HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

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Abstract. The polynomial $\mathbf{P}_b^m(x)$ is a $2m+1$ degree polynomial in $(x,b) \in \mathbb{R}$ defined by a certain polynomial identity for odd-powers x^{2m+1}.

Contents

1. History and evolution of the polynomial $\mathbf{P}_b^m(x)$

Back than, in 2016 being a student of faculty of mechanical engineering, I remember myself playing with finite differences of polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite differences?

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Sources: https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

The answer to this question is definitely *Yes*, utilizing the interpolation principles. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

That time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I thought. That mindset truly inspired me so that my own mathematical journey has been started. Let us begin considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed via summation over n, e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6\cdot 0 \\
\Delta(1^3) &= 1 + 6\cdot 0 + 6\cdot 1 \\
\Delta(2^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 \\
\Delta(3^3) &= 1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 \\
&\vdots
\end{aligned} \tag{1.1}$$

Finally reaching the general one form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{1.2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then ours relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{1.3}$$

However, as it said, I was not the experienced one mathematician back than, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have rearranged the first order finite differences from the table (1) using (1.1) to get the polynomial n^3

$$\begin{aligned} n^3 = {} & [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\ & + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \end{aligned}$$

Then, combining above equation under the summation in terms of $(n-k)$

$$\begin{aligned} n^3 = {} & n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \cdots \\ & \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \end{aligned}$$

Hence, we have successfully reached interpolation of the polynomial n^3

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.4}$$

It is immediately seen that (1.4) holds by observing the table of $6k(n-k)+1$ values

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4]. Sequences such that row sums give the polynomials n^5 and n^7 are also registered in OEIS [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Well, if the relation (1.4) true for the polynomial n^3, then is it true that (1.4) can be generalized for higher powers, e.g. for n^4 or n^5 either?

That was my next question, however without any expectation of the form of generalized relation. Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials n^5, n^7 and n^9 obtained similarly as (1.4). In short, it appeared that relation (1.4) could be generalized for any odd-power $2m+1$ solving certain system of linear equations. It was proposed that case for n^5 has explicit form as

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C \right]$$

where A, B, C are unknown real coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ respectively so that we reach the form of compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Now the potential form generalized of odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we construct the system of linear equations. In its explicit form

$$\begin{aligned} n^5 &= \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2 \end{aligned}$$

Expanding the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$, we get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3) \right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5) \right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true arranging the terms $30k^2(n-k)^2+1$ over $0 \leq k \leq n$ as tabular. See OEIS sequence [5].

Therefore, the relation (1.4) we got previously via interpolation of the polynomial n^3 can be generalized for all odd-powers $2m+1$ by constructing and solving certain system of linear equations, and its generalized form to be

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r(n-k)^r \tag{1.5}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the equation (1.5) is discussed separately in [9, 10].

However, constructing and solving systems of linear equations each time for every odd-power $2m+1$ requires huge effort, there must be a function that generates the set of real coefficients $\mathbf{A}_{m,r}$ for each fixed m, I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural $m \geq 0,\ 0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate $\mathbf{A}_{m,r}$ recursively, taking base case $\mathbf{A}_{m,m}$ evaluating next $\mathbf{A}_{m,m-1}$ via backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula [2]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to note that summation bound is p while binomial coefficient upper bound is $p+1$. It means that we cannot omit summation bounds letting j run over infinity, unless we do some trick as

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} = \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$$
$$= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$$

So that now we consider the derivation of coefficients $\mathbf{A}_{m,r}$. Using the Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}$ we get

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r}$$
$$= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right]$$
$$= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right]$$
$$= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}$$
$$= \sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}$$
$$= \sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}$$

Now, we notice that

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } j=0; \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1}, & \text{if } j>0. \end{cases} \tag{1.6}$$

An elegant proof of the binomial identity (1.6) is presented in [12]. In particular, the equation (1.6) is zero for $0 < t \le j$. So that taking $j=0$ we have

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \left[\sum_{j\ge 1}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right]$$
$$- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

Now let's simplify the double summation applying the identity (1.6)

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j}\binom{r}{2r-j+1} B_j n^{2r+1-j}\right]}_{(\star)}$$

$$- \underbrace{\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r-j+1$ to $(\star)$ and $\ell = r-t$ to $(\diamond)$ we collapse the common terms of the above equation so that we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell}\binom{r}{\ell}\frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

Using the definition of $\mathbf{A}_{m,r}$, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r}\sum_{\text{odd } \ell}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_r \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r}\sum_{d}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_r \mathbf{A}_{m,r}\left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r}\left[\sum_{d}\frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0 \tag{1.7}$$

Taking the coefficient of n^{2m+1} in (1.7), we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$ we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}$$

Continue similarly we can express $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \le r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Finally, the coefficient $\mathbf{A}_{m,r}$ is defined recursively as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ (2r+1)\binom{2r}{r}\sum_{d\ge 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \le r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases} \tag{1.8}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 3. Coefficients $\mathbf{A}_{m,r}$.

