The Complexity of Verifying Boolean Programs as Differentially Private

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Joint work with Mark Bun and Marco Gaboardi

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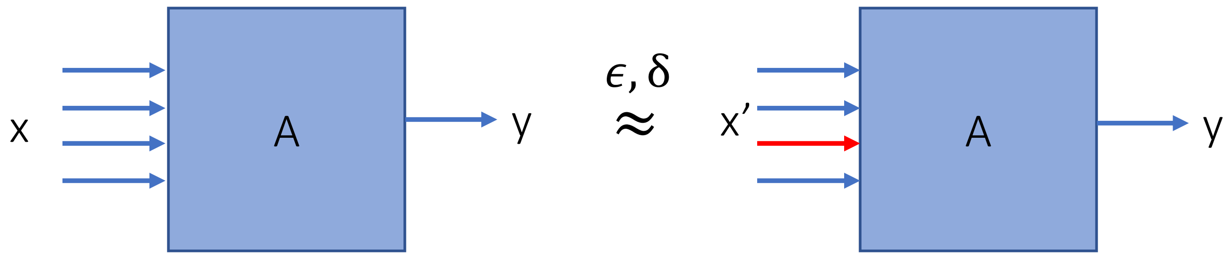
Plan of the talk

1. Prior work
   - How hard is it to verify whether a program is DP for
     - Turing-complete languages
     - Boolean languages with bounded memory without loops
2. Our results and proof ideas
   - BPWhile: Boolean language with loops and finite memory
   - PSPACE-completeness of verifying DP for BPWhile
   - PSPACE-hardness: reduction from TQBF
   - PSPACE algorithm based on computing hitting probabilities in a Markov chain

Differential Privacy



C is (ϵ, δ)-differentially private if for every set of possible outputs O, and for every neighboring x, x':

$$\mathrm{P}[C(x) \in O] \leq e^{\epsilon} \cdot \mathrm{P}[C(x') \in O] + \delta$$

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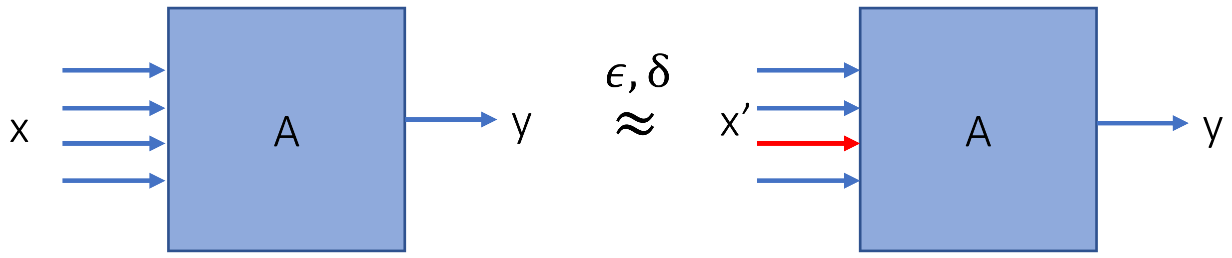

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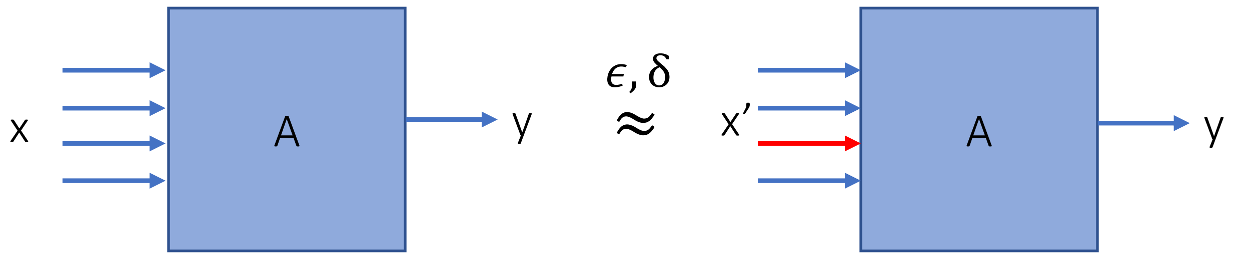

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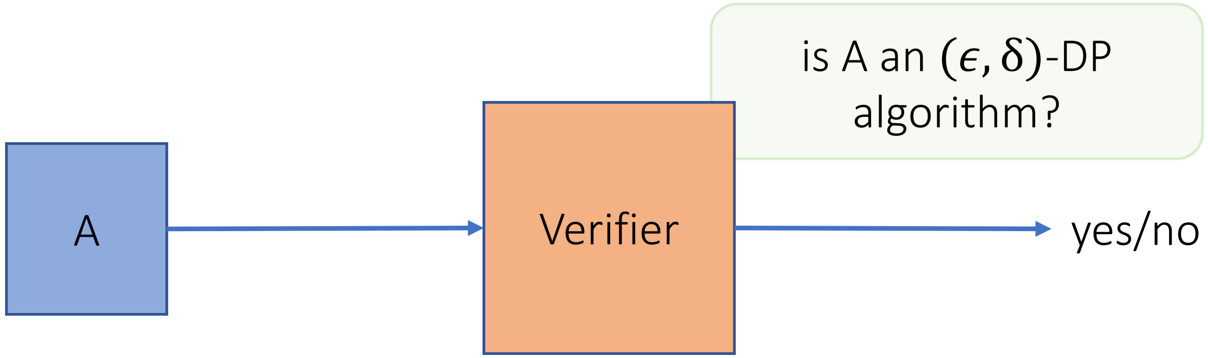
Neighboring relation we consider:

- Inputs are bit strings which differ in one bit
- Can be extended to any bounded polyspace computable relation

Verification of Differential Privacy



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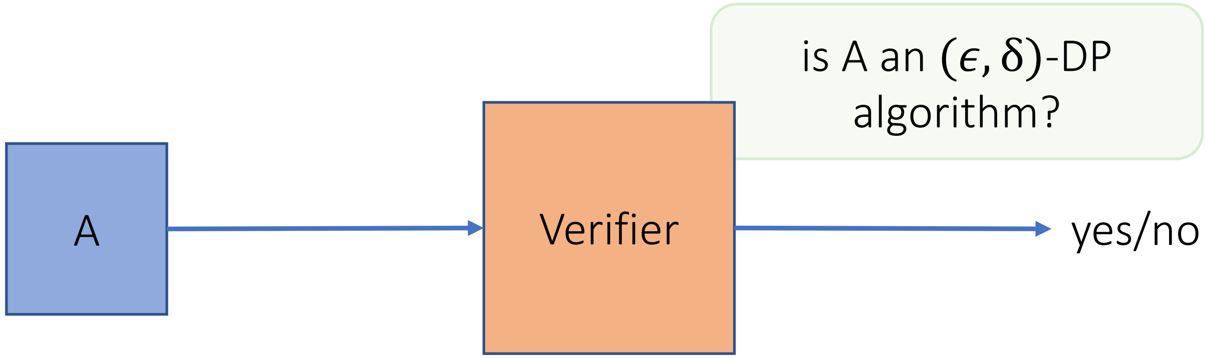

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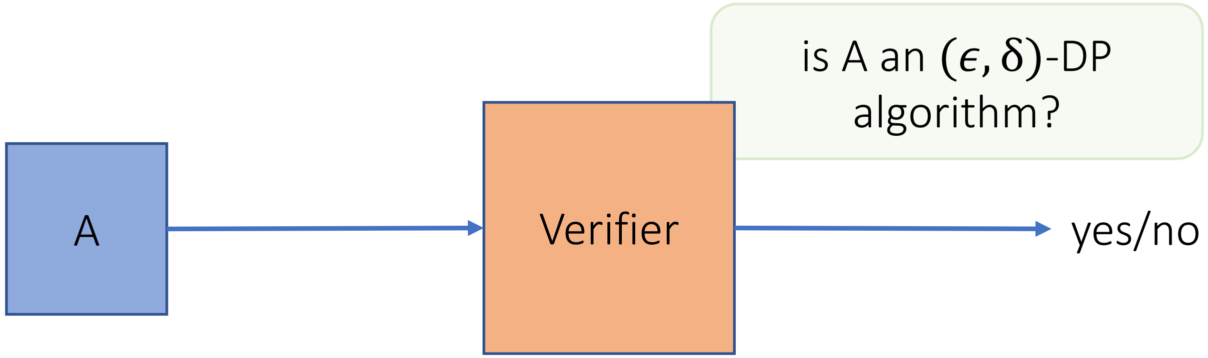

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  - Undecidable for Turing-complete languages

[Barthe, Chadha, Jagannath, Sistla, Viswanathan'20]

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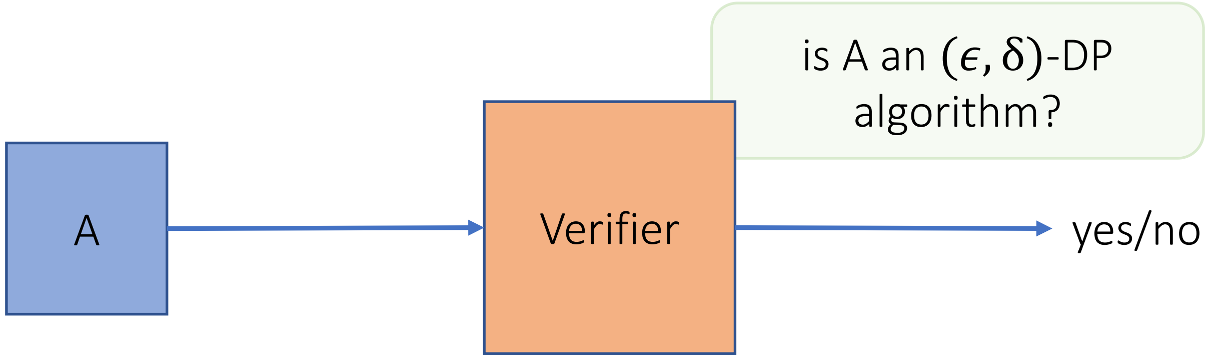

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[Barthe, Chadha, Jagannath, Sistla, Viswanathan'20]

- For a simple Boolean language with **bounded memory,** if statements and **random assignments**, but **without loops**
  - $coNP^{\#P}$-completeness for $(\epsilon, 0)$-DP
    - Reduction from All-Min-SAT
  - $coNP^{\#P}$-hard and in $coNP^{\#P^{\#P}}$ for (ϵ, δ)-DP

[Gaboardi, Nissim, Purser'20]

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In this work: white-box model, full access to the code

BPWhile: Boolean language with While loops

We design a language to meet the following goals:

- Captures classical computations on real computers
- Simple to analyze

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Boolean expressions

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Numbers of variables and input bits are fixed in the definition of the program

⇒ Length of the program is an upper bound on the size of the memory that the program uses

Example of a BPWhile program:

$\texttt{input}(\vec{c}, \epsilon);$
$\vec{k} := \lceil \log(2/\epsilon) \rceil;$
$\vec{d} := (2^{\vec{k}+1} + 1)(2^{\vec{k}} + 1)^{n-1};$
$\vec{u} := \texttt{uniform}(0, \vec{d}];$
$\vec{z} := 0;$
$\vec{r} := n;$
$\texttt{while}\ \vec{z} < \vec{n} \wedge \vec{r} = n\ \texttt{then}$
$\quad \texttt{if}\ \vec{z} < \vec{c}\ \texttt{then}$
$\quad\quad \texttt{if}\ \vec{u} \leq 2^{\vec{k}(\vec{c}-\vec{z})}(2^{\vec{k}} + 1)^{n-(\vec{c}-\vec{z})}$
$\quad\quad \texttt{then}\ \vec{r} := \vec{z}$
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$\quad \texttt{else}$
$\quad\quad \texttt{if}\ \vec{u} \leq d - 2^{\vec{k}(\vec{z}-\vec{c}+1)}(2^{\vec{k}} + 1)^{n-1-(\vec{z}-\vec{c})}$
$\quad\quad \texttt{then}\ \vec{r} := \vec{z}$
$\quad\quad \texttt{else skip}$
$\quad \vec{z} = \vec{z} + 1;$
$\texttt{return}(\vec{z});$

Implementation of the Bounded Geometric Mechanism in finite precision arithmetic

[Ghosh, Roughgarden, Sundararajan'09]

[Balcer, Vadhan'17]

Our results

Main result: if A is a BPWhile program, then the problem of verifying whether A is differentially private is PSPACE-complete.

