Unsupervised Learning From Incomplete Measurements for Inverse Problems

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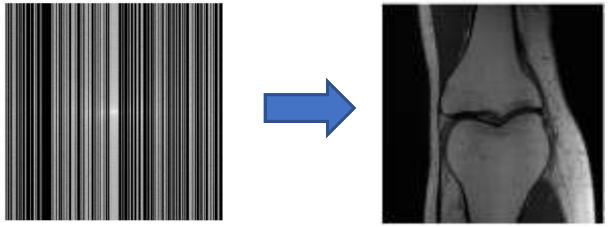
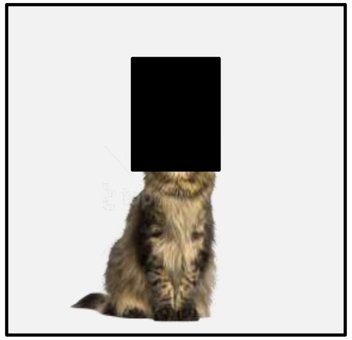
Linear Inverse Problems

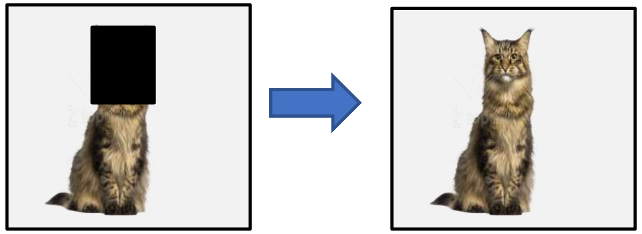
Magnetic Resonance Imaging



measurements reconstruction

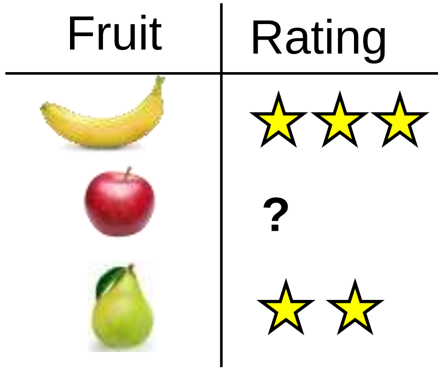
Image Inpainting

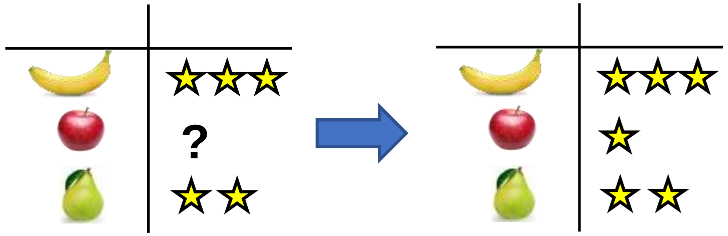




measurements reconstruction

Recommender Systems





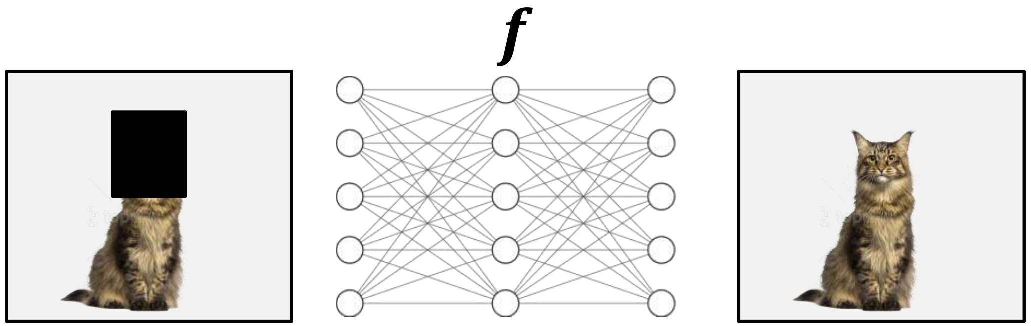
measurements reconstruction

Linear Inverse Problems

$$y = Ax + \epsilon$$

Learning Approach

Train with pairs (x_i, y_i) to directly learn the inversion function



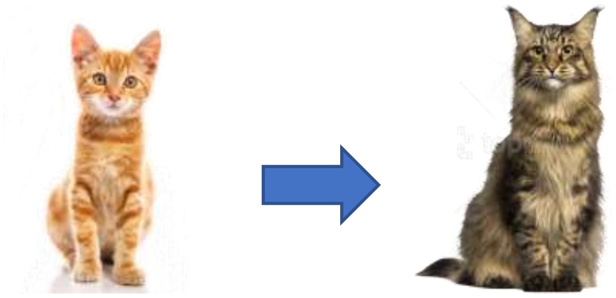
$$\underset{f}{\operatorname{argmin}} \sum_i \left|\left|x_i - f(y_i)\right|\right|^2$$

