On Tseitin formulas, read-once branching programs and treewidth

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Outline

- Branching programs
- Tseitin formulas
- Treewidth and pathwidth
- Lower bounds

Branching program

- BP is a way to represent Boolean function

    - directed graph without cycles;
    - two sinks: labeled with 0 and 1, one source;
    - every internal vertex labeled with a variable and has two outgoing edges: labeled with 0 and 1;
    - the value of function equals label of the sink in the end of corresponding path.
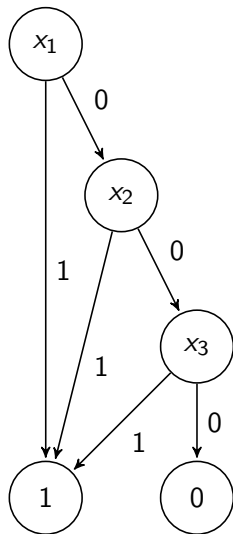
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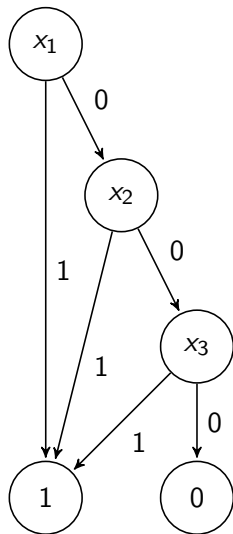   - the value of function equals label of the sink in the end of corresponding path.

- BP size is between circuit and formula sizes

    - $C(f) \leq 3BP(f) \leq O(L^{1+\epsilon}(f))$,
    - $C(f)$ circuit complexity of f, $L(f)$ size of minimal formula, BP size of minimal branching program.

Non-deterministic Branching program

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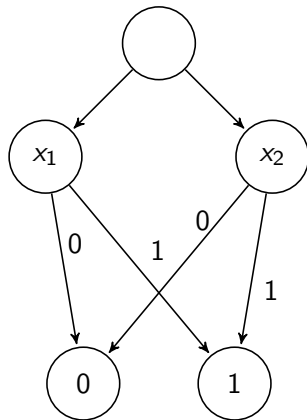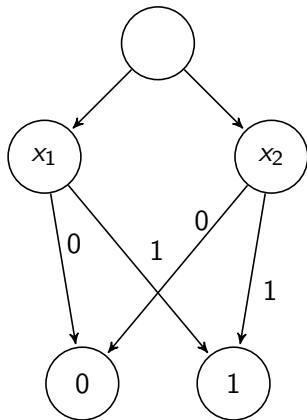
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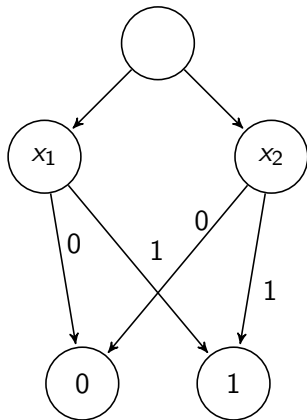- ▶ NBP corresponds to NL/poly

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- OBDD is 1-BP with fixed order for variables in every path
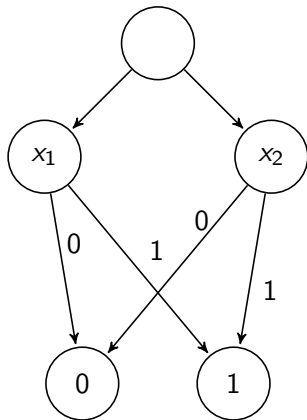
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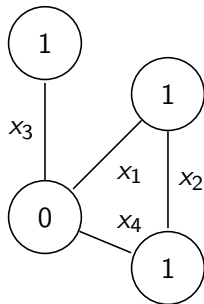 - $TS_{G,c}(x) = 1 \iff$

