Lower bound for read-once nondeterministic branching program for satisfiable Tseitin formula using tree-width

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Outline

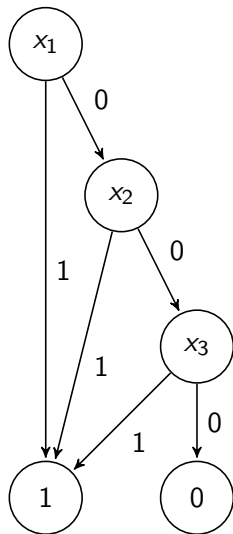- Branching programs
- Tseitin formulas
- Lower bounds

Branching program

- BP is a way to represent Boolean function

  - directed graph without cycles;
  - two sinks: labeled with 0 and 1, one source;
  - every internal vertex labeled with a variable and has two outgoing edges: labeled with 0 and 1;
 - the value of function equals label of the sink in the end of corresponding path.

- BP size is between circuit and formula sizes

  - $C(f) \leq 3BP(f) \leq O(L^{1+\epsilon}(f))$,
  - $C(f)$ circuit complexity of f, $L(f)$ size of minimal formula, BP size of minimal branching program.

Non-deterministic Branching program

- ► NBP additionally has non-deterministic nodes:
  - ► non-deterministic nodes are unlabeled
  - ► the value equals 1 iff there exists a path to 1-sink
- ► BP corresponds to L/poly
- ► NBP corresponds to NL/poly
- ► k-(N)BP if for every path every variable occurs no more than k times
- ► OBDD is 1-BP with fixed order for variables in every path
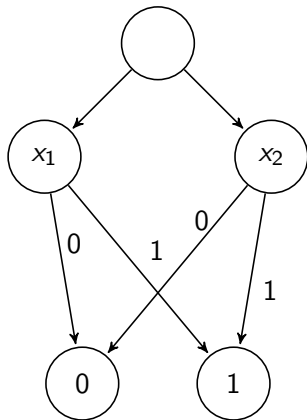
Known lower bounds for k-BPs

- (Borodin, Razborov, Smolensky, 1993)
  $CLIQUE\_ONLY_n(G) : \{0,1\}^{\frac{n \cdot (n-1)}{2}} \to 0,1$
  - $CLIQUE\_ONLY_n(G) = 1$ iff graph G on n vertices is exactly $\frac{n}{2}$-clique
  - 1-NBP($CLIQUE\_ONLY_n$) $= 2^{\Omega(n)}$
  - 2-BP($CLIQUE\_ONLY_n$) $= poly(n)$
- (Borodin, Razborov, Smolensky, 1993) first strongly exponential lower bound
  - f artificially constructed function
  - 1-NBP(f) $= 2^{\Omega(n)}$
- (Thathachar, 1998) an explicit functions f_k for every k:
  - k-NBP(f_k) $= 2^{\Omega(n^{1/k})}$
  - (k+1)-BP(f_k) $= O(n)$

Known lower bounds for k-BPs (2)

- (Duris, Hromkovic, Jukna, Sauerhoff, Schnitger, 2004)
    - $\oplus$ parity of the number of triangles in a graph, $\overline{\triangle}$ is 1 iff graph has no triangles
    - 1-NBP($\oplus$) = $2^{\Omega(n)}$
    - 1-NBP($\overline{\triangle}$) = $2^{\Omega(n)}$
- (Juhna, 1995)
    - f_k characteristic function of error-correcting codes
    - k-NBP(f_k) = $2^{\Omega(\sqrt{n})}$

Tseitin formulas

- Tseitin $TS_{G,c}$ formula defined for a graph
  - every edge is labeled with a variable
  - every vertex has a 0-1 label: $c : V \to 0, 1$
  - $TS_{G,c}(x) = 1 \iff$

    $$\bigwedge_{v \in V} \left(\sum_{e \text{ incident } v} x_e = c(v) \bmod 2 \right)$$

