Math Notes for Compound Interest

1 Relationships

1.1 Debt Owed

First let's calculate the debt after a time period t given a principal P, an interest rate r, a compound frequency f, and a cyclic payment p.

$$
\begin{aligned}
a &= 1 + \frac{r}{n} \\
D_0 &= P \\
D_1 &= aP - p \\
D_2 &= a(aP - p) - p \\
&= a^2P - p(a+1) \\
&\vdots \\
D_{nt} &= a^{nt}P - p\sum_{i=0}^{nt-1} a^i
\end{aligned}
$$

Remember that we can find the solution to the finite exponential series as so

$$
\begin{aligned}
x &= \sum_{i=0}^{nt-1} a^i \\
ax &= \sum_{i=1}^{nt} a^i \\
ax - a^{nt} + 1 &= \sum_{i=0}^{nt-1} a^i
\end{aligned}
$$

$$\begin{aligned}
ax - a^{nt} + 1 &= x \\
(1-a)x &= 1 - a^{nt} \\
x &= \frac{1-a^{nt}}{1-a}
\end{aligned}$$

Plugging this in gives us our debt calculation.

$$\begin{aligned}
D_{nt} &= a^{nt}P - p\frac{1-a^{nt}}{1-a} \\
&= a^{nt}P - p\frac{n(a^{nt}-1)}{r}
\end{aligned}$$

1.2 Payments

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$\begin{aligned}
0 &= D_{nt} \\
0 &= a^{nt}P - p\frac{n(a^{nt}-1)}{r} \\
p &= \frac{ra^{nt}}{n(a^{nt}-1)}P
\end{aligned}$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$\begin{aligned}
p_{\min} &= \lim_{t\to\infty} \frac{ra^{nt}}{n(a^{nt}-1)}P \\
&= \frac{r}{n}P
\end{aligned}$$

1.3 Time

We can also calculate how long it will take to pay off all debts.

$$\begin{aligned}
0 &= a^{nt}P - p\frac{n(a^{nt}-1)}{r} \\
0 &= ra^{nt}P - np(a^{nt}-1) \\
-np &= (rP - np)a^{nt}
\end{aligned}$$

$$
\begin{aligned}
a^{nt} &= \frac{np}{np - rP} \\
nt &= \frac{\ln np - \ln{(np - rP)}}{\ln a} \\
t &= \frac{\ln np - \ln{(np - rP)}}{n \ln a}
\end{aligned}
$$

2 Derivatives

2.1 Time vs Payment

We can also take the derivative of t with respect to p to find the optimal trade-off point.

$$
\begin{aligned}
\frac{\partial t}{\partial p} &= -\delta \\
\frac{\frac{n}{np} - \frac{n}{np-rP}}{n \ln a} &= -\delta \\
\frac{\frac{1}{np} - \frac{1}{np-rP}}{\ln a} &= -\delta \\
\frac{1}{np} - \frac{1}{np - rP} &= -\delta \ln a \\
np - rP - np &= -\delta \ln anp(np - rP) \\
-rP &= -\delta \ln anp(np - rP) \\
n^2p^2 - nrPp &= \frac{rP}{\delta \ln a} \\
n^2p^2 - nrPp - \frac{rP}{\delta \ln a} &= 0 \\
p &= \frac{nrP \pm \sqrt{n^2r^2P^2 + \frac{4n^2rP}{\delta \ln a}}}{2n^2} \\
&= \frac{rP \pm \sqrt{r^2P^2 + \frac{4rP}{\delta \ln a}}}{2n}
\end{aligned}
$$

If we consider δ to be $\frac{\Delta\text{time to payoff}}{\Delta\text{payment}}$ Then $\delta = \frac{1}{5}$ would represent the point where we have to pay \$5 more for a decrease of 1 year in time to payoff.

2.2 Debt Owed vs Principal

We can also consider the case for deriving by the principal.

$$\frac{\partial D_{nt}}{\partial P} = \delta$$
$$a^{nt} = \delta$$
$$p = \frac{ra^{nt}}{n(a^{nt}-1)}P$$
$$p = \frac{r\delta}{n(\delta-1)}P$$

Note that we must have $\delta > 1$. In this case $\delta = 5$ would mean a \$1 increase in principal increases the total paid amount by \$5.

3 Meeting a Specific Return on Investment

Say that we want to only pay a specific amount of return on a debt. For instance we only want to pay \$1200 back on a \$1000 dollar loan. We can say G is what we're willing to pay back, where $G > P$. We then have

$$ntp = G$$
$$nt = \frac{G}{p}$$
$$p = \frac{ra^{G/p}}{n(a^{G/p}-1)}P$$
$$p = \frac{r}{n(1-a^{-G/p})}P$$
$$p - \frac{r}{n(1-a^{-G/p})}P = 0$$

This is easily solvable, but we can use Newton's method to find an answer using a calculator.

$$f(p) = p - \frac{r}{n(1-a^{-G/p})}P$$
$$f'(p) = 1 - \frac{r}{n}\left(-\frac{1}{(1-a^{-G/p})^2}\right)\left(\frac{-\ln(a)a^{-G/p}}{p^2}\right)P$$

$$
\begin{aligned}
&= 1 - \frac{rP\ln(a)a^{-G/p}}{np^2(1-a^{-G/p})^2} \\
&= 1 - \frac{rP\ln(a)}{np^2a^{G/p}\left(\frac{a^{G/p}-1}{a^{Gp}}\right)^2} \\
&= 1 - \frac{rP\ln(a)}{np^2\left(a^{G/p}(a^{G/p}-1)\right)^2} \\
p_{n+1} &= p_n - \frac{f(p_n)}{f'(p_n)}
\end{aligned}
$$

For getting a good p_0 we can note that in the general case G is quite large, allowing to give the approximation.

$$
\begin{aligned}
p_0 &= \lim_{G\to\infty} \frac{r}{n(1-a^{-G/p})}P \\
&= \frac{r}{n}P
\end{aligned}
$$