

# 03. Heaps, Heapsort, and the Sorting Lower Bound

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# Heaps

“Heap” data structure in general

- ▶ like binary search tree, but weaker order condition
- ▶ find min/max fast, usually  $\Theta(1)$
- ▶ insert/delete-min/max fast,  $\Theta(\log n)$  or even  $\Theta(1)$
- ▶ search slow,  $\Theta(n)$

Applications

- ▶ operating system scheduler
- ▶ accelerate algorithms: selection sort (heapsort), Prim's MST algorithm, Dijkstra's shortest paths algorithm, others

## Kinds of Heaps

Like binary search trees and hash tables, there are many competing variations of heap

- ▶ binary heap (covered here)
- ▶ Fibonacci heap
- ▶ quake heap
- ▶ hollow heap

*Heap memory* is a disparate concept and unfortunate naming clash

## Binary Heap Order Invariant

Let parent node  $p$  have children  $l, r$

Recall **BST invariant**:  $l.key < p.key < r.key$

Binary **max-heap invariant**:  $p.key \geq l.key$  and  $p.key \geq r.key$   
(sketch)

Implies

1. BST is totally sorted
2. heap is “halfway” sorted; can tell parent comes before children, but siblings are unsorted
3. root must be overall maximum; that one element is perfectly sorted

## Max-Heap vs. Min-Heap

**Max-heap:** maximum key at root;  $p.key \geq l.key, p.key \geq r.key$

**Min-heap:** minimum key at root;  $p.key \leq l.key, p.key \leq r.key$

No significant difference in implementation; swap  $<$  for  $>$  in **if** statements

Max-heap is more convenient for sorting in non-decreasing order, so we'll focus on that

(Min-heap is more convenient for non-increasing sort, Prim, Dijkstra)

## Binary Heap Balance Invariant

**Height** of a node: distance from bottom (root is highest)

**Depth** of a node: distance from root (leaves are deepest)

Heap is a nearly-perfect binary tree

- ▶ every level except the bottom is completely full
- ▶ bottom level: full on the left side, may be missing elements on the right side (sketch)
- ▶ (N.B. insisting on a completely perfect tree is impractical because it requires  $(n + 1)$  is a power of 2)
- ▶  $\implies$  tree height  $\Theta(\log n)$

## Arrayed Binary Heap

*Arrayed* structure: packs something into an array

- ▶ locality of reference  $\implies$  good cache performance
- ▶ only one allocate/free per structure lifetime  $\implies$  fast constant factors

Arrayed max-heap:

- ▶ partially-filled array  $A[1, \dots, n]$  stores  $k$  heap elements in  $A[1, \dots, k]$  for  $k \leq n$
- ▶ root/maximum always in  $A[1]$
- ▶  $PARENT(i) = \lfloor i/2 \rfloor$
- ▶  $LEFT(i) = 2i$
- ▶  $RIGHT(i) = 2i + 1$
- ▶  $PARENT, LEFT, RIGHT$  are  $\Theta(1)$  and ordinarily only 1-2 CPU instructions each

## Max-Heapify Intro.

*MAX – HEAPIFY*( $A, i$ )

- ▶ assuming *LEFT*( $i$ ) and *RIGHT*( $i$ ) obey the order invariant, ensure  $i$  obeys the invariant
- ▶  $A[i]$  might be OK, or might need to “float down” deeper
- ▶ Delete-max and build are easy once we have *MAX – HEAPIFY*
- ▶  $\Theta(\log n)$  time



## Max-Heapify Pseudocode

```

1: function MAX-HEAPIFY( $A, i$ )
Require:  $A[LEFT(i)]$  and  $A[RIGHT(i)]$  are heap-ordered
Ensure:  $A[i]$  is heap-ordered
2:    $l = LEFT(i), r = RIGHT(i)$ 
3:   if  $l \leq k$  and  $A[l] > A[i]$  then
4:      $largest = l$ 
5:   else
6:      $largest = i$ 
7:   end if
8:   if  $r \leq k$  and  $A[r] > A[largest]$  then
9:      $largest = r$ 
10:  end if
11:  if  $largest \neq i$  then
12:     $swap(A[i], A[largest])$ 
13:     $MAX-HEAPIFY(A, largest)$ 
14:  end if
15: end function

