

CE 311K: Taylor series and Newton Raphson

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1 Catenary vs Parabola

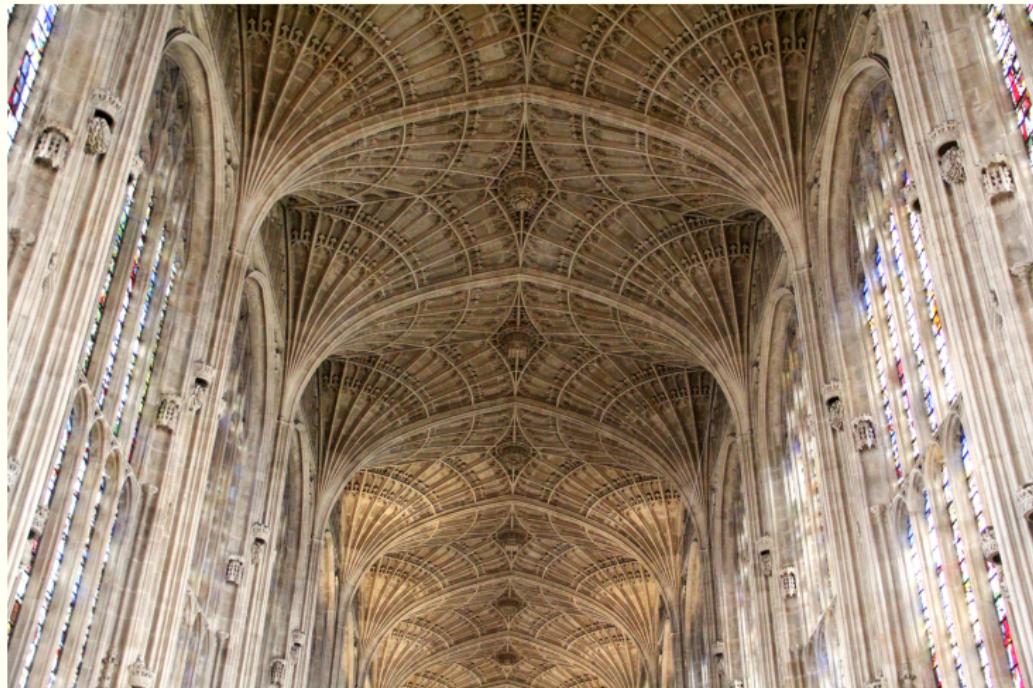
2 Taylor series

3 Newton Raphson

The fan vaults of King's college



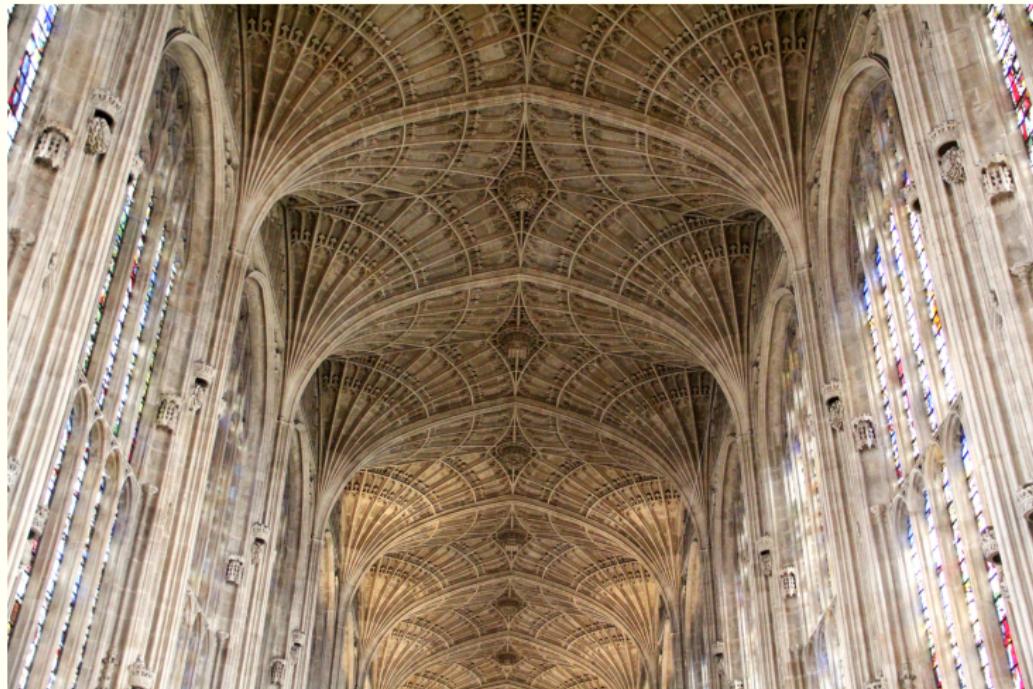
② What is the thickness of the ceiling?



