Math Notes for Compound Interest

First let's calculate the debt after a time period t given a principal P, an interest rate r, a compound frequency f, and a cyclic payment p.

$$a = 1 + \frac{r}{n}$$

$$D_0 = P$$

$$D_1 = aP - p$$

$$D_2 = a(aP - p) - p$$

$$= a^2P - p(a+1)$$

$$\vdots$$

$$D_{nt} = a^{nt}P - p\sum_{i=0}^{nt-1} a^i$$

Remember that we can find the solution to the finite exponential series as so

$$x = \sum_{i=0}^{nt-1} a^i$$

$$ax = \sum_{i=1}^{nt} a^i$$

$$ax - a^{nt} + 1 = \sum_{i=0}^{nt-1} a^i$$

$$ax - a^{nt} + 1 = x$$

$$(1 - a)x = 1 - a^{nt}$$

$$x = \frac{1 - a^{nt}}{1 - a}$$

Plugging this in gives us our debt calculation.

$$D_{nt} = a^{nt}P - p\frac{1 - a^{nt}}{1 - a}$$
$$= a^{nt}P - p\frac{n(a^{nt} - 1)}{r}$$

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$0 = D_{nt}$$

$$0 = a^{nt}P - p\frac{n(a^{nt} - 1)}{r}$$

$$p = \frac{ra^{nt}}{n(a^{nt} - 1)}P$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$p_{\min} = \lim_{t \to \infty} \frac{ra^{nt}}{n(a^{nt} - 1)} P$$
$$= \frac{r}{n} P$$

We can also calculate how long it will take to pay off all debts.

$$0 = a^{nt}P - p\frac{n(a^{nt} - 1)}{r}$$
$$0 = ra^{nt}P - np(a^{nt} - 1)$$
$$-np = (rP - np)a^{nt}$$
$$a^{nt} = \frac{np}{np - rP}$$
$$nt = \frac{\ln np - \ln (np - rP)}{\ln a}$$
$$t = \frac{\ln np - \ln (np - rP)}{n \ln a}$$

We can also take the derivative of t with respect to p to find the optimal trade-off point.

$$\frac{\partial t}{\partial p} = -\delta$$

$$\frac{\frac{n}{np} - \frac{n}{np-rP}}{n \ln a} = -\delta$$

$$\frac{\frac{1}{np} - \frac{1}{np-rP}}{\ln a} = -\delta$$

$$\frac{1}{np} - \frac{1}{np-rP} = -\delta \ln a$$

$$np - rP - np = -\delta \ln anp(np - rP)$$

$$-rP = -\delta \ln anp(np - rP)$$

$$-rP = -\delta \ln anp(np - rP)$$

$$n^2p^2 - nrPp = \frac{rP}{\delta \ln a}$$

$$n^2p^2 - nrPp - \frac{rP}{\delta \ln a} = 0$$

$$p = \frac{nrP \pm \sqrt{n^2r^2P^2 + \frac{4n^2rP}{\delta \ln a}}}{2n^2}$$

$$= \frac{rP \pm \sqrt{r^2P^2 + \frac{4rP}{\delta \ln a}}}{2n}$$

If we consider δ to be $\frac{\Delta \text{time to payoff}}{\Delta \text{payment}}$ Then $\delta = \frac{1}{5}$ would represent the point where we have to pay \$5 more for a decrease of 1 year in time to payoff.

We can also consider the case for deriving by the principal.

$$\frac{\partial D_{nt}}{\partial P} = \delta$$

$$a^{nt} = \delta$$

$$p = \frac{ra^{nt}}{n(a^{nt} - 1)}P$$

$$p = \frac{r\delta}{n(\delta - 1)}P$$

Note that we must have $\delta > 1$. In this case $\delta = 5$ would mean a \$1 increase in principal increases the total paid amount by \$5.