-1][j]$
  - $\leftarrow \equiv c[i][j-1]$
- define

$$b[i][j] \in \{\nwarrow, \uparrow, \leftarrow\}$$

- rewrite $\max(c[i-1][j], c[i][j-1])$ as **if**/**else** so we can update $b[i][j]$

Longest Common Subsequence Step 5

```
function LCS-SOLUTION(X[1..m], Y[1..n])
    Create arrays c[0..m][0..n] and b[1..m][1..n]        ▷ different index ranges
    for i from 0 to m do
        c[i][0] = 0
    end for
    for j from 1 to n do                                  ▷ only initialize c[0][0] once
        c[0][j] = 0
    end for
    for i from 1 to m do
        for j from 1 to n do
            if X[i] == Y[j] then
                c[i][j] = c[i − 1][j − 1] + 1
                b[i][j] =↖
            else if c[i − 1][j] ≥ c[i][j − 1] then
                c[i][j] = c[i − 1][j]
                b[i][j] =↑
            else
                c[i][j] = c[i][j − 1]
                b[i][j] =←
            end if
        end for
    end for
    return LCS-BTRACK(b, X, m, n)
end function
```

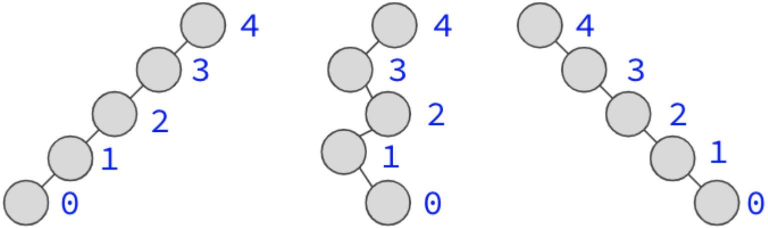
Longest Common Subsequence Step 5

```
function LCS-BTRACK(b[1..m][1..n], X[1..m], i, j)
    if i == 0 or j == 0 then
        return ⟨⟩                                      ▷ empty sequence
    end if
    if b[i][j] == ↖ then
        return LCS-BTRACK(b, X, i − 1, j − 1) + ⟨X[i]⟩  ▷ append
    else if b[i][j] == ↑ then
        return LCS-BTRACK(b, X, i − 1, j)
    else
        return LCS-BTRACK(b, X, i, j − 1)
    end if
end function
```

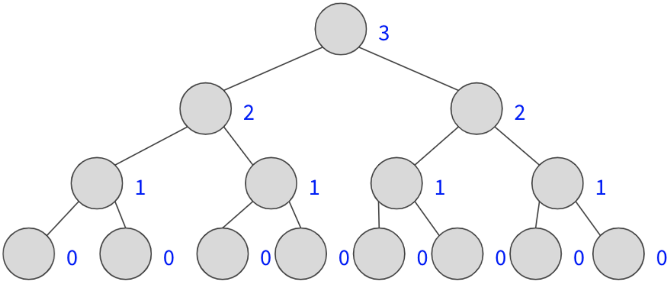
Review: Binary Search Trees

- Recall **Binary Search Tree (BST):** fundamental data structure
- **Depth** of node x = length of path from root to x
- **Height** of tree = maximum depth of any node
- Time of a search = **depth** of search path
- Height
  - worst case = $\Theta(n)$
  - best case = $\Theta(\log n)$
- self-balancing BST maintains $\Theta(\log n)$ height

Worst-Case BSTs



Best-Case BST



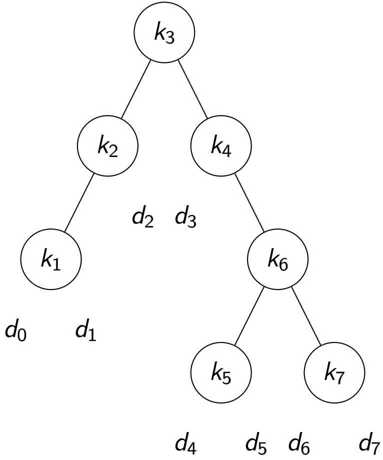
Optimal BSTs

- Fix the sequence of search operations
- **Optimal** BST: minimizes total search time
    - including constant factors
- Total time for n elements and k searches:
    - any self-balancing BST: $O(k \log n)$
    - optimal BST: $O(k \log n)$ with lowest possible constant factor
- Goal
    - frequency-visited elements near root
    - rarely-visited elements near leaves
    - tricky because a path visits multiple nodes; all count

