

Notation: k field, $\Lambda = \Lambda(k\mathbb{I}_1 \oplus \dots \oplus k\mathbb{I}_n)$, $|\mathbb{I}_i| = -1$ graded exterior alg.

$\Lambda^! = k[t_1, \dots, t_n]$, $|t_i| = 1$ graded polynomial alg.

$S = k[x_1, \dots, x_n]$, $|x_i| = 2$ graded polynomial alg. all complex are chronological

classical BGK correspondence

$$\Xi: D^f(\Lambda^!-\text{gr}) \cong D^f(\Lambda-\text{gr})$$

Recall Φ :

$$\text{Define } \Phi: C(\Lambda^!-\text{Gr}) \rightarrow C(\Lambda-\text{Gr})$$

$$\Lambda^!-\text{Gr} \ni M = \bigoplus_{i \in \mathbb{Z}} M_i \xrightarrow{\quad} \dots \rightarrow \Lambda^* \otimes_k M_i \xrightarrow{\partial} \Lambda^* \otimes_R M_{i+1} \rightarrow \dots$$

$$f \otimes m \mapsto (-) \sum_{i=1}^{k+1+m} s_i f \otimes t_i m$$

$$\dots \rightarrow M^{j+1} \xrightarrow{d} M^j \xrightarrow{d} M^{j+1} \rightarrow \dots$$

\$\xrightarrow{\text{tot}}\$

$$\begin{pmatrix} \dots & \downarrow & & \downarrow \\ \dots & \xrightarrow{\Lambda^* \otimes M_i} & \xrightarrow{\partial} & \xrightarrow{\Lambda^* \otimes M_{i+1}} \dots \\ & \downarrow d & & \downarrow d \\ \dots & \xrightarrow{\Lambda^* \otimes M_i^{j+1}} & \xrightarrow{\Lambda^* \otimes M_{i+1}^{j+1}} & \dots \\ & \vdots & & \vdots \end{pmatrix}$$

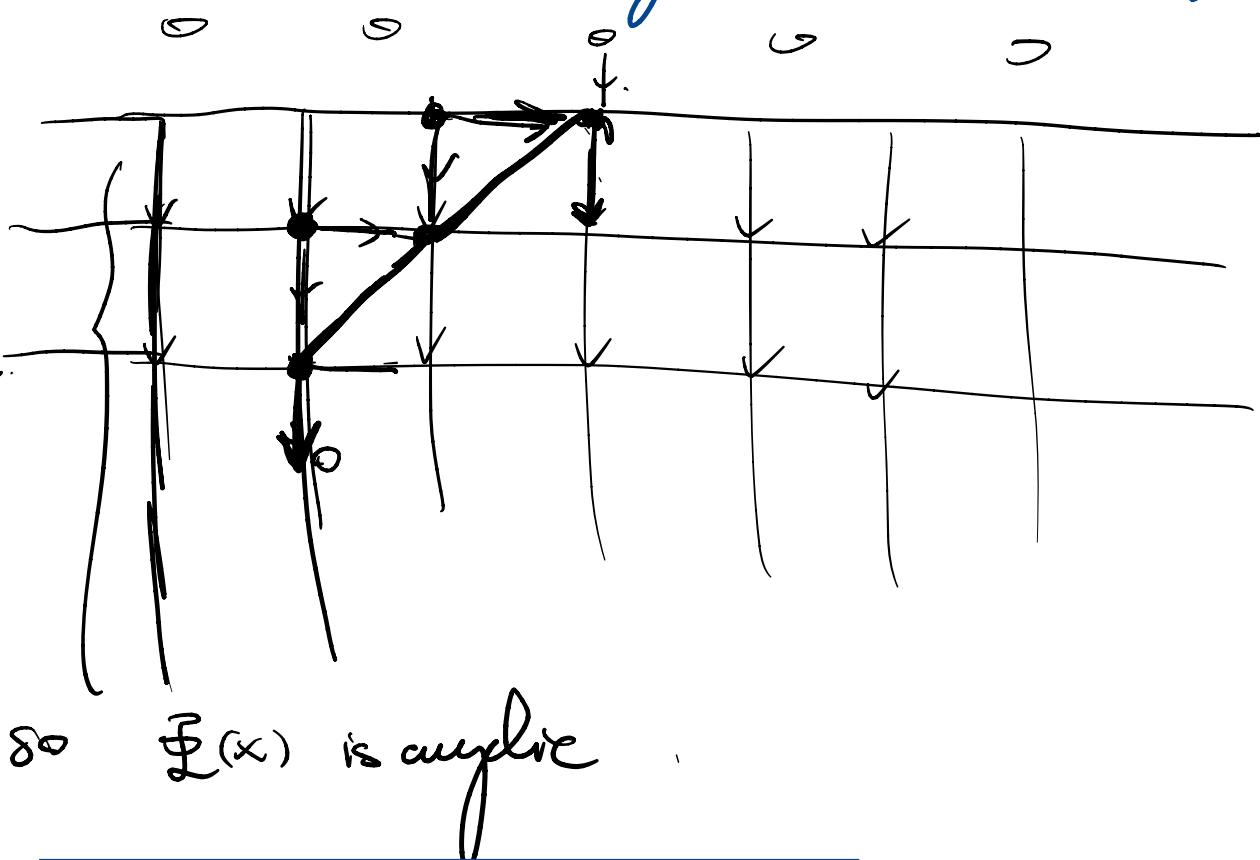
check:
 ① $\partial^2 = 0$
 ② Sign here makes sure that ∂ is Λ -linear.

