CE 311K: Linear System of Equations

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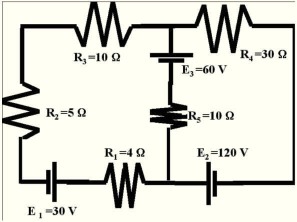
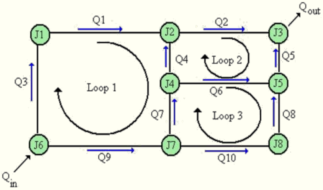
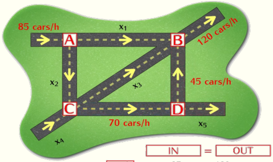
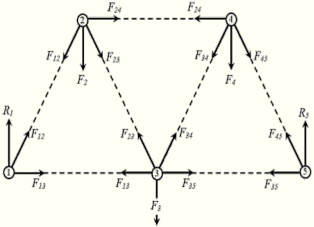
1. Linear System of Equations



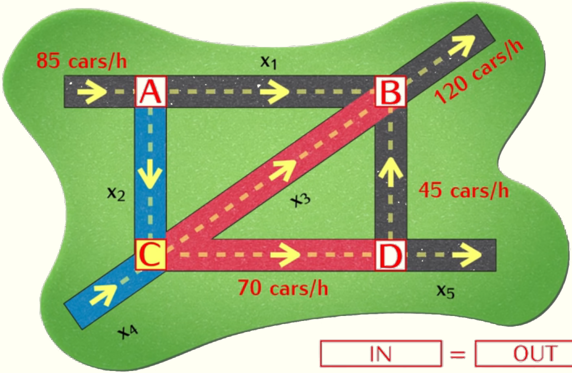




	IN = OUT
total:	$85 + x_4 = 120 + x_5$
@ A:	$85 = x_1 + x_2$
@ B:	$x_1 + x_3 + 45 = 120$
@ C:	$x_2 + x_4 = 70 + x_3$



Linear System of Equations: Traffic flow

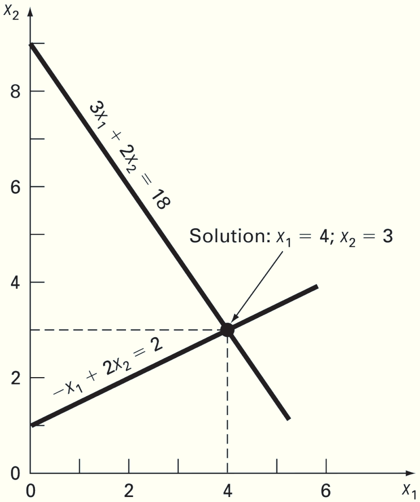


	IN = OUT
total:	$85 + x_4 = 120 + x_5$
@ A:	$85 = x_1 + x_2$
@ B:	$x_1 + x_3 + 45 = 120$
@ C:	$x_2 + x_4 = 70 + x_3$

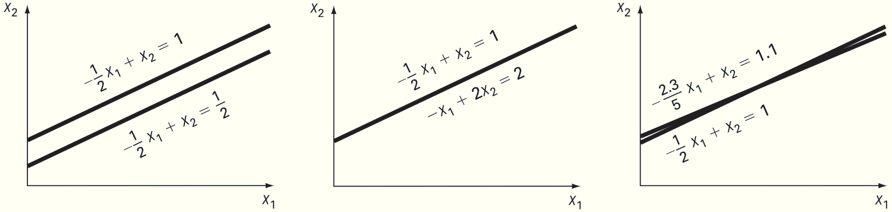
Solving Linear System of Equations

$$
\begin{aligned}
3x_1 + 2x_2 &= 18 \\
-x_1 + 2x_2 &= 2
\end{aligned}
$$

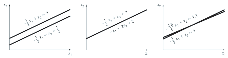
Solving Linear System of Equations



Singularity and Ill-conditioned



Singularity and Ill-conditioned



A singular matrix is a square matrix which is not invertible. Alternatively, a matrix is singular if and only if it has a determinant of 0.

