

# HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

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ABSTRACT. The polynomial  $\mathbf{P}_b^m(x)$  is a polynomial of degree  $2m + 1$  in  $(x, b) \in \mathbb{R}$ , defined by an identity for odd powers, closely linked to Binomial theorem and Faulhaber's formula. The odd-power identity is derived using certain interpolation techniques, including systems of linear equations, recurrence relations, and finite differences. This manuscript offers a comprehensive historical survey of the milestones and evolution of the polynomial  $\mathbf{P}_b^m(x)$ , followed by related works based on it. Notable results in related works include the relation between ordinary and partial derivatives for odd powers and finding polynomial derivatives via a double limit. Finally, the manuscript proposes future research directions.

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6. Addendum 2: Derivation of the coefficients  $\mathbf{A}_{m,r}$ 

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1. HISTORY AND EVOLUTION OF THE POLYNOMIAL  $\mathbf{P}_b^m(x)$ 

Back then, in 2016 being a student at the faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial  $n^3$  over the domain of natural numbers  $n \in \mathbb{N}$  having at most  $0 \leq n \leq 20$  values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

*Is it possible to re-assemble the value of the polynomial  $n^3$  backwards having its finite differences?*

The answer to this question is definitely *Yes*, utilizing the interpolation principles. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton’s fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I thought. That mindset truly inspired me so that my own mathematical journey has been started. Let us begin considering the table of finite differences of the polynomial  $n^3$

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Sources: <https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP>

$n$	$n^3$	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

**Table 1.** Table of finite differences of the polynomial  $n^3$ .

First and foremost, we can observe that finite difference  $\Delta(n^3)$  of the polynomial  $n^3$  can be expressed through summation over  $n$ , e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots
\end{aligned} \tag{1.1}$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^n k \tag{1.2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term  $\sum_{k=0}^n k$  reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^n k = \frac{1}{2}(n + n^2)$$

Then our relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6 \left[ \frac{1}{2}(n + n^2) \right] = 1 + 3n + 3n^2 = \sum_{k=0}^2 \binom{3}{k} n^k \tag{1.3}$$

However, as was said, I was not the experienced one mathematician back then, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have rearranged the first order finite differences from the table (1) using (1.1) to get the polynomial  $n^3$

$$\begin{aligned} n^3 &= [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ &\quad + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned} \quad (1.4)$$

Then, rearranging the terms of the equation (1.4) so that it turns into summation in terms of  $k(n-k)$

$$\begin{aligned} n^3 &= n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \cdots \\ &\quad \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

Gives the interpolation of the polynomial  $n^3$

$$n^3 = \sum_{k=1}^n 6k(n-k) + 1 \quad (1.5)$$

It is immediately seen that (1.5) is true by observing the table of  $6k(n-k) + 1$  values

$n/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

**Table 2.** Values of  $6k(n-k) + 1$ . See the OEIS entry: [A287326](#) [4]. Sequences such that row sums give the polynomials  $n^5$  and  $n^7$  are also registered in OEIS [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial  $n^3$ . Fairly enough that the next curiosity would be

*Well, if the relation (1.5) true for the polynomial  $n^3$ , then is it true that (1.5) can be generalized for higher powers, e.g. for  $n^4$  or  $n^5$  either?*

That was my next question, however without any expectation of the final form of generalized relation.

**1.1. System of linear equations approach.** Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials  $n^5$ ,  $n^7$  and  $n^9$  that were obtained similarly as (1.5). In short, it appears that relation (1.5) could be generalized for any non-negative odd power  $2m + 1$  solving a system of linear equations. It was proposed that the case for  $n^5$  has explicit form

$$n^5 = \sum_{k=1}^n [Ak^2(n-k)^2 + Bk(n-k) + C]$$

where  $A, B, C$  are yet-unknown coefficients. Denote  $A, B, C$  as  $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$  to reach the form of a compact double sum

$$n^5 = \sum_{k=1}^n \sum_{r=0}^2 \mathbf{A}_{2,r} k^r (n-k)^r$$

Observing the equation above, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients  $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$  it is necessary construct and solve a system of linear equations following the process

$$\begin{aligned} n^5 &= \sum_{r=0}^2 \mathbf{A}_{2,r} \sum_{k=1}^n k^r (n-k)^r \\ &= \mathbf{A}_{2,0} \sum_{k=1}^n k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^n k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^n k^2 (n-k)^2 \end{aligned}$$

Expand the terms  $\sum_{k=1}^n k^r (n-k)^r$  applying the Faulhaber's formula [2] to get the equation

