Classical BGG and DG BGG.

Jian Liu 2021.5

Notation: k field, $\Lambda = \Lambda(k\xi_1 \oplus \cdots \oplus k\xi_n)$, $|\xi_i| = -1$ graded exterior alg.

$\Lambda^! = k[t_1, \cdots, t_n]$, $|t_i| = 1$ graded polynomial alg.

$S = k[x_1, \cdots, x_n]$. $|x_i| = 2$. graded polynomial alg.

all complexes are cohomological.

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classical BGG correspondence

$$\Phi: D^f(\Lambda^!\text{-gr}) \cong D^f(\Lambda\text{-gr})$$

Recall Φ:

Define $\Phi: C(\Lambda^!\text{-Gr}) \longrightarrow C(\Lambda\text{-Gr})$

$$\Lambda^!\text{-Gr} \ni M = \bigoplus_{i \in \mathbb{Z}} M_i \longmapsto \cdots \to \Lambda^* \otimes_k M_i \xrightarrow{\partial} \Lambda^* \otimes_k M_{i+1} \to \cdots$$

$$f \otimes m \longmapsto (-1)^{|f|+|m|} \sum_{i=1}^{n} \xi_i f \otimes t_i m$$

$$\cdots \to M^{j-1} \xrightarrow{d} M^j \xrightarrow{d} M^{j+1} \to \cdots \longmapsto \mathrm{tot}\left(
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \Lambda^* \otimes M_i^j \arrow[r, "\partial"] \arrow[d, "d"] & \Lambda^* \otimes M_{i+1}^j \arrow[r] \arrow[d, "d"] & \cdots \\
 & \Lambda^* \otimes M_i^{j+1} \arrow[r] \arrow[d, "d"] & \Lambda^* \otimes M_{i+1}^{j+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}
\right)$$

check: ① $\partial^2 = 0$

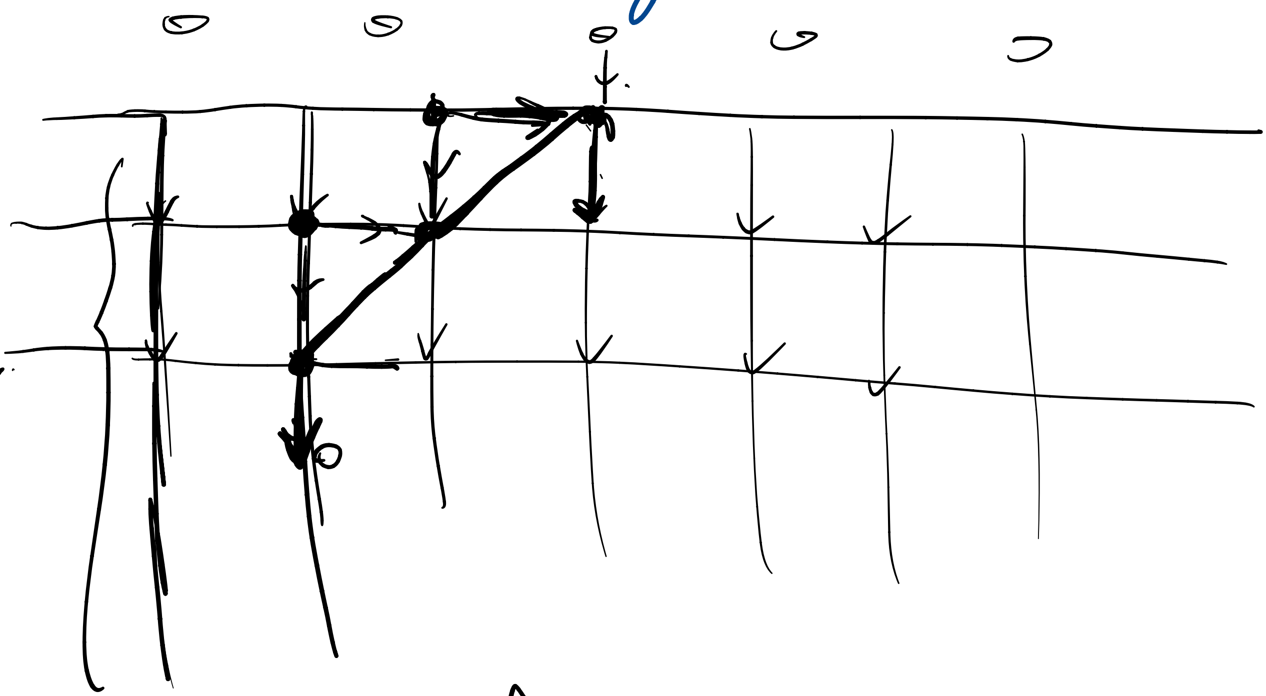
② Sign here makes sure that ∂ is Λ-linear.