Solving Linear System of Equations

1. Direct Methods
    1. Gauss Elimination
    2. Gauss-Jordan Elimination
    3. LU decomposition
2. Iterative Methods
    1. Jacobi iterative
    2. Gauss-Seidel

Direct methods

Consider a system of 3 linear equations for simplicity:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Matrix form is:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

Concise form: $Ax = b$

Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Solve by "back substitution' Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with the last equation 3: $x_3 = b_3/a_{33}$
Equation 2: $a_{22}x_2 + a_{23}x_3 = b_2$ so

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$

Equation 1: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ so

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

General for 'n' systems: $x_n = b_n/a_{nn}$ $x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

```
a = np.array([[2,4,6], [4,11,21], [6, 21, 52]])
b = np.array([24, 72, 158])
x = np.linalg.solve(a, b)
```

Gauss Elimination: Limitations

1. Prone to round off errors, when we have many (> 100) equations.
2. If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

Gauss Seidel Iterative approach

For conciseness, we limit to 3×3 equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$
\begin{aligned}
4x_1 + x_2 + 2x_3 &= 4 \\
3x_1 + 5x_2 + x_3 &= 7 \\
x_1 + x_2 + 3x_3 &= 3
\end{aligned}
$$

Assuming an initial guess of $x_1, x_2, x_3 = 0$. End of first iteration.

$$
\begin{aligned}
x_1 &= \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0 - 0}{4} = 1 \\
x_2 &= \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 1 - 0}{5} = 0.8 \\
x_3 &= \frac{3 - x_1 - x_2}{3} = \frac{3 - 1 - 0.8}{3} = 0.4
\end{aligned}
$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x] End of first iteration:
$x_1, x_2, x_3 = 1, 0.8, 0.4$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.8 - 2 * 0.4}{4} = 0.6$$
$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.6 - 0.4}{5} = 0.96$$
$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.6 - 0.96}{3} = 0.48$$

End of first iteration: $x_1, x_2, x_3 = 0.6, 0.96, 0.48$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.96 - 2 * 0.48}{4} = 0.52$$
$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.52 - 0.48}{5} = 0.992$$
$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.52 - 0.992}{3} = 0.496$$

Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where k, and $k-1$ represents the current and previous iterations

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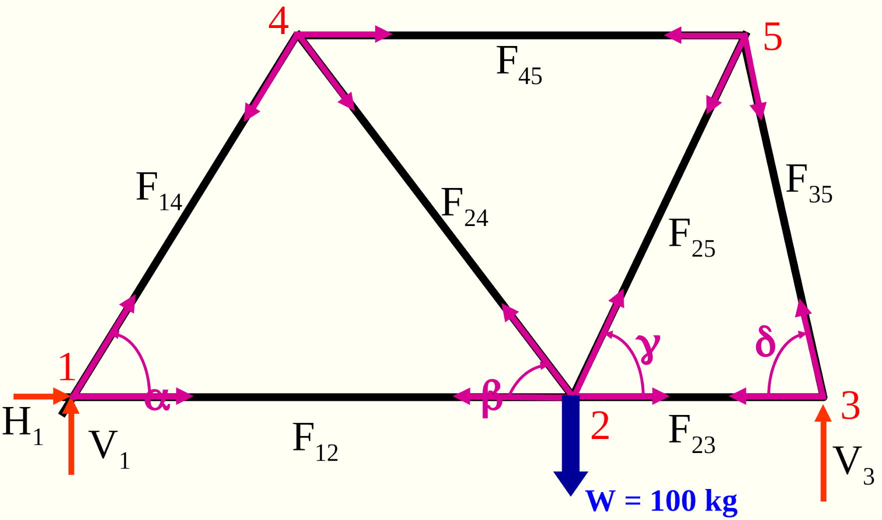
Gauss-Seidel only works for diagonally dominant matrix

Truss analysis



Truss analysis: Force balance

Node 1
$$\begin{cases} \sum F_{y,1} = V_1 + F_{14}\sin\alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14}\sin\alpha = 0 \end{cases}$$

Node 2
$$\begin{cases} \sum F_{y,2} = F_{24}\sin\beta + F_{25}\sin\gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24}\cos\beta + F_{25}\cos\gamma = 0 \end{cases}$$

Node 3
$$\begin{cases} \sum F_{y,3} = V_3 + F_{35}\sin\delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35}\cos\delta = 0 \end{cases}$$

Node 4
$$\begin{cases} \sum F_{y,4} = -F_{14}\sin\alpha - F_{24}\sin\beta = 0 \\ \sum F_{x,4} = -F_{14}\cos\alpha + F_{24}\cos\beta + F_{45} = 0 \end{cases}$$

Node 5
$$\begin{cases} \sum F_{y,5} = -F_{25}\sin\gamma - F_{35}\sin\delta = 0 \\ \sum F_{x,5} = -F_{25}\cos\gamma + F_{35}\cos\delta - F_{45} = 0 \end{cases}$$

Truss analysis: Matrix formulation

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta & 0 \\
0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta & -1
\end{bmatrix}
\begin{Bmatrix}
V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
$$