On Tseitin formulas, read-once branching programs and treewidth

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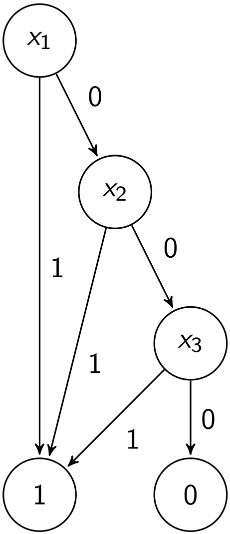
Outline

- Branching programs
- Tseitin formulas
- Tree-width and path-width
- Lower bounds

Branching program

- BP is a way to represent Boolean function
  - directed graph without cycles;
  - two sinks: labeled with 0 and 1, one source;
  - every internal vertex labeled with a variable and has two outgoing edges: labeled with 0 and 1;
  - the value of function equals label of the sink in the end of corresponding path.



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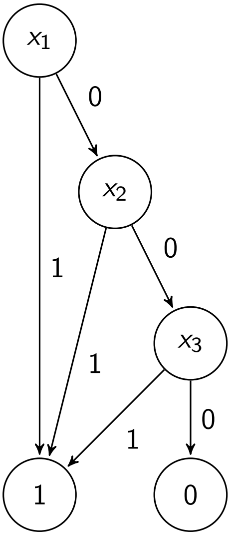
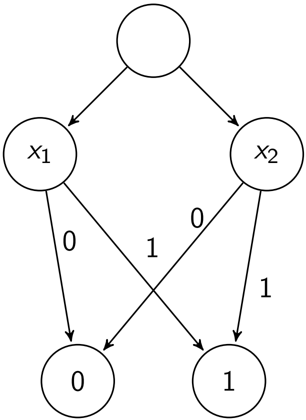
  - the value of function equals label of the sink in the end of corresponding path.
- BP size is between circuit and formula sizes
  - $C(f) \leq 3BP(f) \leq O(L^{1+\epsilon}(f))$,
  - $C(f)$ circuit complexity of f, $L(f)$ size of minimal formula, BP size of minimal branching program.



Non-deterministic Branching program

- NBP additionally has non-deterministic nodes:
  - non-deterministic nodes are unlabeled
  - the value equals 1 iff there exists a path to 1-sink



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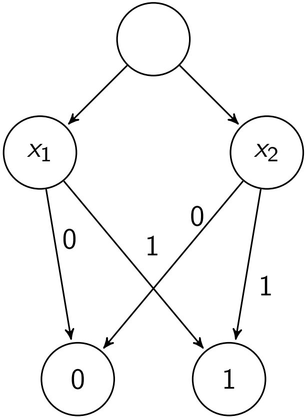

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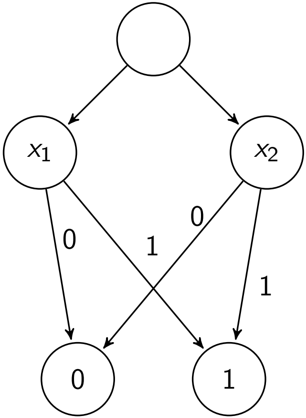

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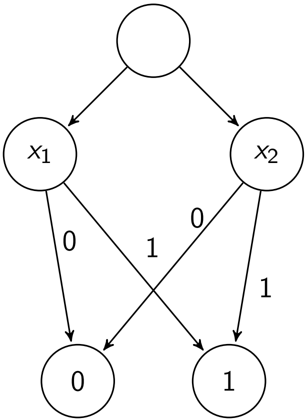
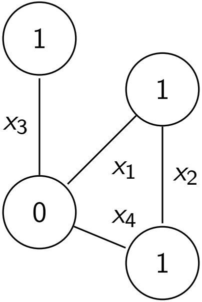
- OBDD is 1-BP with fixed order for variables in every path



Tseitin formulas

- Tseitin $TS_{G,c}$ formula defined for a graph
  - every edge is labeled with a variable
  - every vertex has a 0-1 label: $c : V \to 0, 1$
  - $TS_{G,c}(x) = 1 \iff$
    $$\bigwedge_{v \in V} \left(\sum_{e \text{ incident } v} x_e = c(v) \bmod 2 \right)$$