Φ preserves null-homotopy and cone. so $\Phi: K(\Lambda^!\text{-Gr}) \to K(\Lambda\text{-Gr})$.

Φ induces $D(\Lambda^!\text{-Gr}) \to D(\Lambda\text{-Gr})$??

Fact 1: If X is bounded below and acyclic, then $\Phi(X)$ is acyclic.

Pf:



all column are acyclic

so $\Phi(X)$ is acyclic.

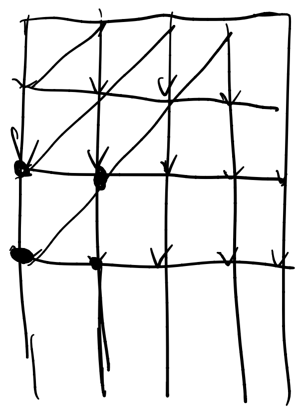
$$\begin{tikzcd} \Phi: D^+(\Lambda^!\text{-Gr}) \arrow[r] & D(\Lambda\text{-Gr}) \\ D^f(\Lambda^!\text{-Gr}) \arrow[u, hook] \arrow[r, dashed, "\Phi"] & D^+(\Lambda\text{-Gr}) \arrow[u, hook] \end{tikzcd}$$

Want: $\Phi: D^f(\Lambda^!\text{-gr}) \longrightarrow D^f(\Lambda\text{-gr})$.

It is enough to show

For $M \in D^b(\Lambda^!\text{-gr})$

$H^i(\Phi(M)) = 0, \ \forall i \gg 0.$



Fact 2: $H^i(\Phi(M))_j = \mathrm{Tor}^{\Lambda^!}_{j-i}(k, M)_j$

Pf: $H^i(\Phi(M))_j =$

$$\cdots \to (\Lambda^*)_{j-i} \otimes M_i \to (\Lambda^*)_{j-i-1} \otimes M_{i+1} \to \cdots$$

homology at here.

you know that:

$$F: \ \cdots \to (\Lambda^*)_n \otimes_k \Lambda^! \to (\Lambda^*)_{n-1} \otimes \Lambda^! \to \cdots \to (\Lambda^*)_0 \otimes_k \Lambda^! \to 0 \quad \xrightarrow{\simeq}_{\Lambda^!} k$$

$$k \overset{L}{\underset{\Lambda^!}{\otimes}} M \cong F \underset{\Lambda^!}{\otimes} M : \ 0 \to (\Lambda^*)_n \otimes_k M \to \cdots \overset{j-i}{\vdots} \cdots \to (\Lambda^*)_0 \otimes_k M \to 0.$$

$$\left(\cdots \to \Lambda^*_{j-i} \otimes M \to \cdots \right)_j \text{-th homogeneous component}$$

For $M \in D^b(\Lambda^!\text{-gr})$, since $\Lambda^!$ is polynomial alg, M has a finite graded free res.

By taking a minimal one, we see $\exists$ only finite many $j, l \in \mathbb{Z}$ s.t. $\mathrm{Tor}^{\Lambda^!}_l(k, M)_j \neq 0$

$\left(\begin{array}{l} l \leftrightarrow \text{projective dim} \\ j \leftrightarrow \text{宽度} \end{array} \right.$

$\Rightarrow$ if $i \gg 0$, $H^i(\Phi(M))_j = \mathrm{Tor}^{\Lambda^!}_{j-i}(k, M)_j = 0$.

