Math Notes for Compound Interest

1 Basic Formula for Debt Owed

First let's calculate the debt after a time period t given a principal P, an interest rate r, a compound frequency n, a cyclic payment p, and a payment frequency m. It is assumed that $n \geq m$.

$$\begin{aligned}
a &= 1 + \frac{r}{n} \\
D_0 &= P \\
D_1 &= aP \\
D_2 &= a^2 P \\
&\vdots \\
D_{n/m} &= a^{n/m} P - p \\
D_{n/m+1} &= a(a^{n/m} P - p) \\
D_{n/m+2} &= a^2(a^{n/m} P - p) \\
&\vdots \\
D_{2n/m} &= a^{n/m}(a^{n/m} P - p) - p \\
&= a^{2n/m} P - a^{n/m} p - p \\
&= a^{2n/m} P - (a^{n/m} + 1)p \\
&\vdots \\
D_{nt} &= a^{nt} P - p \sum_{i=0}^{mt-1} a^{in/m}
\end{aligned}$$

Remember that we can find the solution to the finite exponential series as so

$$
\begin{aligned}
x &= \sum_{i=0}^{n-1} a^i \\
ax &= \sum_{i=1}^{n} a^i \\
ax - a^n + 1 &= \sum_{i=0}^{n-1} a^i \\
ax - a^n + 1 &= x \\
(1-a)x &= 1 - a^n \\
x &= \frac{1-a^n}{1-a}
\end{aligned}
$$

Plugging this in gives us our debt calculation.

$$
\begin{aligned}
D_{nt} &= a^{nt}P - p\frac{1-a^{mt(n/m)}}{1-a^{n/m}} \\
&= a^{nt}P - p\frac{1-a^{nt}}{1-a^{n/m}}
\end{aligned}
$$

2 Calculating Parameters from Known Values

For the following we will always assume $D_{nt} = 0$.

2.1 Principal

$$
\begin{aligned}
0 &= D_{nt} \\
0 &= a^{nt}P - p\frac{1-a^{nt}}{1-a^{n/m}} \\
P &= \frac{a^{-nt}-1}{1-a^{n/m}}p
\end{aligned}
$$

2.2 Payments

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$
\begin{aligned}
P &= \frac{a^{-nt} - 1}{1 - a^{n/m}} p \\
p &= \frac{1 - a^{n/m}}{a^{-nt} - 1} P
\end{aligned}
$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$
\begin{aligned}
p_{\min} &= \lim_{t \to \infty} \frac{1 - a^{n/m}}{a^{-nt} - 1} P \\
&= \frac{1 - a^{n/m}}{-1} P \\
&= (a^{n/m} - 1) P
\end{aligned}
$$

2.3 Time

We can also calculate how long it will take to pay off all debts.

$$
\begin{aligned}
p &= \frac{1 - a^{n/m}}{a^{-nt} - 1} P \\
(a^{-nt} - 1)p &= (1 - a^{n/m}) P \\
a^{-nt} - 1 &= (1 - a^{n/m}) \frac{P}{p} \\
a^{-nt} &= (1 - a^{n/m}) \frac{P}{p} + 1 \\
-nt &= \log_a \left((1 - a^{n/m}) \frac{P}{p} + 1 \right) \\
t &= -\frac{\log_a \left((1 - a^{n/m}) \frac{P}{p} + 1 \right)}{n}
\end{aligned}
$$

3 Recommended Payment

$$\begin{aligned}
T &= mtp \\
\frac{\mathrm{d}T}{\mathrm{d}p} &= -\delta \\
mt &= -\delta \\
t &= -\frac{\delta}{m} \\
p &= \frac{1 - a^{n/m}}{a^{n\delta/m} - 1} P
\end{aligned}$$

4 Meeting a Specific Return on Investment

Say that we want to only pay a specific amount of return on a debt. For instance we only want to pay \$1200 back on a \$1000 dollar loan. We can say G is what we're willing to pay back, where $G > P$. We then have

$$\begin{aligned}
mtp &= G \\
t &= \frac{G}{mp} \\
p &= \frac{1 - a^{n/m}}{a^{-nt} - 1} P \\
&= \frac{1 - a^{n/m}}{a^{-nG/mp} - 1} P
\end{aligned}$$

This is easily solvable, but we can use Newton's method to find an answer using a calculator.

$$\begin{aligned}
f(p) &= \frac{1 - a^{n/m}}{a^{-nG/mp} - 1} P \\
f'(p) &= -\frac{1 - a^{n/m}}{(a^{-nG/mp} - 1)^2} P \left(-\frac{nG\ln(a)}{mp} a^{-nG/mp} \right) \left(-\frac{1}{p^2} \right) \\
&= -\frac{1 - a^{n/m}}{p^2(a^{-nG/mp} - 1)^2} \left(\frac{nG\ln(a)}{mp} a^{-nG/mp} \right) P \\
&= \frac{a^{n/m} - 1}{p^2(a^{-nG/mp} - 1)^2} \left(\frac{nG\ln(a)}{mp} a^{-nG/mp} \right) P
\end{aligned}$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$