

Math Notes for Compound Interest

First let's calculate the debt after a time period t given a principal P , an interest rate r , a compound frequency f , and a cyclic payment p .

$$a = 1 + \frac{r}{n}$$

$$D_0 = P$$

$$D_1 = aP - p$$

$$\begin{aligned} D_2 &= a(aP - p) - p \\ &= a^2P - p(a + 1) \end{aligned}$$

$$\vdots$$

$$D_{nt} = a^{nt}P - p \sum_{i=0}^{nt-1} a^i$$

Remember that we can find the solution to the finite exponential series as so

$$x = \sum_{i=0}^{nt-1} a^i$$

$$ax = \sum_{i=1}^{nt} a^i$$

$$ax - a^{nt} + 1 = \sum_{i=0}^{nt-1} a^i$$

$$ax - a^{nt} + 1 = x$$

$$(1 - a)x = 1 - a^{nt}$$

$$x = \frac{1 - a^{nt}}{1 - a}$$

Plugging this in gives us our debt calculation.

$$\begin{aligned} D_{nt} &= a^{nt}P - p \frac{1 - a^{nt}}{1 - a} \\ &= a^{nt}P - p \frac{n(a^{nt} - 1)}{r} \end{aligned}$$

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$\begin{aligned} 0 &= D_{nt} \\ 0 &= a^{nt}P - p \frac{n(a^{nt} - 1)}{r} \\ p &= \frac{ra^{nt}}{n(a^{nt} - 1)}P \end{aligned}$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$\begin{aligned} p_{\min} &= \lim_{t \rightarrow \infty} \frac{ra^{nt}}{n(a^{nt} - 1)}P \\ &= \frac{r}{n}P \end{aligned}$$

We can also calculate how long it will take to pay off all debts.

$$\begin{aligned} 0 &= a^{nt}P - p \frac{n(a^{nt} - 1)}{r} \\ 0 &= ra^{nt}P - np(a^{nt} - 1) \\ -np &= (rP - np)a^{nt} \\ a^{nt} &= \frac{np}{np - rP} \\ nt &= \frac{\ln np - \ln(np - rP)}{\ln a} \\ t &= \frac{\ln np - \ln(np - rP)}{n \ln a} \end{aligned}$$

We can also take the derivative of t with respect to p to find the optimal trade-off point.

$$\frac{\partial t}{\partial p} = -\delta$$

$$\begin{aligned}
\frac{\frac{n}{np} - \frac{n}{np-rP}}{n \ln a} &= -\delta \\
\frac{\frac{1}{np} - \frac{1}{np-rP}}{\ln a} &= -\delta \\
\frac{1}{np} - \frac{1}{np-rP} &= -\delta \ln a \\
np - rP - np &= -\delta \ln a np (np - rP) \\
-rP &= -\delta \ln a np (np - rP) \\
n^2 p^2 - nrPp &= \frac{rP}{\delta \ln a} \\
n^2 p^2 - nrPp - \frac{rP}{\delta \ln a} &= 0 \\
p &= \frac{nrP \pm \sqrt{n^2 r^2 P^2 + \frac{4n^2 rP}{\delta \ln a}}}{2n^2} \\
&= \frac{rP \pm \sqrt{r^2 P^2 + \frac{4rP}{\delta \ln a}}}{2n}
\end{aligned}$$

If we consider δ to be $\frac{\Delta \text{time to payoff}}{\Delta \text{payment}}$ Then $\delta = \frac{1}{5}$ would represent the point where we have to pay \$5 more for a decrease of 1 year in time to payoff.