



# ECON 3818

## Expectations

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Expectations

# Outline

## Expectation

- Definition
- Properties

## Variance

- Definiton
- Properties

# What are Expectations?

We are familiar with expectations. For example,

- What salary can I expect to earn after graduating from college?
- I will not invest in Facebook because I expect its stock price to fall in the future
- If you complete your homework, you should expect to do well on exams

In statistics, the idea of expectation has a formal, mathematical definition.

Most of econometrics revolves around finding the best *conditional expectation* (we'll discuss this concept later in the course)

# Expectation: Discrete Case

First we give the mathematical definition of **expectation** for discrete random variables.

Let  $X$  be a discrete random variable with pmf  $p_X(x)$ .

The **expectation** of  $X$ , denoted  $E(X)$  or  $\mu$ , is:

$$E(X) = \sum_{x \in S} x \cdot p_X(x),$$

where  $S$  is the sample space and  $x$  is a realization of the random variable  $X$

The expectation is essentially the **weighted average** of the possible outcomes of  $X$ , or the *mean*.

# Example of Discrete Expectation

Suppose we roll a six-sided *unfair* die. The probability of each outcome is provided below:

$x$	$p_x(x)$
1	0.1
2	0.1
3	0.1
4	0.1
5	0.5
6	0.1

What is the expected outcome of a single throw? By definition:

$$\begin{aligned} E(X) &= \sum_{x=1}^6 x \cdot p_x(x) \\ &= (1 \cdot 0.1) + (2 \cdot 0.1) + (3 \cdot 0.1) + (4 \cdot 0.1) + (5 \cdot 0.5) + (6 \cdot 0.1) \\ &= 4.1 \end{aligned}$$

# Clicker Question -- Midterm Example

Assume  $X$  can only obtain the values 1, 2, 3, and 5. Given the following PMF, what is  $E(X)$ ?

**Unfair Die**

$x$	$P_X(x)$
1	0.1
2	0.2
3	0.2
5	?

- a. 1.1
- b. 3.6
- c. 3.1
- d. Cannot be determined from information

# Expectation: Continuous Case

How would you define an expectation of a continuous random variable?

Let  $Y$  be the continuous random variable with pdf  $f_Y(u)$ . Then the expectation of  $Y$  is

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy$$

This is still a weighted average!  $y$  is the value you observe and  $f_Y(y)$  is the density which is just the relative likelihood or *weight*.

*Don't forget to multiply by  $y$ !*



# Example of Continuous Expectation

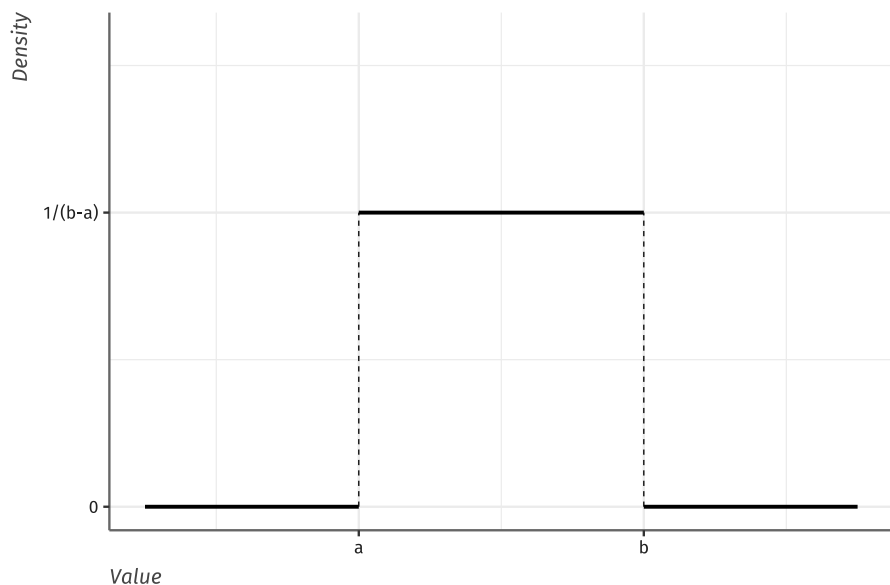
Suppose  $Y$  is a continuous random variable with the pdf  $f_Y(y) = 3y^2$  for  $0 < y < 1$ . Then:

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) \, dy = \int_0^1 y \cdot 3y^2 \, dy = \int_0^1 3y^3 \, dy = \left. \frac{3}{4}y^4 \right|_0^1 = \frac{3}{4}$$

# Clicker Question

Suppose  $X$  has a **uniform distribution** over  $(a, b)$  denoted

$X \sim U(a, b)$ . That is,  $f_x(x) = \frac{1}{b-a}$  for  $a < x < b$ . What is  $E(X)$ ?



- a.  $\frac{b-a}{2}$
- b.  $\frac{b+a}{2}$
- c.  $\frac{1}{b-a}$
- d. **1**

# Midterm Example

Consider the probability distribution for random variable  $X$ :

$$f(x) = .08x, \quad 0 \leq x \leq 5$$

Find  $E[X]$

# Properties of Expectation

A trick that will make our lives easier is that expectations are a **linear operator** which means you can move constants in and out of the  $E$  function.

