

CE 311K: Linear System of Equations

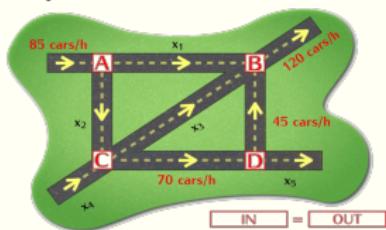
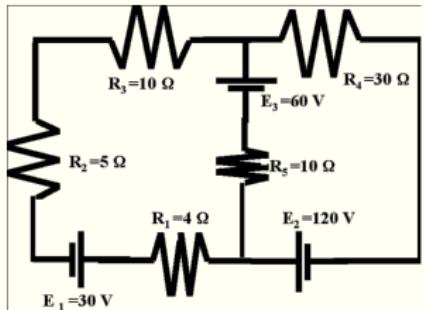
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1 Linear System of Equations

Linear System of Equations

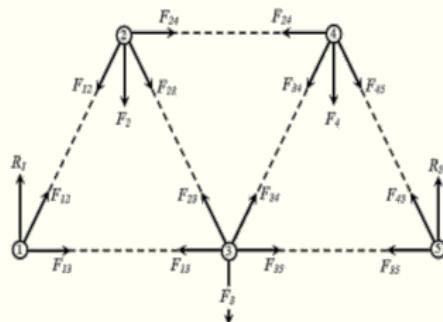
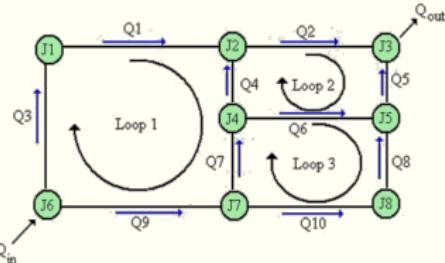