where $f: \mathbb{R}^m \mapsto \mathbb{R}^n$ is parameterized as a deep neural network.

Pitfalls

Main disadvantage: Obtaining training signals x_i can be expensive or impossible.

- Medical and scientific imaging
- Solves problems which we already solved
- Risk of training with signals from a different distribution



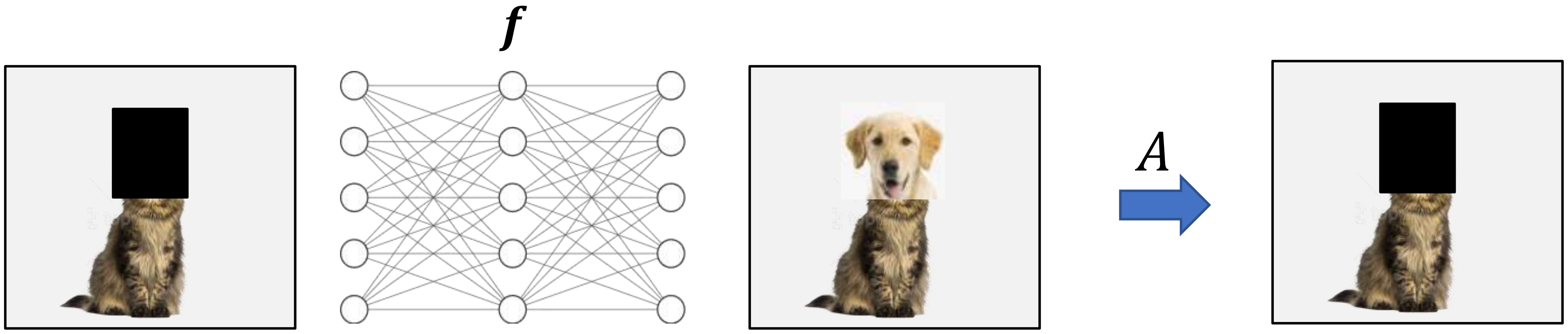
train *test?*

Learning Approach

Learning from only measurements y?

$$\operatorname*{argmin}_f \sum_i ||y_i - Af(y_i)||^2$$

Proposition: Any reconstruction function $f(y) = A^{\dagger}y + g(y)$ where $g: \mathbb{R}^m \mapsto \mathcal{N}_A$ is any function whose image belongs to the nullspace of A.

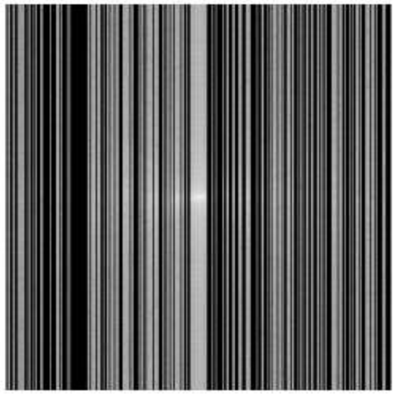


Learning from Measurements

How to learn from only $\boldsymbol{y}$?

