NBER Summer Institute Econometrics Methods Lecture:

GMM and Consumption-Based Asset Pricing

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GMM and Consumption-Based Models: Themes

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- Themes are important in choosing *which* methods to use.

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- Consumption-based asset pricing: concluding thoughts

- Economic model implies set of r population moment restrictions

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GMM Review (Hansen, 1982)

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- $\mathbf{w}_t$ is an $h \times 1$ vector of variables known at t
- $\boldsymbol{\theta}$ is an $a \times 1$ vector of coefficients
- Idea: choose $\boldsymbol{\theta}$ to make the sample moment as close as possible to the population moment.

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- Sample moments:

$$\underbrace{\mathbf{g}(\boldsymbol{\theta};\mathbf{y}_T)}_{(r\times 1)} \equiv (1/T)\sum_{t=1}^{T}\mathbf{h}\left(\boldsymbol{\theta},\mathbf{w}_t\right),$$

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$$Q\left(\boldsymbol{\theta};\boldsymbol{y}_{\mathbf{T}}\right) = \underset{(1\times r)}{[\mathbf{g}(\boldsymbol{\theta};\mathbf{y}_T)]'}\underset{(r\times r)}{\mathbf{W}_T}\underset{(r\times 1)}{[\mathbf{g}(\boldsymbol{\theta};\boldsymbol{y}_T)]}, \tag{1}$$

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- $\{\mathbf{W}_T\}_{T=1}^{\infty}$ a sequence of $r \times r$ positive definite matrices which may be a function of the data, $\mathbf{y}_T$.
- If $r = a$, $\boldsymbol{\theta}$ estimated by setting each $\mathbf{g}(\boldsymbol{\theta};\mathbf{y}_T)$ to zero.
- GMM refers to use of (1) to estimate $\boldsymbol{\theta}$ when $r > a$.

- Asym. properties (Hansen 1982): $\widehat{\theta}$ consistent, asym. normal.

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- Optimal weighting $\mathbf{W}_T = \mathbf{S}^{-1}$

$$\underset{r\times r}{\mathbf{S}} = \sum_{j=-\infty}^{\infty} E\left\{ \underbrace{\left[\mathbf{h}\left(\boldsymbol{\theta}_o, \mathbf{w}_t\right)\right]}_{r\times 1} \underbrace{\left[\mathbf{h}\left(\boldsymbol{\theta}_o, \mathbf{w}_{t-j}\right)\right]'}_{1\times r} \right\}.$$

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- In many asset pricing applications, it is inappropriate to use $\mathbf{W}_T = \mathbf{S}^{-1}$ (see below).

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1. Obtain an initial estimate of $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}_T^{(1)}$, by minimizing $Q(\boldsymbol{\theta}; \boldsymbol{y}_{\mathbf{T}})$ subject to arbitrary weighting matrix, e.g., $\mathbf{W} = \mathbf{I}$.

2. Use $\widehat{\boldsymbol{\theta}}_T^{(1)}$ to obtain initial estimate of $\mathbf{S} = \widehat{\mathbf{S}}_T^{(1)}$.

3. Re-minimize $Q(\boldsymbol{\theta}; \boldsymbol{y}_{\mathbf{T}})$ using initial estimate $\widehat{\mathbf{S}}_T^{(1)}$; obtain new estimate $\widehat{\boldsymbol{\theta}}_T^{(2)}$.

4. Continue iterating until convergence, or stop. (Estimators have same asym. dist. but finite sample properties differ.)

Classic Example: Hansen and Singleton (1982)

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$$\begin{cases} u\left(C_t\right) = \frac{C_t^{1-\gamma}}{1-\gamma} & \gamma > 0 \\ u\left(C_t\right) = \ln(C_t) & \gamma = 1 \end{cases} \tag{2}$$

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- N assets $=>$ N first-order conditions

$$C_t^{-\gamma} = \beta E_t \left\{ \left(1 + \Re_{i,t+1}\right) C_{t+1}^{-\gamma} \right\} \qquad i = 1, ..., N. \tag{3}$$

Asset Pricing Example: Hansen and Singleton (1982)