Φ preserves null-homotopy and cone. so $\Phi: K(\Lambda^!-\text{Gr}) \rightarrow K(\Lambda-\text{Gr})$.

Φ induces $D(\Lambda^!-\text{Gr}) \rightarrow D(\Lambda-\text{Gr})$??

Fact 1: If X is bounded below and acyclic, then $\mathbb{E}(X)$ is acyclic.

Pf :



$$\mathbb{E}: D^+(\Lambda^!-\text{gr}) \rightarrow D(\Lambda-\text{gr}).$$

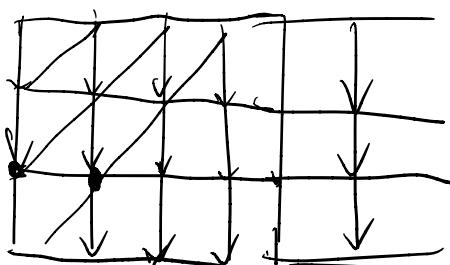
$$D^f(\Lambda^!-\text{gr}) \xrightarrow{\mathbb{E}} D^+(\Lambda-\text{gr})$$

Want: $\mathbb{E}: D^f(\Lambda^!-\text{gr}) \rightarrow D^f(\Lambda-\text{gr})$.

It is enough to show

For $N \in D^b(\Lambda^!-\text{gr})$
 $H^i(\mathbb{E}(N)) = 0, \forall i > 0$.

By induction, it is enough to show it is true for graded module



$$\text{Fact 2 : } H^i(\Phi(M))_j = \text{Tor}_{j-i}^{A^!}(k, M)_j, \forall M \in A^!-\text{gr}$$

$$\text{Pf: } H^i(\Phi(M))_j =$$

$$\cdots \rightarrow (\wedge^*)_{j-i} \otimes M_i \rightarrow (\wedge^*)_{j-i} \otimes M_{(i)} \rightarrow \cdots$$

homology at here.

You know that:

$$F: \cdots \rightarrow (\wedge^*)_n \otimes_k A^! \rightarrow (\wedge^*)_m \otimes_k A^! \rightarrow \cdots \rightarrow (\wedge^*)_0 \otimes_k A^! \rightarrow \xrightarrow{\cong} k$$

$$k \underset{A^!}{\otimes} M \cong F \underset{A^!}{\otimes} M: \cdots \rightarrow (\wedge^*)_n \otimes_k M \rightarrow \cdots \rightarrow (\wedge^*)_0 \otimes_k M \rightarrow 0.$$

\downarrow

$j-i$

($\cdots \rightarrow \wedge^*_{j-i} \otimes M \rightarrow \cdots$) $\underset{j\text{-th homogeneous}}{\longrightarrow}$

For $M \in A^!-\text{gr}$. since $A^!$ is polynomial alg, component \blacksquare

M has a finite graded free res.

