

10. Integer Linear Programming

CPSC 535

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Recall: General LP Problem

general-form linear programming problem

input:

- ▶ Boolean for whether f is maximized/minimized
- ▶ vector $c \in \mathbb{R}^n$
- ▶ vector $b \in \mathbb{R}^m$
- ▶ vector $o \in \{\leq, =, \geq\}^m$
- ▶ $m \times n$ matrix A of real numbers

output: one of

1. “unbounded”;
2. “infeasible”; or
3. “solution” with a vector $x \in \mathbb{R}^n$ maximizing the objective function

Recall: Reduction Algorithm

```
1: function SOLVE-A(input-for-A)
2:   input-for-B = pre-process input-for-A
3:   solution-for-B = solve-B(input-for-B)
4:   solution-for-A = post-process solution-for-B
5:   return solution-for-A
6: end function
```

Algorithms that Reduce to LP

```
1: function SOLVE-A(input-for-A)
2:   program = convert input-for-A to linear program
3:   result = solve-B(input-for-B)
4:   if result is "unbounded" then
5:     return output-for-unbounded-case
6:   else if result is "infeasible" then
7:     return output-for-infeasible-case
8:   else
9:     solution-for-A = convert LP solution
10:    return solution-for-A
11:  end if
12: end function
```

Integer Linear Programming

- ▶ **integer linear programming**: like general form, but all variables are integers instead of real
- ▶ i.e. each $x_i \in \mathbb{Z}$
- ▶ *Mixed Integer Programming (MIP)*: mixture of real and integer variables
- ▶ i.e. a subset $I \subseteq \{x_1, \dots, x_n\}$ of variables are restricted to integers

MIP problem

mixed-integer programming problem (MIP)

input:

- ▶ Boolean for whether f is maximized/minimized
- ▶ vector $c \in \mathbb{R}^n$
- ▶ vector $b \in \mathbb{R}^m$
- ▶ vector $o \in \{\leq, =, \geq\}^m$
- ▶ $m \times n$ matrix A of real numbers
- ▶ set $I \subset \{1, \dots, n\}$

output: one of

1. “unbounded”;
2. “infeasible”; or
3. “solution” with a vector $x \in \mathbb{R}^n$ maximizing the objective function; if $i \in I$ then $x_i \in \mathbb{Z}$

MIP Applications

- ▶ **discrete variables:** can formulate a business-logic whole number concept with
 - ▶ variable $x_i, i \in I$
 - ▶ example: you can buy 3 or 4 airplanes but not 3.7
- ▶ **true/false decision:** can formulate a true/false choice with
 - ▶ variable $x_i, i \in I$
 - ▶ constraints $0 \leq x_i$ and $x_i \leq 1$
- ▶ **choose among k alternatives:** more generally, can formulate a choice from $\{a, \dots, b\} \subset \mathbb{Z}$ with
 - ▶ variable $x_i, i \in I$
 - ▶ constraints $a \leq x_i$ and $x_i \leq b$

MIP Hardness

- ▶ Recall: hardness of general LP is an open question
- ▶ not proven in P , not proven NP -hard
- ▶ MIP **is** NP -complete
- ▶ specifying integer variables seems to make the problem substantially harder
- ▶ worst-case MIP programs are intractible
- ▶ **but** MIP solvers use lots of clever heuristics
- ▶ so specific MIP formulations are often computationally feasible in practice

Vertex Cover

vertex cover problem

input: an undirected graph $G = (V, E)$

output: a vertex cover C of minimum size

vertex cover: a subset $C \subseteq V$ such that, if $(u, v) \in E$, then $u \in C$ or $v \in C$ (or both)

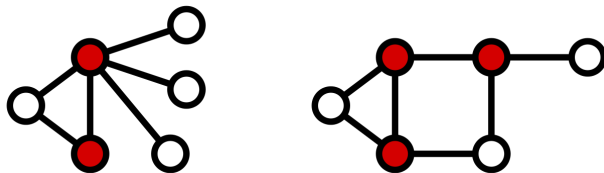


Image credit: <https://commons.wikimedia.org/wiki/File:Minimum-vertex-cover.svg>

Formulating Vertex Cover

Recall:

- ▶ vertex cover is NP -complete
- ▶ if vertex cover can be formulated as a MIP problem, then MIP is NP -hard

“Rules” to represent:

- ▶ each vertex is either in C or not
- ▶ each edge has at least one end in C
- ▶ minimize $|C|$

Formulating Vertex Cover

Variables: for each $v \in V$, create an integer variable x_v such that

$$x_v = 1 \Leftrightarrow v \in C$$

Objective: minimize

$$\sum_{v \in V} x_v$$

Constraints:

$$\begin{array}{ll} 0 \leq x_v \leq 1 & \forall v \in V \quad (0 \text{ or } 1 \text{ indicator}) \\ x_u + x_v \geq 1 & \forall (u, v) \in E \quad (\text{each edge is covered}) \end{array}$$

Vertex Cover Outcomes

▶ **Infeasible:**

- ▶ never happens
- ▶ \exists a solution: setting all $x_v = 1$ satisfies all constraints

▶ **Unbounded:**

- ▶ never happens
- ▶ objective is bounded: the objective function is to minimize

$$\sum_{v \in V} x_v;$$

since every $x_v \geq 0$, the minimum objective value is zero, which is finite, so the program is never unbounded

