10. The Linear Programming Problem

CPSC 535

Kevin A. Wortman

CALIFORNIA STATE UNIVERSITY FULLERTON

Big Ideas

- duality — same problem from different perspectives
- formulations, reductions
- visualizing high geometric dimensions

Overview

- *programming* in math involves finding some kind of optimal solution subject to mathematically-codified constraints
  - (not coding e.g. C++ programming)
- *linear programming (LP):* optimize a linear *objective function* subject to inequalities
- very general framework
- pioneered by Soviet economist Leonid Kantorovich circa 1930s; goal was to optimize supply/demand in a communist in lieu of prices
- now used in business *(operations research)*
  - scheduling UPS deliveries, optimizing farm production, allocating investment portfolios, etc.

Computational Complexity

- many tough problems in P, including max-flow, reduce to LP
- on the border of P
- simplex algorithm technically takes $O(2^n)$ worst-case time, but is fast polynomial on most practical inputs
- we have pseudopolynomial algorithms with e.g. $O(n^{2.5}W)$ runtime and expensive constant factors
- open question whether there is a strongly polynomial LP algorithm with runtime e.g. $O(n^3)$, not a function of W

Standard Form

- *standard form:* restricted/simplified LP, easier for algorithms to solve
- later: *general form* which is more convenient for end-user formulations
- general reduces to standard with constant overhead
- similar situation to max-flow and robust max-flow
- actual solver algorithm sees a simplified standard form; reduction algorithm “frontend” accepts a generalized problem that is more convenient for end-users

Standard Form

standard form with n variables and m constraints:

maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_{1,n} & \leq & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_{2,n} & \leq & b_2 \\ \vdots & & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_{m,n} & \leq & b_m \\ x_1, x_2, \ldots, x_n & \geq & 0 \end{array}$$

variables: $x_1, \ldots, x_n \in \mathbb{R}$
objective function defined by coefficients $c_1, \ldots, c_n \in \mathbb{R}$
constraints defined by coefficients $a_{i,j}, b_i \in \mathbb{R}$

Standard Form Example

maximize $2x_1 + x_2 - \frac{1}{3}x_3$
subject to

$$\begin{array}{rcl} x_1 + x_2 & \leq & 10 \\ -x_3 & \leq & -2 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$$

Standard Form Matrix Notation

- more compact math notation
- collect:
  - variables into vector $x = \langle x_1, \ldots, x_n \rangle$
  - objective coefficients into vector $c = \langle c_1, \ldots, c_n \rangle$
  - r.h.s. of inequalities into vector $b = \langle b_1, \ldots, b_m \rangle$
  - $a_{i,j}$ coefficients into matrix A
- LP can be written in terms of dot-product and matrix-vector multiplication as (and note the transpose c^T):

maximize $c^T x$
subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Possible Outcomes

LPs are not always solvable!

there are three outcomes:

1. **solution**: concrete values for $x_1, \ldots, x_n$ that maximize $c^T x$ (good, usually the goal)
2. **unbounded**: objective can be made arbitrarily large i.e. $+\infty$ (bad, usually means there is a bug in your LP that makes it nonsensical)
3. **infeasible**: impossible to satisfy all constraints simultaneously (bad, usually means that either your LP is nonsensical; or your LP makes sense but meeting all your goals is impossible)

Standard-Form LP Problem

standard-form linear programming problem
input: vector $c \in \mathbb{R}^n$, vector $b \in \mathbb{R}^m$, and $m \times n$ matrix A of real numbers
output: one of

1. "unbounded";
2. "infeasible"; or
3. "solution" with a vector $x \in \mathbb{R}^n$ maximizing the objective function

Exploring the Three Outcomes

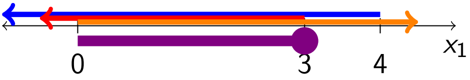
- we will explore unbounded/infeasible/solution in 1D, then 2D
- *dimension* of an LP: #variables n
- *feasible region:* space of x vectors that satisfy all constraints
- *halfspace:* half of all geometric space,
  - 1D: one side of a point on the number line e.g. $x = 3$
  - 2D: one side of a line e.g. $y = 3x + 2$
  - 3D: one side of a plane e.g. $2x + 3y - z = 5$
- each new constraint limits the feasible region to a halfspace
- as we go, make note of
  - the shape of the feasible region
  - optimal solutions are found at extreme points ("corners") of halfspaces
  - unbounded $\Leftrightarrow$ feasible region extends out infinitely
  - infeasible $\Leftrightarrow$ empty feasible region

1D Solution

maximize $2x_1$

subject to

$$\begin{array}{ccc} x_1 & \leq & 4 \\ x_1 & \leq & 3 \\ x_1 & \geq & 0 \end{array}$$

