HISTORY AND OVERVIEW OF POLYNOMIAL $\mathbf{P}_B^M(X)$

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Abstract. This manuscript gives a comprehensive history survey and evolution of work done over the topics of polynomials $\mathbf{P}_b^m(x)$ such that are polynomials in $(x, b) \in \mathbb{R},\ m \in \mathbb{N}$. Motivation of this manuscript is to mitigate degree of chaos (what do you mean by chaos?) in that topic (what topic?), need to provide clear unambitious explanation (explanation of what?) and context as a whole. We start our journey from the very beginning, e.g interpolation (interpolation of what?) approach to reach the definition of $\mathbf{P}_b^m(x)$ continuing further with results based on $\mathbf{P}_b^m(x)$, for instance various polynomial identities, differential equations etc. Also, this manuscript provides further research directions connected with the polynomials $\mathbf{P}_b^m(x)$. (Why these polynomials are of interest at all?)

Contents

1. History and evolution of the topic

Back than, in 2016 being a student of faculty of mechanical engineering, I remember myself playing with finite differences of polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$. Looking to the values in difference table, the first and very naive question that came to my mind was *Is it possible to assemble the value of n^3 backwards having finite differences?*

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Definitely, answer to this question is *Yes.* Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental work on the topic that is nowadays known as foundation of classical interpolation theory (citations). That time, in 2016, due to lack of knowledge and perspective of view I started re-inventing the interpolation formula by myself, fueled by purest interest and sense of mystery. All mathematical laws and relations exist till the very beginning, but we only find and describe them, I was inspired by that mindset and started my own journey. So let's begin considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed via summation over n, e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k
\end{aligned} \tag{1.1}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then ours above relation $\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$ immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{1.2}$$

However, as it said, I was not experienced mathematician back than, so that it appeared out for me from a little bit different perspective. Not following the convenient solution (1.2), I rearranged the finite differences from table (1) explicitly to get

$$\begin{aligned} n^3 = {} & [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ & + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

And then combined under the summation in terms of $(n - k)$

$$n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + (n-2) \cdot 6 \cdot 2 + \cdots + 1 \cdot 6 \cdot (n-1)$$

Therefore, the polynomial n^3 can be considered as follows

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.3}$$

It is immediately seen that (1.3) holds looking to the table of its values

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that next curiosity would be, for example *Well, if we have a relation* (1.3) *for polynmial* n^3 *then is it true that* (1.3) *can be generalized, say for* n^4 *or* n^5 *either?* That was my next question, however without any expectation of the form of generalized relation. Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [5, 6] showing the cases for polynomials n^5, n^7 and n^9. In short, it appeared that relation (1.3) could be generalized for any odd-power solving certain system of linear equations. It was proposed that case for n^5 has explicit form as

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C \right]$$

where A, B, C are unknown real coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ respectively so that we reach the form of compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Now the potential form generalized of odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct the system of linear equations. In its explicit form

$$n^5 = \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r$$
$$= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2$$

Expanding the terms $\sum_{k=1}^{n} k^r (n-k)^r$ via Faulhaber's formula (citation) we get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] - n^5 = 0$$

Multiplying by 30 right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true evaluating the terms $30k^2(n-k)^2+1$ over $0 \leq k \leq n$ as it is shown at [7].

Therefore, the relation (1.3) we got previously by interpolation can be generalized for all odd-powers by constructing and solving certain system of linear equations, and its generalized form to be

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, it is discussed separately in [8, 9]. However, constructing and solving systems of linear equations each time for every odd-power $2m+1$ requires huge effort, there must be a function that generates the set of real coefficients $\mathbf{A}_{m,r}$, I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [10], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural $m \geq 0,\ 0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate $\mathbf{A}_{m,r}$ recursively, taking base case $\mathbf{A}_{m,m}$ evaluating next $\mathbf{A}_{m,m-1}$ via backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot skip summation bounds unless we do some trick as

$$\begin{aligned}\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}\end{aligned}$$

So that now we consider the derivation of coefficients $\mathbf{A}_{m,r}$. Using the Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\begin{aligned}
\sum_{k=1}^{n} k^r (n-k)^r &= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r} \\
&= \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1}\right] \\
&= \sum_{t=0}^{r} \binom{r}{t} \left[\frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t}\right] \\
&= \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\
&= \sum_j \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \\
&= \sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} - \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}
\end{aligned}$$

Now, we notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j=0; \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1}, & \text{if } j>0. \end{cases} \tag{1.4}$$

An elegant proof of the above binomial identity is provided at [11]. In particular, the equation (1.4) is zero for $0 < t \leq j$. So that taking $j=0$ we have

$$\begin{aligned}
\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} &+ \left[\sum_{j\geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] \\
&- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]
\end{aligned}$$

Now let's simplify the double summation applying the identity (1.4)

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j}\binom{r}{2r-j+1} B_j n^{2r+1-j}\right]}_{(\star)}$$
$$- \underbrace{\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms of the above equation so that we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell}\binom{r}{\ell}\frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r} \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{r} \mathbf{A}_{m,r} \sum_{d} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_{r} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_{r} \mathbf{A}_{m,r} \left[\sum_{d} \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0 \tag{1.5}$$

Taking the coefficient of n^{2m+1} in (1.5), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.6}$$

where B_t are Bernoulli numbers [12]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 3. Coefficients $\mathbf{A}_{m,r}$.

The coefficients $\mathbf{A}_{m,r}$ are also registered in the OEIS [13, 14]. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$ give powers of 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Therefore, we have successfully shown and generalized previously obtained equation (1.3)

Theorem 1.1. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.6).

Theorem (1.1) can be considered as succeeded interpolation of the polynomial x^{2m+1} over the domains $n \in \mathbb{R},\ m \in \mathbb{N}$.

Finally, we got our road to the main definition of polynomial $\mathbf{P}_b^m(x)$ as it is closely related to the equation (1.1). So let's define it without further hesitation

Definition 1.2.

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.7}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.6), $m \in \mathbb{N}$ and $(x, b) \in \mathbb{R}$.

2. Related works

3. Conclusions

Conclusions of your manuscript.

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