- Re-write moment conditions

$$0 = E_t \left\{ 1 - \beta \left[\left(1 + \Re_{i,t+1}\right) \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right] \right\}. \tag{4}$$

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- $\mathbf{x}_t \subset \mathbf{x}_t^*$. Conditional model (5) $=>$ unconditional model:

$$0 = E \left\{ \left[1 - \left\{ \beta \left(1 + \Re_{i,t+1}\right) \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right\} \right] \mathbf{x}_t \right\} \qquad i = 1, \ldots N \tag{6}$$

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- Let $\mathbf{x}_t$ be $M \times 1$. Then $r = N \cdot M$ and,

$$\underset{r\times 1}{\mathbf{h}\left(\boldsymbol{\theta}, \mathbf{w}_t\right)} = \begin{bmatrix} \left[1-\beta\left\{\left(1+\Re_{1,t+1}\right)\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\right\}\right]\mathbf{x}_t \\ \left[1-\beta\left\{\left(1+\Re_{2,t+1}\right)\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\right\}\right]\mathbf{x}_t \\ \cdot \\ \cdot \\ \cdot \\ \left[1-\beta\left\{\left(1+\Re_{N,t+1}\right)\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\right\}\right]\mathbf{x}_t \end{bmatrix} \quad (7)$$

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- Take sample mean of (7) to get $\mathbf{g}(\boldsymbol{\theta}; \mathbf{y}_T)$, minimize

$$\min_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}; \mathbf{y}_T) = \underbrace{[\mathbf{g}(\boldsymbol{\theta}; \mathbf{y}_T)]'}_{1\times r} \underset{r\times r}{\mathbf{W}_T} \underbrace{[\mathbf{g}(\boldsymbol{\theta}; \boldsymbol{y}_T)]}_{r\times 1}$$

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- Model cannot capture predictable variation in *excess returns* over commercial paper $\Rightarrow$
- Researchers have turned to other models of preferences.

More GMM results: Euler Equation Errors

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- Let $M_{t+1} = \beta(C_{t+1}/C_t)^{-\gamma}$. Define Euler equation errors:

$$e_R^j \equiv E[M_{t+1}R_{t+1}^j] - 1$$

$$e_X^j \equiv E[M_{t+1}(R_{t+1}^j - R_{t+1}^f)]$$

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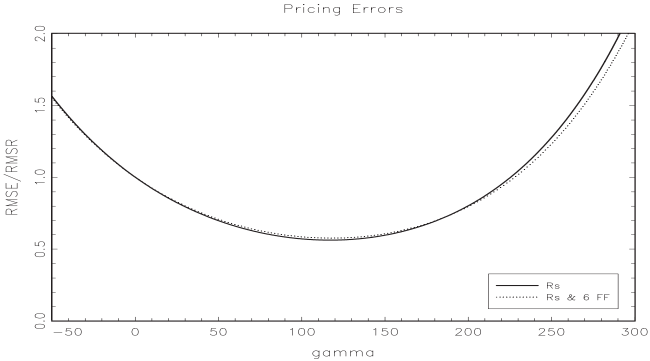
- Choose params: $\min_{\beta,\gamma} \mathbf{g}_T' \mathbf{W}_T \mathbf{g}_T$ where jth element of $\mathbf{g}_T$

$$g_{j,t}(\gamma,\beta) = \frac{1}{T}\sum_{t=1}^{T} e_{R,t}^j$$

$$g_{j,t}(\gamma) = \frac{1}{T}\sum_{t=1}^{T} e_{X,t}^j$$

Unconditional Euler Equation Errors, Excess Returns

- $e_X^j \equiv E[\beta(C_{t+1}/C_t)^{-\gamma}(R_{t+1}^j - R_{t+1}^f)] \qquad j = 1, ..., N$
- $RMSE = \sqrt{\frac{1}{N}\sum_{j=1}^N [e_X^j]^2}, \quad RMSR = \sqrt{\frac{1}{N}\sum_{j=1}^N [E(R_{t+1}^j - R_{t+1}^f)]^2}$



Source: Lettau and Ludvigson (2009). *Rs* is the excess return on CRSP-VW index over 3-Mo T-bill rate. *Rs* & 6 FF refers to this return plus 6 size and book-market sorted portfolios provided by Fama and French. For each value of γ, β is chosen to minimize the Euler equation error for the T-bill rate. U.S. quarterly data, 1954:1-2002:1.

