CE 311K: Taylor series and Newton Raphson

Krishna Kumar

University of Texas at Austin

krishnak@utexas.edu

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The fan vaults of King's college

? What is the thickness of the ceiling?

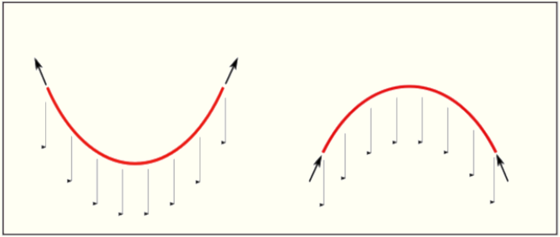
From arches to fan vaults

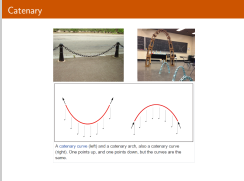
The fan vaults of King's college

King's college fan vaults were built in 1500s

Catenary



A catenary curve (left) and a catenary arch, also a catenary curve (right). One points up, and one points down, but the curves are the same.



The English architect Robert Hooke was the first to study the catenary mathematically and in 1675 published an anagram (in Latin) of : "As hangs the flexible line, so but inverted will stand the rigid arch." The arch above Wembley Stadium has the shape of a catenary and Christopher Wren also intended to use it in St. Paul's dome

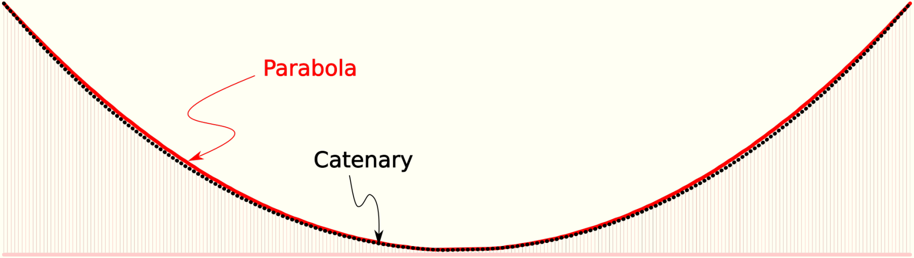
Sagrada Familia

Antoni Gaudi i Cornet (1852-1926) was a well-known architect from Spain. Gaudi used catenary arches in many of his projects. One of Gaudi's greatest works however became the Temple of the Holy Family (the Sagrada Familia) in Barcelona.

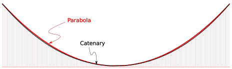
Catenary, caternary . . . everywhere

? What is this shape?

Catenary vs Parabola



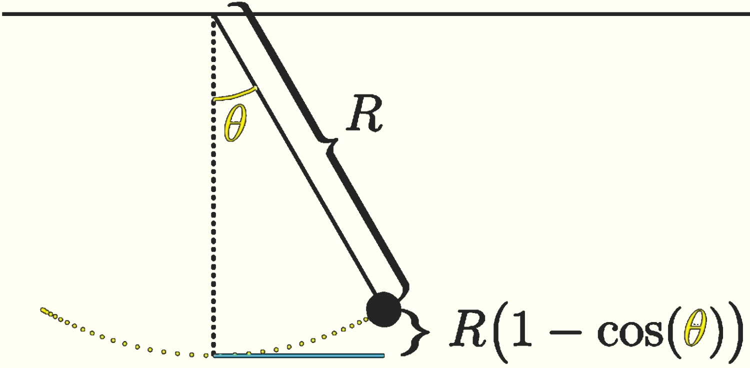
Catenary vs Parabola

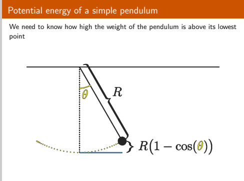


Parabola $y = x^2$ for a parabola, a hyperbolic function like $\cosh(x) \approx= 1 + \frac{x^2}{2}$

Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point





Taylor series approximation $\cos(\theta) \approx$ for a parabola, a hyperbolic function like $\cosh(x) \approx 1 + \frac{x^2}{2}$, which gives $h = R(1 - (1 + \frac{\theta^2}{2})) = R\frac{\theta^2}{2}$