total: $85 + x_4 = 120 + x_3$

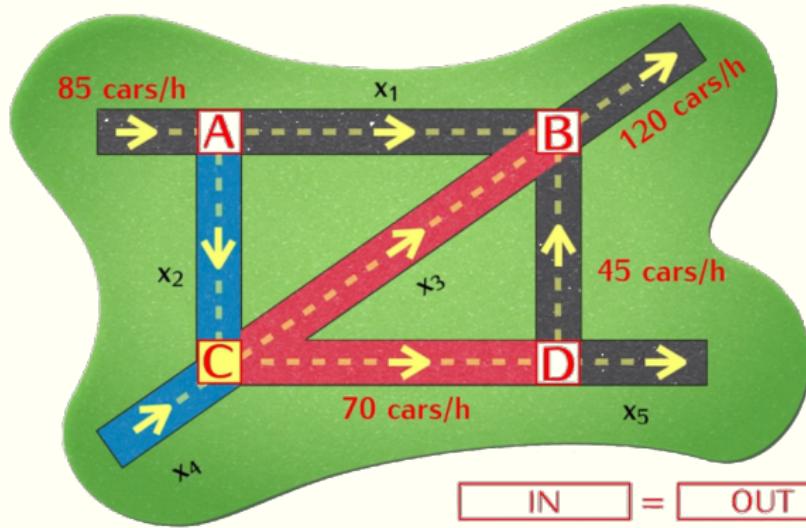
@ A: $85 = x_1 + x_2$

@ B: $x_1 + x_3 + 45 = 120$

@ C: $x_2 + x_4 = 70 + x_3$



Linear System of Equations: Traffic flow



$$\boxed{\text{IN}} = \boxed{\text{OUT}}$$

total: $85 + x_4 = 120 + x_5$

@ A: $85 = x_1 + x_2$

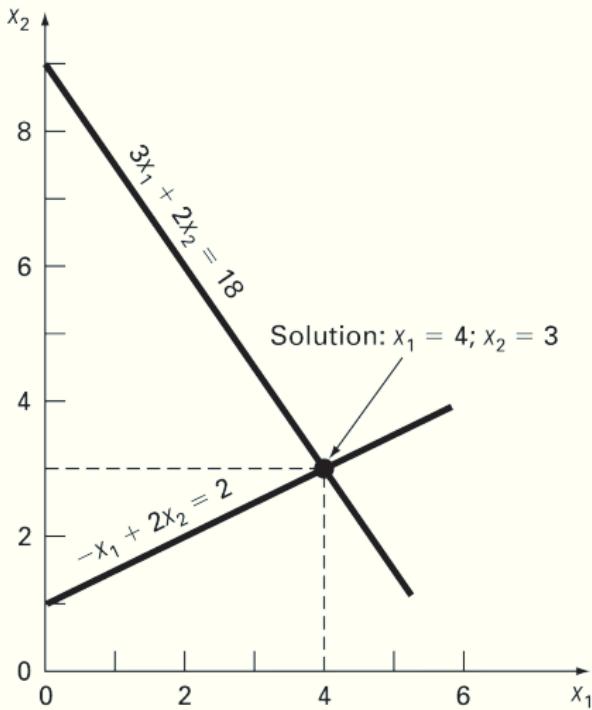
@ B: $x_1 + x_3 + 45 = 120$

@ C: $x_2 + x_4 = 70 + x_3$

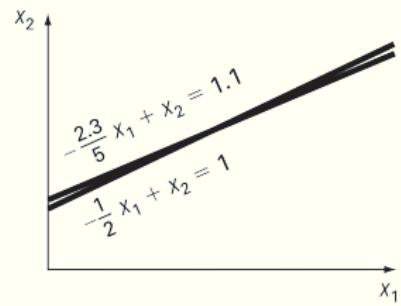
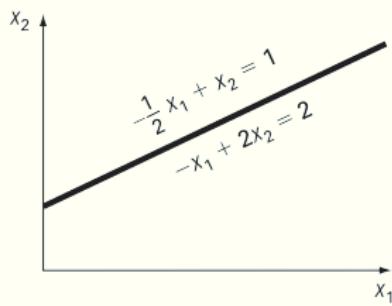
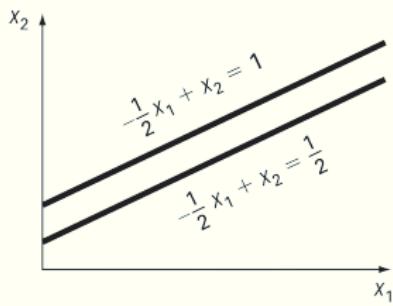
Solving Linear System of Equations

$$\begin{aligned}3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2\end{aligned}$$

Solving Linear System of Equations



Singularity and Ill-conditioned





A singular matrix is a square matrix which is not invertible. Alternatively, a matrix is singular if and only if it has a determinant of 0.

Solving Linear System of Equations

① Direct Methods

- ① Gauss Elimination
- ② Gauss-Jordan Elimination
- ③ LU decomposition

② Iterative Methods

- ① Jacobi iterative
- ② Gauss-Seidel

Direct methods

Consider a system of 3 linear equations for simplicity:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form: $Ax = b$

Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Solve by “back substitution’ Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with the last equation 3: $x_3 = b_3/a_{33}$

Equation 2: $a_{22}x_2 + a_{23}x_3 = b_2$ so

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$

Equation 1: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ so

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

General for ‘n’ systems: $x_n = b_n/a_{nn}$ $x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij}x_j}{a_{ii}}$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$$x_3 = 2, x_2 = 2, x_1 = 2$$

CE 311K: Linear Systems

└ Linear System of Equations

└ Gauss Elimination

```
a = np.array([[2,4,6], [4,11,21], [6, 21, 52]])  
b = np.array([24, 72, 158])  
x = np.linalg.solve(a, b)
```

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 0 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 56 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

Gauss Elimination: Limitations

- ① Prone to round off errors, when we have many (> 100) equations.
- ② If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

Gauss Seidel Iterative approach

For conciseness, we limit to 3×3 equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$4x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 5x_2 + x_3 = 7$$

$$x_1 + x_2 + 3x_3 = 3$$

Assuming an initial guess of $x_1, x_2, x_3 = 0$. End of first iteration.

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0 - 0}{4} = 1$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 1 - 0}{5} = 0.8$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 1 - 0.8}{3} = 0.4$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x] End of first iteration:

$$x_1, x_2, x_3 = 1, 0.8, 0.4$$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.8 - 2 * 0.4}{4} = 0.6$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.6 - 0.4}{5} = 0.96$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.6 - 0.96}{3} = 0.48$$

End of first iteration: $x_1, x_2, x_3 = 0.6, 0.96, 0.48$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.96 - 2 * 0.48}{4} = 0.52$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.52 - 0.48}{5} = 0.992$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.52 - 0.992}{3} = 0.496$$

Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where k , and $k - 1$ represents the current and previous iterations