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that the odd-power identity holds

$$n^5 = \sum_{k=1}^n 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true by arranging the terms  $30k^2(n-k)^2 + 1$  over  $0 \leq k \leq n$  as tabular. See the OEIS sequence [\[5\]](#)

$n/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

**Table 3.** Values of  $30k^2(n-k)^2 + 1$ . See the OEIS entry [A300656](#).

Now we can see that the relation (1.5) we got via interpolation of cubes can be generalized for all non-negative odd-powers  $2m+1$  by constructing and solving a certain system of linear equations. Therefore, the generalized form of odd-power identity has the form

$$n^{2m+1} = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=1}^n k^r (n-k)^r \quad (1.6)$$

where  $\mathbf{A}_{m,r}$  are real coefficients. In more details, the identity (1.6) is discussed separately in [9, 10].

As one more example, let be  $m = 3$  so that we have the following relation defined by (1.6)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1} \left[ \frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[ \frac{1}{30}(-n + n^5) \right] + \mathbf{A}_{m,3} \left[ \frac{1}{420}(-10n + 7n^3 + 3n^7) \right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n + n^3) + 14\mathbf{A}_{2,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Expanding brackets and rearranging the terms gives

$$\begin{aligned} 420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 \\ - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0 \end{aligned}$$

Combining the common terms yields

$$\begin{aligned} & n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3}) \\ & + n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0 \end{aligned}$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (1.6) holds

$$n^7 = \sum_{k=1}^n 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms  $140k^3(n-k)^3 - 14k(n-k) + 1$  over  $0 \leq k \leq n$  as the OEIS sequence [A300785](#) [6] shows.



$n/k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

**Table 4.** Values of  $140k^3(n-k)^3 - 14k(n-k) + 1$ . See the OEIS entry [A300785](#).

**1.2. Recurrence relation approach.** However, constructing and solving a system of linear equations for every odd-power  $2m+1$  requires a huge effort, there must be a formula that generates a set of real coefficients  $\mathbf{A}_{m,r}$  for each fixed  $m$ , I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient  $\mathbf{A}_{m,r}$  for each natural  $m, r$  such that  $m \geq 0$  and  $0 \leq r \leq m$ . The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the  $\mathbf{A}_{m,r}$  recursively, taking the base case  $\mathbf{A}_{m,m}$  and then evaluating the next coefficient  $\mathbf{A}_{m,m-1}$  by using backtracking, continuing similarly up to  $\mathbf{A}_{m,0}$ . Before we consider the derivation of the recurrent formula for coefficients  $\mathbf{A}_{m,r}$ , a few words must be said regarding the Faulhaber's formula [2]

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to notice that iteration step  $j$  is bounded by the value of power  $p$ , while the upper index of the binomial coefficient  $\binom{p+1}{j}$  is  $p+1$ . It means that we cannot omit summation bounds letting  $j$  run over infinity, unless we perform the following action on the

Faulhaber's formula

$$\begin{aligned} \sum_{k=1}^n k^p &= \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j} = \left[ \frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\ &= \left[ \frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \end{aligned} \quad (1.7)$$

At this point we are good to go through the entire derivation of the recurrent formula for coefficients  $\mathbf{A}_{m,r}$ . By applying Binomial theorem  $(n-k)^r = \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} k^t$  and Faulhaber's formula (1.7) to the equation (1.6) we get

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \\ &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \left[ \frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right] \\ &= \sum_{t=0}^r \binom{r}{t} \left[ \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right] \\ &= \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \\ &= \left[ \sum_j \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned}$$

Rearranging terms yields

$$\left[ \sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \quad (1.8)$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1) \binom{2r}{r}} & \text{if } j = 0 \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \quad (1.9)$$

An elegant proof of the binomial identity (1.9) is presented in [12]. In particular, the equation (1.9) is zero for  $0 < t \leq j$ . So that moving  $j = 0$  out of summation in (1.8) we

have

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[ \sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\ &\quad - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \end{aligned}$$

Simplifying above equation by using (1.9) yields

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[ \sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)} \\ &\quad - \underbrace{\left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \end{aligned}$$

Hence, introducing  $\ell = 2r - j + 1$  to  $(\star)$  and  $\ell = r - t$  to  $(\diamond)$  we collapse the common terms in the equation above so that

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[ \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[ \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Assuming that  $\mathbf{A}_{m,r}$  is defined by (1.6), we obtain the following relation for polynomials in

$n$

$$\sum_{r=0}^m \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd  $\ell$  by  $d$  we get

$$\sum_{r=0}^m \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{d \geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

Therefore,

$$\sum_{r=0}^m \mathbf{A}_{m,r} \left[ \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r=0}^m \mathbf{A}_{m,r} \left[ \sum_{d \geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] \quad (1.10)$$

$$\equiv n^{2m+1}$$

Taking the coefficient of  $n^{2m+1}$  we set iteration steps  $r = m$  and  $d = m$  in (1.10) to get

$$\mathbf{A}_{m,m} = (2m+1) \binom{2m}{m}$$

Note that  $\binom{r}{2d+1} = 0$  in (1.10) having  $r = m$  and  $d = m$ .