From arches to fan vaults



The fan vaults of King's college



CE 311K: Taylor series

- └ Catenary vs Parabola

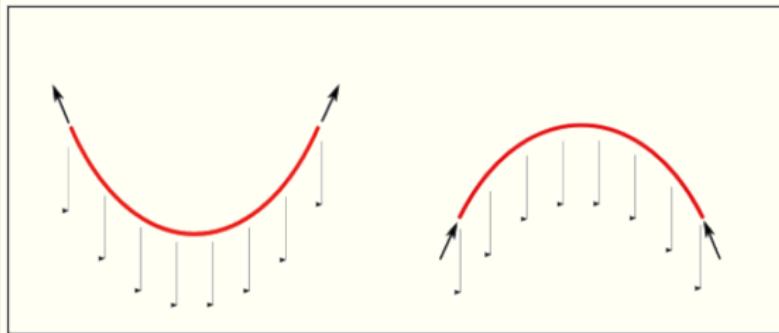
- └ The fan vaults of King's college

The fan vaults of King's college



King's college fan vaults were built in 1500s

Catenary

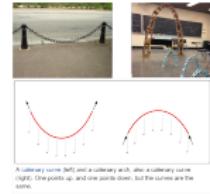


A [catenary curve](#) (left) and a catenary arch, also a catenary curve (right). One points up, and one points down, but the curves are the same.

CE 311K: Taylor series

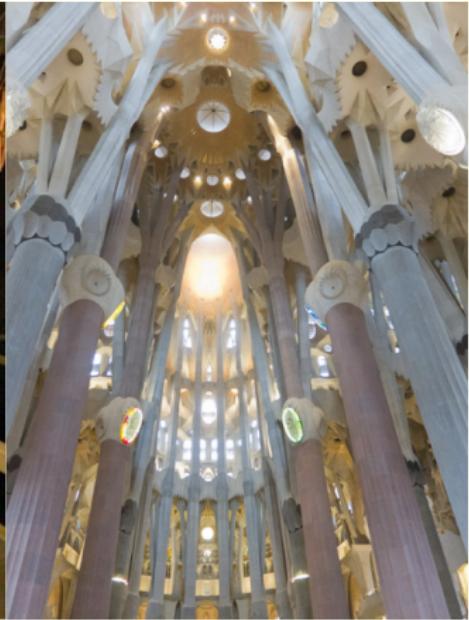
└ Catenary vs Parabola

└ Catenary



The English architect Robert Hooke was the first to study the catenary mathematically and in 1675 published an anagram (in Latin) of : "As hangs the flexible line, so but inverted will stand the rigid arch." The arch above Wembley Stadium has the shape of a catenary and Christopher Wren also intended to use it in St. Paul's dome

Sagrada Familia



- CE 311K: Taylor series
- └ Catenary vs Parabola
- └ Sagrada Familia



Antoni Gaudi i Cornet (1852-1926) was a well-known architect from Spain. Gaudi used catenary arches in many of his projects. One of Gaudi's greatest works however became the Temple of the Holy Family (the Sagrada Familia) in Barcelona.

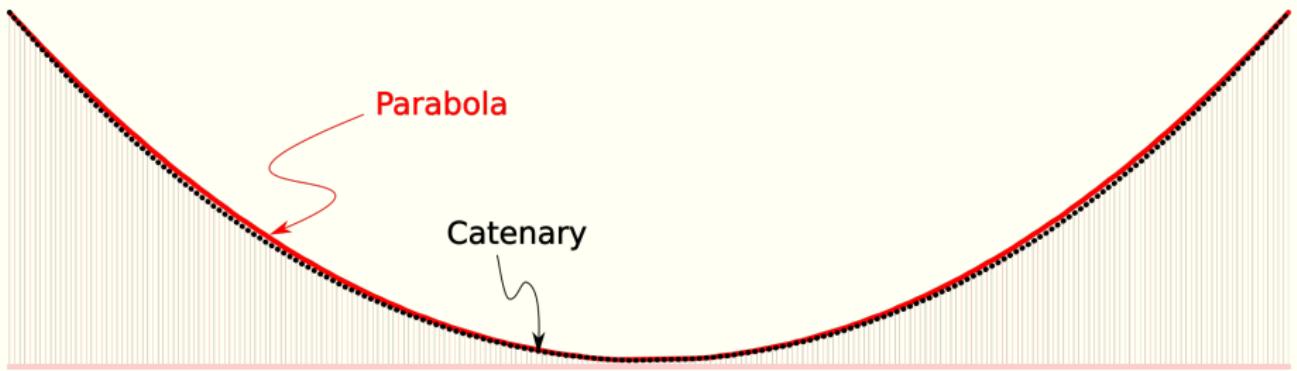
Catenary, catenary . . . everywhere



② What is this shape?