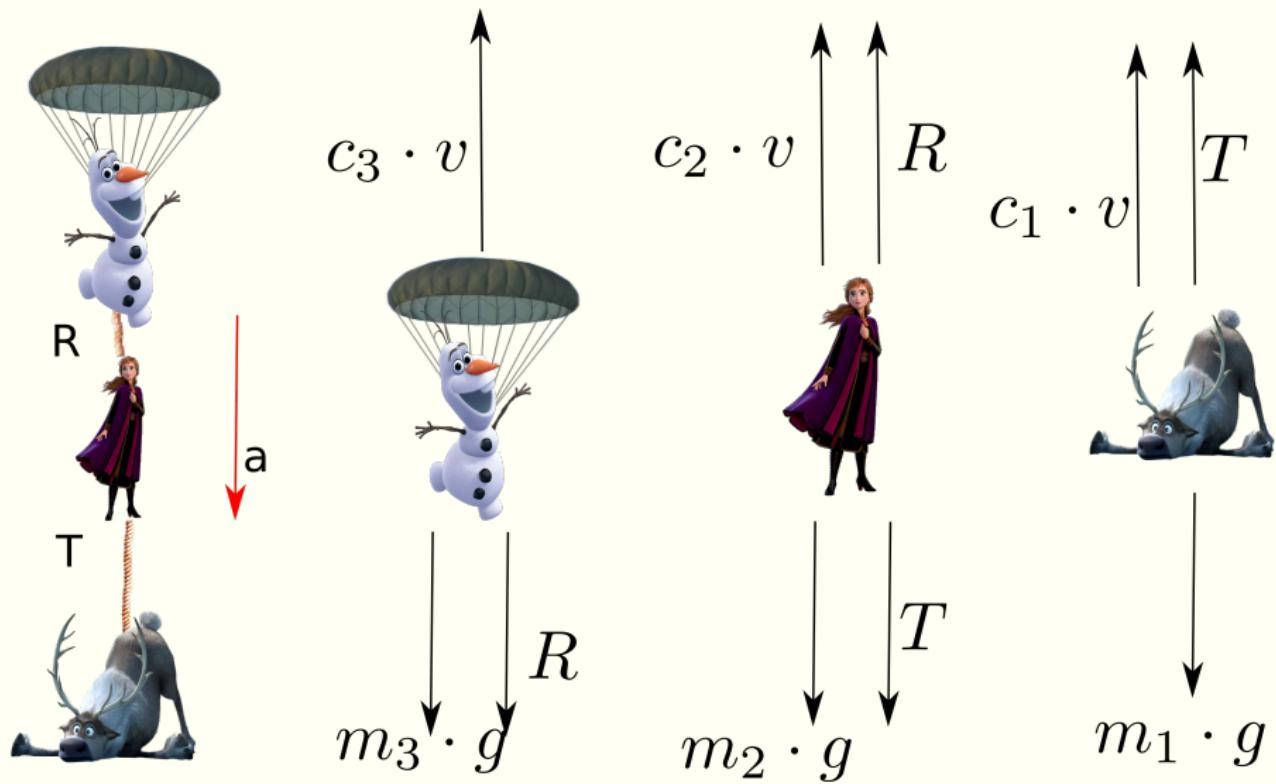
Convergence can be checked using the relative error.

$$|\epsilon_{ik}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \times 100\% \right| < \epsilon_k$$

where k , and $k - 1$ represents the current and previous iterations

Gauss-Seidel only works for diagonally dominant matrix

Gauss Seidel: Disney force diagram



Gauss Seidel: Disney force diagram

Character	Mass (kg)	Drag coefficient (kg/s)
Sven	100	40
Anna	50	5
Olaf	15	10

Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity $v = 5m/s$. compute the acceleration and forces. Unknowns are a, T, R .