The nominators and denominators of the coefficients $\mathbf{A}_{m,r}$ are also registered as sequences in OEIS [14, 15], respectively. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$

give powers of 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Therefore, we have successfully generalized previously obtained equation (1.4) for each natural $m \geq 0$

Theorem 1.1. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.8).

Theorem (1.1) can be considered as succeeded interpolation of the polynomial x^{2m+1} over the domains $n \in \mathbb{R},\ m \in \mathbb{N}$.

Finally, we got our road to the main definition of the polynomial $\mathbf{P}_b^m(x)$. Introducing the parameter b to the upper summation bound to (1.1), we have the definition

Definition 1.2. *(Polynomial $\mathbf{P}_b^m(x)$of degree $2m+1$.)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.9}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.8), $m \in \mathbb{N}$ and $(x, b) \in \mathbb{R}$. A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ as well as its properties can be found at [16]. In 2023, Albert Tkaczyk yet again extended the theorem (1.1) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17].

2. Related works

In this section let's give a short overview of related works that are based onto definition of polynomials $\mathbf{P}_b^m(x)$. In [18] is given a relation in terms of partial differential differential equations such that ordinary derivative of odd-power $2m+1$ can be reached in terms of partial derivatives of $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$

of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u,u)$ evaluate in point (u,u) plus partial derivative $(f_v)'_z(u,u)$ evaluate in point (u,u)

$$\frac{d}{dx} g_v(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{2.1}$$

where $f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$. Afterward, the equation (2.1) is generalized over the timescales $\mathbb{T} \times \mathbb{T}$ providing its dynamic equation analog in [19].

Second article [20] gives another perspective of ordinary derivatives of polynomials expressing them via double limit as

$$\lim_{h \to 0} \mathbf{P}_{x+h}^m(x) = x^{2m+1}$$

that opens such opportunity.

In [21] based on (1.7), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.

Few sequences were contributed to OEIS [22, 23, 24] showing the coefficients of the polynomial $\mathbf{P}_b^m(x)$ having fixed points m, b while $x \in \mathbb{R}$.

3. Future research and activities

- Differential equation (2.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [19] could be generalized further.
- Theorem (1.1) gives an opportunity to express odd-power identity in terms of multiplication of certain matrices.
- There are Taylor series and Maclaurin series versions in terms of $\mathbf{P}_b^m(x)$.
- The summation bounds of definition (1.9) can be altered so that k runs over $1 \le k \le b$, by symmetry.
- Prove that $\mathbf{P}_b^m(x)$ is an integer valued polynomial in (x, b).

- Definition (1.9) is closely related to discrete convolution because

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

where $\sum_{k=0}^{b-1} k^r (x-k)^r$ is the discrete convolution of x^r. It is worth to get a closer look at it so that new relations in terms of discrete convolution may be found.

- All kinds of derivatives e.g forward, backward and central, including the derivatives on time-scales can be expressed as double limit of $\mathbf{P}_b^m(x)$ extending the results of [20].

- Introducing the definition of coefficient $\left\llbracket {m,n \atop k} \right\rrbracket$

$$\left\llbracket {m,n \atop k} \right\rrbracket = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

the novel identities can be reached, for example

$$\left\llbracket {m, 2t+1 \atop 1} \right\rrbracket = \left\llbracket {m, t+2 \atop 2} \right\rrbracket$$
$$\left\llbracket {m, n \atop k} \right\rrbracket = \left\llbracket {m, n \atop n-k} \right\rrbracket$$
$$\left\llbracket {m, 2t-3r \atop r} \right\rrbracket = \left\llbracket {m, t \atop 2r} \right\rrbracket = \left\llbracket {m, 2t-3r \atop 2t-4r} \right\rrbracket$$

so that combinatorial sense of above is also a topic to research.

