In[1]:= (* List of fast Deleglise-Rivat alpha factors for x ≤ 10^21 found by
running pi(x) benchmarks using the find_fastest_alpha.sh script *)

alphaDegliseRivat = { (* {x, alpha} *) {1, 1}, {10^1, 1},
  {10^2, 1}, {10^3, 1}, {10^4, 1}, {10^5, 1}, {10^6, 1.172}, {10^7, 1.561},
  {10^8, 1.865}, {10^9, 2.255}, {10^10, 2.854}, {10^11, 4.365},
  {10^12, 7.422}, {10^13, 10.164}, {10^14, 12.806}, {10^15, 20.604},
  {10^16, 29.676}, {10^17, 51.527}, {10^18, 62.914} , {10^19, 68.930}}

Out[1]= {{1, 1}, {10, 1}, {100, 1}, {1000, 1}, {10 000, 1}, {100 000, 1}, {1 000 000, 1.172},
  {10 000 000, 1.561}, {100 000 000, 1.865}, {1 000 000 000, 2.255}, {10 000 000 000, 2.854},
  {100 000 000 000, 4.365}, {1 000 000 000 000, 7.422}, {10 000 000 000 000, 10.164},
  {100 000 000 000 000, 12.806}, {1 000 000 000 000 000, 20.604},
  {10 000 000 000 000 000, 29.676}, {100 000 000 000 000 000, 51.527},
  {1 000 000 000 000 000 000, 62.914}, {10 000 000 000 000 000 000, 68.93}}

In[2]:= ListLogLinearPlot[alphaDegliseRivat, Filling → Bottom, Joined → True]

Out[2]=
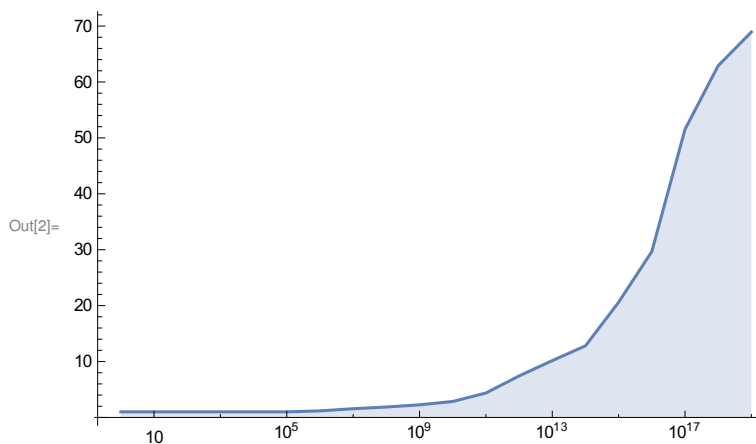
In[3]:=
(* alpha is a tuning factor that balances the compuation
 of the easy special leaves and the hard special leaves. The
 formula below is used in the file src/primesum.cpp to
 calculate a fast alpha factor for the computation of pi(x). *)

NonlinearModelFit[alphaDegliseRivat,
 a (Log[x])^3 + b (Log[x])^2 + c Log[x] + d, {a, b, c, d}, x]

Out[3]= FittedModel[$-0.5712 + 0.964706 \, \text{Log}[x] - 0.0975921 \ll 1 \gg^2 + 0.00261288 \, \text{Log}[x]^3$]

In[4]:= %3["BestFit"]

Out[4]= $-0.5712 + 0.964706 \, \text{Log}[x] - 0.0975921 \, \text{Log}[x]^2 + 0.00261288 \, \text{Log}[x]^3$

```
(* List of fast Deleglise-Rivat alpha factors for x
   greater than 10^21 found by running pi(x) benchmarks. A larger
   alpha reduces CPU cache misses for large pi(x) computations. *)

alphaDegliseRivatLarge =
 { (* {x, alpha} *) {1, 1}, {10^1, 1}, {10^2, 1}, {10^3, 1}, {10^4, 1},
   {10^5, 1}, {10^6, 1.172}, {10^7, 1.861}, {10^8, 2.778}, {10^10, 5.426},
   {10^11, 7.795}, {10^12, 10.960}, {10^13, 15.22}, {10^16, 34.80},
   {10^20, 79.68}, {10^21, 96.86}, {10^22, 109.61}, {10^23, 130.33}, {10^24, 154.69}}
```

{{1, 1}, {10, 1}, {100, 1}, {1000, 1}, {10 000, 1}, {100 000, 1}, {1 000 000, 1.172},
 {10 000 000, 1.861}, {100 000 000, 2.778}, {10 000 000 000, 5.426},
 {100 000 000 000, 7.795}, {1 000 000 000 000, 10.96}, {10 000 000 000 000, 15.22},
 {10 000 000 000 000 000, 34.8}, {100 000 000 000 000 000 000, 79.68},
 {1 000 000 000 000 000 000 000, 96.86}, {10 000 000 000 000 000 000 000, 109.61},
 {100 000 000 000 000 000 000 000, 130.33}, {1 000 000 000 000 000 000 000 000, 154.69}}

```
ListLogLinearPlot[alphaDegliseRivatLarge, Filling → Bottom, Joined → True]
```

```
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"], {{{},
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       7, 6, 5, 4, 3, 2}}]]]}, {}, {}}, {{}, {},
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      19}]}}}], {}},
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     Part[#, 2]}& ), "CopiedValueFunction" -> ({
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```
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      " ", "computation", " ", "of", " ", "pi",
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     RowBox[{{"b", " ",
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  3.634323497831388*^9, 3.6344056317660627`*^9, 3.652340523756771*^9,
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