ECON 3818

Expectations

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Expectations

Outline

Expectation

- Definition
- Properties

Variance

- Definiton
- Properties

What are Expectations?

We are familiar with expectations. For example,

- What salary can I expect to earn after graduating from college?
- I will not invest in Facebook because I expect its sock price to fall in the future
- If you complete your homework, you should expect to do well on exams

In statistics, the idea of expectation has a formal, mathematical definition.

Most of econometrics revolves around finding the best *conditional expectation* (we'll discuss this concept later in the course)

Expectation: Discrete Case

First we give the mathematical definition of expectation for discrete random variables.

Let X be a discrete random variable with pmf $p_X(x)$.

The **expectation** of X, denoted $E(X)$ or μ, is:

$$E(X) = \sum_{x \in S} x \cdot p_X(x),$$

where S is the sample space and x is a realization of the random variable X

The expectation is essentially the **weighted average** of the possible outcomes of X, or the *mean*.

Example of Discrete Expectation

Suppose we roll a six-sided *unfair* die. The probability of each outcome is provided below:

X	P_X(X)
1	0.1
2	0.1
3	0.1
4	0.1
5	0.5
6	0.1

What is the expected outcome of a single throw? By definition:

$$
\begin{aligned}
E(X) &= \sum_{x=1}^{6} x \cdot p_x(x) \\
&= (1 \cdot 0.1) + (2 \cdot 0.1) + (3 \cdot 0.1) + (4 \cdot 0.1) + (5 \cdot 0.5) + (6 \cdot 0.1) \\
&= 4.1
\end{aligned}
$$

Clicker Question -- Midterm Example

Assume X can only obtain the values $1, 2, 3,$ and 5. Given the following PMF, what is $E(X)$?

Unfair Die

X	P_X(X)
1	0.1
2	0.2
3	0.2
5	?

a. 1.1

b. 3.6

c. 3.1

d. Cannot be determined from information

Expectation: Continuous Case

How would you define an expectation of a continuous random variable?

Let Y be the continuous random variable with pdf $f_Y(u)$. Then the expectation of Y is

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy$$

This is still a weighted average! y is the value you observe and $f_Y(y)$ is the density which is just the relative likelihood or *weight*.

Don't forget to multiply by y!

Example of Continuous Expectation

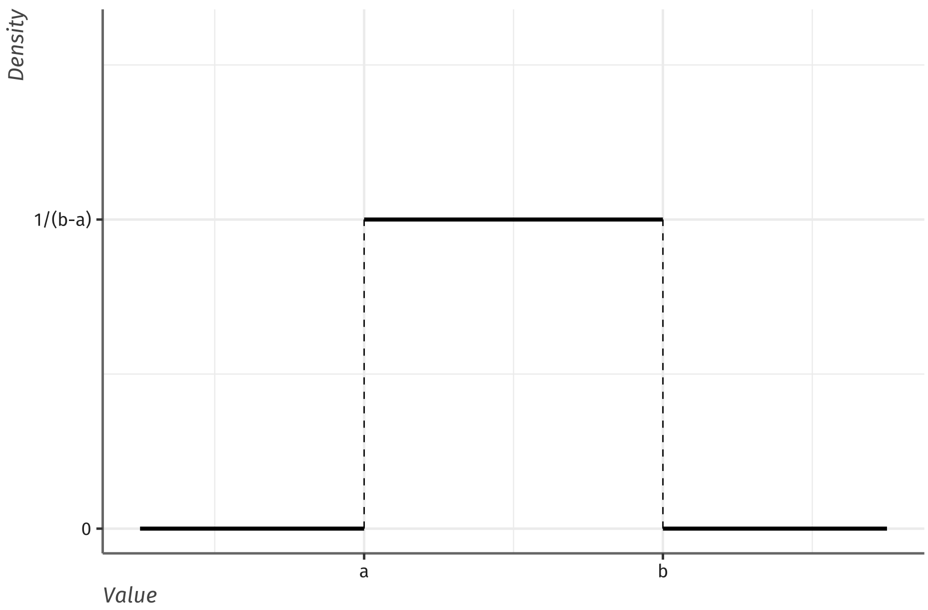
Suppose Y is a continuous random variable with the pdf $f_Y(y) = 3y^2$ for $0 < y < 1$. Then:

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_0^1 y \cdot 3y^2\, dy = \int_0^1 3y^3\, dy = \frac{3}{4}y^4 \Big|_0^1 = \frac{3}{4}$$

Clicker Question

Suppose X has a uniform distribution over (a, b) denoted

$X \sim U(a, b)$. That is, $f_x(x) = \frac{1}{b-a}$ for $a < x < b$. What is $E(X)$?



a. $\frac{b-a}{2}$
b. $\frac{b+a}{2}$
c. $\frac{1}{b-a}$
d. 1

Midterm Example

Consider the probability distribution for random variable X:

$$f(x) = .08x,\ 0 \leq x \leq 5$$

Find $E[X]$

Properties of Expectation

A trick that will make our lives easier is that expectations are a linear operator which means you can move constants in and out of the E function.