Problem Setup

- Given:
  - ordered keys $K = \langle k_1, k_2, \ldots, k_n \rangle$
  - "dummy" values $d_0, d_1, \ldots, d_n$ represent values of failed searches, between keys
- for a given search and index i,
  - p_i = probability that this is a successfull search for k_i
  - q_i = probability that this is a failed search for value d_i

Problem Setup



Expected Search Cost

Every search ends in a key k_i or dummy d_i, so

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1.$$

For tree T,

$$\begin{aligned} E[\text{search in } T] &= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^{n} (\text{depth}_T(d_i)) \cdot q_i \\ &= 1 + \sum_{i=1}^{n} (\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^{n} (\text{depth}_T(d_i)) \cdot q_i \end{aligned}$$

Have: probabilities $p_1, \ldots, p_n$ and $q_0, \ldots, q_n$
Need: shape T to minimize sum

Optimal BST Problem

Optimal Binary Search Tree (BST) solution problem
input: keys $K = \langle k_1, k_2, \ldots, k_n \rangle$; successfull-search probabilities $p_1, p_2, \ldots, p_n$; and failed-search probabilities $q_0, q_1, \ldots, q_n$
output: a BST T that contains K with minimum expected search cost

Optimal Binary Search Tree (BST) value problem
input: successfull-search probabilities $p_1, p_2, \ldots, p_n$; and failed-search probabilities $q_0, q_1, \ldots, q_n$
output: the minimum expected search cost of a tree that contains K

(Note: keys K unneeded for value problem.)

Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
    4.1 **top-down** alternative: add table base case (**memoization**)
    4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

Optimal BST Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.

- **solution:** a BST T
- **value:** $E[\text{search in } T] = 1 + \sum_{i=1}^{n}(\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^{n}(\text{depth}_T(d_i)) \cdot q_i$
- goal is value

Optimal BST Step 2

2. Derive a **recurrence** for an optimal value.

- Make one decision and recurse for the rest
- Decision: **choose some key to be root**
- Define $e[i,j] = E[\text{search in optimal tree containing } k_i, \ldots, k_j]$
- Denote empty tree with $j = i - 1$
- Base case: empty tree; cost is q_{i-1}
- General case:
  - choose a split index r
  - recursively compute left subtree $e[i, r-1]$
  - recursively compute right subtree $e[r+1, j]$
  - add root on top; increases depths of subtrees

Optimal BST Step 2

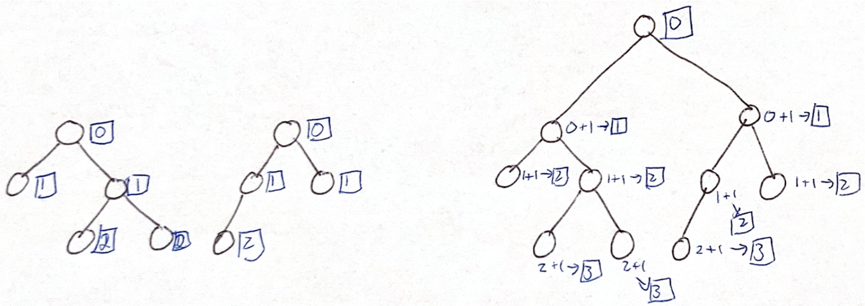
- Place root atop two subtrees
- $+1$ to path length of every descendant
- Recall

$$
\begin{aligned}
E[\text{search in } T] &= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^{n} (\text{depth}_T(d_i)) \cdot q_i \\
&= 1 + \sum_{i=1}^{n} (\text{depth}_T(k_i)) \cdot p_i + \sum_{i=0}^{n} (\text{depth}_T(d_i)) \cdot q_i
\end{aligned}
$$

- $+1$ to each path increases $E[\text{search in } T]$ by $\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} \cdot q_i$
- Define

$$
w(i,j) = \sum_{k=1}^{i} p_k + \sum_{k=0}^{j} \cdot q_k
$$

Adding a Root Increments Path Lengths



Optimal BST Step 2

For a chosen root index r,

$$e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j)$$

Optimize by choosing whichever root has minimal total cost:

$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1 \\ \min_{r \in [i:j]} (e[i,r-1] + e[r+1,j] + w(i,j)) & \text{if } i \le j \end{cases}$$