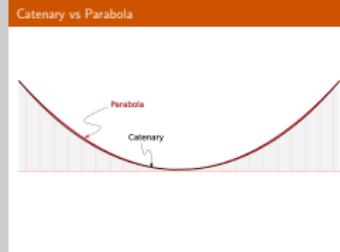
Catenary vs Parabola



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- Catenary vs Parabola

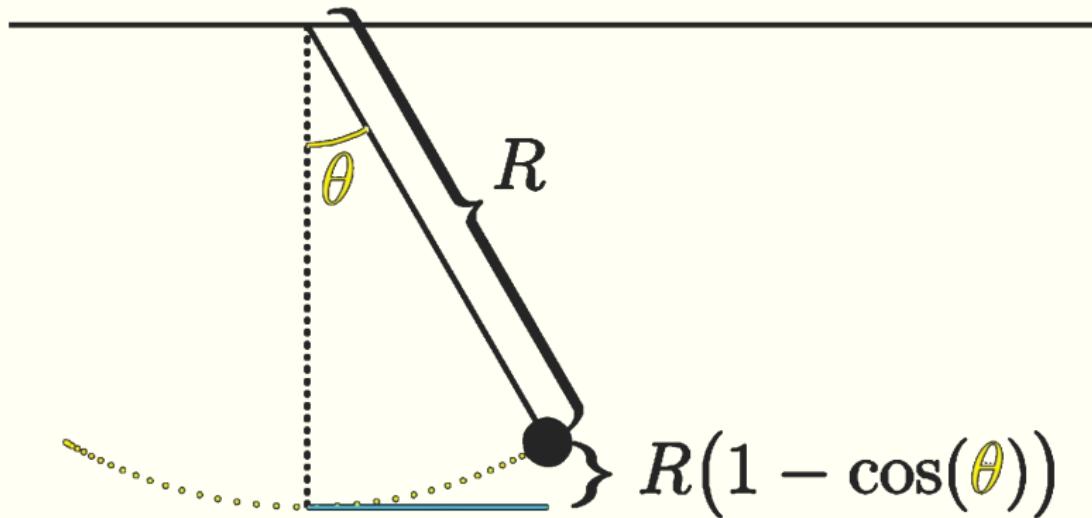
- Catenary vs Parabola



Parabola $y = x^2$ for a parabola, a hyperbolic function like $\cosh(x) \approx 1 + \frac{x^2}{2}$

Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point



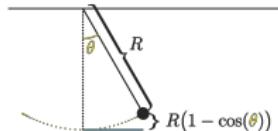
CE 311K: Taylor series

- Catenary vs Parabola

- Potential energy of a simple pendulum

Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point.



Taylor series approximation $\cos(\theta) \approx$ for a parabola, a hyperbolic function like $\cosh(x) \approx 1 + \frac{x^2}{2}$, which gives $h = R(1 - (1 + \frac{\theta^2}{2})) = R\frac{\theta^2}{2}$

The cosine function made the problem awkward and unwieldy. If we approximate using $\cos(\theta) \approx \frac{\theta^2}{2}$ everything fell into place. An approximation like that might seem completely out of left field. Let's graph these functions. They do look close to each other. But how do we think to make this approximation? and how do we even get that particular quadratic?