Taking the coefficient of  $n^{2d+1}$  for an integer  $d$  in the range  $\frac{m}{2} \leq d < m$ , we get

$$\mathbf{A}_{m,d} = 0$$

because  $\binom{r}{2d+1}$  is zero for every  $r \leq m$  and  $2d+1$  having  $\frac{m}{2} \leq d < m$ . For example  $\binom{r}{m+1} = 0$  having  $r \leq m$  and  $d = \frac{m}{2}$ .

Taking the coefficient of  $n^{2d+1}$  for  $d$  in the range  $\frac{m}{4} \leq d < \frac{m}{2}$  we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute  $\mathbf{A}_{m,r}$  for each integer  $r$  in range  $\frac{m}{2s+1} \leq r < \frac{m}{2s}$ , iterating consecutively over  $s = 1, 2, \dots$  by using previously determined values of  $\mathbf{A}_{m,d}$  as follows

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient  $\mathbf{A}_{m,r}$

**Definition 1.1.** (Definition of coefficient  $\mathbf{A}_{m,r}$ .)

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1) \binom{2r}{r} & \text{if } r = m \\ (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \quad (1.11)$$

where  $B_t$  are Bernoulli numbers [13]. It is assumed that  $B_1 = \frac{1}{2}$ . For example,

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 5.** Coefficients  $\mathbf{A}_{m,r}$ . See OEIS sequences [14, 15].

The nominators and denominators of the coefficients  $\mathbf{A}_{m,r}$  are also registered as sequences in OEIS [14, 15]. It is as well interesting to notice that row sums of the  $\mathbf{A}_{m,r}$  give powers of 2

$$\sum_{r=0}^m \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Let be a theorem

**Theorem 1.2.** *For every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are  $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ , such that*

$$n^{2m+1} = \sum_{k=1}^n \sum_{r=0}^m \mathbf{A}_{m,r} k^r (n-k)^r$$

where  $\mathbf{A}_{m,r}$  is a real coefficient defined recursively by (1.11).

Finally, we got our road to definition of  $2m+1$ -degree polynomial  $\mathbf{P}_b^m(x)$ . Introducing the parameter  $b$  to the upper summation bound of the equation (1.2), we have the definition

**Definition 1.3.** *(Polynomial  $\mathbf{P}_b^m(x)$  of degree  $2m+1$ .)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \quad (1.12)$$

where  $\mathbf{A}_{m,r}$  are real coefficients (1.11). A comprehensive discussion on the polynomial  $\mathbf{P}_b^m(x)$  as well as its properties can be found at [16]. In 2023, Albert Tkaczyk yet again extended the theorem (1.2) to the so-called three dimension case so that it gives polynomials of the form  $n^{3l+2}$  at [17]. For example,

$$\mathbf{P}_b^0(x) = b,$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5$$