```

## Max-Heapify Analysis

- ▶ Suppose the subtree rooted at  $i$  contains  $n$  elements
- ▶ Everything except recursion takes  $\Theta(1)$  time
- ▶ One recursive call on one child
- ▶ Worst-case: the child subtree we recurse into has more elements
- ▶ Balance invariant  $\implies$  at least  $1/3$  elements on right side  
 $\implies$  at at most  $2/3$  elements in worst case
- ▶  $T(n) = T(\frac{2}{3}n) + \Theta(1)$
- ▶  $T(n) \in \Theta(\log n)$  by master theorem case 2
- ▶ **Pushing the envelope:** for any fraction  $f < 1$ ,  
 $T(fn) + \Theta(1) \in O(\log n)$ ; not just  $f = \frac{1}{2}$

## Create Heap, Get Maximum, Increase Key, Pseudocode

1: **function** CREATE-MAX-HEAP( $A$ )

2:      $heapsize = 0$

3: **end function**

1: **function** MAX-HEAP-MAXIMUM( $A$ )

**Require:**  $A$  is a valid heap,  $n > 0$

2:     **return**  $A[1]$

3: **end function**

1: **function** MAX-HEAP-INCREASE-KEY( $A, i, key$ )

**Require:**  $A$  is a valid heap,  $1 \leq i \leq n$ ,  $key > A[i]$

2:      $A[i] = key$

3:     **while**  $i > 1$  and  $A[PARENT(i)] < A[i]$  **do**

4:          $swap(A[i], A[PARENT(i)])$

5:          $i = PARENT(i)$

6:     **end while**

7: **end function**

## Create Heap, Get Maximum, Increase Key, Analysis

Create Heap, Get Maximum: clearly  $\Theta(1)$

Increase Key:

- ▶  $A[i]$  “floats up” until it is either in heap-order, or becomes the root
- ▶  $\Theta(\text{depth of } A[i]) = \Theta(\log n)$  time

## Insert

1: **function** MAX-HEAP-INSERT( $A$ ,  $key$ )

**Require:**  $A$  is a valid heap,  $heapsize < n$

2:      $heapsize = heapsize + 1$

3:      $A[heapsize] = -\infty$

4:      $MAX - HEAP - INCREASE - KEY(A, heapsize, key)$

5: **end function**

Analysis:  $\Theta(1)$  plus  $MAX - HEAP - INCREASE - KEY$ , so  $\Theta(\log n)$

## Delete-Max

Idea: grab rightmost node on bottom level, move it to root, heapify root (sketch)

1: **function** MAX-HEAP-DELETE-MAX( $A$ )

**Require:**  $A$  is a valid heap,  $heapsize > 0$

2:      $max = A[1]$

3:      $A[1] = A[heapsize]$

4:      $heapsize = heapsize - 1$

5:      $MAX - HEAPIFY(A, 1)$

6:     **return**  $max$

7: **end function**

Analysis:  $\Theta(1)$  plus  $MAX - HEAPIFY$  so  $\Theta(\log n)$

## Build-Heap

**Online/incremental construction:** build structure one element at a time

**Offline construction/build:** given  $n$  elements all at once, build valid data structure

Offline is often faster, by constant factors or even asymptotically

Leaves are trivially in heap order and live in  $A[\lfloor n/2 \rfloor + 1, \dots, n]$

Parents might be out of heap-order and live in  $A[1, \dots, \lfloor n/2 \rfloor]$

Just heapify all the parents!

## Build-Heap Pseudocode

```
1: function BUILD-MAX-HEAP( $A, i$ )  
Require:  $A[1, \dots, n]$  may or may not be in heap-order  
Ensure:  $A[1, \dots, n]$  is in heap-order  
2:    $heapsize = n$   
3:   for  $i$  from  $\lfloor n/2 \rfloor$  down to 1 do  
4:      $MAX - HEAPIFY(A, i)$   
5:   end for  
6: end function
```



## Build-Heap Analysis

Loose analysis: if  $A[i]$  is at height  $h$ ,  $MAX - HEAPIFY$  takes  $\Theta(h)$ ;  $h \in O(\log n)$ , so each call is  $O(\log n)$ ;  $n$  calls; so  $O(n \log n)$   
(Correct but loose upper bound.)

