HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

PETRO KOLOSOV

Abstract. Polynomial $\mathbf{P}_b^m(x)$ is a $2m+1,\ m \in \mathbb{N}$ degree polynomial in $(x, b) \in \mathbb{R}$, such that derived applying certain methods of interpolation, so that initially we reach the base case for $m = 1$ generalizing it up to $m \in \mathbb{N}$ afterwards. In particular, the polynomial $\mathbf{P}_b^m(x)$ may be successfully applicable for polynomial interpolation and approximation approaches. This manuscript provides a comprehensive historical survey of the milestones and evolution of $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

Contents

1. History and evolution of the $\mathbf{P}_b^m(x)$

Back than, in 2016 being a student of faculty of mechanical engineering, I remember myself playing with finite differences of polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$. Looking to the values in difference table, the first and very naive question that

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came to my mind was *Is it possible to assemble the value of* n^3 *backwards having finite differences?* Definitely, answer to this question is *Yes*, via interpolation. Interpolation is a method of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental work on the topic that is nowadays known as foundation of classical interpolation theory [1]. That time, in 2016, due to lack of knowledge and perspective of view I started re-inventing the interpolation formula by myself,fueled by purest interest and sense of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I was inspired by that mindset and started my own journey. So let's begin considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed via summation over n, e.g

$$\begin{aligned}
&\Delta(0^3) = 1 + 6\cdot 0 \\
&\Delta(1^3) = 1 + 6\cdot 0 + 6\cdot 1 \\
&\Delta(2^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 \\
&\Delta(3^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 \\
&\qquad\vdots \\
&\Delta(n^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n = 1 + 6\sum_{k=0}^{n} k
\end{aligned} \tag{1.1}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then ours above relation $\Delta(n^3) = 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + \cdots + 6\cdot n = 1 + 6\sum_{k=0}^{n} k$ immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2}\binom{3}{k} n^k \tag{1.2}$$

However, as it said, I was not experienced mathematician back than, so that it appeared out for me from a little bit different perspective. Not following the convenient solution (1.2), I rearranged the finite differences from table (1) explicitly to get

$$\begin{aligned}
n^3 = {} & [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2] + \cdots \\
& + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot (n-1)]
\end{aligned}$$

And then combined under the summation in terms of $(n-k)$

$$n^3 = n + (n-0)\cdot 6\cdot 0 + (n-1)\cdot 6\cdot 1 + (n-2)\cdot 6\cdot 2 + \cdots + 1\cdot 6\cdot (n-1)$$

Therefore, the polynomial n^3 can be considered as follows

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1 \tag{1.3}$$

It is immediately seen that (1.3) holds looking to the table of its values

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4]. Sequences for n^5 and n^7 polynomials are at [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that next curiosity would be, for example *Well, if we have a relation* (1.3) *for polynmial* n^3 *then is it true that* (1.3) *can be generalized, say for* n^4 *or* n^5 *either?* That was my next question, however without any expectation of the form of generalized relation. Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials n^5, n^7 and n^9 obtained similarly as (1.3). In short, it appeared that relation (1.3) could be generalized for any odd-power $2m+1$ solving certain system of linear equations. It was proposed that case for n^5 has explicit form as

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are unknown real coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ respectively so that we reach the form of compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Now the potential form generalized of odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct the system of linear equations. In its explicit form

$$\begin{aligned} n^5 &= \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2 \end{aligned}$$

Expanding the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$, we get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true arranging the terms $30k^2(n-k)^2+1$ over $0 \leq k \leq n$ in table as it is shown in OEIS sequence [5].

Therefore, the relation (1.3) we got previously via interpolation of the polynomial n^3 can be generalized for all odd-powers $2m+1$ by constructing and solving certain system of linear equations, and its generalized form to be

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r(n-k)^r \tag{1.4}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the equation (1.4) is discussed separately in [9, 10]. However, constructing and solving systems of linear equations each time for every odd-power $2m+1$ requires huge effort, there must be a function that generates the set of real coefficients $\mathbf{A}_{m,r}$, I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural $m \geq 0,\ 0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate $\mathbf{A}_{m,r}$ recursively, taking base case $\mathbf{A}_{m,m}$ evaluating next $\mathbf{A}_{m,m-1}$ via backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot omit summation bounds letting j run over infinity, unless we do some trick as

