Modern Algorithmic Game Theory

Martin Schmid
Department of Applied Mathematics
Charles University in Prague

November 26, 2025

Sequence Form Representation

Sequence Form

- The conversion to normal-form games allows us to solve extensive-form games using the techniques we already know, e.g. linear programming in two-player zero-sum games
- Unfortunately, as we have seen, the constructed normal-form game can be exponentially large, making it unusable beyond smaller games
- One downside of pure strategies is that they require specifying an action for **every** information state of a player, even the ones that become unreachable given the player's previous moves
- Instead of representing all possible plans of actions, we can represent all paths in the tree, called **sequences**

Sequences & Perfect Recall

- A sequence of moves of player i is the sequence of its actions (ignoring actions of other players) on the unique path from the root to the history h, denoted by $\sigma_i(h)$
- Consider the history $\{\texttt{K}, \texttt{J}, \texttt{check}, \texttt{bet}, \texttt{call}\}$ in Kuhn Poker:
  - $\sigma_1(\texttt{K}, \texttt{J}, \texttt{check}, \texttt{bet}, \texttt{call}) = (\texttt{check}, \texttt{call})$
  - $\sigma_2(\texttt{K}, \texttt{J}, \texttt{check}, \texttt{bet}, \texttt{call}) = (\texttt{bet})$
- Using the notion of sequences, we can now state a formal definition of **perfect recall**

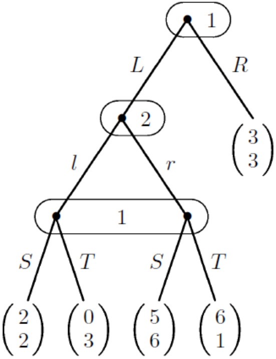
Definition: Perfect Recall

A game satisfies **perfect recall** if and only if, for all players $i \in \mathcal{N}$ and for all pairs of histories $h_1, h_2 \in \mathcal{S}_i$, it holds that $\sigma_i(h_1) = \sigma_i(h_2)$

Sequences

- As we only consider games with perfect recall, we can denote the unique sequence of moves leading to an information state $s \in \mathcal{S}_i$ as σ_s
- Formally, we can define the set Σ_i of all sequences of player i as $\Sigma_i = \{\emptyset\} \cup \{\sigma_s a \mid s \in \mathcal{S}_i, a \in \mathcal{A}_i(s)\}$
- We can see that the size of Σ_i is the total number of unique moves player i can perform (plus 1 for the empty sequence), i.e. $1 + \sum_{s \in \mathcal{S}_i} |\mathcal{A}_i(s)|$, which is **linear in the size of the game tree**!

Normal-Form vs. Sequence-Form Representation



$$A = \begin{array}{c} \\ \end{array}\begin{array}{cc|l} l & r & \\ \hline 2 & 5 & \langle L,S\rangle \\ 0 & 6 & \langle L,T\rangle \\ 3 & 3 & \langle R,S\rangle \\ 3 & 3 & \langle R,T\rangle \end{array} \qquad B = \begin{array}{cc|l} l & r & \\ \hline 2 & 6 & \langle L,S\rangle \\ 3 & 1 & \langle L,T\rangle \\ 3 & 3 & \langle R,S\rangle \\ 3 & 3 & \langle R,T\rangle \end{array}$$

$$A = \begin{array}{ccc|l} \emptyset & l & r & \\ \hline & & & \emptyset \\ & & & L \\ 3 & & & R \\ & 2 & 5 & LS \\ & 0 & 6 & LT \end{array} \qquad B = \begin{array}{ccc|l} \emptyset & l & r & \\ \hline & & & \emptyset \\ & & & L \\ 3 & & & R \\ & 2 & 6 & LS \\ & 3 & 1 & LT \end{array}$$

Figure: Up: Normal-form representation; Down: Sequence-form representation

Realization Probabilities & Expected Utilities

- Given a behavioral strategy π_i, we define the **realization probability** of a sequence σ_s as $\pi_i[\sigma_s] = \prod_{a \in \sigma_s} \pi_i(a)$, where $\pi_i(a)$ is the behavioral strategy at a corresponding information state along the sequence
- Given behavioral strategies for all players (including the chance player), we can express the expected utility of player i in terms of realization probabilities as

$$u_i(\pi) = \sum_{h \in \mathcal{Z}} u_i(h) \pi_c[\sigma_c(h)] \pi_1[\sigma_1(h)] \pi_2[\sigma_2(h)]$$

- However, such an expression is non-linear in individual behavioral probabilities $\pi_i(a)$ which is unsuitable for a linear program
- Instead, we will consider the realization probabilities as functions of sequences σ_s directly

