The Role of Pseudorandomness in (Computational) Differential Privacy

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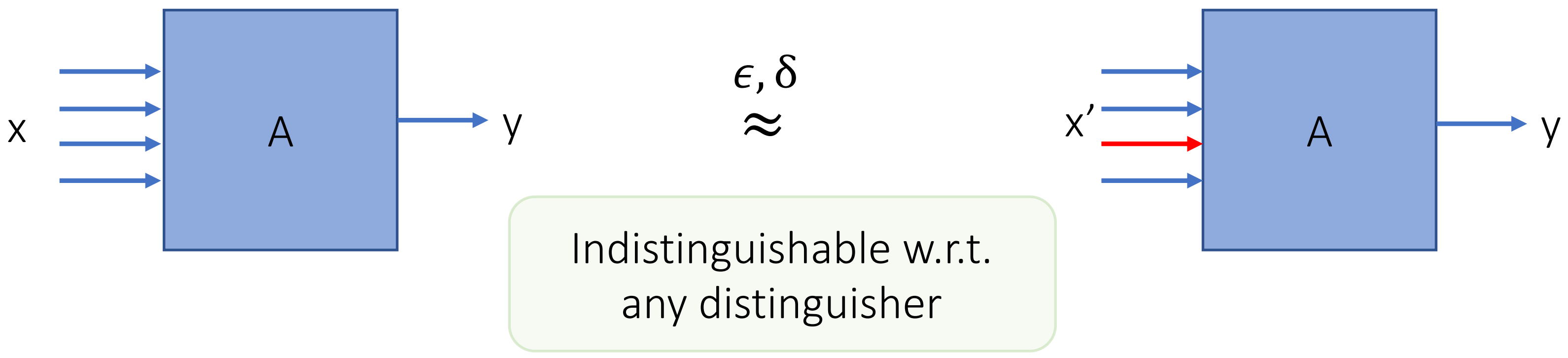
Plan of the talk

1. Overview of the papers
    1. Computational Differential Privacy
        1. Dense Model Theorem
    2. SV-sources for DP algorithms
    3. PRGs for Local DP algorithms
2. Proof of the equivalence result
    1. Simple Direction: hybrid argument
    2. Hard direction: dense model theorem
3. Possible extensions and future work

Differential Privacy

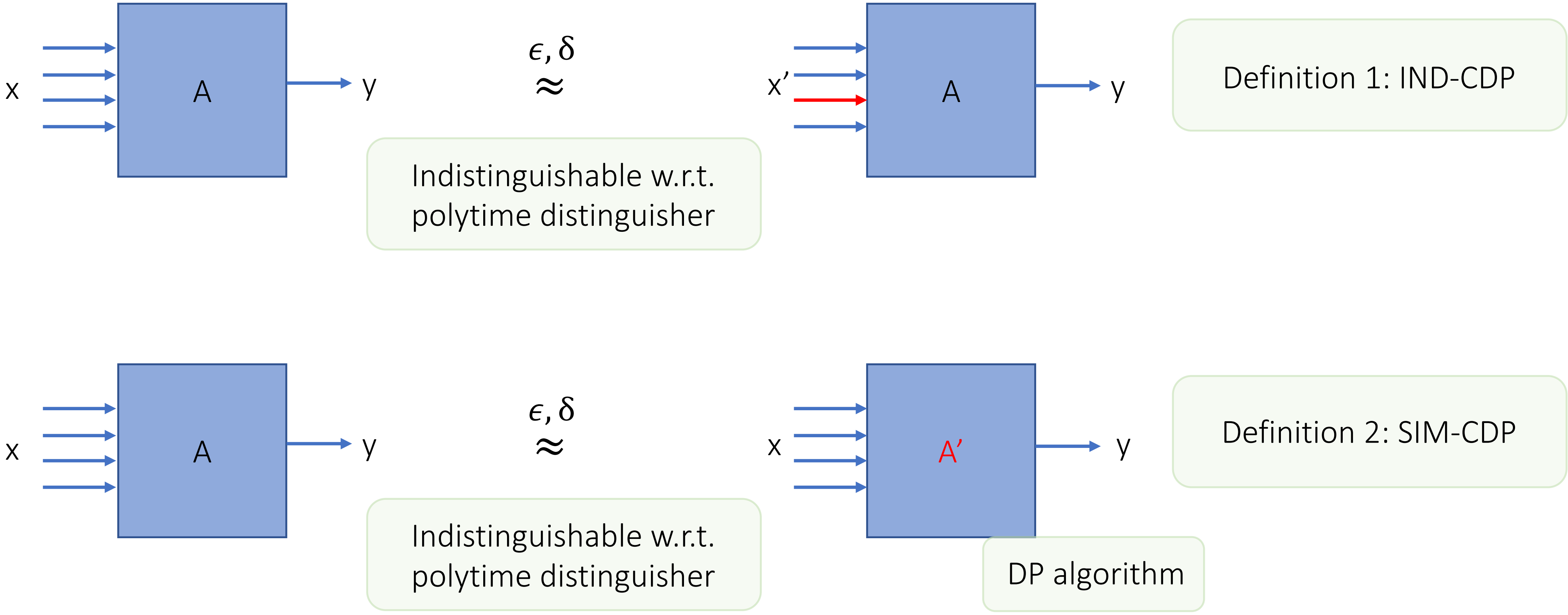


A is (ϵ, δ)-differentially private if for every set of possible outputs *O*, and for every neighboring *x, x'*:

$$\mathrm{P}[A(x) \in O] \leq e^{\epsilon} \cdot \mathrm{P}[A(x') \in O] + \delta$$

Computational Differential Privacy

If a real adversary is a polysize probabilistic circuit, can we relax the definition of Differential Privacy?