Fact about balanced binary trees:

- ▶ Sum of all node **depths** is  $\Theta(n \log n)$
- ▶ Sum of all node **heights** is  $\Theta(n)$
- ▶ Observe that  $1/2$  of nodes are at height 0,  $1/4$  at height 1,  $1/8$  at height 2, etc.
- ▶ (sketch)

$n$  calls to  $MAX - HEAPIFY$  take time

$$\Theta(\sum_i (\text{height of } i)) = \Theta(n)$$

$\therefore BUILD - MAX - HEAP$  takes  $\Theta(n)$  time

## Arrayed Binary Max-Heap Summary

### Operation

Create empty heap

Find maximum element

Insert one element

Delete and return maximum element

Increase key of previously-inserted element

Build  $n$ -element heap offline

### Time Compl.

$\Theta(1)$

$\Theta(1)$

$\Theta(\log n)$

$\Theta(\log n)$

$\Theta(\log n)$

$\Theta(n)$

## Heapsort Intro

- ▶ Reduction to max-heap operations
- ▶ Selection sort: find maximum unsorted element, place at back of sorted array, repeat until done
- ▶ Use max-heap to accelerate "find maximum" step
- ▶ Convenient: if heap holds  $k \leq n$  elements, it occupies  $A[1, \dots, k]$
- ▶ Remaining  $n - k$  elements  $A[k + 1, \dots, n]$  are free to hold sorted elements

### High-level heapsort

1. Build-heap; all  $n$  elements hold a valid heap
2. Delete-max; removes maximum element from heap zone, vacates one element in array
3. Move the old max into the vacancy
4. Repeat until done

## Heapsort Pseudocode

```

1: function HEAPSORT( $A[1, \dots, n]$ )
Ensure:  $A$  is in non-decreasing order
2:   BUILD – MAX – HEAP( $A$ )
3:   for  $i$  from  $n$  down to 2 do
4:      $A[i] = \text{MAX – HEAP – DELETE – MAX}(A)$ 
5:   end for
6: end function

```

(Observe: no need for  $i = 1$  iteration.)

### Analysis

- ▶ build-heap =  $\Theta(n)$
- ▶  $(n - 1)$  iterations of loop  $\times \Theta(\log n)$  each
- ▶ =  $\Theta(n \log n)$  total

## Sorting Lower Bound

So far all our sorts have compared elements to each other, e.g.

$$A[i] < A[j]$$

(insertion, selection, merge, heap sort; also quick sort)

Q: what is the minimum number of comparisons adequate to sort?

A: enough to decide which of the  $n!$  permutations of  $A$  would be in order

Binary search through a set of  $N$  things takes  $\lceil \log_2 N \rceil$  steps

$\implies$  correct sort makes  $\geq \lceil \log_2 n! \rceil$  comparison operations

$= \Omega(n \log n)$  comparisons

$\therefore$  every comparison-based sorting algorithm takes  $\Omega(n \log n)$  time  
(**sorting lower bound**)

## Optimal Algorithms

**Optimal Algorithm:** time complexity matches problem's lower bound

E.g.

- ▶ Proven  $\Omega(n \log n)$  lower bound for sorting
- ▶ Merge sort, heap sort take  $\Theta(n \log n)$  time
- ▶  $\implies$  merge sort, heap sort are **optimal**

(Heap sort is theoretically superior because it is in-place.)

End-goal of algorithm design and lower-bound analysis.

## Epilogue — Sorted Order vs. Heap Order

We can sort by building a heap/BST and then retrieving elements in order.

Sorting lower bound  $\implies$  that process **must** be  $\Omega(n \log n)$

Phase of sorting	BST	heap
Build structure	$\Theta(n \log n)$	$\Theta(n)$
Retrieve in order	$\Theta(n)$	$\Theta(n \log n)$

Whack-a-mole: a  $\Omega(n \log n)$  phase inevitably pops up somewhere!

Heap-order is not insightful enough to be asymptotically significant, which is why it can be  $o(n \log n)$ , and maintain a stronger balance invariant than BSTs.