

02. Algorithm Fundamentals

CPSC 535

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Problems

Computational problem: definition of input and desired output

Each is a mathematical object that could be stored in a computer data structure.

Sorting problem

input: A sequence of n numbers $\langle a_1, a_2, \dots, a_n \rangle$.

output: A permutation (reordering) $\langle a'_1, a'_2, \dots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \dots \leq a'_n$.

Algorithms

Instance (of a problem): Concrete input datum

Example

$\langle 71, 14, 31, 2, 82 \rangle$

Algorithm: well-defined computational procedure that unerringly transforms input to output

Motivation

Why do we care about algorithms, or algorithmic efficiency?

- ▶ Algorithms are automating major parts of the economy: operations research, high frequency trading, machine learning, etc.
- ▶ Efficiency can mean the difference between computations being viable, sustainable, for human use versus impractical.
- ▶ The principle of not wasting product.
- ▶ Intriguing mathematical questions are worth studying in their own right.

Data Structures

Data Structure: method for storing, organizing data

- ▶ Data members e.g. head pointer, tail pointer
- ▶ *Invariant(s)* defining how the structure must be organized to remain valid, e.g. head points to first node, tail points to last node
- ▶ Defined *operations*, each operation is an algorithm that operates on the structure.

Pseudocode

Pseudocode: code-like notation for conveying algorithms

- ▶ Goal is clear communication to a human audience
- ▶ Not compiled, so no need to be syntactically perfect
- ▶ Not software engineering; no need for error checking, modularity, encapsulation, etc.

Algorithm implementer is a specific role and skill set, bridging the gap between scholarly pseudocode and industrial coding practices.

Insertion Sort

```
1: function INSERTION-SORT(A)
2:   for  $j = 2$  to  $A.length$  do
3:      $key = A[j]$ 
4:     // Insert  $A[j]$  into the sorted sequence  $A[1 \dots j-1]$ .
5:      $i = j - 1$ 
6:     while  $i > 0$  and  $A[i] > key$  do
7:        $A[i+1] = A[i]$ 
8:        $i = i-1$ 
9:     end while
10:     $A[i+1] = key$ 
11:  end for
12: end function
```

Pseudocode Observations

- ▶ Algorithm is a function/procedure, input is argument(s)
- ▶ No global variables
- ▶ Code-like but not compile-able code
- ▶ Arrays start at 1
- ▶ No error checking or modularity
- ▶ Translatable into practically any programming language

Analysis

Analysis: establish how efficient an algorithm is

- ▶ Usually a mathematical proof (alternatively empirical evidence)
- ▶ Usually analyze for time spent (or disk I/O, space, energy, randomness, etc.)
- ▶ Usually summarize resource use by *order of growth* in asymptotic notation; $O(n)$, $\Theta(n^2)$, etc.

RAM model

Computational model: defines how a computer executes an algorithm, specifically enough to measure time (or other resources)

Random Access Machine (RAM):

- ▶ "default" computational model, approximates a generic real-world CPU and memory
- ▶ CPU has instructions for integer arithmetic, floating point arithmetic, control (jump, call, return, if), logic (or, and, not), data copying.
- ▶ one step $\equiv O(1)$ instructions; each pseudocode statement counts as 1 step (except function calls)
- ▶ CPU has some $O(1)$ word size, e.g. 32 or 64 bit; one instruction is limited to writing that many bits
- ▶ cannot "cheat" by packing $\Theta(n)$ information in one word

Worst-Case Analysis

- ▶ In a time analysis, we need to prove how much time insertion sort takes when run
- ▶ depends on the type of input, e.g. pre-sorted, completely jumbled, in between
- ▶ convention: analyze the *worst case* for the algorithm at hand
- ▶ generous to skeptics, conservative for software engineers
- ▶ as an exception, sometimes analyze *average case* of deliberately randomized algorithms

Claim: The worst-case time complexity of insertion sort is $\Theta(n^2)$.