Let  $X, Y$  be random variables and  $a, b$  be constants. Then:

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(X + b) = E(X) + b$
- $E(aX + b) = aE(X) + b$
- $E(X + Y) = E(X) + E(Y)$

# Clicker Question

Suppose that the expected income in Boulder is \$90,000 and the tax rate is 10%. Then the post-tax income is  $Y=0.9X$  where  $X$  is annual income of a Boulder resident. What is the expected post-tax income for a Boulder resident,  $E(Y)$ ?

- a. \$8,000
- b. \$90,000
- c. \$9,000
- d. \$81,000

# Definition of Variance

Recall the variance of a sample is

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

We can use expectations to define a similar expression for the population variance.

Let  $X$  be a random variable with  $E(X) < \infty$ . Then the **variance** of  $X$  is:

$$\text{Var}(X) = E[(X - E(X))^2]$$

Imagine writing this out as an integral and solving it. Would not be fun.  
*Instead, we will be using a much simpler equation to calculate variance...*

# Alternate Form

Note that we can write,

$$\begin{aligned} \text{Var}(X) &= E[(X - E(X))^2] = E[X^2 - 2X \cdot E(X) + [E(X)]^2] \\ &= E(X^2) - 2E(X) \cdot E(X) + [E(X)]^2 \\ &= E(X^2) - 2[E(X)]^2 + [E(X)]^2 \\ &= E(X^2) - [E(X)]^2, \end{aligned}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

is a much simpler expression of variance (and the one you should use!)

# Properties of Expectation

We can compose  $E$  with a function of a random variable,  $g(X)$ .

$$E[g(X)] = \sum_{x \in S} g(x) \cdot p_x(x)$$

The result is the expected value of  $g(X)$ . We will use this property to calculate  $E[X^2]$



# Example

While  $E(X)$  is called the **first moment**, we call  $E(X^2)$  the **second moment** of  $X$ .

Therefore, we calculate  $E[X^2]$  using the following equation:

$$E[X^2] = \sum x^2 * p_X(x)$$

or in the continuous case

$$\int_{-\infty}^{\infty} x^2 f_X(x) dx$$

# Example

Let  $X$  be the number of heads in two coin flips.

$$E(X) = 0 \cdot \left(\frac{1}{4}\right) + 1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$E(X^2) = 0^2 \cdot \left(\frac{1}{4}\right) + 1^2 \cdot \left(\frac{1}{2}\right) + 2^2 \cdot \left(\frac{1}{4}\right) = \frac{1}{2} + 1 = \frac{3}{2}^*$$

This means that

$$V(X) = E(X^2) - [E(X)]^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$$

\***Never** square probabilities when solving  $E(X^2)$ . Only square  $(X)$  values

# Properties of Variance

There are several important properties of the variance operator:

- $Var(a) = 0$
- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \cdot Cov(X, Y).$ <sup>\*</sup>

<sup>\*</sup> We will discuss more about covariance later, when  $(X)$  and  $(Y)$  are independent then the covariance is zero

# Clicker Question

Recall that in Boulder the mean income is \$90,000 and the tax rate is 10%. Suppose that the variance of income in Boulder is \$1,000. What is the variance of post-tax income,  $Y = 0.9X$ ?

- a. \$900
- b. \$1000
- c. \$810
- d. \$800

# Midterm Example

Consider the probability distribution for random variable  $Y$

$$f(x) = .08x, 0 \leq x \leq 5$$

Find  $V[X]$ :

$$\begin{aligned} V[X] &= \int_{-\infty}^{\infty} x^2 * f(x) dx - \left[ \int_{-\infty}^{\infty} x * f(x) dx \right]^2 \\ &= \int_0^5 x^2 * .08x dx - \left[ \int_0^5 x * .08x dx \right]^2 = \int_0^5 .08x^3 dx - \left[ \int_0^5 . \right. \\ &= \frac{.08x^4}{4} \Big|_0^5 - \left( \frac{.08x^3}{3} \Big|_0^5 \right)^2 = 12.5 - (3.3)^2 = 1.61 \end{aligned}$$

# Midterm Example

Consider the probability distribution for random variable  $X$ :

$x$	$P_X(x)$
2	0.19
4	0.07
6	0.35
8	$p$

1. Find the expectation of  $X$
2. Find the variance of  $X$
3. A friend says, the expected value of  $X$  is 5. To justify this he says, "Well  $X$  could be 2,4,6, or 8, so the average is  $\frac{2+4+6+8}{4} = 5$ ". Why is your friend wrong? Why isn't  $E[X] = 5$ ? (1-2) sentences.
4. Let  $Y \sim N(4, 1)$ . Random variable  $W$  is defined as  $W = 2X + 3Y$ . Given your information about the random variables  $X$  and  $Y$ . What is  $E[W]$ ? What is  $Var[W]$ ? (assuming  $X$  &  $Y$  are independent) }

# Midterm Example

Consider the probability distribution for random variable  $Y$ :

$$f(y) = 8y, 0 \leq y \leq \frac{1}{2}$$

1. Find  $E[Y]$
2. Find  $P(Y < \frac{1}{3})$
3. Find  $P(Y = \frac{1}{4})$
4. Find  $V[Y]$
5. Find  $P(\frac{3}{4} < Y < 1)$