Rmk: $\mathrm{Tor}_\ell^{\Lambda^!}(k,M)_j = \beta_M^{\ell j}$ the graded betti number.

CM regularity of M: $\mathrm{reg}(M) = \max\{ n \mid \underset{\underset{\beta_M^{i,i+n}}{\parallel}}{\mathrm{Tor}_i^R(k,M)_{i+n}} \neq 0 \text{ for some } i \}$

① By Fact 2, we know

$\Phi(M)_{\geq i}$ is acyclic except degree i $\Leftrightarrow$ $\mathrm{reg}(M) \leq i$.

every f.g. graded module over graded polynomial alg has finite regularity

[Eisenbud-Goto] Let M be a f.g. graded module over $k[t_0, \cdots, t_n]$, $|t_i| = 1$.

$M_{\geq d}$ has d-linear resolution $\Leftrightarrow$ $d \geq \mathrm{reg}(M)$

② For any f.g. graded module M over $\Lambda^! = k[t_0, \cdots, t_n]$, $|t_i| = 1$.

$M_{\geq d}$ has d-linear resolution if $d \gg 0$.

Bernstein-Gelfand-Gelfand 1978:

$$\Phi : D^f(\Lambda^!\text{-gr}) \xrightarrow{\cong} D^f(\Lambda\text{-gr}).$$

DG BGG Correspondence

Notation: $\Lambda = \Lambda(k\xi_1 \oplus \cdots \oplus k\xi_n)$ $|\xi_i| = -1$.

$S = k[x_1, \dots, x_n]$. $|x_i| = 2$ $\quad (\,(-)^* = \mathrm{Hom}_k(-, k))$

Want: $D^f(S) \xrightarrow{\cong} D^f(\Lambda)$ (DG BGG correspondence).

First, define the functor $D(S) \longrightarrow D(\Lambda)$

Define (J, ∂^J) $\quad J^\natural = \Lambda^* \otimes_k S$

$$\partial^J(f \otimes s) = \sum_{i=1}^{n} \xi_i \cdot f \otimes x_i \cdot s.$$

$\rightsquigarrow$ J is a DG $\Lambda \otimes S$-bimodule.

Fact 1: J is perfect DG S-module and $J \xrightarrow[S]{\simeq} k$ is quasi-iso

Pf: First, J is the total complex of the following

$$0 \to (\Lambda^*)_n \otimes_k S \to (\Lambda^*)_{n-1} \otimes S \to \cdots \to (\Lambda^*)_0 \otimes_k S \to 0$$

$f \otimes s \longmapsto \sum \xi_i f \otimes x_i \cdot s$ $\quad$ [taking mapping cone].

Second, the above complex is Koszul complex with homology k. $\square$

(taking total complex over S doesn't change the S-module structure)

In particular, $J \otimes_S - : D(S) \longrightarrow D(\Lambda)$.

$\boxed{\text{Fact 2: } J \otimes_S S = J \text{ is homotopy injective res. of } k \text{ over } \Lambda.}$

$\rightsquigarrow J \otimes_S - : \operatorname{thick}_{D(S)}(S) \longrightarrow \operatorname{thick}_{D(\Lambda)}(k)$

$$\begin{array}{ccc} \operatorname{thick}_{D(S)}(S) & & \operatorname{thick}_{D(\Lambda)}(k) \\ \| & & \| \\ D^f(S) & & D^f(\Lambda). \end{array}$$

Pf: ① J is homotopy injective.

$J \underset{\Lambda}{\cong} J^{**}$

$\operatorname{Hom}_\Lambda(-, J) \cong \operatorname{Hom}_\Lambda(-, J^{**}) \cong \operatorname{Hom}_k(J^* \otimes_\Lambda -, k)$.

It is enough to show J^* is semi-free DG Λ-module.