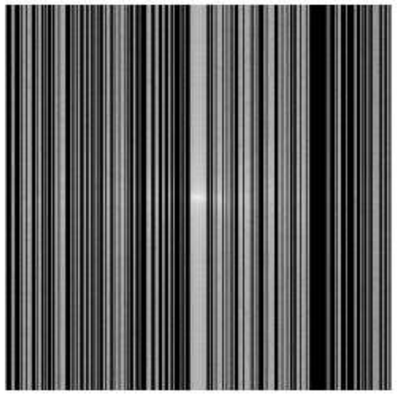
- Access multiple operators $y_i = A_{g_i} x_i$ with $g \in \{1, \ldots, G\}$
- Each A_g with different nullspace

$A_1 x$



$A_2 x$

$A_3 x$



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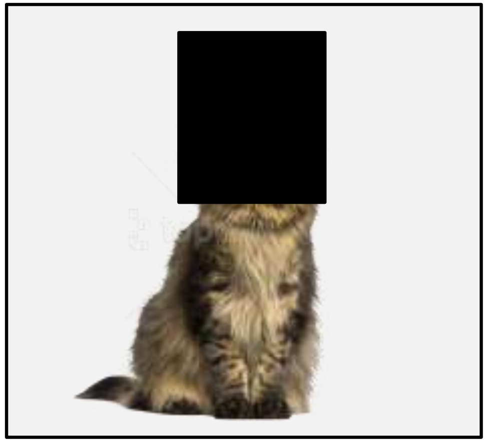

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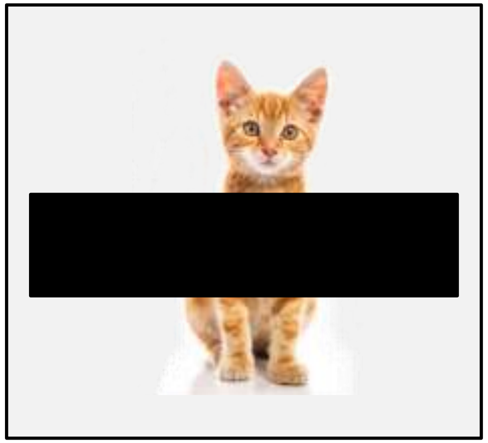

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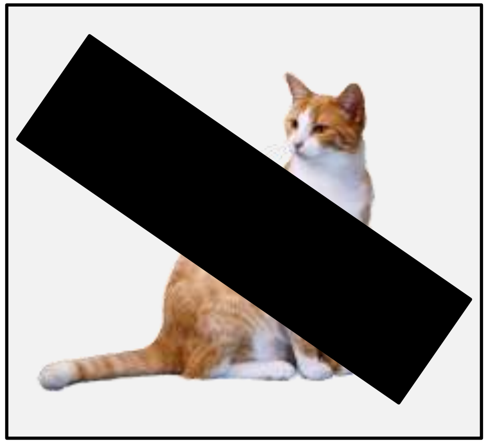

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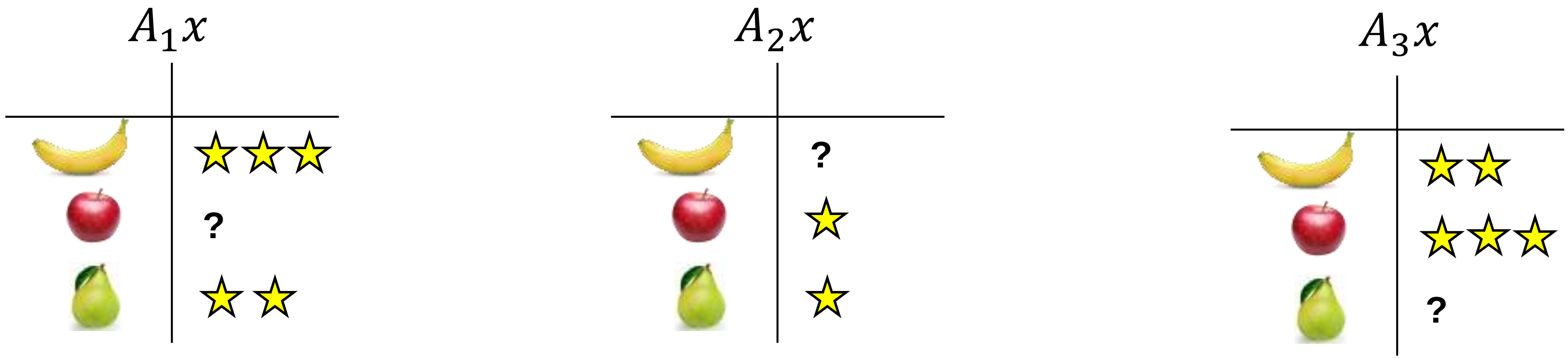