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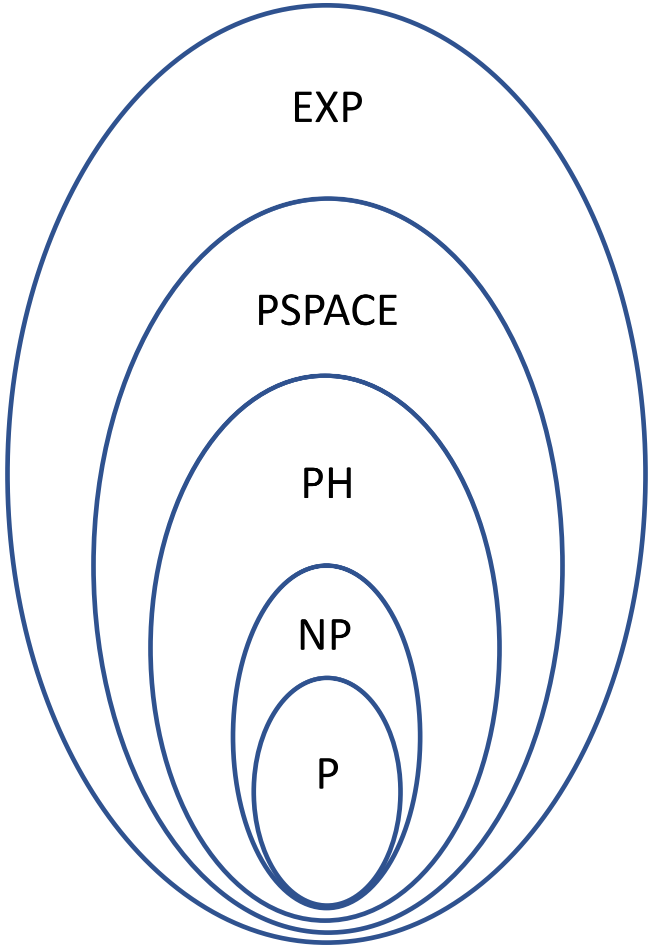
It holds for the following notions of differential privacy:

- $(\epsilon, 0)$-DP
- (ϵ, δ)-DP
- (ϵ, δ)-DP parameters approximation
- Renyi-DP
- Zero-Concentrated-DP
- Truncated Concentrated-DP

PSPACE-completeness

PSPACE-completeness of a problem A implies:



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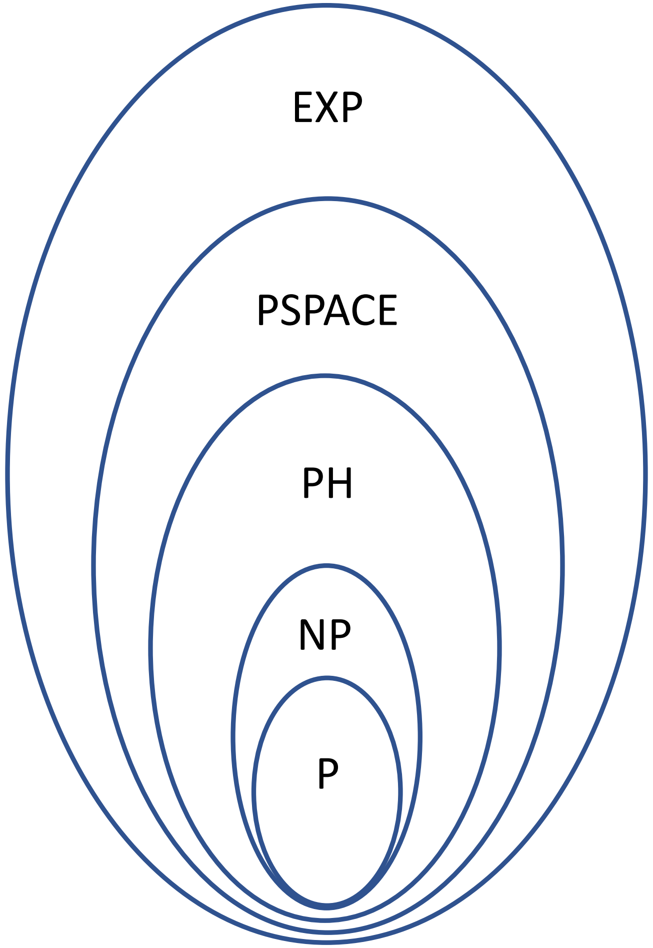

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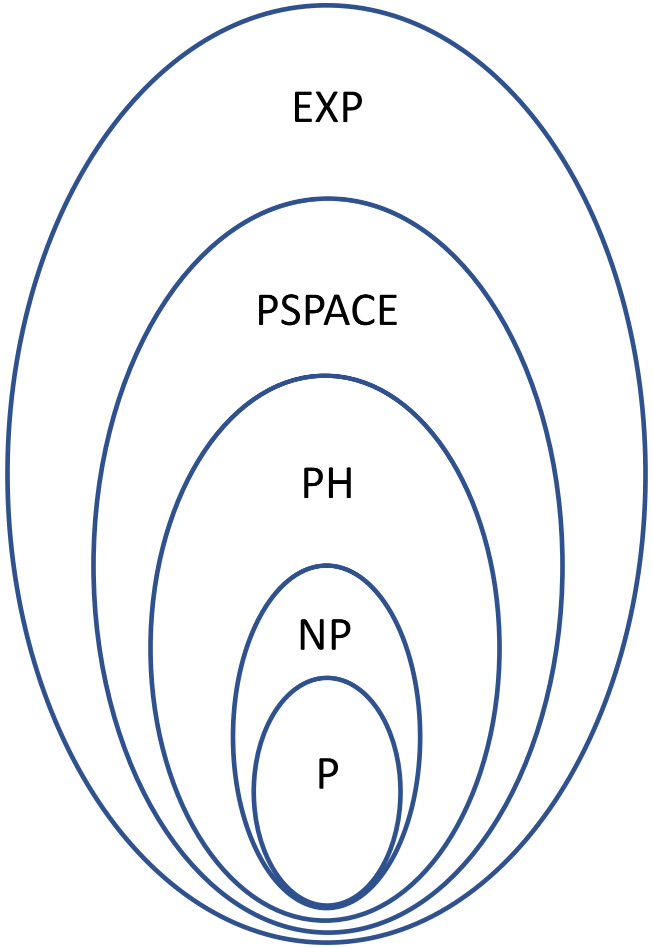

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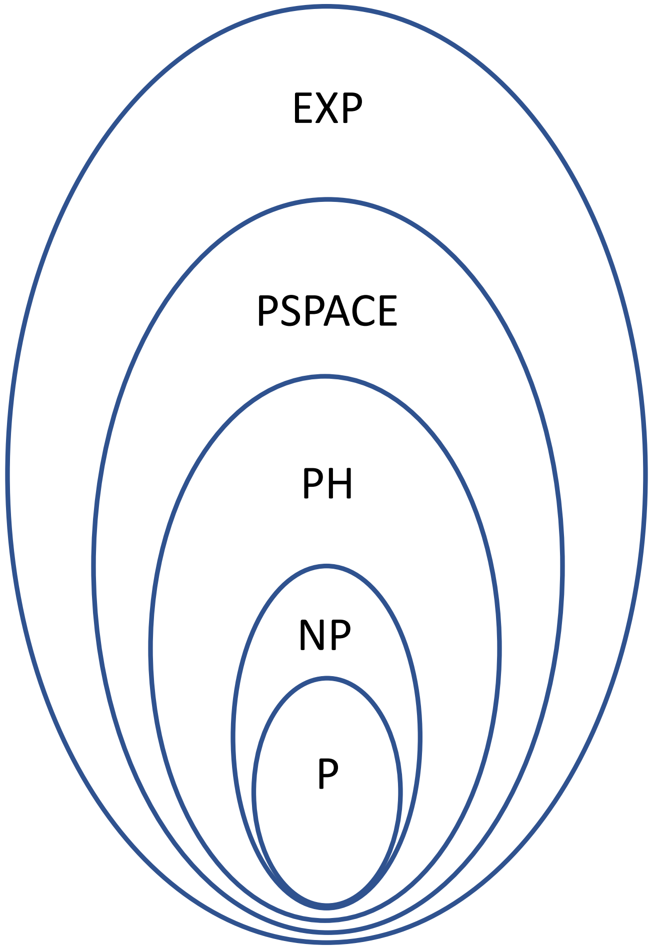

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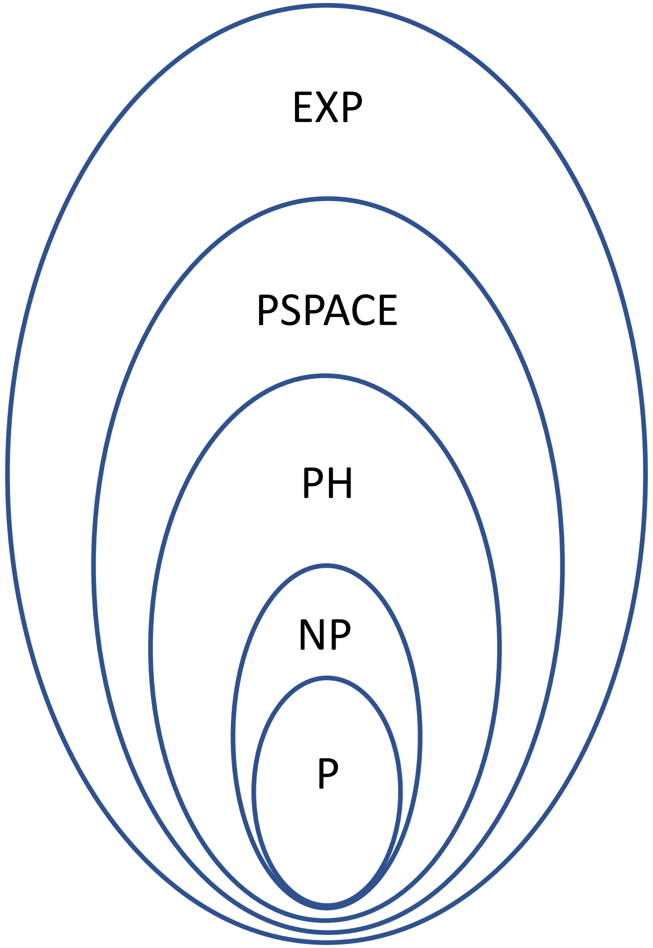
- No polytime algorithm for A, unless P = PSPACE
    - That is widely believed not to be true



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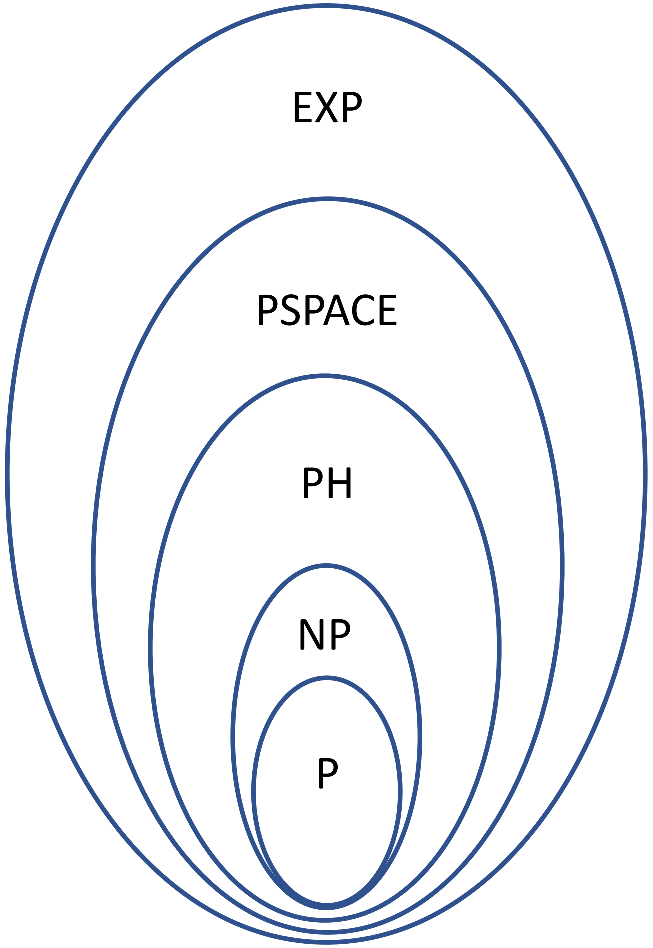

