11. Dynamic Programming Introduction

CPSC 535

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Dynamic Programming

- pattern for designing algorithms
- *programming:*
  - optimize subject to constraints
  - (same as Linear Programming)
  - **not** writing programs
- *dynamic:* curious buzzword
- specialized tool
  - dynamic programming only applies to problems with *overlapping subproblems*
  - rare
  - huge speedup over naive algorithms for such problems

Big Ideas

- important algorithm design approach in its own right
- problem solving to view a problem in a different way
- **time-space trade-off**
  - speedup costs space
- **efficiency-complexity trade-off**
  - top-down, bottom-up variants
  - top-down is simpler to design and implement
  - bottom-up has faster constant factors

Optimization, Value, Solution

- dynamic programming usually applies to **optimization** problems
  - correct output "minimizes" or "maximizes" something
- **value:** quality of the solution
  - quantity to minimize/maximize
- designing a dynamic programming algorithm to. . .
  - ...calculate optimal **value** is simpler
  - ...calculate optimal **solution** is more complicated
- $\therefore$ some examples and exercises only involves **values**
- algo's for **solutions** are more practical but difficult

Example: Vertex Cover

vertex cover problem
input: an undirected graph $G = (V, E)$
output: a vertex cover C of minimum size

- solution = a set of vertices C
- value = size of C
- optimal = minimize $|C|$

vertex cover value problem
input: an undirected graph $G = (V, E)$
output: the minimum size of a vertex cover of G

- note: output data type is an integer, not a set

Example: Bipartite Matching

bipartite maximum matching problem

input: an undirected bipartite graph $G = (V, E)$ with parts $V = L \cup R$

output: a matching $M \subseteq E$ where the number of matched vertices is maximum

- solution = a set of edges M
- value = size of M

bipartite maximum matching value problem

input: an undirected bipartite graph $G = (V, E)$ with parts $V = L \cup R$

output: the maximum number of edges in a matching of G

- note: output data type is an integer, not a set

Ties

- we say **an** optimal solution
- not **the** optimal solution
- multiple solutions may have same value
- any of these are correct
- examples:
  - vertex cover: "a vertex cover C of minimum size"
  - bipartite matching: "a matching $M \subseteq E$ where the number of matched vertices is maximum"
- not worrying about ties simplifies dynamic programming algorithms

The Main Idea

- dynamic programming works on a problem where. . .
  - a solution has a **recursive structure**
  - so we *could* design a naive divide-and-conquer algorithm
  - **but,** subproblems **overlap**
  - so divide-and-conquer would do the same work repeatedly
  - would be slow (often exponential time)
- idea: store subproblem solutions in a **table** (array or hash dictionary)
- only solve subproblems not already in table
- $\therefore$ each subproblem is solved **only once**
- fast polynomial time, often $\Theta(n)$ or $\Theta(n^2)$

Top-Down versus Bottom Up

- two ways to write the pseudocode
- **top-down**
  - improvement to divide-and-conquer pseudocode
  - add a base case that checks for a solution in the table
  - simple to derive from divide-and-conquer algorithm
  - usually depends on a hash dictionary data structure, so expected time
- **bottom-up**
  - clean-sheet redesign
  - nested loops explicitly solve problems in sorted order
  - base case, larger subproblems, . . . , full problem
  - store subproblems in array (not hash table)
  - no recursion or hash table $\Rightarrow$ faster constant factors

Design Process

1. Identify the problem's **solution** and **value**, and note which is our **goal**.
2. Derive a **recurrence** for an optimal value.
3. Design a divide-and-conquer algorithm that computes an **optimal value**.
4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)
   4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion
5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

Rod Cutting Problem

story:

- have a rod of metal n inches long
- can chop it into pieces of size $1, 2, \ldots, n$
- total length of all pieces $= n$
- market price of a i-inch piece is p_i
- market price of a 0-inch piece is 0
- goal: maximize total price of the pieces

rod cutting value problem
input: an array of non-negative prices $P = \langle p_1, \ldots, p_n \rangle$
output: the maximum total price that can be achieved by cutting an n-inch rod into pieces

Example with $n = 4$

i	1	2	3	4
p_i	3	7	8	11

Ways of cutting □□□□:

1. □□□□ : $p_4 = \$11$
2. □ | □□□ : $p_1 + p_3 = \$3 + \$8 = \$11$
3. □□ | □□ : $p_2 + p_2 = \$7 + \$7 = \$14$
4. □□□ | □ : $p_3 + p_1 = \$8 + \$3 = \$11$
5. □ | □ | □□ : $p_1 + p_1 + p_2 = \$3 + \$3 + \$7 = \$13$
6. □ | □□ | □ : $p_1 + p_2 + p_1 = \$3 + \$7 + \$3 = \$13$
7. □□ | □ | □ : $p_2 + p_1 + p_1 = \$7 + \$3 + \$3 = \$13$
8. □ | □ | □ | □ : $p_1 + p_1 + p_1 + p_1 = \$3 + \$3 + \$3 + \$3 = \$12$

