

CE 311K: Taylor series

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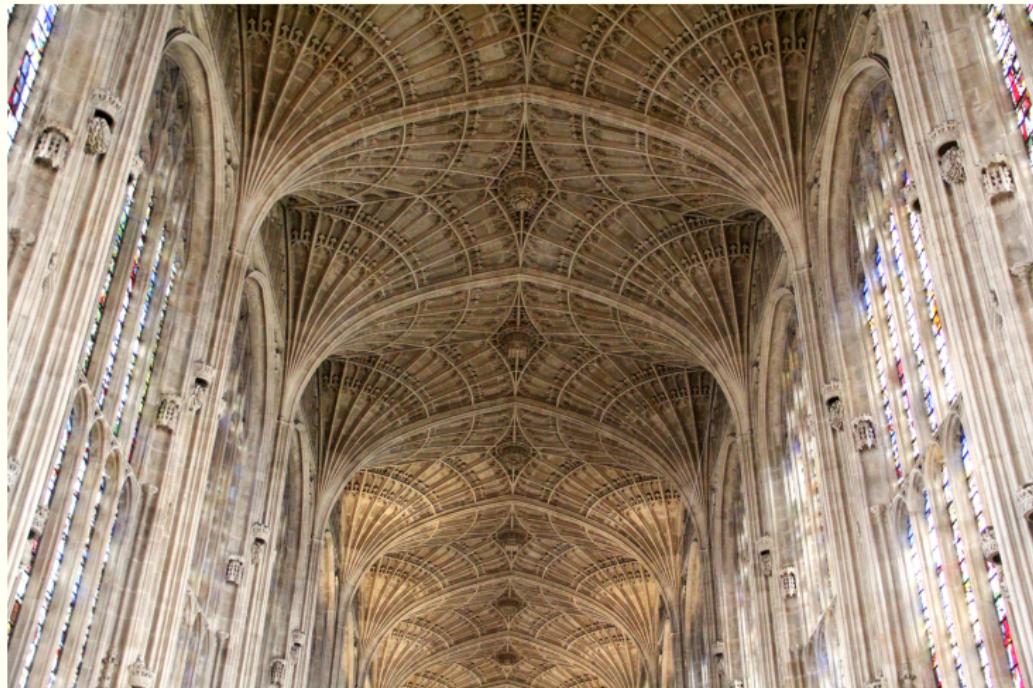
November 3, 2019

1 Catenary vs Parabola

The fan vaults of King's college



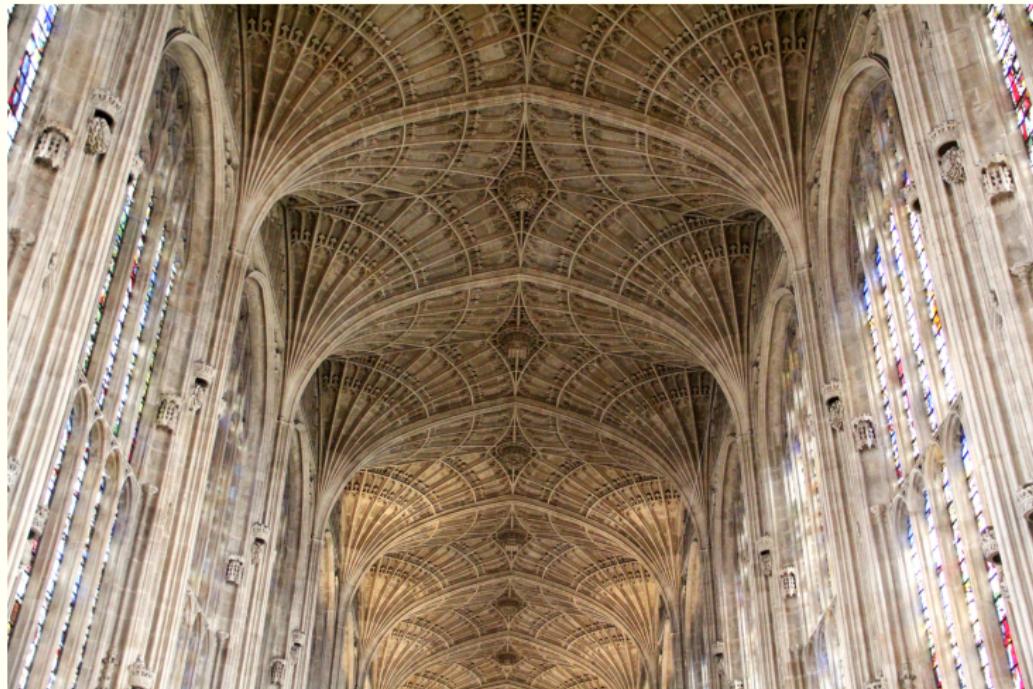
② What is the thickness of the ceiling?



