

# CE 311K: Linear System of Equations

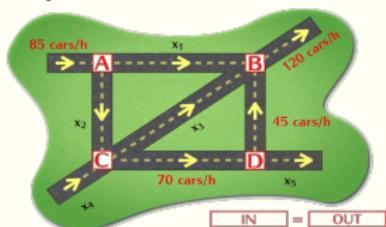
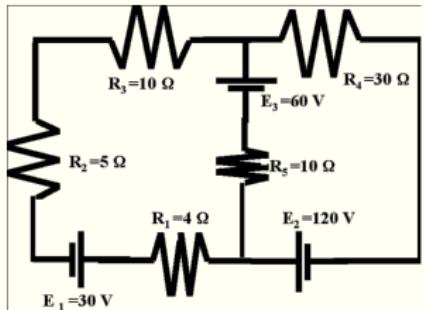
Krishna Kumar

University of Texas at Austin

krishnak@utexas.edu

# 1 Linear System of Equations

# Linear System of Equations

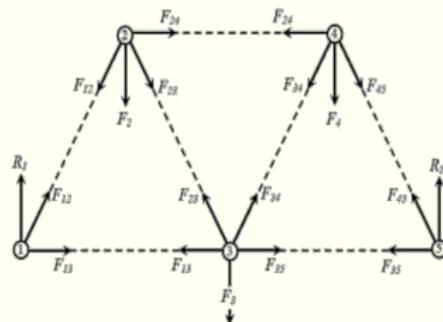
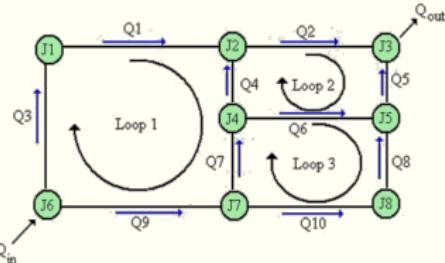


**total:**  $85 + x_4 = 120 + x_3$

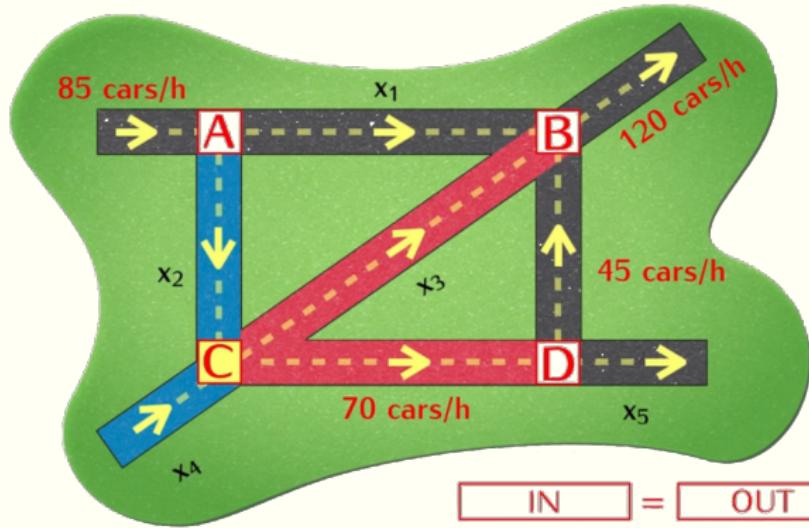
**@ A:**  $85 = x_1 + x_2$

**@ B:**  $x_1 + x_3 + 45 = 120$

**@ C:**  $x_2 + x_4 = 70 + x_3$



# Linear System of Equations: Traffic flow



$$\boxed{\text{IN}} = \boxed{\text{OUT}}$$

total:

$$85 + x_4 = 120 + x_5$$

@ A:

$$85 = x_1 + x_2$$

@ B:

$$x_1 + x_3 + 45 = 120$$

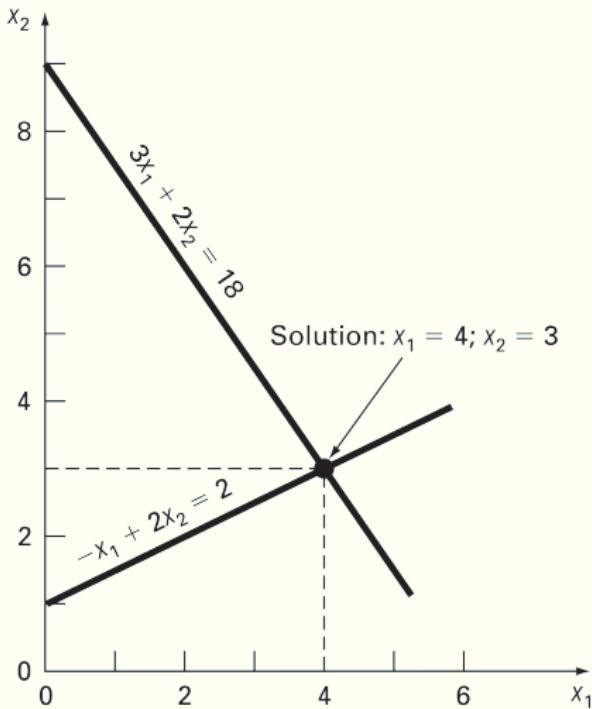
@ C:

$$x_2 + x_4 = 70 + x_3$$

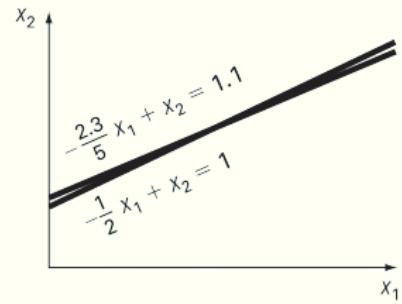
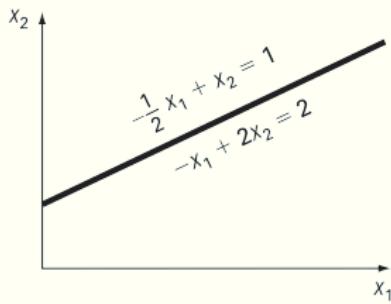
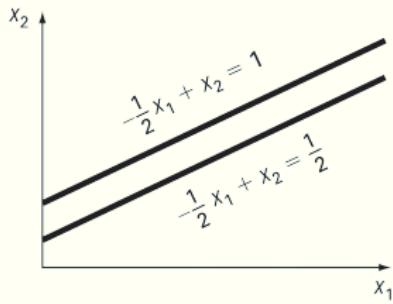
# Solving Linear System of Equations

$$\begin{aligned}3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2\end{aligned}$$

# Solving Linear System of Equations



# Singularity and Ill-conditioned





A singular matrix is a square matrix which is not invertible. Alternatively, a matrix is singular if and only if it has a determinant of 0.

# Solving Linear System of Equations

## ① Direct Methods

- ① Gauss Elimination
- ② Gauss-Jordan Elimination
- ③ LU decomposition

## ② Iterative Methods

- ① Jacobi iterative
- ② Gauss-Seidel

# Direct methods

Consider a system of 3 linear equations for simplicity:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form:  $Ax = b$

# Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

## Solve by “back substitution’ Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with the last equation 3:  $x_3 = b_3/a_{33}$

Equation 2:  $a_{22}x_2 + a_{23}x_3 = b_2$  so

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$

Equation 1:  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$  so

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

General for ‘n’ systems:  $x_n = b_n/a_{nn}$   $x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij}x_j}{a_{ii}}$

# Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$$x_3 = 2, x_2 = 2, x_1 = 2$$

# CE 311K: Linear Systems

## └ Linear System of Equations

### └ Gauss Elimination

```
a = np.array([[2,4,6], [4,11,21], [6, 21, 52]])  
b = np.array([24, 72, 158])  
x = np.linalg.solve(a, b)
```

#### Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 0 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 56 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

# Gauss Elimination: Limitations

- ① Prone to round off errors, when we have many ( $> 100$ ) equations.
- ② If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

## Gauss Seidel Iterative approach

For conciseness, we limit to  $3 \times 3$  equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

# Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$4x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 5x_2 + x_3 = 7$$

$$x_1 + x_2 + 3x_3 = 3$$

Assuming an initial guess of  $x_1, x_2, x_3 = 0$ . End of first iteration.

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0 - 0}{4} = 1$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 1 - 0}{5} = 0.8$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 1 - 0.8}{3} = 0.4$$

## Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x] End of first iteration:

$$x_1, x_2, x_3 = 1, 0.8, 0.4$$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.8 - 2 * 0.4}{4} = 0.6$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.6 - 0.4}{5} = 0.96$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.6 - 0.96}{3} = 0.48$$

End of first iteration:  $x_1, x_2, x_3 = 0.6, 0.96, 0.48$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0.96 - 2 * 0.48}{4} = 0.52$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 0.52 - 0.48}{5} = 0.992$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 0.52 - 0.992}{3} = 0.496$$

# Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where  $k$ , and  $k - 1$  represents the current and previous iterations