Divide-and-conquer

1. **divide** input into several smaller instances of the same problem (often, divide input in half)
2. **"conquer"** by recursively solving all the sub-problems; may involve a simple base case
3. **combine** the many solutions into one coherent solution for the original problem

Merge sort

Merge sort: classical sorting algorithm using divide-and-conquer

divide: chop list of n unsorted elements into two lists of $n/2$ elements each

conquer: merge-sort each unsorted list; if $n \leq 1$, nothing to do

combine: *merge* two sorted lists of $n/2$ elements, into one sorted list of n elements

Merge pseudocode

Ensure: $A[p \dots r]$ is sorted

```

1: function MERGE-SORT( $A, p, r$ )
2:   if  $p < r$  then
3:      $q = \lfloor (p + r) / 2 \rfloor$ 
4:     MERGE-SORT( $A, p, q$ )
5:     MERGE-SORT( $A, q + 1, r$ )
6:     MERGE( $A, p, q, r$ )
7:   end if
8: end function

```

Require: $p \leq q < r$, $A[p \dots q]$ is sorted, $A[q + 1 \dots r]$ is sorted

Ensure: $A[p \dots r]$ is sorted

```

1: function MERGE( $A, p, q, r$ )
2:    $n_1 = (q - p + 1)$ ,  $n_2 = (r - q)$ 
3:   let  $L[1 \dots n_1 + 1]$  and  $R[1 \dots n_2 + 1]$  be new arrays
4:    $L[1 \dots n_1] = A[p \dots q]$ 
5:    $R[1 \dots n_2] = A[q + 1 \dots r]$ 
6:    $L[n_1 + 1] = R[n_2 + 1] = \infty$ 
7:    $i = j = 1$ 
8:   for  $k = p$  to  $r$  do
9:     if  $L[i] \leq R[j]$  then
10:       $A[k] = L[i]$ 
11:       $i = i + 1$ 
12:     else
13:       $A[k] = R[j]$ 
14:       $j = j + 1$ 
15:     end if
16:   end for
17: end function

```

Merge sort analysis

The worst-case time complexity of merge sort is given by the *recurrence relation*

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

Claim: $T \in \Theta(n \log n)$

Claim: Merge sort uses $\Theta(n)$ extra space for L and R . (Observe that at most one merge is happening at any moment, and the largest merge uses $n + 2$ extra array elements.)

Insertion sort versus merge sort

Insertion sort: $\Theta(n^2)$ time, $\Theta(1)$ space

Merge sort: $\Theta(n \log n)$ time, $\Theta(n)$ space

Merge sort is subjectively more complicated

Space vs. time tradeoff (typical)

Efficiency vs. complication tradeoff (typical)

Refactoring complication into algorithm design is usually a win

Asymptotic notation

Asymptotic notation: $\Theta(n^2)$, $O(n^2)$, $\Omega(n^2)$, etc.

Solves some problems with algorithm analysis

- ▶ Whole point is how algorithms perform on very large n
- ▶ \implies ignore small n
- ▶ We count abstract "steps" so constant factors are not meaningful
- ▶ i.e. $4n^2$ and $5n^2$ should count as essentially the same
- ▶ Want to prioritize speeding up the part of the algorithm that actually accounts for the most time
- ▶ Running time functions usually include multiple terms; the asymptotically fastest-growing term *dominates* the running time

Example: insertion sort

Suppose insertion sort's run time is

$$\begin{aligned} T(n) &= (\text{inner loop steps}) + (\text{outer loop steps}) + (\text{outside loop}) \\ &= 3n^2 + 4n + 1 \end{aligned}$$

Let $n = 2$,

$$T(n) = 3(2)^2 + 4(2) + 1 = 12 + 8 + 1 = 21.$$

Let $n = 1,000$,

$$T(n) = 3(1,000)^2 + 4(1,000) + 1 = 3,000,000 + 4,000 + 1.$$

\implies inner loop dominates time, focus on speeding that up

Θ notation

"big-theta": intuitively,

$$\Theta(g(n)) = \{f(n) : f(n) \text{ grows asymptotically the same as } g(n)\}$$

