IML S2017: MANDATORY ASSIGNMENT 2

This is the second mandatory assignment. You must *hand in your solution on Tuesday, May 30, 2017, for marking, and you* must *have your solution to this assignment (and the previous) approved in order to take the final exam.*

Problem 1 (5 points)

Write a formula $\varphi(z, x, y)$ in the Language Of Set Theory (LOST) expressing the fact that $z = (x, y)$ where (x, y), i.e., the ordered pair, is considered to be an abbreviation for $\{\{x\}, \{x, y\}\}$.

Hint: (1) It pays off, big time, to be systematic. (2) You could consider starting by finding a formula $\psi(w, x, y)$ that expresses that $w = \{x, y\}$ (the unordered pair).

Problem 2 (5 points)

For this problem, we will need the following definitions: Recall that a sentence is just a wff without free variables.

Definition 1. Fix a language $\mathcal{L}$.

(1) Two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are called *elementarily equivalent*, written $\mathfrak{A} \equiv \mathfrak{B}$ if any only if they satisfy the same sentences (of $\mathcal{L}$).
(2) We write $\operatorname{Th}\mathfrak{A}$ for the set of sentences (of $\mathcal{L}$) true in $\mathfrak{A}$, also called the *theory of* $\mathfrak{A}$.
(3) We say a set of Γ of sentences (in $\mathcal{L}$) is *satisfiable* if and only if Γ has a model, that is, if there is an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\Gamma \subseteq \operatorname{Th}\mathfrak{A}$.

For this problem, use the following theorem (we will prove it in the lecture next week):

Theorem 1 (Compactness)**.** *Let Γ be a set of sentences (in a language $\mathcal{L}$). Then Γ is satisfiable if and only if every finite subset of Γ is satisfiable.*

Let $\mathcal{L}$ be the language of elementary number theory (see p. 70 in Enderton), but without the symbol $\mathbf{E}$. Let $\mathfrak{N}$ be the 'obvious' (or *intended*) $\mathcal{L}$-structure described in Enderton, p. 91.

(1) Let $\mathcal{L}_c$ be $\mathcal{L}$ with a new constant symbol c added. Show that $\Gamma = \operatorname{Th}\mathfrak{N} \cup \{\theta_n \colon n \in \mathbb{N}\}$ is satisfiable, where θ_n is the sentence $\underbrace{\mathbf{S}\ldots\mathbf{S}}_{n \text{ times}}\mathbf{0} < c$.
(2) Show that there is a structure $\mathfrak{N}'$ which is elementarily equivalent to $\mathfrak{N}$ but not isomorphic to $\mathfrak{N}$. Such a structure is called a *non-standard model of arithmetic.*[1]
(3) For $\mathfrak{N}'$ as above, show that there is an injective homomorphism (also called an *embedding*) of $\mathfrak{N}$ into $\mathfrak{N}'$. The range of this homomorphism, which is obviously isomorphic to $\mathfrak{N}$, is called the *standard part of* $\mathfrak{N}'$ and each element of $|\mathfrak{N}'|$ not in the standard part is called a *non-standard number.* Give an example for a non-standard number in $\mathfrak{N}'$.

[1]Hint: First find a $\mathcal{L}_c$-structure; then find an $\mathcal{L}$-structure with the desired properties.

As you might be interested in hearing, it can be shown that $+^{\mathfrak{N}'}$, $\times^{\mathfrak{N}'}$, $\mathbf{E}^{\mathfrak{N}}$ must be rather complicated on nonstandard numbers. But this is not something we will show in this course.

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