By taking a monomial one, we see \exists only finitely many j, l $\in \mathbb{Z}$

$$\text{sit } \text{Tor}_l^{A^!}(k, M)_j \neq 0$$

$\begin{cases} l \mapsto \text{projective dim} \\ j \mapsto \text{degree} \end{cases}$

$$\Rightarrow \text{if } i > 0, H^i(\Phi(M))_j = \text{Tor}_{j-i}^{A^!}(k, M)_j = 0.$$

Rmk: $\text{Tor}_i^R(k, M)_j = \beta_M^{ij}$ the graded betti number

All regularities of M : $\text{reg}(M) = \max \{ n \mid \text{Tor}_{n+1}^R(k, M) \neq 0 \text{ for some } i \}$

① By Fact 2, we know

$\underline{\Phi}(M)_{\geq i}$ is acyclic except degree $i \Leftrightarrow \text{reg}(M) \leq i$.

every f.g. graded module over graded polynomial alg has finite regularity.

[Eisenbud-Goto] let M be a f.g. graded module over $k[t_0, \dots, t_n]$
 $|t_i| = 1$.

$M_{\geq d}$ has d -linear resolution $\Leftrightarrow d \geq \text{reg}(M)$

② for any f.g. graded module M over $\Lambda^! = k[t_0, \dots, t_n], |t_i| = 1$.

$M_{\geq d}$ has d -linear resolution if $d > 0$.

Bernstein-Gelfand-Gelfand 1978:

$$\underline{\Phi}: D^f(\Lambda^!-\text{gr}) \xrightarrow{\sim} D^f(\Lambda-\text{gr}).$$

DG, BGB Correspondence

Notation : $\Lambda = \Lambda(kz_1 \otimes \dots \otimes kz_n)$ ($z_i^2 = -1$).

$$S = k[x_1, \dots, x_n] \quad |x_i| = z \quad (-)^* = \text{Hom}_k(-, k)$$

Want: $D^f(S) \xrightarrow{\cong} D^f(\Lambda)$ (Der PBW correspondence)

First, define the functor $D(S) \rightarrow D(\Lambda)$

Define (J, δ^J) $J^\sharp = \Lambda^* \otimes_k S$

$$\delta^J(f \otimes s) = \sum_{i=1}^n z_i \cdot f \otimes x_i \cdot s.$$

$\rightsquigarrow J$ is a DG $\Lambda \otimes S$ -bimodule.

Fact 1: J is perfect DG S -module and $J \xrightarrow[S]{\cong} k$ is quasi-iso

Pf: First, J is take total complex of the following

$$\cdots \rightarrow (\Lambda^*)_n \otimes_k S \rightarrow (\Lambda^*)_{n+1} \otimes_k S \rightarrow \cdots \rightarrow (\Lambda^*)_0 \otimes_k S \rightarrow 0$$

$$f \otimes s \mapsto \sum z_i f \otimes x_i \cdot s \quad [\text{taking mapping cone}]$$

Second, The above complex is kernel complex with homology k . (⊗ over)

(⊗ total complex
doesn't change the module structure)

In particular, $J \otimes_S^\sharp : D(S) \rightarrow D(\Lambda)$.

Fact 2: $J \otimes S = J$ is homotopy injective res. of k over Λ .

$$\hookrightarrow \int_{S^-}^{\otimes -} : \text{thick}_{D(S)}(S) \longrightarrow \text{thick}_{D(\Lambda)}(k)$$

$$Df(S) \quad \quad \quad Df(\Lambda).$$

Pf: ① J is homotopy injective.

$$J \xrightarrow{\cong} J^{**}$$

$$\text{Hom}_\Lambda(-, J) \cong \text{Hom}_\Lambda(-, J^{**}) \cong \text{Hom}_k(J^{*\otimes \Lambda} -, k).$$

It is enough to show J^* is semi-free DG Λ -module.

