HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

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ABSTRACT. The polynomial $\mathbf{P}_b^m(x)$ is a polynomial of degree $2m+1$ in $(x,b) \in \mathbb{R}$, defined by an identity for odd powers, closely linked to Binomial theorem and Faulhaber's formula. The odd-power identity is derived using certain interpolation techniques, including systems of linear equations, recurrence relations, and finite differences. This manuscript offers a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$, followed by related works based on it. Notable results in related works include the relation between ordinary and partial derivatives for odd powers and finding polynomial derivatives via a double limit. Finally, the manuscript proposes future research directions.

CONTENTS

1. History and evolution of the polynomial $\mathbf{P}_b^m(x)$

Back then, in 2016 being a student at the faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite

differences?

The answer to this question is definitely *Yes*, utilizing the interpolation principles. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I thought. That mindset truly inspired me so that my own mathematical journey has been started. Let us begin considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed through summation over n, e.g

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \qquad (1.1)$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \qquad (1.2)$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then our relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \qquad (1.3)$$

However, as was said, I was not the experienced one mathematician back then, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have rearranged the first order finite differences from the table (1) using (1.1) to get the polynomial n^3

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \qquad (1.4)$$

Then, rearranging the terms of the equation (1.4) so that it turns into summation in terms of $k(n-k)$

$$n^3 = n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2] + \cdots$$

$$\cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)]$$

Gives the interpolation of the polynomial n^3

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.5}$$

It is immediately seen that (1.5) is true by observing the table of $6k(n-k)+1$ values

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4]. Sequences such that row sums give the polynomials n^5 and n^7 are also registered in OEIS [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Well, if the relation (1.5) true for the polynomial n^3, then is it true that (1.5) can be generalized for higher powers, e.g. for n^4 or n^5 either?

That was my next question, however without any expectation of the final form of generalized relation. Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials n^5, n^7 and n^9 that were obtained similarly as (1.5). In short, it appears that relation (1.5) could be generalized for

any non-negative odd power $2m + 1$ solving a system of linear equations. It was proposed that the case for n^5 has explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C \right]$$

where A, B, C are yet-unknown coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the form of a compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Observing the equation above, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ it is necessary construct and solve a system of linear equations following the process

$$n^5 = \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r$$

$$= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2$$

Expand the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula [2] to get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n + n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n + n^5) \right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that the odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true by arranging the terms $30k^2(n-k)^2 + 1$

over $0 \le k \le n$ as tabular. See the OEIS sequence [5]

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $30k^2(n-k)^2 + 1$. See the OEIS entry A300656.

Now we can see that the relation (1.5) we got via interpolation of cubes can be generalized

for all non-negative odd-powers $2m+1$ by constructing and solving a certain system of linear

equations. Therefore, the generalized form of odd-power identity has the form

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r(n-k)^r \tag{1.6}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the identity (1.6) is discussed separately in [9, 10].

However, constructing and solving a system of linear equations for every odd-power $2m+1$ requires a huge effort, there must be a formula that generates a set of real coefficients $\mathbf{A}_{m,r}$ for each fixed m, I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural m, r such that $m \geq 0$ and $0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula [2]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to notice that iteration step j is bounded by the value of power p, while the upper index of the binomial coefficient $\binom{p+1}{j}$ is $p + 1$. It means that we cannot omit summation bounds letting j run over infinity, unless we perform the following action on the Faulhaber's formula

$$\begin{aligned}
\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} &= \left[\frac{1}{p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1} \\
&= \left[\frac{1}{p+1} \sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}
\end{aligned} \tag{1.7}$$

At this point we are good to go through the entire derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$. By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t \binom{r}{t} n^{r-t} k^t$ and

Faulhaber's formula (1.7) to the equation (1.6) we get

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r}$$

$$= \sum_{t=0}^{r}(-1)^t \binom{r}{t} n^{r-t} \left[\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right]$$

$$= \sum_{t=0}^{r} \binom{r}{t} \left[\frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right]$$

$$= \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

$$= \left[\sum_{j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Rearranging terms yields

$$\left[\sum_{j} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \qquad (1.8)$$

We can notice that

$$\sum_{t} \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\[2ex] \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \qquad (1.9)$$