1 Catenary vs Parabola

2 Taylor series

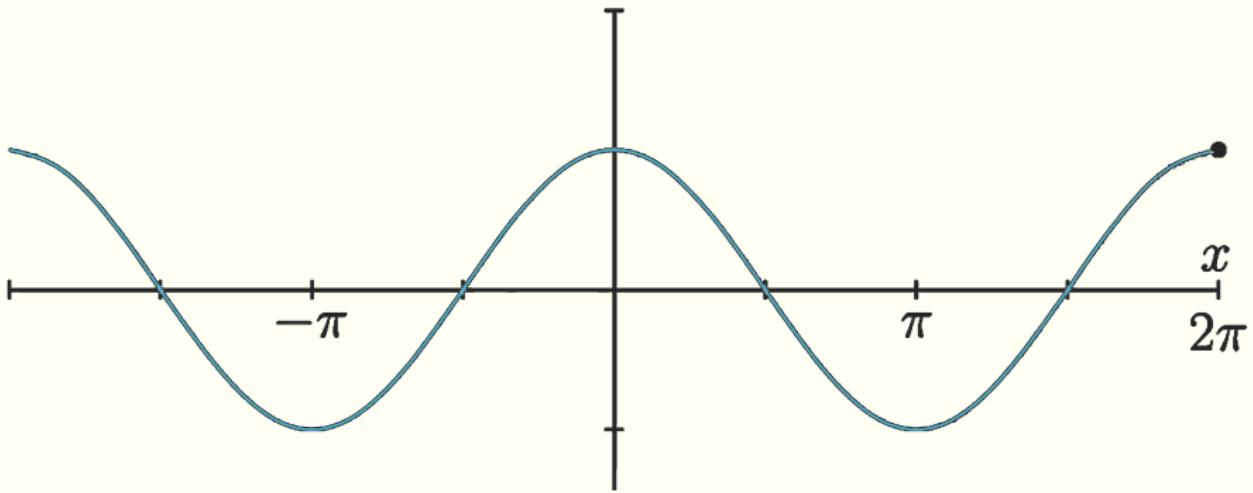
3 Newton Raphson

Taylor series

- Taylor series is one the best tools maths has to offer for approximating functions.
- Taylor series is about taking non-polynomial functions and finding polynomials that approximate at some input.
- The motive here is the polynomials tend to be much easier to deal with than other functions, they are easier to compute, take derivatives, integrate, just easier overall.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

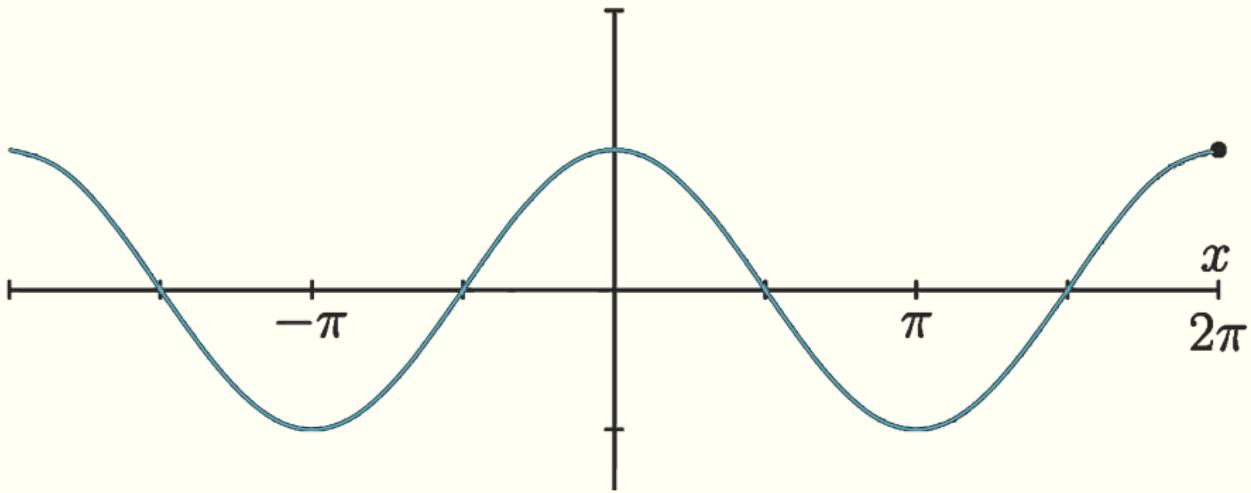
Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$$\cos(x) \xrightarrow{x=0} 1$$