From arches to fan vaults



The fan vaults of King's college



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- └ Catenary vs Parabola

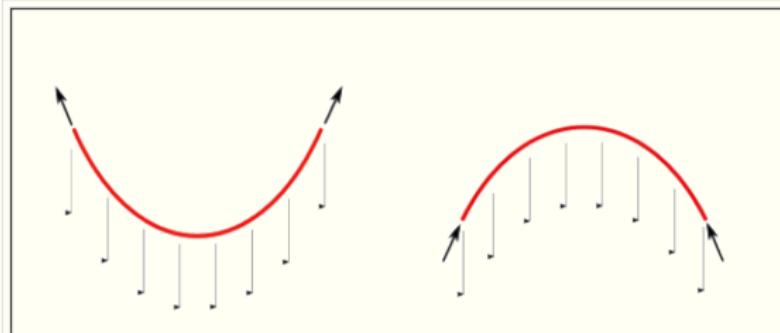
- └ The fan vaults of King's college

The fan vaults of King's college



King's college fan vaults were built in 1500s

Catenary

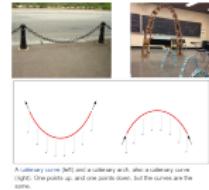


A [catenary curve](#) (left) and a catenary arch, also a catenary curve (right). One points up, and one points down, but the curves are the same.

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└ Catenary vs Parabola

└ Catenary

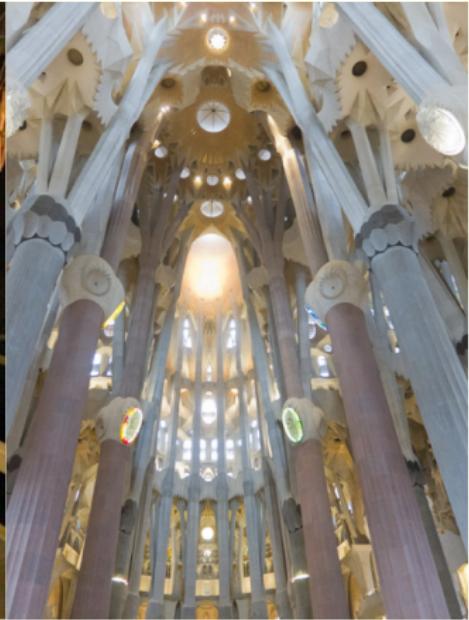


The English architect Robert Hooke was the first to study the catenary mathematically and in 1675 published an anagram (in Latin) of : "As hangs the flexible line, so but inverted will stand the rigid arch." The arch above Wembley Stadium has the shape of a catenary and Christopher Wren also intended to use it in St. Paul's dome

Sagrada Familia



catenary design



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└ Catenary vs Parabola

└ Sagrada Familia



Antoni Gaudi i Cornet (1852-1926) was a well-known architect from Spain. Gaudi used catenary arches in many of his projects. One of Gaudi's greatest works however became the Temple of the Holy Family (the Sagrada Familia) in Barcelona.