An elegant proof of the binomial identity (1.9) is presented in [12]. In particular, the equation (1.9) is zero for $0 < t \leq j$. So that moving $j = 0$ out of summation in (1.8) we have

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \geq 1} B_j n^{2r+1-j} \sum_{t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right]$$

$$- \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Simplifying above equation by using (1.9) yields

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)}$$

$$- \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms in the equation above so that

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell \right]$$

$$- \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^\ell \right]$$

$$= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (1.6), we obtain the following relation for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^\ell \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{d\geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

Therefore,

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{d\geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] \tag{1.10}$$

$$\equiv n^{2m+1}$$

Taking the coefficient of n^{2m+1} we set iteration steps $r = m$ and $d = m$ in (1.10) to get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Note that $\binom{r}{2d+1} = 0$ in (1.10) having $r = m$ and $d = m$.

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

because $\binom{r}{2d+1}$ is zero for every $r \leq m$ and $2d+1$ having $\frac{m}{2} \leq d < m$. For example $\binom{r}{m+1} = 0$ having $r \leq m$ and $d = \frac{m}{2}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d \geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d \geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.11}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 4. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [14, 15].

The nominators and denominators of the coefficients $\mathbf{A}_{m,r}$ are also registered as sequences in OEIS [14, 15]. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$ give powers of 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Let be a theorem

Theorem 1.2. *For every $n \geq 1$, $n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.11).

Finally, we got our road to the main definition of the polynomial $\mathbf{P}_b^m(x)$. Introducing the parameter b to the upper summation bound of the equation (1.2), we have the definition

Definition 1.3. *(Polynomial $\mathbf{P}_b^m(x)$ of degree $2m+1$.)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.12}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.11). A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ as well as its properties can be found at [16]. In 2023, Albert Tkaczyk yet again

extended the theorem (1.2) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17]. For example,

$$\mathbf{P}_b^0(x) = b,$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5$$
$$- 15b^2x + 30b^3x - 15b^4x$$
$$+ 5bx^2 - 15b^2x^2 + 10b^3x^2$$

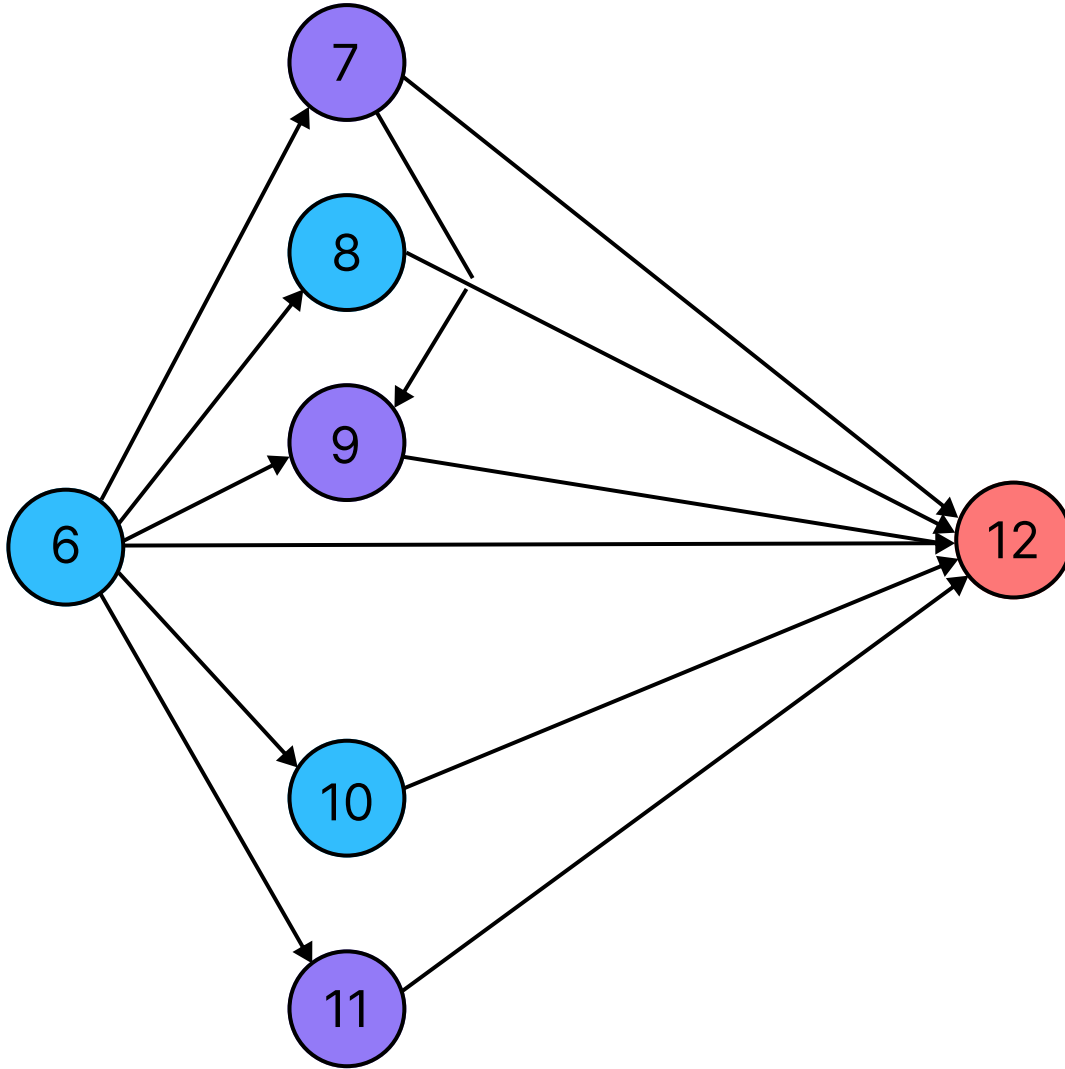$$\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7$$
$$+ 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x$$
$$+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2$$
$$+ 35b^2x^3 - 70b^3x^3 + 35b^4x^3$$