$$m_1g - T - c_1 * v = m_1a$$

$$m_2g + T - c_2 * v - R = m_2a$$

$$m_3g - c_3 * v + R = m_3a$$

$$m_1a + T = m_1g - c_1 * v$$

$$m_2a - T + R = m_2a - c_2 * v$$

$$m_3a - R = m_3a - c_3 * v$$

$$100a + T = 1000 - 40 * 5 = 800$$

$$50a - T + R = 500 - 5 * 5 = 475$$

$$15a - R = 150 - 10 * 5 = 100$$

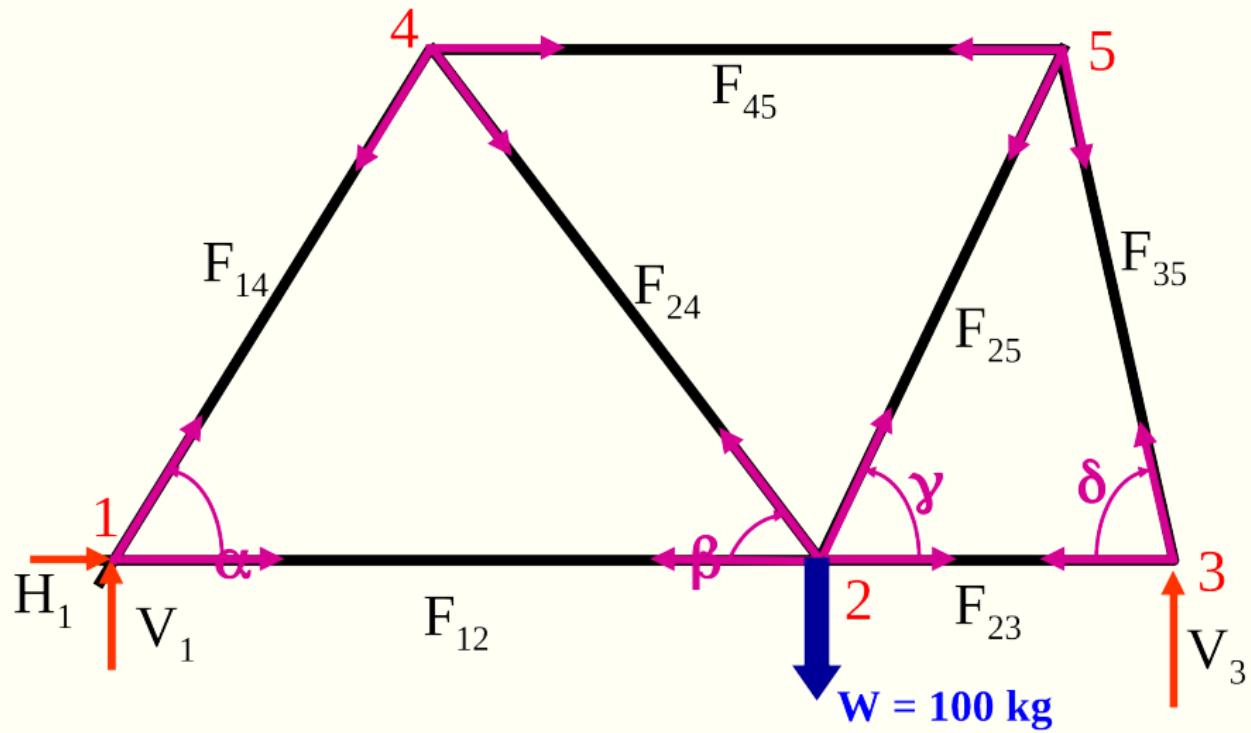
Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity $v = 5m/s$. compute the acceleration and forces. Unknowns are a, T, R .

$$\begin{bmatrix} 100 & 1 & 0 \\ 50 & -1 & 1 \\ 15 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ T \\ R \end{bmatrix} = \begin{bmatrix} 800 \\ 475 \\ 100 \end{bmatrix}$$

The acceleration $a = 7.8571428m/s^2$, $T = 14.29N$, and $R = 17.86N$.

Truss analysis



Truss analysis: Force balance

Node 1

$$\begin{cases} \sum F_{y,1} = V_1 + F_{14} \sin \alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14} \cos \alpha = 0 \end{cases}$$

Node 2

$$\begin{cases} \sum F_{y,2} = F_{24} \sin \beta + F_{25} \sin \gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24} \cos \beta + F_{25} \cos \gamma = 0 \end{cases}$$

Node 3

$$\begin{cases} \sum F_{y,3} = V_3 + F_{35} \sin \delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35} \cos \delta = 0 \end{cases}$$

Node 4

$$\begin{cases} \sum F_{y,4} = -F_{14} \sin \alpha - F_{24} \sin \beta = 0 \\ \sum F_{x,4} = -F_{14} \cos \alpha + F_{24} \cos \beta + F_{45} = 0 \end{cases}$$

Node 5

$$\begin{cases} \sum F_{y,5} = -F_{25} \sin \gamma - F_{35} \sin \delta = 0 \\ \sum F_{x,5} = -F_{25} \cos \gamma + F_{35} \cos \delta - F_{45} = 0 \end{cases}$$

Truss analysis: Matrix formulation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta \\ 0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta \end{bmatrix} \begin{pmatrix} V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$