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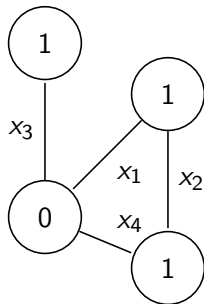
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- A Tseitin formula is satisfiable iff for every connected component the sum of labels is even
- Unsatisfiable Tseitin formulas are classical hard examples for proof systems:
  - Resolution [Tseitin, 1968], [Urquhart, 1987]
  - Bounded depth Frege [Ben-Sasson, 2002], [Pitassi, Rossman, Servedio, Tan, 2016]
  - Polynomial Calculus over field with char $!= 2$ [Alekhnovich, Razborov, 2001]
  - Tree-like Lovasz-Schrijver [Itsykson, Kojevnikov, 2006]
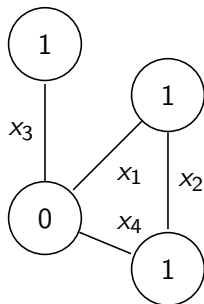
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Tseitin formulas (2)

- (Itsykson, Knop, Romashchenko, Sokolov, 2017) Exponential lower bound for OBDD(join) proof systems for unsatisfiable Tseitin formulas
  - key step: satisfiable Tseitin formulas are hard for OBDDs.
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  - $OBDD(TS_{G,c}) = 2^{\Omega(n)}$
- OBDD is a partial case of 1-BP
  - (Bryant, 1991) (Sieling, Wegener, 1995)
    $f : OBDD(f) = 2^{\Omega(n)}$, 1-BP(f)=$poly(n)$

The goal: what is the size of 1-BP for satisfiable Tseitin formulas?

Lower bound for Tseitin formula on complete graph

Theorem:
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Generalized theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V|-k_G(l)-k_G(|E|-l)+1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deletion of l edges.

Treewidth and pathwidth

Tree/path decomposition of a graph
$G = (V, E)$ is a graph T that is a tree/path
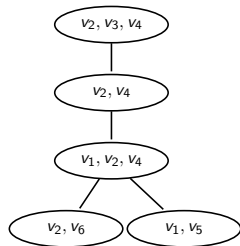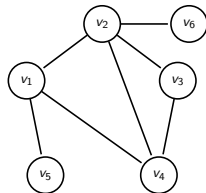
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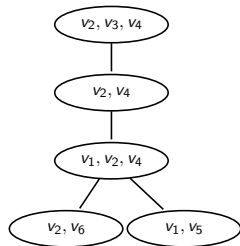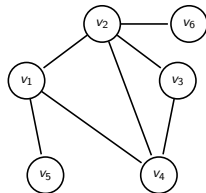3. All bags with the same vertex form a tree/path

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Width of a decomposition is the size of a maximal bag in it minus one.
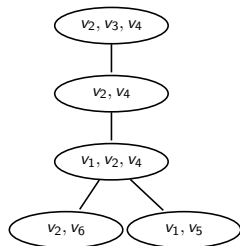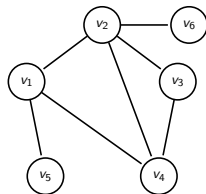
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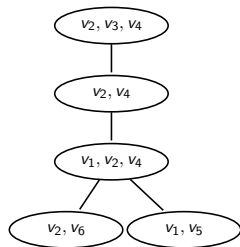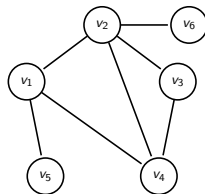1. Vertices of T (called "bags") correspond to sets of vertices of G

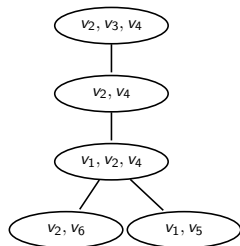2. If $(a, b) \in E$ then there is a bag with a and b
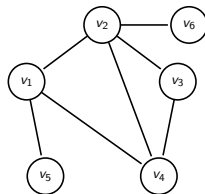
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Treewidth/**pathwidth** of a graph is the minimal width among all its tree/path decompositions.
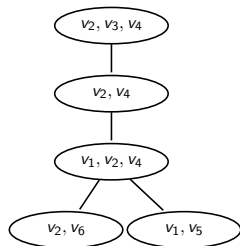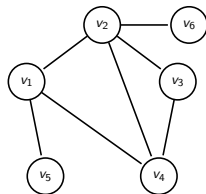
Lower and upper bounds

Lower bound via treewidth: 1-NBP for satisfiable Tseitin formula on a graph G has size $\Omega(2^{t^\delta})$, where t is a treewidth of G, $\delta < \frac{1}{36}$.