Let X, Y be random variables and a, b be constants. Then:

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(X + b) = E(X) + b$
- $E(aX + b) = aE(X) + b$
- $E(X + Y) = E(X) + E(Y)$

Clicker Question

Suppose that the expected income in Boulder is \$90,000 and the tax rate is 10%. Then the post-tax income is Y=0.9X where X is annual income of a Boulder resident. What is the expected post-tax income for a Boulder resident, E(Y)?

a. \$8,000
b. \$90,000
c. \$9,000
d. \$81,000

Definition of Variance

Recall the variance of a sample is

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

We can use expectations to define a similar expression for the population variance.

Let X be a random variable with $E(X) < \infty$. Then the **variance** of X is:

$$Var(X) = E[(X - E(X))^2]$$

Imagine writing this out as an integral and solving it. Would not be fun. *Instead, we will be using a much simpler equation to calculate variance...*

Alternate Form

Note that we can write,

$$\begin{aligned} Var(X) = E[(X - E(X))^2] &= E[X^2 - 2X \cdot E(X) + [E(X)]^2] \\ &= E(X^2) - 2E(X) \cdot E(X) + [E(X)]^2 \\ &= E(X^2) - 2[E(X)]^2 + [E(X)]^2 \\ &= E(X^2) - [E(X)]^2, \end{aligned}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

is a much simpler expression of variance (and the one you should use!)

Properties of Expectation

We can compose E with a function of a random variable, g(X).

$$E[g(X)] = \sum_{x \in S} g(x) \cdot p_x(x)$$

The result is the expected value of g(X). We will use this property to calculate $E[X^2]$

Example

While $E(X)$ is called the first moment, we call $E(X^2)$ the second moment of X.

Therefore, we calculate $E[X^2]$ using the following equation:

$$E[X^2] = \sum x^2 * p_X(x)$$

or in the continuous case

$$\int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Example

Let X be the number of heads in two coin flips.

$$E(X) = 0 \cdot (\frac{1}{4}) + 1 \cdot (\frac{1}{2}) + 2 \cdot (\frac{1}{4}) = \frac{1}{2} + \frac{1}{2} = 1$$

$$E(X^2) = 0^2 \cdot (\frac{1}{4}) + 1^2 \cdot (\frac{1}{2}) + 2^2 \cdot (\frac{1}{4}) = \frac{1}{2} + 1 = \frac{3}{2}^{*}$$

This means that

$$V(X) = E(X^2) - [E(X)]^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$$

*__Never__ square probabilities when solving (E(X^2)). Only square (X) values

Properties of Variance

There are several important properties of the variance operator:

- $Var(a) = 0$
- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \cdot Cov(X, Y)$.*

* We will discuss more about covariance later, when `(X)` and `(Y)` are independent then the covariance is zero

Clicker Question

Recall that in Boulder the mean income is $90,000 and the tax rate is 10%. Suppose that the variance of income in Boulder is $1,000. What is the variance of post-tax income, $Y = 0.9X$?

a. $900

b. $1000

c. $810

d. $800

Midterm Example

Consider the probability distribution for random variable Y

$$f(x) = .08x, 0 \leq x \leq 5$$

Find $V[X]$:

$$V[X] = \int_{-\infty}^{\infty} x^2 * f(x)\, dx - \left[\int_{-\infty}^{\infty} x * f(x)\, dx\right]^2$$

$$= \int_0^5 x^2 * .08x\, dx - \left[\int_0^5 x * .08x\, dx\right]^2 = \int_0^5 .08x^3\, dx - \left[\int_0^5 .08x^2\, dx\right]^2$$

$$= \frac{.08x^4}{4}\Big|_0^5 - \left(\frac{.08x^3}{3}\Big|_0^5\right)^2 = 12.5 - (3.3)^2 = 1.61$$

Midterm Example

Consider the probability distribution for random variable X:

X	P_X(X)
2	0.19
4	0.07
6	0.35
8	p

1. Find the expectation of X
2. Find the variance of X
3. A friend says, the expected value of X is 5. To justify this he says, "Well X could be 2,4,6, or 8, so the average is $\frac{2+4+6+8}{4} = 5$". Why is your friend wrong? Why isn't $E[X] = 5$? (1-2) sentences.
4. Let $Y \sim N(4,1)$. Random variable W is defined as $W = 2X + 3Y$. Given your information about the random variables X and Y. What is $E[W]$? What is $Var[W]$? (assuming X & Y are independent) }

Midterm Example

Consider the probability distribution for random variable Y:

$$f(y) = 8y, 0 \leq y \leq \frac{1}{2}$$

1. Find $E[Y]$
2. Find $P(Y < \frac{1}{3})$
3. Find $P(Y = \frac{1}{4})$
4. Find $V[Y]$
5. Find $P(\frac{3}{4} < Y < 1)$