Main Theorem of this talk

[Mironov, Pandey, Reingold, Vadhan'09]

Definition 1: IND-CDP

$\Updownarrow$

Proof uses Dense Model Theorem

Definition 3: $\mathrm{SIM}_{\forall\exists}$-CDP

Relaxation of Def 2

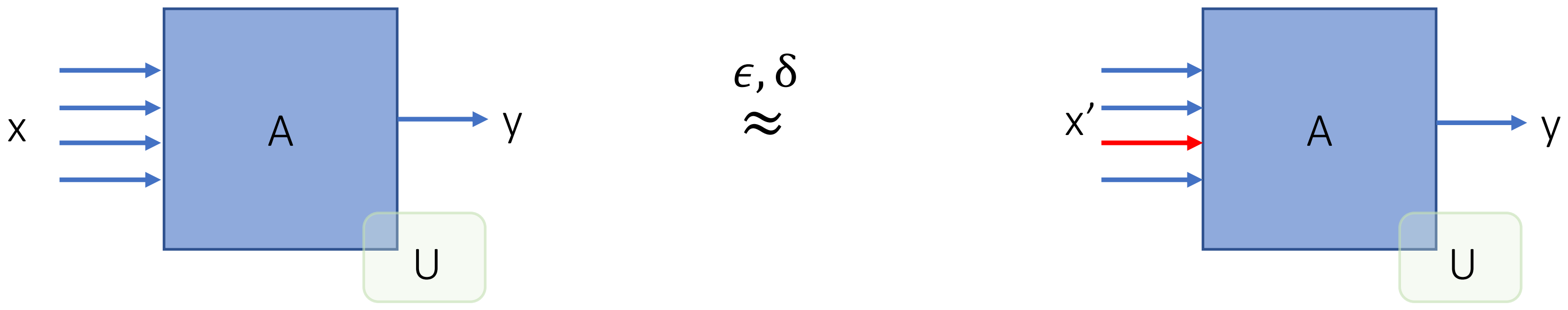
$\Uparrow$

Definition 2: SIM-CDP

Differential Privacy and Randomness Sources

All non-trivial DP algorithms should be randomized

- Standard algorithms sample from the uniform distribution



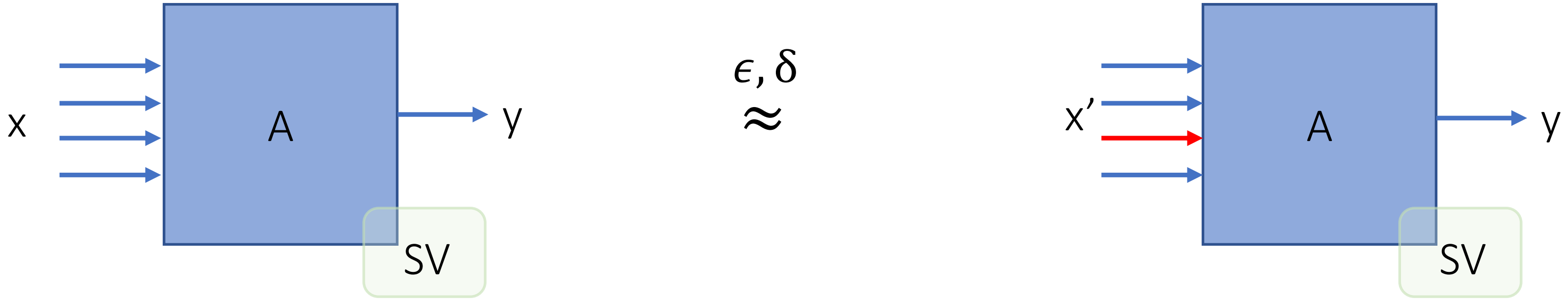
Can we use Santha-Vazirani random sources instead of the Uniform distribution?

- Every i-th bit in the gamma-SV sequence has bias gamma
- Obstacles:
  - SV sources are non-extractable
  - Cannot construct signatures and other basic “privacy” protocols out of it

Differential Privacy with SV sources of Randomness

Can we build DP protocols that uses SV sources of randomness?

[Dodis, Lopez-Alt, Mironov, Vadhan'09]

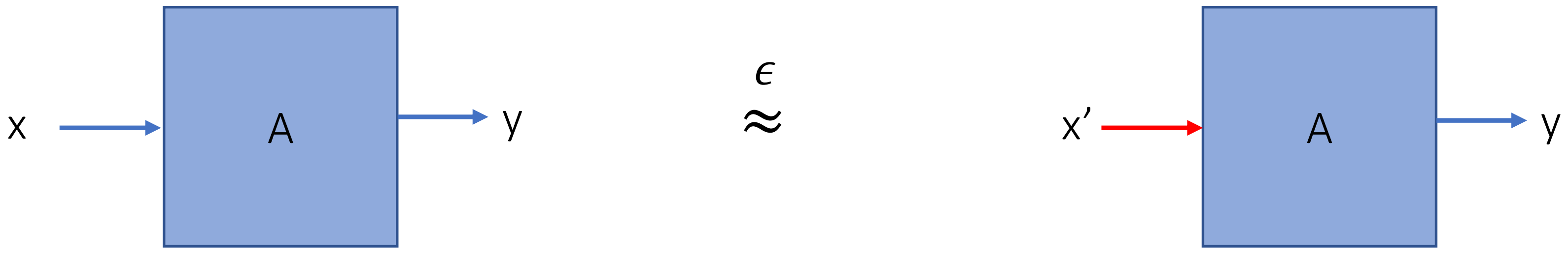


- No, if we use additive noise DP algorithms
  - A(x)=f(x)+random_noise, where f is a true, non-DP answer
- Yes, if we use additive noise DP algorithms with the discretization
  - The main intuition is that sets of probabilities of getting the same output values should be almost the same for all neighboring datasets