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Graph minor

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Grid Minor Theorem [Robertson, Seymour 1986], [Chuzhoy 2015]:
Every graph G of a tree-width t has a grid minor of size $t^{\delta} \times t^{\delta}$, where $\delta < \frac{1}{36}$.

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- Show that 1-NBP for $t^\delta \times t^\delta$ grid graph has size $\Omega(2^{t^\delta})$

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  No exponential lower bounds for semantic 1-NBP for a boolean function
  All known results with exponential lower bounds are for functions with non-boolean domain [Cook, Edmonds, Medabalimi, Pitassi, 2016], [Jukna, 2009]

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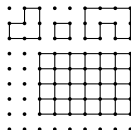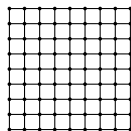
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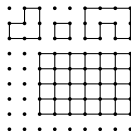- We set $I = \frac{|E|}{2}$

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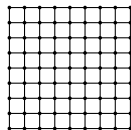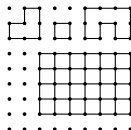
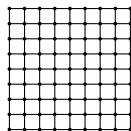▶ Use theorem: 1-NBP for a satisfiable Tseitin formula on a graph G has at least $2^{|V| - k_G(I) - k_G(|E|-I) + 1}$ nodes

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▶ After that we get a lower bound $2^{\Omega(2\varepsilon\sqrt{|V|})} = 2^{\Omega(s)}$
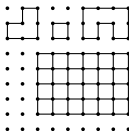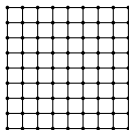
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- **Deletion of edge** x_i: substitute any value that will not break the satisfiability of the formula
- **Deletion of vertex** v: delete all connected edges one-by-one
- **Contraction of edge** x_i: the same as the satisfiability of the formula $\exists x_i : Ts_{G,w}(x)$
  - in 1-NBP all nodes labeled with x_i should be changed to non-deterministic nodes

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Known lower bounds for k-BPs

- (Borodin, Razborov, Smolensky, 1993)
  $CLIQUE\_ONLY_n(G) : \{0,1\}^{\frac{n \cdot (n-1)}{2}} \to 0, 1$
  - $CLIQUE\_ONLY_n(G) = 1$ iff graph G on n vertices is exactly $\frac{n}{2}$-clique
  - $1\text{-NBP}(CLIQUE\_ONLY_n) = 2^{\Omega(n)}$
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  - 1-NBP(f) = $2^{\Omega(n)}$
- (Thathachar, 1998) an explicit functions f_k for every k:
  - k-NBP(f_k) = $2^{\Omega(n^{1/k})}$
  - (k+1)-BP(f_k) = $O(n)$

Known lower bounds for k-BPs (2)

- (Duris, Hromkovic, Jukna, Sauerhoff, Schnitger, 2004)
  - $\oplus$ parity of the number of triangles in a graph, $\overline{\triangle}$ is 1 iff graph has no triangles
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- (Jukna, 1995)
    - f_k characteristic function of error-correcting codes
    - k-NBP(f_k) = $2^{\Omega(\sqrt{n})}$

The comparison with other results

Satisfiable Tseitin formulas $TS_{G,w}$ on (n, d, α)-expander can be represented by:

- CNF($TS_{G,c}$)=$O(n)$
- 1-NBP($\rceil TS_{G,c}$)=$O(n)$
- 2-BP($TS_{G,c}$)=$O(n)$

- 1-NBP($TS_{G,c}$)=$2^{\Omega(n)}$

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Best previously known gaps:

CNF(f)=$O(n^3)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004)
1-NBP($\lceil f$)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Jukna, 2009) explicit construction
1-NBP($\lceil f$)=$O(n)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004) probabilistic construction
2-BP(f)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Thathachar, 1998)

Idea of the proof

Theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V|-k_G(l)-k_G(|E|-l)+1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deletion of l edges.

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Theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V|-k_G(I)-k_G(|E|-I)+1}$ nodes, where the value $k_G(I)$ denotes the maximal number of connected components that can be obtained from G by deletion of I edges.

 Idea of the proof:

Choose level I of the 1-NBP

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Idea of the proof:

Choose level l of the 1-NBP

- show that there are at least $2^{C_1 n}$ different non-zero paths to level l

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Get that at least $2^{(C_1-C_2)n}$ different nodes are on the l-th level, so is in the 1-NBP