CE 311K: Linear System of Equations

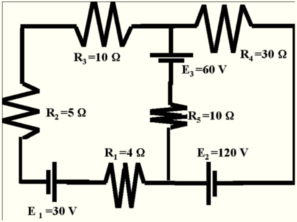
Krishna Kumar

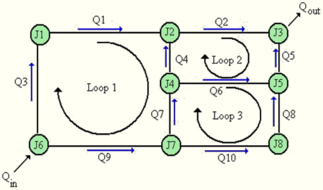
University of Texas at Austin

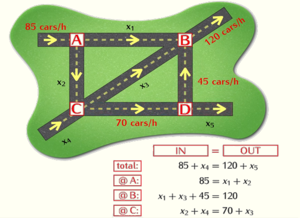
krishnak@utexas.edu

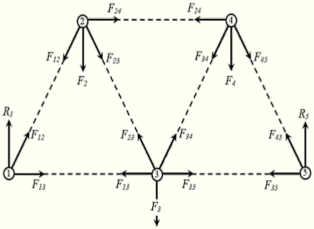
1. Linear System of Equations

Linear System of Equations

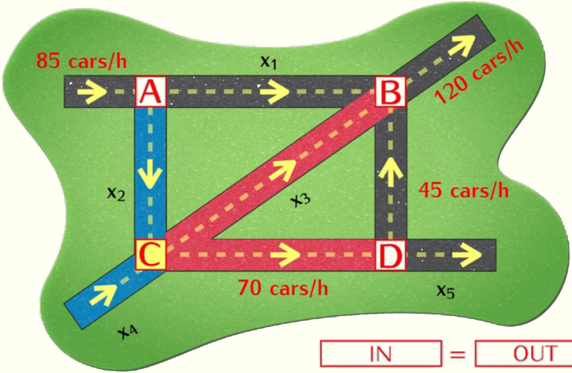








Linear System of Equations: Traffic flow



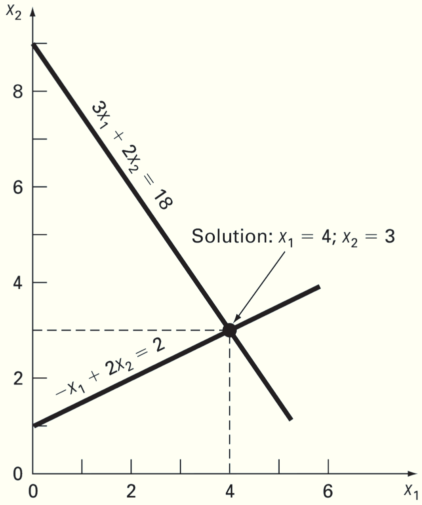
IN = OUT

total:	$85 + x_4 = 120 + x_5$
@ A:	$85 = x_1 + x_2$
@ B:	$x_1 + x_3 + 45 = 120$
@ C:	$x_2 + x_4 = 70 + x_3$

Solving Linear System of Equations

$$
\begin{aligned}
3x_1 + 2x_2 &= 18 \\
-x_1 + 2x_2 &= 2
\end{aligned}
$$

Solving Linear System of Equations



Singularity and Ill-conditioned

Solving Linear System of Equations

1. Direct Methods
    1. Gauss Elimination
    2. Gauss-Jordan Elimination
    3. LU decomposition
2. Iterative Methods
    1. Jacobi iterative
    2. Gauss-Seidel

Direct methods

Consider a system of 3 linear equations for simplicity:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form: $Ax = b$

Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Solve by "back substitution' Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Gauss Elimination: Limitations

1. Prone to round off errors, when we have many (> 100) equations.
2. If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

Gauss Seidel Iterative approach

For conciseness, we limit to 3×3 equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$
\begin{aligned}
4x_1 + x_2 + 2x_3 &= 4 \\
3x_1 + 5x_2 + x_3 &= 7 \\
x_1 + x_2 + 3x_3 &= 3
\end{aligned}
$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

End of first iteration: $x_1, x_2, x_3 = 1, 0.8, 0.4$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} =$$
$$x_2 = \frac{7 - 3x_1 - x_3}{5} =$$
$$x_3 = \frac{3 - x_1 - x_2}{3} =$$

End of first iteration: $x_1, x_2, x_3 = 0.6, 0.96, 0.48$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} =$$
$$x_2 = \frac{7 - 3x_1 - x_3}{5} =$$
$$x_3 = \frac{3 - x_1 - x_2}{3} =$$

Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where k, and $k-1$ represents the current and previous iterations

Gauss Seidel: Disney force diagram

Gauss Seidel: Disney force diagram

Character	Mass (kg)	Drag coefficient (kg/s)
Sven	100	40
Anna	50	5
Olaf	15	10

Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity $v = 5m/s$. compute the acceleration and forces. Unknowns are a, T, R.

$$\begin{aligned} m_1 g - T - c_1 * v &= m_1 a \\ m_2 g + T - c_2 * v - R &= m_2 a \\ m_3 g - c_3 * v + R &= m_3 a \end{aligned}$$

$$\begin{aligned} m_1 a + T &= m_1 g - c_1 * v \\ m_2 a - T + R &= m_2 a - c_2 * v \\ m_3 a - R &= m_3 a - c_3 * v \end{aligned}$$

$$\begin{aligned} 100a + T &= 1000 - 40 * 5 = 800 \\ 50a - T + R &= 500 - 5 * 5 = 475 \\ 15a - R &= 150 - 10 * 5 = 100 \end{aligned}$$

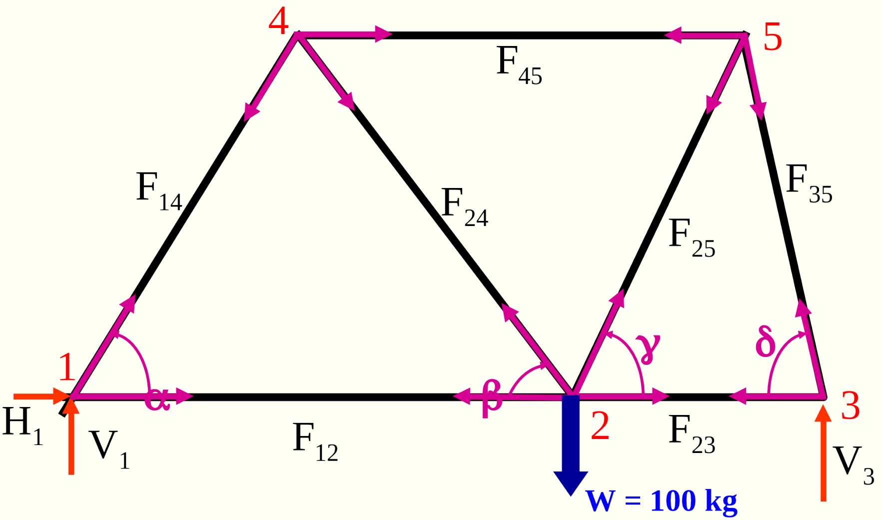
Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity $v = 5m/s$. compute the acceleration and forces. Unknowns are a, T, R.

$$\begin{bmatrix} 100 & 1 & 0 \\ 50 & -1 & 1 \\ 15 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ T \\ R \end{bmatrix} = \begin{bmatrix} 800 \\ 475 \\ 100 \end{bmatrix}$$

The acceleration $a = 7.8571428m/s^2$, $T = 14.29N$, and $R = 17.86N$.

Truss analysis



Truss analysis: Force balance

Node 1
$$\begin{cases} \sum F_{y,1} = V_1 + F_{14} \sin\alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14} \sin\alpha = 0 \end{cases}$$

Node 2
$$\begin{cases} \sum F_{y,2} = F_{24} \sin\beta + F_{25} \sin\gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24} \cos\beta + F_{25} \cos\gamma = 0 \end{cases}$$

Node 3
$$\begin{cases} \sum F_{y,3} = V_3 + F_{35} \sin\delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35} \cos\delta = 0 \end{cases}$$

Node 4
$$\begin{cases} \sum F_{y,4} = -F_{14} \sin\alpha - F_{24} \sin\beta = 0 \\ \sum F_{x,4} = -F_{14} \cos\alpha + F_{24} \cos\beta + F_{45} = 0 \end{cases}$$

Node 5
$$\begin{cases} \sum F_{y,5} = -F_{25} \sin\gamma - F_{35} \sin\delta = 0 \\ \sum F_{x,5} = -F_{25} \cos\gamma + F_{35} \cos\delta - F_{45} = 0 \end{cases}$$

Truss analysis: Matrix formulation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta & 0 \\ 0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta & -1 \end{bmatrix} \begin{Bmatrix} V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$