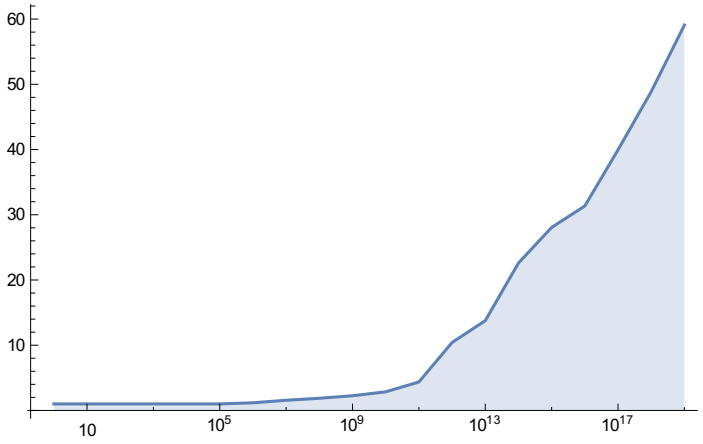
In[1]:=
```
(* List of fast Deleglise-Rivat alpha factors for x ≤ 10^21 found by
  running pi(x) benchmarks using the find_fastest_alpha.sh script *)

alphaDelegliseRivat = { (* {x, alpha} *) {1, 1}, {10^1, 1},
  {10^2, 1}, {10^3, 1}, {10^4, 1}, {10^5, 1}, {10^6, 1.172}, {10^7, 1.561},
  {10^8, 1.865}, {10^9, 2.255}, {10^10, 2.854}, {10^11, 4.365},
  {10^12, 10.422}, {10^13, 13.764}, {10^14, 22.599}, {10^15, 28.055},
  {10^16, 31.346}, {10^17, 39.948}, {10^18, 48.867} , {10^19, 59.083}}
```

Out[1]=
```
{{1, 1}, {10, 1}, {100, 1}, {1000, 1}, {10 000, 1}, {100 000, 1}, {1 000 000, 1.172},
 {10 000 000, 1.561}, {100 000 000, 1.865}, {1 000 000 000, 2.255}, {10 000 000 000, 2.854},
 {100 000 000 000, 4.365}, {1 000 000 000 000, 10.422}, {10 000 000 000 000, 13.764},
 {100 000 000 000 000, 22.599}, {1 000 000 000 000 000, 28.055},
 {10 000 000 000 000 000, 31.346}, {100 000 000 000 000 000, 39.948},
 {1 000 000 000 000 000 000, 48.867}, {10 000 000 000 000 000 000, 59.083}}
```

In[2]:=
```
ListLogLinearPlot[alphaDelegliseRivat, Filling → Bottom, Joined → True]
```

Out[2]=



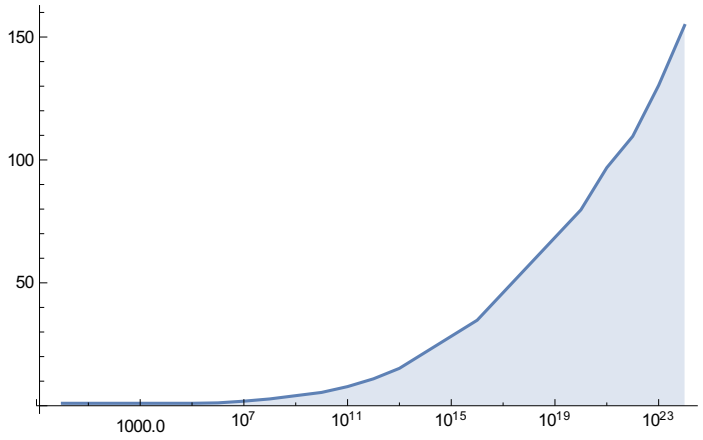
In[3]:=
```
(* alpha is a tuning factor that balances the compuation
 of the easy special leaves and the hard special leaves. The
 formula below is used in the file src/primesum.cpp to
 calculate a fast alpha factor for the computation of pi(x). *)

NonlinearModelFit[alphaDelegliseRivat,
 a (Log[x])^3 + b (Log[x])^2 + c Log[x] + d, {a, b, c, d}, x]
```