$$- 15b^2x + 30b^3x - 15b^4x$$

$$+ 5bx^2 - 15b^2x^2 + 10b^3x^2$$

$$\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7$$

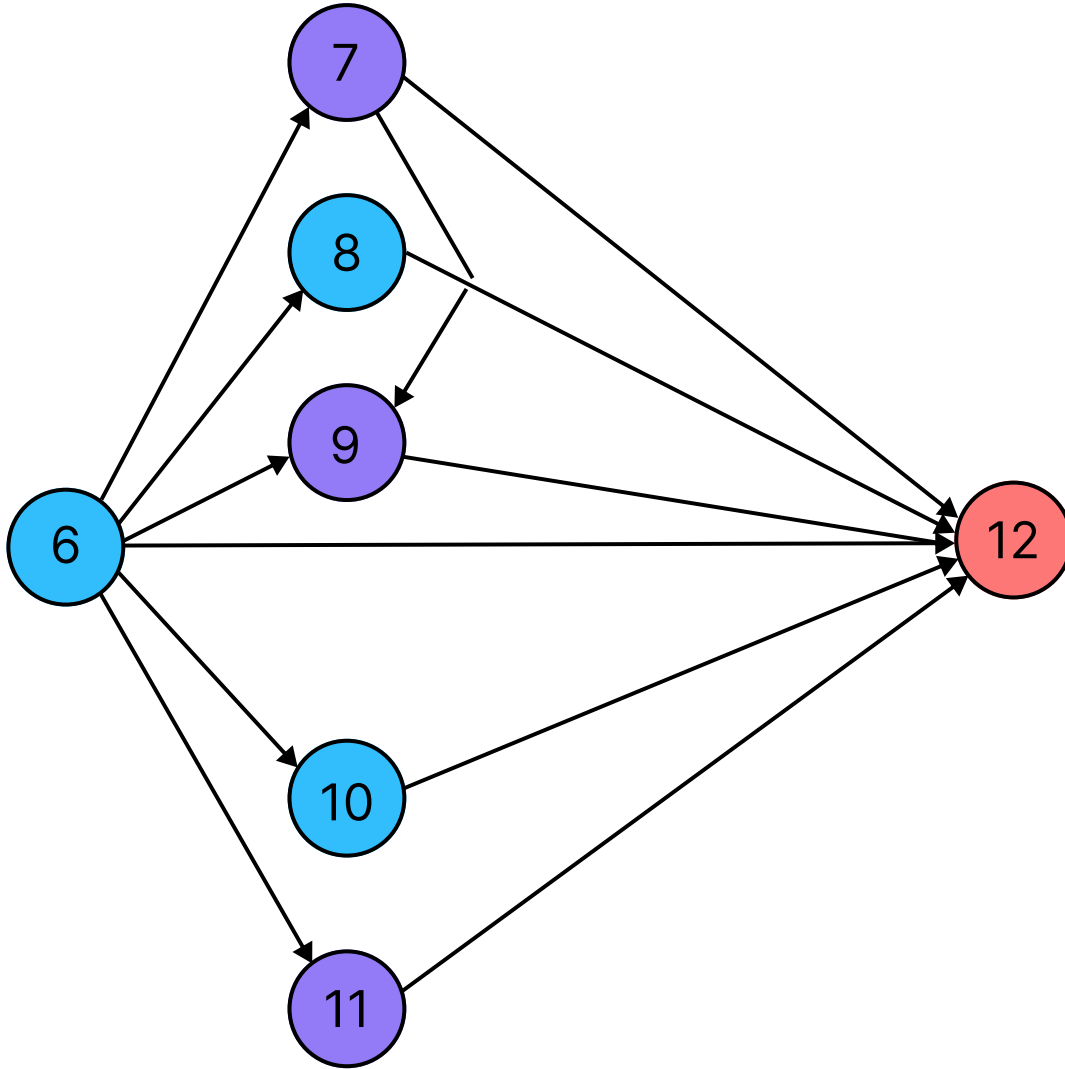
$$+ 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x$$

$$+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2$$

$$+ 35b^2x^3 - 70b^3x^3 + 35b^4x^3$$

## 2. RELATED WORKS

In this section we provide a short overview of related works that are based onto the main results of [16] mainly focusing on the polynomial  $\mathbf{P}_b^m(x)$



**Figure 1.** Related works graph.

- In *On the link between binomial theorem and discrete convolution* [16] which is denoted as 6: Let  $\mathbf{P}_b^m(x)$  be a  $2m + 1$ -degree polynomial in  $x$  and  $b \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r$$

where  $\mathbf{A}_{m,r}$  are real coefficients. In this manuscript, we introduce the polynomial  $\mathbf{P}_b^m(x)$  and study its properties, establishing a polynomial identity for odd-powers in terms of this polynomial. Based on mentioned polynomial identity for odd-powers, we explore the connection between the Binomial theorem and discrete convolution of

odd-powers, further extending this relation to the multinomial case. All findings are verified using Mathematica programs.

- In *A study on partial dynamic equation on time scales involving derivatives of polynomials* [19] which is denoted as 7: This work derives and discusses an identity that connects the timescale derivative of odd-powered polynomial with partial derivatives of polynomial  $\mathbf{P}_b^m(x)$  evaluated in particular points. For every  $t \in \mathbb{T}_1$  and  $(x, b) \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m, b, x)}{\Delta b}(m, t, t)$$

such that  $\sigma(t) > t$  is forward jump operator. In addition, we discuss various derivative operators in the context of the partial cases of above equation, We show finite difference, classical derivative,  $q$ -derivative,  $q$ -power derivative on behalf of it.