2. Related works

In this section we provide a short overview of related works that are based onto the main results of [16] mainly focusing on the polynomial $\mathbf{P}_b^m(x)$

Figure 1. Related works graph.

- In *On the link between binomial theorem and discrete convolution* [16] which is denoted as 6: Let $\mathbf{P}_b^m(x)$ be a $2m + 1$-degree polynomial in x and $b \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In this manuscript, we introduce the polynomial $\mathbf{P}_b^m(x)$ and study its properties, establishing a polynomial identity for odd-powers in terms of this polynomial. Based on mentioned polynomial identity for odd-powers, we explore the connection between the Binomial theorem and discrete convolution of

odd-powers, further extending this relation to the multinomial case. All findings are verified using Mathematica programs.

- In *A study on partial dynamic equation on time scales involving derivatives of polynomials* [19] which is denoted as 7: This work derives and discusses an identity that connects the timescale derivative of odd-powered polynomial with partial derivatives of polynomial $\mathbf{P}_b^m(x)$ evaluated in particular points. For every $t \in \mathbb{T}_1$ and $(x, b) \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m, b, x)}{\Delta b}(m, t, t)$$

such that $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in the context of the partial cases of above equation, We show finite difference, classical derivative, $q-$derivative, $q-$power derivative on behalf of it.

- In *106.37 An unusual identity for odd-powers* [9] which is denoted as 8: Explores and proves the polynomial identity for odd-powers

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

- In *Another approach to get derivative of odd-power* [18] which is denoted as 9: is given a relation in terms of partial differential equations such that ordinary derivative of odd-power $2m + 1$ can be reached in terms of partial derivatives of $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)_x'(u, u)$ evaluate in point (u, u) plus partial derivative $(f_v)_z'(u, u)$ evaluate in point (u, u)

$$\frac{d}{dx} g_v(u) = (f_v)_x'(u, u) + (f_v)_z'(u, u) \tag{2.1}$$

where $f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$.

- In *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [10] which is denoted as 10: proves that for every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients

$\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0}k^0(n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m}k^m(n-k)^m$$

which is a direct consequence of the definition of $\mathbf{P}_b^m(x)$, Binomial theorem and Faulhaber's formula.

