常用的无穷级数

1.三角函数

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\begin{aligned}\tan x &= \sum_{n=0}^{\infty} \frac{U_{2n+1} x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n} x^{2n-1}}{(2n)!} \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\csc x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 (2^{2n-1}-1) B_{2n} x^{2n-1}}{(2n)!} \\ &= \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots, \qquad \text{for } 0 < |x| < \pi\end{aligned}$$

$$\begin{aligned}\sec x &= \sum_{n=0}^{\infty} \frac{U_{2n} x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n E_n x^{2n}}{(2n)!} \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\cot x &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!} \\ &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots, \qquad \text{for } 0 < |x| < \pi\end{aligned}$$

U_n是 n 次上／下数，

B_n是 n 次伯努利数，

E_n（下面的）是 n 次欧拉数

2.反三角函数

$$\begin{aligned}\arcsin z &= z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{z^{2n+1}}{(2n+1)}; \qquad |z| \le 1\end{aligned}$$

$$\begin{aligned}\arccos z &= \frac{\pi}{2} - \arcsin z \\ &= \frac{\pi}{2} - \left(z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots\right) \\ &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{z^{2n+1}}{(2n+1)}; \qquad |z| \le 1\end{aligned}$$

$$\begin{aligned}\arctan z &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}; \qquad |z| \le 1 \qquad z \ne i, -i\end{aligned}$$

$$\begin{aligned}\operatorname{arccot} z &= \frac{\pi}{2} - \arctan z \\ &= \frac{\pi}{2} - \left(z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots\right) \\ &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}; \qquad |z| \le 1 \qquad z \ne i, -i\end{aligned}$$

$$\begin{aligned}\operatorname{arcsec} z &= \arccos\left(z^{-1}\right) \\ &= \frac{\pi}{2} - \left(z^{-1} + \left(\frac{1}{2}\right)\frac{z^{-3}}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^{-5}}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^{-7}}{7} + \cdots\right) \\ &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{z^{-(2n+1)}}{(2n+1)}; \qquad |z| \ge 1\end{aligned}$$

$$\begin{aligned}\operatorname{arccsc} z &= \arcsin\left(z^{-1}\right) \\ &= z^{-1} + \left(\frac{1}{2}\right)\frac{z^{-3}}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^{-5}}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^{-7}}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{z^{-(2n+1)}}{2n+1}; \qquad |z| \ge 1\end{aligned}$$

欧拉发现了反正切的更有效的级数：

$$\arctan x = \frac{x}{1+x^2}\sum_{n=0}^{\infty}\prod_{k=1}^{n}\frac{2kx^2}{(2k+1)(1+x^2)}.$$

注意对 $n=0$ 在和中的项是空积 1。

3.对数函数

$$\ln z = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n}(z-1)^n$$

更有效率的级数是：

$$\ln z = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{z-1}{z+1}\right)^{2n+1}$$

对于任何其他底数β，我们使用：

$$\log_\beta x = \frac{\ln x}{\ln \beta}$$

4.计算特殊常数的

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9}\right)$$

$$\pi = \frac{426880\sqrt{10005}}{\sum_{k=0}^{\infty} \frac{(6n)!}{(n!)^3}\frac{(545140134n+13591409)}{(3n)!\,(-640320)^{3n}}}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

5.特殊的函数

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k$$