$$(J^*)^\wedge \cong \Lambda \otimes_k S^*$$

$$\Lambda \otimes_k (S^*)_{\geq 0} \hookrightarrow \Lambda \otimes_k (S^*)_{\geq 2} \hookrightarrow \Lambda \otimes_k (S^*)_{\geq 4} \hookrightarrow \dots \subseteq J^*$$

$$F_0 \quad \quad \quad F_1 \quad \quad \quad F_2.$$

F_i/F_{i+2} is graded free Λ -module. $\Rightarrow J^*$ is semi-free

$$\textcircled{2} \quad \Lambda \xrightarrow{\pi} k \xrightarrow{\epsilon} S.$$

$$k \xrightarrow{\pi \otimes \text{id}} J \xrightarrow{\cong} k \quad \Rightarrow \quad k \xrightarrow{\epsilon} J \text{ is also quasi-iso}$$

(check maps)



Fact 3 : $S \xrightarrow{\cong} \text{Hom}_\Lambda(J, J)$
 $s \longmapsto (x \mapsto xs)$

Pf: $S \xrightarrow{m} \text{Hom}_\Lambda(J, J) \quad l: k \xrightarrow{\cong} J$

$$\begin{array}{ccc} & \downarrow \cong & \\ \text{Hom}_\Lambda(k, J^\otimes) & = & \text{Hom}_k(k, J) \\ \curvearrowleft & & \downarrow \cong \\ \text{Hom}_\Lambda(k, \Lambda^{\otimes_k} S) & \cong & \text{Hom}_\Lambda(k, \text{Hom}_k(\Lambda, S)) \end{array}$$

$\forall f: k \rightarrow J$ A -linear
means $l \circ f$ is in the socle of J
 $k \xrightarrow{f} J \xrightarrow{l} J$ must be in degree 0 .
Composition is 0 .

$$\cong \text{Hom}_k(\Lambda \otimes k, S)$$

$$\cong S$$

□

Diagram commutes : $S \xrightarrow{(f \otimes x \mapsto f \otimes xs)}$
 \downarrow
 $(1 \mapsto 1 \otimes S) \xrightarrow{\quad} (1 \mapsto 1 \otimes S)$

$D^f(S) \xrightarrow{\substack{f \otimes - \\ R\text{Hom}_\Lambda(J, -)}} D^f(\Lambda)$ and $S \xrightarrow{\cong} \text{End}_\Lambda(J) \Rightarrow$ Two functors are equivalence.

Making use of Beilinson Lemma. $\text{Hom}_{D(S)}(S, S[i]) \longrightarrow \text{Hom}_{D(\Lambda)}(J, J[i])$

$\forall i \in \mathbb{Z}$, if $i \neq 0$, both sides are 0 .

$$i=0 \quad \text{Hom}_{D(S)}(S, \mathbb{S}) \xrightarrow{\cong} \text{Hom}_{D(\Lambda)}(J, J)$$

by Beilinson lemma: $J_S^\otimes : D^f(S) \xrightarrow{S} D^f(\Lambda)$ is fully faithful multiplication

Similarly you can check $R\text{Hom}_\Lambda(J, -)$ is fully faithful.

so they are equivalences.

Classical BGr vs DG BGr

Recall: Let A be a graded alg.

$$\text{Tot}: C(A\text{-gr}) \longrightarrow C(A)$$

$$M: \dots \rightarrow M^d \xrightarrow{d} N^{d+1} \dots \longrightarrow \text{Tot}(M)$$

$$\text{Tot}(M)^1 = \bigoplus_{i+j=n} M^i_j$$

$$\forall m \in M^i_j. \quad \partial_{\text{Tot}(M)}(m) = (-)^j d(m) \quad \text{Tot}(M) \text{ is a } \Delta\text{-Gr module}$$

$$a \circ m = a \cdot m$$

It is clear that Tot is an exact functor.

Tot preserves $[1]$, cone and homotopy maps. so \exists

$$\Delta\text{-functor} \quad \text{Tot}: K(A\text{-gr}) \rightarrow K(A)$$

$$\text{Tot}: D(A\text{-gr}) \longrightarrow D(A)$$

If A is noetherian, then $\text{Tot}: D^f(A\text{-gr}) \longrightarrow D^f(A)$.