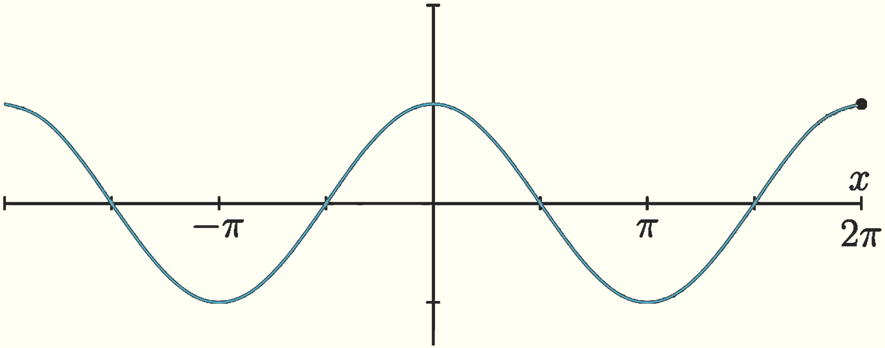
The cosine function made the problem awkward and unweildy. If we approximate using $\cos(\theta) \approx \frac{\theta^2}{2}$ everything fell into place. An approximation like that might seem completely out of left field. Let's graph these functions. They do look close to each other. But how do we think to make this approximation? and how do we even get that particular quadratic?

Taylor series

- Taylor series is one the best tools maths has to offer for approximating functions.
- Taylor series is about taking non-polynomial functions and finding polyomials that approximate at some input.
- The motive here is the polynomials tend to be much easier to deal with than other functions, they are easier to compute, take derivatives, integrate, just easier overall.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Taylor series of cos(x)

$\cos(x)$



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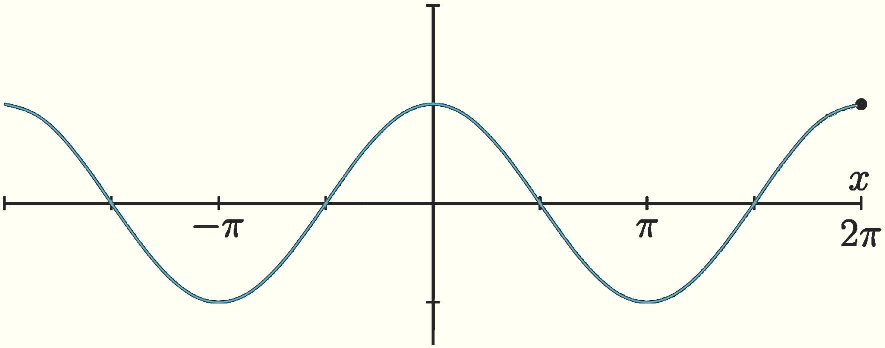

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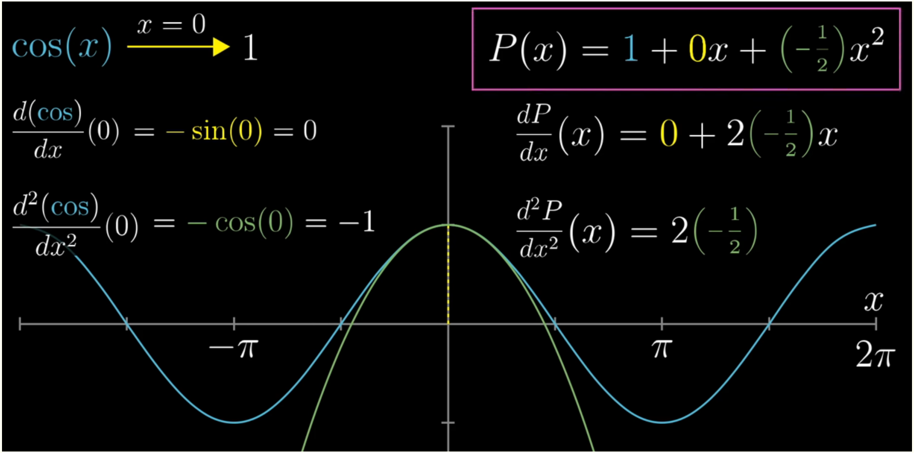