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- Anomaly is striking b/c early evidence (e.g., Hansen & Singleton) that the classic model's Euler equations were violated provided the impetus for developing these newer models.
- Results imply data on consumption and asset returns not jointly lognormal!

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- Consider two estimated models of SDF, e.g.,
  1. CCAPM: $M_{t+1}^{(1)} = \beta(C_{t+1}/C_t)^{-\gamma}$, OID restricts not rejected
  2. CAPM: $M_{t+1}^{(2)} = a + bR_{m,t+1}$, OID restricts rejected

GMM Asset Pricing With Non-Optimal Weighting

Comparing specification error: Hansen and Jagannathan, 1997

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- May we conclude Model 1 is superior?
- No. Hansen's J-test of OID restricts depends on model specific $\mathbf{S}$: $J = \mathbf{g}_T' \mathbf{S}^{-1} \mathbf{g}_T$.

- Asset pricing applications often require $\mathbf{W}_T \neq \mathbf{S}^{-1}$. Why?
- One reason: assessing specification error, comparing models.
- Consider two estimated models of SDF, e.g.,
  1. CCAPM: $M_{t+1}^{(1)} = \beta(C_{t+1}/C_t)^{-\gamma}$, OID restricts not rejected
  2. CAPM: $M_{t+1}^{(2)} = a + bR_{m,t+1}$, OID restricts rejected
- May we conclude Model 1 is superior?
- No. Hansen's J-test of OID restricts depends on model specific $\mathbf{S}$: $J = \mathbf{g}_T'\mathbf{S}^{-1}\mathbf{g}_T$.
- Model 1 can look better simply b/c the SDF and pricing errors $\mathbf{g}_T$ are more volatile, not b/c pricing errors are lower.

- HJ: compare models $M_t(\boldsymbol{\theta})$ using distance metric:

$$\text{Dist}_T(\boldsymbol{\theta}) = \sqrt{\min_{\boldsymbol{\theta}} \mathbf{g}_T(\boldsymbol{\theta})' \, \mathbf{G}_T^{-1} \mathbf{g}_T(\boldsymbol{\theta})}, \qquad \mathbf{G}_T \equiv \frac{1}{T}\sum_{t=1}^{T} \underbrace{R_t R_t'}_{N\times N}$$

$$\mathbf{g}_T(\boldsymbol{\theta}) \equiv \frac{1}{T}\sum_{t=1}^{T} \left[M_t(\boldsymbol{\theta}) \mathbf{R}_t - \mathbf{1}_N \right]$$

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GMM Asset Pricing With Non-Optimal Weighting

Comparing specification error: Hansen and Jagannathan, 1997

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- Dist_T is a measure of model misspecification:
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  - Gives maximum pricing error of any portfolio formed from the N assets.

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- Important problem: how to compare HJ distances statistically?
- One possibility developed in Chen and Ludvigson (2009): White's reality check method.

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1. Take benchmark model, e.g., model with smallest squared distance $d_{1,T}^2 \equiv \min\{d_{j,T}^2\}_{j=1}^K$.
2. Null: $d_{1,T}^2 - d_{2,T}^2 \leq 0$, where $d_{2,T}^2$ is competing model with the next smallest squared distance.
3. Test statistic $\mathcal{T}^W = \sqrt{T}(d_{1,T}^2 - d_{2,T}^2)$.
4. If null is true, test statistic should not be unusually large, given sampling error.
5. Given distribution for $\mathcal{T}^W$, reject null if historical value $\widehat{\mathcal{T}}^W$ is $>$ 95th percentile.

GMM Asset Pricing With Non-Optimal Weighting

Statistical comparison of HJ distance: Chen and Ludvigson, 2009

- Chen and Ludvigson (2009) compare HJ distances among K competing models using White's reality check method.
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  5. Given distribution for $\mathcal{T}^W$, reject null if historical value $\widehat{\mathcal{T}}^W$ is $>$ 95th percentile.
- Method applies generally to **any stationary** law of motion for data, **multiple** competing possibly **nonlinear**, SDF models.