Formally,

$$\Theta(g(n)) = \{f(n) : \exists \text{ positive constants } c_1, c_2, n_0 \text{ such that } 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \quad \forall n \geq n_0\}.$$

This is set notation, so e.g.

$$3n^2 + 4n + 1 \in \Theta(n^2).$$

Θ -notation indicates a *tight bound*: set of functions upper- **and** lower-bounded by $g(n)$

$O, \Omega, o,$ and ω

Intuitively,

▶ "big-oh":

$$O(g(n)) = \{f(n) : f(n) \text{ grows asymptotically } \leq g(n)\}$$

▶ "big-omega":

$$\Omega(g(n)) = \{f(n) : f(n) \text{ grows asymptotically } \geq g(n)\}$$

▶ "little-oh":

$$o(g(n)) = \{f(n) : f(n) \text{ grows asymptotically } < g(n)\}$$

▶ "little-omega":

$$\omega(g(n)) = \{f(n) : f(n) \text{ grows asymptotically } > g(n)\}$$

Knuth-style notation abuse

An asymptotic notation expression such a $O(n^2)$ is, technically, a **set** of function objects

\implies a run-time function may be an **element** of one of these sets

Mathematically precise:

$$3n^2 + 4n + 1 \in O(n^2).$$

Some others (notably Knuth) abuse notation to use $=$, which is technically incorrect since the operands are different types, but is arguably more readable:

$$3n^2 + 4n + 1 = O(n^2).$$

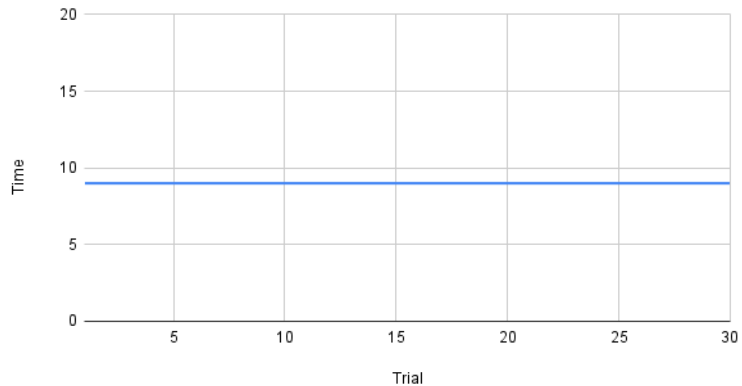
Kinds of Time Bounds

Suppose algorithm A takes...

- ▶ $\Theta(n)$ **worst-case time**: for every input, A takes $\Theta(n)$ time
- ▶ $\Theta(n)$ **amortized time**: for any sequence of k executions, the total time is $\Theta(k \cdot n)$, so the average time of a single execution is $\Theta(n)$
- ▶ $\Theta(n)$ **expected time**: A 's run time is a random variable whose expected value is $\Theta(n)$
- ▶ $\Theta(n)$: no explicit qualifier; by default, this means $\Theta(n)$ worst-case time

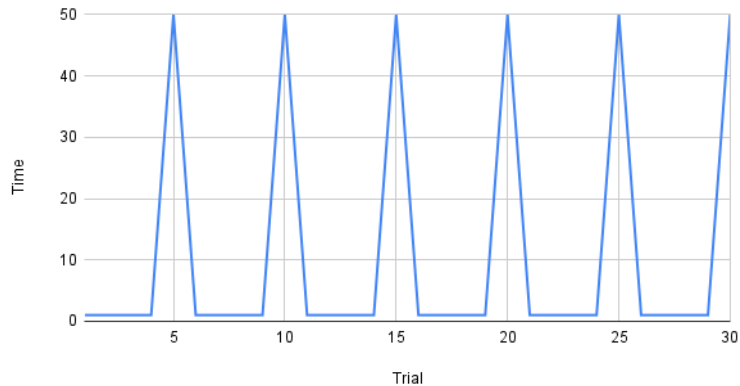
Repeated Trials of $O(1)$ Worst-Case Time