- In *106.37 An unusual identity for odd-powers* [9] which is denoted as 8: Explores and proves the polynomial identity for odd-powers

$$n^{2m+1} = \sum_{k=1}^n \sum_{r=0}^m \mathbf{A}_{m,r} k^r (n-k)^r$$

- In *Another approach to get derivative of odd-power* [18] which is denoted as 9: is given a relation in terms of partial differential equations such that ordinary derivative of odd-power  $2m+1$  can be reached in terms of partial derivatives of  $\mathbf{P}_b^m(x)$ . Let be a fixed point  $v \in \mathbb{N}$ , then ordinary derivative  $\frac{d}{dx}g_v(u)$  of the odd-power function  $g_v(x) = x^{2v+1}$  evaluate in point  $u \in \mathbb{R}$  equals to partial derivative  $(f_v)'_x(u, u)$  evaluate in point  $(u, u)$  plus partial derivative  $(f_v)'_z(u, u)$  evaluate in point  $(u, u)$

$$\frac{d}{dx}g_v(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \quad (2.1)$$

where  $f_y(x, z) = \sum_{k=1}^z \sum_{r=0}^y \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$ .

- In *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [10] which is denoted as 10: proves that for every  $n \geq 1$ ,  $n, m \in \mathbb{N}$  there are coefficients



$\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^n \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \dots + \mathbf{A}_{m,m} k^m (n-k)^m$$

which is a direct consequence of the definition of  $\mathbf{P}_b^m(x)$ , Binomial theorem and Faulhaber's formula.

- In *Finding the derivative of polynomials via double limit* [20] which is denoted as 11: gives another perspective of ordinary derivatives of polynomials allowing expressing them via double limit, because

$$\lim_{h \rightarrow 0} \mathbf{P}_{x+h}^m(x) = x^{2m+1}$$

- This manuscript is denoted as 12.
- In [21] based on (1.10), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form  $1^m(n-1)^m + 2^m(n-2)^m + \dots + (n-1)^m 1^m$  for positive integers  $m$  and  $n$ .
- Three sequences were contributed to the OEIS [22, 23, 24] showing the coefficients of the polynomial  $\mathbf{P}_b^m(x)$  having fixed points  $m, b$  while  $x \in \mathbb{R}$ .
- OEIS sequences such that row sums give odd-powers [4, 5, 6].
- OEIS sequences related to the coefficients  $\mathbf{A}_{m,r}$  [14, 15].

The indexes of the related works graph are same that are on the website

[kolosovpetro.github.io/math](https://kolosovpetro.github.io/math)

### 3. FUTURE RESEARCH AND ACTIVITIES

- Differential equation (2.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [19] could be generalized further.
- Theorem (1.2) provides an opportunity to express odd-power identity in terms of multiplication of certain matrices.

- There are Taylor series and Maclaurin series versions in terms of  $\mathbf{P}_b^m(x)$ .
- The summation bounds of definition (1.12) can be altered so that  $k$  runs over  $1 \leq k \leq b$ , by symmetry.
- Prove that  $\mathbf{P}_b^m(x)$  is an integer valued polynomial in  $(x, b)$ .
- The definition (1.12) is closely related to discrete convolution because

$$\mathbf{P}_b^m(x) = \sum_{r=0}^m \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

where  $\sum_{k=0}^{b-1} k^r (x-k)^r$  is the discrete convolution of  $x^r$ . It is worth to get a closer look into it so that new relations in terms of discrete convolution may be found.

- All kinds of derivatives e.g. forward, backward and central, including the derivatives on time-scales can be expressed as double limit of  $\mathbf{P}_b^m(x)$  extending the results of [20].
- Introducing the definitions of the coefficients  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_m$  and  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_m$

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_m = \sum_{r=0}^m \mathbf{A}_{m,r} k^r (n-k)^r$$

$$\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}_m = \sum_{k=0}^{n-1} \mathbf{A}_{m,r} k^r (n-k)^r$$

