

# Math Notes for Compound Interest

First let's calculate the debt after a time period  $t$  given a principal  $P$ , an interest rate  $r$ , a compound frequency  $f$ , and a cyclic payment  $p$ .

$$a = 1 + \frac{r}{n}$$

$$D_0 = P$$

$$D_1 = aP - p$$

$$\begin{aligned} D_2 &= a(aP - p) - p \\ &= a^2P - p(a + 1) \end{aligned}$$

$$\vdots$$

$$D_{nt} = a^{nt}P - p \sum_{i=0}^{nt-1} a^i$$

Remember that we can find the solution to the finite exponential series as so

$$x = \sum_{i=0}^{nt-1} a^i$$

$$ax = \sum_{i=1}^{nt} a^i$$

$$ax - a^{nt} + 1 = \sum_{i=0}^{nt-1} a^i$$

$$ax - a^{nt} + 1 = x$$

$$(1 - a)x = 1 - a^{nt}$$

$$x = \frac{1 - a^{nt}}{1 - a}$$

Plugging this in gives us our debt calculation.

$$D_{nt} = a^{nt}P - p \frac{1 - a^{nt}}{1 - a}$$

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$\begin{aligned} 0 &= D_{nt} \\ 0 &= a^{nt}P - p \frac{1 - a^{nt}}{1 - a} \\ p &= \frac{a^{nt}}{\frac{1 - a^{nt}}{1 - a}} P \\ &= \frac{a^{nt}(1 - a)}{1 - a^{nt}} P \\ &= \frac{a^{nt}(1 - a)}{1 - a^{nt}} P \end{aligned}$$

Taking the limit of  $t$  we can also find the minimum payment needed to maintain debt.

$$\begin{aligned} p_{\min} &= \lim_{t \rightarrow \infty} \frac{a^{nt}(1 - a)}{1 - a^{nt}} P \\ &= (a - 1)P \end{aligned}$$

We can also calculate how long it will take to pay off all debts.

$$\begin{aligned} 0 &= a^{nt}(1 - a)P - p(1 - a^{nt}) \\ p &= ((1 - a)P + p)a^{nt} \\ a^{nt} &= \frac{p}{(1 - a)P + p} \\ nt &= \frac{\ln p - \ln((1 - a)P + p)}{\ln a} \\ t &= \frac{\ln p - \ln((1 - a)P + p)}{n \ln a} \end{aligned}$$