- In *Finding the derivative of polynomials via double limit* [20] which is denoted as 11: gives another perspective of ordinary derivatives of polynomials allowing expressing them via double limit, because

$$\lim_{h \to 0} \mathbf{P}_{x+h}^m(x) = x^{2m+1}$$

- This manuscript is denoted as 12.
- In [21] based on (1.10), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.
- Three sequences were contributed to the OEIS [22, 23, 24] showing the coefficients of the polynomial $\mathbf{P}_b^m(x)$ having fixed points m, b while $x \in \mathbb{R}$.
- OEIS sequences such that row sums give odd-powers [4, 5, 6].
- OEIS sequences related to the coefficients $\mathbf{A}_{m,r}$ [14, 15].

The indexes of the related works graph are same that are on the website

kolosovpetro.github.io/math

3. FUTURE RESEARCH AND ACTIVITIES

- Differential equation (2.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [19] could be generalized further.
- Theorem (1.2) provides an opportunity to express odd-power identity in terms of multiplication of certain matrices.

- There are Taylor series and Maclaurin series versions in terms of $\mathbf{P}_b^m(x)$.

- The summation bounds of definition (1.12) can be altered so that k runs over $1 \leq k \leq b$, by symmetry.

- Prove that $\mathbf{P}_b^m(x)$ is an integer valued polynomial in (x, b).

- The definition (1.12) is closely related to discrete convolution because

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x - k)^r$$

where $\sum_{k=0}^{b-1} k^r (x - k)^r$ is the discrete convolution of x^r. It is worth to get a closer look into it so that new relations in terms of discrete convolution may be found.

- All kinds of derivatives e.g. forward, backward and central, including the derivatives on time-scales can be expressed as double limit of $\mathbf{P}_b^m(x)$ extending the results of [20].

- Introducing the definitions of the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_m$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m$

$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n - k)^r$$

$$\left\{ \begin{matrix} n \\ r \end{matrix} \right\}_m = \sum_{k=0}^{n-1} \mathbf{A}_{m,r} k^r (n - k)^r$$

the novel identities can be reached, for example

$$\begin{bmatrix} 2t + 1 \\ 1 \end{bmatrix}_m = \begin{bmatrix} t + 2 \\ 2 \end{bmatrix}_m$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \begin{bmatrix} n \\ n - k \end{bmatrix}_m$$

$$\begin{bmatrix} 2t - 3r \\ r \end{bmatrix}_m = \begin{bmatrix} t \\ 2r \end{bmatrix}_m = \begin{bmatrix} 2t - 3r \\ 2t - 4r \end{bmatrix}_m$$

so that combinatorial sense of above is also a topic to research.

- Contribute new OEIS sequences related to $\begin{bmatrix} n \\ k \end{bmatrix}_m$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_m$.