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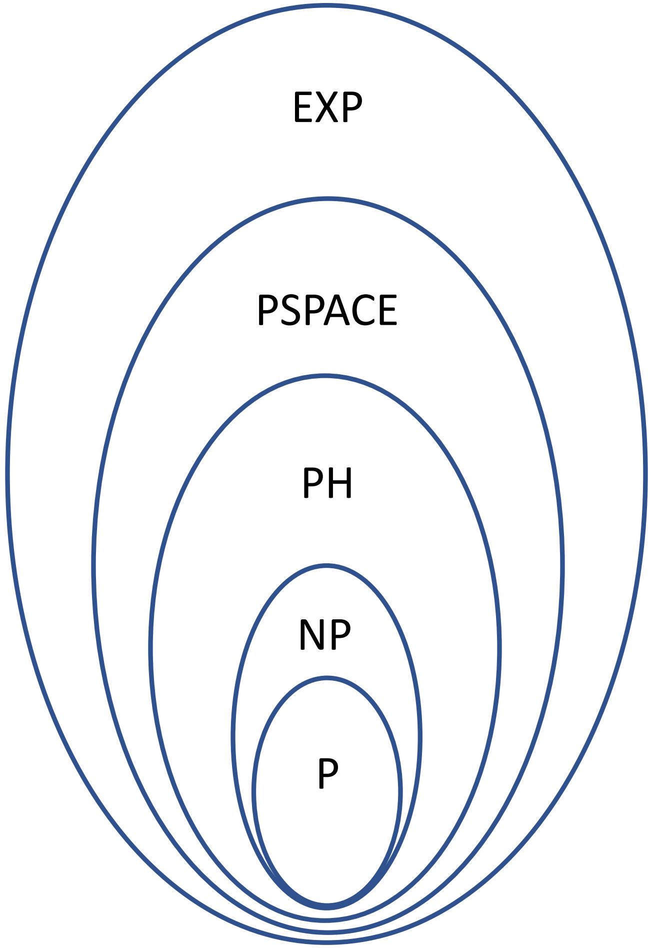
1. Show hardness: construct sequence of reductions from TQBF



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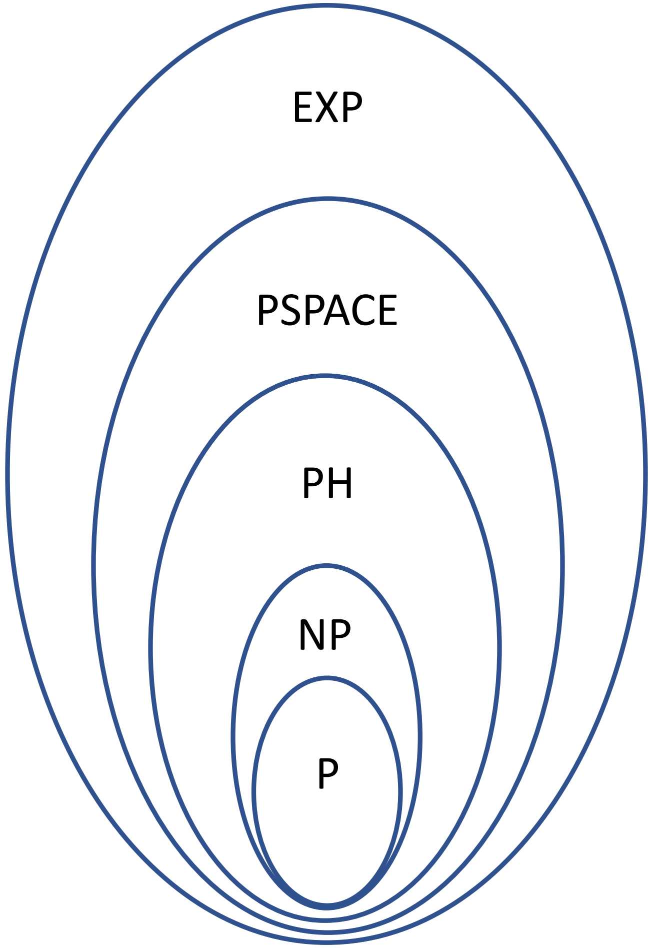
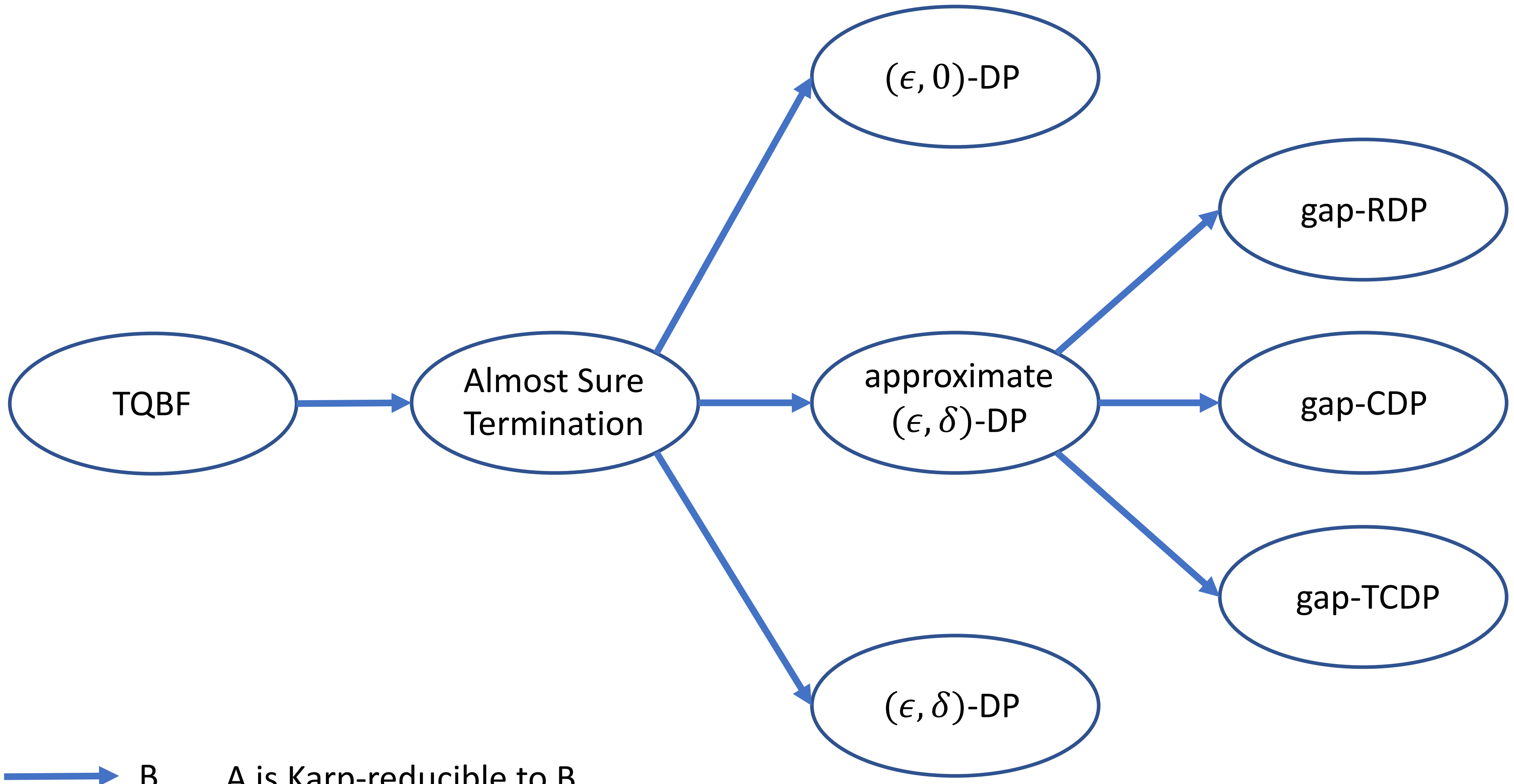


To show PSPACE completeness we need:

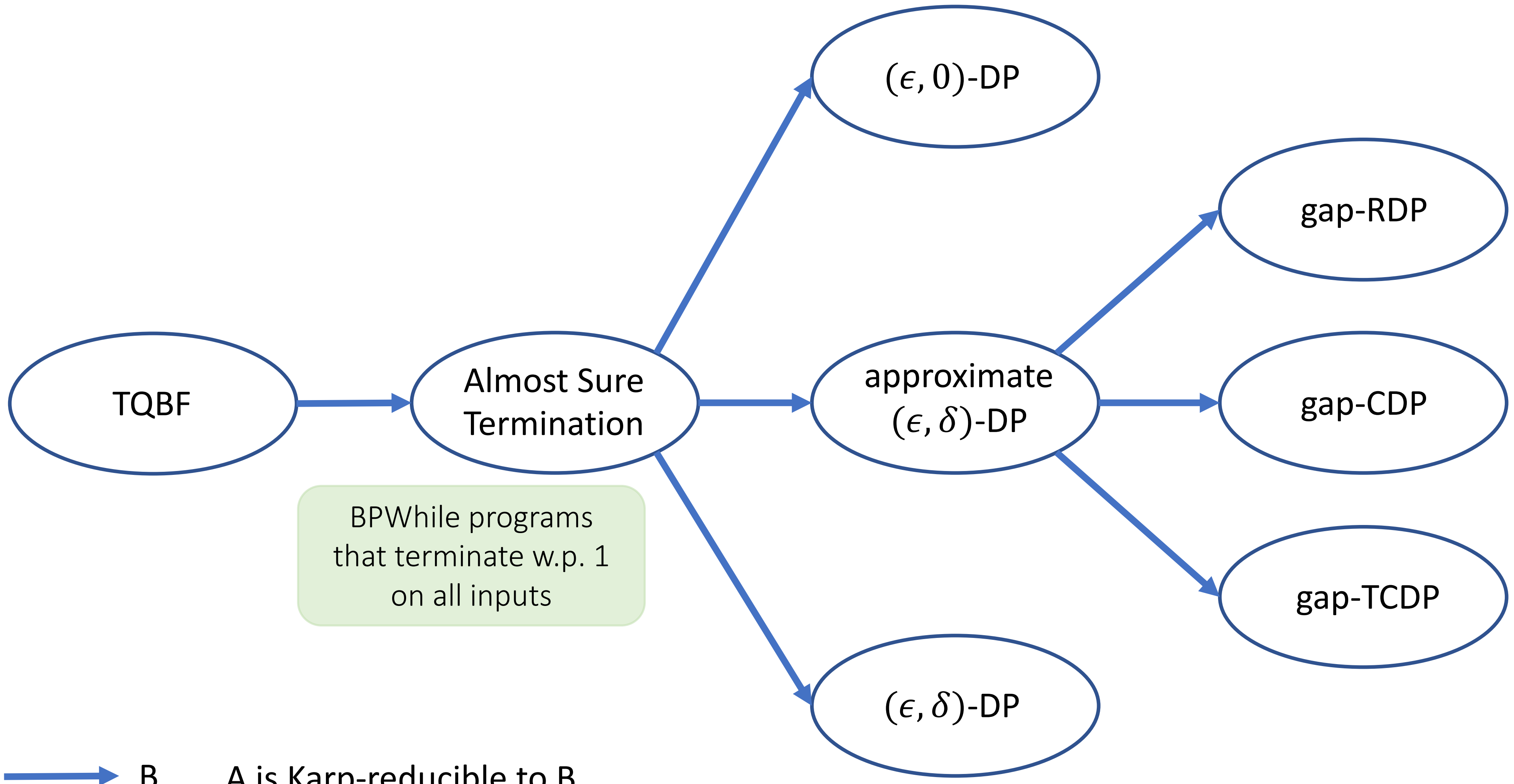
1. Show hardness: construct sequence of reductions from TQBF
2. Construct polynomial-space algorithm: analyze Markov chain based on the state graph of the program