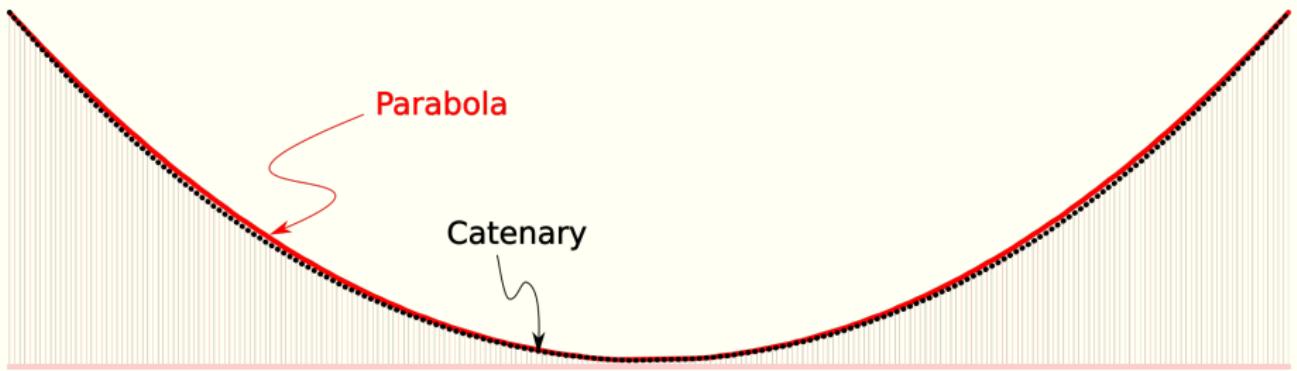
Catenary, catenary . . . everywhere



② What is this shape?



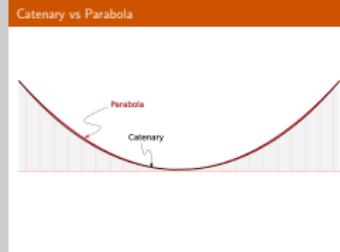
Catenary vs Parabola



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- Catenary vs Parabola

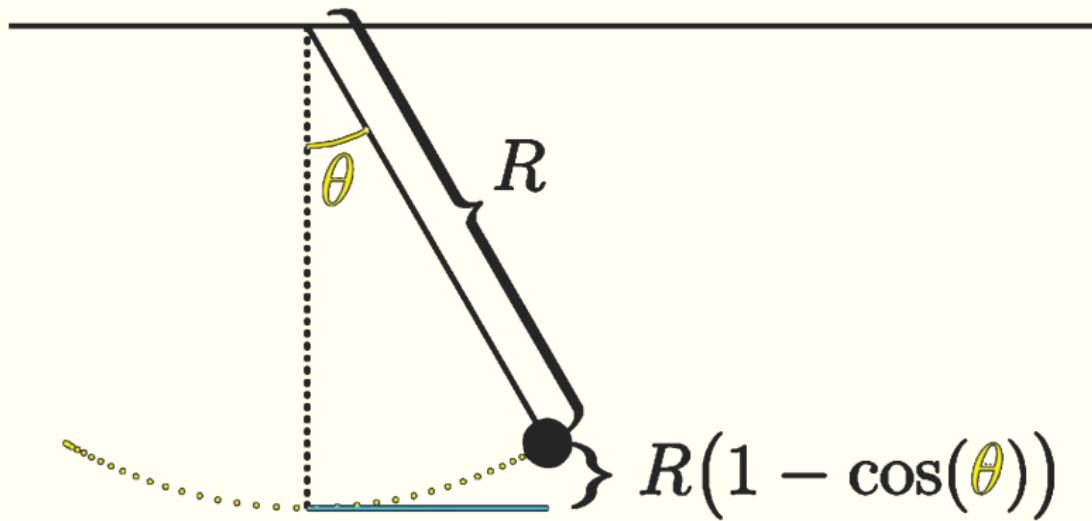
- Catenary vs Parabola



Parabola $y = x^2$ for a parabola, a hyperbolic function like $\cosh(x) \approx 1 + \frac{x^2}{2}$

Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point

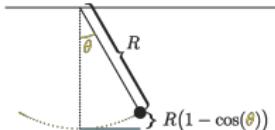


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└ Catenary vs Parabola

└ Potential energy of a simple pendulum

Potential energy of a simple pendulum
We need to know how high the weight of the pendulum is above its lowest point.



Taylor series approximation $\cos(\theta) \approx$ for a parabola, a hyperbolic function like $\cosh(x) \approx 1 + \frac{x^2}{2}$, which gives $h = R(1 - (1 + \frac{\theta^2}{2})) = R\frac{\theta^2}{2}$

