THE $\mathcal{G}_0$ DICHOTOMY (APRÈS MILLER & HAGA)

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Abstract. This note presents a proof of the $\mathcal{G}_0$ *dichotomy theorem* by Kechris, Solecki and Todorčević (1996; published 1999 in Advances in Mathematics). The proof we give is a reorganized version of a proof due to Ben Miller (circa 2007). The reorganized proof, with its emphasis on trees and its division along the wellfoundedness/illfoundedness line, was worked out in detail by Karen Bakke Haga. We use the $\mathcal{G}_0$ dichotomy to derive *Silver's dichotomy theorem* (1980), a famous result about coanalytic equivalence relations, as a corollary.

1. Introduction

Recall that a *binary relation* on a set X is a subset of $X \times X$. A binary relation is called a *graph* if it is symmetric and irreflexive. An *equivalence relation* is a binary relation which is reflexive, symmetric and transitive. If X is a Polish space and $R \subseteq X \times X$ is a binary relation, then we call R open, closed, Borel, analytic, co-analytic[1], etc., if it is open, closed, Borel, analytic, co-analytic, etc., in the product topology on $X \times X$.

The following is a famous theorem in descriptive set theory:

Theorem 1 (Silver's dichotomy theorem, 1980)**.** *Let E be a co-analytic equivalence relation on a Polish space X. Then exactly one of the following hold:*

(I) *E has countably many equivalence classes.*
(II) *There is a Cantor set $C \subseteq X$ which meets each E equivalence class in at most one point.*

Alternative (II) above is usually described by saying "E has perfectly many classes". Silver's theorem shows that in a very strong way, the cardinality of the set of equivalence classes of a co-analytic equivalence relation is unaffected by the status of the Continuum Hypothesis: The cardinality is either at most $\aleph_0$, or it is $2^{\aleph_0}$.

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[1] A set is co-analytic if it is the complement of an analytic set.

The original proof of Silver's theorem was very long and technical, and sometimes people say that it used "all the set-theoretic technology known at the time". (This is an overstatement, but it gives you an idea of how difficult it was.) Just a few years later, Leo Harrington (a colleague of Jack Silver at Berkeley) gave a much easier and shorter proof. Harrington's proof, which can be found in [1], is only about three pages in total, but it relies on the machinery of *effective descriptives set theory*, an elaborate refinement of classical descriptive which would need its own course to be developed.

Around 2007-2008, Ben Miller found a classical proof of Silver's theorem, that is, a proof relying only on classical descriptive set theory. This proof comes about by first giving a classical proof of *another* dichotomy theorem, due to Kechris, Solecki and Todorčević. This theorem is known as the $\mathcal{G}_0$ *dichotomy theorem*, or sometimes simply as "KST".

2. The $\mathcal{G}_0$ dichotomy theorem

KST is a theorem about analytic graphs on Polish spaces (as opposed to Silver's theorem, which is a theorem about co-analytic equivalence relations). To state it, we need to define a few graph-theoretic notions.

Definition 1 (*Discrete sets, colourings, homomorphisms*)**.** Let $\mathcal{G}$ be a graph on a set X.

1. A set $A \subseteq X$ is $\mathcal{G}$*-discrete* if
$$(\forall x, x' \in A)\ x \not\mathcal{G}\ x'.$$
2. An ω*-colouring* of $\mathcal{G}$ is a function $c : X \to \omega$ such that
$$(\forall x, x' \in X)\ x\ \mathcal{G}\ x' \implies c(x) \neq c(x').$$
3. If $\mathcal{H}$ is a graph on a set Y, then a *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a map $h : X \to Y$ such that
$$(\forall x, x' \in X)\ x\ \mathcal{G}\ x' \implies h(x)\ \mathcal{H}\ h(x').$$

Notice that if c is an ω-colouring of $\mathcal{G}$ then $c^{-1}(\{i\})$ is a $\mathcal{G}$-discrete set for every $i \in \omega$.

Exercise 1. (A) Find a Borel graph on 2^ω which admits no ω-colouring. (This is easier than you think.)

(B) Let X be a set and $\mathcal{G}$ a graph on X. A *path* from $x \in X$ to $y \in X$ in $\mathcal{G}$ is a sequence $x_0, \dots, x_n$ such that $x_i\, \mathcal{G}\, x_{i+1}$ for all $i < n$, and $x = x_0$ and $y = x_n$. The *connected component* of $x \in X$ is the set
$$\{y \in X : \text{there is a path from } x \text{ to } y\}.$$
Show that if $\mathcal{G}$ is a graph with all components countable, then $\mathcal{G}$ has an ω-colouring.