PSPACE-hardness



A $\longrightarrow$ B A is Karp-reducible to B

PSPACE-hardness



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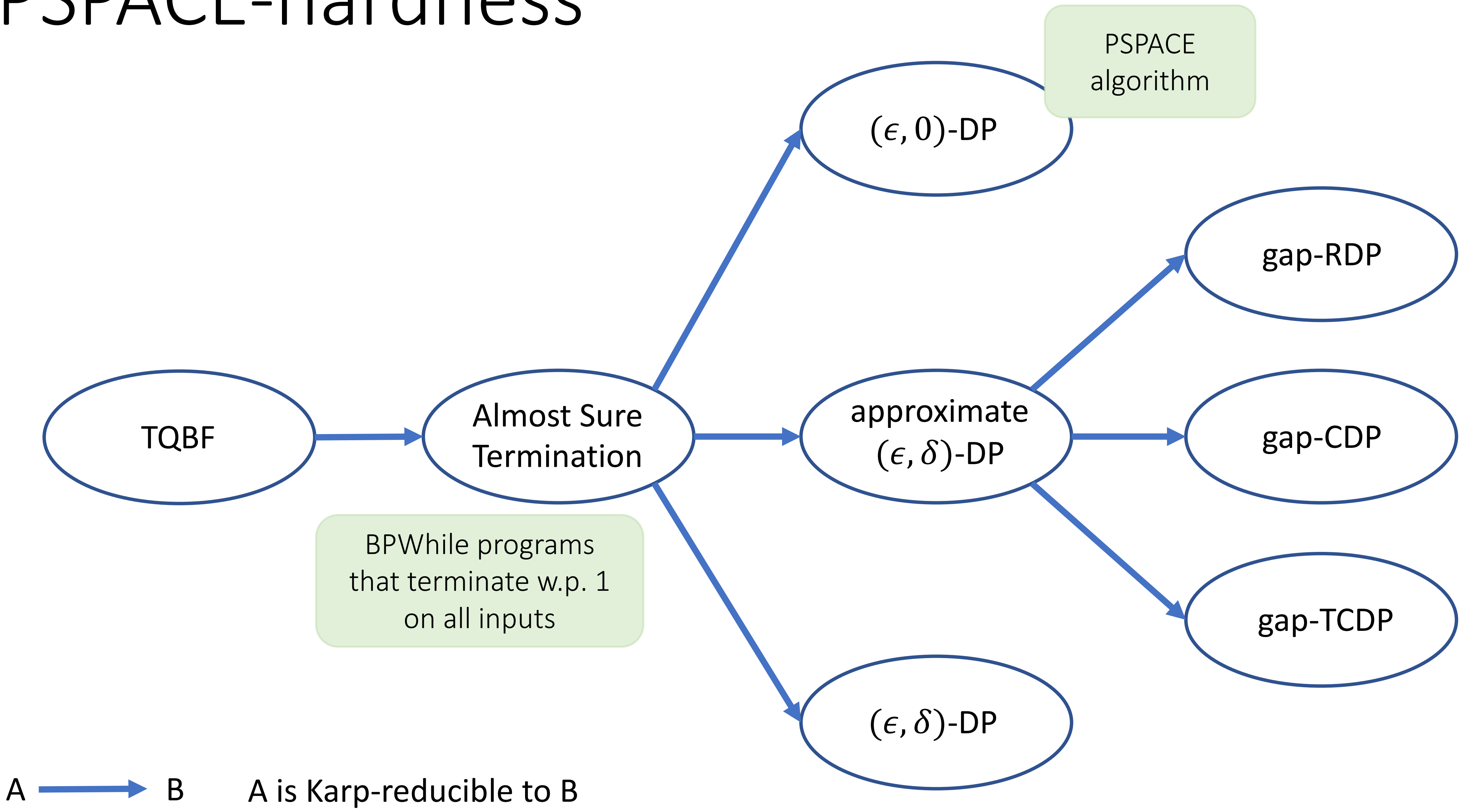

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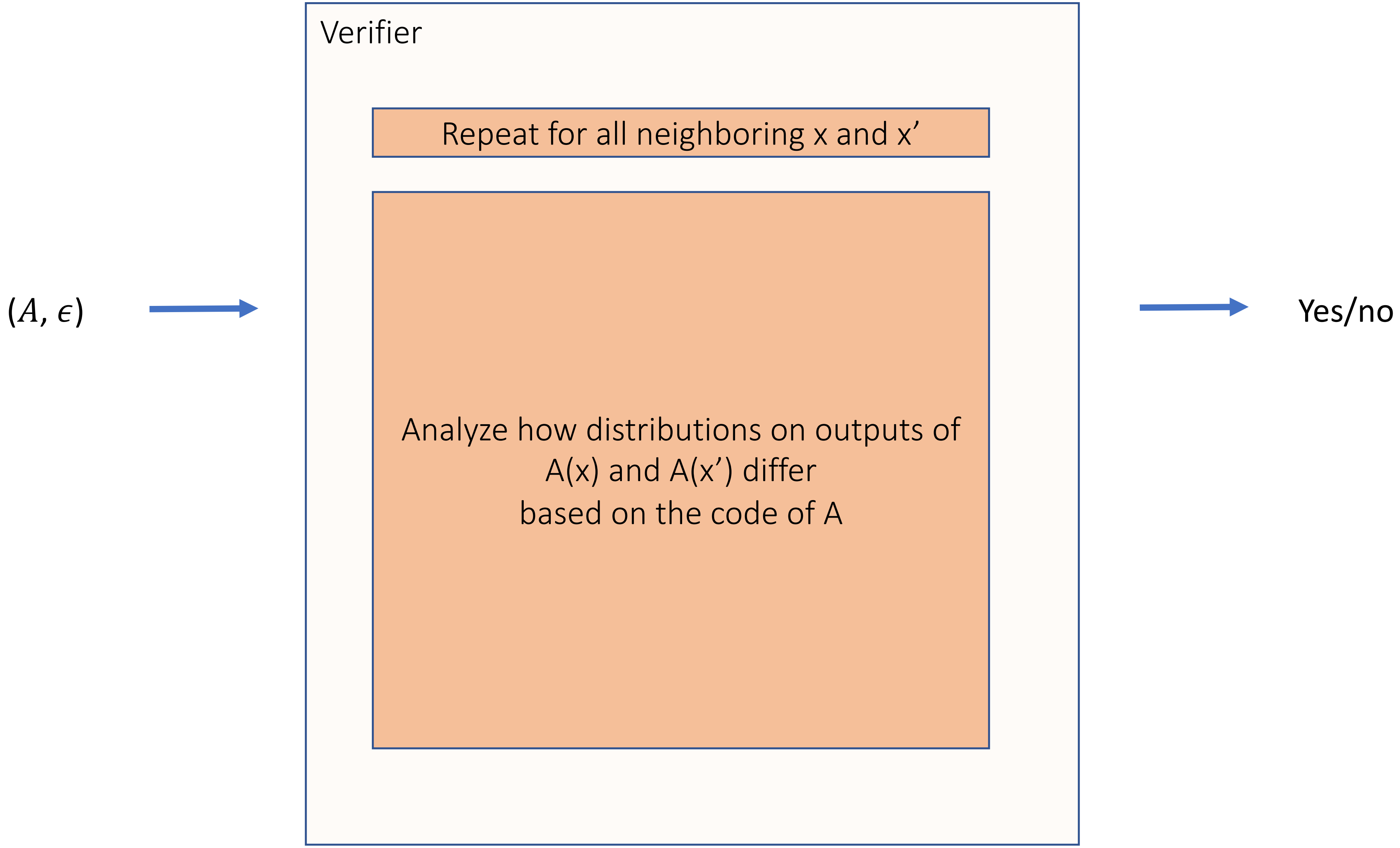
Polyspace membership: algorithm for (ϵ,0)-DP

(A, ϵ) → Verifier → Yes/no

Verifier

Repeat for all neighboring x and x’