the novel identities can be reached, for example

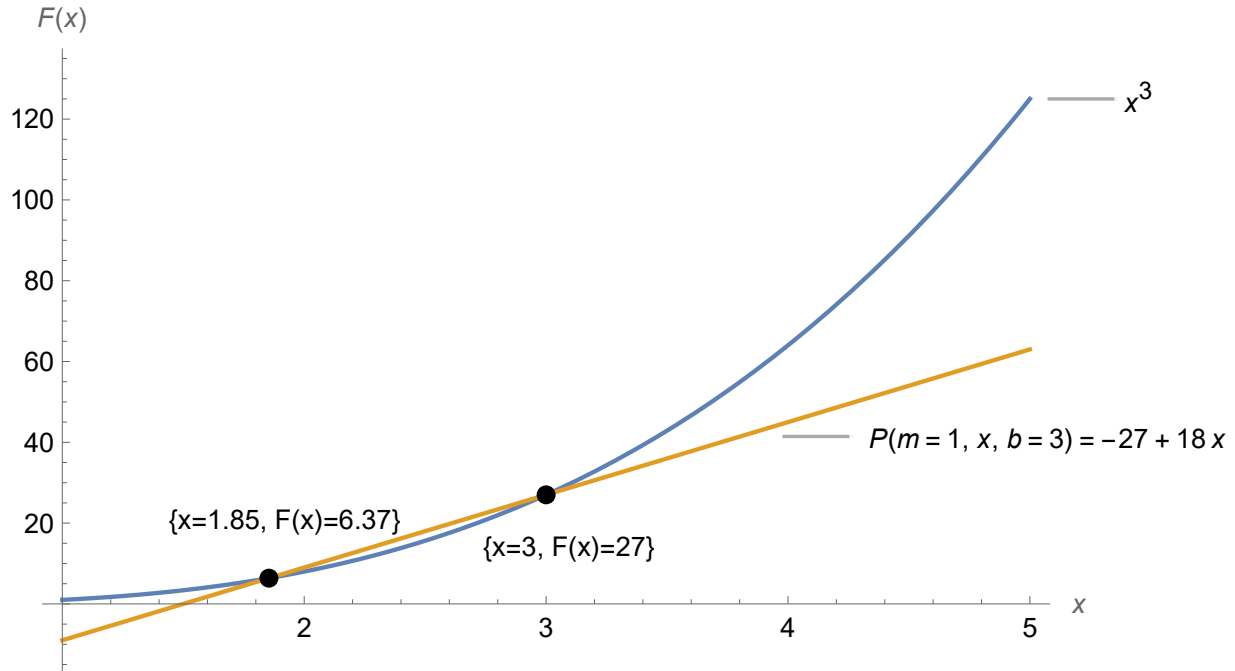
$$\left[ \begin{smallmatrix} 2t+1 \\ 1 \end{smallmatrix} \right]_m = \left[ \begin{smallmatrix} t+2 \\ 2 \end{smallmatrix} \right]_m$$

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_m = \left[ \begin{smallmatrix} n \\ n-k \end{smallmatrix} \right]_m$$

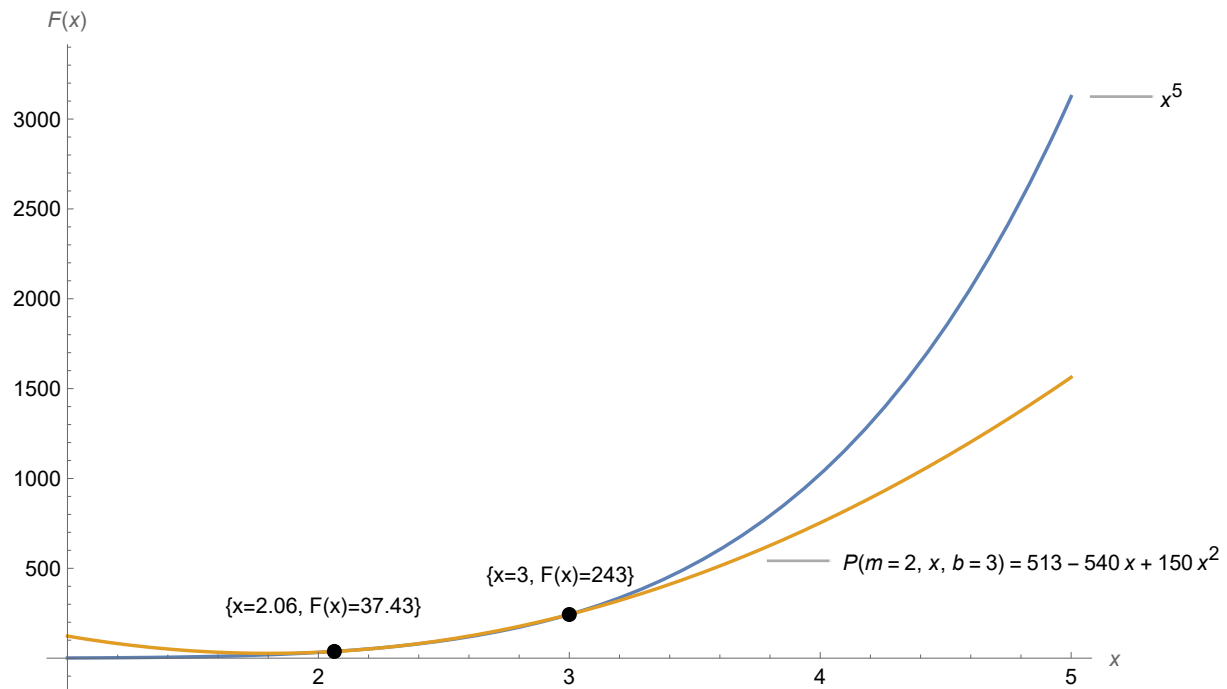
$$\left[ \begin{smallmatrix} 2t-3r \\ r \end{smallmatrix} \right]_m = \left[ \begin{smallmatrix} t \\ 2r \end{smallmatrix} \right]_m = \left[ \begin{smallmatrix} 2t-3r \\ 2t-4r \end{smallmatrix} \right]_m$$