(C) Find a Borel graph $\mathcal{G}$ on 2^ω such that each connected component of the graph is countable, but $\mathcal{G}$ admits no **Baire measurable** ω-colouring. (This is a bit harder (maybe a lot harder) than (A), but $\mathcal{G}_0$ introduced below is an example, as Corollary 1 shows. But before you look at the definition of $\mathcal{G}_0$ below, try to cook up your own example. Attempting to do so, even if you fail, will help you understand how Kechris, Solecki and Todorčević arrived at the definition of $\mathcal{G}_0$.)

If $\mathcal{G}$ is a graph, then we say that x and y are *adjacent* if $x \,\mathcal{G}\, y$.

Lemma 1. *Let $\mathcal{G}$ be an analytic graph on a Polish space X, and suppose $A \subseteq X$ is analytic and $\mathcal{G}$-discrete. Then there is a Borel set $B \supseteq A$ which is $\mathcal{G}$-discrete.*

Proof. Let

$$A' = \{x \in X : (\exists y \in A)\ x \,\mathcal{G}\, y\}.$$

Then A' is analytic and $A' \cap A = \emptyset$. By Lusin's separation theorem (14.7 in Kechris' book, [2]) there is a Borel set $B_0 \supseteq A$ such that $B_0 \cap A' = \emptyset$. No point in B_0 is adjacent to anything in A, but even so, B_0 may not be $\mathcal{G}$-discrete. To remedy this, let

$$A'' = \{x \in X : (\exists y \in B_0)\ x \,\mathcal{G}\, y\}.$$

Then $A'' \cap A = \emptyset$ since no point in A is adjacent to anything in B_0. So we may apply Lusin separation again to get a Borel set $B_1 \supseteq A$ such that $B_1 \cap A'' = \emptyset$. Let $B = B_0 \cap B_1$. $\square$

Exercise 2. Show that A' and A'' in the proof of the previous lemma are analytic sets. (Hint: Projection maps are continuous.)

Sequences and notation. Recall that $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ is the set of all finite binary sequences (i.e., functions $s : n \to 2$ for some n). More generally, we define $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$, the set of all finite sequences of natural numbers (so clearly $2^{<\omega} \subseteq \omega^{<\omega}$). If $s \in \omega^{<\omega}$ and $t \in \omega^{<\omega} \cup \omega^\omega$, then we write $s \subseteq t$ if t extends the finite sequence s, we write $t \restriction n$ for the restriction of t to the first n entries (where $n \in \omega$), and we write $\mathrm{lh}(t)$ for the *length* of the sequence t. We also write $s^\frown t$ for the sequence where t is appended to the end of the (finite) sequence[2] s.

The following exercise should help you get comfortable with the above notation.

Exercise 3. With notation as in the previous paragraph, explain why (1) $\mathrm{lh}(s) = \mathrm{dom}(s)$, (2) if $n \geq \mathrm{lh}(t)$ then $t \restriction n = t$, and (3) $\mathrm{lh}(s^\frown t) = \mathrm{lh}(s) + \mathrm{lh}(t)$.

[2]As is usual throughout mathematics, we identify the $i \in \omega$ with the one element sequence (i), so that $s^\frown i$ means the same as $s^\frown(i)$.

(The latter even works when t is infinite if we use ordinal addition; if you don't know ordinal addition, just assume t is a finite sequence.)

Definition 2. A set $I \subseteq 2^{<\omega}$ is called *dense* if for every $s \in 2^{<\omega}$ there is $t \in I$ such that $s \subseteq t$. In other words, I is dense if every $s \in 2^{<\omega}$ has an extension which is in I.

Exercise 4. There is a sequence $(s_i)_{i\in\omega}$ such that $s_i \in 2^i$ for all $i \in \omega$, and $I_0 = \{s_i : i \in \omega\}$ is dense in $2^{<\omega}$.

We now fix once and for all some sequence $(s_i)_{i\in\omega}$ as in the previous exercise. (The specific choice is unimportant as long as it is fixed.)

Definition 3. The graph $\mathcal{G}_0$ on 2^ω is defined by

$$x \; \mathcal{G}_0 \; x' \iff (\exists i \in \omega)(\exists j \in 2) \; s_i{}^\frown j \subseteq x \wedge s_i{}^\frown(1-j) \subseteq x' \wedge (\forall n > i) \; x_n = x'_n.$$

In other words, $\mathcal{G}_0$ consists of all the pairs $(s_i{}^\frown j^\frown z, s_i{}^\frown(1-j)^\frown z)$, where $i \in \omega$, $j \in \{0,1\}$ and $z \in 2^\omega$.