Polyspace membership: algorithm for (ϵ,0)-DP



PSPACE membership: state graph

State graph depends on the input values

D(b=1):

```
input(b);
if b == 1 then
   r = rand();
   if r == 0 then
      while true
         skip;
   else skip;
else skip;
return(1)
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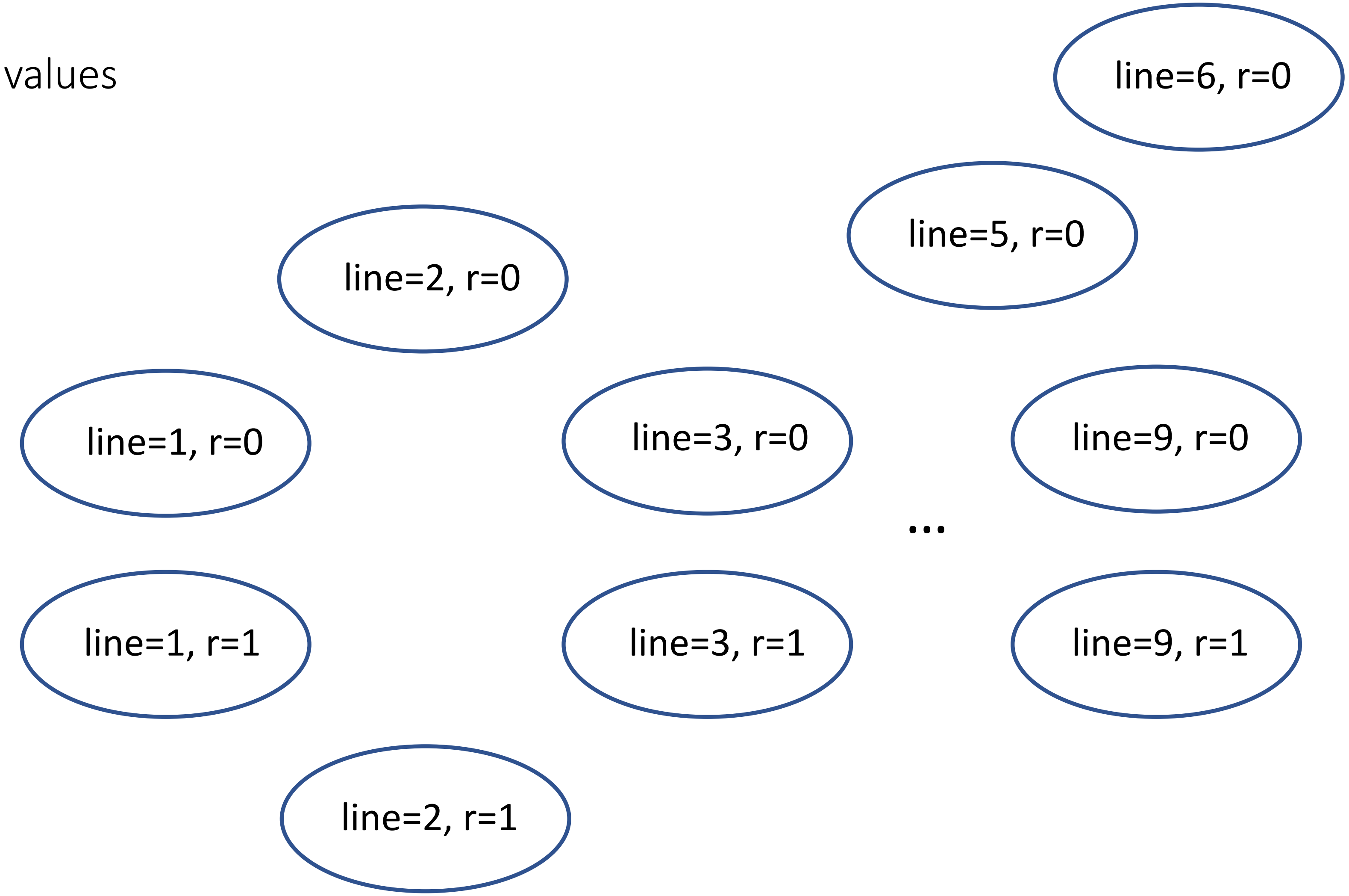

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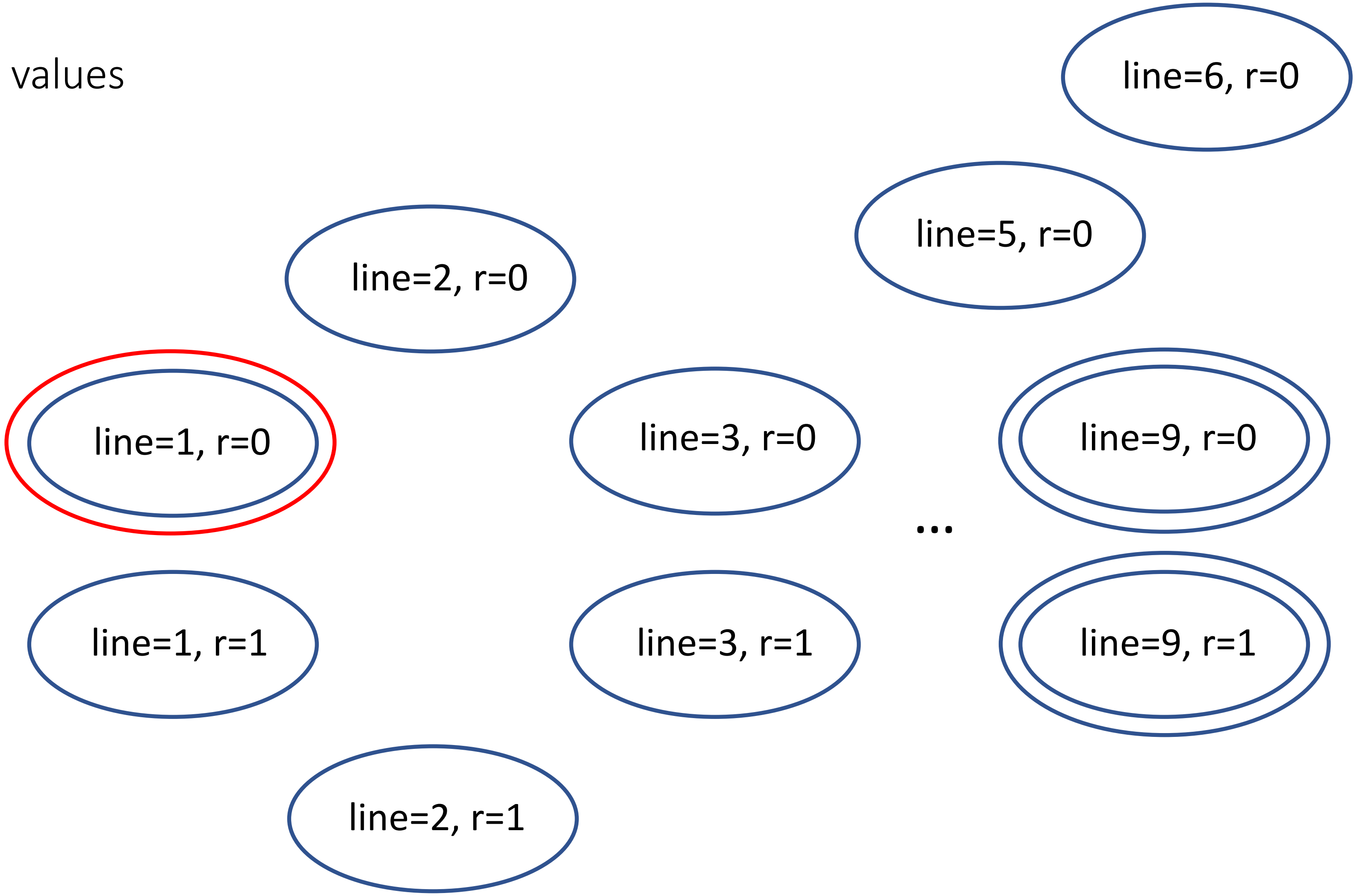

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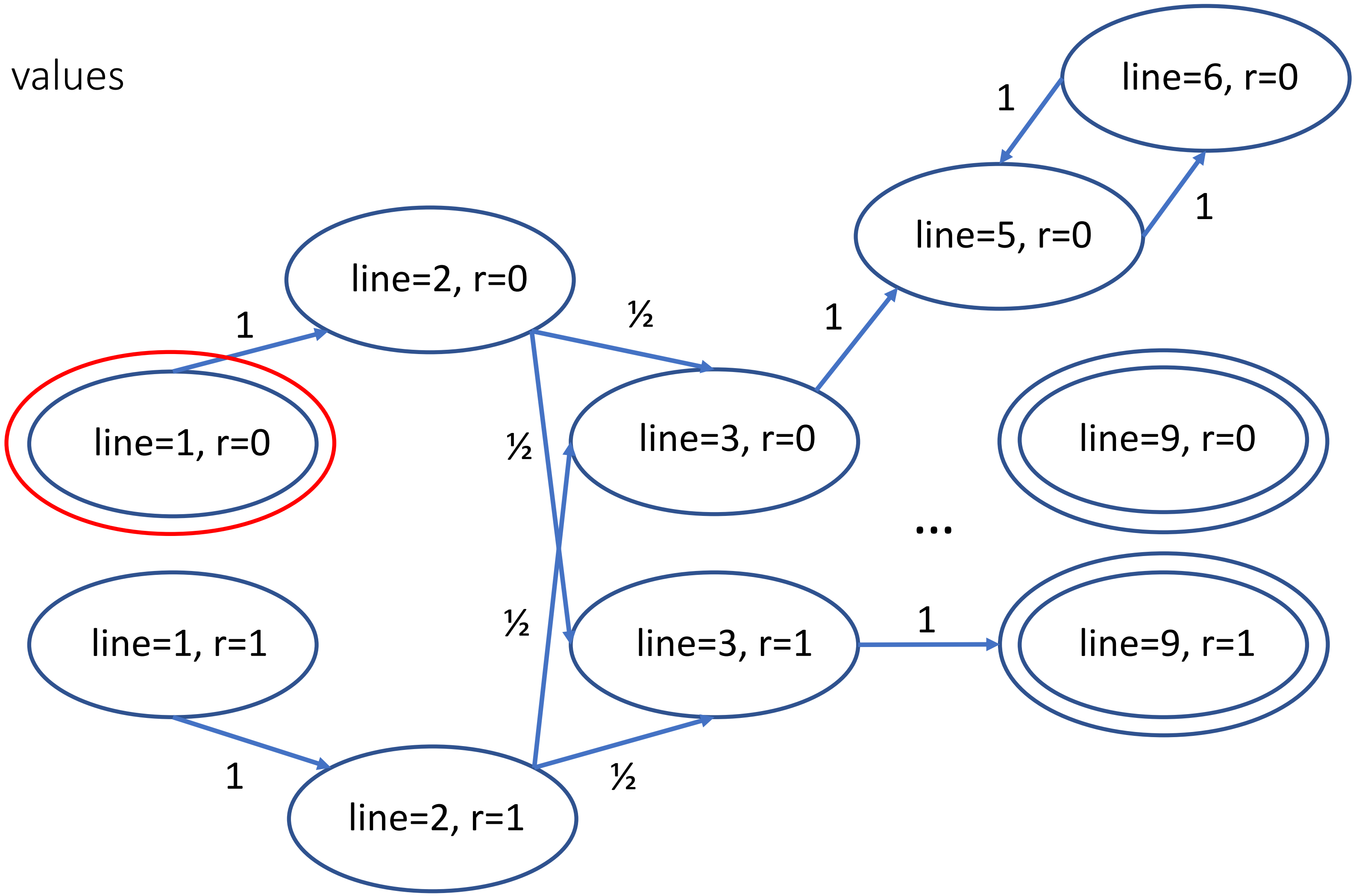

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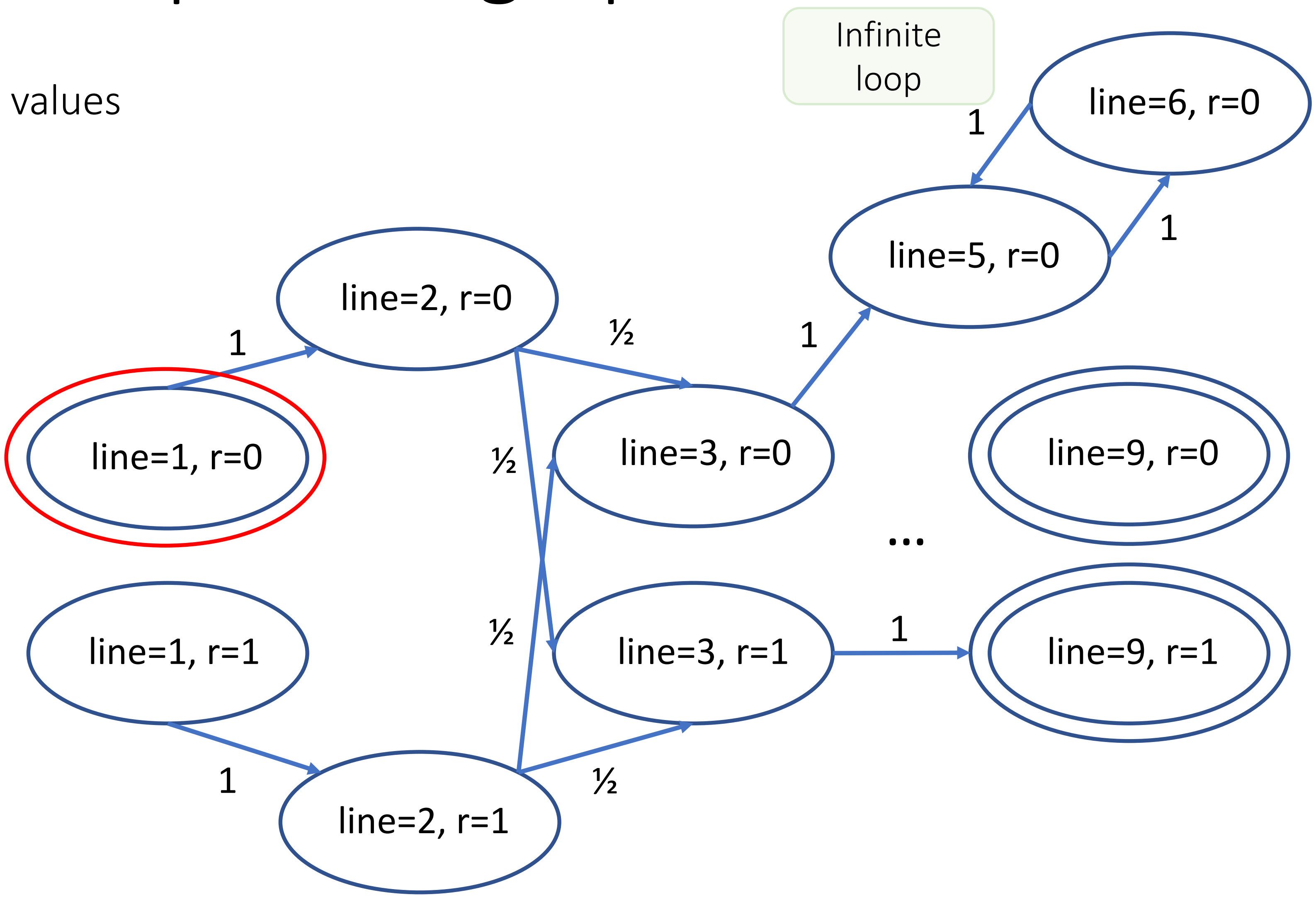

PSPACE membership: state graph

D(b=0):

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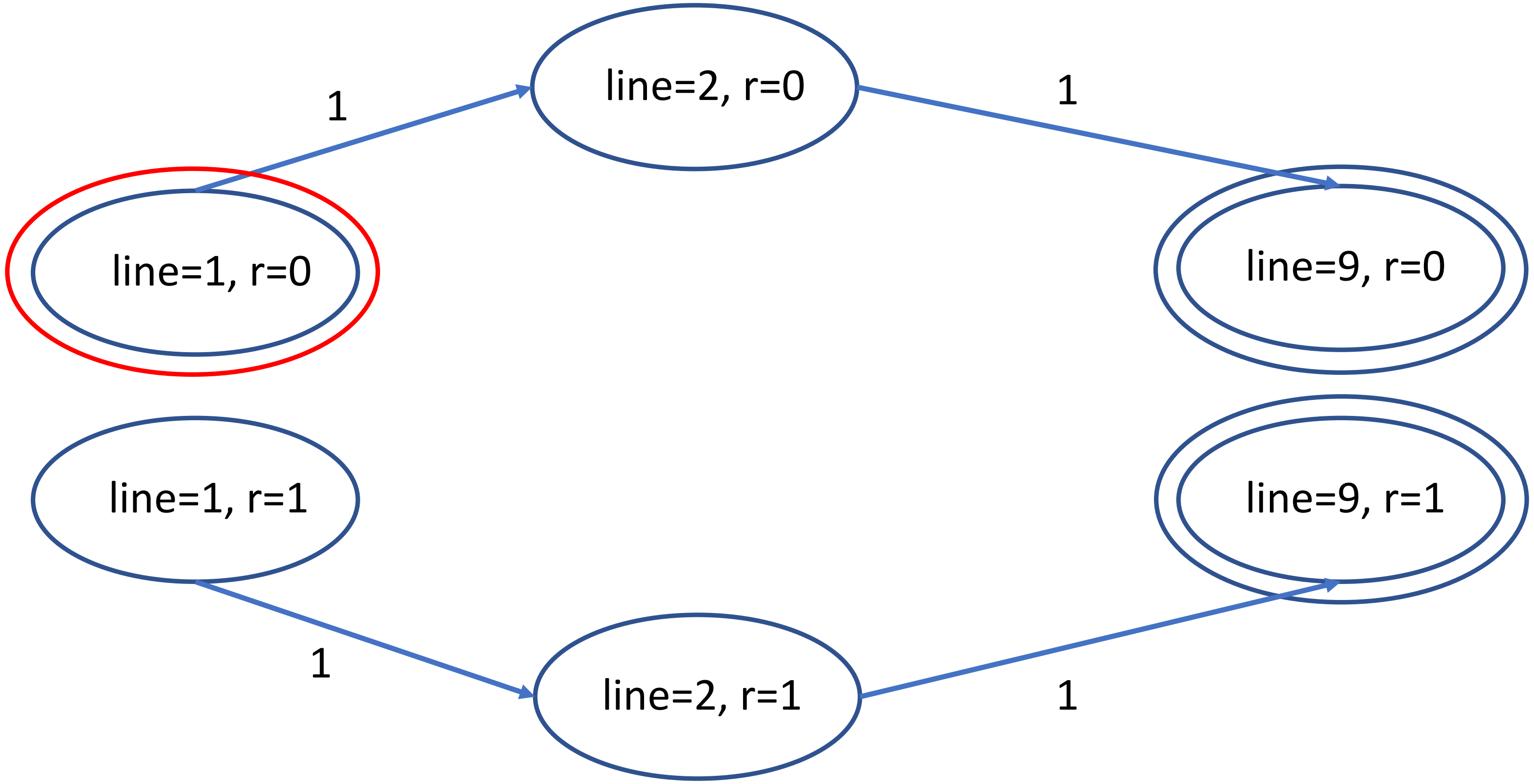
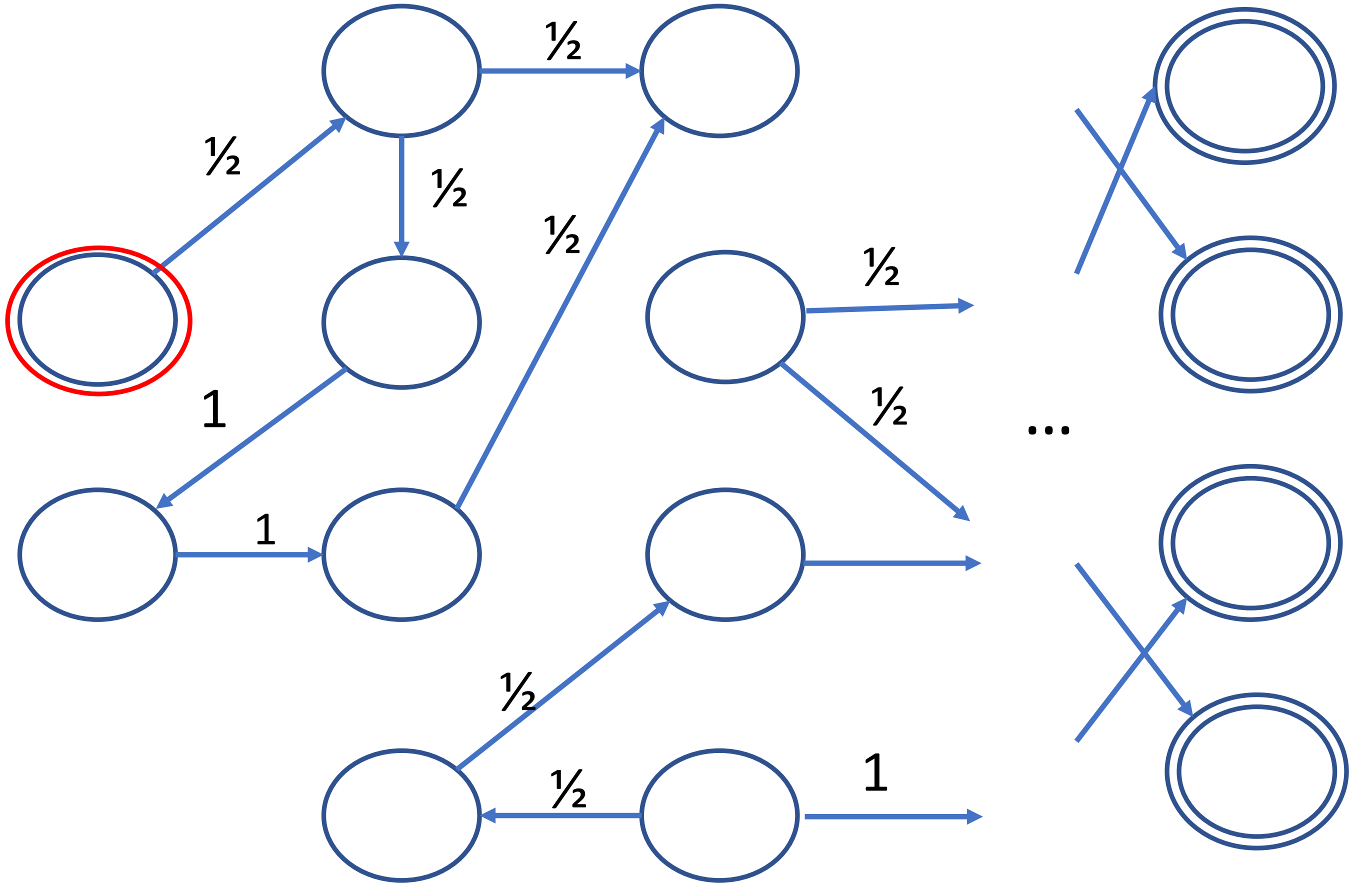


PSPACE membership: algorithm for (ϵ,0)-DP

For a program D and all neighboring inputs x, x':

- Construct the Markov chain for $D(x)$ and $D(x')$
- Compute and compare hitting probabilities



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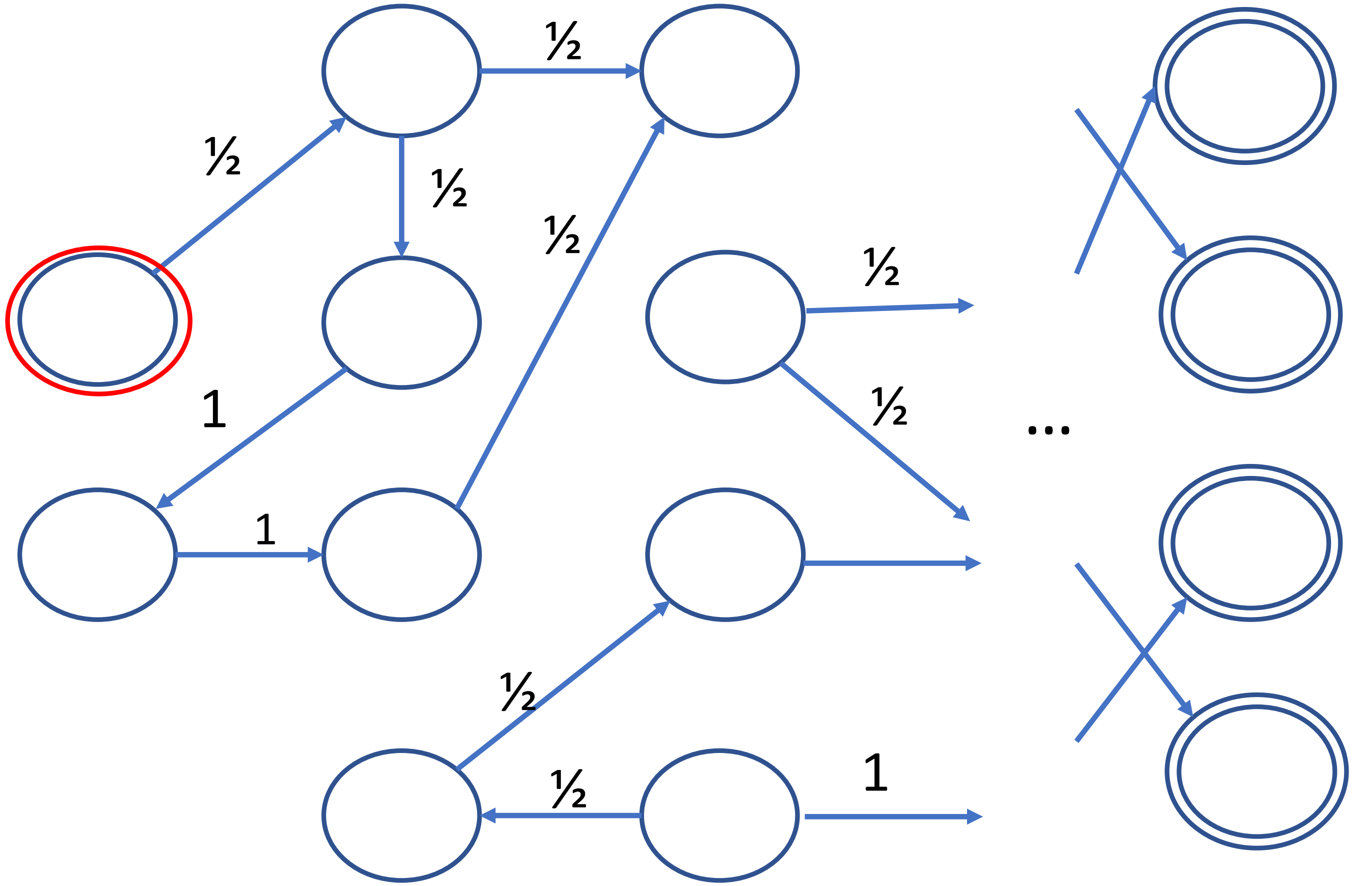

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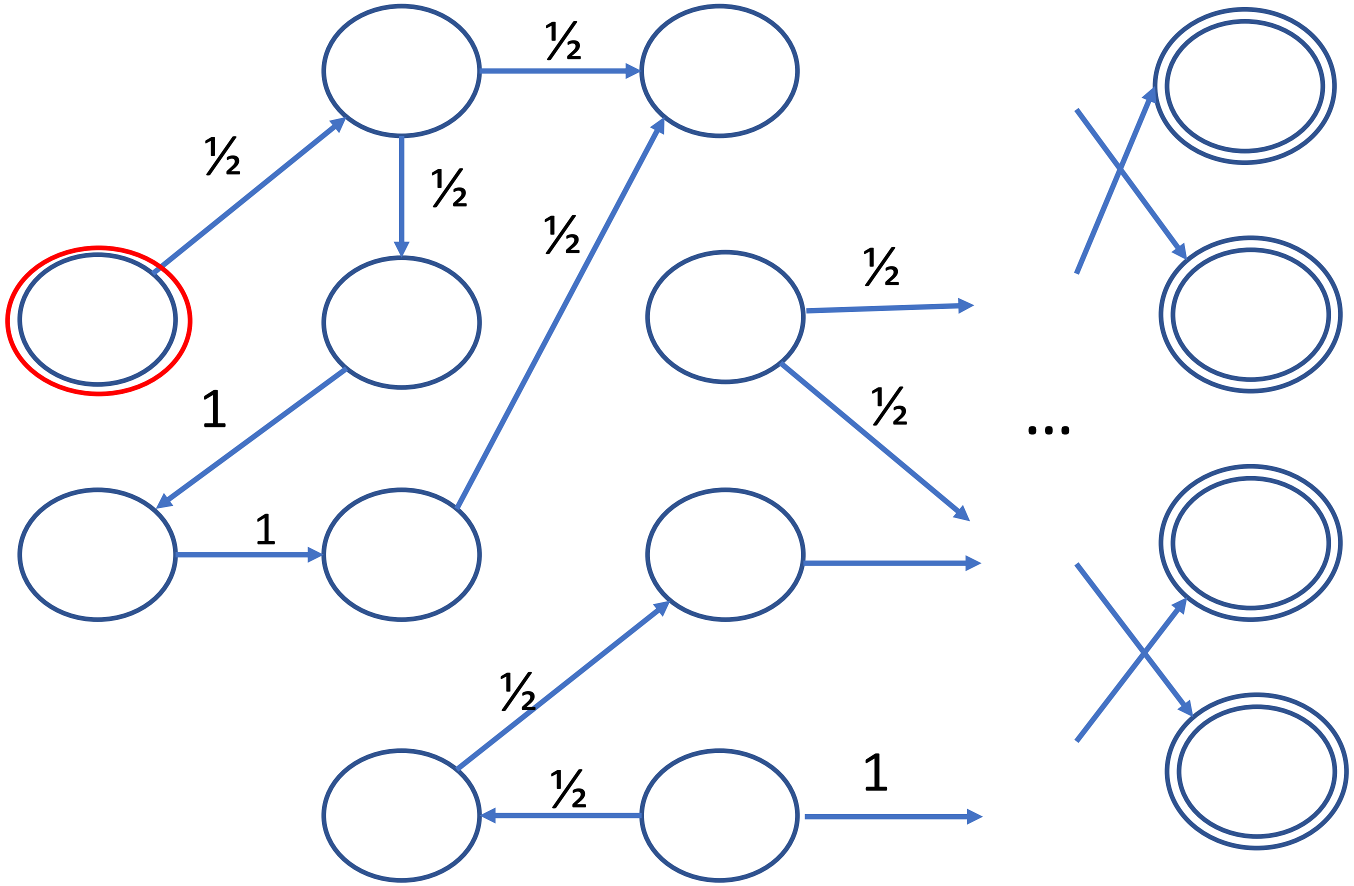
Need **space-efficient** algorithm for computing hitting probabilities with **implicit access** to the Markov chain