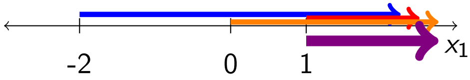


- feasible region = intersection of all arrows = is line segment $[0, 3]$
- solution is $x_1 = 3$
- optimal objective function value is $2x_1 = 2(3) = 6$

1D Unbounded

maximize $2x_1$
subject to

$$\begin{aligned} -x_1 &\leq 2 \\ -x_1 &\leq -1 \\ x_1 &\geq 0 \end{aligned}$$



- feasible region = intersection of all arrows = open interval $[1, +\infty)$
- solution is undefined
- optimal objective function value is $2x_1 = 2(\infty) = \infty$

1D Infeasible

maximize $2x_1$
subject to

$$\begin{aligned} x_1 &\leq 1 \\ -x_1 &\leq -3 \\ x_1 &\geq 0 \end{aligned}$$

- feasible region = intersection of all arrows = $\emptyset$
- solution is undefined
- cannot evaluate objective function

2D Solution

maximize x_2
subject to

$$\begin{aligned} \frac{1}{4}x_1 + x_2 &\leq 2 \\ -\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Sidebar: Math Definition of a Line

- recall
  - slope-intercept form $y = mx + b$
  - 2D LP constraint is $c_1 x_1 + c_2 x_2 \leq b$
- substitute $x_1 = x, x_2 = y$, rearrange to slope-intercept:

$$
\begin{aligned}
c_1 x_1 + c_2 x_2 &\leq b \\
c_1(x) + c_2(y) &\leq b \\
-(c_1 x) & -(c_1 x) \\
c_2 y &\leq -c_1 x + b
\end{aligned}
$$

if $c_2 > 0$ then

$$y \leq -\frac{c_1}{c_2} x + \frac{b}{c_2}$$

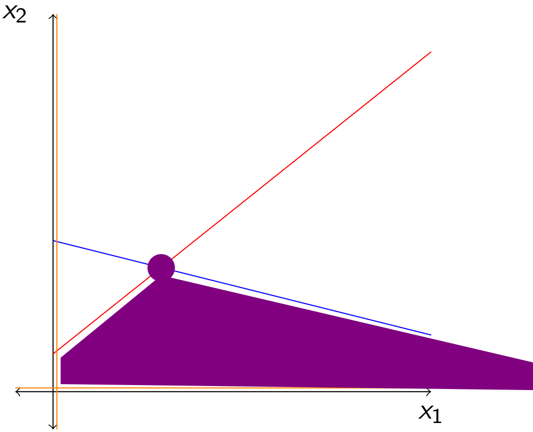
else, $c_2 < 0$, dividing by c_2 flips $\leq$ to $\geq$, and

$$y \geq -\frac{c_1}{c_2} x + \frac{b}{c_2}$$

2D Solution

maximize x_2
subject to

$$\begin{aligned} \frac{1}{4}x_1 + x_2 &\leq 2 \\ -\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

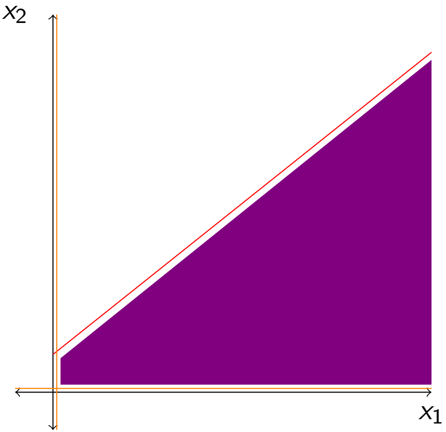


- feasible region is intersection of halfspaces $\Leftrightarrow$ polygon
- optimal solution is intersection of lines at $x_1 \approx 1.43, x_2 \approx 1.64$

2D Unbounded

maximize x_2
subject to

$$\begin{aligned} -\frac{4}{5}x_1 + x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

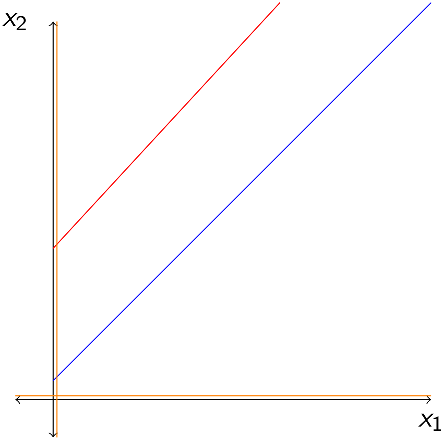


- feasible region is intersection of halfspaces ⇔ some polygon sides, one infinite side
- optimal solution undefined

2D Infeasible

maximize x_2
subject to

$$\begin{aligned} -x_1 + x_2 &\leq .25 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- feasible region is intersection of halfspaces $\Leftrightarrow$ empty set
- optimal solution undefined