Optimal BST Step 3 – core function

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
function OBST-REC(p[1..n], q[0..n], i, j)
    if j == (i − 1) then
        return q[i − 1]
    end if
    e = ∞
    for r from i to j do
        t = OBST-REC(p, q, i, r − 1) + OBST-REC(p, q, r + 1, j) + W(p, q, i, j)
        if t < e then
            e = t
        end if
    end for
    return e
end function
```

Optimal BST Step 3 – helper functions

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
function OBST-DC(p[1..n], q[0..n])
    return OBST-DC-REC(p, q, 1, n)
end function
function W(p[1..n], q[0..n], i, j)
    w = 0
    for k from i to j do
        w = w + p[k]
    end for
    for k from i − 1 to j do
        w = w + q[k]
    end for
    return w
end function
```

Optimal BST Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)

- create hash table T
- use pair (i, j) as key in table T, storing OBST-REC(p, q, i, j)

Optimal BST Step 4.a – helper functions

```
function OBST-MEMOIZED(p[1..n], q[0..n])
    HASH-TABLE-CREATE(T)
    return OBST-DC-REC(p, q, T, 1, n)
end function
function W(p[1..n], q[0..n], i, j)
    w = 0
    for k from i to j do
        w = w + p[k]
    end for
    for k from i − 1 to j do
        w = w + q[k]
    end for
    return w
end function
```

Optimal BST Step 4.a – core function

```
function OBST-M(p[1..n], q[0..n], T, i, j)
    q = HASH-TABLE-SEARCH(T, (i, j))
    if q ≠ NIL then
        return q
    end if
    if j == (i − 1) then
        return q[i − 1]
    end if
    e = ∞
    for r from i to j do
        t = OBST-M(p, q, T, i, r − 1) + OBST-M(p, q, T, r + 1, j) + W(p, q, i, j)
        if t < e then
            e = t
        end if
    end for
    e.key = (i, j)
    HASH-TABLE-INSERT(e)
    return e
end function
```

Optimal BST Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)
   4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion

- create 2D array e where $e[i][j] = \text{OBST-REC}(p, q, i, j)$
- **bottom-up:** write an explicit **for** loop that computes and stores every general case

Optimal BST Step 4.b

```
function OBST-BU(p[1..n], q[0..n])
    Create array e[1..n+1][0..n]                          ▷ unusual index range
    for i from 1 to n+1 do
        e[i][i−1] = q[i−1]                                ▷ base cases
    end for
    for ℓ from 1 to n do
        for i from 1 to n−ℓ+1 do
            j = i+ℓ−1
            e[i][j] = ∞
            for r = i to j do
                t = e[i][r−1] + e[r+1][j] + W(p,q,i,j)
                if  t < e[i][j] then
                    e[i][j] = t
                end if
            end for
        end for
    end for
    return e[1][n]
end function
```

Optimal BST Bottom-Up Analysis

- Create array e: $\Theta(n^2)$
- Base cases: $\Theta(n)$
- General cases:
  - **for** loop over ℓ: $\Theta(n)$ iterations
  - nested **for** loop over i: $\Theta(n)$ iterations
  - nested **for** loop over r: $\Theta(n)$ iterations
  - call $W(p, q, i, j)$: $\Theta(n)$ time
- total $\Theta(n^4)$ time
- bottleneck is calls to W
- can precompute and cache W values in their own table

Optimal BST Final Draft

```
function OBST-BU(p[1..n], q[0..n])
    Create array e[1..n+1][0..n]                          ▷ unusual index range
    Create array w[1..n+1][0..n]                          ▷ w[i][j] = W(p, q, i, j)
    for i from 1 to n+1 do
        e[i][i−1] = q[i−1]                                ▷ base cases
        w[i][i−1] = q[i−1]
    end for
    for ℓ from 1 to n do
        for i from 1 to n−ℓ+1 do
            j = i + ℓ − 1
            e[i][j] = ∞
            w[i][j] = w[i][j−1] + p[j] + q[j]
            for r = i to j do
                t = e[i][r−1] + e[r+1][j] + w[i][j]
                if t < e[i][j] then
                    e[i][j] = t
                end if
            end for
        end for
    end for
    return e[1][n]
end function
```

Optimal BST Final Draft Analysis

- three nested loop
- body of innermost loop is now only $\Theta(1)$
- OBST-BU takes $\Theta(n^3)$ time