Realization Probabilities & Plans

- Let us denote the realization probability of a sequence σ_s as $x(\sigma_s)$
- Then, a vector $x \in \mathbb{R}^{|\Sigma_i|}$ consisting of realization probabilities of all sequences $\sigma_s \in \Sigma_i$ is called a **realization plan**
- To ensure that each $x(\sigma_s)$ behaves as the product of behavioral strategies corresponding to σ_s, we will require each realization plan to satisfy the following linear constraints

$$x(\emptyset) = 1$$

$$\sum_{a \in \mathcal{A}_i(s)} x(\sigma_s a) = x(\sigma_s) \;\; \forall s \in \mathcal{S}_i$$

- Since we will solve the LP using realization plans, we need to ensure that realization plans and behavioral strategies are equivalent representations

From Realization Plans to Behavioral Strategies

- A realization plan $x(\sigma_s)$ uniquely defines the behavioral strategy π_i of player i
- For each information set $s \in \mathcal{S}_i$ and action $a \in \mathcal{A}_i(s)$, we define

$$\pi_i(s, a) = \frac{x(\sigma_s a)}{x(\sigma_s)}$$

- This ratio is valid because realization plans satisfy the consistency condition

$$x(\sigma_s) = \sum_{a \in \mathcal{A}_i(s)} x(\sigma_s a)$$

- If $x(\sigma_s) = 0$, i.e. the information set is unreachable, the behavioral strategy at s can be chosen arbitrarily because it does not influence any expected utilities

From Behavioral Strategies to Realization Plans

- Any behavioral strategy π_i induces a realization plan $x(\sigma_s)$
- For a sequence $\sigma_s = (a_1, a_2, \ldots, a_k)$ of player i's actions, taken at information sets $s_1, s_2, \ldots, s_k$, the realization probability is

$$x(\sigma_s) = \prod_{j=1}^{k} \pi_i(s_j, a_j)$$

- This construction ensures that realization probabilities behave exactly as the product of the behavioral choices along the path
- Thus, behavioral strategies and realization plans are equivalent, but realization plans are linear and therefore suitable for LP formulations

Sequence-Form Payoff Matrix & Expected Utilities

- For an extensive-form game with perfect recall and a fixed strategy π_c of the chance player, we define the entries of the **sequence-form payoff matrix** corresponding to a pair of sequences (σ, τ) as

$$A_{\sigma,\tau} = \sum_{h \in \mathcal{Z}\colon\, \sigma_1(h)=\sigma, \sigma_2(h)=\tau} u_i(h) \pi_c[\sigma_c(h)]$$

- This matrix is **sparse**; it contains zeros whenever a pair of sequences (σ, τ) does not lead to a terminal history
- To compute the expected payoff given the realization plans for both players, we just need to go through all the terminal nodes and weight them accordingly

$$\sum_{h \in \mathcal{Z}} u_i(h) \pi_c[\sigma_c(h)] x_1(\sigma_1(h)) x_2(\sigma_2(h))$$

- Or equivalently, we can express it in the familiar form $x^\top A y$

Realization Plan Constraints in Matrix Form

- Consider the realization plan constraints again

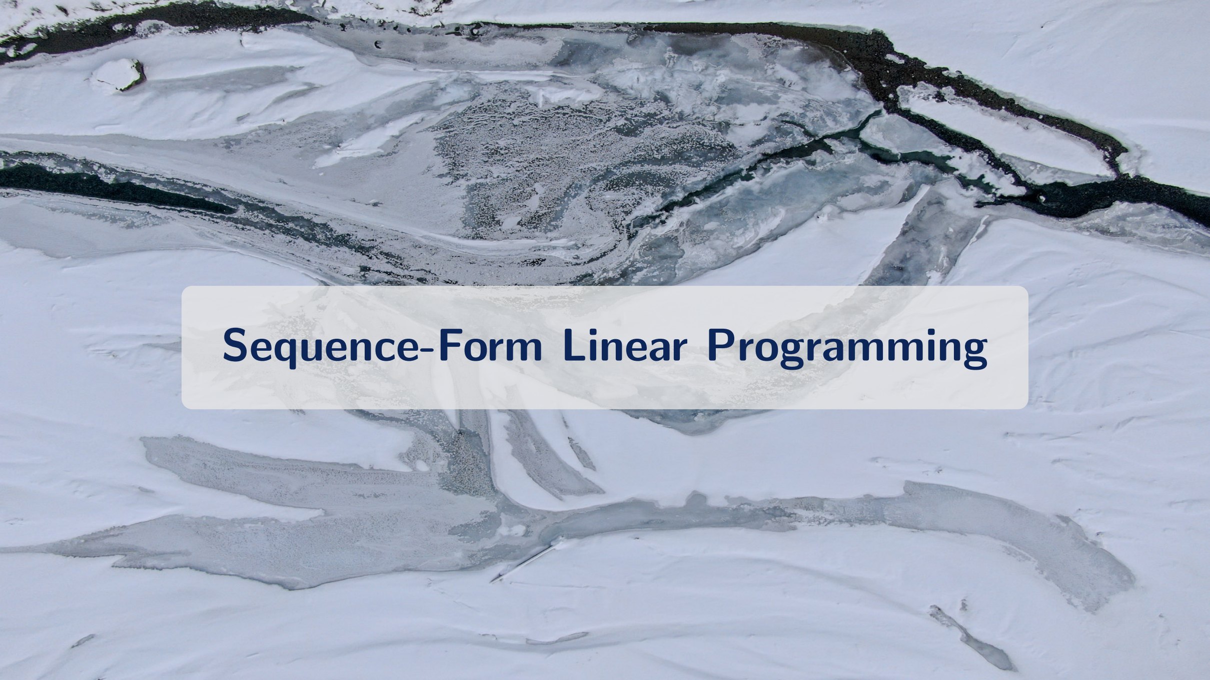
$$x(\emptyset) = 1 \;\; \text{and} \sum_{a \in \mathcal{A}_i(s)} x(\sigma_s a) = x(\sigma_s) \;\; \forall s \in \mathcal{S}_i$$