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  - $\mathcal{T}^W$ has complicated limiting distribution.
- Bootstrap works only if have a multivariate, joint, continuous, limiting distribution under null.
- Proof of limiting distributions exists for applications to most asset pricing models:
  - For parametric models (Hansen, Heaton, Luttmer '95)
  - For semiparametric models (Ai and Chen '07).

GMM Asset Pricing With Non-Optimal Weighting

Reasons to use identity matrix: econometric problems

- Econometric problems: near singular $\mathbf{S}_T^{-1}$ or $\mathbf{G}_T^{-1}$.
    - Asset returns are highly correlated.
    - We have large N and modest T.
    - If $T < N$ covariance matrix for N asset returns is singular. Unless $T >> N$, matrix can be near-singular.

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- Using $\mathbf{W}_T = \mathbf{S}_T^{-1}$ or $\mathbf{G}_T^{-1}$ same as using $\mathbf{W}_T = \mathbf{I}$ and doing GMM on *re-weighted portfolios* of original test assets.
  - Triangular factorization $\mathbf{S}^{-1} = (\mathbf{P}'\mathbf{P})$, $\mathbf{P}$ lower triangular

$$\min \mathbf{g}_T' \mathbf{S}^{-1} \mathbf{g}_T \Leftrightarrow (\mathbf{g}_T' \mathbf{P}') \mathbf{I} (\mathbf{P} \mathbf{g}_T)$$

GMM Asset Pricing With Non-Optimal Weighting

Reasons to use identity matrix: economically interesting portfolios

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- May imply implausible long and short positions in test assets.

GMM Asset Pricing With Non-Optimal Weighting

Reasons not to use $\mathbf{W}_T = \mathbf{I}$: objective function dependence on test asset choice

- Using $\mathbf{W}_T = [E_T(\mathbf{R}'\mathbf{R})]^{-1}$, GMM objective function is invariant to initial choice of test assets.

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- Form a portfolio, $\mathbf{AR}$ from initial returns $\mathbf{R}$. (Note, portfolio weights sum to 1 so $\mathbf{A1}_N = \mathbf{1}_N$).

$$
\begin{aligned}
& \left[E\left(M\mathbf{R}\right) - \mathbf{1}_N\right]' E\left(\mathbf{R}\mathbf{R}'\right)^{-1}\left[E\left(M\mathbf{R} - \mathbf{1}_N\right)\right] \\
= \quad & \left[E\left(M\mathbf{A}\mathbf{R}\right) - \mathbf{A1}_N\right]' E\left(\mathbf{A}\mathbf{R}\mathbf{R}'\mathbf{A}\right)^{-1}\left[E\left(M\mathbf{A}\mathbf{R} - \mathbf{A1}_N\right)\right].
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$$

- With $\mathbf{W}_T = \mathbf{I}$ or other fixed weighting, GMM objective depends on choice of test assets.

More Complex Preferences: Scaled Consumption-Based Models

- Consumption-based models may be approximated:

$$M_{t+1} \approx a_t + b_t \Delta c_{t+1}, \qquad c_{t+1} \equiv \ln(C_{t+1})$$

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- Example: Classic CCAPM with CRRA utility

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \quad \Rightarrow \quad M_{t+1} \approx \underbrace{\beta}_{a_t = a_0} - \underbrace{\beta\gamma}_{b_t = b_0} \Delta c_{t+1}$$

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- Model with habit and time-varying risk aversion: Campbell and Cochrane '99, Menzly et. al '04

$$u(C_t, S_t) = \frac{(C_t S_t)^{1-\gamma}}{1-\gamma}, \qquad S_{t+1} \equiv \frac{C_t - X_t}{C_t}$$

$$\Rightarrow M_{t+1} \approx \beta \Big(\underbrace{1 - \gamma(\phi - 1)(s_t - \bar{s})}_{a_t} - \underbrace{\gamma(1 + \psi(s_t))}_{b_t} \Delta c_{t+1} \Big)$$

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- Proxies for time-varying risk-premia should be good proxies for time-variation in a_t and b_t.