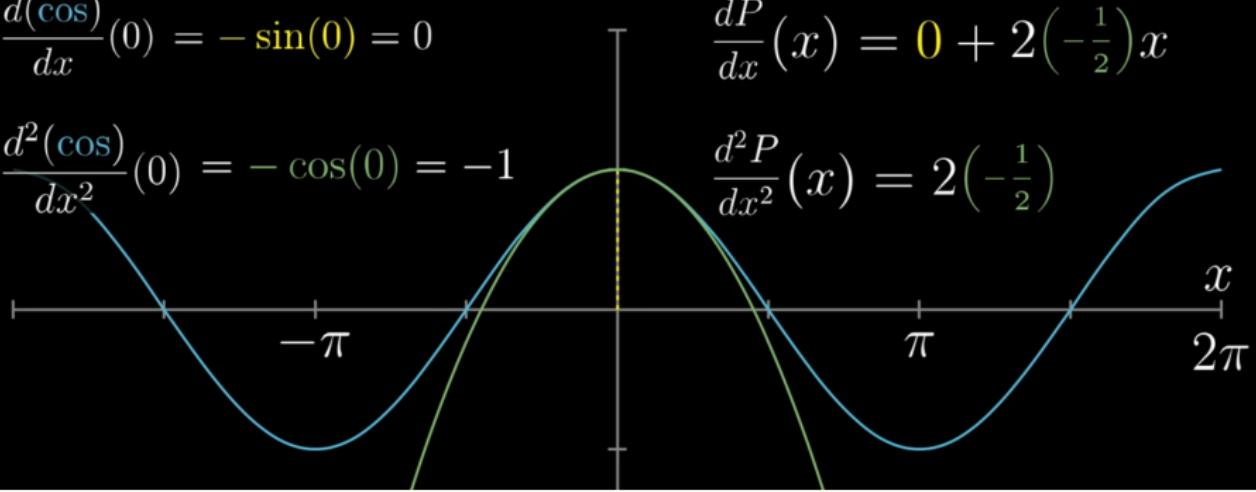
$$P(x) = 1 + 0x + \left(-\frac{1}{2}\right)x^2$$

$$\frac{d(\cos)}{dx}(0) = -\sin(0) = 0$$

$$\frac{d^2(\cos)}{dx^2}(0) = -\cos(0) = -1$$

$$\frac{dP}{dx}(x) = 0 + 2\left(-\frac{1}{2}\right)x$$

$$\frac{d^2P}{dx^2}(x) = 2\left(-\frac{1}{2}\right)$$



Taylor series of $\cos(x)$: 4th derivative

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$\begin{aligned}\frac{d^4 P}{dx^4}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{24} \\ &= 24 \cdot \frac{1}{24}\end{aligned}$$

To find the coefficient of n^{th} term:

$$\begin{aligned}\frac{d^8}{dx^8}(c_8 x^8) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot c_8 \\ &= 8!\end{aligned}$$

$$\text{Set } c_8 = \frac{\text{desired derivative value}}{8!}$$

Taylor series of $\cos(x)$

$$\cos(0) = 1$$



$$-\sin(0) = 0$$



$$-\cos(0) = -1$$



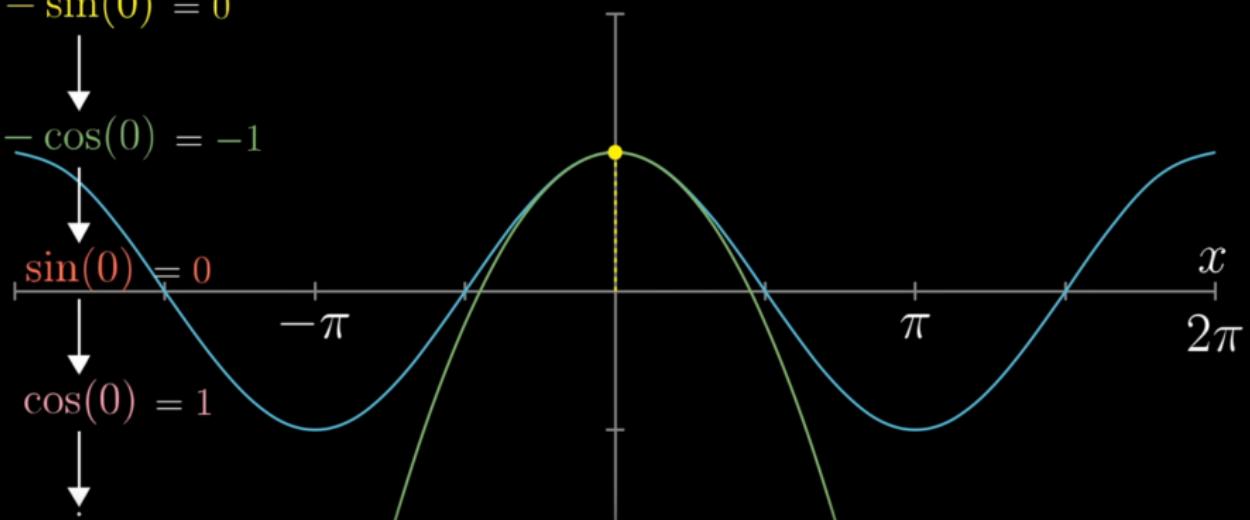
$$\sin(0) = 0$$



$$\cos(0) = 1$$



$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \dots$$



Taylor series: Generalization

$$P(x) = 1 + \left(-\frac{1}{2}\right)x^2 + c_4x^4$$

Doesn't affect previous terms

$$\frac{d^2P}{dx^2}(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4c_4(0)^2$$