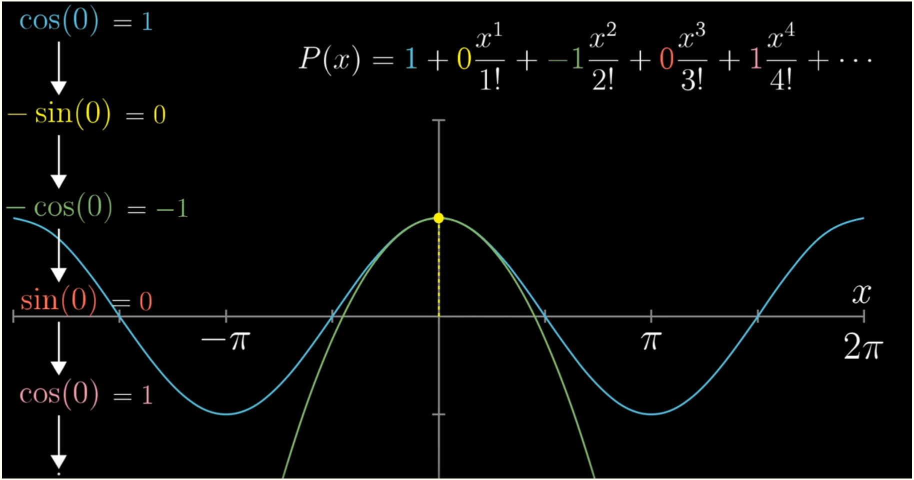
Taylor series of cos(x): 4th derivative

$$
\begin{aligned}
P(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \\
\frac{d^4P}{dx^4}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{24} \\
&= 24 \cdot \frac{1}{24}
\end{aligned}
$$

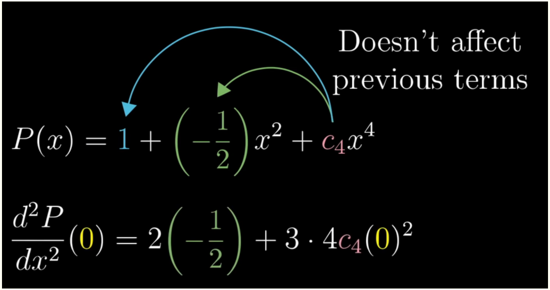
To find the coefficient of n^{th} term:

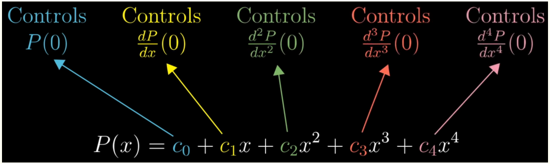
$$
\begin{aligned}
\frac{d^8}{dx^8}(c_8 x^8) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot c_8 \\
&= 8! \\
Set\ c_8 &= \frac{\text{desired derivative value}}{8!}
\end{aligned}
$$

Taylor series of cos(x)



Taylor series: Generalization





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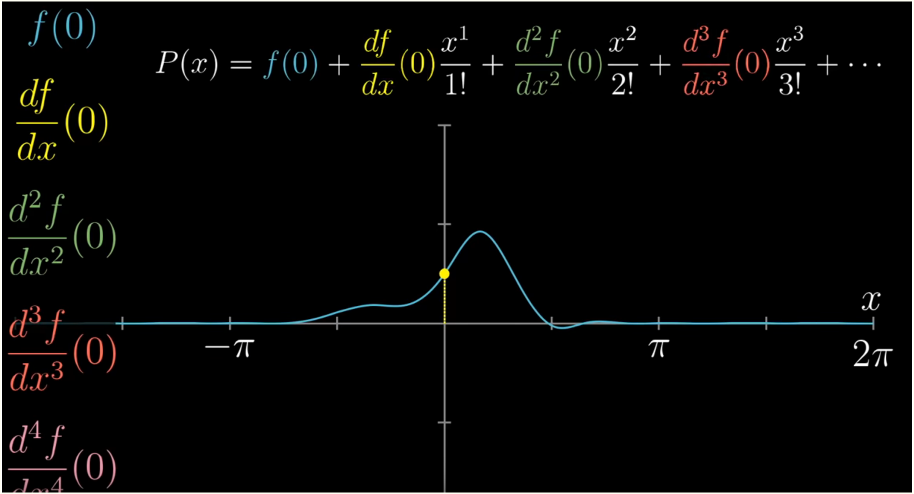

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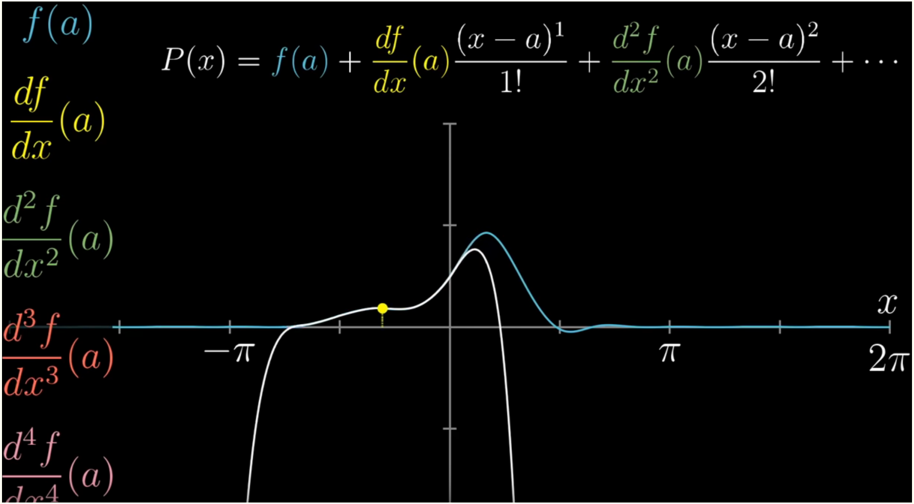