Local Differential Privacy

Users may not trust a centralized database,

- Then, they can add noise to their data guarantee their privacy



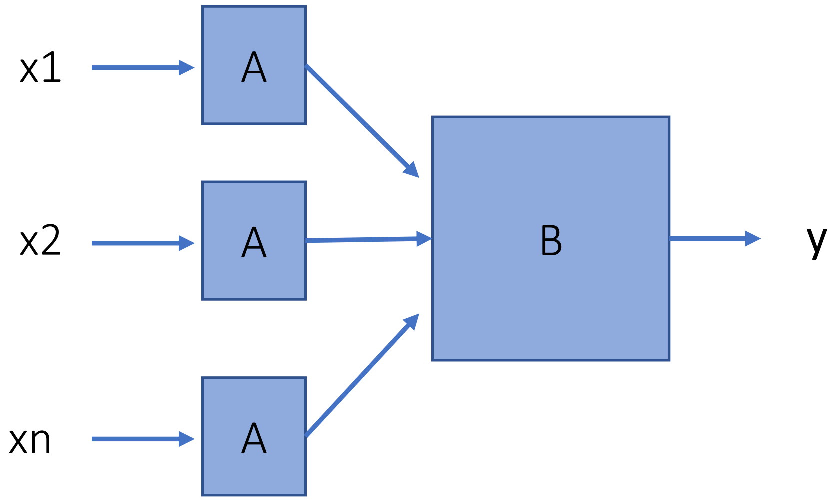
For every user, for every pair of values *x, x’* that this user may have, for every possible output *o*:

$$\mathrm{P}[A(x) = o] \leq e^{\epsilon} \cdot \mathrm{P}[A(x') = o]$$

Local DP: communication overhead

Each user adds noise, encodes, sends their data to the centralized server

- Many users (millions in case of Google, Apple, Facebook) send their data
- Amount of information in each message is small due to noise
- If encoding is not efficient, then communication overhead is huge



What if result of each algorithm A could be encoded using an output of a PRG, so we would need to send only a seed of such PRG?

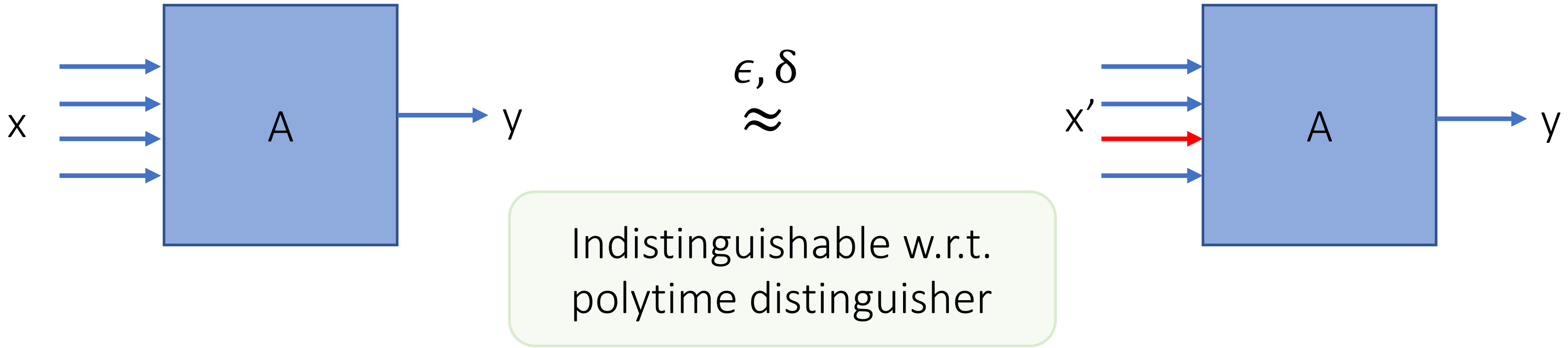
Can significantly decrease communication

- For element counting
- For mean estimation

[Feldman, Talwar'21]

Proof of $\mathrm{SIM}_{\forall\exists}$-CDP $\Leftrightarrow$ IND-CDP

CDP: two definitions



Definition $\mathrm{SIM}_{\forall\exists}$-CDP:

Distribution on outputs of *A* is indistinguishable by polysized circuits from a distribution on outputs of a family of DP algorithm

Definition IND-CDP:

Distribution on outputs of *A* on different inputs is indistinguishable

Equivalence result

Theorem:

A mechanism *B* is IND-CDP if and only if *B* is $\mathrm{SIM}_{\forall\exists}$-CDP

Proof idea:

$\mathrm{SIM}_{\forall\exists}$-CDP => IND-CDP

Hybrid argument

$\mathrm{SIM}_{\forall\exists}$-CDP <= IND-CDP

Dense Model Theorem

This talk

Dense, Pseudodense, and Indist Distributions

Distribution X is e^{ϵ}-dense in Y if

- X, Y are distribution over the same set R,
- $\forall x \in R \ \Pr[\mathrm{X} = \mathrm{x}] \leq e^{\epsilon} \cdot \Pr[Y = x]$.