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} = \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$$
$$= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$$

So that now we consider the derivation of coefficients $\mathbf{A}_{m,r}$. Using the Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\
&= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\
&= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\
&= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t} \\
&= \sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t} \\
&= \sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}
\end{aligned}$$

Now, we notice that

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j = 0; \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1}, & \text{if } j > 0. \end{cases} \tag{1.5}$$

An elegant proof of the above binomial identity is provided at [12]. In particular, the equation (1.5) is zero for $0 < t \leq j$. So that taking $j = 0$ we have

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} &+ \left[\sum_{j\geq 1}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \\
&- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}$$

Now let's simplify the double summation applying the identity (1.5)

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r+1-j} \right]}_{(\star)}$$

$$- \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms of the above equation so that we get

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r} \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r} \mathbf{A}_{m,r} \sum_{d} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_{r} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r} \mathbf{A}_{m,r} \left[\sum_{d} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] - n^{2m+1} = 0 \tag{1.6}$$

Taking the coefficient of n^{2m+1} in (1.6), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$ we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \le r < m/2^s$ (iterating consecutively $s = 1, 2, \dots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \le r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.7}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 3. Coefficients $\mathbf{A}_{m,r}$.

The coefficients $\mathbf{A}_{m,r}$ are also registered in the OEIS [14, 15]. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$ give powers of 2

$$\sum_{r=0}^{m}\mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Therefore, we have successfully shown and generalized previously obtained equation (1.3)

Theorem 1.1. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.7).

Theorem (1.1) can be considered as succeeded interpolation of the polynomial x^{2m+1} over the domains $n \in \mathbb{R},\ m \in \mathbb{N}$.

Finally, we got our road to the main definition of polynomial $\mathbf{P}_b^m(x)$ as it is closely related to the equation (1.1). So let's define it without further hesitation

Definition 1.2.

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.8}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.7), $m \in \mathbb{N}$ and $(x, b) \in \mathbb{R}$. A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ and its properties can be found at [16]. In 2023, Albert Tkaczyk yet again extended the theorem (1.1) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17].

2. Related works

In this section let's give a short overview of related works that are based onto definition of polynomials $\mathbf{P}_b^m(x)$. In [18] is given a relation in terms of partial differential differential equations such that ordinary derivative of odd-power $2m+1$ can be reached in terms of partial derivatives of $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u,u)$ evaluate in point (u,u) plus partial derivative $(f_v)'_z(u,u)$ evaluate in point (u,u)

$$\frac{d}{dx} g_v(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{2.1}$$

where $f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$. Afterwards, the equation (2.1) is generalized over the timescales $\mathbb{T} \times \mathbb{T}$ providing its dynamic equation analog in [19].

Second article [20] gives another perspective of ordinary derivatives of polynomials expressing them via double limit as

$$\lim_{b\to x}\mathbf{P}_b^m(x)=x^{2m+1}$$

that opens such opportunity.

In [21] based on (1.6), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, a Faulhaber-like formula for sums of the form $1^m(n-1)^m+2^m(n-2)^m+\cdots+(n-1)^m1^m$ for positive integers m and n.

3. Future research

- Differential equation (2.1) can be expressed in terms of backward and central differentials, as well as its dynamic equation analogs [19]
- Definition (1.8) is closely related to discrete convolution, probably some new identities in term of discrete convolution may be found
- All kind of derivatives (forward, backward, central), including time scale ones can be expressed as double limit similarly to [20]
- Equation (1.8) approximates odd-power $2m+1$ in some neighborhood of fixed point a as it shown on graphs

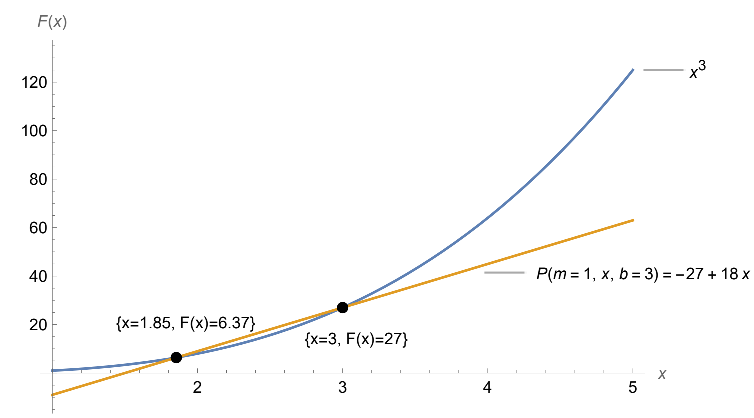

Figure 1. Approximation of x^3.