Newton Raphson

Assuming r is a root of f and that f is continuously differentiable in the vicinity of r with $f'(r) \neq 0$, then a sequence (x_n) that converges to r for $n \to \infty$ can be found using the Taylor expansion of f:

$$f(r) = f(x_n + \varepsilon_n) = f(x_n) + f'(x_n)\varepsilon_n + \frac{f''(x_n)}{2!}\varepsilon_n^2 \ldots$$

$$\varepsilon_n \approx -\frac{f(x_n)}{f'(x_n)}$$

$$r = x_n + \varepsilon_n \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

in other words $x_n - \frac{f(x_n)}{f'(x_n)}$ is the next iteration of r, and hence we write:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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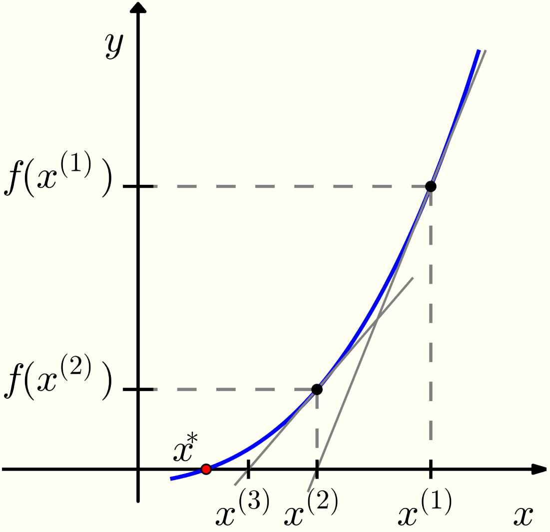
We wish to find roots of $f(x)$ using a converging sequence (x_n). But we want to do it faster.

Newton's original method (1685) was purely algebraic, which he applied only to polynomials and used a sequence of polynomials instead of successive approximations x_n.

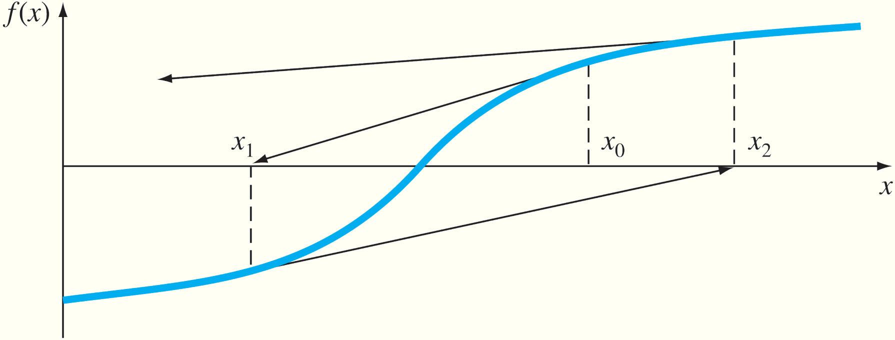
Raphson's simplified version (1690) was also only algebraic and he applied it only to polynomials but used x_n approximations.

Simpson gave the form used today 50 years later (1740), along with other important results in the same paper.

Newton-Raphson graphical expression



Newton-Raphson failure



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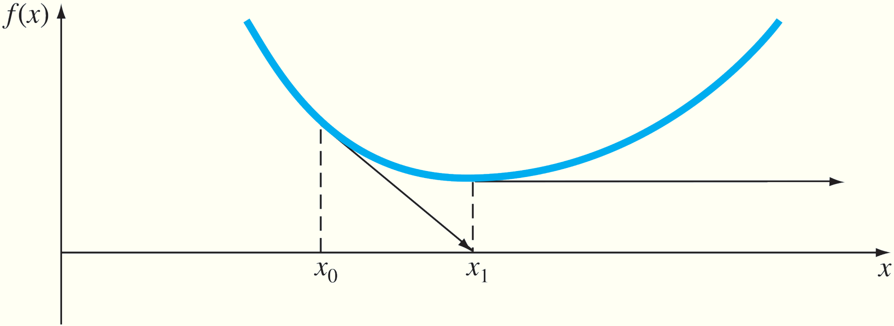

Newton-Raphson failure