Math Notes for Compound Interest

First let's calculate the debt after a time period t given a principal P, an interest rate r, a compound frequency f, and a cyclic payment p.

$$
\begin{aligned}
a &= 1 + \frac{r}{n} \\
D_0 &= P \\
D_1 &= aP - p \\
D_2 &= a(aP - p) - p \\
&= a^2 P - p(a + 1) \\
&\vdots \\
D_{nt} &= a^{nt} P - p \sum_{i=0}^{nt-1} a^i
\end{aligned}
$$

Remember that we can find the solution to the finite exponential series as so

$$
\begin{aligned}
x &= \sum_{i=0}^{nt-1} a^i \\
ax &= \sum_{i=1}^{nt} a^i \\
ax - a^{nt} + 1 &= \sum_{i=0}^{nt-1} a^i \\
ax - a^{nt} + 1 &= x \\
(1 - a)x &= 1 - a^{nt} \\
x &= \frac{1 - a^{nt}}{1 - a}
\end{aligned}
$$

Plugging this in gives us our debt calculation.

$$
\begin{aligned}
D_{nt} &= a^{nt}P - p\frac{1-a^{nt}}{1-a} \\
&= a^{nt}P - p\frac{n(a^{nt}-1)}{r}
\end{aligned}
$$

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$
\begin{aligned}
0 &= D_{nt} \\
0 &= a^{nt}P - p\frac{n(a^{nt}-1)}{r} \\
p &= \frac{ra^{nt}}{n(a^{nt}-1)}P
\end{aligned}
$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$
\begin{aligned}
p_{\min} &= \lim_{t\to\infty} \frac{ra^{nt}}{n(a^{nt}-1)}P \\
&= \frac{r}{n}P
\end{aligned}
$$

We can also calculate how long it will take to pay off all debts.

$$
\begin{aligned}
0 &= a^{nt}P - p\frac{n(a^{nt}-1)}{r} \\
0 &= ra^{nt}P - np(a^{nt}-1) \\
-np &= (rP-np)a^{nt} \\
a^{nt} &= \frac{np}{np-rP} \\
nt &= \frac{\ln np - \ln{(np-rP)}}{\ln a} \\
t &= \frac{\ln np - \ln{(np-rP)}}{n\ln a}
\end{aligned}
$$

We can also take the derivative of t with respect to p to find the optimal trade-off point.

$$
\frac{\partial t}{\partial p} = -\delta
$$

$$\frac{\frac{n}{np} - \frac{n}{np-rP}}{n \ln a} = -\delta$$
$$\frac{\frac{1}{np} - \frac{1}{np-rP}}{\ln a} = -\delta$$
$$\frac{1}{np} - \frac{1}{np-rP} = -\delta \ln a$$
$$np - rP - np = -\delta \ln a np(np - rP)$$
$$-rP = -\delta \ln a np(np - rP)$$
$$n^2p^2 - nrPp = \frac{rP}{\delta \ln a}$$
$$n^2p^2 - nrPp - \frac{rP}{\delta \ln a} = 0$$
$$p = \frac{nrP \pm \sqrt{n^2r^2P^2 + \frac{4n^2rP}{\delta \ln a}}}{2n^2}$$
$$= \frac{rP \pm \sqrt{r^2P^2 + \frac{4rP}{\delta \ln a}}}{2n}$$

If we consider δ to be $\frac{\Delta\text{time to payoff}}{\Delta\text{payment}}$ Then $\delta = \frac{1}{5}$ would represent the point where we have to pay \$5 more for a decrease of 1 year in time to payoff.

We can also consider the case for deriving by the principal.

$$\frac{\partial D_{nt}}{\partial P} = \delta$$
$$a^{nt} = \delta$$
$$p = \frac{ra^{nt}}{n(a^{nt} - 1)}P$$
$$p = \frac{r\delta}{n(\delta - 1)}P$$

Note that we must have $\delta > 1$. In this case $\delta = 5$ would mean a \$1 increase in principal increases the total paid amount by \$5.