- We can express these in a matrix form as $Ex = e$, where each row in E represents the left-hand side of the constraint for the corresponding sequence of Player 1
- Similarly, we can use $Fy = f$ for Player 2
- Player 1's realization plan constraints from the example game are

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & 1 & & \\ & -1 & 1 & 1 & \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- The sequence-form payoff matrices, together with the realization plan constraints define the **sequence form** of an extensive-form game

Sequence-Form Linear Programming



Sequence-Form LP Derivation

- Now that we have defined the sequence-form of an extensive-form game and the realization plans, we are ready to derive the **Sequence-form LP**
- It is now easy to see that for a fixed realization plan y of our opponent, we can compute a best response using the following linear program

$$\begin{aligned} \max_x x^\top Ay \\ Ex = e \\ x \geq 0 \end{aligned}$$

Sequence-Form LP Derivation

- The dual LP to the LP from the previous slide looks as follows

$$\min_{u} u^\top e$$
$$E^\top u \geq Ay$$

- Both LPs have feasible solutions and by the strong duality theorem, their optimal values coincide
- We can intuitively view the dual LP as a *certificate* of how large the best response value can be as it upper-bounds $x^\top Ay$

Sequence-Form LP Derivation

- We know that a Minimax strategy of Player 2 is the solution to the following expression
$$\min_y \max_x x^\top A y$$
- However, by the Strong Duality Theorem $\max_x x^\top A y = \min_u u^\top e$ subject to $E^\top u \geq Ay$ for a fixed y
- Substituting this to the above expression, we get
$$\min_{y,u} u^\top e$$
$$E^\top u \geq Ay$$
- Even after adding y as a variable, the above LP is still **linear** in both y and u

Sequence-Form LP

- Finally, we add the realization plan constraints for y and arrive at the LP on the left-hand side which finds a Minimax strategy for Player 2
- Dualizing this LP leads to the LP on the right-hand side for finding a Maximin strategy for Player 1 and looks as follows

$$\begin{aligned} \min_{y,u} u^\top e \\ Fy = f \\ E^\top u \geq Ay \\ y \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max_{x,v} v^\top f \\ Ex = e \\ F^\top v \leq A^\top x \\ x \geq 0 \end{aligned}$$

- As we have previously proven, a pair of Maximin strategies corresponds to a Nash equilibrium in zero-sum games!

Sequence-Form LP

- We have derived an LP formulation that finds a Nash equilibrium of a zero-sum extensive-form game by reformulating the game in terms of its sequence form
- This means that there is a polynomial time algorithm for finding Nash equilibria for two-player zero-sum extensive-form games
- This formulation leads to **exponentially smaller** representation compared to the induced normal-form representation
- It is possible to follow a very similar line of reasoning even for general-sum games
- However, the resulting problem is no longer a linear program but instead a **linear complementarity problem**
- The perhaps most well-known method for solving linear complementarity problems is **Lemke's algorithm**, which has exponential running time in the worst case

Week 8 Homework

You can find more detailed descriptions of homework tasks in the GitHub repository.

1. Extensive-form to normal-form games conversion
2. Finding Nash equilibria in zero-sum games using sequence-form linear programming