Distribution X is δ-indistinguishable from Y w.r.t. a family of predicates $\boldsymbol{A}: R \to \{0,1\}$ if

- $\forall A \in \boldsymbol{A} \quad |\Pr[A(\mathrm{X}) = 1] - \Pr[A(Y) = 1]| \leq \delta$.

Distribution X is $(\mathrm{e}^{\epsilon}, \delta)$-pseudodense in Y w.r.t. a family of predicates $\boldsymbol{A}: R \to \{0,1\}$ if

- $\forall A \in \boldsymbol{A} \quad \Pr[A(\mathrm{X}) = 1] \leq e^{\epsilon} \cdot \Pr[A(Y) = 1] + \delta$.

DP Definitions via Pseudodensity

$f: \boldsymbol{D} \rightarrow R$ is ϵ-DP if and only if for all neighboring $D, D' \in \boldsymbol{D}$

- $\mathrm{f(D)}$ is e^{ϵ}-dense in $\mathrm{f(D')}$.

$\{f_k\}: \boldsymbol{D} \rightarrow R$ is ϵ_k-IND-CDP if and only if for all

- Exists $\mathrm{s(k)} = k^{\omega(1)}$
- For all neighboring $D, D' \in \boldsymbol{D}$ of size $\leq \mathrm{s(k)}$ $\mathrm{f}_k\mathrm{(D)}$ is $(\mathrm{e}^{\epsilon}, \frac{1}{s(k)})$-pseudodense in $\mathrm{f}_k\mathrm{(D')}$
  - w.r.t. circuits of size $\leq \mathrm{s(k)}$

$\{f_k\}: \boldsymbol{D} \rightarrow R$ is ϵ_k- $\mathrm{SIM}_{\forall\exists}$-CDP if and only if for all sequences pf neighboring inputs $\{(D_k, {D_k}'\}$

- Exists $\mathrm{s(k)} = k^{\omega(1)}$
- Exists $\{F_k\}$, such that every F_k is ϵ_k-DP
- $\mathrm{f}_k\mathrm{(D)}$ is $\frac{1}{s(k)}$ -indistinguishable from $\mathrm{F}_k(D)$.
  - w.r.t. circuits of size $\leq \mathrm{s(k)}$

Dense Model Theorem

[Reingold, Trevisan, Tulsiani, Vadhan'08]

Theorem:

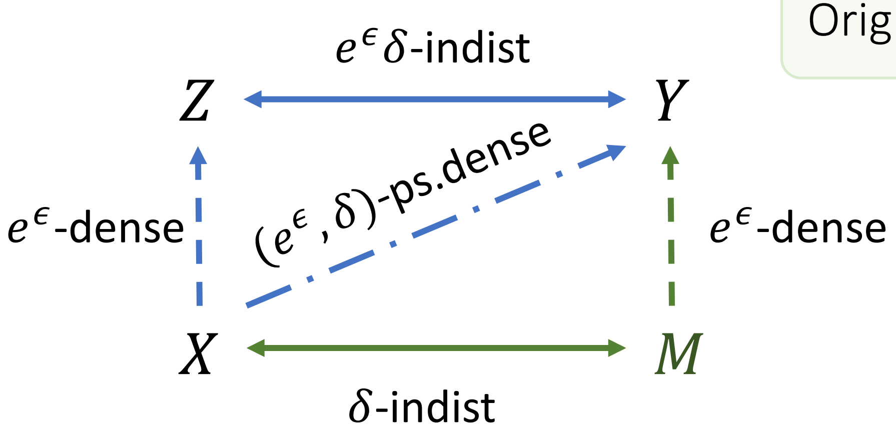
X, Y, and Z are distributions over finite set R

- X is e^{ϵ}-dense in Z
- $A = \{a_i\}$ is a set of predicates $a_i: R \to \{0,1\}$

If all distributions that are e^{ϵ}-dense in Y are δ-distinguishable from X,

Then Z is $\Omega(e^{\epsilon}\delta)$-distinguishable from Y.

Original proof for $Y = U_X$



New Dense Model Theorem

[Mironov, Pandey, Reingold, Vadhan'09]

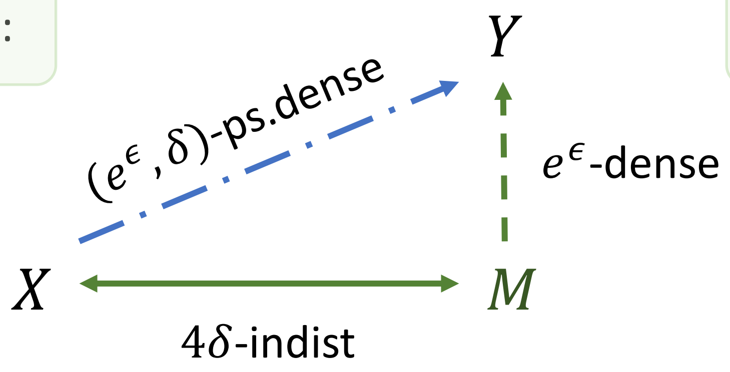
Theorem:

X, Y are distributions over finite set R

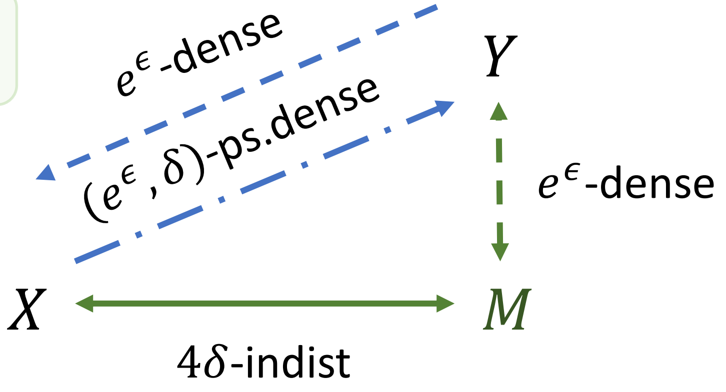
- X is (e^{ϵ}, δ)-pseudodense in Y
- $G(A)$ is a family of threshold predicates

1. Then there exists a distribution M that is e^{ϵ}-dense in Y and 4δ-distinguishable from X with respect to family of predicates A;
2. If Y is e^{ϵ}-dense in X, then Y is e^{ϵ}-dense in M.

Statement 1:

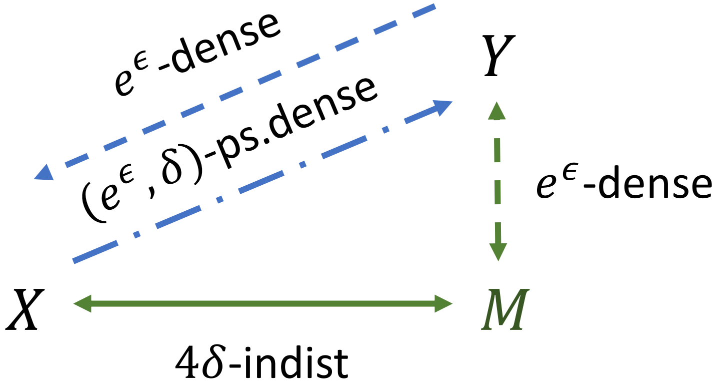


Statement 2:



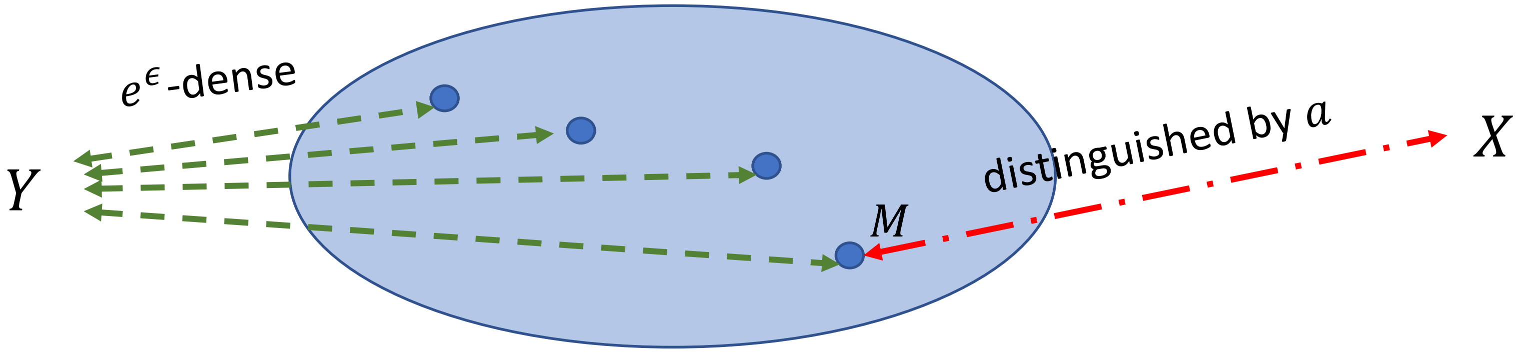
Proof of New Dense Model Theorem

Theorem:



Proof by contradiction:

Assume for any M that is e^{ϵ}-dense in Y exists $a \in \boldsymbol{A}$ that distinguishes M from X w.p. $\geq \mu = 4\delta$.



