The Complexity of Verifying Boolean Programs as Differentially Private

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Joint work with Mark Bun and Marco Gaboardi
Boston University

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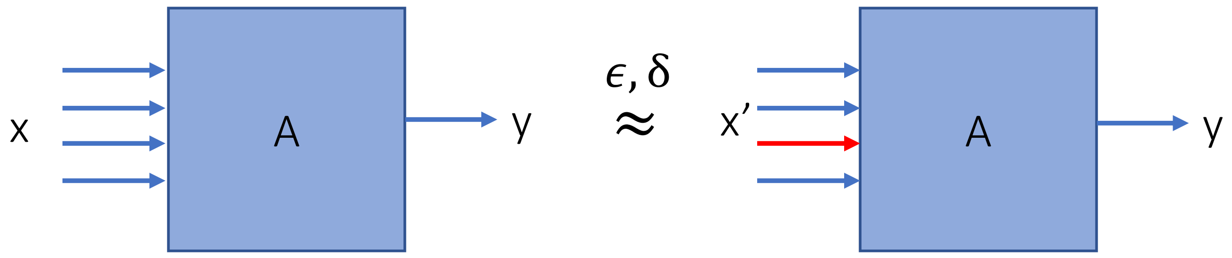
Plan of the talk

1. Prior work
    - How hard is it to verify whether a program is DP for
        - Turing-complete languages
        - Boolean languages with bounded memory without loops
2. Our results and proof ideas
    - BPWhile: Boolean language with loops and finite memory
    - PSPACE-completeness of verifying DP for BPWhile
    - PSPACE-hardness: reduction from TQBF
    - PSPACE algorithm based on computing hitting probabilities in a Markov chain

Differential Privacy



C is (ϵ, δ)-differentially private if for every set of possible outputs O, and for every neighboring x, x':

$$\mathrm{P}[C(x) \in O] \leq e^{\epsilon} \cdot \mathrm{P}[C(x') \in O] + \delta$$

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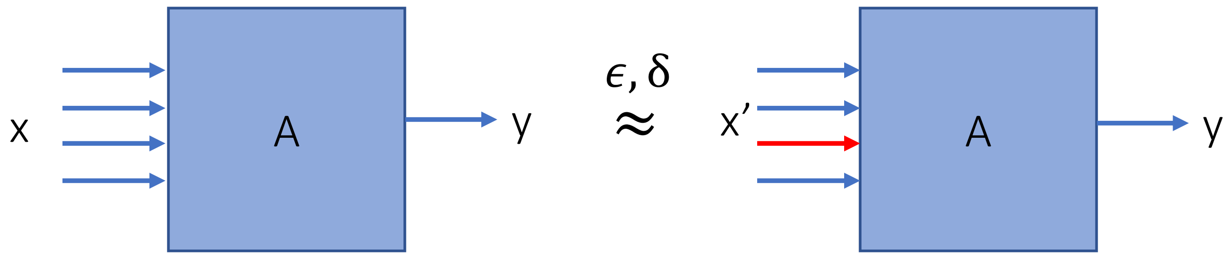

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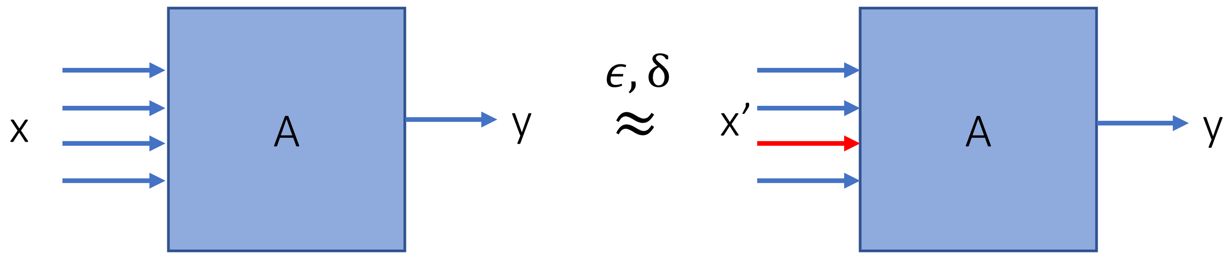

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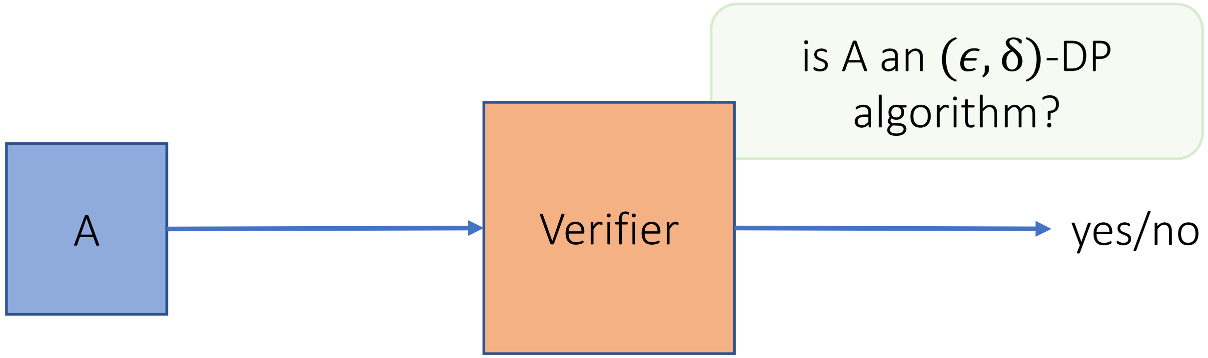
Neighboring relation we consider:

- Inputs are bit strings which differ in one bit
- Can be extended to any bounded polyspace computable relation

Verification of Differential Privacy



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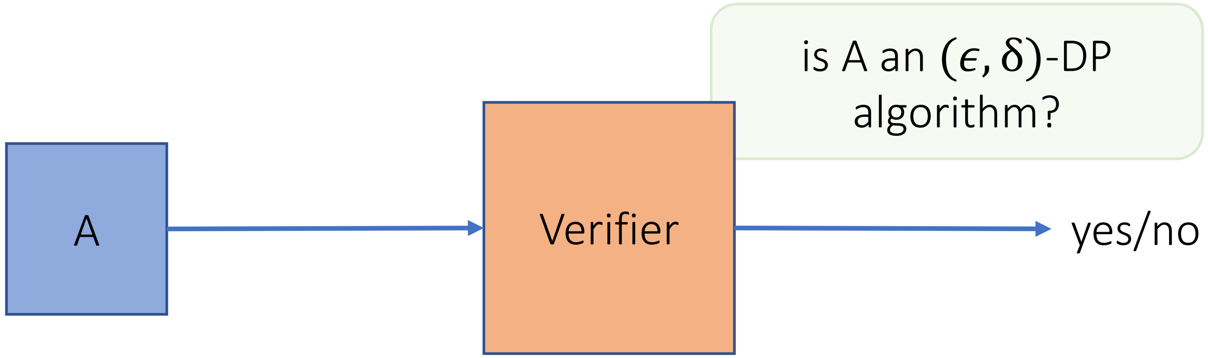

Complexity depends on the expressivity of the language and the type of access to the code:

- **Black box:** no information about the algorithm, query access
  - Impossible to verify $(\epsilon, 0)$-DP

[Gilbert, McMillan'19]

Verification of DP: white box

- Linear-time algorithm in the size of automaton
  - For pure-DP

[Chadha, Sistla, Viswanathan'21]

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- #P-hardness for approximating parameters in labelled Markov chains
  - For approximate-DP
  - Undecidable to compute exactly

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- For a simple Boolean language with **bounded memory,** if statements and **random assignments**, but **without loops**
  - $coNP^{\#P}$-completeness for $(\epsilon, 0)$-DP
    - Reduction from All-Min-SAT
  - $coNP^{\#P}$-hard and in $coNP^{\#P^{\#P}}$ for (ϵ, δ)-DP

[Gaboardi, Nissim, Purser'20]

BPWhile: Boolean language with While loops

We design a language to meet the following goals:

- Captures classical computations on real computers
- Simple to analyze

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Boolean expressions

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c ::= skip | x := b | c; c | if b then c else c | **while b then c** Commands

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Numbers of variables and input bits are fixed in the definition of the program

⇒ Length of the program is an upper bound on the size of the memory that the program uses

Example of a BPWhile program:

$$
\begin{array}{l}
\texttt{input}(\vec{c}, \epsilon); \\
\vec{k} := \lceil \log(2/\epsilon) \rceil; \\
\vec{d} := (2^{\vec{k}+1} + 1)(2^{\vec{k}} + 1)^{n-1}; \\
\vec{u} := \texttt{uniform}(0, \vec{d}]; \\
\vec{z} := 0; \\
\vec{r} := n; \\
\texttt{while}\ \vec{z} < \vec{n} \wedge \vec{r} = n\ \texttt{then} \\
\quad \texttt{if}\ \vec{z} < \vec{c}\ \texttt{then} \\
\quad\quad \texttt{if}\ \vec{u} \leq 2^{\vec{k}(\vec{c}-\vec{z})}(2^{\vec{k}} + 1)^{n-(\vec{c}-\vec{z})} \\
\quad\quad \texttt{then}\ \vec{r} := \vec{z} \\
\quad\quad \texttt{else skip} \\
\quad \texttt{else} \\
\quad\quad \texttt{if}\ \vec{u} \leq d - 2^{\vec{k}(\vec{z}-\vec{c}+1)}(2^{\vec{k}} + 1)^{n-1-(\vec{z}-\vec{c})} \\
\quad\quad \texttt{then}\ \vec{r} := \vec{z} \\
\quad\quad \texttt{else skip} \\
\quad \vec{z} = \vec{z} + 1; \\
\texttt{return}(\vec{z});
\end{array}
$$

Implementation of the Bounded Geometric Mechanism in finite precision arithmetic

[Ghosh, Roughgarden, Sundararajan'09]

[Balcer, Vadhan'17]

Our results

Main result: if A is a BPWhile program, then the problem of verifying whether A is differentially private is PSPACE-complete.

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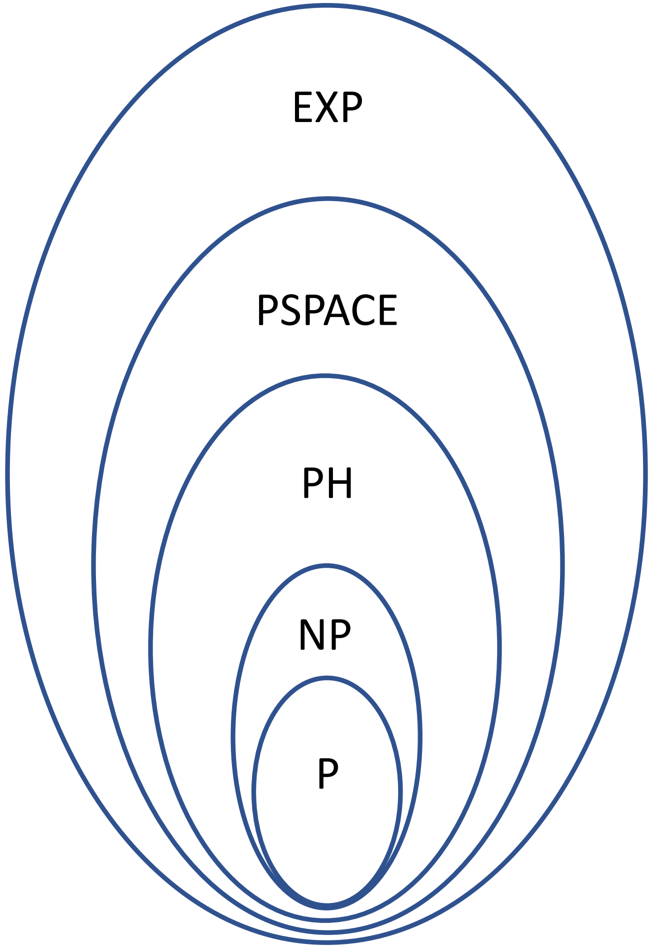
It holds for the following notions of differential privacy:

- $(\epsilon, 0)$-DP
- (ϵ, δ)-DP
- (ϵ, δ)-DP parameters approximation
- Renyi-DP
- Zero-Concentrated-DP
- Truncated Concentrated-DP

PSPACE-completeness

PSPACE-completeness of a problem A implies:



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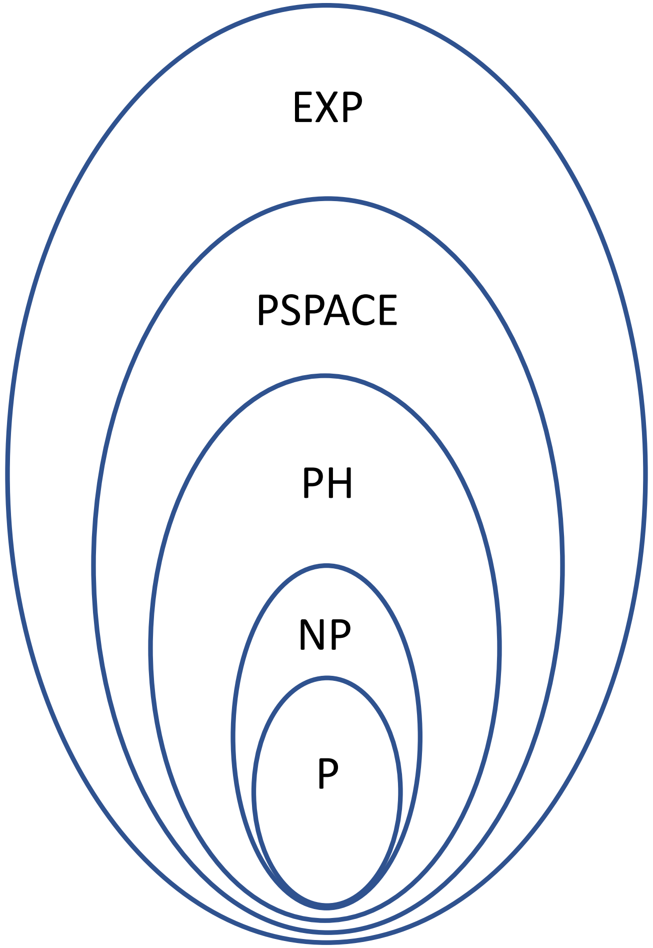
PSPACE-completeness of a problem A implies:

- A is solvable by a TM that uses polynomial space



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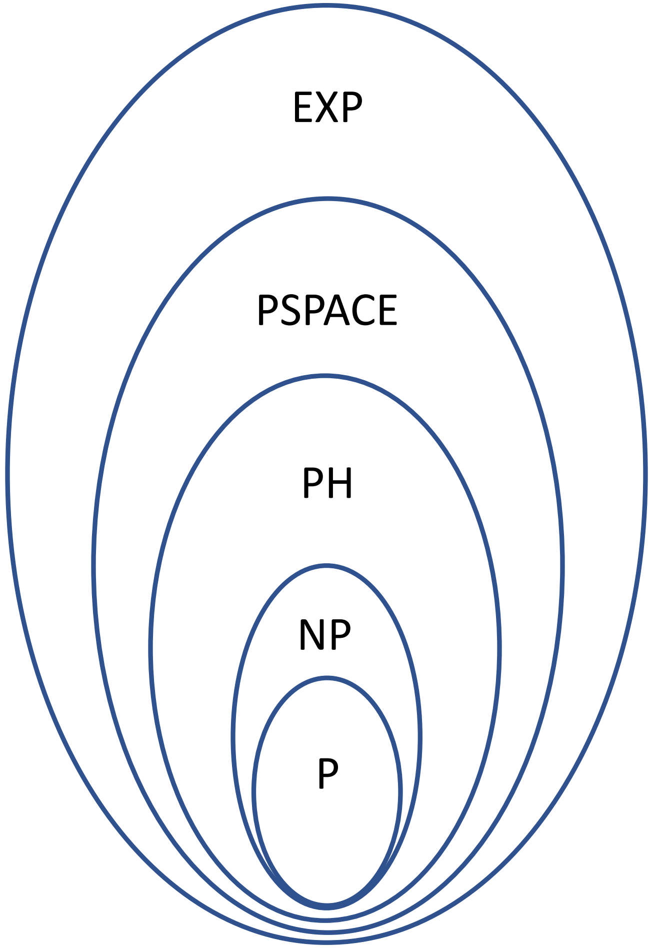

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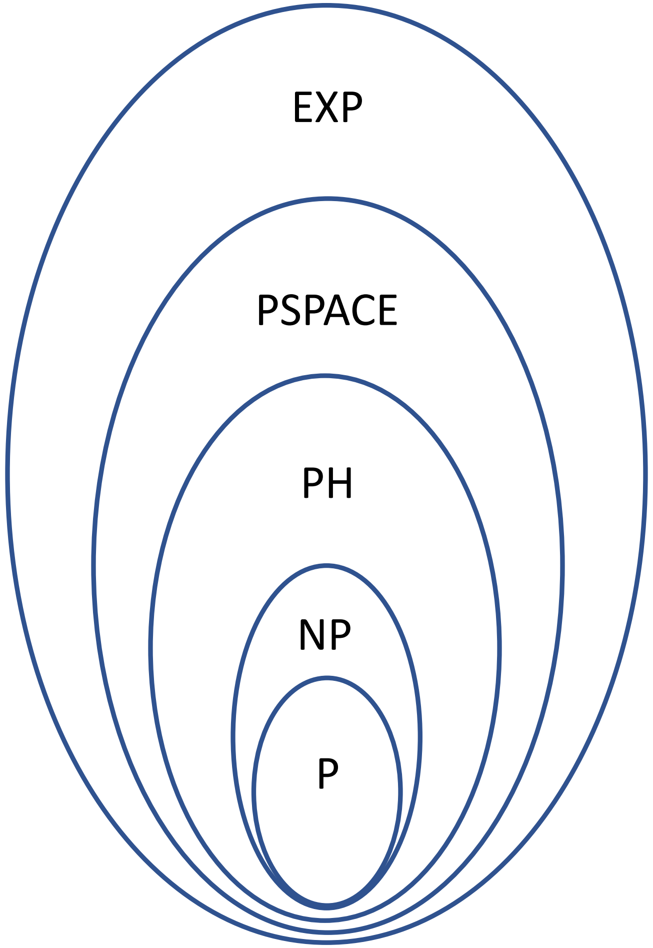

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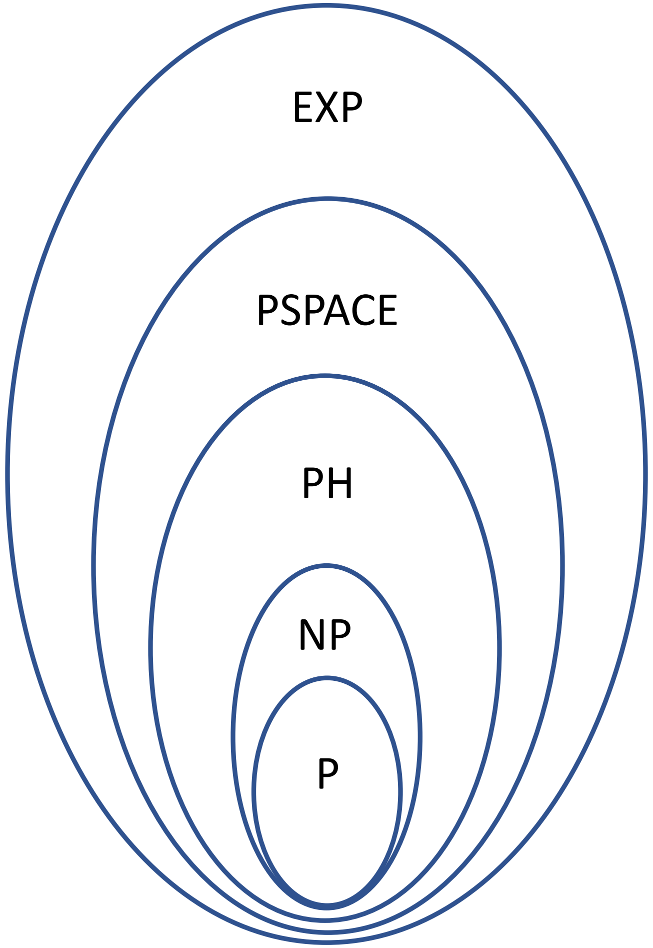
- A is at least as hard as any problem solvable in polyspace
- No polytime algorithm for A, unless P = PSPACE
  - That is widely believed not to be true



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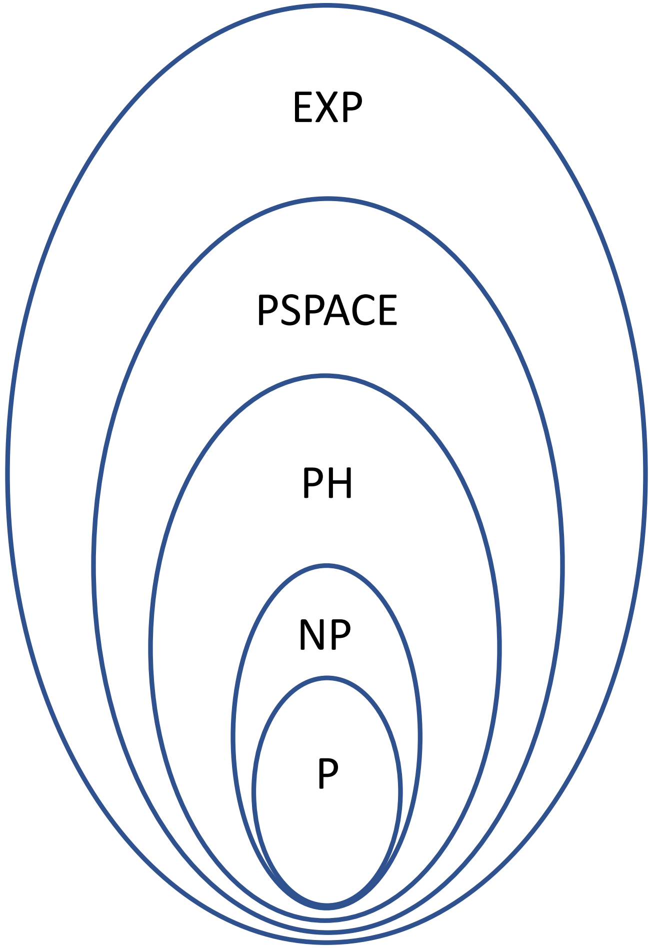
To show PSPACE completeness we need:



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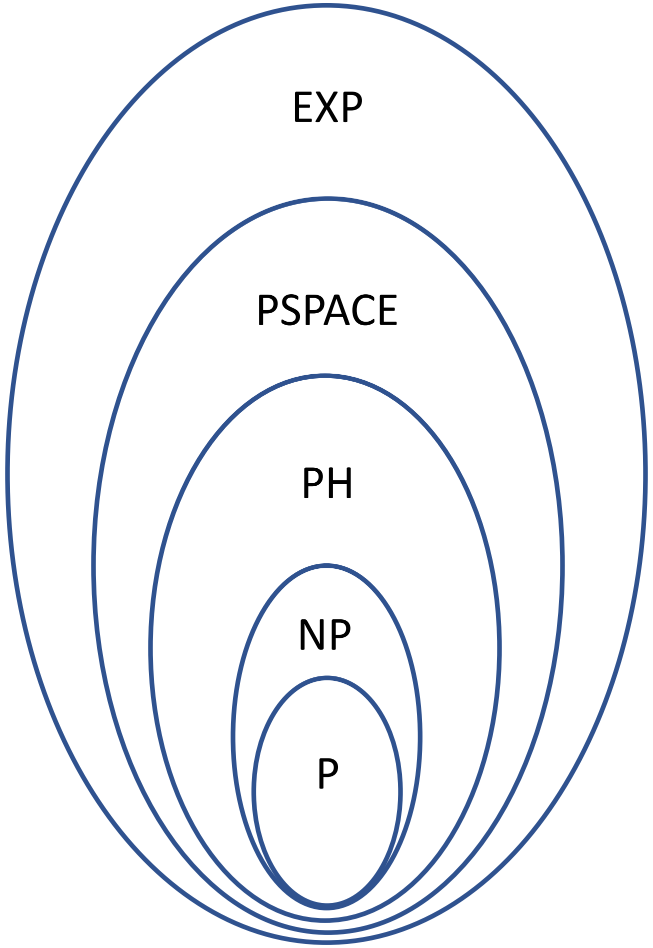
1. Show hardness: construct sequence of reductions from TQBF