Model Identification

Can we *uniquely* identify the set of signals $\mathcal{X} \subset \mathbb{R}^n$ from the observed measurement sets $\{\mathcal{Y}_g = A_g \mathcal{X}\}_{g=1}^{G}$?

Necessary Conditions

Proposition: Recovering $\mathcal{X}$ from observed measurement sets $\{\mathcal{Y}_g = A_g\mathcal{X}\}_{g=1}^{G}$ possible only if

$$\operatorname{rank}\left(\begin{bmatrix} A_1 \\ \vdots \\ A_G \end{bmatrix}\right) = n$$

and thus, if $m \geq n/G$.

Sufficient Conditions

Additional assumption: The model is **low-dimensional**

- Box-counting dimension of $\mathcal{X}$ is $k \ll n$

Examples: Sparse dictionaries, manifold models, generative models, etc.

Theorem: Identifying a k-dimensional $\mathcal{X}$ from observed sets $\{\mathcal{Y}_g = A_g\mathcal{X}\}_{g=1}^{G}$ is possible by almost every $A_1, \ldots A_G \in \mathbb{R}^{n\times m}$ if

$$m > k + \frac{n}{G}$$

Proposed Objective

Neural Network: $\hat{x} = f(y, A_g)$

- Pseudo-inverse $f(y, A_g) = f(A_g^{\dagger} y)$
- Unrolled optimization [Gregor and LeCun, 2010]

Proposed unsupervised loss: $\underset{f}{\operatorname{argmin}}\, \mathcal{L}_{MOI}(f)$

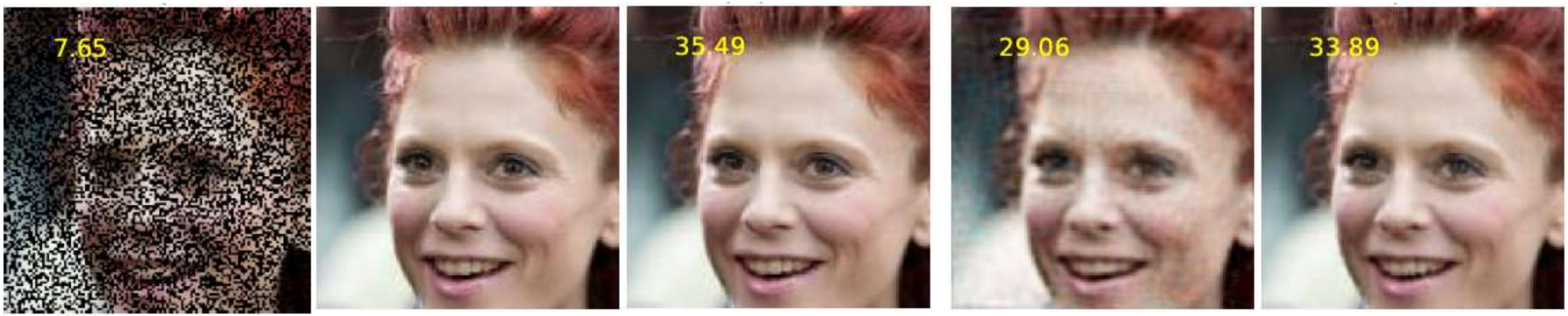
$$\mathcal{L}_{MOI}(f) = \sum_i \left|\left|y_i - A_{g_i} f(y_i, A_{g_i})\right|\right|^2 + \sum_s \left|\left|f(A_s \hat{x}_i, A_s) - \hat{x}_i\right|\right|^2$$