Time vs. Trial



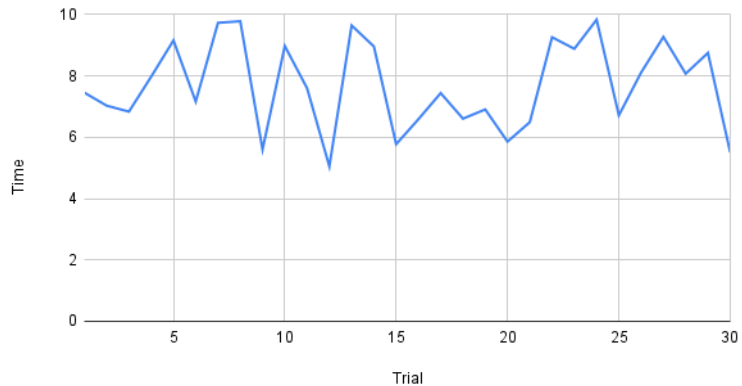
Repeated Trials of $O(1)$ Amortized Time

Time vs. Trial



Repeated Trials of $O(1)$ Expected Time

Time vs. Trial



Kinds of Time Bounds and Goal Setting

Predictability is desirable, so in general, our preference is

1. worst-case
2. amortized
3. expected

Improving the kinds of time bounds is a research area

Example:

- ▶ Conventional chained hash tables take $O(1)$ expected time to search, insert, or delete
- ▶ *Cuckoo hash tables* take $O(1)$ worst-case time to search or delete, and $O(1)$ expected time to insert

Combining Different Kinds of Time Bounds

In general:

an "amortized" or "expected" qualifier is "contagious"

Examples:

- ▶ $\Theta(n)$ worst-case + $\Theta(n)$ amortized = $\Theta(n)$ amortized
- ▶ $\Theta(n)$ worst-case + $\Theta(n)$ expected = $\Theta(n)$ expected
- ▶ $\Theta(n)$ amortized + $\Theta(n)$ expected = $\Theta(n)$ amortized expected
- ▶ $\Theta(n)$ worst-case \times $\Theta(n)$ expected = $\Theta(n^2)$ expected

Exception: Dropping "Amortized"

- ▶ Recall: $\Theta(n)$ **amortized time**: for any sequence of k executions, the total time is $\Theta(k \cdot n)$, so the average time of a single execution is $\Theta(n)$
- ▶ Observe: the total time is worst-case
- ▶ So **when an amortized operation is repeated $\omega(1)$ times, the total time drops "amortized" to become worst-case**
- ▶ Examples:
 - ▶ n repetitions of $O(1)$ amortized = $O(n)$ worst-case
 - ▶ n repetitions of $O(\log n)$ amortized = $O(n \log n)$ worst-case
 - ▶ 2 repetitions of $O(\log n)$ amortized = $O(\log n)$ amortized

Exception: Dropping "Expected"

- ▶ A randomized algo. has both
 - ▶ a faster expected-time bound, and
 - ▶ a slower worst-case time bound
- ▶ Example: quick sort takes $\Theta(n \log n)$ expected time and $\Theta(n^2)$ worst-case time
- ▶ We may choose to use either one
- ▶ When adding an expected time bound to a non-expected one,
choose the randomized algo's worst-case bound if that wouldn't be the dominating term

Example: Dropping "Expected"

Algo.: insertion sort then quick sort

Choosing quick sort's expected time bound:

$$\Theta(n^2) + \Theta(n \log n) \text{ expected} = \Theta(n^2) \text{ expected}$$

Choosing quick sort's worst-case time bound:

$$\Theta(n^2) + \Theta(n^2) = \Theta(n^2)$$

Second choice is more favorable