Scaled Consumption-Based Models

- $M_{t+1} \approx a_t + b_t \Delta c_{t+1}$
- Empirical specification: Lettau and Ludvigson (2001a, 2001b):
  - $a_t = a_0 + a_1 z_t, \quad b_t = b_0 + b_1 z_t$
  - $z_t = cay_t \equiv c_t - \alpha_a a_t - \alpha_y y_t$, (cointegrating residual)
  - cay_t related to log consumption-(aggregate) wealth ratio.
  - cay_t strong predictor of excess stock market returns

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- Other examples: including housing consumption

$$U(C_t, H_t) = \frac{\widetilde{C}_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \qquad \widetilde{C}_t = \left[\chi C_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\chi) H_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

$$\Rightarrow \ln M_{t+1} \approx a_t + b_t \Delta \ln C_{t+1} + d_t \Delta \ln S_{t+1}, \qquad S_{t+1} \equiv \frac{p_t^C C_t}{p_t^C C_t + p_t^H H_t}$$

- Lustig and Van Nieuwerburgh '05 (incomplete markets):
  - $a_t = a_0 + a_1 z_t, \quad b_t = b_0 + b_1 z_t, \quad d_t = d_0 + d_1 z_t$
  - $z_t =$ housing collateral ratio (measures quantity of risk sharing)

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$$\begin{aligned} M_{t+1} &= \mathbf{b}_t' \mathbf{f}_{t+1} \text{ with } \mathbf{b}_t = \mathbf{b}_0 + \mathbf{b}_1 \mathbf{z}_t \\ &= \mathbf{b}'\left(\mathbf{f}_{t+1} \otimes (1\ \mathbf{z}_t)'\right) \end{aligned}$$

Scaled Consumption-Based Models

Distinguishing two types of conditioning, or state dependence

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- Scaled consumption-based models have been *tested* on unconditional moments, $E\left[M_{t+1}R_{t+1}\right] = 1$
  - $\Rightarrow$ NO scaled *returns*.

- Scaled CCAPM turns a *single* factor model with *state-dependent* weights into *multi-factor* model $\mathbf{f_t}$ with *constant* weights:

$$\begin{aligned} M_{t+1} &= (a_0 + a_1 z_t) + (b_0 + b_1 z_t)\,\Delta \ln C_{t+1} \\ &= a_0 + a_1 \underbrace{z_t}_{f_{1,t+1}} + b_0 \underbrace{\Delta \ln C_{t+1}}_{f_{2,t+1}} + b_1 (\underbrace{z_t \Delta \ln C_{t+1}}_{f_{3,t+1}}) \end{aligned}$$

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- Scaled consumption models have *multiple, constant* betas for each factor, rather than a single time-varying beta for $\Delta \ln C_{t+1}$.

Deriving the "beta"-representation

Let $\mathbf{F} = (1 \;\; \mathbf{f}')'$, $M = \mathbf{b}'\mathbf{F}$, ignore time indices

$$\begin{aligned} 1 &= E[MR^i] \\ &= E[R^i\mathbf{F}']\mathbf{b} \quad \Rightarrow \quad \text{unconditional moments} \\ &= E[R^i]E[\mathbf{F}']\mathbf{b} + Cov(R^i, \mathbf{F}')\mathbf{b} \Rightarrow \end{aligned}$$

$$\begin{aligned} E[R^i] &= \frac{1 - Cov(R^i, \mathbf{F}')\mathbf{b}}{E[\mathbf{F}']\mathbf{b}} \\ &= \frac{1 - Cov(R^i, \mathbf{f}')\overline{\mathbf{b}}}{E[\mathbf{F}']\mathbf{b}} \\ &= \frac{1 - Cov(R^i, \mathbf{f}')Cov(\mathbf{f}, \mathbf{f}')^{-1}Cov(\mathbf{f}, \mathbf{f}')\overline{\mathbf{b}}}{E[\mathbf{F}']\mathbf{b}} \\ &= R^0 - R^0\boldsymbol{\beta}'Cov(\mathbf{f}, \mathbf{f}')\overline{\mathbf{b}} \\ &= R^0 - \boldsymbol{\beta}'\boldsymbol{\lambda} \quad \Rightarrow \quad \text{multiple, constant betas} \end{aligned}$$

- Estimate cross-sectional model using Fama-MacBeth (see Brandt lecture).