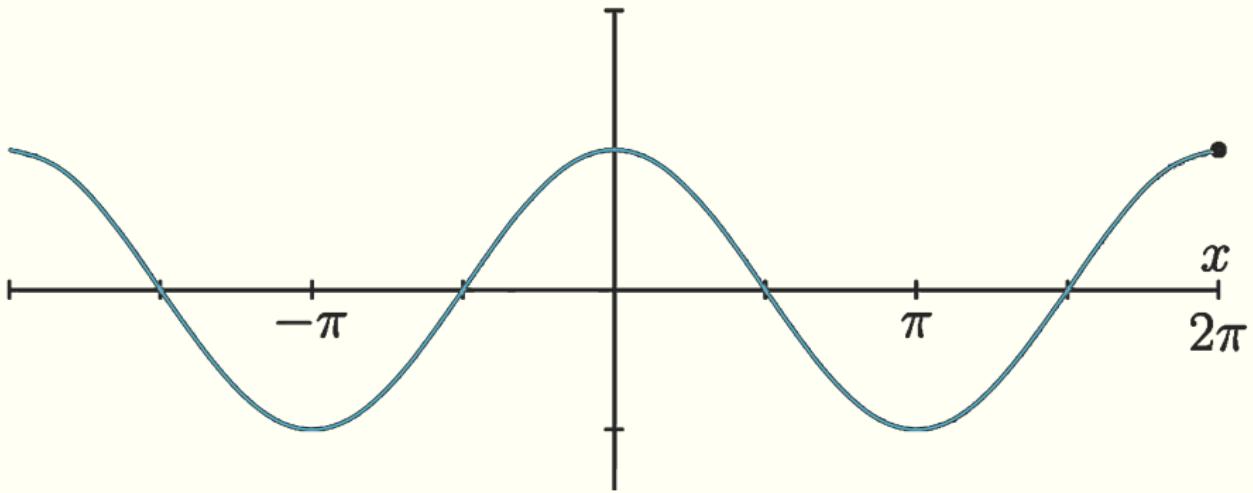
The cosine function made the problem awkward and unwieldy. If we approximate using $\cos(\theta) \approx \frac{\theta^2}{2}$ everything fell into place. An approximation like that might seem completely out of left field. Let's graph these functions. They do look close to each other. But how do we think to make this approximation? and how do we even get that particular quadratic?

Taylor series

- Taylor series is one the best tools maths has to offer for approximating functions.
- Taylor series is about taking non-polynomial functions and finding polynomials that approximate at some input.
- The motive here is the polynomials tend to be much easier to deal with than other functions, they are easier to compute, take derivatives, integrate, just easier overall.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

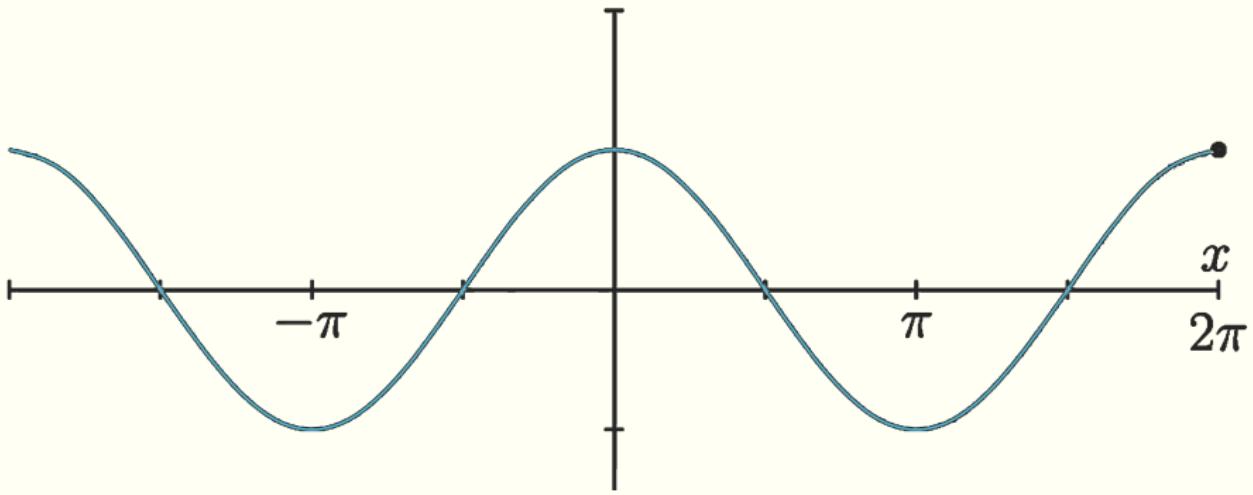
Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$$\cos(x) \xrightarrow{x=0} 1$$