Lemma: there exists a diagram :

$$\begin{array}{ccc} D^f(A^!\text{-gr}) & \xrightarrow{\cong} & D^f(A\text{-gr}) \\ \downarrow c & & \downarrow \text{Tot} \\ D^f(S\text{-gr}) & & \\ \downarrow \text{Tot} & & \\ D^f(S) & \xrightarrow[\cong]{j_S^*} & D^f(N) \end{array}$$

$$\begin{aligned} \text{where } l: A^!\text{-gr} &\rightarrow S\text{-gr} \\ \bigoplus_{i \in \mathbb{Z}} M_i &\mapsto \bigoplus_{i \in \mathbb{Z}} N_i \\ N_{2i} = N_i, N_{2i+1} = & \end{aligned}$$

$$x_i \cdot n = b_i \cdot m.$$

Def: for $M \in \Lambda^k$ -gr

$$M \xrightarrow{\quad} \cdots \xrightarrow{\quad} \Lambda^k \otimes M_j \xrightarrow{d} \Lambda^k \otimes M_{j+1} \xrightarrow{\quad} \cdots$$

\downarrow

N

\downarrow

$N_{2j} = M_0$

$N_{2l+1} = 0$

\downarrow

N

$$\xrightarrow{\quad} \mathcal{J}_S^{\otimes N} \cong \Lambda_k^{\otimes N}.$$

$f \otimes m : \Lambda^k \otimes M_j \hookrightarrow \text{Tot}(\mathcal{J}_S^{\otimes N})^{l+i+j}$

$\partial(f \otimes m) = (-1)^{i+l} d(f \otimes m) = \sum \xi_i f \otimes \text{dim}_i m.$

$(\Lambda_{l-k}^{\otimes l} \otimes M_j) \subseteq (\Lambda^k \otimes N)^{\otimes j}$

differential $f \otimes m \in (\Lambda^k \otimes M_j)$

$\partial(f \otimes m) = \sum \xi_i f \otimes \text{dim}_i m$

$$M^i \xrightarrow{d} M^{i+1} \xrightarrow{\text{Tot}} \text{Tot} \left(\begin{array}{c} \Lambda^k \otimes M_j^i \xrightarrow{\quad} \Lambda^k \otimes M_{j+1}^{i+1} \xrightarrow{\quad} \cdots \\ \downarrow d \\ (-1)^i \leftarrow \Lambda^k \otimes M_j^{i+1} \xrightarrow{\quad} \Lambda^k \otimes M_{j+1}^{i+2} \xrightarrow{\quad} \cdots \end{array} \right)$$

\downarrow

N

\downarrow

$\text{Tot}(N)$

$m \in M_j^i$

$\partial(m) = (-1)^{i+l} d_{\text{Tot}}(f \otimes m)$

$= (-1)^{i+l} \left((-1)^{i+l} \cancel{\sum \xi_i f \otimes \text{dim}_i m} + (-1)^i f \otimes dm \right)$

$f \otimes m \in (\Lambda_{l-k}^{\otimes l} \otimes M_j^i) \subseteq (\mathcal{J}_S^{\otimes N})^{l+i+2j}$

$\partial(f \otimes m) = \sum \xi_i f \otimes \text{dim}_i m + (-1)^l f \otimes dm$

□

References

- [Eisenbud-Goto] J. Alg. 1984.
Linear free resolutions and minimal multiplicity.
- [Eisenbud-Fløystad-Schreyer] TRANS. AMS. 2003
Sheaf cohomology and free resolutions over exterior algs.
- [Beilinson] 1978
Coherent sheaves on \mathbb{P}^n and problems of linear alg.
- [Bernstein-Gelfand-Gelfand] 1978
Algebraic bundles over \mathbb{P}^n and problems of linear alg.
- [Araman-Buchweitz-Iyengar-Miller] Adv. Math. 2010.
Homology of perfect complexes.
- [Benson-Iyengar-Krause] Ann. Math. 2011.
Stratifying modular representations of finite groups