Fama-MacBeth Methodology: Preview–See Brandt

Step 1: Estimate β's in time-series regression for each portfolio i:

$$\boldsymbol{\beta}_i \equiv Cov(\mathbf{f}_{t+1}, \mathbf{f}'_{t+1})^{-1} Cov(\mathbf{f}_{t+1}, R_{i,t+1})$$

Step 2: Cross-sectional regressions (T of them):

$$R_{i,t+1} - R_{0,t} = \alpha_{i,t} + \boldsymbol{\beta}'_i \boldsymbol{\lambda}_t$$

$$\boldsymbol{\lambda} = 1/T \sum_{t=1}^{T} \boldsymbol{\lambda}_t; \qquad \sigma^2(\boldsymbol{\lambda}) = 1/T \sum_{t=1}^{T} (\boldsymbol{\lambda}_t - \boldsymbol{\lambda})^2$$

Note: report Shanken t-statistics (corrected for estimation error of betas in first stage)

Scaled Consumption-Based Models

Distinguishing two types of conditioning, or state dependence

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not in Euler equation: $E(M\mathbf{R}) = \mathbf{1}_N$.

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- Unlikely the same time-varying beta as obtained from modeling conditional mean $E_t(M_{t+1}\mathbf{R}_{t+1}) = 1$.

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- Partial solution: summarize information in large number of time-series with few estimated dynamic factors (e.g., Ludvigson and Ng '07, '09).

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- Bottom lines:

  1. Conditional moments of $M_{t+1}\mathbf{R}_{t+1}$ difficult to model $\Rightarrow$ reason to focus on unconditional moments $E[M_{t+1}\mathbf{R}_{t+1}] = 1$.
  2. Models are misspecified: interesting question is whether state-dependence of M_{t+1} on consumption growth $\Rightarrow$ less misspecification than standard, fixed-weight CCAPM.
  3. As before, can compare models on basis of HJ distances, using White "reality check" method to compare statistically.

Asset Pricing Models With Recursive Preferences

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- Two reasons recursive utility is of interest:
  - More flexibility as regards attitudes toward risk and intertemporal substitution.
  - Preferences deliver an added risk factor for explaining asset returns.
- But, only a small amount of econometric work on recursive preferences $\Rightarrow$ gap in the literature.

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- In (1) DGP is left *unrestricted*, as is joint distribution of consumption and returns (distribution-free estimation procedure).

- In (2) DGP *and* distribution of shocks explicitly modeled.

- Recursive utility (Epstein, Zin ('89, '91) & Weil ('89)):

$$V_t = \left[(1-\beta) C_t^{1-\rho} + \beta \mathcal{R}_t \left(V_{t+1} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}}$$

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- Special case: $\rho = \theta \Rightarrow$ CRRA separable utility $V_t = \beta \frac{C_t^{1-\theta}}{1-\theta}$.

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  where $R_{w,t}$ *proxied* by stock market return.
- Problem: $R_{w,t+1}$ represents a claim to future C_t, itself unobservable.

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- Points to need for estimation method feasible under less restrictive assumptions.

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- Semiparametric approach is **sieve minimum distance** (SMD) procedure.

- First order conditions for optimal consumption choice:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}}{\mathcal{R}_t\left(\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)}\right)^{\rho-\theta}R_{i,t+1}-1\right]=0 \tag{8}$$

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- N test asset returns, $\{R_{i,t+1}\}_{i=1}^{N}$. (9) is a x-sect asset pricing model.
- Moment restrictions (9) form the basis of empirical investigation.
- Empirical model is *semiparametric*: $\delta \equiv (\beta, \theta, \rho)'$ denote finite dimensional parameter vector; V_t/C_t unknown function.

- Assume $\frac{V_t}{C_t}$ an unknown function F: $\mathbb{R}^2 \to \mathbb{R}$ of form

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- With a nonlinear Markov process for x_t, $F(\cdot)$ can display *nonmonotonicities* in both arguments.

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- **Note**: Markov assumption only a *motivation* for arguments of $F(\cdot)$. Econometric methodology itself leaves LOM for $\Delta \ln C_t$ unspecified.

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- With GMM we average and then square.
- With SMD, we square and then average.