$(J^*)^\natural \cong \Lambda \otimes_k S^*$

$$\Lambda \otimes_k (S^*)_{\geq 0} \hookrightarrow \Lambda \otimes_k (S^*)_{\geq -2} \hookrightarrow \Lambda \otimes_k (S^*)_{\geq -4} \hookrightarrow \cdots\cdots \quad \subseteq J^*$$

$$\begin{array}{ccc} \| & \| & \| \\ F_0 & F_1 & F_2. \end{array}$$

F_i/F_{i-1} is graded free Λ-module. $\Rightarrow$ J^* is semi-free

② $\Lambda \xrightarrow{\pi} k \quad k \xrightarrow{\ell} S.$

$k \xrightarrow[\Lambda]{\pi^* \otimes \ell} J \xrightarrow[S]{\simeq} k \quad \Rightarrow \quad k \xrightarrow[\Lambda]{} J$ is also quasi-iso $\square$

chain maps.

Fact 3: $S \xrightarrow{\simeq} \mathrm{Hom}_\Lambda(J, J)$

$s \longmapsto (x \longmapsto xs)$

pf:

$$\begin{tikzcd}
S \arrow[r, "m"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] \arrow[dd, bend right] & \mathrm{Hom}_\Lambda(J, J) \arrow[d, "\cong"] \\
\mathrm{Hom}_\Lambda(k, J^{\#}) \arrow[r, equal] & \mathrm{Hom}_\Lambda(k, J) \\
\mathrm{Hom}_\Lambda(k, \Lambda^* \otimes_k S) &
\end{tikzcd}$$

$\iota : k \xrightarrow[\Lambda]{\simeq} J$

$\forall f : k \to J$ Λ-linear means $\mathrm{Im} f$ is in the socle of J, must be in degree 0, $k \to J \xrightarrow{\partial} J$ composition is 0.

$\mathrm{Hom}_\Lambda(k, \Lambda^* \otimes_k S) \cong \mathrm{Hom}_\Lambda(k, \mathrm{Hom}_k(\Lambda, S))$

$\cong \mathrm{Hom}_k(\Lambda \otimes_\Lambda k, S)$

$\cong S$

$\boxed{\text{Th}}$

diagram commutes:

$$\begin{tikzcd}
s \arrow[r, mapsto] \arrow[d] & (f \otimes x \mapsto f \otimes xs) \arrow[d] \\
(1 \mapsto 1 \otimes s) \arrow[r, mapsto] & (1 \mapsto 1 \otimes s)
\end{tikzcd}$$

$D^f(S) \underset{R\mathrm{Hom}_\Lambda(J, -)}{\overset{J \otimes_S -}{\rightleftarrows}} D^f(\Lambda)$ and $S \xrightarrow{\simeq} \mathrm{End}_\Lambda(J)$. $\Rightarrow$ Two functors are equivalence.

Making use of Beilinson Lemma. $\mathrm{Hom}_{D(S)}(S, S[i]) \longrightarrow \mathrm{Hom}_{D(\Lambda)}(J, J[i])$

$\forall i \in \mathbb{Z}$, if $i \neq 0$, both sides are 0.

$i = 0$ $\quad \mathrm{Hom}_{D(S)}(S, S) \xrightarrow{\cong} \mathrm{Hom}_{D(\Lambda)}(J, J)$ right multiplication

$\phantom{i = 0 \quad \mathrm{Hom}_{D(S)}(} \overset{\shortparallel}{S}$

by Beilinson lemma; $J \otimes_S - : D^f(S) \longrightarrow D^f(\Lambda)$ is fully faithful

Similarly you can check $R\mathrm{Hom}_\Lambda(J, -)$ is fully faithful. $\boxed{\text{so They are equivalence.}}$

Classical BGG vs DG BGG

Recall: Let A be a graded alg.

$$Tot : C(A\text{-}Gr) \longrightarrow C(A)$$

$$M : \cdots \to M^i \xrightarrow{d} M^{i+1} \to \cdots \longmapsto Tot(M)$$

$$Tot(M)^n = \bigoplus_{i+j=n} M^i_j$$

$\forall m \in M^i_j$. $\quad \partial_{Tot(M)}(m) = (-1)^j d(m)$

$a \circ m = a \cdot m$

Sign here makes sure $Tot(M)$ is a DG module.