- Following the results of https://arxiv.org/pdf/1603.02468v15.pdf, the equation (1.12) approximates the odd-power polynomial x^{2m+1} around given points x_i as it may be observed from the following plots
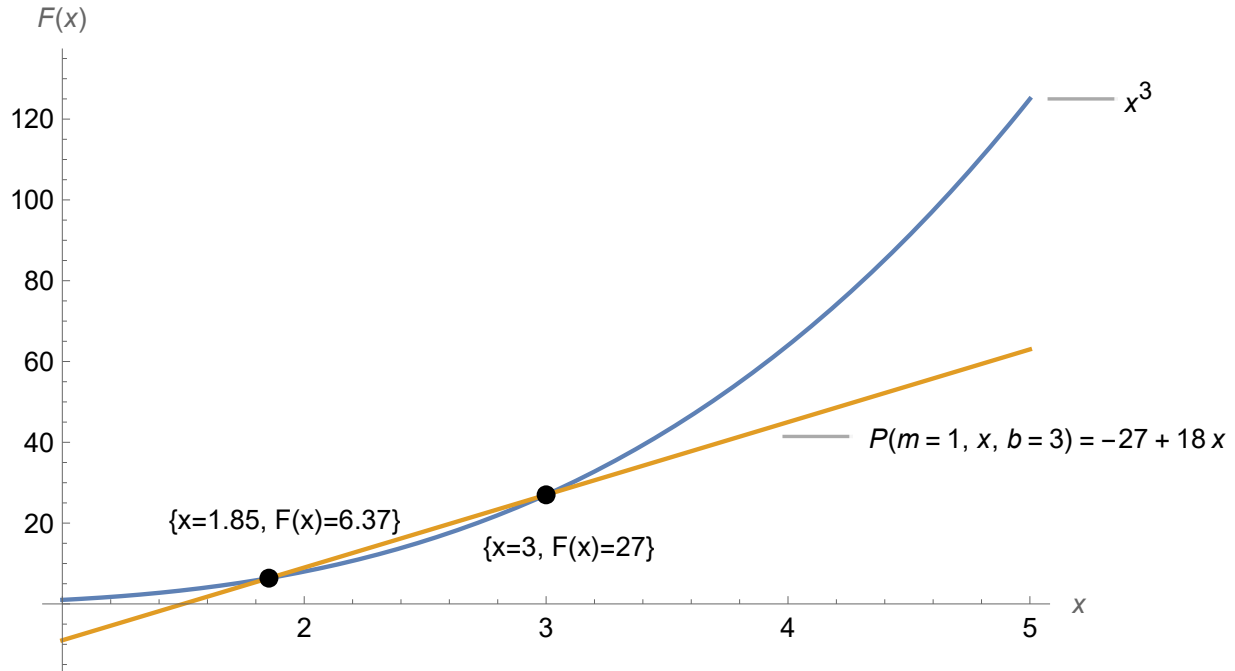
Figure 2. Approximation of x^3.
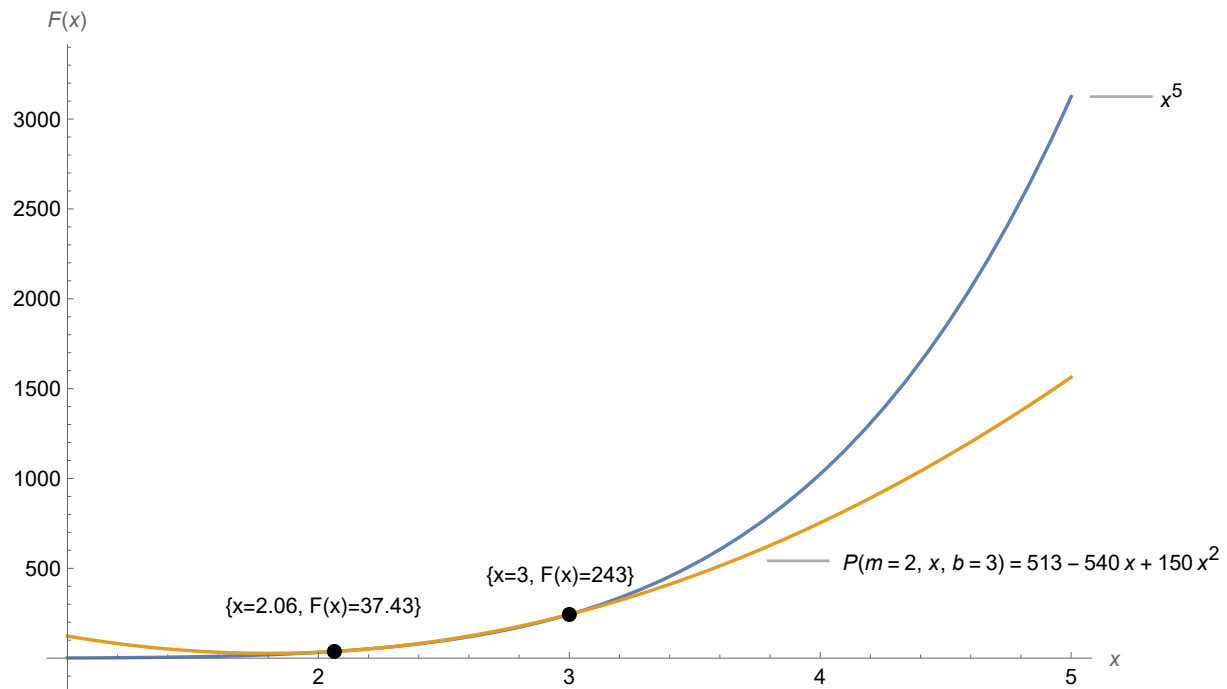
Figure 3. Approximation of x^5.

- English grammar reviews and improvements are welcome.