Greedy Fails

- greedy heuristics are **not correct** for this problem
- note that there is no requirement that prices p_i obey "common sense" market dynamics
- e.g. it is allowed for $p_4 > p_5$
- example of an **incorrect** greedy heuristic: find length i with highest unit price p_i/i, then make $\lceil n/i \rceil$ pieces of length i
- fails when the leftover $n - \lceil n/i \rceil$ inches could be used better
- Recall: the designer of a greedy algorithm has the burden of **proving** their heuristic is correct
- **Tip:** if you are told to design a dynamic programming algorithm, don't waste time with greedy algorithms

Rod Cutting Step 1

1. Identify the problem's **solution** and **value**, and note which is our **goal**.

rod cutting value problem
input: an array of non-negative prices $P = \langle p_1, \ldots, p_n \rangle$
output: the maximum total price that can be achieved by cutting an n-inch rod into pieces

- **solution:** list of piece lengths e.g. $\langle 2, 2 \rangle$
- **value:** total price e.g. \$14
- goal: **value**

Rod Cutting Step 2

2. Derive a **recurrence** for an optimal value.

- define r_i = the maximum total price starting from i inches
- base case: $r_0 = 0$
- general case:
  - **think** divide-and-conquer; define r_i in terms of $r_{<i}$
  - make the problem **one piece** smaller
  - try to make one cut, then recursively use the remaining inches
  - try every option and keep the optimal one

$$r_i = \max_{1 \leq i \leq n} (p_i + r_{n-i})$$

Rod Cutting Step 3

3. Design a divide-and-conquer algorithm that computes an **optimal value**.

```
function CUT-ROD-DC(P, n)
    if n ==0 then
        return 0
    end if
    q = −∞
    for i from 1 to n do
        q = max(q, P[i] + CUT-ROD-DC(P, n-i))
    end for
    return q
end function
```

Sidebar: Analysis of CUT-ROD-DC

- CUT-ROD-DC corresponds directly to the r_i defition
- **but** it is very slow
- fundamental problem: CUT-ROD-DC calls itself many times
  - each iteration of the **for** loop is a recursive call
  - each of those has a **for** loop with recursive calls...
- recall: fast divide-and-conquer algorithms usually call themselves 1–2 times
- *Claim:* The time complexity of CUT-ROD-DC is $O(2^{n-1})$.
- dynamic programming will circumvent all this recursion

Rod Cutting Step 4.a

4. Design a dynamic programming algorithm that computes an **optimal value**.
    4.1 **top-down** alternative: add table base case (**memoization**)

- **memoization:** use a hash dictionary to make a "memo" of pre-calculated solutions
- use i as key in table T (same API as hash tables in deck 4)
- after we compute an r_i, set $r_i.key = i$ and insert r_i into T
- if T does not contain key i, then we haven't computed r_i yet
- need two functions
    - public non-recursive function to create T and start recursion
    - private recursive function that expects T to exist

Rod Cutting Step 4.a

```
function CUT-ROD-MEMOIZED(P, n)
    HASH-TABLE-CREATE(T)
    return CUT-ROD-MEMO-REC(T, P, n)
end function
function CUT-ROD-MEMO-REC(T, P, n)
    q = HASH-TABLE-SEARCH(T, n)
    if q ≠ NIL then
        return q
    end if
    if n == 0 then
        q = 0
    else
        q = −∞
        for i from 1 to n do
            q = max(q, P[i] + CUT-ROD-MEMO-REC(T, P, n - i))
        end for
    end if
    q.key = n
    HASH-TABLE-INSERT(q)
    return q
end function
```

Rod Cutting Step 4.b

4. Design a dynamic programming algorithm that computes an **optimal value**.
   4.1 **top-down** alternative: add table base case (**memoization**)
   4.2 **bottom-up** alternative: rewrite to use bottom-up loops instead of recursion

- observe: in CUT-ROD-MEMOIZED, keys are inserted into T in order $0, 1, \ldots, n$
- **bottom-up:** write an explicit **for** loop that computes and stores every general case r_i in order $r_1, \ldots, r_n$
- base case is computed and stored before the loop
- convenient to use an array instead of hash table
- define $R[i] = r_i$
- no more recursion, just loops

Rod Cutting Step 4.b

```
function CUT-ROD-BU(P[1..n])
    Create array R[0..n]
    R[0] = 0
    for j from 1 to n do
        q = −∞
        for i from 1 to n do
            q = max(q, P[i] + R[j − i])
        end for
        R[j] = q
    end for
    return R[n]
end function
```

Bottom-Up Analysis

- CUT-ROD-BU is clearly $\Theta(n^2)$ time
- (Note: easy analysis)

