

CE 311K: Linear System of Equations

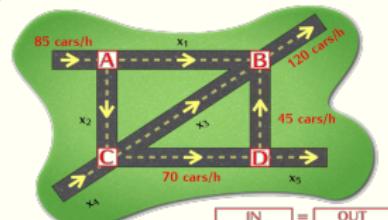
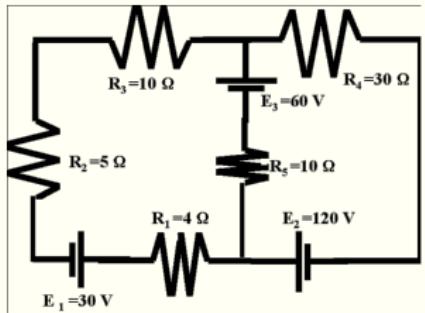
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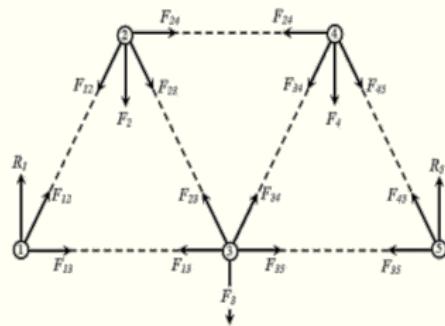
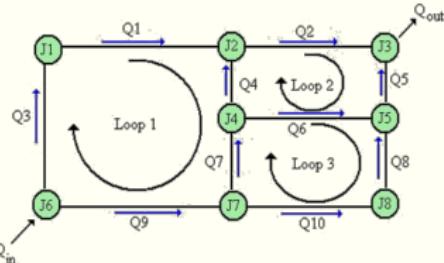
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1 Linear System of Equations

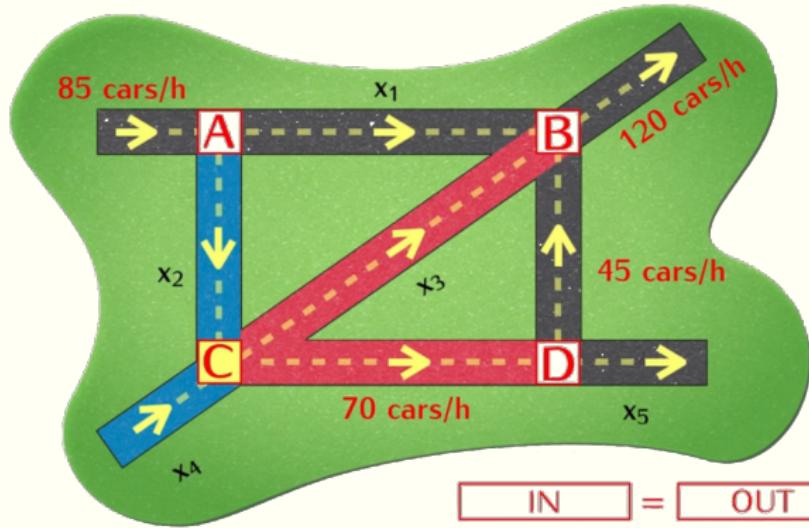
Linear System of Equations



total: $85 + x_4 = 120 + x_5$
(@ A): $85 = x_1 + x_2$
(@ B): $x_1 + x_3 + 45 = 120$
(@ C): $x_2 + x_4 = 70 + x_3$



Linear System of Equations: Traffic flow



$$\boxed{\text{IN}} = \boxed{\text{OUT}}$$

total:

$$85 + x_4 = 120 + x_5$$

@ A:

$$85 = x_1 + x_2$$

@ B:

$$x_1 + x_3 + 45 = 120$$

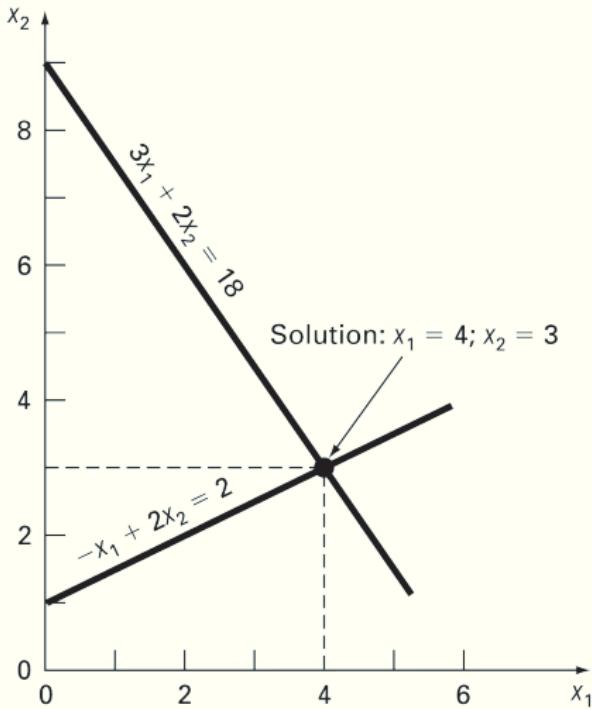
@ C:

$$x_2 + x_4 = 70 + x_3$$

Solving Linear System of Equations

$$\begin{aligned}3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2\end{aligned}$$

Solving Linear System of Equations



Singularity and Ill-conditioned

Solving Linear System of Equations

① Direct Methods

- ① Gauss Elimination
- ② Gauss-Jordan Elimination
- ③ LU decomposition

② Iterative Methods

- ① Jacobi iterative
- ② Gauss-Seidel

Direct methods

Consider a system of 3 linear equations for simplicity:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form: $Ax = b$

Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Solve by “back substitution” Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Gauss Elimination: Limitations

- ① Prone to round off errors, when we have many (> 100) equations.
- ② If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

Gauss Seidel Iterative approach

For conciseness, we limit to 3×3 equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$4x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 5x_2 + x_3 = 7$$

$$x_1 + x_2 + 3x_3 = 3$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

End of first iteration: $x_1, x_2, x_3 = 1, 0.8, 0.4$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} =$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} =$$

$$x_3 = \frac{3 - x_1 - x_2}{3} =$$

End of first iteration: $x_1, x_2, x_3 = 0.6, 0.96, 0.48$

$$x_1 = \frac{4 - x_2 - 2x_3}{4} =$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} =$$

$$x_3 = \frac{3 - x_1 - x_2}{3} =$$

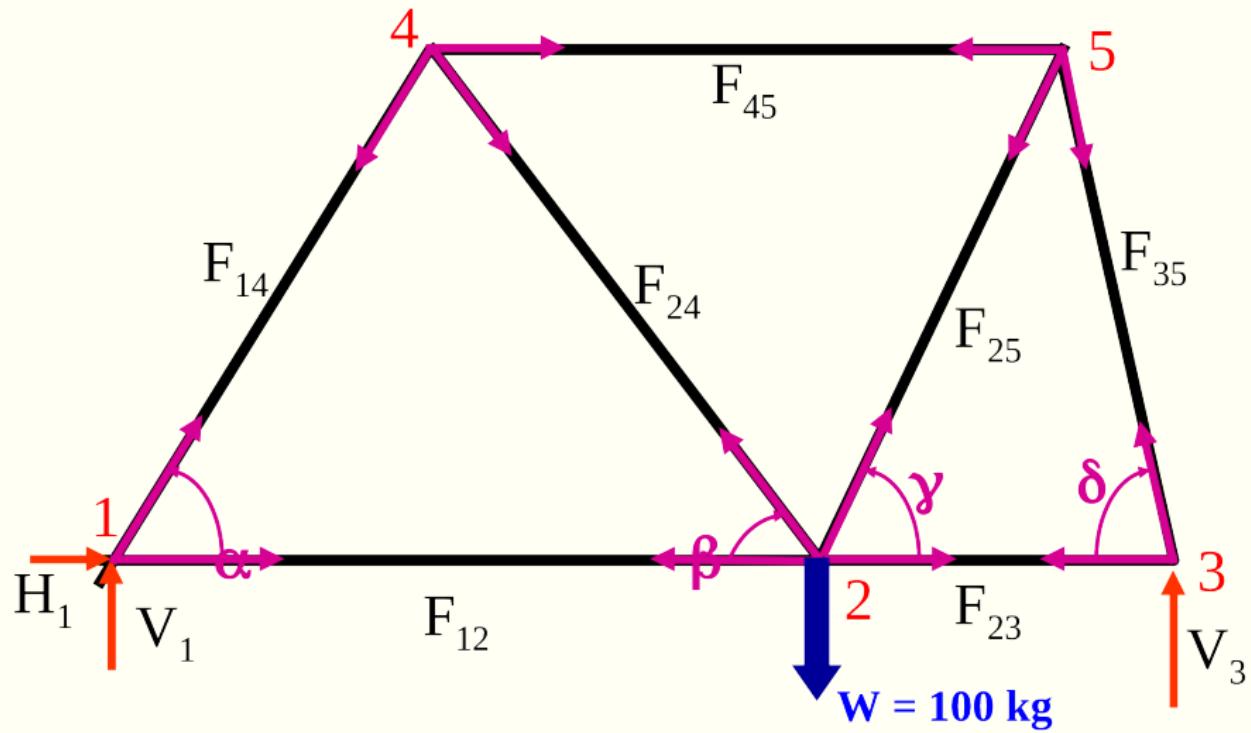
Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where k , and $k - 1$ represents the current and previous iterations

Truss analysis



Truss analysis: Force balance

- Node 1** $\begin{cases} \sum F_{y,1} = V_1 + F_{14} \sin \alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14} \sin \alpha = 0 \end{cases}$
- Node 2** $\begin{cases} \sum F_{y,2} = F_{24} \sin \beta + F_{25} \sin \gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24} \cos \beta + F_{25} \cos \gamma = 0 \end{cases}$
- Node 3** $\begin{cases} \sum F_{y,3} = V_3 + F_{35} \sin \delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35} \cos \delta = 0 \end{cases}$
- Node 4** $\begin{cases} \sum F_{y,4} = -F_{14} \sin \alpha - F_{24} \sin \beta = 0 \\ \sum F_{x,4} = -F_{14} \cos \alpha + F_{24} \cos \beta + F_{45} = 0 \end{cases}$
- Node 5** $\begin{cases} \sum F_{y,5} = -F_{25} \sin \gamma - F_{35} \sin \delta = 0 \\ \sum F_{x,5} = -F_{25} \cos \gamma + F_{35} \cos \delta - F_{45} = 0 \end{cases}$

Truss analysis: Matrix formulation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\beta & \sin\gamma & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cos\beta & \cos\gamma & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos\delta \\ 0 & 0 & 0 & 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & 0 & \cos\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\gamma & \sin\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\gamma & \cos\delta \end{bmatrix} \begin{pmatrix} V_1 \\ H_1 \\ V_3 \\ F_{12} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{25} \\ F_{35} \\ F_{45} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$