so that combinatorial sense of above is also a topic to research.

- Contribute new OEIS sequences related to  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_m$  and  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_m$ .
- Following the results of <https://arxiv.org/pdf/1603.02468v15.pdf>, the equation (1.12) approximates the odd-power polynomial  $x^{2m+1}$  around given points  $x_i$  as it may be observed from the following plots



**Figure 2.** Approximation of  $x^3$ .



**Figure 3.** Approximation of  $x^5$ .

- English grammar reviews and improvements are welcome.

- Improvements and suggestions to current manuscript under open-source initiatives at <https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP>

#### 4. CONCLUSIONS

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial  $\mathbf{P}_b^m(x)$  as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

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5. ADDENDUM 1: EXAMPLES OF THE POLYNOMIAL  $\mathbf{P}_b^m(x)$ 

$$\mathbf{P}_b^0(x) = b$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2$$

$$\begin{aligned} \mathbf{P}_b^3(x) = & -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ & + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3 \end{aligned}$$

$$\begin{aligned} \mathbf{P}_b^4(x) = & -60b^2 + 180b^3 - 294b^5 + 420b^7 - 315b^8 + 70b^9 + 60bx - 270b^2x + 735b^4x - 1470b^6x \\ & + 1260b^7x - 315b^8x + 90bx^2 - 630b^3x^2 + 1890b^5x^2 - 1890b^6x^2 + 540b^7x^2 + 210b^2x^3 \\ & - 1050b^4x^3 + 1260b^5x^3 - 420b^6x^3 - 21bx^4 + 210b^3x^4 - 315b^4x^4 + 126b^5x^4 \end{aligned}$$

$$\begin{aligned} \mathbf{P}_b^5(x) = & -693b^2 + 2068b^3 - 330b^4 - 2640b^5 + 2772b^7 - 2310b^9 + 1386b^{10} - 252b^{11} + 693bx \\ & - 3102b^2x + 660b^3x + 6600b^4x - 9702b^6x + 10395b^8x - 6930b^9x + 1386b^{10}x + 1034bx^2 \\ & - 330b^2x^2 - 5940b^3x^2 + 12936b^5x^2 - 18480b^7x^2 + 13860b^8x^2 - 3080b^9x^2 + 2310b^2x^3 \\ & - 8085b^4x^3 + 16170b^6x^3 - 13860b^7x^3 + 3465b^8x^3 - 330bx^4 + 2310b^3x^4 - 6930b^5x^4 \\ & + 6930b^6x^4 - 1980b^7x^4 - 231b^2x^5 + 1155b^4x^5 - 1386b^5x^5 + 462b^6x^5 \end{aligned}$$

$$\begin{aligned} \mathbf{P}_b^6(x) = & -10920b^2 + 33306b^3 - 9009b^4 - 36036b^5 + 37752b^7 - 22022b^9 + 12012b^{11} - 6006b^{12} + 924b^{13} \\ & + 10920bx - 49959b^2x + 18018b^3x + 90090b^4x - 132132b^6x + 99099b^8x - 66066b^{10}x + 36036b^{11}x \\ & - 6006b^{12}x + 16653bx^2 - 9009b^2x^2 - 84084b^3x^2 + 180180b^5x^2 - 180180b^7x^2 + 150150b^9x^2 \\ & - 90090b^{10}x^2 + 16380b^{11}x^2 + 36036b^2x^3 - 120120b^4x^3 + 168168b^6x^3 - 180180b^8x^3 \\ & + 120120b^9x^3 - 24024b^{10}x^3 - 6006bx^4 + 40040b^3x^4 - 84084b^5x^4 + 120120b^7x^4 - 90090b^8x^4 \\ & + 20020b^9x^4 - 6006b^2x^5 + 21021b^4x^5 - 42042b^6x^5 + 36036b^7x^5 - 9009b^8x^5 + 286bx^6 \\ & - 2002b^3x^6 + 6006b^5x^6 - 6006b^6x^6 + 1716b^7x^6 \end{aligned}$$