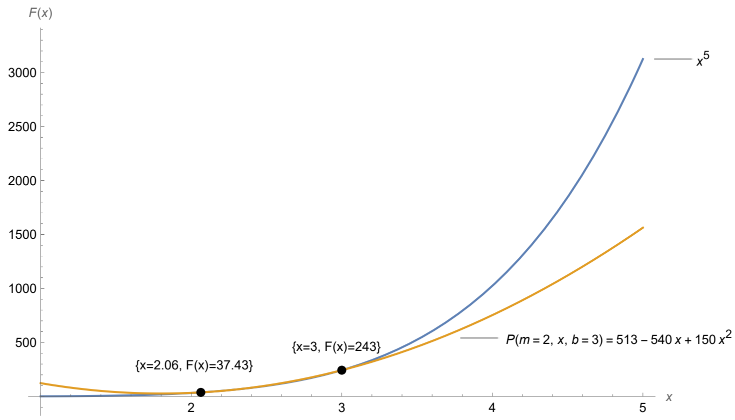

Figure 2. Approximation of x^5.

4. Conclusions

Conclusions of your manuscript.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Alan F Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996.

[3] Milton Abramowitz, Irene A Stegun, and Robert H Romer. Handbook of mathematical functions with formulas, graphs, and mathematical tables, 1988.

[4] Petro Kolosov. Numerical triangle, row sums give third power, Entry A287326 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A287326, 2017.

[5] Petro Kolosov. Numerical triangle, row sums give fifth power, Entry A300656 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A300656, 2018.

[6] Petro Kolosov. Numerical triangle, row sums give seventh power, Entry A300785 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A300785, 2018.

[7] Tkaczyk, Albert. About the problem of a triangle developing the polynomial function. Published electronically at LinkedIn, 2018.

[8] Tkaczyk, Albert. On the problem of a triangle developing the polynomial function - continuation. Published electronically at LinkedIn, 2018.

[9] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[10] Petro Kolosov. Polynomial identity involving Binomial Theorem and Faulhaber's formula. Published electronically at https://kolosovpetro.github.io/pdf/PolynomialIdentityInvolvingBTandFaulhaber.pdf, 2023.

[11] Alekseyev, Max. MathOverflow answer 297916/113033. Published electronically at https://mathoverflow.net/a/297916/113033, 2018.

[12] Scheuer, Markus. MathStackExchange answer 4724343/463487. Published electronically at https://math.stackexchange.com/a/4724343/463487, 2023.

[13] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[14] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A302971, 2018.

[15] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A304042, 2018.

[16] Petro Kolosov. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[17] Albert Tkaczyk. On three-dimensional expansions of the polynomial function f(n)=nm̂, September 2023. https://doi.org/10.5281/zenodo.8371454.

[18] Petro Kolosov. Another approach to get derivative of odd-power. *arXiv preprint arXiv:2310.07804*, 2023. https://arxiv.org/abs/2310.07804.

[19] Petro Kolosov. A study on partial dynamic equation on time scales involving derivatives of polynomials. *arXiv preprint arXiv:1608.00801*, 2016. https://arxiv.org/abs/1608.00801.

[20] Petro Kolosov. Finding the derivative of polynomials via double limit, January 2024. https://doi.org/10.5281/zenodo.10575485.

[21] J Fernando Barbero G, Juan Margalef-Bentabol, and Eduardo JS Villaseñor. A two-sided Faulhaber-like formula involving Bernoulli polynomials. *Comptes Rendus. Mathématique*, 358(1):41–44, 2020. https://doi.org/10.5802/crmath.10.

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Software Developer, DevOps Engineer

Email address: `kolosovp94@gmail.com`

URL: `https://kolosovpetro.github.io`