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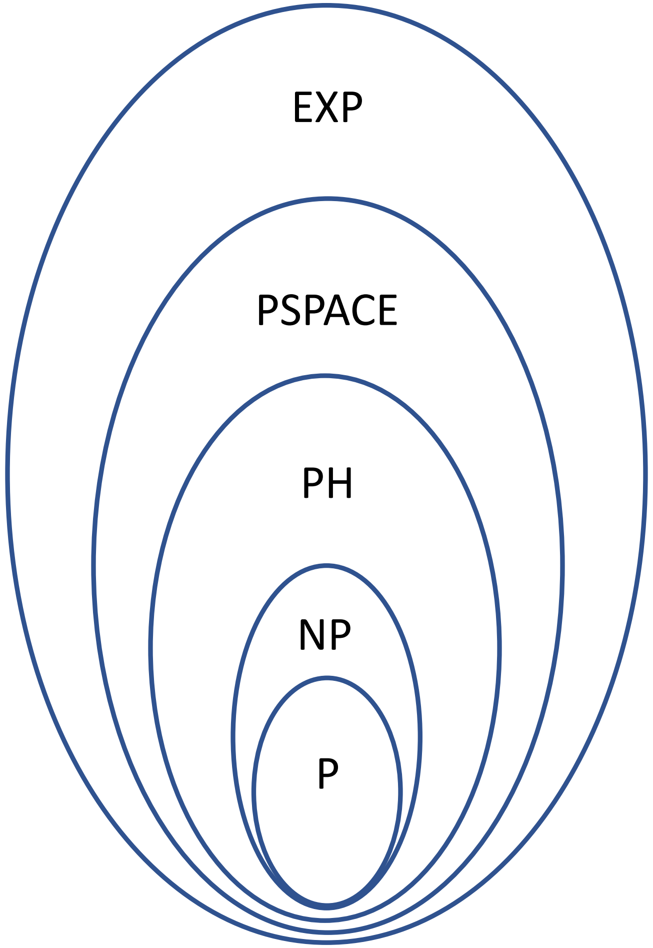
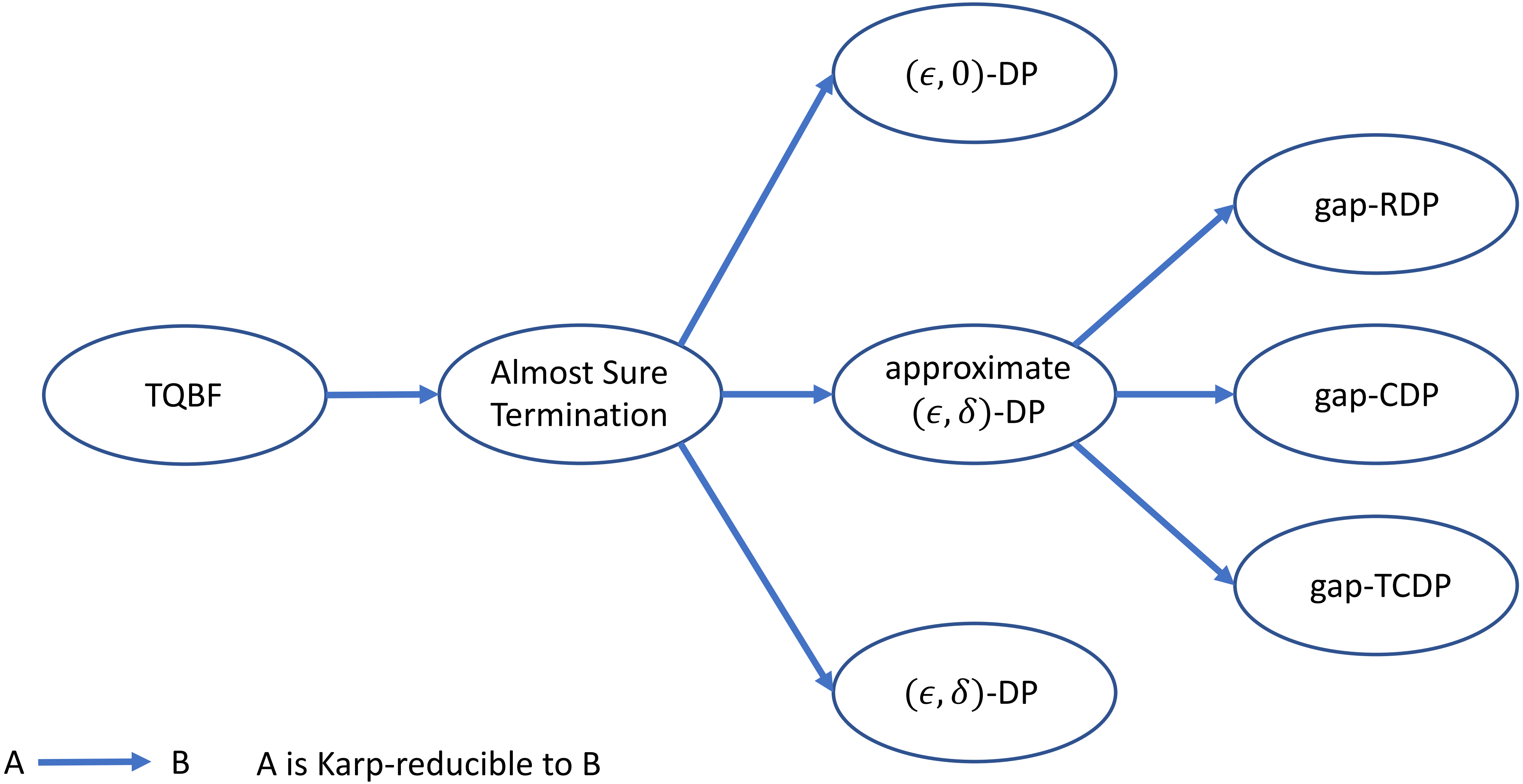
To show PSPACE completeness we need:

1. Show hardness: construct sequence of reductions from TQBF
2. Construct polynomial-space algorithm: analyze Markov chain based on the state graph of the program