Polyspace algorithm for computing hitting probabilities in a Markov chain

Lemma [Simon'81]: If M is a Markov chain with at most 2^L states

- the initial distribution places all mass on one state,
- there is a set F of final states each with only one self-transition,
- every non-final state has outgoing transition probability 0 or ½.

Then there is an $O(L^6)$-space deterministic algorithm that computes the hitting probability of every state in F.

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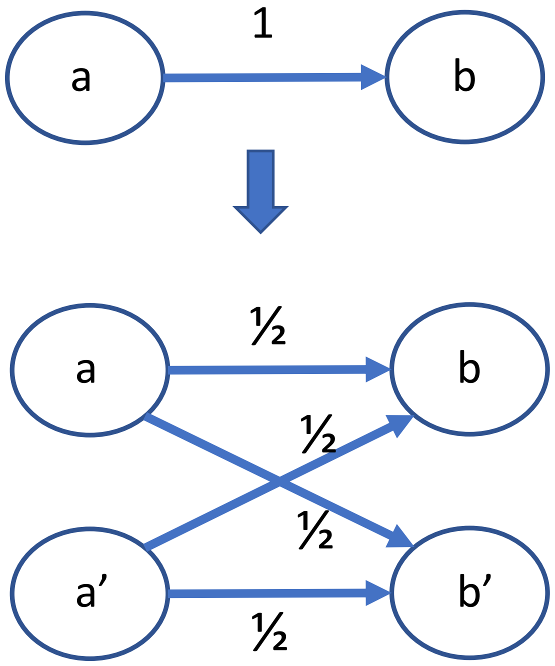


Note: to use the algorithm, we need to replace all transitions labelled by 1 in the state graph of the BPWhile program:

- Clone all states
- For each state *a* with outgoing edge w.p. 1 replace it by two edges:
  - Edge (a,b) with weight ½ to original state
  - Edge (a,b') with weight ½ to the clone-state b' of b

PSPACE membership: exponentially long numbers

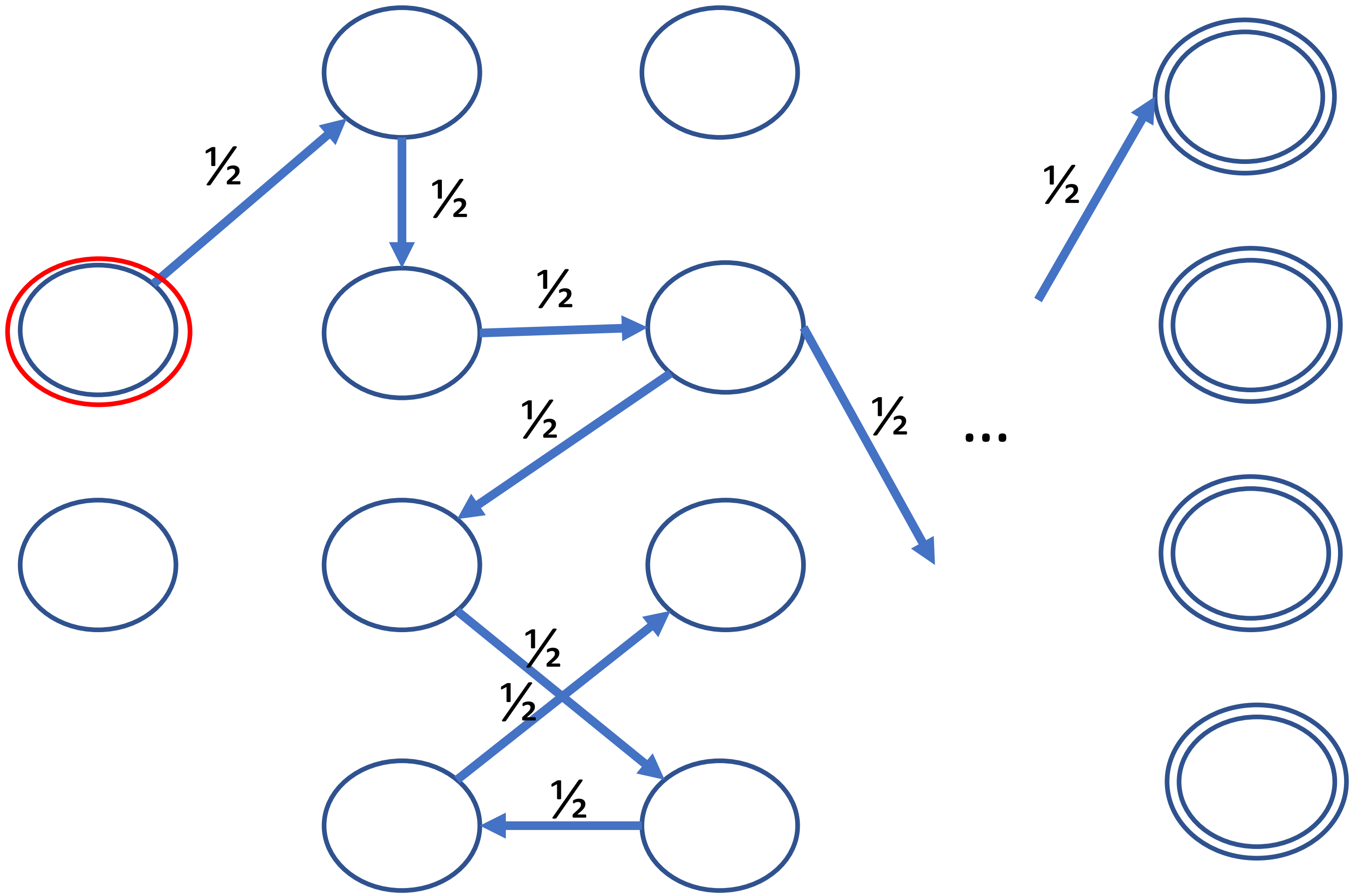
For a program D

for all neighboring inputs x, x':

- Construct the Markov chain for $D(x)$ and $D(x')$
- Compare hitting probabilities of the final states with the same values

Problem:

Markov chain has exp-many states