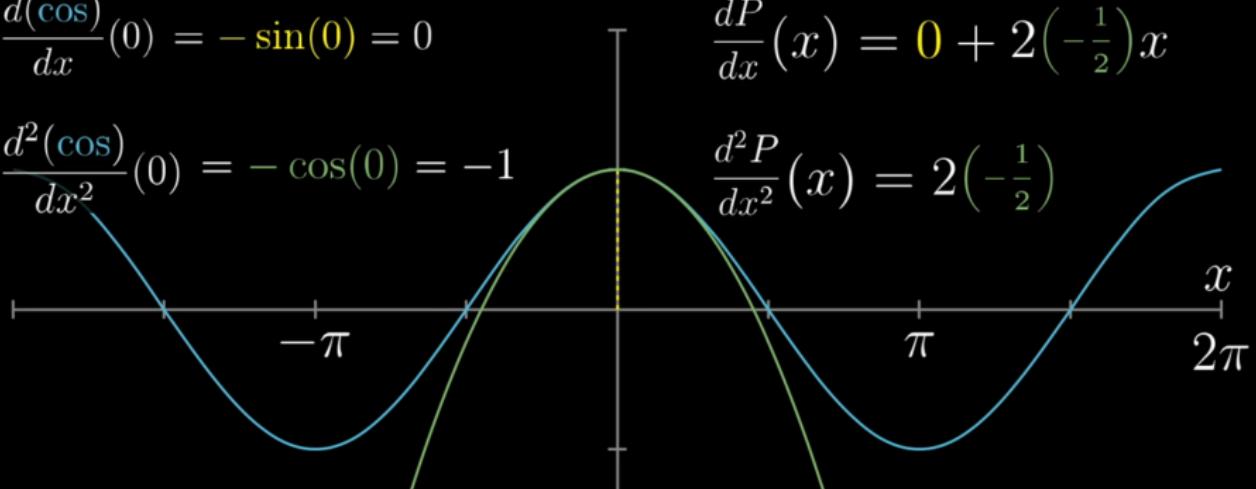
$$P(x) = 1 + 0x + \left(-\frac{1}{2}\right)x^2$$

$$\frac{d(\cos)}{dx}(0) = -\sin(0) = 0$$

$$\frac{d^2(\cos)}{dx^2}(0) = -\cos(0) = -1$$

$$\frac{dP}{dx}(x) = 0 + 2\left(-\frac{1}{2}\right)x$$

$$\frac{d^2P}{dx^2}(x) = 2\left(-\frac{1}{2}\right)$$



Taylor series of cos(x): 4th derivative

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$\begin{aligned}\frac{d^4 P}{dx^4}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{24} \\ &= 24 \cdot \frac{1}{24}\end{aligned}$$

To find the coefficient of n^{th} term:

$$\begin{aligned}\frac{d^8}{dx^8}(c_8 x^8) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot c_8 \\ &= 8!\end{aligned}$$