▶ **Solution:** Construct C as

$$C = \{v \mid v \in V \text{ and } x_v = 1\}$$

TSP

traveling salesperson problem (TSP)

input: a complete, weighted, undirected graph $G = (V, E)$

output: a tour T of minimum weight

tour: a sequence of vertices $\langle t_1, \dots, t_n \rangle$ that visits each vertex exactly once *Hamiltonian cycle*

Define:

$$n \equiv |V|$$

$$w(u, v) \equiv \text{the weight of the edge from } u \text{ to } v$$

Formulating TSP

“Rules” to represent:

- ▶ each vertex is visited exactly once
- ▶ minimize total weight

Formulating TSP

Variables: for each $u \in V$ and $v \in V$, create an integer variable $x_{u,v}$ such that

$$x_{u,v} = 1 \Leftrightarrow \text{the tour steps from } u \text{ to } v$$

Objective: minimize

$$\sum_{u,v \in V} w(u, v) \cdot x_{u,v}$$

Constraints:

$$\begin{array}{lll} 0 \leq x_{u,v} \leq 1 & \forall u, v \in V & (0 \text{ or } 1 \text{ indicator}) \\ \sum_{u \in V} x_{u,v} = 1 & \forall v \in V & (\text{each vertex is entered once}) \\ \sum_{v \in V} x_{u,v} = 1 & \forall u \in V & (\text{each vertex is exited once}) \\ \sum_{u,v \in V} x_{u,v} = n & & (\text{tour has } n \text{ edges}) \end{array}$$

TSP Outcomes

► **Infeasible:**

- never happens
- \exists a solution: G is complete, so certainly contains at least one tour

► **Unbounded:**

- never happens
- objective is bounded: observe that $\sum_{u,v \in V} w(u,v) \cdot x_{u,v}$ is minimized when every $x_{u,v}$ is zero; so the minimum objective value is zero; which is finite.

► **Solution:** Construct $T = \langle t_1, \dots, t_n \rangle$ as

$$t_i = \begin{cases} \text{an arbitrary } v \in V & i = 1 \\ v \text{ such that } x_{t_{i-1},v} = 1 & i > 1 \end{cases}$$

Formulating Sudoku

Sudoku: input is a 9×9 grid, some cells are integers $\{1, \dots, 9\}$, others are blank

			2	6		7		1
6	8			7			9	
1	9				4	5		
8	2		1				4	
		4	6		2	9		
	5				3		2	8
		9	3				7	4
	4			5			3	6
7		3		1	8			

Rules:

1. Objective: fill every blank
2. Each row contains $\{1, \dots, 9\}$
3. Each column contains $\{1, \dots, 9\}$
4. Each 3×3 subgrid contains $\{1, \dots, 9\}$
5. (implies none of these regions has duplicates)

Formulating Sudoku: Variables

Create binary decision variables

$$x_{ijv} = 1 \Leftrightarrow \text{row } i, \text{ column } j, \text{ is assigned value } v$$

Specify that every x_{ijv} is an integer variable.

Add constraints for the variables to be used properly:

$$\begin{array}{ll} 0 \leq x_{ijv} \leq 1 & \forall i, j, v \in \{1, \dots, 9\} \quad (0 \text{ or } 1 \text{ indicator}) \\ \sum_{v=1}^9 x_{ijv} = 1 & \forall i, j \in \{1, \dots, 9\} \quad (\text{each cell has exactly one value}) \end{array}$$

Rule 1: Pre-Filled Cells

For each pre-filled cell at row i , column j , filled with value v , add one constraint

$$x_{ijv} = 1$$

Rules 2, 3: Each Row, Column is Filled Properly

“Row i is filled in properly” \Leftrightarrow each value v appears exactly once
in row i
(and for columns, resp.)

Add constraints:

$$\begin{aligned}\sum_{j=1}^9 x_{ijv} &= 1 & \forall i, v \in \{1, \dots, 9\} & \text{ rows are filled properly} \\ \sum_{i=1}^9 x_{ijv} &= 1 & \forall j, v \in \{1, \dots, 9\} & \text{ columns are filled properly}\end{aligned}$$

Rule 4: Each Subgrid is Filled Properly

For $r, c \in \{1, 2, 3\}$, let

$G(r, c) = \{(i, j) : i, j \in \{1, \dots, 9\} \text{ and } (i, j) \text{ is a cell of subgrid } r, c\}$.

Add constraints:

$$\sum_{(i,j) \in G(r,c)} x_{ijv} = 1 \quad \forall v \in \{1, \dots, 9\}; r, c \in \{1, 2, 3\} \quad \text{subgrids}$$

Objective Function

- ▶ Those constraints model all the rules of Sudoku!
- ▶ Still need an objective function
- ▶ Sudoku does not involve minimizing or maximizing anything
- ▶ Any arbitrary objective function works
- ▶ Define objective:
 maximize 0

Outcomes of MLP

- ▶ **Infeasible:**
 - ▶ it is impossible to fill the grid without breaking a rule
 - ▶ the pre-filled cells must break a rule and be invalid
- ▶ **Unbounded:** the objective function maximize 0 is a constant function, so is certainly bounded. So our MIP will never be unbounded.
- ▶ **Solution:** To fill in the grid: for each row i and column j , search for the v such that

$$x_{ijv} = 1$$

and then write v into cell (i, j) .

References

https://en.wikipedia.org/wiki/Integer_programming

http://profs.sci.univr.it/~rrizzi/classes/PLS2015/sudoku/doc/497_Olszowy_Wiktor_Sudoku.pdf

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