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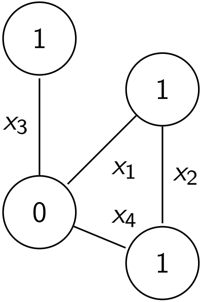

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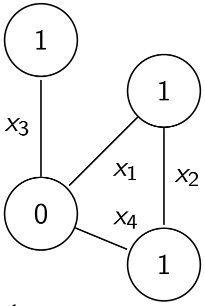
- Unsatisfiable Tseitin formulas are classical hard examples for proof systems:
  - Resolution [Tseitin, 1968], [Urquhart, 1987]
  - Bounded depth Frege [Ben-Sasson, 2002], [Pitassi, Rossman, Servedio, Tan, 2016]
  - Polynomial Calculus over field with char != 2 [Alekhnovich, Razborov, 2001]
  - Tree-like Lovasz-Schrijver [Itsykson, Kojevnikov, 2006]



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Tseitin formulas (2)

- (Itsykson, Knop, Romashchenko, Sokolov, 2017) Exponential lower bound for OBDD(join) proof systems for unsatisfiable Tseitin formulas
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  - $OBDD(TS_{G,c}) = 2^{\Omega(n)}$
- OBDD is a partial case of 1-BP
  - (Bryant, 1991) (Sieling, Wegener, 1995)
    $f : OBDD(f) = 2^{\Omega(n)}$, 1-BP(f)=$poly(n)$

The goal: what is the size of 1-BP for satisfiable Tseitin formulas?

Lower bound for Tseitin formula on an expander

Theorem:
1-NBP for satisfiable Tseitin formula for (n, d, α)-expander is $2^{\Omega(n)}$, where

- $\alpha < \frac{1}{3}$
- (n, d, α)-expander is a d-regular graph on n vertices with absolute value of the second largest eigenvalue $\leq d \cdot \alpha$

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Generalized theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V| - k_G(l) - k_G(|E| - l) + 1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deletion of l edges.

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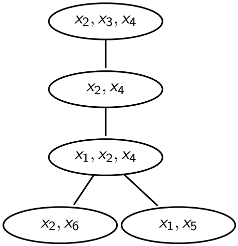
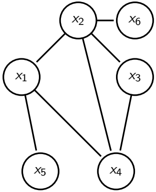
Tree (path) decomposition of a graph $G = (V, E)$ is a graph T that is a tree (path)





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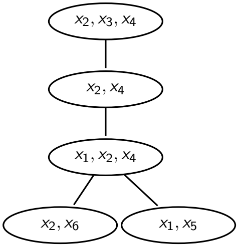

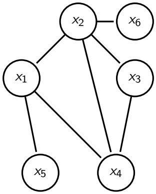

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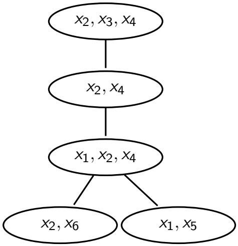

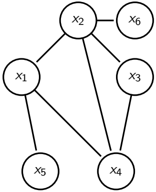

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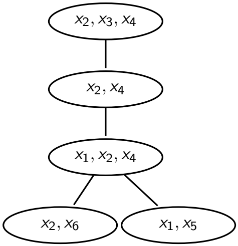

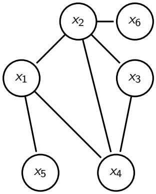

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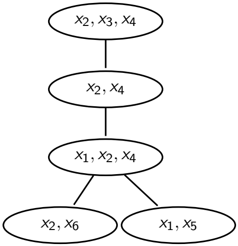

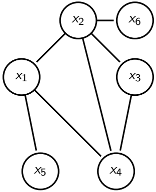

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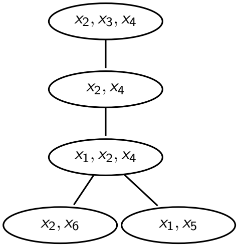
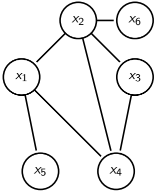
Tree-width (**path-width**) of a graph is the minimal width among all its tree (path) decompositions.





