<u>Problem 1</u>. Unique Minimum Spanning Trees

1. **Show that a graph G=(V, E) has a unique minimum spanning tree T, if, for every cut of G, there is a unique minimum weight edge crossing the cut.**

Answer:

To prove this, we can use some assumption that the graph has 2 distinct MST's G1 and G2.

A minimum weight edge crossing the cut is also called a **Light Edge**.

Let's also assume that we have an edge that is in G1 but not in G2 – (u, v). If we remove that edge (u, v), then we said we cut the tree G1 into two other trees.

Let's refer to the component containing the vertex v as Gv and to the component containing vertex u as Gu. In that case, the edge (u, v) would be the minimum cost edge in the cut Gv-Gu. Now, if we go back and consider the Tree of G2, we know that it has an edge (u1, v1) that is a minimum cost edge of the cut between Gv and Gu. BUT, the edges (u, v) and (u1, v1) are not equal and we must have their costs to be the same. By doing that, we contradict the fact that for every cut of the graph, the edge with the smallest cost across the cut will be unique.

2. **Show the converse it is not true by providing a counterexample.**

Answer:

It is easy to come up with counterexample in this case.

Consider a graph with three nodes { A, B, C } and edges { [4,(A, B)], [4, (A, C)] }. *[weight, (node, node)]*

We see that the graph itself is the only spanning tree, but the cut with A and B,C contain two cost 4 edges, hence the minimum weight edge is not unique in this cut.