where $\hat{x}_i = f(y_i, A_{g_i})$

Inpainting

- U-Net network
- CelebA dataset
- A_g are inpainting masks

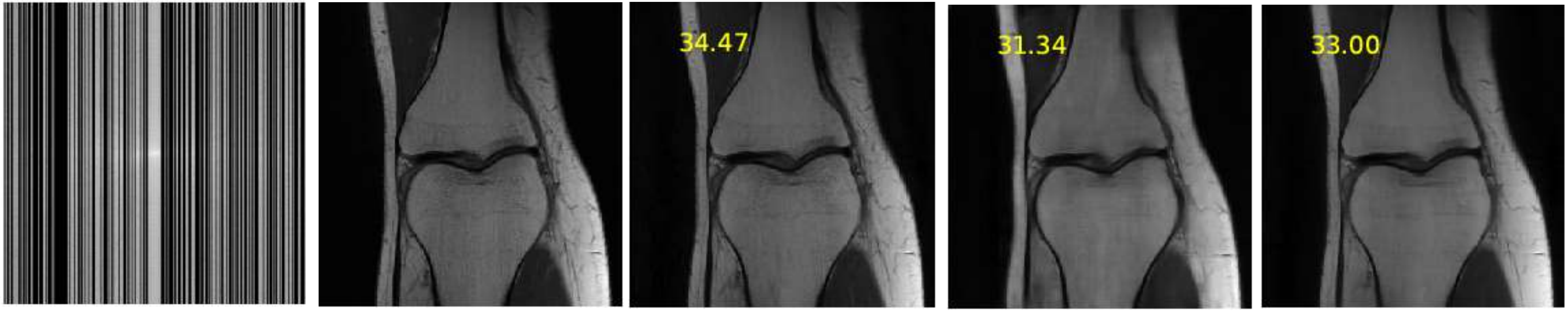
Measurements y | Signal x | Supervised | AmbientGAN [Bora et al, 2018] | **Proposed**



Magnetic Resonance Imaging

- Unrolled network
- FastMRI dataset
- A_g are subsets of Fourier measurements (x4 downsampling)

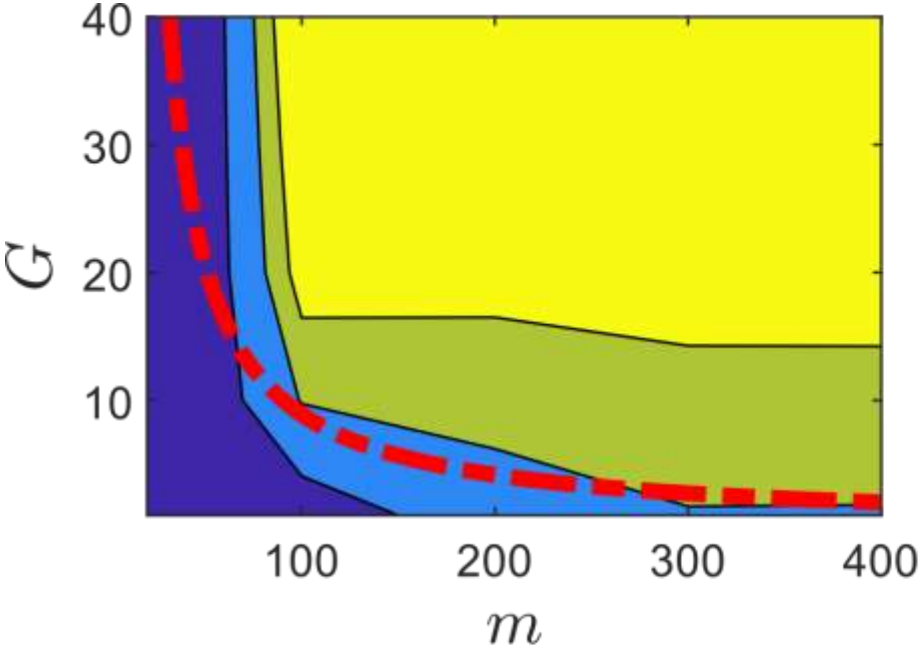
Measurements y | Signal x | Supervised | Measurement Splitting [Yaman et al., 2020] | **Proposed**



