

On Tseitin formulas, read-once branching programs and treewidth

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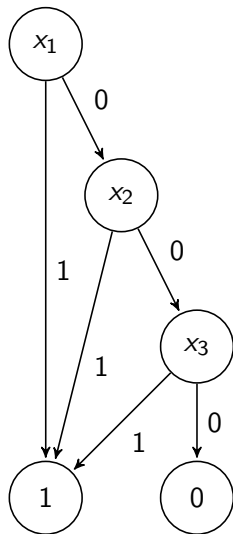
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Outline

- ▶ Branching programs
- ▶ Tseitin formulas
- ▶ Tree-width and path-width
- ▶ Lower bounds

Branching program

- ▶ BP is a way to represent Boolean function
 - ▶ directed graph without cycles;
 - ▶ two sinks: labeled with 0 and 1, one source;
 - ▶ every internal vertex labeled with a variable and has two outgoing edges: labeled with 0 and 1;
 - ▶ the value of function equals label of the sink in the end of corresponding path.



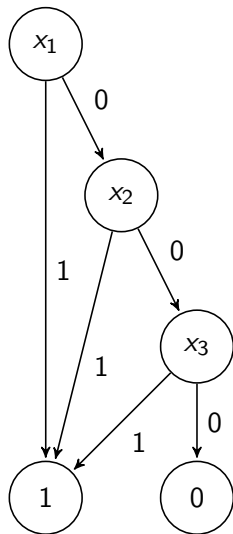
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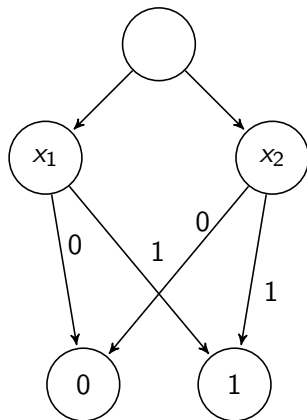
- ▶ BP size is between circuit and formula sizes

- ▶ $C(f) \leq 3BP(f) \leq O(L^{1+\epsilon}(f))$,
- ▶ $C(f)$ circuit complexity of f , $L(f)$ size of minimal formula, BP size of minimal branching program.



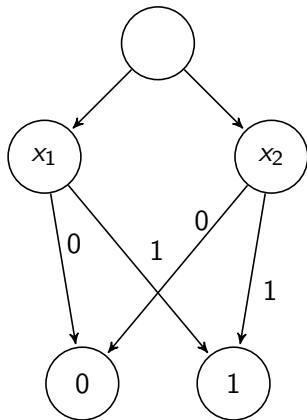
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- ▶ NBP additionally has non-deterministic nodes:
 - ▶ non-deterministic nodes are unlabeled
 - ▶ the value equals 1 iff there exists a path to 1-sink



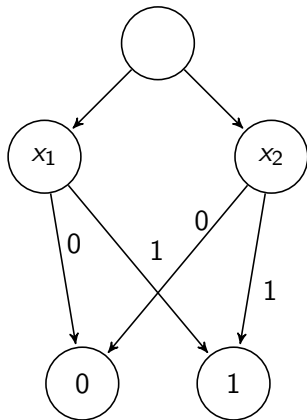
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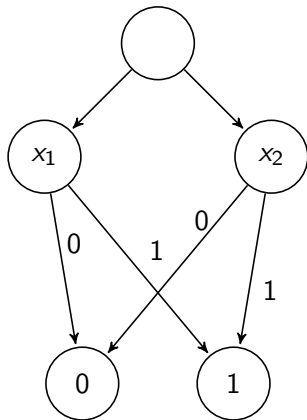
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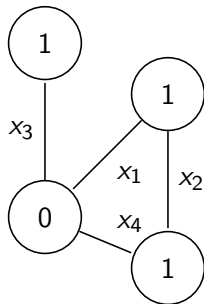
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- ▶ $k\text{-(N)BP}$ if for every path every variable occurs no more than k times
- ▶ OBDD is 1-BP with fixed order for variables in every path



Tseitin formulas

- ▶ Tseitin $TS_{G,c}$ formula defined for a graph
 - ▶ every edge is labeled with a variable
 - ▶ every vertex has a 0-1 label: $c : V \rightarrow 0, 1$
 - ▶ $TS_{G,c}(x) = 1 \iff$

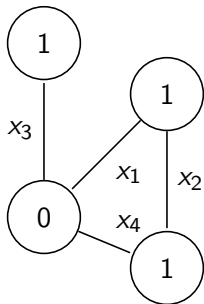
$$\bigwedge_{v \in V} \left(\sum_{e \text{ incident } v} x_e = c(v) \bmod 2 \right)$$