Lower and upper bounds

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Graph minor

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Grid Minor Theorem [Robertson, Seymour 1986], [Chuzhoy 2015]:
Every graph G of a tree-width t has a grid minor of size $t^\delta \times t^\delta$, where $\delta < \frac{1}{36}$.

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- Show that 1-NBP for $t^\delta \times t^\delta$ grid graph has size $2^{\Omega(t^\delta)}$

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No exponential lower bounds for semantic 1-NBP for a boolean function

All known results with exponential lower bounds are for functions with non-boolean domain [Cook, Edmonds, Medabalimi, Pitassi, 2016], [Jukna, 2009]

Lower bound for a grid graph

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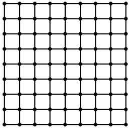

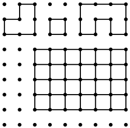

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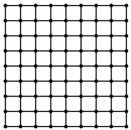

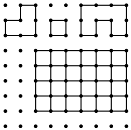

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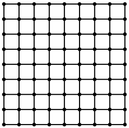

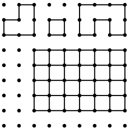

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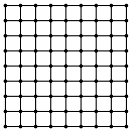

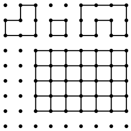

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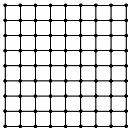
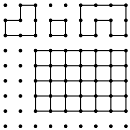
- After that we get a lower bound $2^{\Omega(2\varepsilon\sqrt{|V|})} = 2^{\Omega(s)}$





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- **Deletion of vertex** v: delete all connected edges one-by-one
- **Contraction of edge** x_i: the same as the satisfiability of the formula $\exists x_i : Ts_{G,w}(x)$
  - in 1-NBP all nodes labeled with x_i should be changed to non-deterministic nodes

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Known lower bounds for k-BPs

- (Borodin, Razborov, Smolensky, 1993)
  $CLIQUE\_ONLY_n(G) : \{0,1\}^{\frac{n\cdot(n-1)}{2}} \rightarrow 0,1$
  - $CLIQUE\_ONLY_n(G) = 1$ iff graph G on n vertices is exactly $\frac{n}{2}$-clique
  - 1-NBP$(CLIQUE_ONLY_n) = 2^{\Omega(n)}$
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- (Thathachar, 1998) an explicit functions f_k for every k:
  - k-NBP(f_k) $= 2^{\Omega(n^{1/k})}$
  - (k+1)-BP(f_k) $= O(n)$

Known lower bounds for k-BPs (2)

- (Duris, Hromkovic, Jukna, Sauerhoff, Schnitger, 2004)
  - $\oplus$ parity of the number of triangles in a graph, $\overline{\triangle}$ is 1 iff graph has no triangles
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  - 1-NBP($\overline{\triangle}$) $= 2^{\Omega(n)}$
- (Jukna, 1995)
  - f_k characteristic function of error-correcting codes
  - k-NBP(f_k) $= 2^{\Omega(\sqrt{n})}$

The comparison with other results

Satisfiable Tseitin formulas $TS_{G,w}$ on (n, d, α)-expander can be represented by:

- CNF($TS_{G,c}$)=$O(n)$
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Best previously known gaps:

CNF(f)=$O(n^3)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004)
1-NBP($\rceil f$)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Jukna, 2009) explicit construction
1-NBP($\rceil f$)=$O(n)$	1-NBP(f)=$2^{\Omega(n)}$ (Duris et al, 2004) probabilistic construction
2-BP(f)=$O(n)$	1-NBP(f)=$2^{\Omega(\sqrt{n})}$ (Thathachar, 1998)

Idea of the proof

Theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V|-k_G(l)-k_G(|E|-l)+1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deletion of l edges.

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Get that at least $2^{(C_1-C_2)n}$ different nodes are on the l-th level, so is in the 1-NBP