- Following the results of `https://arxiv.org/pdf/1603.02468v15.pdf`, the equation (1.9) approximates the odd-power polynomial x^{2m+1} around given points x_i as it may be observed from the following plots

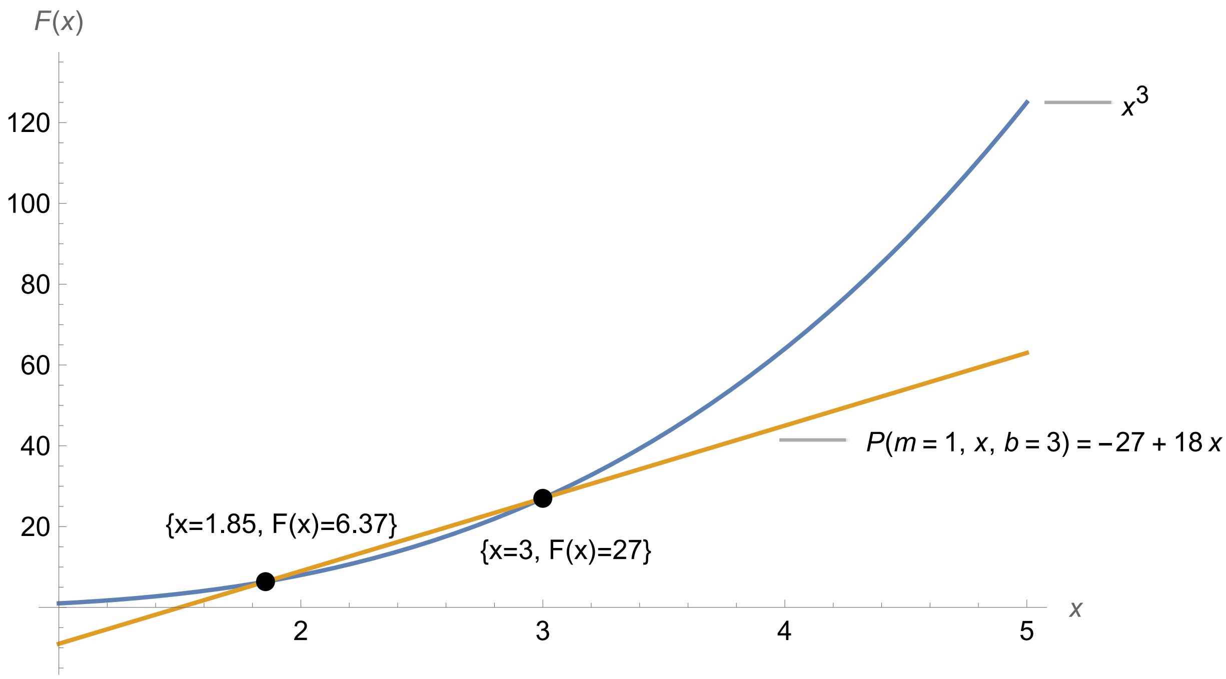

Figure 1. Approximation of x^3.