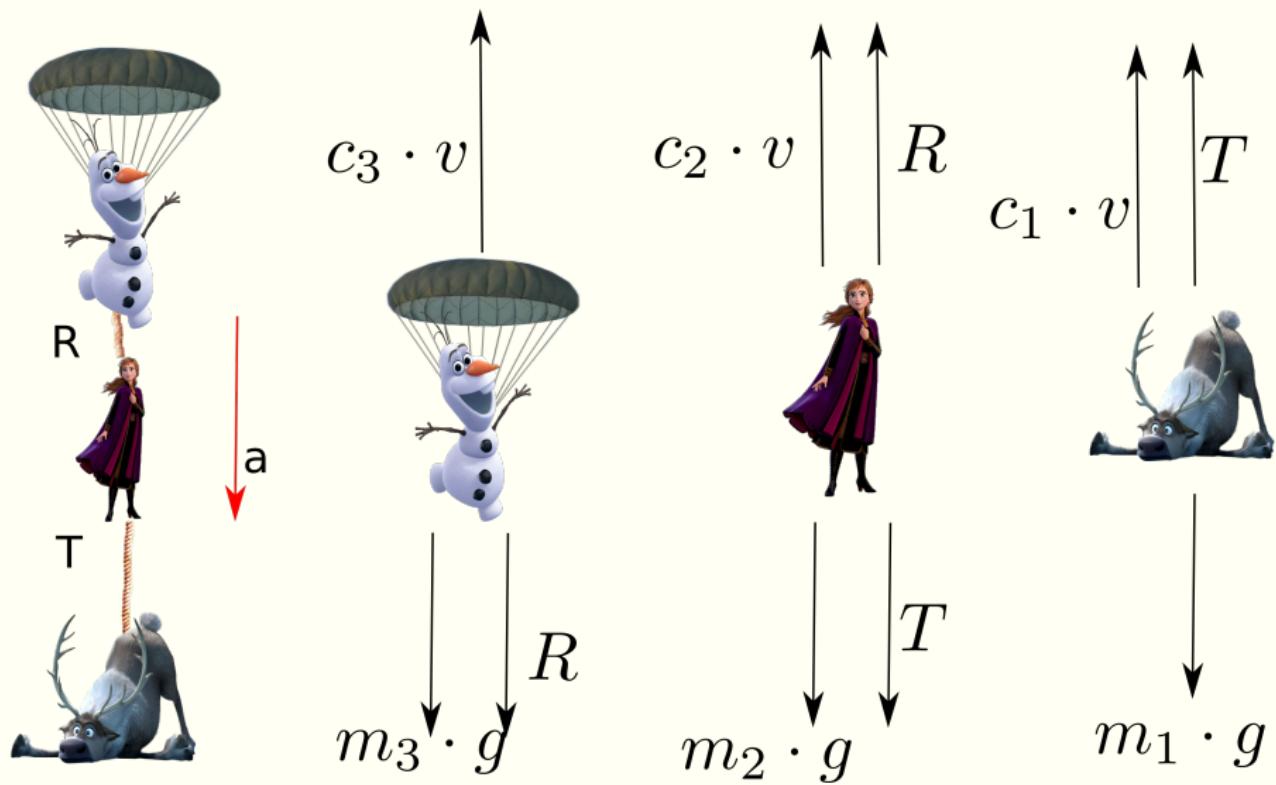
Convergence can be checked using the relative error.

$$|\epsilon_{ik}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \times 100\% \right| < \epsilon_k$$

where  $k$ , and  $k - 1$  represents the current and previous iterations

Gauss-Seidel only works for diagonally dominant matrix

# Gauss Seidel: Disney force diagram



# Gauss Seidel: Disney force diagram



# Gauss Seidel: Disney force diagram

Character	Mass (kg)	Drag coefficient (kg/s)
Sven	100	40
Anna	60	25
Olaf	15	10

## Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity  $v = 5m/s$ . compute the acceleration and forces. Unknowns are  $a, T, R$ .