- Improvements and suggestions to current manuscript under open-source initiatives at https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

4. Conclusions

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

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5. Addendum 1: Examples of the polynomial $\mathbf{P}_b^m(x)$

$\mathbf{P}_b^0(x) = b$

$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x$

$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2$

$\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x$

$\qquad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3$

$\mathbf{P}_b^4(x) = -60b^2 + 180b^3 - 294b^5 + 420b^7 - 315b^8 + 70b^9 + 60bx - 270b^2x + 735b^4x - 1470b^6x$

$\qquad + 1260b^7x - 315b^8x + 90bx^2 - 630b^3x^2 + 1890b^5x^2 - 1890b^6x^2 + 540b^7x^2 + 210b^2x^3$

$\qquad - 1050b^4x^3 + 1260b^5x^3 - 420b^6x^3 - 21bx^4 + 210b^3x^4 - 315b^4x^4 + 126b^5x^4$

$\mathbf{P}_b^5(x) = -693b^2 + 2068b^3 - 330b^4 - 2640b^5 + 2772b^7 - 2310b^9 + 1386b^{10} - 252b^{11} + 693bx$

$\qquad - 3102b^2x + 660b^3x + 6600b^4x - 9702b^6x + 10395b^8x - 6930b^9x + 1386b^{10}x + 1034bx^2$

$\qquad - 330b^2x^2 - 5940b^3x^2 + 12936b^5x^2 - 18480b^7x^2 + 13860b^8x^2 - 3080b^9x^2 + 2310b^2x^3$

$\qquad - 8085b^4x^3 + 16170b^6x^3 - 13860b^7x^3 + 3465b^8x^3 - 330bx^4 + 2310b^3x^4 - 6930b^5x^4$

$\qquad + 6930b^6x^4 - 1980b^7x^4 - 231b^2x^5 + 1155b^4x^5 - 1386b^5x^5 + 462b^6x^5$

$\mathbf{P}_b^6(x) = -10920b^2 + 33306b^3 - 9009b^4 - 36036b^5 + 37752b^7 - 22022b^9 + 12012b^{11} - 6006b^{12} + 924b^{13}$

$\qquad + 10920bx - 49959b^2x + 18018b^3x + 90090b^4x - 132132b^6x + 99099b^8x - 66066b^{10}x + 36036b^{11}x$

$\qquad - 6006b^{12}x + 16653bx^2 - 9009b^2x^2 - 84084b^3x^2 + 180180b^5x^2 - 180180b^7x^2 + 150150b^9x^2$

$\qquad - 90090b^{10}x^2 + 16380b^{11}x^2 + 36036b^2x^3 - 120120b^4x^3 + 168168b^6x^3 - 180180b^8x^3$

$\qquad + 120120b^9x^3 - 24024b^{10}x^3 - 6006bx^4 + 40040b^3x^4 - 84084b^5x^4 + 120120b^7x^4 - 90090b^8x^4$

$\qquad + 20020b^9x^4 - 6006b^2x^5 + 21021b^4x^5 - 42042b^6x^5 + 36036b^7x^5 - 9009b^8x^5 + 286bx^6$

$\qquad - 2002b^3x^6 + 6006b^5x^6 - 6006b^6x^6 + 1716b^7x^6$

6. Addendum 2: Derivation of the coefficients $\mathbf{A}_{m,r}$

Consider the definition (1.11) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$
\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\[2mm] \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\[2mm] 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}
$$

Therefore, let be a definition of the real coefficient $T(d,r)$

Definition 6.1. *Real coefficient $T(d,r)$*

$$
T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}
$$

Example 6.2. *Let be $m=2$ so first we get $\mathbf{A}_{2,2}$*

$$
\mathbf{A}_{2,2} = 5\binom{4}{2} = 30
$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$
\mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) = \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0)
$$

$$
= 30 \cdot \frac{1}{30} = 1
$$

Example 6.3. *Let be $m=3$ so that first we get $\mathbf{A}_{3,3}$*

$$
\mathbf{A}_{3,3} = 7\binom{6}{3} = 140
$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$
\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14
$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d \geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 6.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 6.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d \geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\mathbf{A}_{5,0} = \sum_{d \geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0)$$

$$= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1$$

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