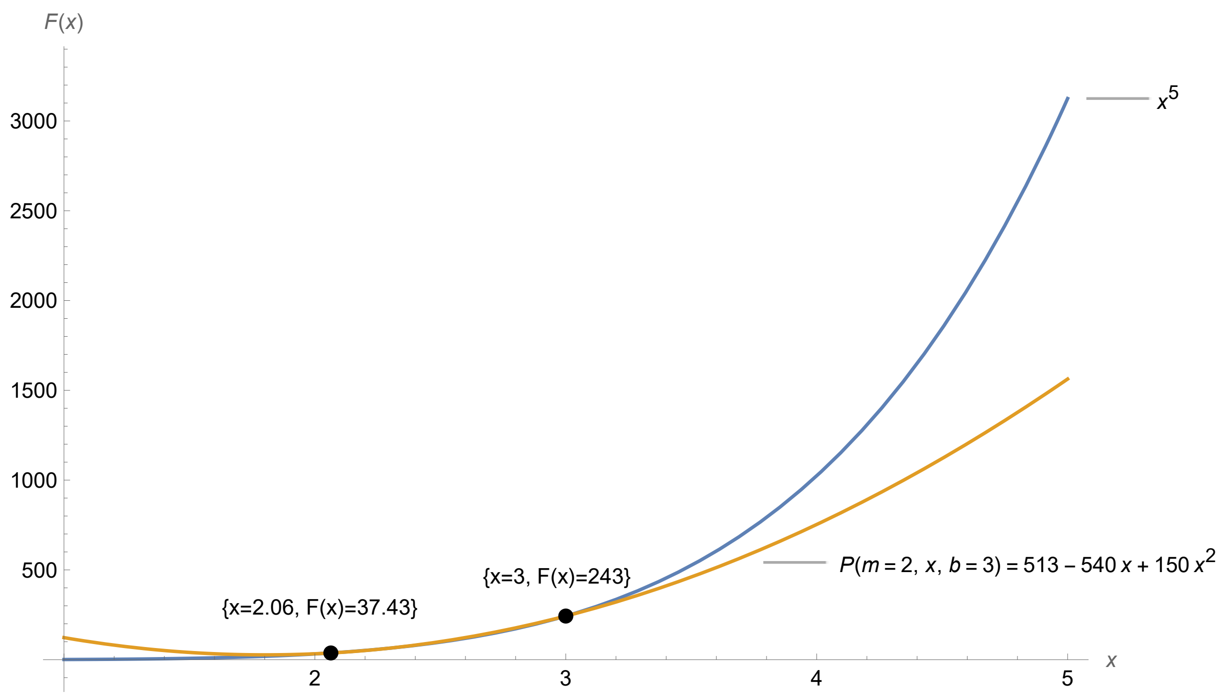

Figure 2. Approximation of x^5.

- English grammar reviews and improvements are welcome.

- Improvements and suggestions to current manuscript under open-source initiatives at https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

4. Conclusions

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

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5. Addendum 1: Examples of the polynomial $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^0(x) = b$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2$$

$$\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x$$
$$+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3$$

$$\mathbf{P}_b^4(x) = -60b^2 + 180b^3 - 294b^5 + 420b^7 - 315b^8 + 70b^9 + 60bx - 270b^2x + 735b^4x - 1470b^6x$$
$$+ 1260b^7x - 315b^8x + 90bx^2 - 630b^3x^2 + 1890b^5x^2 - 1890b^6x^2 + 540b^7x^2 + 210b^2x^3$$
$$- 1050b^4x^3 + 1260b^5x^3 - 420b^6x^3 - 21bx^4 + 210b^3x^4 - 315b^4x^4 + 126b^5x^4$$

$$\mathbf{P}_b^5(x) = -693b^2 + 2068b^3 - 330b^4 - 2640b^5 + 2772b^7 - 2310b^9 + 1386b^{10} - 252b^{11} + 693bx$$
$$- 3102b^2x + 660b^3x + 6600b^4x - 9702b^6x + 10395b^8x - 6930b^9x + 1386b^{10}x + 1034bx^2$$
$$- 330b^2x^2 - 5940b^3x^2 + 12936b^5x^2 - 18480b^7x^2 + 13860b^8x^2 - 3080b^9x^2 + 2310b^2x^3$$
$$- 8085b^4x^3 + 16170b^6x^3 - 13860b^7x^3 + 3465b^8x^3 - 330bx^4 + 2310b^3x^4 - 6930b^5x^4$$
$$+ 6930b^6x^4 - 1980b^7x^4 - 231b^2x^5 + 1155b^4x^5 - 1386b^5x^5 + 462b^6x^5$$

$$\mathbf{P}_b^6(x) = -10920b^2 + 33306b^3 - 9009b^4 - 36036b^5 + 37752b^7 - 22022b^9 + 12012b^{11} - 6006b^{12} + 924b^{13}$$
$$+ 10920bx - 49959b^2x + 18018b^3x + 90090b^4x - 132132b^6x + 99099b^8x - 66066b^{10}x + 36036b^{11}x$$
$$- 6006b^{12}x + 16653bx^2 - 9009b^2x^2 - 84084b^3x^2 + 180180b^5x^2 - 180180b^7x^2 + 150150b^9x^2$$
$$- 90090b^{10}x^2 + 16380b^{11}x^2 + 36036b^2x^3 - 120120b^4x^3 + 168168b^6x^3 - 180180b^8x^3$$
$$+ 120120b^9x^3 - 24024b^{10}x^3 - 6006bx^4 + 40040b^3x^4 - 84084b^5x^4 + 120120b^7x^4 - 90090b^8x^4$$
$$+ 20020b^9x^4 - 6006b^2x^5 + 21021b^4x^5 - 42042b^6x^5 + 36036b^7x^5 - 9009b^8x^5 + 286bx^6$$
$$- 2002b^3x^6 + 6006b^5x^6 - 6006b^6x^6 + 1716b^7x^6$$

6. Addendum 2: Derivation of the coefficients $\mathbf{A}_{m,r}$

Consider the definition (1.8) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition of the real coefficient $T(d,r)$

Definition 6.1. *Real coefficient $T(d,r)$*

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 6.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$\begin{aligned} \mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) &= \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0) \\ &= 30 \cdot \frac{1}{30} = 1 \end{aligned}$$

Example 6.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 6.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 6.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d\geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

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