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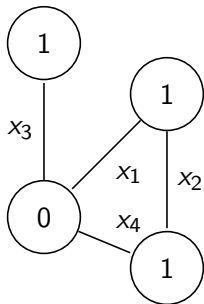
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- ▶ A Tseitin formula is satisfiable iff for every connected component the sum of labels is even



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- ▶ A Tseitin formula is satisfiable iff for every connected component the sum of labels is even
- ▶ Unsatisfiable Tseitin formulas are classical hard examples for proof systems:
 - ▶ Resolution [Tseitin, 1968], [Urquhart, 1987]
 - ▶ Bounded depth Frege [Ben-Sasson, 2002], [Pitassi, Rossman, Servedio, Tan, 2016]
 - ▶ Polynomial Calculus over field with char $\neq 2$ [Alekhnovich, Razborov, 2001]
 - ▶ Tree-like Lovasz-Schrijver [Itsykson, Kojevnikov, 2006]



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Tseitin formulas (2)

- ▶ (Itsykson, Knop, Romashchenko, Sokolov, 2017) Exponential lower bound for OBDD(join) proof systems for unsatisfiable Tseitin formulas
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 - ▶ $OBDD(TS_{G,c}) = 2^{\Omega(n)}$

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 - ▶ key step: satisfiable Tseitin formulas are hard for OBDDs.
 - ▶ $OBDD(TS_{G,c}) = 2^{\Omega(n)}$
- ▶ OBDD is a partial case of 1-BP
 - ▶ (Bryant, 1991) (Sieling, Wegener, 1995)
 $f : OBDD(f) = 2^{\Omega(n)}, 1\text{-BP}(f) = \text{poly}(n)$

The goal: what is the size of 1-BP for satisfiable Tseitin formulas?

Lower bound for Tseitin formula on an expander

Theorem:

1-NBP for satisfiable Tseitin formula for (n, d, α) -expander is $2^{\Omega(n)}$, where

- ▶ $\alpha < \frac{1}{3}$
- ▶ (n, d, α) -expander is a d -regular graph on n vertices with absolute value of the second largest eigenvalue $\leq d \cdot \alpha$

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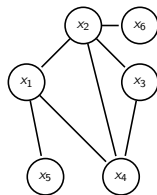
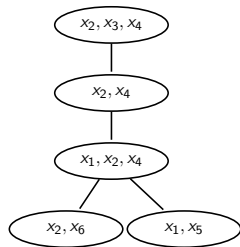
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Generalized theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V| - k_G(l) - k_G(|E| - l) + 1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deleting of l edges.

Tree-width and path-width

Tree (path) decomposition of a graph

$G = (V, E)$ is a graph T that is a tree (path)

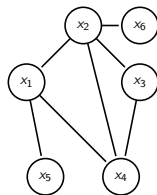
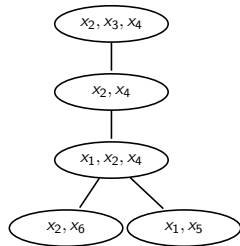


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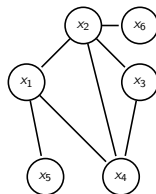
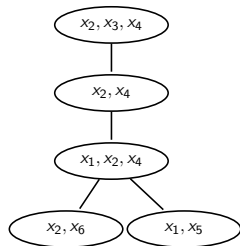


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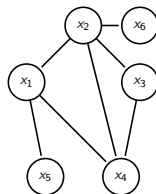
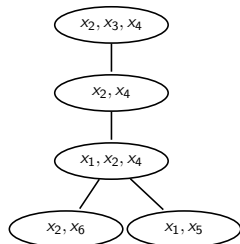


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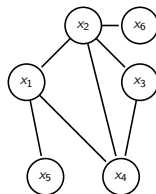
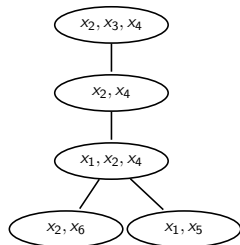


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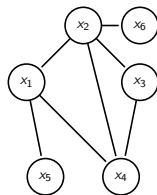
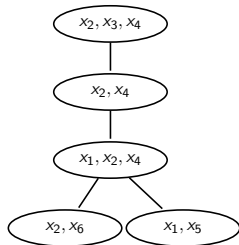
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Tree-width (path-width) of a graph is the minimal width among all its tree (path) decompositions minus 1.