$$m_1g - T - c_1 * v = m_1a$$

$$m_2g + T - c_2 * v - R = m_2a$$

$$m_3g - c_3 * v + R = m_3a$$

$$m_1a + T = m_1g - c_1 * v$$

$$m_2a - T + R = m_2a - c_2 * v$$

$$m_3a - R = m_3a - c_3 * v$$

$$100a + T = 1000 - 40 * 5 = 800$$

$$60a - T + R = 600 - 25 * 5 = 475$$

$$15a - R = 150 - 10 * 5 = 100$$

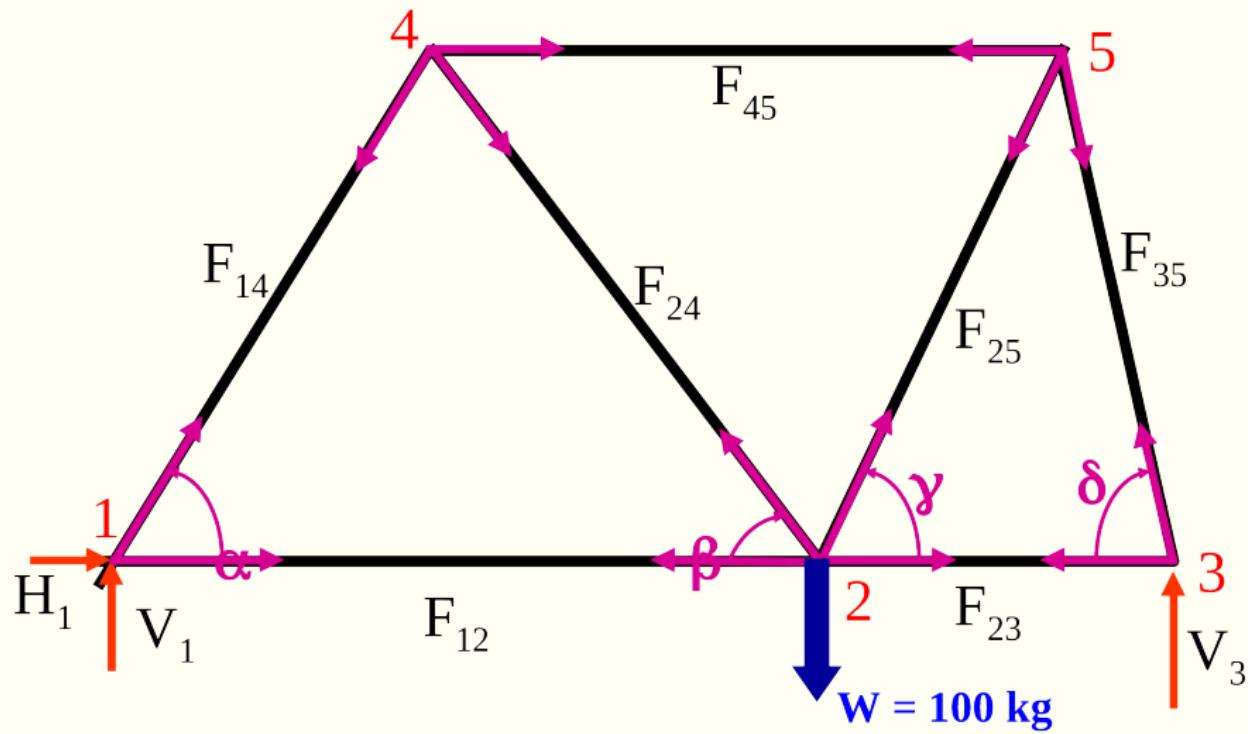
## Gauss Seidel: Disney force diagram

Free-body diagram, moving with a constant velocity  $v = 5m/s$ . compute the acceleration and forces. Unknowns are  $a, T, R$ .

$$\begin{bmatrix} 100 & 1 & 0 \\ 60 & -1 & 1 \\ 15 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ T \\ R \end{bmatrix} = \begin{bmatrix} 800 \\ 475 \\ 100 \end{bmatrix}$$

The acceleration  $a = 7.8571428m/s^2$ ,  $T = 14.29N$ , and  $R = 17.86N$ .

# Truss analysis



# Truss analysis: Force balance

**Node 1**

$$\begin{cases} \sum F_{y,1} = V_1 + F_{14} \sin \alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14} \cos \alpha = 0 \end{cases}$$

**Node 2**

$$\begin{cases} \sum F_{y,2} = F_{24} \sin \beta + F_{25} \sin \gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24} \cos \beta + F_{25} \cos \gamma = 0 \end{cases}$$

**Node 3**

$$\begin{cases} \sum F_{y,3} = V_3 + F_{35} \sin \delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35} \cos \delta = 0 \end{cases}$$

**Node 4**

$$\begin{cases} \sum F_{y,4} = -F_{14} \sin \alpha - F_{24} \sin \beta = 0 \\ \sum F_{x,4} = -F_{14} \cos \alpha + F_{24} \cos \beta + F_{45} = 0 \end{cases}$$

**Node 5**

$$\begin{cases} \sum F_{y,5} = -F_{25} \sin \gamma - F_{35} \sin \delta = 0 \\ \sum F_{x,5} = -F_{25} \cos \gamma + F_{35} \cos \delta - F_{45} = 0 \end{cases}$$

# Truss analysis: Matrix formulation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta \\ 0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta \end{bmatrix} \begin{pmatrix} V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$