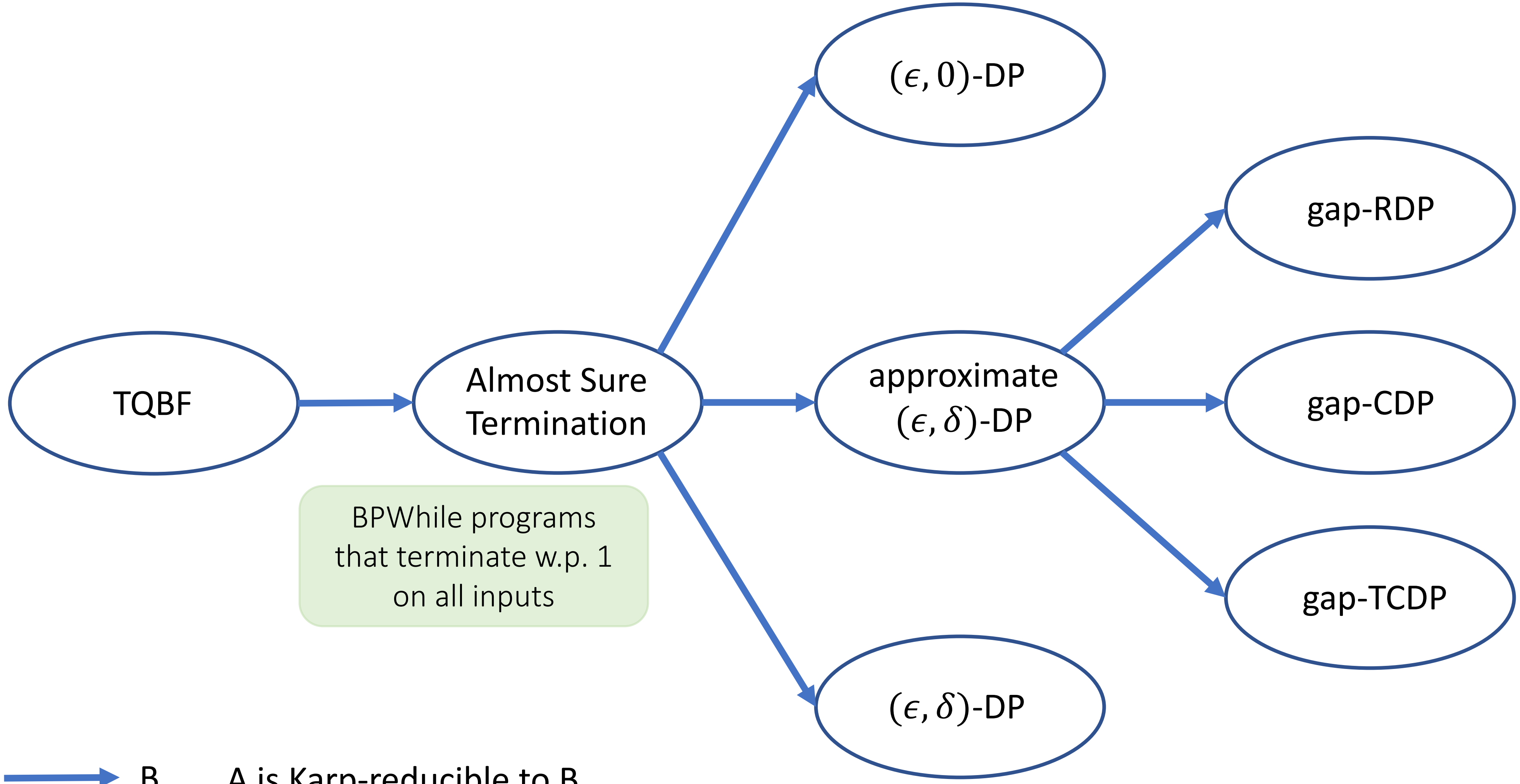


PSPACE-hardness



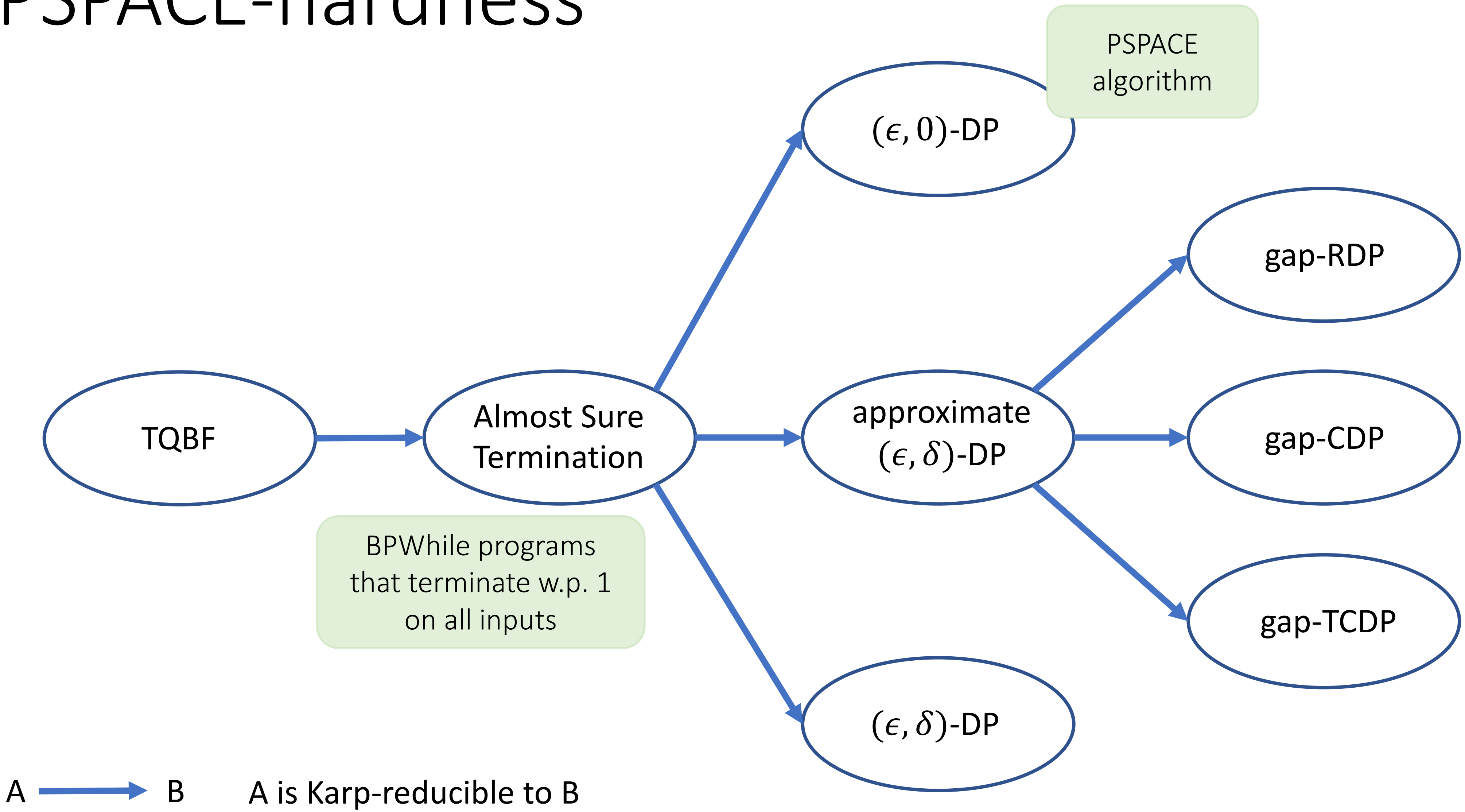
A is Karp-reducible to B

PSPACE-hardness



A → B A is Karp-reducible to B

PSPACE-hardness



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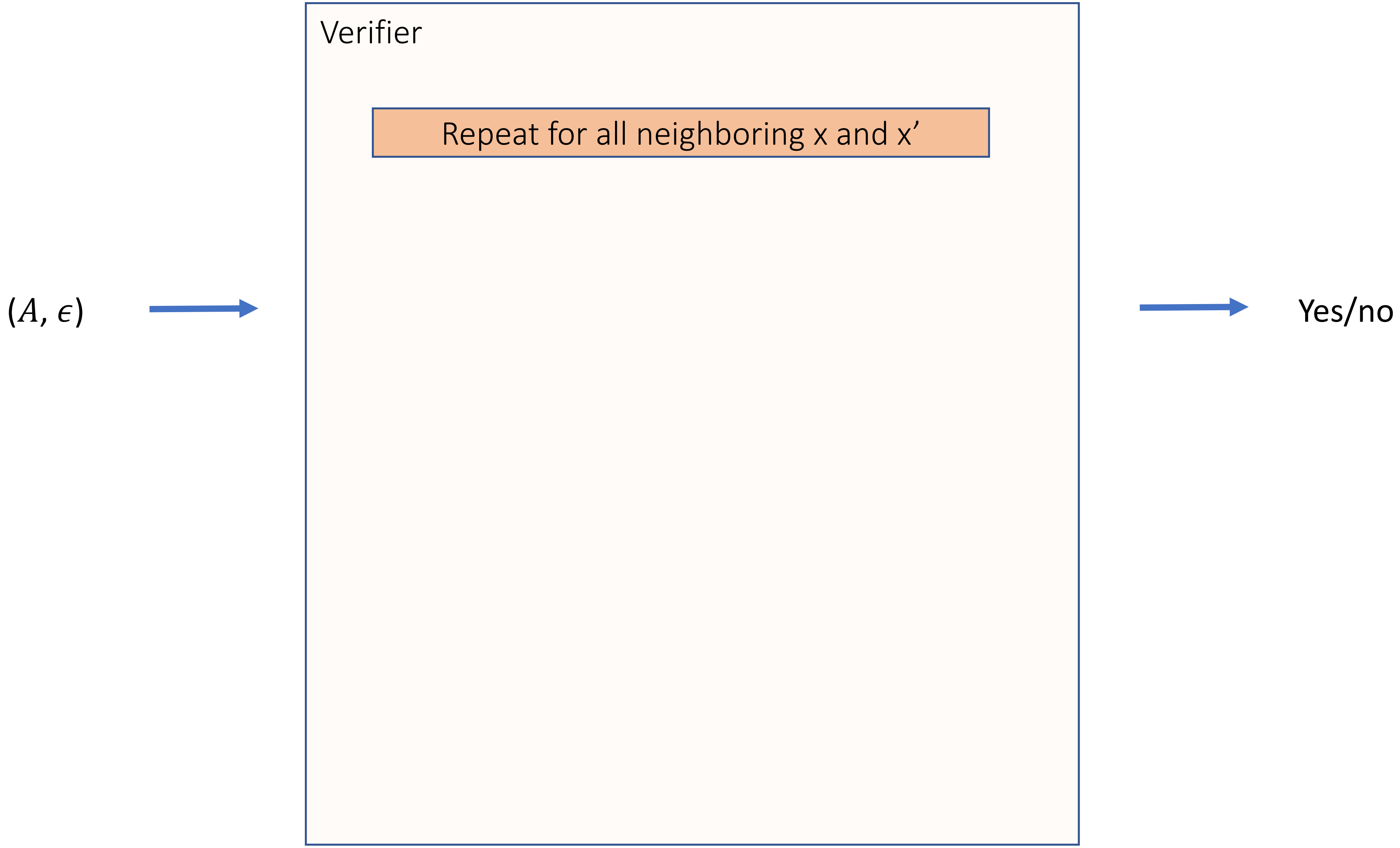
Polyspace membership: algorithm for (ϵ,0)-DP



Polyspace membership: algorithm for (ϵ,0)-DP

(A, ϵ) →

Verifier

Repeat for all neighboring x and x’

Analyze how distributions on outputs of
A(x) and A(x’) differ
based on the code of A

→ Yes/no

PSPACE membership: state graph

State graph depends on the input values

D(b=1):

```
input(b);
if b == 1 then
    r = rand();
    if r == 0 then
        while true
            skip;
    else skip;
else skip;
return(1)
```

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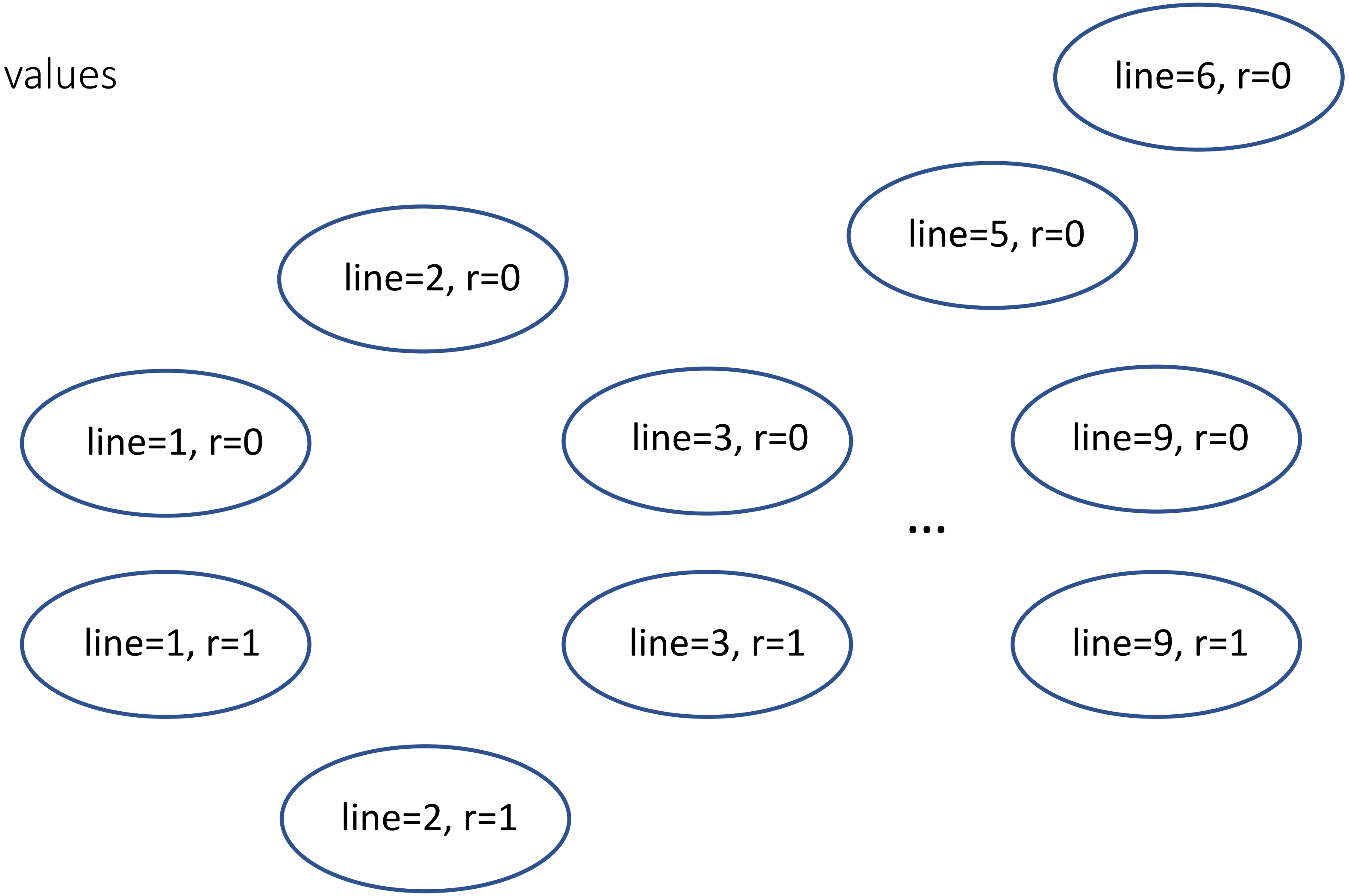

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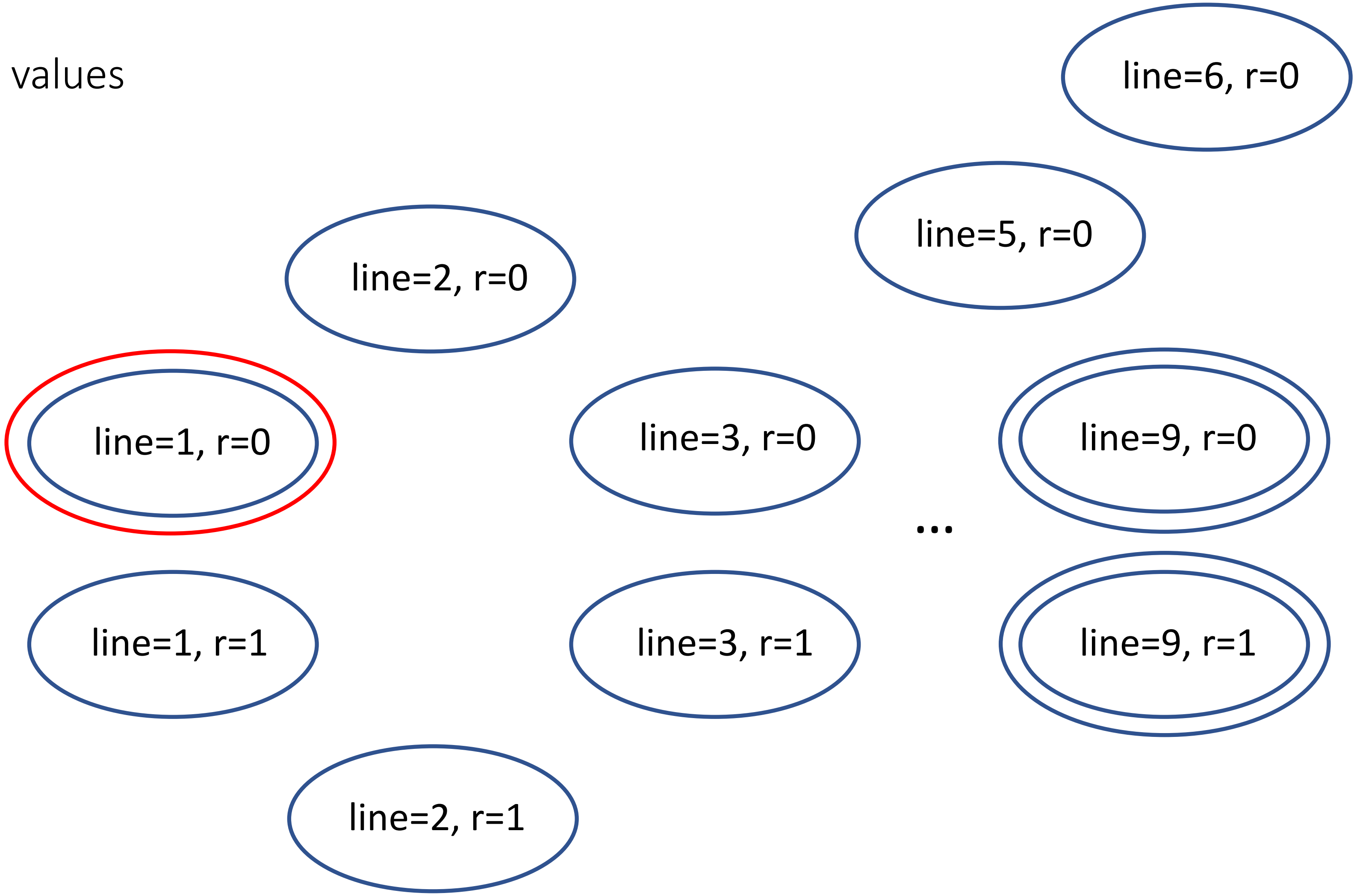