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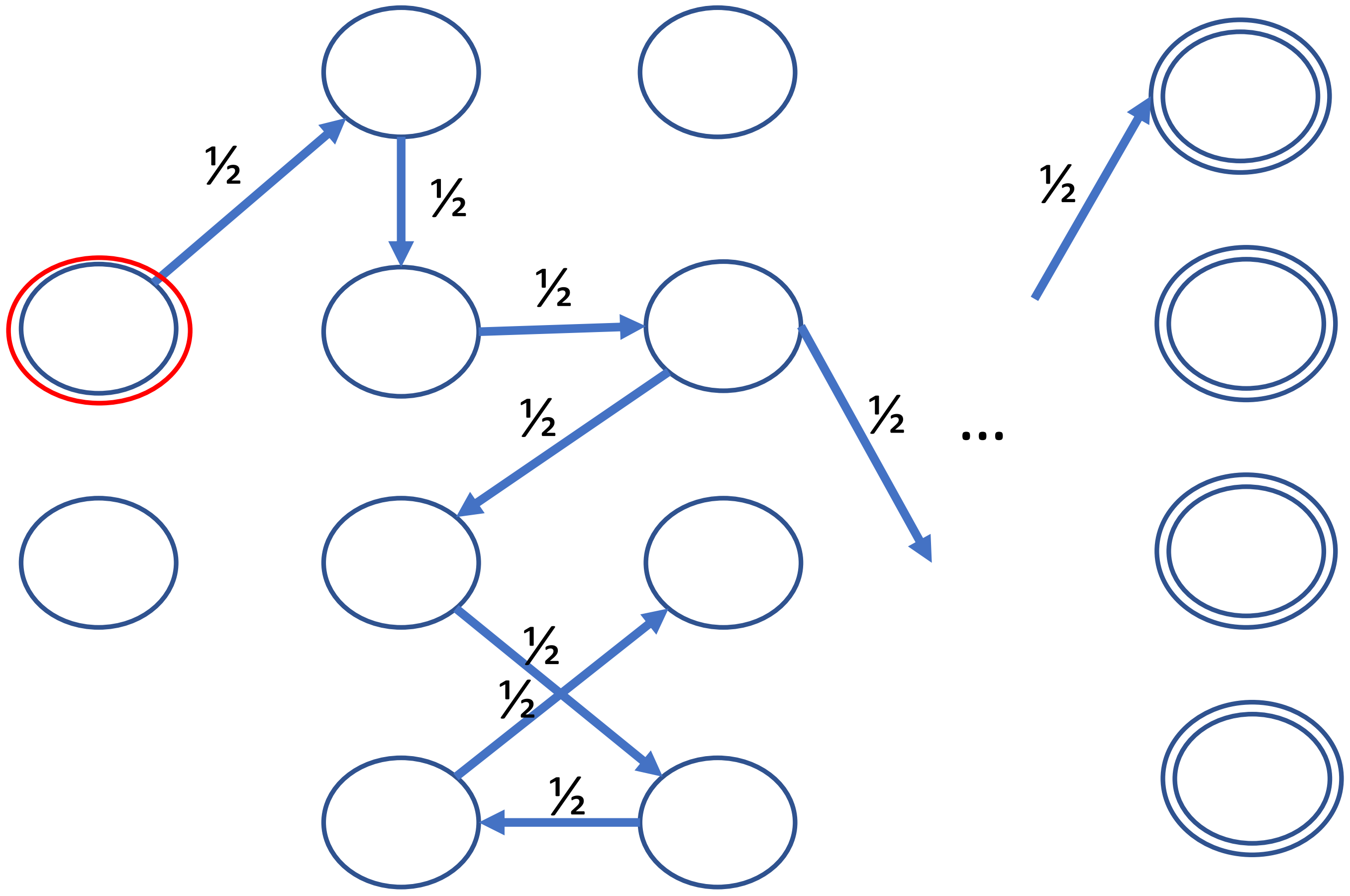
Markov chain has exp-many states $\Rightarrow$ hitting probabilities can be as small as $\frac{1}{2^{exp}}$



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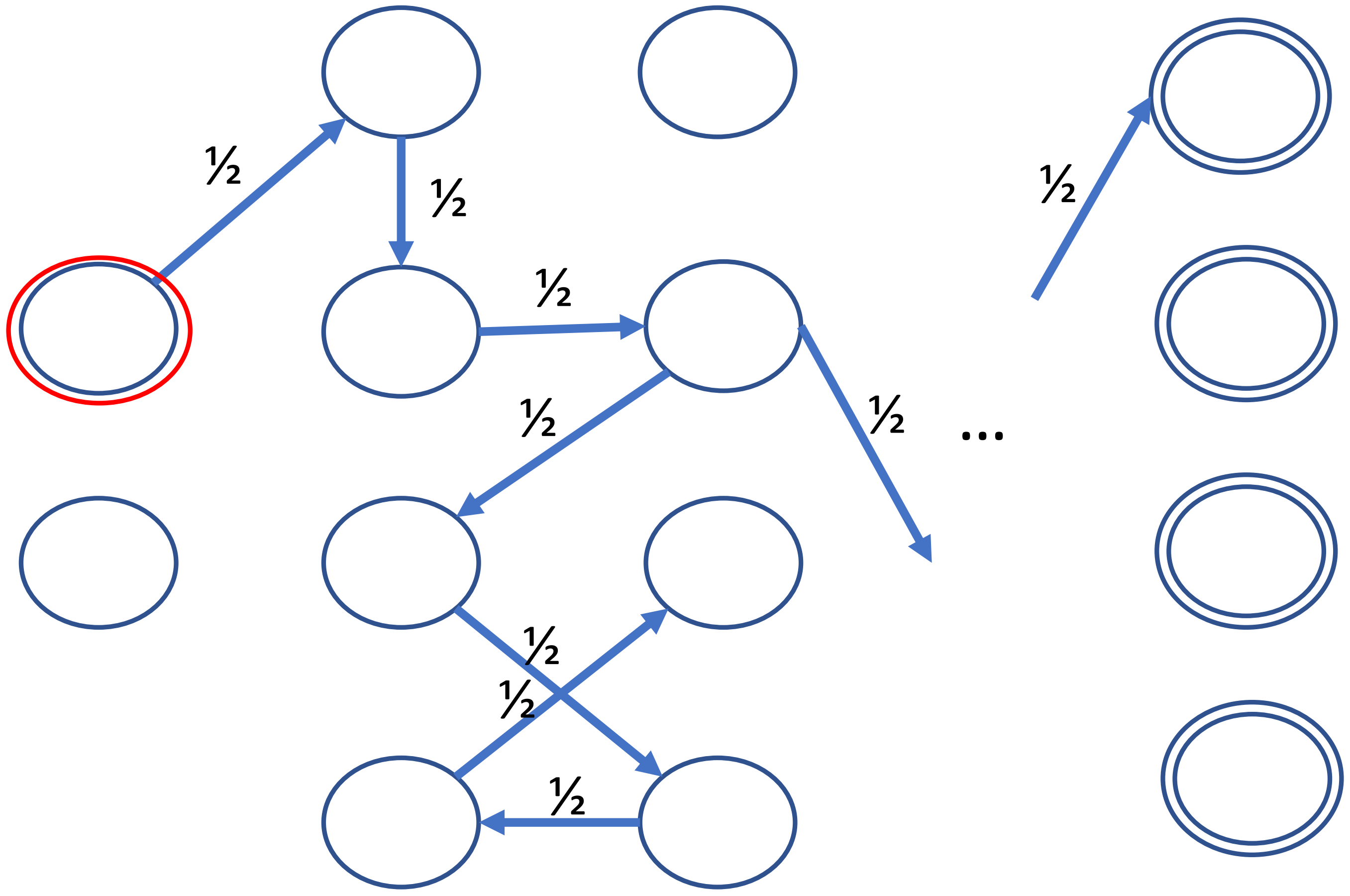
Markov chain has exp-many states $\Rightarrow$ hitting probabilities can be as small as $\frac{1}{2^{exp}}$

$\Rightarrow$ numbers are exponentially long



Operations with exponentially long numbers

	1	0	1	1	1	…	1	0	0	1
+										
	0	0	1	0	1	…	1	1	1	1
	1	1	1	0	1	…	1	0	0	0

Operations with exponentially long numbers

	1	**0**	**1**	**1**	**1**	**…**	**1**	**0**	**0**	**1**
+										
	0	**0**	**1**	**0**	**1**	**…**	**1**	**1**	**1**	**1**
	1	**1**	**1**	**0**	**1**	**…**	**1**	**0**	**0**	**0**

Uniform family of log-depth circuits:

- One logspace algorithm provides implicit access to the circuits
- Each circuit has log-depth and poly size

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+										
	0	**0**	**1**	**0**	**1**	**…**	**1**	**1**	**1**	**1**
	1	**1**	**1**	**0**	**1**	**…**	**1**	**0**	**0**	**0**

Uniform family of log-depth circuits:

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Lemma:

Uniform families of log-depth circuits exist for:

- Comparison