$$P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

Controls $P(0)$

Controls $\frac{dP}{dx}(0)$

Controls $\frac{d^2P}{dx^2}(0)$

Controls $\frac{d^3P}{dx^3}(0)$

Controls $\frac{d^4P}{dx^4}(0)$

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graph TD; P0["Controls P(0)"] --> P0Term["P(0)"]; dPdx["Controls dP/dx(0)"] --> dPdxTerm["dP/dx(0)"]; d2Pdx2["Controls d^2P/dx^2(0)"] --> d2Pdx2Term["d^2P/dx^2(0)"]; d3Pdx3["Controls d^3P/dx^3(0)"] --> d3Pdx3Term["d^3P/dx^3(0)"]; d4Pdx4["Controls d^4P/dx^4(0)"] --> d4Pdx4Term["d^4P/dx^4(0)"];
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Taylor series: Generalization

$$f(0)$$

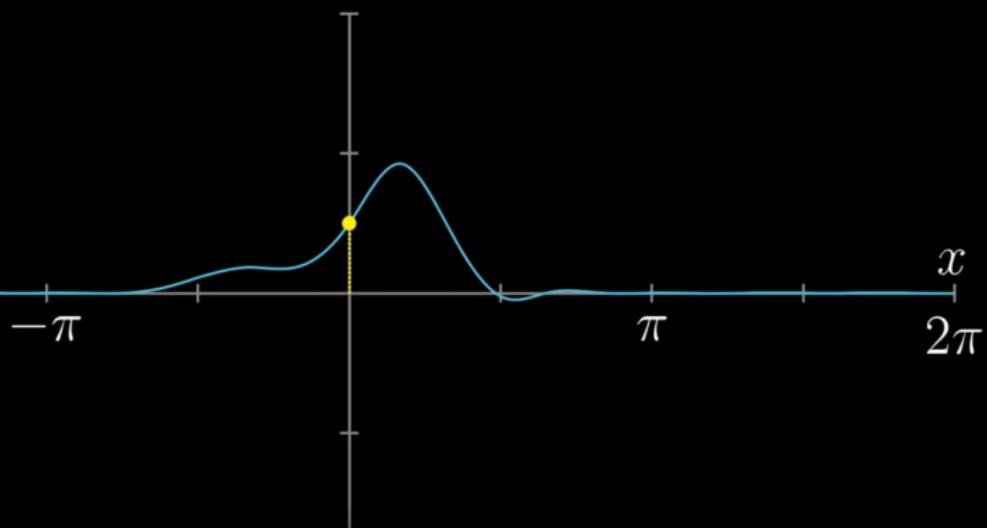
$$\frac{df}{dx}(0)$$

$$\frac{d^2f}{dx^2}(0)$$

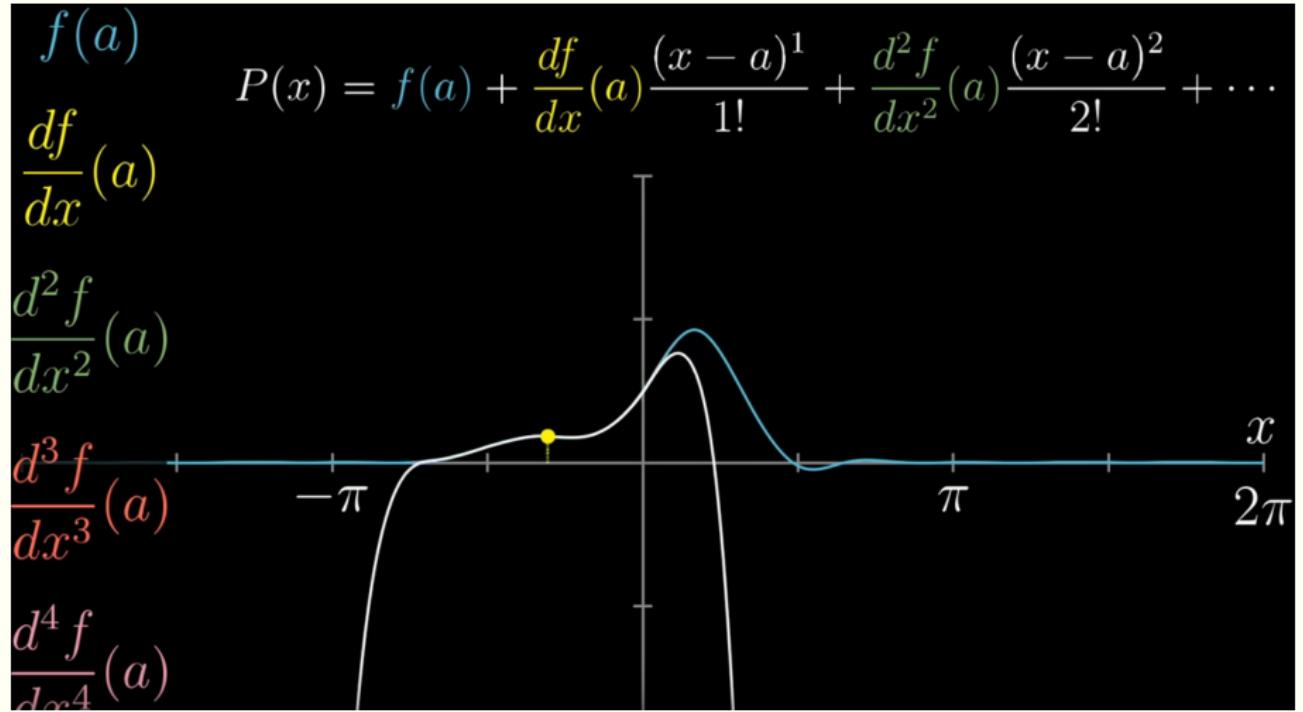
$$\frac{d^3f}{dx^3}(0) +$$

$$\frac{d^4f}{dx^4}(0)$$

$$P(x) = f(0) + \frac{df}{dx}(0) \frac{x^1}{1!} + \frac{d^2f}{dx^2}(0) \frac{x^2}{2!} + \frac{d^3f}{dx^3}(0) \frac{x^3}{3!} + \dots$$



Taylor series: Generalization



Taylor series: Two variables

The Taylor series of f (expanded about $(x, y) = (a, b)$) is:

$$\begin{aligned}(x, y) = f(a, b) &+ \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b) \\ &+ \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2}(a, b) \cdot (x - a)^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 f}{\partial xy}(a, b) \cdot (x - a)(y - b) + \right. \\ &\quad \left. \frac{\partial^2 f}{\partial y^2}(a, b) \cdot (y - b)^2 \right] + \dots\end{aligned}$$

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Newton Raphson

Assuming r is a root of f and that f is continuously differentiable in the vicinity of r with $f'(r) \neq 0$, then a sequence (x_n) that converges to r for $n \rightarrow \infty$ can be found using the Taylor expansion of f :

$$f(r) = f(x_n + \varepsilon_n) = f(x_n) + f'(x_n)\varepsilon_n + \frac{f''(x_n)}{2!}\varepsilon_n^2 \dots$$

$$\varepsilon_n \approx -\frac{f(x_n)}{f'(x_n)}$$

$$r = x_n + \varepsilon_n \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

in other words $x_n - \frac{f(x_n)}{f'(x_n)}$ is the next iteration of r , and hence we write:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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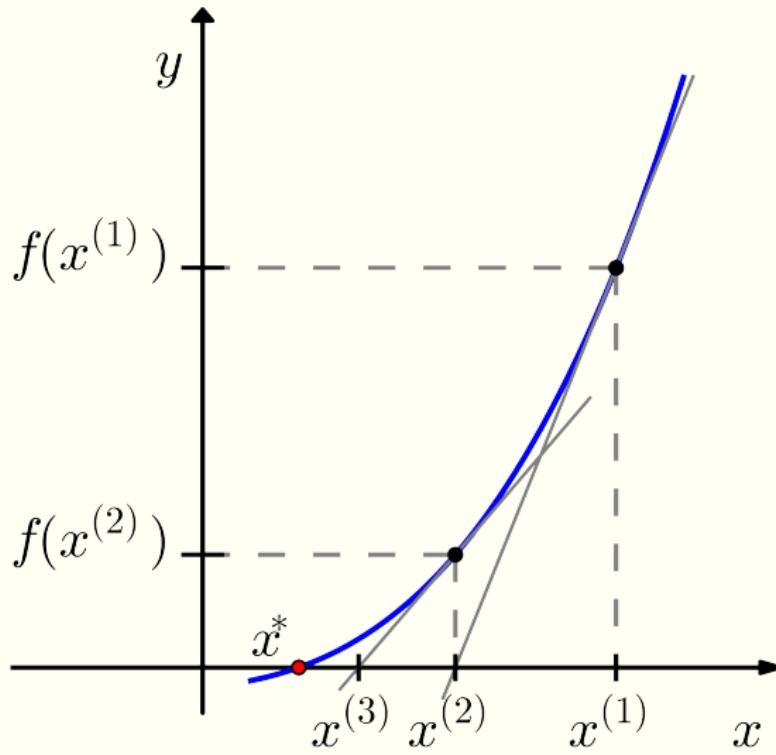
We wish to find roots of $f(x)$ using a converging sequence (x_n) . But we want to do it faster.