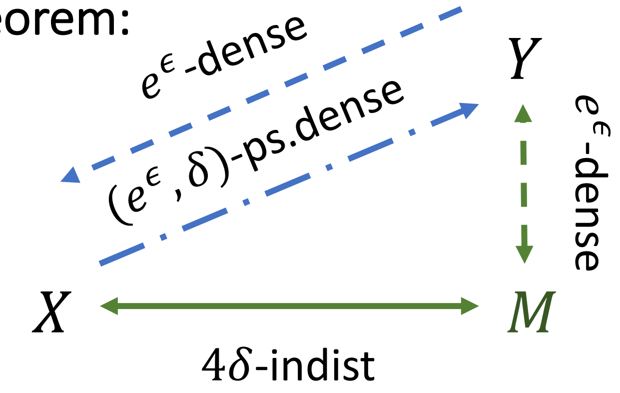
Proof of NDMT

Set of all such M is convex

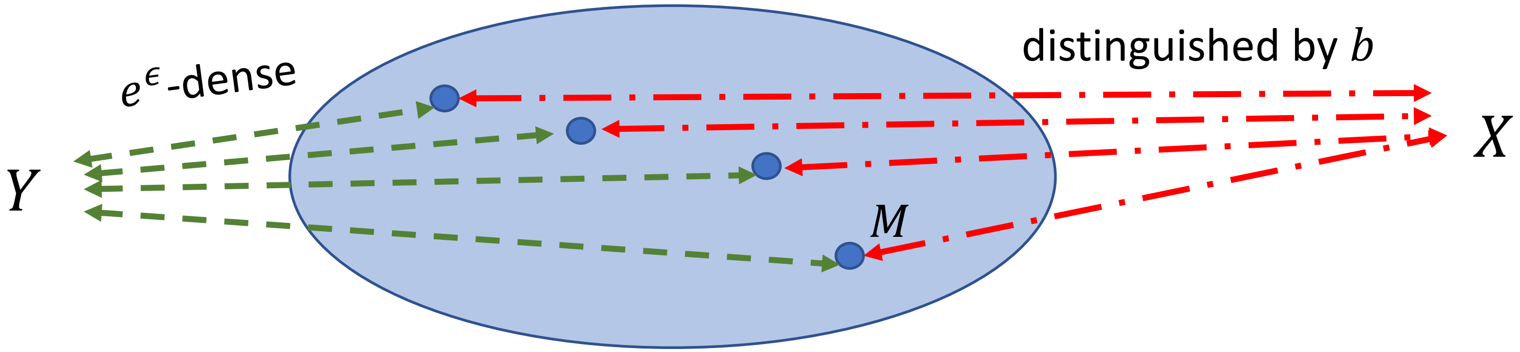
Idea: apply MinMax principle to get a new predicate b that distinguishes every M from X w.p. $\geq \mu = 4\delta$

- Player A picks a predicate a
- Player B picks a distribution D
- The payoff is $\mathrm{E}[a(D)] - \mathrm{E}[a(X)]$
- $\exists$ mixed strategy b: $\forall M$ $\mathrm{E}[b(M)] - \mathrm{E}[b(X)] \geq \gamma$
- $\exists$ mixed strategy M': $\forall a$ $\mathrm{E}[a(M')] - \mathrm{E}[a(X)] \leq \gamma$

Theorem:



Distribution over dense distributions is a dense-distribution => $\gamma \geq \mu$



Proof of NDMT

Consider b: $\forall M\ \mathrm{E}[b(M)] - \mathrm{E}[b(X)] \geq \mu$

Sort all elements $x \in R$ in decreasing order of $\Pr[b(x) = 1]$

x_1	x_2	x_3	x_4	x_5	x_6	…	$x_{\|R\|-1}$	$x_{\|R\|}$

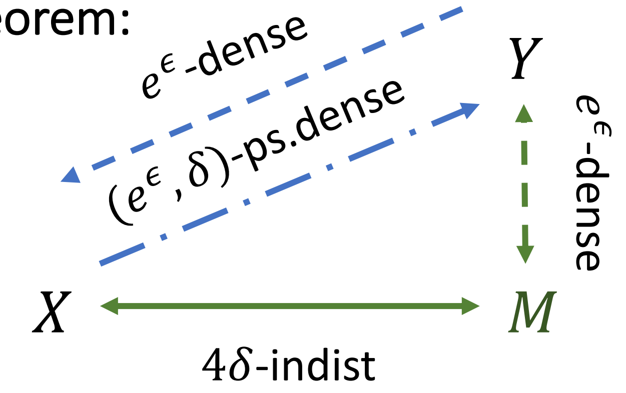
$$\underbrace{x_1\ x_2\ x_3\ x_4\ x_5}_{S:\ \Pr[Y \in S] = \frac{1}{1+e^{\epsilon}}}$$

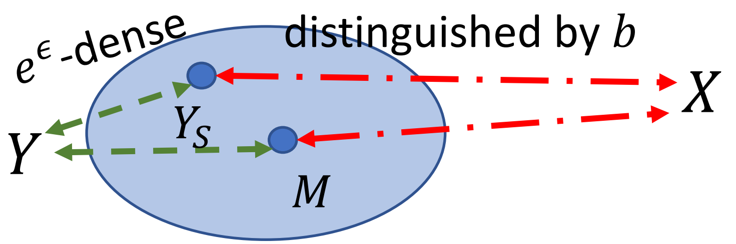
Define a new distribution Y_S:

$$\Pr[Y_S = y] = \Pr[Y = y] \cdot \begin{cases} e^{\epsilon}, \text{if } y \in S \\ e^{-\epsilon}, \text{otherwise} \end{cases}$$

Y_S and Y are e^{ϵ}-dense => b distinguishes Y_S from X

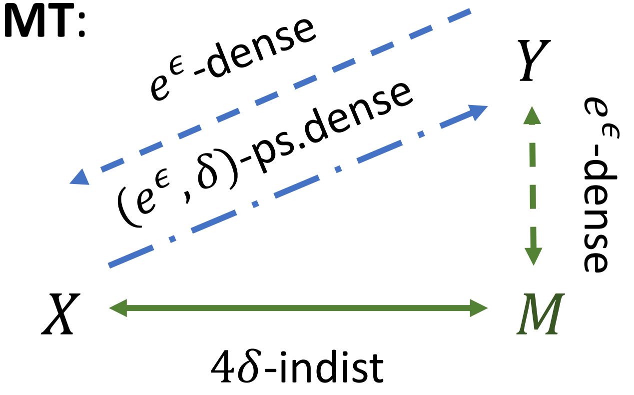
Theorem:





Proof of NDMT

NDMT:



Lemma:

$F: X \to [0,1]$

- Z and W are distributions, such that $\mathrm{E}[F(Z)] \geq \mathrm{E}[F(W)] + \mu$