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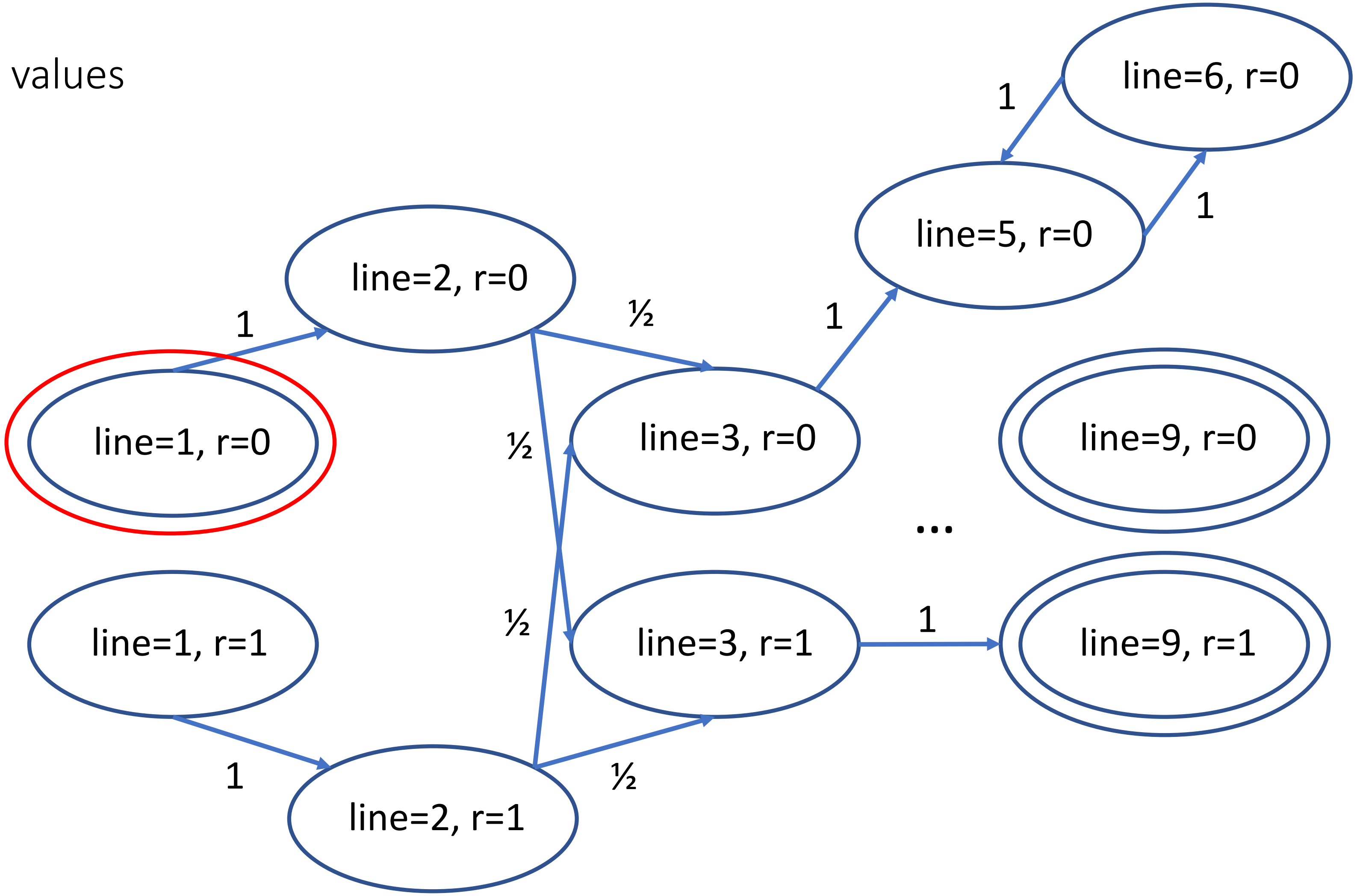

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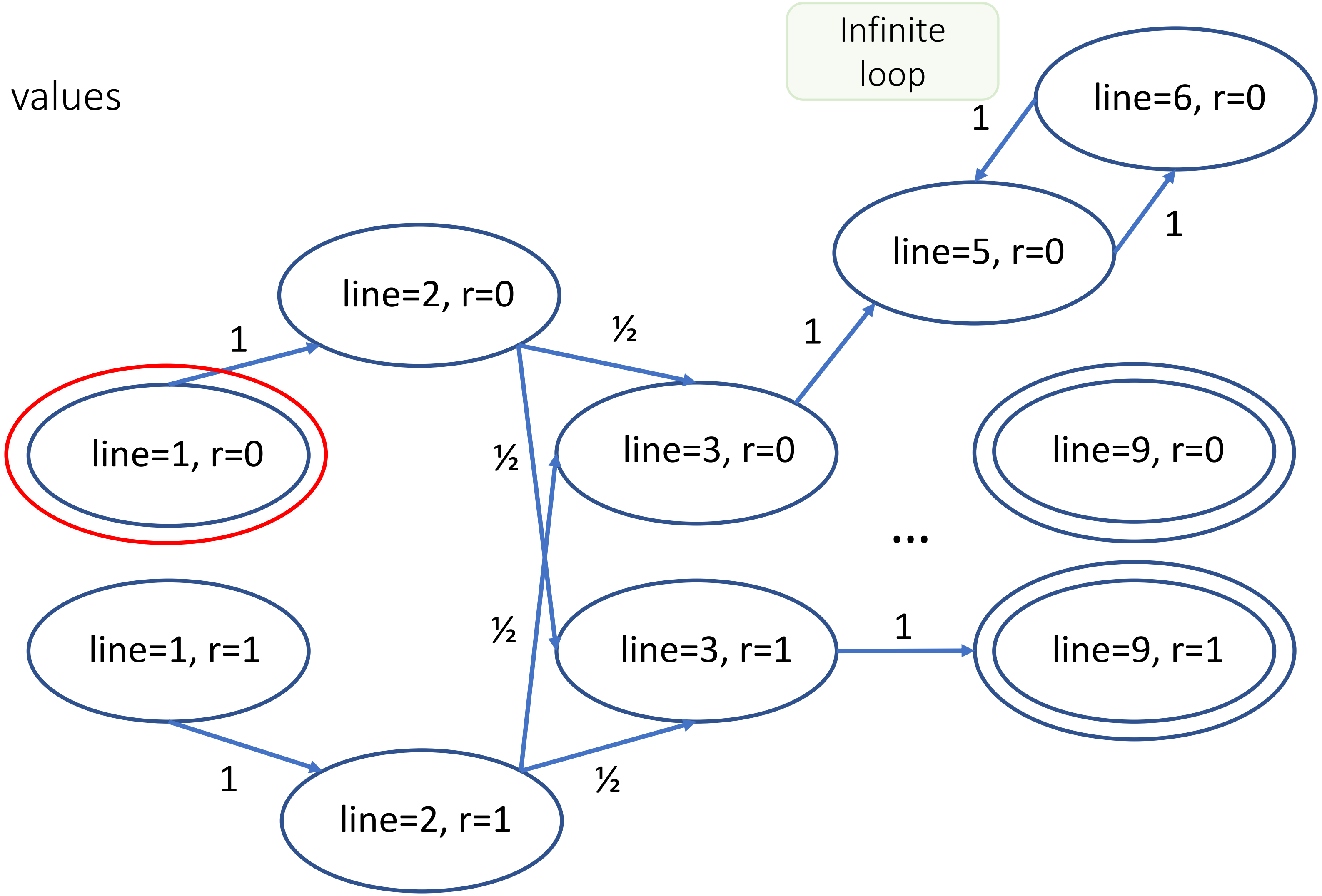