Newton's original method (1685) was purely algebraic, which he applied only to polynomials and used a sequence of polynomials instead of successive approximations x_n .

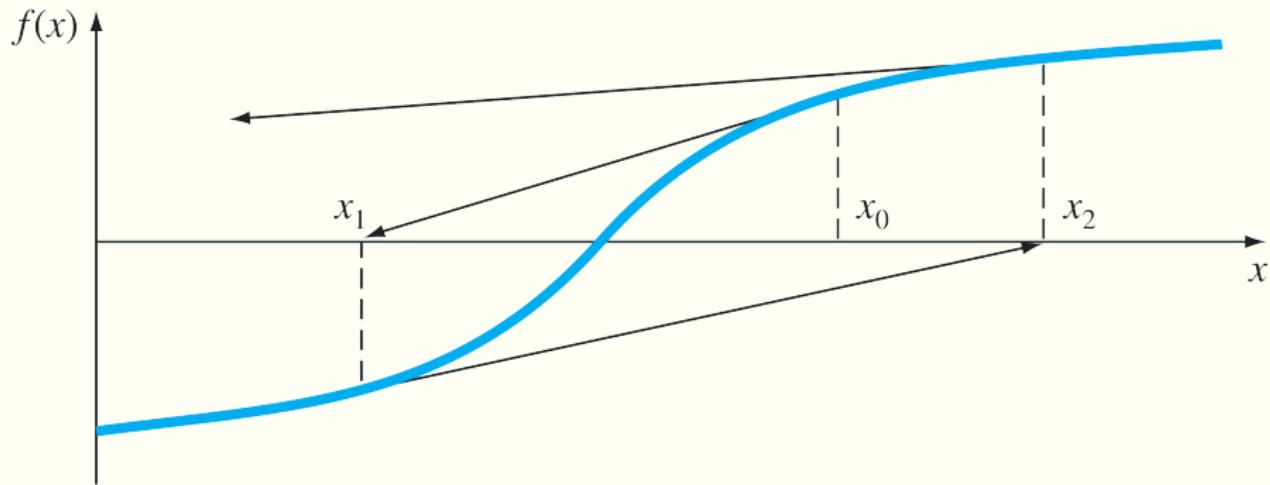
Raphson's simplified version (1690) was also only algebraic and he applied it only to polynomials but used x_n approximations.

Simpson gave the form used today 50 years later (1740), along with other important results in the same paper.

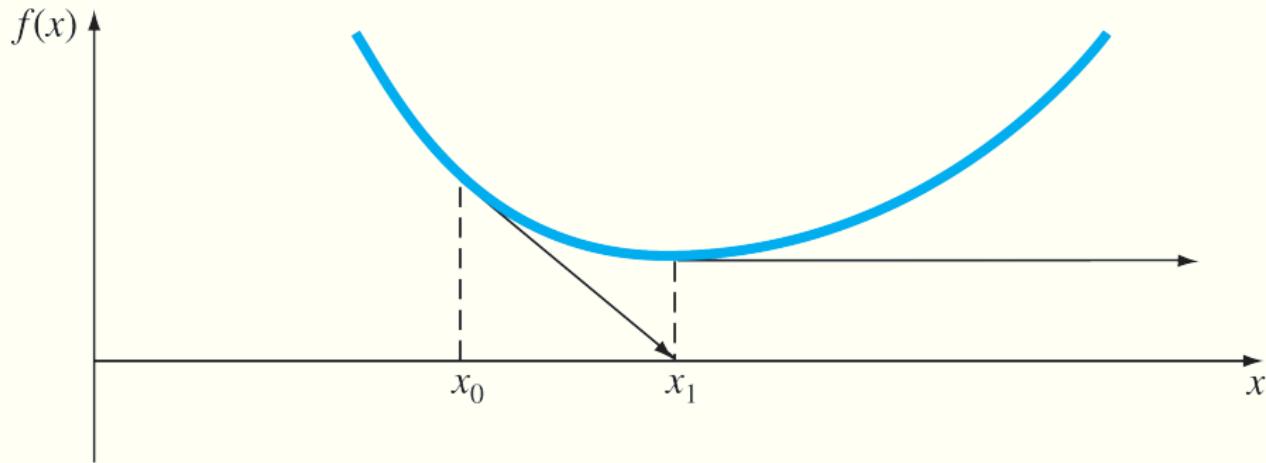
Newton-Raphson graphical expression



Newton-Raphson failure



Newton-Raphson failure



Newton-Raphson failure