Graph minor

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Grid Minor Theorem [Robertson, Seymour 1986], [Chuzhoy 2015]:

Every graph G of a tree-width t has a grid minor of size t^δ , where $\delta < \frac{1}{36}$.

Lower bound for any graph based on its tree-width

Lower bound using tree-width: 1-NBP for satisfiable Tseitin formula on a graph G has size $\Omega(2^{t^\delta})$, where t is a tree-width of G , $\delta < \frac{1}{36}$.

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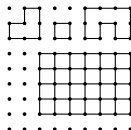
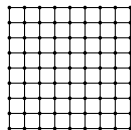
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- ▶ Show that 1-NBP for $s \times s$ grid graph has size $2^{\Omega(s)}$

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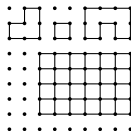
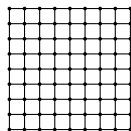
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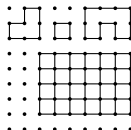
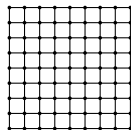
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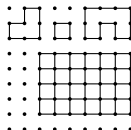
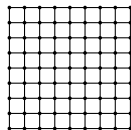
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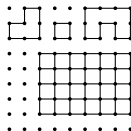
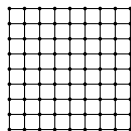
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- ▶ After that we get a lower bound $2^{\Omega(2\varepsilon \sqrt{|V|})} = 2^{\Omega(s)}$



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- ▶ **Deletion of edge x_i** : substitute any value that will not break the satisfiability of the formula
- ▶ **Deletion of vertex v** : delete all connected edges one-by-one
- ▶ **Contraction of edge x_i** : the same as the satisfiability of the formula $\exists x_i : Ts_{G,w}(x)$
 - ▶ in 1-NBP all nodes labeled with x_i should be changed to non-deterministic nodes

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No exponential lower bounds for semantic 1-NBP for a boolean function

All known results with exponential lower bounds are for functions with non-boolean domain [Cook, Edmonds, Medabalimi, Pitassi, 2016], [Jukna, 2009]

Upper bound using path-width

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Known lower bounds for k-BPs

- ▶ (Borodin, Razborov, Smolensky, 1993)

$$CLIQUE_ONLY_n(G) : \{0, 1\}^{\frac{n \cdot (n-1)}{2}} \rightarrow 0, 1$$

- ▶ $CLIQUE_ONLY_n(G) = 1$ iff graph G on n vertices is exactly $\frac{n}{2}$ -clique
- ▶ $1\text{-NBP}(CLIQUE_ONLY_n) = 2^{\Omega(n)}$
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- ▶ (Thathachar, 1998) an explicit functions f_k for every k :
 - ▶ $k\text{-NBP}(f_k) = 2^{\Omega(n^{1/k})}$
 - ▶ $(k+1)\text{-BP}(f_k) = O(n)$

Known lower bounds for k-BPs (2)

- ▶ (Duris, Hromkovic, Jukna, Sauerhoff, Schnitger, 2004)
 - ▶ \oplus parity of the number of triangles in a graph, $\overline{\Delta}$ is 1 iff graph has no triangles
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- ▶ (Jukna, 1995)
 - ▶ f_k characteristic function of error-correcting codes
 - ▶ $k\text{-NBP}(f_k) = 2^{\Omega(\sqrt{n})}$

The comparison with other results

Satisfiable Tseitin formulas $TS_{G,w}$ on (n, d, α) -expander can be represented by:

- ▶ $\text{CNF}(TS_{G,c}) = O(n)$
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Best previously known gaps:

$\text{CNF}(f) = O(n^3)$	$1\text{-NBP}(f) = 2^{\Omega(n)}$ (Duris et al, 2004)
$1\text{-NBP}(\lceil f \rceil) = O(n)$	$1\text{-NBP}(f) = 2^{\Omega(\sqrt{n})}$ (Jukna, 2009) explicit construction
$1\text{-NBP}(\lceil f \rceil) = O(n)$	$1\text{-NBP}(f) = 2^{\Omega(n)}$ (Duris et al, 2004) probabilistic construction
$2\text{-BP}(f) = O(n)$	$1\text{-NBP}(f) = 2^{\Omega(\sqrt{n})}$ (Thathachar, 1998)

Idea of the proof

Theorem: 1-NBP for a satisfiable Tseitin formula on a connected graph G has at least $2^{|V| - k_G(l) - k_G(|E| - l) + 1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from G by deleting of l edges.

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Get that at least $2^{(C_1-C_2)n}$ different nodes are on the l -th level, so is in the 1-NBP