PSPACE membership: state graph

D(b=0):

```
1.  input(b);
2.  if b == 1 then
3.      r = rand();
4.      if r == 0 then
5.          while true
6.              skip;
7.      else skip;
8.  else skip;
9.  return(1)
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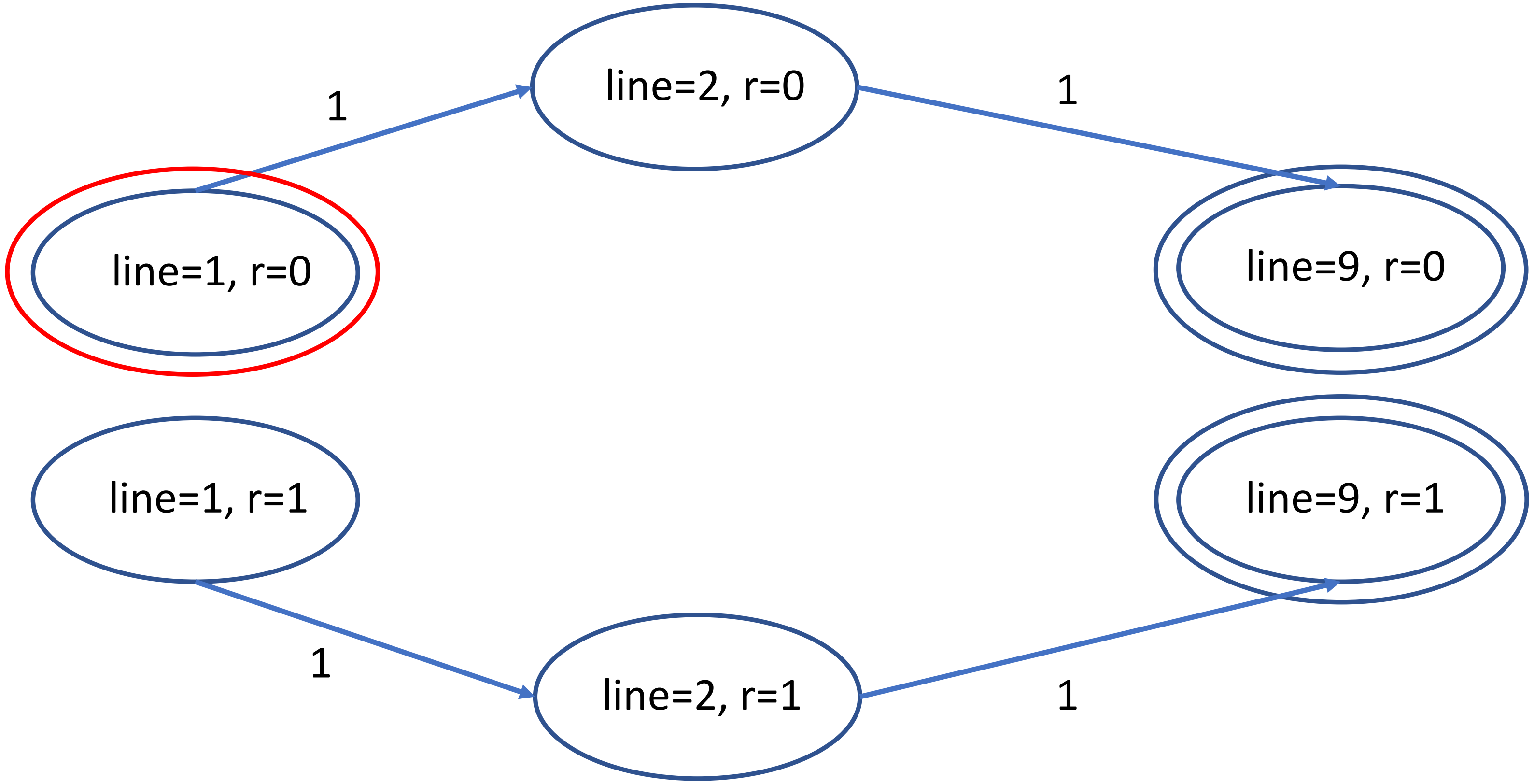
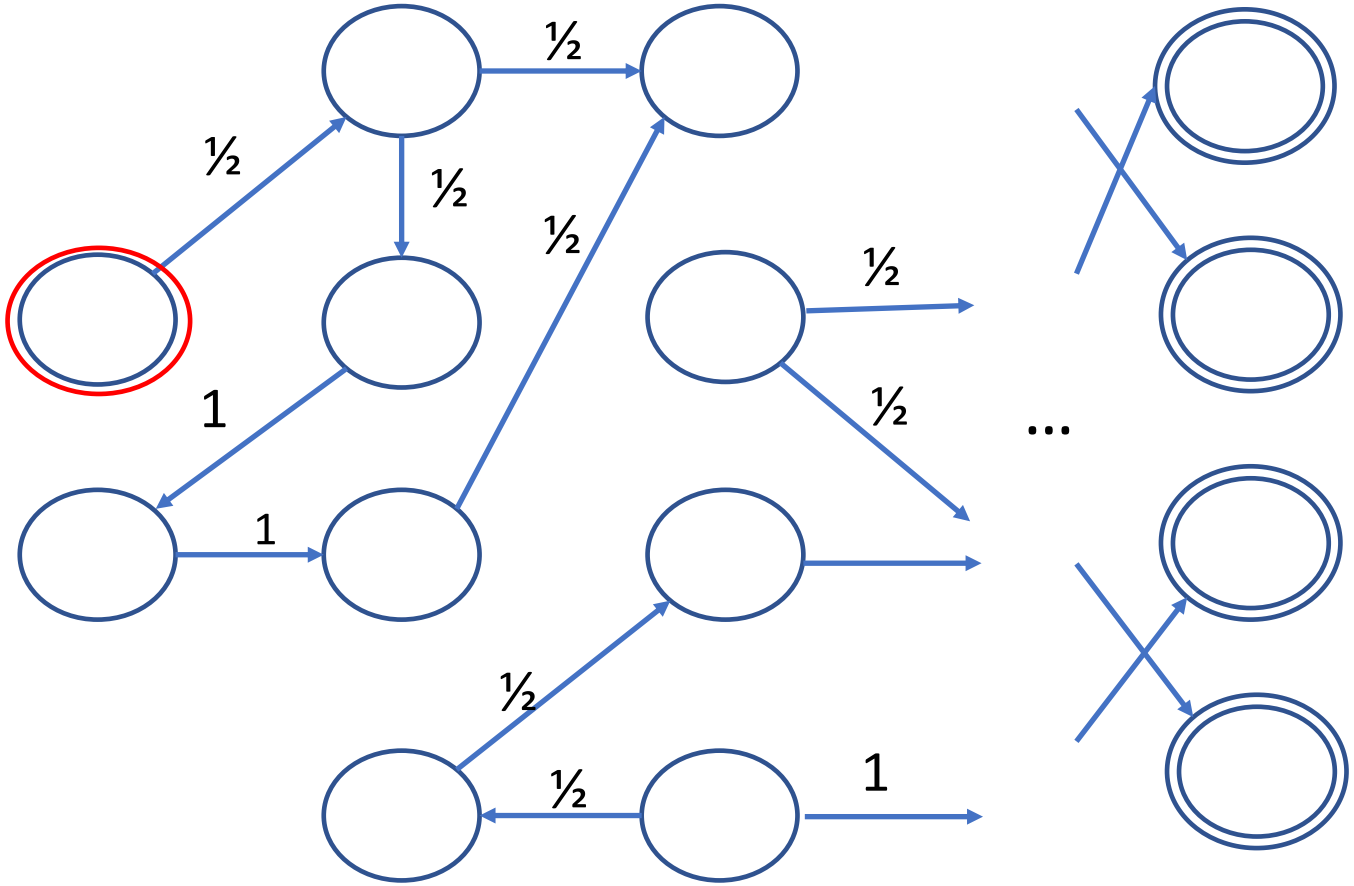


PSPACE membership: algorithm for (ϵ,0)-DP

For a program D and all neighboring inputs x, x':

- Construct the Markov chain for $D(x)$ and $D(x')$
- Compute and compare hitting probabilities



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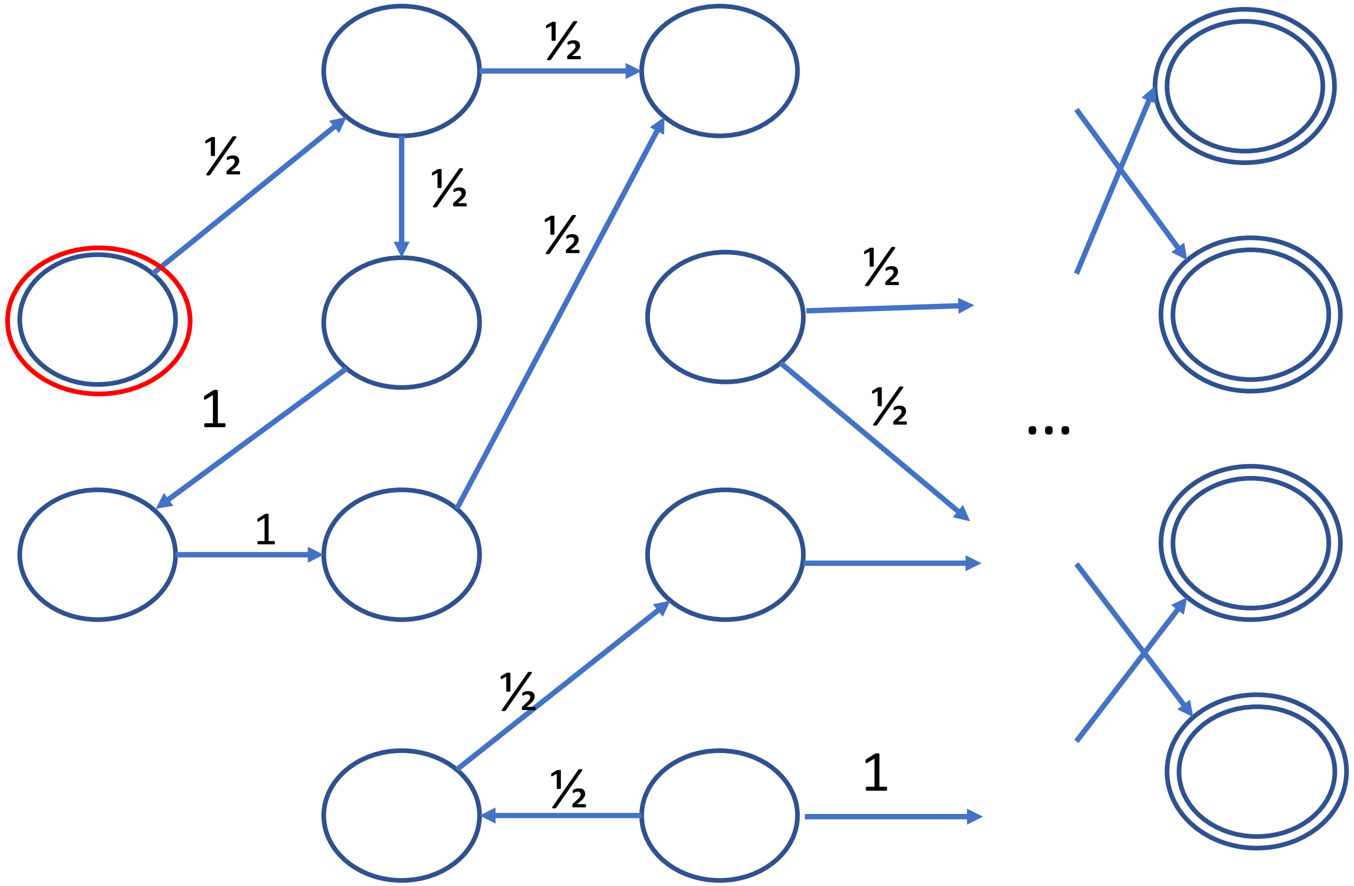

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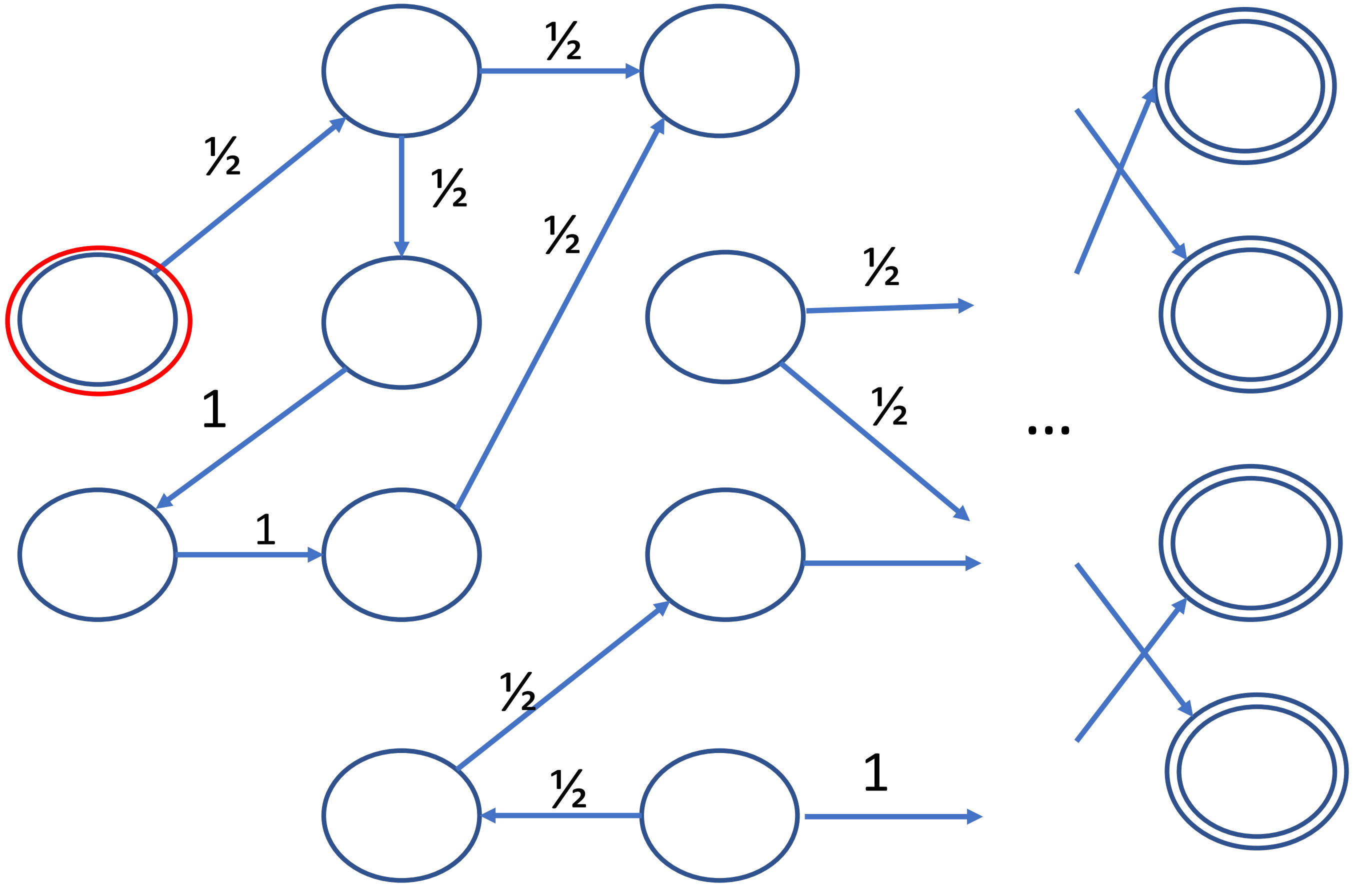
Need **space-efficient** algorithm for computing hitting probabilities with **implicit access** to the Markov chain



Polyspace algorithm for computing hitting probabilities in a Markov chain

Lemma [Simon'81]: If M is a Markov chain with at most 2^L states

- the initial distribution places all mass on one state,
- there is a set F of final states each with only one self-transition,
- every non-final state has outgoing transition probability 0 or ½.

Then there is an $O(L^6)$-space deterministic algorithm that computes the hitting probability of every state in F.

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Note: to use the algorithm, we need to replace all transitions labelled by 1 in the state graph of the BPWhile program:

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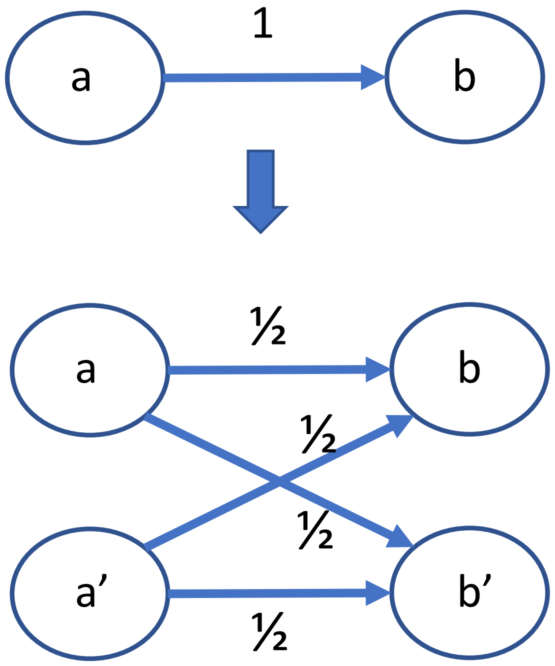


Note: to use the algorithm, we need to replace all transitions labelled by 1 in the state graph of the BPWhile program:

- Clone all states
- For each state *a* with outgoing edge w.p. 1 replace it by two edges:
  - Edge (a,b) with weight ½ to original state
  - Edge (a,b') with weight ½ to the clone-state b' of b

PSPACE membership: exponentially long numbers

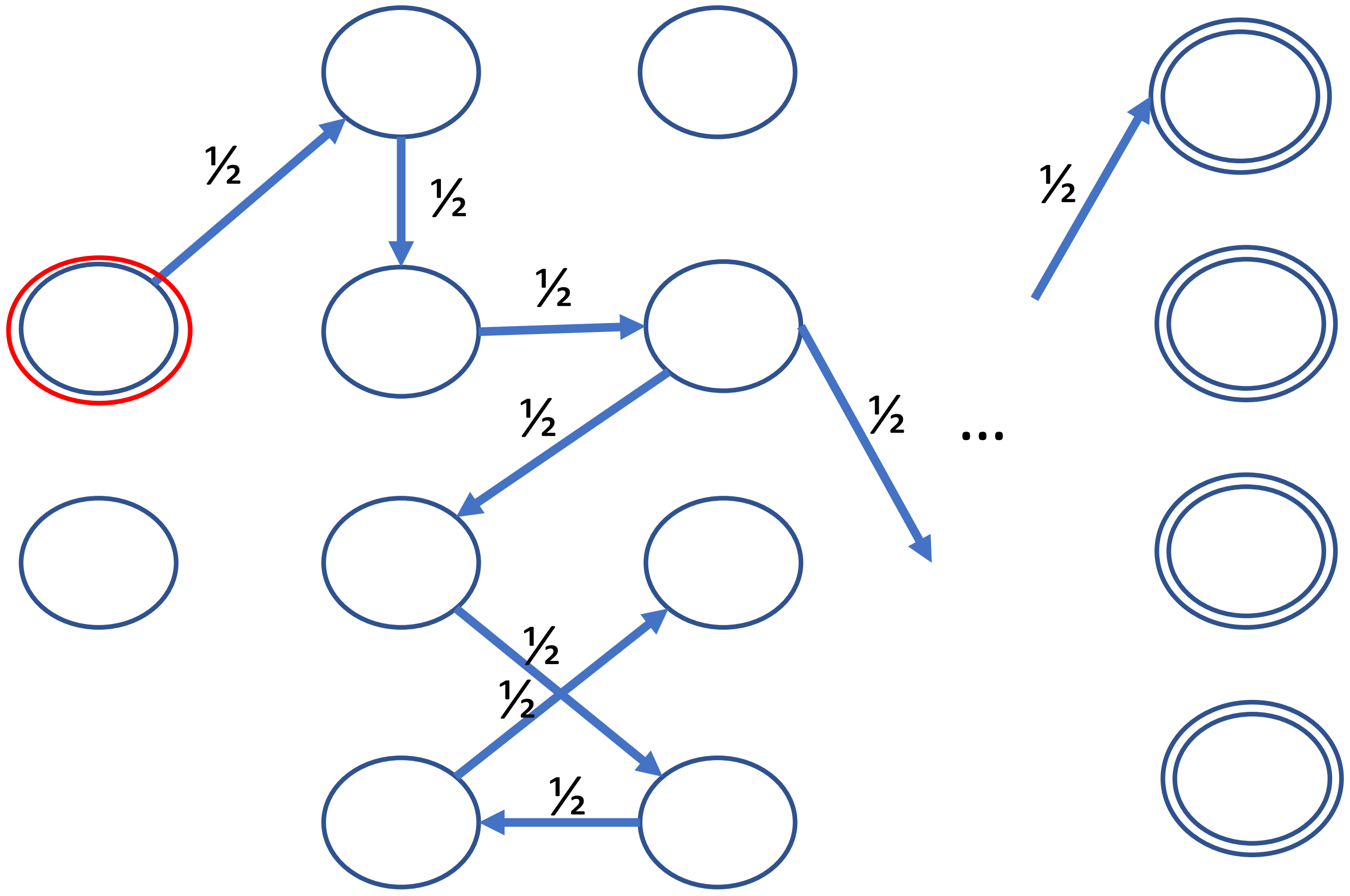
For a program D

for all neighboring inputs x, x':

- Construct the Markov chain for $D(x)$ and $D(x')$
- Compare hitting probabilities of the final states with the same values

Problem:

Markov chain has exp-many states



PSPACE membership: exponentially long numbers

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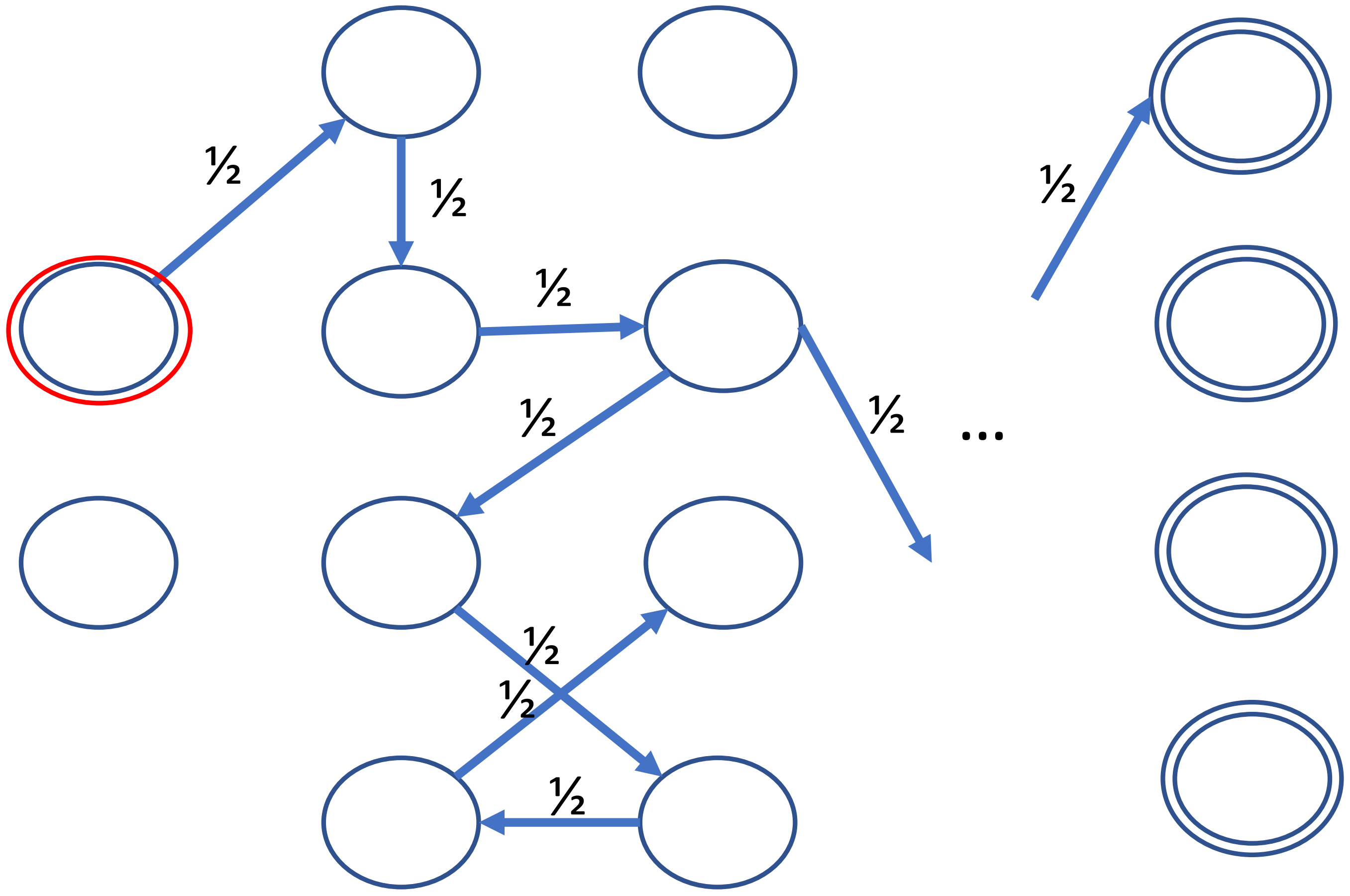
Problem:

Markov chain has exp-many states $\Rightarrow$ hitting probabilities can be as small as $\frac{1}{2^{exp}}$



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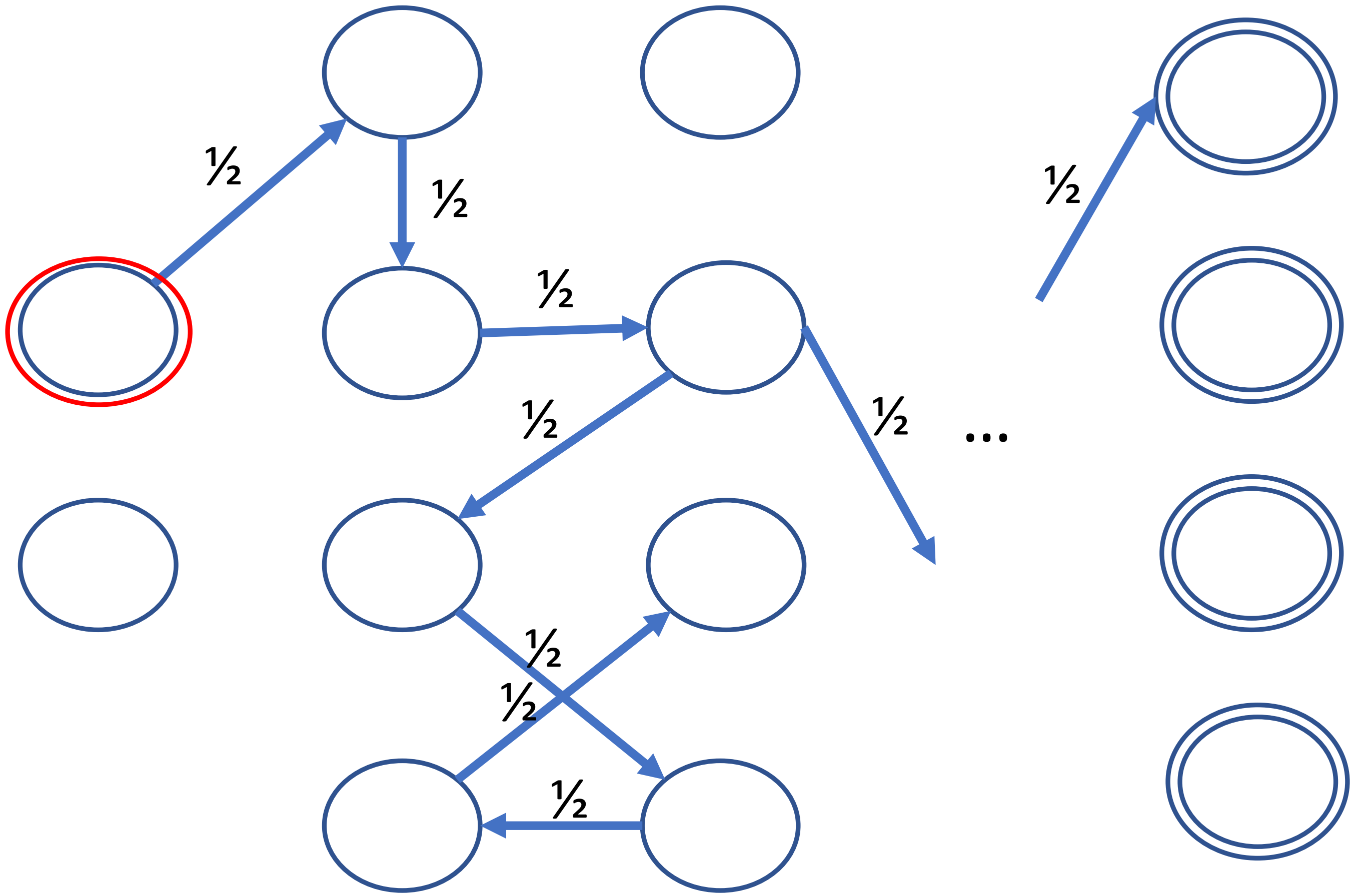