- True parameters δ_o and $F_o(\cdot, \delta_o)$ solve:

$$\min_{\delta \in \mathcal{D}} \inf_{F \in \mathcal{V}} E\left[m(\mathbf{w}_t, \delta, F)' m(\mathbf{w}_t, \delta, F)\right],$$

- where $m(\mathbf{w}_t, \delta, F) = E\{\gamma(\mathbf{z}_{t+1}, \delta, F) | \mathbf{w}_t\}$
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- It is clear that $F_o(\mathbf{z}_t, \delta_o) = F^*(\mathbf{z}_t, \delta_o)$

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EZW Recursive Preferences: Two-Step Procedure

Unrestricted Dynamics, Distribution-Free Estimation: Chen, Favilukis, Ludvigson '07

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- Second step: estimates of δ_o is obtained by solving a sample minimum distance problem such as GMM.

- Approximate $\frac{V_t}{C_t} = F\left(\frac{V_{t-1}}{C_{t-1}}, \frac{C_t}{C_{t-1}}; \delta\right)$ with a bivariate sieve:

$$F\left(\frac{V_{t-1}}{C_{t-1}}, \frac{C_t}{C_{t-1}}; \delta\right) \approx F_{K_T}(\cdot, \delta) = a_0(\delta) + \sum_{j=1}^{K_T} a_j(\delta) B_j\left(\frac{V_{t-1}}{C_{t-1}}, \frac{C_t}{C_{t-1}}\right)$$

- Sieve coefficients $\{a_0, a_1, ..., a_{K_T}\}$ depend on δ
- Basis functions $\{B_j(\cdot, \cdot) : j = 1, ..., K_T\}$ have known functional forms independent of δ
- Initial value for $\frac{V_t}{C_t}$ at time $t = 0$, denoted $\frac{V_0}{C_0}$, taken as a unknown scalar parameter to be estimated.
- Given $\frac{V_0}{C_0}$, $\{a_j\}_{j=1}^{K_T}$, $\{B_j\}_{j=1}^{K_T}$ and data on consumption $\left\{\frac{C_t}{C_{t-1}}\right\}_{t=1}^{T}$, use F_{K_T} to generate a sequence $\left\{\frac{V_i}{C_i}\right\}_{i=1}^{T}$.

- Recall $m(\mathbf{w}_t, \delta_o, F^*(\cdot, \delta_o)) \equiv E\left\{\gamma(\mathbf{z}_{t+1}, \delta_o, F^*\left(\cdot, \delta_o\right)) | \mathbf{w}_t\right\} = 0.$

- First-step SMD estimate $\widehat{F}\left(\cdot\right)$ for $F^*\left(\cdot\right)$ based on

$$\widehat{F}\left(\cdot, \delta\right) = \arg\min_{F_{K_T}} \frac{1}{T}\sum_{t=1}^{T} \widehat{m}(\mathbf{w}_t, \delta, F_{K_T}(\cdot, \delta))'\widehat{m}(\mathbf{w}_t, \delta, F_{K_T}(\cdot, \delta)),$$

- $\widehat{m}(\mathbf{w}_t, \delta, F_{K_T}(\cdot, \delta))$ any nonpara. estimator of m.

- Do this for a three dimensional grid of values of $\delta = (\beta, \theta, \rho)'$.

- Example of nonparametric estimator of m:
- Let $\{p_{0j}(\mathbf{w}_t),\ j = 1, 2, ..., J_T\}$, $\mathbb{R}^{d_w} \to \mathbb{R}$ be *instruments*. $p^{J_T}(\cdot) \equiv (p_{01}(\cdot), ..., p_{0J_T}(\cdot))'$
- Define $T \times J_T$ matrix $\mathbf{P} \equiv \left(p^{J_T}(w_1), ..., p^{J_T}(w_T)\right)'$. Then:

$$\widehat{m}(\mathbf{w}, \delta, F) = \left(\sum_{t=1}^{T} \gamma(\mathbf{z}_{t+1}, \delta, F) p^{J_T}(\mathbf{w}_t)' (\mathbf{P}'\mathbf{P})^{-1}\right) p^{J_T}(\mathbf{w})$$

- $\widehat{m}(\cdot)$ a sieve LS estimator of $m(\mathbf{w}, \delta, F)$.
- Procedure equivalent to regressing each γ_i on instruments and taking fitted values as estimate of conditional mean.