$$\text{Set } c_8 = \frac{\text{desired derivative value}}{8!}$$

Taylor series of $\cos(x)$

$$\cos(0) = 1$$



$$-\sin(0) = 0$$



$$-\cos(0) = -1$$



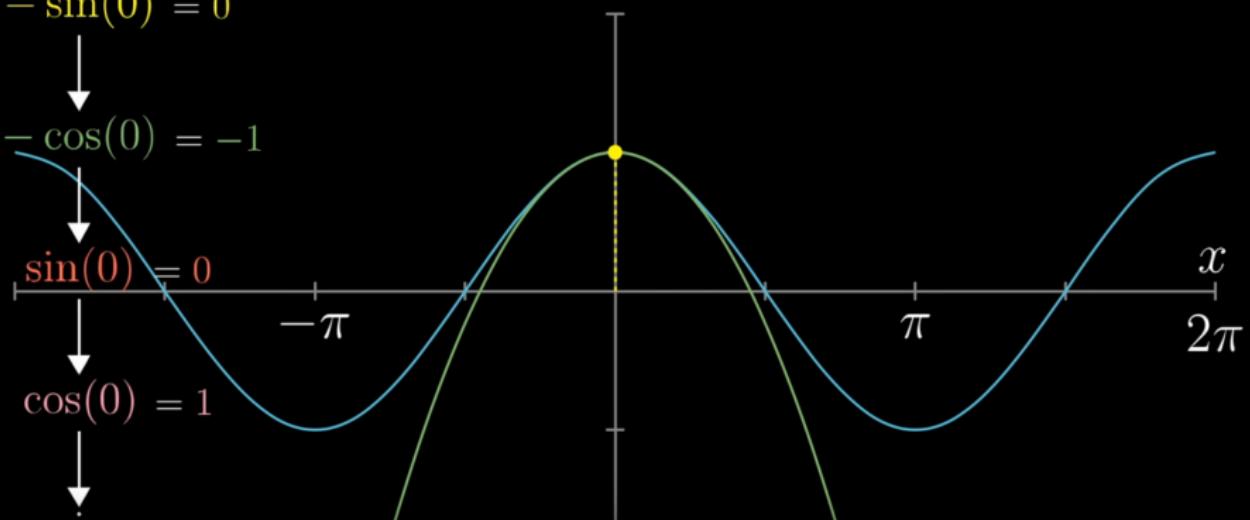
$$\sin(0) = 0$$



$$\cos(0) = 1$$



$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \dots$$



Taylor series: Generalization

$$P(x) = 1 + \left(-\frac{1}{2}\right)x^2 + c_4x^4$$

Doesn't affect previous terms

$$\frac{d^2P}{dx^2}(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4c_4(0)^2$$

$$P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

Controls $P(0)$

Controls $\frac{dP}{dx}(0)$

Controls $\frac{d^2P}{dx^2}(0)$

Controls $\frac{d^3P}{dx^3}(0)$

Controls $\frac{d^4P}{dx^4}(0)$

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graph TD; P0["Controls P(0)"] --> P0Term["P(0)"]; dPdx["Controls dP/dx(0)"] --> dPdxTerm["dP/dx(0)"]; d2Pdx2["Controls d^2P/dx^2(0)"] --> d2Pdx2Term["d^2P/dx^2(0)"]; d3Pdx3["Controls d^3P/dx^3(0)"] --> d3Pdx3Term["d^3P/dx^3(0)"]; d4Pdx4["Controls d^4P/dx^4(0)"] --> d4Pdx4Term["d^4P/dx^4(0)"];
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Taylor series: Generalization

$$f(0)$$

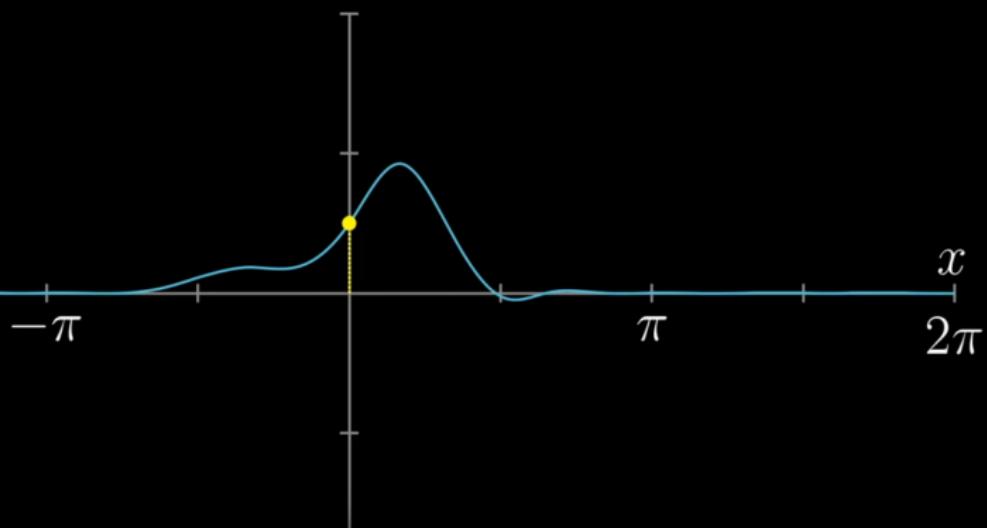
$$\frac{df}{dx}(0)$$

$$\frac{d^2f}{dx^2}(0)$$

$$\frac{d^3f}{dx^3}(0) +$$

$$\frac{d^4f}{dx^4}(0)$$

$$P(x) = f(0) + \frac{df}{dx}(0) \frac{x^1}{1!} + \frac{d^2f}{dx^2}(0) \frac{x^2}{2!} + \frac{d^3f}{dx^3}(0) \frac{x^3}{3!} + \dots$$



Taylor series: Generalization