Problem:

Markov chain has exp-many states
⇒ hitting probabilities can be as small as $\frac{1}{2^{exp}}$

⇒ numbers are exponentially long



Operations with exponentially long numbers

1	**0**	**1**	**1**	**1**	**…**	**1**	**0**	**0**	**1**

+

0	**0**	**1**	**0**	**1**	**…**	**1**	**1**	**1**	**1**

1	**1**	**1**	**0**	**1**	**…**	**1**	**0**	**0**	**0**

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	1	**0**	**1**	**1**	**1**	**…**	**1**	**0**	**0**	**1**
+										
	0	**0**	**1**	**0**	**1**	**…**	**1**	**1**	**1**	**1**
	1	**1**	**1**	**0**	**1**	**…**	**1**	**0**	**0**	**0**

Uniform family of log-depth circuits:

- One logspace algorithm provides implicit access to the circuits
- Each circuit has log-depth and poly size

Operations with exponentially long numbers

1	**0**	**1**	**1**	**1**	**…**	**1**	**0**	**0**	**1**

+

0	**0**	**1**	**0**	**1**	**…**	**1**	**1**	**1**	**1**

1	**1**	**1**	**0**	**1**	**…**	**1**	**0**	**0**	**0**

Uniform family of log-depth circuits:

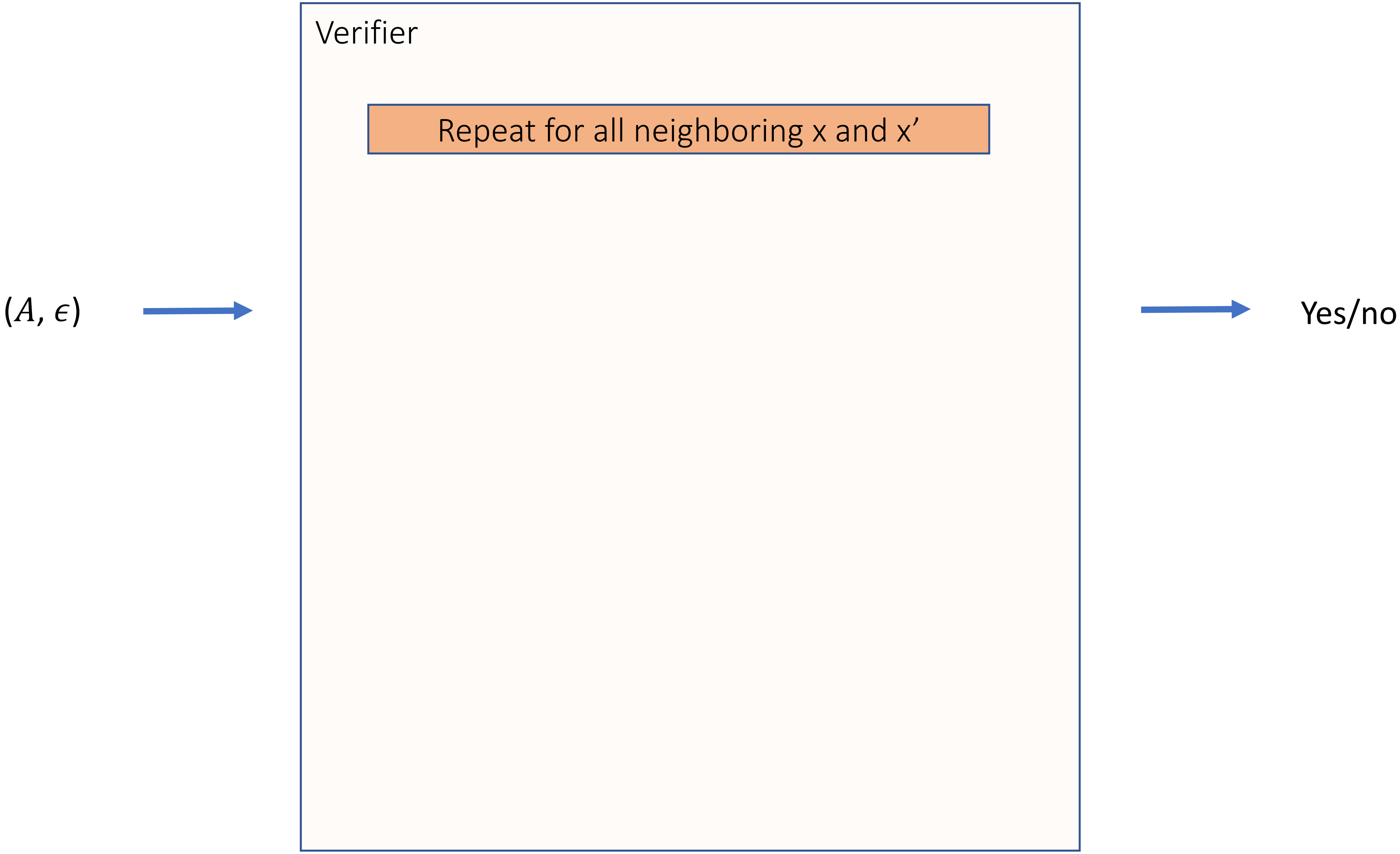
- One logspace algorithm provides implicit access to the circuits
- Each circuit has log-depth and poly size

Lemma:

Uniform families of log-depth circuits exist for:

- Comparison
- Addition
- Multiplication by a fixed rational number
- Multiplication [Ofman’62]
- Square roots [Reif’86]

Polyspace membership algorithm for (ϵ,0)-DP



Polyspace membership algorithm for (ϵ,0)-DP

(A, ϵ) →

Verifier

Repeat for all neighboring x and x'

Provide access to Markov chain corresponding to A(x) and A(x')

→ Yes/no

Polyspace membership algorithm for (ϵ,0)-DP

(A, ϵ) →

Verifier

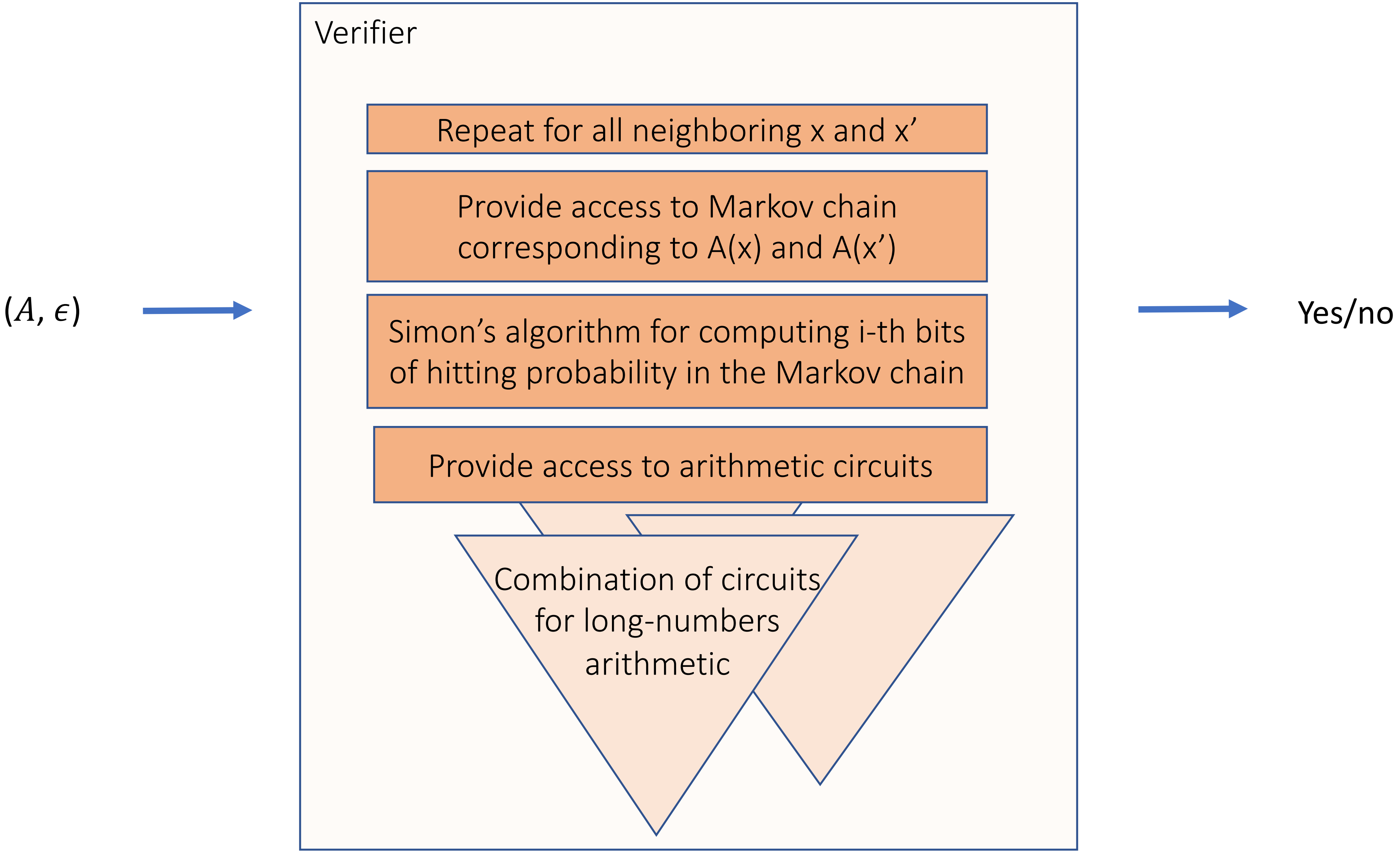
Repeat for all neighboring x and x'

Provide access to Markov chain corresponding to A(x) and A(x')

Simon's algorithm for computing i-th bits of hitting probability in the Markov chain

→ Yes/no

Polyspace membership algorithm for (ϵ,0)-DP



Extending the result to (ϵ, δ)-DP, RDP, zCDP, TCDP

For verifying (ϵ, δ)-DP we use a point-wise definition

- Additionally use a summation of an exponentially long numbers

$$\sum_{o \in \{0,1\}^l \cup \{\perp\}} \left(\delta_{x,x'}(o)\right) \leq \delta$$

$$\delta_{x,x'}(o) = \max(\Pr[C(x) = o] - e^{\epsilon} \Pr[C(x') = o])$$

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- Need to compute Rényi divergence
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    - Exponentiations to rational degrees

$$D_\alpha(P|Q) = \frac{1}{\alpha - 1} \log \sum \frac{p_i^\alpha}{q_i^{\alpha-1}}$$