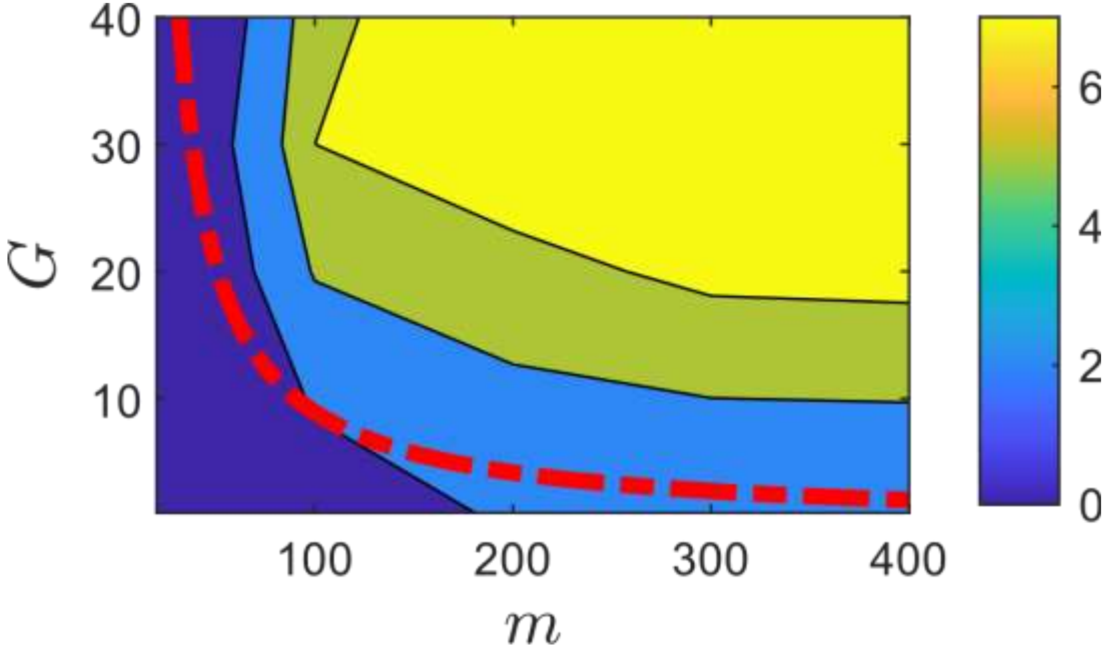
Verifying Bounds

- MNIST dataset: $n = 28^2$, $k \approx 12$ [Hein and Audibert, 2005]
- Red line: sufficient condition $m > k + n/G$

Compressed Sensing



Inpainting



Thanks for your attention!

[1] "Equivariant Imaging: Learning Beyond the Range Space", Chen, Tachella and Davies, *ICCV 2021 (Oral)*

[2] "Robust Equivariant Imaging: a fully unsupervised framework for learning to image from noisy and partial measurements", Chen, *2022 (Oral)*

[3] "Sensing Theorems for Unsupervised Learning in Inverse Problems", Tachella, Chen and Davies, *Arxiv 2022*.

Tachella.github.io

- ✓ Codes
- ✓ Presentations
- ✓ … and more