- A Tseitin formula is satisfiable iff for every connected component the sum of labels is even
- Unsatisfiable Tseitin formulas are classical hard examples for proof systems:
  - Resolution [Tseitin, 1968], [Urquhart, 1987]
  - Bounded depth Frege [Ben-Sasson, 2002], [Pitassi, Rossman, Servedio, Tan, 2016]
  - Polynomial Calculus over field with char $!= 2$ [Alekhnovich, Razborov, 2001]
  - Tree-like Lovasz-Schrijver [Itsykson, Kojevnikov, 2006]
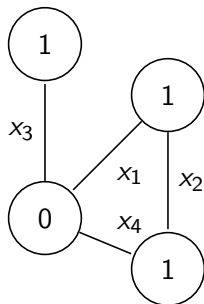
$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_3 + x_4 = 0 \\ x_2 + x_4 = 1 \\ x_3 = 1 \end{cases}$$

Tseitin formulas (2)

- (Itsykson, Knop, Romashchenko, Sokolov, 2017) Exponential lower bound for OBDD(join) proof systems for unsatisfiable Tseitin formulas
    - key step: satisfiable Tseitin formulas are hard for OBDDs.
    - $OBDD(TS_{G,c}) = 2^{\Omega(n)}$
- OBDD is a partial case of 1-BP
    - (Bryant, 1991) (Sieling, Wegener, 1995)
      $f : OBDD(f) = 2^{\Omega(n)}$, 1-BP(f)=$poly(n)$

The goal: what is the size of 1-BP for satisfiable Tseitin formulas?

Lower bound for Tseitin formula on an expander

Theorem:
1-NBP for satisfiable Tseitin formula for (n, d, α)-expander is $2^{\Omega(n)}$, where

- $\alpha < \frac{1}{3}$
- (n, d, α)-expander is a d-regular graph on n vertices with absolute value of the second largest eigenvalue $\leq d \cdot \alpha$

Generalized theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V| - k_G(l) - k_G(|E| - l) + 1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deleting of l edges.

The comparison with other results

Satisfiable Tseitin formulas $TS_{G,w}$ on (n, d, α)-expander can be represented by:

- CNF($TS_{G,c}$)=$O(n)$
- 1-NBP($\rceil TS_{G,c}$)=$O(n)$
- 1-NBP($TS_{G,c}$)=$2^{\Omega(n)}$
- 2-BP($TS_{G,c}$)=$O(n)$

Best previously known gaps:

CNF(f)=$O(n^3)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004)
1-NBP($\rceil$f)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Juhna, 2009) explicit construction
1-NBP($\rceil$f)=$O(n)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004) probabilistic construction
2-BP(f)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Thathachar, 1998)

Idea of the proof

Theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V|-k_G(l)-k_G(|E|-l)+1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deleting of l edges.

**Idea of the proof:**
Choose level l of the 1-NBP

- show that there are at least $2^{C_1 n}$ different non-zero paths to level l
- show that for every node v on the l-th lever only $2^{C_2 n}$ non-zero paths go from source to the same node v

Get that at least $2^{(C_1-C_2)n}$ different nodes are on the l-th level, so is in the 1-NBP
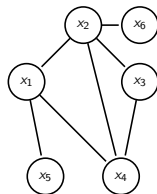
Tree-width and path-width

Tree (path) decomposition of a graph:

1. Vertices (called "bags") correspond to sets of vertices of $G = (V, E)$
2. If $(a, b) \in E$ then there is a bag with a and b
3. All bags with the same vertex form a tree (path)
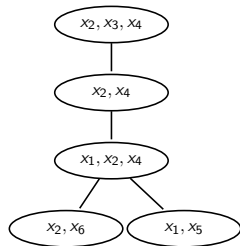
Width of a decomposition is the size of a maximal bag in it.

Tree-width (**path-width**) of a graph is the minimal width among all its tree (path) decompositions.

Graph minor

Minor of a graph G is a subgraph of G that can be obtained from G by sequence of edge contractions, edge and vertex deletion.

Grid Minor Theorem [Robertson, Seymour 1986], [Chuzhoy 2015]:
Every graph G of a tree-width t has a grid minor of size t^δ, where $\delta < \frac{1}{36}$.