Before stating the KST theorem, we motivate the definition of $\mathcal{G}_0$ by observing two basic facts about $\mathcal{G}_0$. That the following lemma and corollary motivate the definition of $\mathcal{G}_0$ should be further amplified if the reader keeps (C) of Exercise 1 in mind.

Lemma 2. *If $A \subseteq 2^\omega$ is $\mathcal{G}_0$-discrete and has the Baire Property (BP), then A is meagre.*

Proof. Suppose not. Then there is $s \in 2^{<\omega}$ such that A is comeagre in the basic open set $N_s = \{x \in 2^\omega : s \subseteq x\}$. Since $\{s_i : i \in \omega\}$ is dense in $2^{<\omega}$, we can find $s_i \supseteq s$. Then A is comeagre in $N_{s_i{}^\frown 0}$ and $N_{s_i{}^\frown 1}$, and so the sets

$$C_j = \{x \in 2^\omega : s_i{}^\frown j^\frown x \in A\},$$

$j \in \{0,1\}$, are comeagre. Taking $z \in C_0 \cap C_1$ we get that $s_i{}^\frown 0^\frown z, s_i{}^\frown 1^\frown z \in A$, contradicting that A is $\mathcal{G}_0$-discrete. □

Corollary 1. *$\mathcal{G}_0$ does not admit a Baire measurable ω-colouring.*

Proof. Suppose $c : 2^\omega \to \omega$ were a Baire measurable ω-colouring. Since $2^\omega = \bigcup_{i\in\omega} c^{-1}(\{i\})$, there would be $i \in \omega$ such that $c^{-1}(\{i\})$ is non-meagre. But $c^{-1}(\{i\})$, which is $\mathcal{G}_0$-discrete, has the BP when c is Baire measurable, contradicting the previous lemma. □

The reader can now easily verify that $\mathcal{G}_0$ is a solution to (C) of Exercise 1. (If you came up with something like $\mathcal{G}_0$ by yourself to solve Exercise 1(C), then good on you!)

We now state the $\mathcal{G}_0$ dichotomy theorem. It says that for analytic graphs, $\mathcal{G}_0$ stands as *the* single obstruction to having a Borel ω-colouring. Quite remarkable!

Theorem 2 (Kechris-Solecki-Todorčević, 1996.)**.** *Let $\mathcal{G}$ be an analytic graph on a Polish space X. Then exactly one of the following hold:*

(I) *$\mathcal{G}$ admits a Borel ω-colouring.*

(II) *There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

In the proof we will present, which was worked out by Karen Bakke Haga on the basis of Ben Miller's proof, we will assume that $X = \omega^\omega$. This has the advantage that the combinatorics of the proof can be organized using trees, and in particular, we can exploit that analytic sets in ω^ω and $\omega^\omega \times \omega^\omega$ can be represented using trees. Once the theorem is established in the case $X = \omega^\omega$, a standard trick (see Exercise 7) can be used to extend it to all Polish X.

Definition 4. 1. A *tree* on $\omega^k = \omega \times \cdots \times \omega$ (k factors) is a set

$$T \subseteq \bigcup_{n \in \omega} \omega^n \times \cdots \times \omega^n$$

(k factors) such that for all $(t^0, \ldots, t^{k-1}) \in T$ and $m \in \omega$ we have

$$(t^0 \restriction m, \ldots, t^{k-1} \restriction m) \in T.$$

(That is, a tree on ω^k is a set of k-tuples of sequences of consistent length, which is closed under initial segments.)

2. If T is a tree on ω^k then we define

$$[T] = \{(x^0, \ldots, x^{k-1}) \in (\omega^\omega)^k : (\forall n \in \omega)(x^0 \restriction n, \ldots, x^{k-1} \restriction n) \in T\}.$$

In other words, $[T]$ is the set of *infinite branches* through T.

Exercise 5. (a) Show that $F \subseteq (\omega^\omega)^k$ is closed if and only if $F = [T]$ for some tree T on ω^k. Moreover, we can ask that T be *pruned* which means that if $t \in T$, there is $x \in [T]$ such that $t \subseteq x$.

(b) Let $p : (\omega^\omega)^{k+1} \to (\omega^\omega)^k : (x^0, \ldots, x^k) \mapsto (x^0, \ldots, x^{k-1})$ be the projection onto the first k coordinates. Prove that every analytic set on $(\omega^\omega)^k$ has the form[3] $p[T]$ for some tree T on $\omega \times \cdots \times \omega$ ($k+1$ factors).