Top-Down Analysis

- trickier analysis
- observe: hash **if** statement guarantees that each subproblem is solved exactly once
- solving subproblem i, not counting recursion: $\Theta(i)$ time due to **for** loop
- total of all subproblems is $\sum_{i=1}^{n} i \in \Theta(n^2)$
- hash operations add "expected" qualifier
- $\therefore$ CUT-ROD-MEMOIZED takes $\Theta(n^2)$ expected time
- with effort we could replace the hash table with an array for $\Theta(n^2)$ worst-case time
- CUT-ROD-MEMOIZED has worse constant factors due to the overhead of recursive function calls

Trade-Offs

Factor	**Naive**	**TDDP**	**BUDP**
Ease of design	easiest	difficult	very difficult
Ease of analysis	medium	difficult	easy
Time efficiency	$O(2^{n-1})$	$\Theta(n^2)$ exp.	$\Theta(n^2)$ w/ fast const.
Space overhead	n/a	$O(n)$ hashtable	$O(n)$ array

- according to principles, **bottom up dynamic programming** is superior
- but top-down dynamic programming is a close second

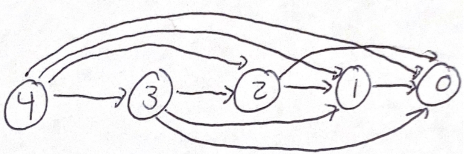
Subproblem Graphs

- solutions have a recursive structure

$$r_i = \max_{1 \leq i \leq n} (p_i + r_{n-i})$$

- a general-case solution depends on other solution(s)
- algorithm must compute solutions in an order that satisfies dependencies
- memoization automates this, with overhead
- bottom-up loops must be designed carefully to iterate in satisfactory order
- visualize dependencies in a **subproblem graph**

Subproblem Graphs



- vertex i = subproblem i
- directed edge (i, j) = computing i requires solution to j
- subproblem i must wait until all outgoing neighbors have been computed
- top-down manages with hashtable
- bottom up manages with loop iteration order

Rod Cutting Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

rod cutting value problem
input: an array of non-negative prices $P = \langle p_1, \ldots, p_n \rangle$
output: the maximum total price that can be achieved by cutting an n-inch rod into pieces

rod cutting problem
input: an array of non-negative prices $P = \langle p_1, \ldots, p_n \rangle$
output: the list of cut-lengths of maximum total price for an n-inch rod

Storing All Subproblem Solutions Is Expensive

- solution to rod cutting problem: a list of cut-lengths; $O(n)$ space each
- our algorithms compute $n + 1$ solutions
- storing all subproblem solutions would takes $O((n+1) \times n) = O(n^2)$ space, **expensive**
- instead, store only $O(n)$ information

Backtracking

- algorithm computes optimal value, and **logs (records) how it made each decision**
- after all optimal values have been computed, follow a "trail" to create solution object
- trail ends at the optimal solution
- each log entry says how to go one step backwards
- follow them until we get to the start (a base case)
- traverses solution in backwards order; reverse it if order matters
- backtracking is usually only $O(n)$ time, and $O(n)$ space overhead

Rod Cutting Step 5

5. (if goal is a solution algo.) Design a dynamic programming algorithm that computes an **optimal solution**.

- bottom-up algo. makes optimal choices with

  $$q = \max(q, P[i] + R[j - i])$$

  step
- i.e. it chooses how many inches to cut right now
- **log** these choices in another array
- recall $R[j]$ = maximum total price starting from j inches
- define $S[j]$ = size of the first optimal cut starting from j inches
- need to update pseudocode to
  - create S
  - update S inside the loops
  - at the end, backtrack S to compute a list of lengths

Rod Cutting Step 5 – Pseudocode

```
function CUT-ROD-SOLUTION(P[1..n])
    Create arrays R[0..n] and S[0..n]
    R[0] = 0
    for j from 1 to n do
        q = −∞
        for i from 1 to n do
            if q < (P[i] + R[j − 1]) then
                q = P[i] + R[j − i]
                S[j] = i
            end if
        end for
        R[j] = q
    end for
    soln = empty list
    j = n
    while j > 0 do
        soln.add(S[j])
        j = j − S[j]
    end while
    reverse soln
    return soln
end function
```

Analysis

- CUT-ROD-SOLUTION solves the *rod cutting problem*
  - it returns a list of cut-lengths, not a price
- analysis is actually straightforward
- time efficiency:
  - nested **for** loops: $\Theta(n^2)$
  - backtracking: **while** loop iterates at most n times $\Rightarrow \Theta(n)$ time
  - reverse soln: $\Theta(n)$
  - total $\Theta(n^2 + n + n) = \Theta(n^2)$ time
- space efficiency: R and S take $\Theta(n + n) = \Theta(n)$ space
- (same as the step-4 algorithms)