Math Notes for Compound Interest

First let's calculate the debt after a time period t given a principal P, an interest rate r, a compound frequency f, and a cyclic payment p.

$$
\begin{aligned}
a &= 1 + \frac{r}{n} \\
D_0 &= P \\
D_1 &= aP - p \\
D_2 &= a(aP - p) - p \\
&= a^2 P - p(a + 1) \\
&\vdots \\
D_{nt} &= a^{nt} P - p \sum_{i=0}^{nt-1} a^i
\end{aligned}
$$

Remember that we can find the solution to the finite exponential series as so

$$
\begin{aligned}
x &= \sum_{i=0}^{nt-1} a^i \\
ax &= \sum_{i=1}^{nt} a^i \\
ax - a^{nt} + 1 &= \sum_{i=0}^{nt-1} a^i \\
ax - a^{nt} + 1 &= x \\
(1 - a)x &= 1 - a^{nt} \\
x &= \frac{1 - a^{nt}}{1 - a}
\end{aligned}
$$

Plugging this in gives us our debt calculation.

$$D_{nt} = a^{nt}P - p\frac{1 - a^{nt}}{1 - a}$$

From here we can calculate the minimum payment needed to pay off all debts in a certain time frame.

$$\begin{aligned}
0 &= D_{nt} \\
0 &= a^{nt}P - p\frac{1 - a^{nt}}{1 - a} \\
p &= \frac{a^{nt}}{\frac{1-a^{nt}}{1-a}}P \\
&= \frac{a^{nt}(1 - a)}{1 - a^{nt}}P \\
&= \frac{a^{nt}(1 - a)}{1 - a^{nt}}P
\end{aligned}$$

Taking the limit of t we can also find the minimum payment needed to maintain debt.

$$\begin{aligned}
p_{\min} &= \lim_{t\to\infty} \frac{a^{nt}(1 - a)}{1 - a^{nt}}P \\
&= (a - 1)P
\end{aligned}$$

We can also calculate how long it will take to pay off all debts.

$$\begin{aligned}
0 &= a^{nt}(1 - a)P - p(1 - a^{nt}) \\
p &= ((1 - a)P + p)a^{nt} \\
a^{nt} &= \frac{p}{(1 - a)P + p} \\
nt &= \frac{\ln p - \ln\big((1 - a)P + p\big)}{\ln a} \\
t &= \frac{\ln p - \ln\big((1 - a)P + p\big)}{n \ln a}
\end{aligned}$$

$$\frac{\partial t}{\partial p} = -\frac{1}{2}$$

$$
\begin{aligned}
\frac{\frac{1}{p} - \frac{1}{(1-a)P+p}}{n \ln a} &= -\frac{1}{2} \\
\frac{1}{p} - \frac{1}{(1-a)P+p} &= -\frac{n \ln a}{2} \\
\frac{(1-a)P}{p((1-a)P+p)} &= -\frac{n \ln a}{2} \\
p((1-a)P+p) &= -\frac{2(1-a)P}{n \ln a} \\
p^2 + (1-a)Pp + \frac{2(1-a)P}{n \ln a} &= 0 \\
p &= \frac{-(1-a)P \pm \sqrt{(1-a)^2P^2 - \frac{8(1-a)P}{n \ln a}}}{2}
\end{aligned}
$$