6. ADDENDUM 2: DERIVATION OF THE COEFFICIENTS  $\mathbf{A}_{m,r}$ 

Consider the definition (1.11) of the coefficients  $\mathbf{A}_{m,r}$ , it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1) \binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \underbrace{(2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition of the real coefficient  $T(d, r)$

**Definition 6.1.** *Real coefficient  $T(d, r)$*

$$T(d, r) = (2r+1) \binom{2r}{r} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

**Example 6.2.** *Let be  $m = 2$  so first we get  $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5 \binom{4}{2} = 30$$

*Then  $\mathbf{A}_{2,1} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $1 \leq d < 2$ . Finally, the coefficient  $\mathbf{A}_{2,0}$  is*

$$\begin{aligned} \mathbf{A}_{2,0} &= \sum_{d \geq 1}^2 \mathbf{A}_{2,d} \cdot T(d, 0) = \mathbf{A}_{2,1} \cdot T(1, 0) + \mathbf{A}_{2,2} \cdot T(2, 0) \\ &= 30 \cdot \frac{1}{30} = 1 \end{aligned}$$

**Example 6.3.** *Let be  $m = 3$  so that first we get  $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7 \binom{6}{3} = 140$$

*Then  $\mathbf{A}_{3,2} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $2 \leq d < 3$ . The  $\mathbf{A}_{3,1}$  coefficient is non-zero and calculated as*

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^3 \mathbf{A}_{3,d} \cdot T(d, 1) = \mathbf{A}_{3,3} \cdot T(3, 1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$



Finally, the coefficient  $\mathbf{A}_{3,0}$  is

$$\begin{aligned}\mathbf{A}_{3,0} &= \sum_{d \geq 1}^3 \mathbf{A}_{3,d} \cdot T(d, 0) = \mathbf{A}_{3,1} \cdot T(1, 0) + \mathbf{A}_{3,2} \cdot T(2, 0) + \mathbf{A}_{3,3} \cdot T(3, 0) \\ &= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1\end{aligned}$$

**Example 6.4.** Let be  $m = 4$  so that first we get  $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9 \binom{8}{4} = 630$$

Then  $\mathbf{A}_{4,3} = 0$  and  $\mathbf{A}_{4,2} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $2 \leq d < 4$ . The value of the coefficient  $\mathbf{A}_{4,1}$  is non-zero and calculated as

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^4 \mathbf{A}_{4,d} \cdot T(d, 1) = \mathbf{A}_{4,3} \cdot T(3, 1) + \mathbf{A}_{4,4} \cdot T(4, 1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient  $\mathbf{A}_{4,0}$  is

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^4 \mathbf{A}_{4,d} \cdot T(d, 0) = \mathbf{A}_{4,1} \cdot T(1, 0) + \mathbf{A}_{4,4} \cdot T(4, 0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

**Example 6.5.** Let be  $m = 5$  so that first we get  $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11 \binom{10}{5} = 2772$$

Then  $\mathbf{A}_{5,4} = 0$  and  $\mathbf{A}_{5,3} = 0$  because  $\mathbf{A}_{m,d}$  is zero in the range  $m/2 \leq d < m$  means that zero for  $d$  in  $3 \leq d < 5$ . The value of the coefficient  $\mathbf{A}_{5,2}$  is non-zero and calculated as

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^5 \mathbf{A}_{5,d} \cdot T(d, 2) = \mathbf{A}_{5,5} \cdot T(5, 2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient  $\mathbf{A}_{5,1}$  is non-zero and calculated as

$$\begin{aligned}\mathbf{A}_{5,1} &= \sum_{d \geq 3}^5 \mathbf{A}_{5,d} \cdot T(d, 1) = \mathbf{A}_{5,3} \cdot T(3, 1) + \mathbf{A}_{5,4} \cdot T(4, 1) + \mathbf{A}_{5,5} \cdot T(5, 1) \\ &= 2772 \cdot \left(-\frac{1}{2}\right) = -1386\end{aligned}$$

Finally, the coefficient  $\mathbf{A}_{5,0}$  is

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d \geq 1}^5 \mathbf{A}_{5,d} \cdot T(d, 0) = \mathbf{A}_{5,1} \cdot T(1, 0) + \mathbf{A}_{5,2} \cdot T(2, 0) + \mathbf{A}_{5,5} \cdot T(5, 0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

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