It is clear that Tot is an exact functor.
Tot preserves $[1]$, cone and homotopy maps. So $\exists$

Δ-functor $Tot : K(A\text{-}Gr) \to K(A)$

$$Tot : D(A\text{-}Gr) \longrightarrow D(A).$$

If A is noetherian, then $Tot : D^f(A\text{-}gr) \longrightarrow D^f(A)$.

Lemma: there exists a diagram:

$$\begin{tikzcd}
D^f(A^!\text{-}gr) \arrow[rr, "\Phi", "\cong"'] \arrow[d, "\ell"'] & & D^f(A\text{-}gr) \arrow[dd, "Tot"] \\
D^f(S\text{-}gr) \arrow[d, "Tot"] & & \\
D^f(S) \arrow[rr, "J\otimes_S -", "\cong"'] & & D^f(\Lambda)
\end{tikzcd}$$

where $\ell : A^!\text{-}Gr \longrightarrow S\text{-}Gr$

$$\bigoplus_{j\in\mathbb{Z}} M_j \longmapsto \bigoplus_{i\in\mathbb{Z}} N_i$$

$N_{2i} = M_i$, $N_{2i+1} = 0$

$x_i \circ m = t_i \cdot m$.

Pf: for $M \in \Lambda^!$-gr

$$\begin{tikzcd}
M \arrow[r, mapsto] \arrow[d, mapsto] & \cdots \to \Lambda^* \otimes M_j^i \arrow[r, "d"] \arrow[d, mapsto, "\mathrm{Tot}"] & \Lambda^* \otimes M_{j+1}^i \to \cdots \\
N & \mathrm{Tot} &
\end{tikzcd}$$

$N.\quad N_{2l} = M_i,\quad N_{2l+1} = 0$

$f\otimes m \in (\Lambda^*_l \otimes M_j) \subseteq \mathrm{Tot}(\,)^{l+i+j}$

$\partial(f\otimes m) = (-1)^{j+l} d(f\otimes m) = \sum \xi_i f\otimes t_i m$.

$$\begin{tikzcd}
N \arrow[r, mapsto, "\mathrm{Tot}"] & N \arrow[r, mapsto] & J\otimes_S N \cong \Lambda^* \otimes_k N.
\end{tikzcd}$$

$(\Lambda^*)_l \otimes_k M_j^i \subseteq (\Lambda^* \otimes N)^{2j+l}$

differential $f\otimes m \in (\Lambda^*)_l \otimes M_j^i$

$\partial(f\otimes m) = \sum \xi_i f \otimes t_i m$

$$\begin{tikzcd}
M^i \arrow[r, "d"] & M^{i+1} \arrow[r, mapsto] & \mathrm{Tot}\left(\begin{array}{ccc} \Lambda^*\otimes M_j^i & \longrightarrow & \Lambda^*\otimes M_{j+1}^i \to \cdots \\ \downarrow d & & \downarrow d \\ \Lambda^*\otimes M_j^{i+1} & \longrightarrow & \Lambda^*\otimes M_{j+1}^{i+1} \to \cdots \end{array}\right)
\end{tikzcd}$$

$(-1)^i$

$(\Lambda^*)_l \otimes M_j^i \subseteq (\quad)^{l+i+2j}$

$\partial(f\otimes m) = (-1)^{j+l} d_{\mathrm{Tot}}(f\otimes m)$

$= (-1)^{j+l}\left((-1)^{j+l} \sum \xi_i f\otimes t_i m + (-1)^j f\otimes dm\right)$

$$\begin{tikzcd}
M \arrow[d, mapsto] \\
N \arrow[d, "\mathrm{Tot}"] \\
\mathrm{Tot}(N) \arrow[r, mapsto] & J\otimes_S N \cong J \otimes_k N
\end{tikzcd}$$

$m \in M_j^i$

$\partial(m) = (-1)^{2j} d(m)$

$= d(m)$

$f\otimes m \in (\Lambda^*)_l \otimes M_j^i \subseteq (J\otimes_S N)^{l+i+2j}$

$\partial(f\otimes m) = \sum \xi_i f\otimes t_i m + (-1)^l f\otimes dm$.

$\square$

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