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where $\mathbf{y}^T = \left(\mathbf{z}'_{T+1}, ...\mathbf{z}'_2, \mathbf{w}'_T, ...\mathbf{w}'_1\right)'$ denotes vector of all obs and

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- Weighting can be understood intuitively by noting that variation in conditional mean $m(\mathbf{w}_t, \delta, F)$ is what identifies $F^*(\cdot, \delta)$.

EZW Preferences: Second Step GMM Est of δ_o

Unrestricted Dynamics, Distribution-Free Estimation: Chen, Favilukis, Ludvigson '07

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  3. Linear combinations may imply implausible long and short positions, do not necessarily deliver a large spread in unconditional mean returns.

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- Test whether HJ distances of competing models are statistically different (White reality check–Chen and Ludvigson '09).

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- Cash flow dynamics in BGT version of LRR model:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_{c,t} + \sigma_t \varepsilon_{c,t+1} \\
\Delta d_{t+1} &= \mu_d + \phi_x \underbrace{x_{c,t}}_{\text{LR risk}} + \phi_s s_t + \sigma_{\varepsilon_d} \sigma_t \varepsilon_{d,t+1} \\
x_{c,t} &= \phi x_{c,t-1} + \sigma_{\varepsilon_x} \sigma \varepsilon_{xc,t} \\
\sigma_t^2 &= \sigma^2 + \nu(\sigma_{t-1}^2 - \sigma^2) + \sigma_w w_t
\end{aligned}$$

$$s_t = (\mu_d - \mu_c) + d_t - c_t$$

$$\varepsilon_{c,t+1}, \varepsilon_{d,t+1}, \varepsilon_{xc,t}, w_t \sim \mathbf{N.i.i.d}\,(\mathbf{0}, \mathbf{1})$$

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  5. Choose value ρ_d that most closely "matches" moments between dist of simulated and historical data ("match" made precise below.)

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- QMLE estimator of auxiliary model on historical data

$$\tilde{\alpha} = \arg\max_{\alpha} \mathcal{L}_T(\alpha, \{\tilde{y}_t\}_{t=1}^{T})$$

$$\mathcal{L}_T(\alpha, \{\tilde{y}_t\}_{t=1}^{T}) = \frac{1}{T}\sum_{t=L+1}^{T}\ln f(\tilde{y}_t|\tilde{y}_{t-L},...,\tilde{y}_{t-1},\alpha)$$

- First-order-condition:

$$\frac{\partial}{\partial \alpha}\mathcal{L}_T(\tilde{\alpha}, \{\tilde{y}_t\}_{t=1}^T) = 0 \quad \text{or,} \quad \frac{1}{T}\sum_{t=L+1}^{T} s_f(\tilde{y}_t | \tilde{y}_{t-L}, ..., \tilde{y}_{t-1}, \tilde{\alpha}) = 0.$$

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- $\Rightarrow$ use Monte Carlo compute expect. of $s(\cdot)$ under $p(\cdot|\rho_d)$.

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- Thus, if data *do* follow the structural model $p(\cdot|\rho_d)$, then $m(\rho_d^o, \alpha^o) = 0$, forms basis of a specification test.
- Summary: solve model for many values of ρ_d, store long simulations of model each time, do one-time estimation of auxiliary f-model. Choose ρ_d to minimize GMM criterion above.

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- Big issue: are these the economically interesting moments? Regards both choice of moments, and weighting function.

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  - ...models are imperfect and will never fit data infallibly.

  - Argue here for need to move away from testing if models are *true*, towards comparison of models based on *magnitude of misspecification*.

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- Rather than ask whether scaled models are true...
- ...ask whether allowing for *state-dependence* of SDF on consumption growth *reduces misspecification* over the analogous non-state-dependent model.

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- In which case, they will always perform at least as well, *or better than*, mismeasured macro factors from true M_t $\Rightarrow$

- Not sensible to run horse races between financial factor models and macro models.

- Goal: not to find better factors, but rather to *explain* financial factors from deeper economic models.