- Addition
- Multiplication by a fixed rational number
- Multiplication [Ofman'62]
- Square roots [Reif'86]

Polyspace membership algorithm for (ϵ,0)-DP

(A, ϵ) →

Verifier

Repeat for all neighboring x and x’

→ Yes/no

Polyspace membership algorithm for (ϵ,0)-DP

(A, ϵ) → Verifier → Yes/no

Verifier

Repeat for all neighboring x and x'

Provide access to Markov chain corresponding to A(x) and A(x')

Polyspace membership algorithm for (ϵ,0)-DP

(A, ϵ) →

Verifier

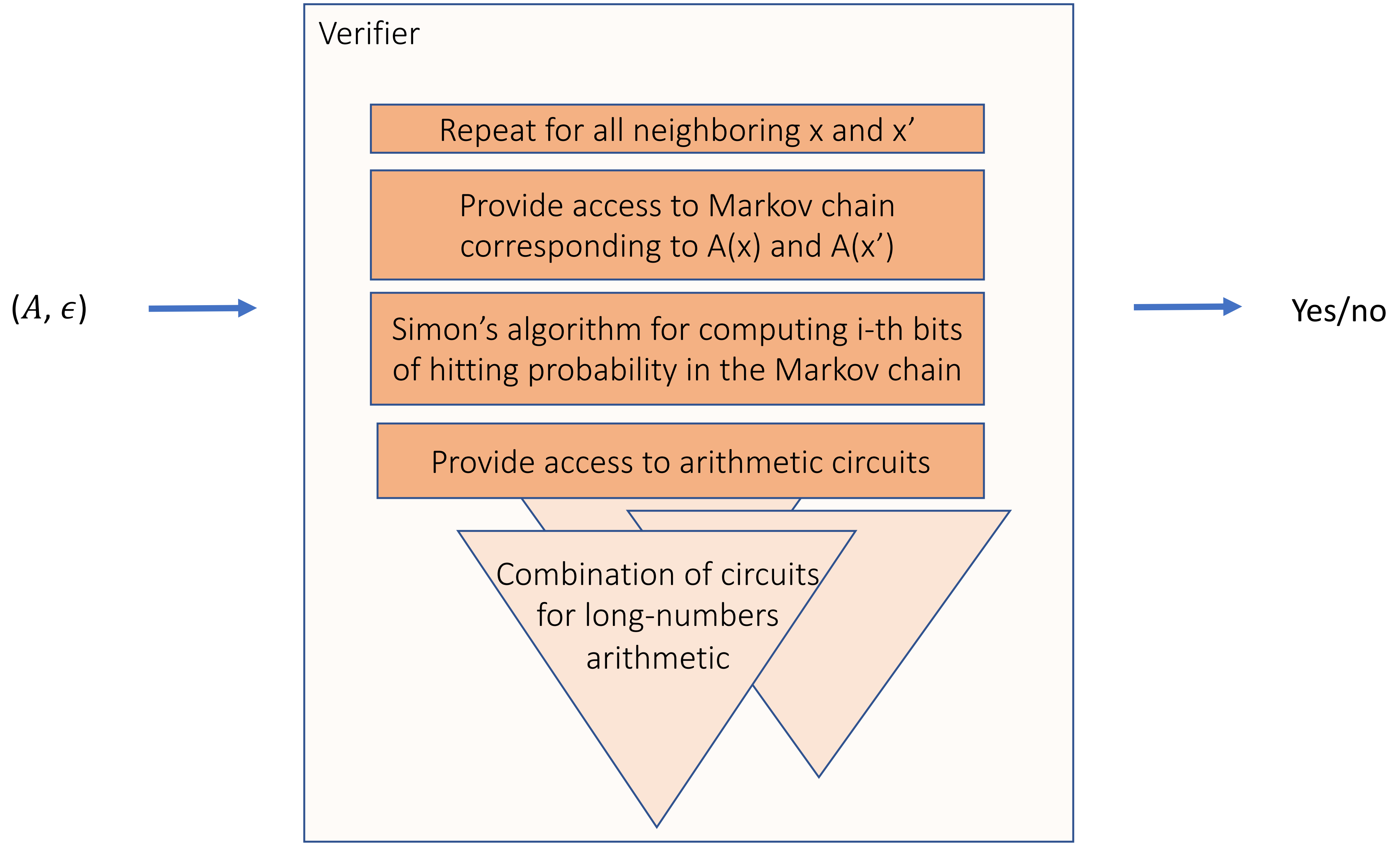
Repeat for all neighboring x and x'

Provide access to Markov chain corresponding to A(x) and A(x')

Simon's algorithm for computing i-th bits of hitting probability in the Markov chain

→ Yes/no

Polyspace membership algorithm for (ϵ,0)-DP



Results and future work

- We showed PSPACE-completeness for the problems of checking:
  - Pure-DP
  - Approximate-DP
  - Gap-RDP
  - Gap-zCDP
  - Gap-TCDP

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- Possibly can extend the result to show PSPACE-completeness of verifying accuracy
- Improve the exact polynomial in the space complexity of the algorithm
  - Improved analysis and more efficient algorithms for Markov chains analysis and long-numbers arithmetic operations are needed for tighter results