Then exists $t \in [\frac{\mu}{2}, 1]$, such that

- $\Pr[F(Z) > t] \geq \Pr\left[F(W) \geq t - \frac{\mu}{2}\right] + \frac{\mu}{2}$

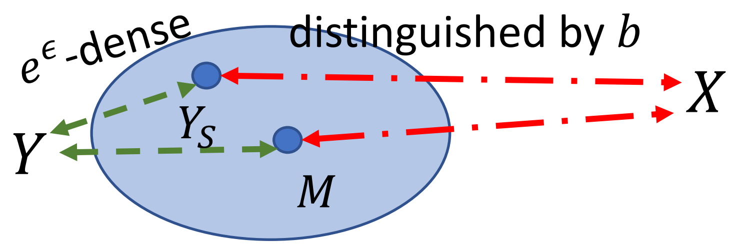
Applying this lemma for $F(x) = \Pr[b(x) = 1]$, X, and Y_S we get:

- $\Pr\left[\Pr[b(X) = 1] > t + \frac{\mu}{2}\right] \geq \Pr[\Pr[b(Y_S) = 1] \geq t] + \frac{\mu}{2}$

If $\geq t + \frac{\mu}{2}$: $b'(x) = 1$
"looks like X"

If $<$ t: $b'(x) = 0$
"looks like Y_S"

new classifier, based on
the output probability of b



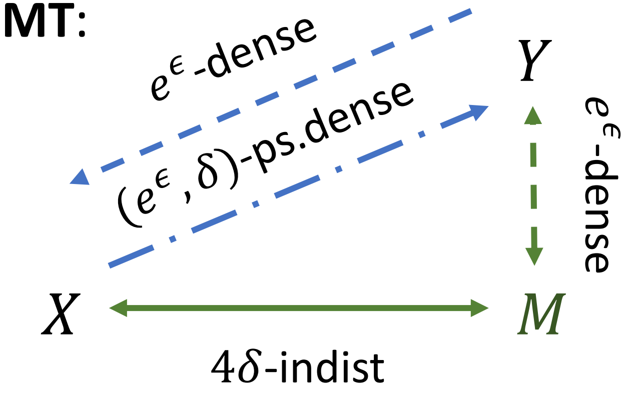
Proof of NDMT

$*$ $\Pr\left[\underbrace{\Pr[b(X) = 1] > t + \frac{\mu}{2}}\right] \geq \Pr[\underbrace{\Pr[b(Y_S) = 1] \geq t}] + \frac{\mu}{2}$

If $\geq t + \frac{\mu}{2}$: $b'(x) = 1$
"looks like X"

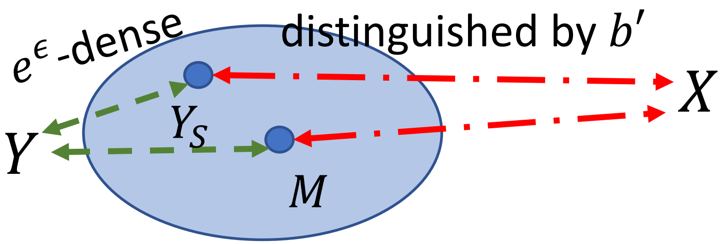
If $<$ t: $b'(x) = 0$
"looks like Y_S"

NDMT:



Claim: $b'(y) = 0\ \forall y \notin S$

Proof by contradiction:
By construction $b'(y) \neq 0\ \forall y \in S$



Y is dense in X => for all $y \notin S$

- $\Pr[Y_S = y] = e^{-\epsilon} \cdot \Pr[Y = y] \leq e^{-\epsilon} e^{\epsilon} \cdot \Pr[X = y] = \Pr[X = y]$

$$\Pr[b'(Y_S) = 0] = \sum_{\substack{y \notin S \\ b(y)=0}} \Pr[Y_s = y] \leq \sum_{\substack{y \notin S \\ b(y)=0}} \Pr[X = y] = \Pr[b'(X) = 0]$$

Contradiction with *:
$\Pr[b'(X) = 0] \leq 1 - \Pr[b'(X) = 1]$
$\leq 1 - \Pr[b'(Y) \neq 0] - \frac{\mu}{2} = \Pr[b'(Y) = 0] - \frac{\mu}{2}$

Final part of the proof of NDMT

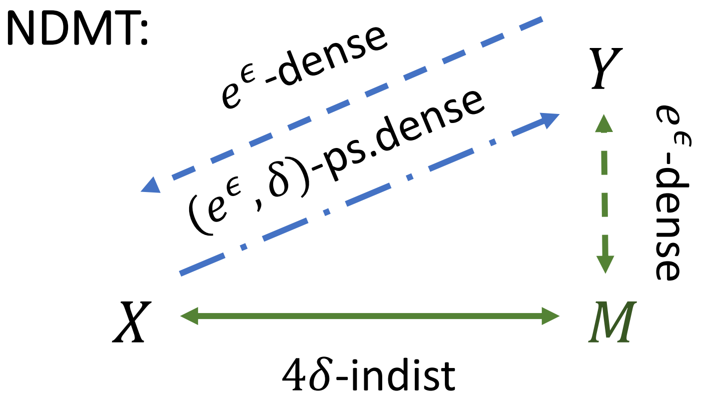
Using the fact that $b'(y) = 0 \ \forall y \notin S$:

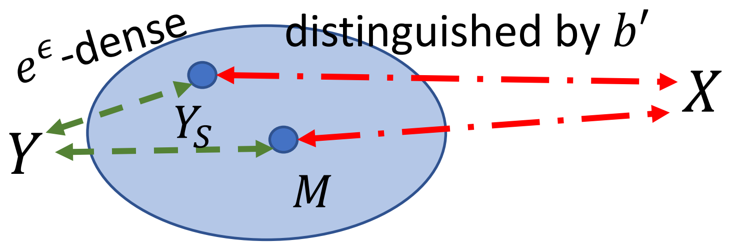
- $\Pr[b'(Y) \neq 0] = \mathrm{e}^{-\epsilon} \cdot \Pr[b'(Y_S) \neq 0] < \mathrm{e}^{-\epsilon}(\Pr[b'(X) = 1] - \frac{\mu}{2})$
- $\Pr[b'(X) = 1] > \mathrm{e}^{\epsilon} \cdot \Pr[b'(Y) \neq 0] + \frac{\mu}{2}$

This could a contradiction with a pseudosensity of X in Y if b' was from the original family of predicates

Lemma:
$F: R \to [0,1]$ is a convex combination of bounded functions from G
- Z_1, Z_2 distributions of G
- $\alpha, \beta > 0$

Then for $k = O(\frac{1}{\alpha^2} \cdot \log \frac{1}{\beta})$ there are $f_1, f_2, \ldots, f_k \in G$ such that
- $\Pr\left[\ \left|F(Z_i) - \frac{1}{k} \cdot (f_1(Z_i) + f_2(Z_i) + \cdots + f_k(Z_i)\right| > \alpha\right] \leq \beta$, for $i = 1,2$





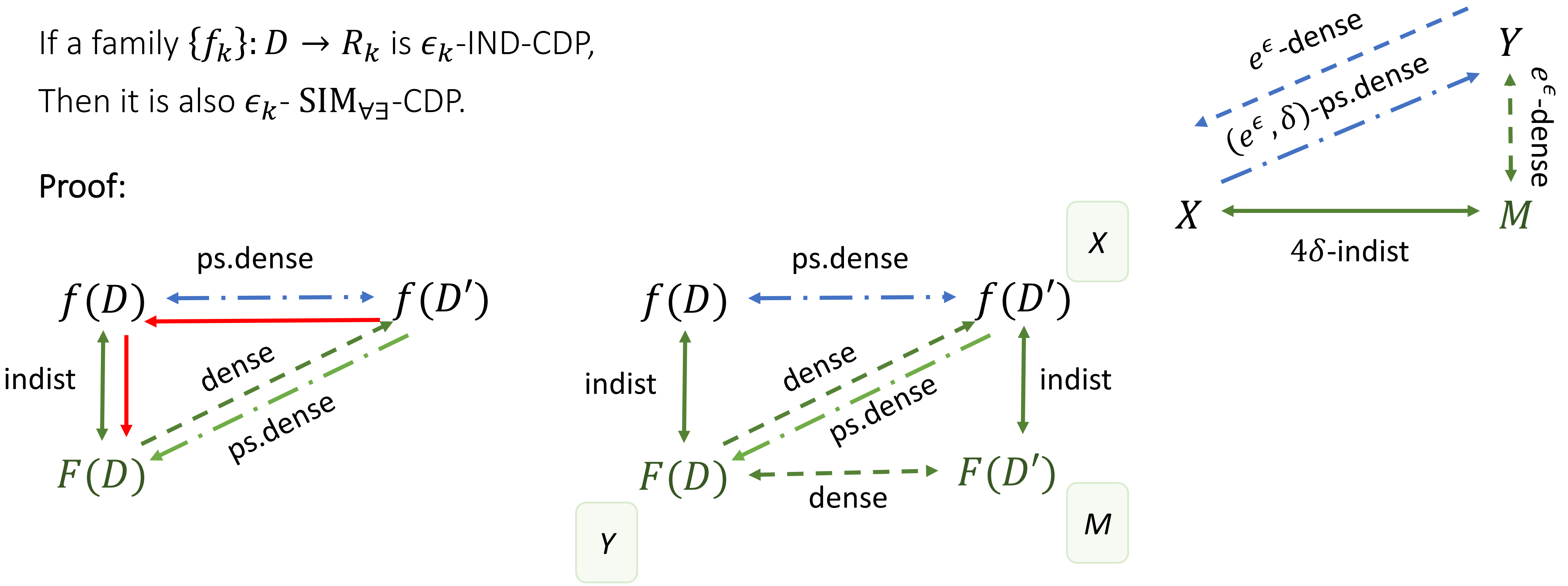
IND-CDP => $\mathrm{SIM}_{\forall\exists}$-CDP using NDMT

Theorem:

If a family $\{f_k\}: D \to R_k$ is ϵ_k-IND-CDP,

Then it is also ϵ_k- $\mathrm{SIM}_{\forall\exists}$-CDP.

Proof:



Conclusion

- Pseudorandomness can help to construct new DP algorithms
  - Can use PRGs to reduce communication complexity for Local-DP algorithms
  - Can use SV-sources with imperfect randomness to construct DP algorithms
  - Can use PRGs and computational indistinguishability to construct various Computationally DP algorithms
- Technical result that we discussed today:
  - Proof of the New Dense Model Theorem
  - Its application to showing that IND-CDP => $\mathrm{SIM}_{\forall\exists}$-CDP

Open Problems

- The separation between $\mathrm{SIM}_{\forall\exists}$-CDP from SIM-CDP remains an open question
- How else, could we apply the new Dense Model Theorem?
- What other random sources (with imperfect randomness) could we use to build DP algorithms?
- Can we unconditionally compress communication in other Local-DP algorithms?