Out[3]=
```
FittedModel[ 1.54658 - 0.0754525 Log[x] - 0.0156173 ≪1≫^2 + 0.00109086 Log[x]^3 ]
```

```
(* List of fast Deleglise-Rivat alpha factors for x
  greater than 10^21 found by running pi(x) benchmarks. A larger
  alpha reduces CPU cache misses for large pi(x) computations. *)

alphaDelegliseRivatLarge =
 { (* {x, alpha} *) {1, 1}, {10^1, 1}, {10^2, 1}, {10^3, 1}, {10^4, 1},
  {10^5, 1}, {10^6, 1.172}, {10^7, 1.861}, {10^8, 2.778}, {10^10, 5.426},
  {10^11, 7.795}, {10^12, 10.960}, {10^13, 15.22}, {10^16, 34.80},
  {10^20, 79.68}, {10^21, 96.86}, {10^22, 109.61}, {10^23, 130.33}, {10^24, 154.69}}
```

```
{{1, 1}, {10, 1}, {100, 1}, {1000, 1}, {10 000, 1}, {100 000, 1}, {1 000 000, 1.172},
 {10 000 000, 1.861}, {100 000 000, 2.778}, {10 000 000 000, 5.426},
 {100 000 000 000, 7.795}, {1 000 000 000 000, 10.96}, {10 000 000 000 000, 15.22},
 {10 000 000 000 000 000, 34.8}, {100 000 000 000 000 000 000, 79.68},
 {1 000 000 000 000 000 000 000, 96.86}, {10 000 000 000 000 000 000 000, 109.61},
 {100 000 000 000 000 000 000 000, 130.33}, {1 000 000 000 000 000 000 000 000, 154.69}}
```

```
ListLogLinearPlot[alphaDelegliseRivatLarge, Filling → Bottom, Joined → True]
```




```
(* alpha is a tuning factor that balances the compuation
 of the easy special leaves and the hard special leaves. The
 formula below is used in the file src/primesum.cpp to
 calculate a fast alpha factor for the computation of pi(x). *)

NonlinearModelFit[alphaDelegliseRivatLarge,
 a (Log[x])^3 + b (Log[x])^2 + c Log[x] + d, {a, b, c, d}, x]
```

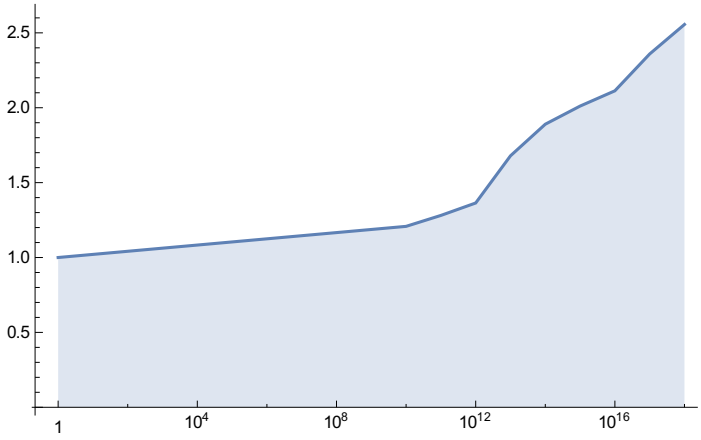
$$\text{FittedModel}\left[0.591972+0.282139\,\text{Log}[x]-0.0375705\,\text{Log}[x]^2+0.00149066\,\text{Log}[x]^3\right]$$

```
(* Below is another formula which is quite accurate for calculating the
   Deleglise-Rivat alpha factor in primesum. The constant 2200 has
   been obtained by running many pi(10^20) benchmarks. *)

alpha[x_] := (Log[x])^3 / (2200 (Log[Log[10^20]] / Log[Log[x]])^3)
```

```
(* List of fast Lagarias-Miller-
 Odlyzko alpha factors found by running pi(x) benchmarks. *)

alphaLMO = {(* {x, alpha} *) {1, 1}, {10^10, 1.208},
  {10^11, 1.281}, {10^12, 1.364}, {10^13, 1.679}, {10^14, 1.890},
  {10^15, 2.011}, {10^16, 2.113}, {10^17, 2.359}, {10^18, 2.556}}
```

```
{{1, 1}, {10 000 000 000, 1.208}, {100 000 000 000, 1.281},
 {1 000 000 000 000, 1.364}, {10 000 000 000 000, 1.679}, {100 000 000 000 000, 1.89},
 {1 000 000 000 000 000, 2.011}, {10 000 000 000 000 000, 2.113},
 {100 000 000 000 000 000, 2.359}, {1 000 000 000 000 000 000, 2.556}}
```

```
ListLogLinearPlot[alphaLMO, Filling → Bottom, Joined → True]
```




```
(* alpha is a tuning factor that balances the compuation
 of the easy special leaves and the hard special leaves. The
 formula below is used in the file src/primesum.cpp to
 calculate a fast alpha factor for the computation of pi(x). *)

NonlinearModelFit[alphaLMO, a (Log[x])^2 + b Log[x] + c, {a, b, c}, x]
```

$$\text{FittedModel}\left[0.990948 - 0.0261411\,\text{Log}[x] + 0.00156512\,\text{Log}[x]^2\right]$$