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For verifying RDP, zCDP, TCDP

- Need to compute Rényi divergence
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  - Exponentiations to rational degrees
- Get infinite fractions
  - Computations with a fixed precision η
  - Consider gap-versions of the problem

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Results and future work

- We showed PSPACE-completeness for the problems of checking:
    - Pure-DP
    - Approximate-DP
    - Gap-RDP
    - Gap-zCDP
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Results and future work

- We showed PSPACE-completeness for the problems of checking:
  - Pure-DP
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  - Gap-TCDP

- Possibly can extend the result to show PSPACE-completeness of verifying accuracy
- Improve the exact polynomial in the space complexity of the algorithm
  - Improved analysis and more efficient algorithms for Markov chains analysis and long-numbers arithmetic operations are needed for tighter results

Polyspace algorithm for RDP, zCDP, TCDP

An algorithm is **RDP/zCDP/TCDP** if for a privacy parameter ρ and a fixed/any/bounded $\alpha > 1$ Rényi divergence for any neighboring inputs x, x' is at most $\rho\alpha$

[Mironov'17],[Dwork-Rothblum'16,Bun-Steinke'16], [Bun,Dwork,Rothblum,Steinke'18]

$$D_\alpha(P|Q) = \frac{1}{\alpha - 1} \log \sum \frac{p_i^\alpha}{q_i^{\alpha-1}}$$

$$D_\alpha(C(x)|C(x')) \leq \rho\alpha$$

New problems:

- To compute Rényi divergence we compute
  - Logarithms
  - Exponentiations to rational degrees
- Hence, get infinite fractions

Solution:

- Computations with a fixed precision η
- Consider gap-versions of the problem

We define Gap-RDP on (C, ρ, α, η) as follows:

- Yes-instance, if for all neighboring x, x'

$$D_\alpha(C(x)|C(x')) \leq \rho\alpha$$

- No-instance, if for at least one neighboring x, x'

$$D_\alpha(C(x)|C(x')) \geq \rho\alpha + \frac{1}{2^\eta}$$

Polyspace algorithm for zCDP

Another problem: an algorithm is **zCDP** if **for all values of α** Rényi divergence for any neighboring inputs x, x' is bounded by $\rho\alpha$

Solution: showed that it is sufficient to check values of α from **the bounded range**:

Lemma:

C is ρ-zCDP, then for all neighboring x, x': $D_\alpha(C(x) | C(x')) \leq \rho\alpha$

C is not $(\rho + 2^{-\eta})$-zCDP, then exist neighboring x, x', exists $\alpha \in (1, 1 + 2^{poly(n)}/\rho)$:

- α is a multiple of $2^{-\eta}$
- $D_\alpha(C(x) | C(x')) \geq \rho\alpha + 2^{-\eta-1}$

Polyspace membership algorithm for Gap-zCDP