Lower bound for any graph based on its tree-width

Lower bound using tree-width: 1-NBP for satisfiable Tseitin formula on a graph G has size $\Omega(2^{t^{\delta}})$, where t is a tree-width of G, $\delta < \frac{1}{36}$.

Sketch of the proof:

▶ Take the smallest 1-NBP for a Tseitin formula on a graph G
▶ Apply sequence of edge contractions, edge and vertex deletions that could lead only to a smaller 1-NBP
▶ Obtain 1-NBP for a $t \times t$ grid-minor of a graph G
▶ Show that 1-NBP for $t \times t$ grid graph has size $2^{\Omega(t)}$

Lower bound for a grid graph

We want to show that 1-NBP for $t \times t$ grid graph has size $2^{\Omega(t)}$

- Use theorem: 1-NBP for a satisfiable Tseitin formula on a graph G has at least $2^{|V|-k_G(l)-k_G(|E|-l)+1}$ nodes
- We set $l = \frac{|E|}{2}$
- We want to show that the number of connected components that can be obtained after deleting of half of edges is at most $\frac{|V|}{2} - \varepsilon\sqrt{|V|}$
- After that we get a lower bound $2^{\Omega(2\varepsilon\sqrt{|V|})} = 2^{\Omega(t)}$
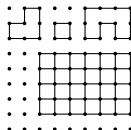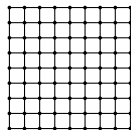
Reducing NBP to its minor

We need to check that all operations can only reduce the size of a 1-NBP

- **Deletion of edge** x_i: substitute any value that will not break the satisfiability of the formula
- **Deletion of vertex** v: delete all connected edges one-by-one
- **Contraction of edge** x_i: the same as the satisfiability of the formula $\exists x_i : Ts_{G,w}(x)$
  - in 1-NBP all nodes labeled with x_i should be changed to non-deterministic nodes

Upper bound using path-width

Theorem: a satisfiable Tseitin formula on graph G can be computed by an OBDD of size $O(m2^{p+1})$, where m is the number of edges and p is the path-width of G.

Plan of the proof: We build OBDD layer-by-layer, each layer corresponds to one edge. Each edge corresponds to the first bag where it appears in the decomposition.
For every edge and its bag we add at most 2^{p+1} vertices that correspond to all possible values of parity in vertices from that bag.
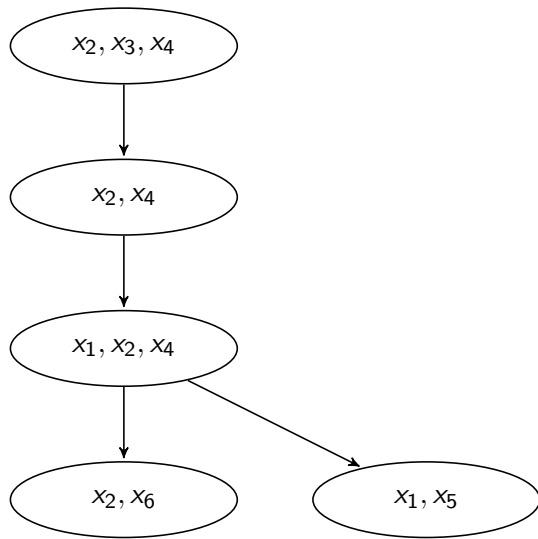
Results and open questions

- Currently we have a lower bound using tree-width and upper bound using path-width. Can we close this gap?
  - $tw(G) \leq pw(G) \leq tw(G) \cdot \log(tw(G))$
- Prove non-trivial lower bound for semantic 1-NBP
  - only consistent paths should contain every variable no more than once
  - that means that all paths that consist repeated variable should not be reachable by any assignment

No exponential lower bounds for semantic 1-NBP for a boolean function

All known results with exponential lower bounds are for functions with non-boolean domain [Cook, Edmonds, Medabalimi, Pitassi, 2016], [Jukna, 2009]

Results and open questions

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