Illfounded, wellfounded, recursively defined things. For our proof of KST we need the notion of a wellfounded (and illfounded) relation, and recursion on such a relation.

If X is a set and $\prec$ is a strict (i.e., irreflexive) relation on X, we say that $\prec$ is *wellfounded* if every non-empty set $A \subseteq X$ has a $\prec$-minimal element, i.e., an element $a \in A$ such that $(\forall x \in A)\ x \not\prec a$. (The minimal element needn't be unique in general.) Equivalently, a strict relation $\prec$ is

[3] N.b.: We usually write $p[T]$ for $p([T])$ in a vain effort to keep the number of parenthesis in check.

wellfounded if there are no infinite $\prec$-descending sequences, i.e., no infinite sequence $x_0, x_1, \ldots \in X$ such that

$$x_0 \succ x_1 \succ \cdots .$$

A relation which is not wellfounded is called *illfounded.*

It turns out that we can do recursion (and induction) on wellfounded relations in much the same way we can on ω (with the usual ordering), or on an ordinal. The reader who is unfamiliar with this method can either take it on faith that this does work as intended (it does), or can consult e.g. [3] for a more thorough treatment of the topic.

Exercise 6. Let T be a tree on ω, and define on T the relation $s \prec t$ iff $t \subseteq s \wedge s \neq t$ (the s and the t being switched around is *not* a typo!) Prove that $\prec$ is illfounded if and only if T has an infinite branch (i.e., $[T] \neq \emptyset$).

Proof of the $\mathcal{G}_0$ dichotomy (*Theorem 2*). Corollary 1 shows that (I) and (II) can't both hold at the same time (since if they did, we could compose the homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$ with the ω-colouring and get a Borel ω-colouring of $\mathcal{G}_0$), so our task is only to show that (I) and (II) can't both *fail* at the same time either. We assume that $X = \omega^\omega$, which is justified by Exercise 7 below.

So assume that $\mathcal{G}$ is an analytic graph on ω^ω. Fix a pruned tree T on $\omega \times \omega \times \omega$ such that $p[T] = \mathcal{G}$. The *domain* of $\mathcal{G}$,

$$\mathrm{dom}(\mathcal{G}) = p(\mathcal{G}) = \{x \in \omega^\omega : (\exists y)\ x \,\mathcal{G}\, y\},$$

is analytic (see Exercise 8 below), and we fix a tree S on $\omega \times \omega$ such that $p[S] = \mathrm{dom}(\mathcal{G})$.

Define $\mathbb{P}$ to be the set of all pairs $p = ((t_u)_{u \in 2^n}, (g_{v,k})_{v \in 2^{n-k-1}, k<n})$ such that

(1) $t_u \in S$ and $g_{v,k} \in \omega^{<\omega}$;
(2) $\mathrm{lh}(t_u) = \mathrm{lh}(g_{v,k}) = n$;
(3) $(t^0_{s_k{}^\frown 0^\frown v}, t^0_{s_k{}^\frown 1^\frown v}, g_{v,k}) \in T$. **(???)**

(Notice that t_u, being in S, is a pair of sequences, $t_u = (t^0_u, t^1_u)$). We call n above the *height* of p, written $\mathrm{ht}(p)$.

Define an ordering $\prec$ on $\mathbb{P}$ by letting $p \prec q$, where

$$p = ((t^p_u)_{u \in 2^m}, (g^p_{v,k})_{v \in 2^{m-k-1}, k<m})$$

and

$$q = ((t^q_u)_{u \in 2^n}, (g^q_{v,k})_{v \in 2^{n-k-1}, k<n}),$$

just in case $m > n$, and

(a) $t^p_u \restriction n = t^q_{u \restriction n}$ for all $u \in 2^m$;
(b) $g^p_{v,k} \supseteq g^q_{v \restriction (n-k-1), k}$ for all $k < n$ and $v \in 2^{m-k-1}$.

The remainder of the proof goes as follows: Either $\prec$ is illfounded, and we get a homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$; or $\prec$ is wellfounded, and we build recursively a Borel ω-colouring of $\mathcal{G}$.

Case 1: $\prec$ is illfounded, in which case there is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

In this case, let $p_0 \succ p_1 \succ p_2 \succ \cdots$ be an infinite descending sequence such that $\mathrm{ht}(p_i) = i$. (a) above guarantees that $t^{p_n}_{x\restriction n} \subseteq t^{p_m}_{x\restriction m}$ for any $x \in 2^\omega$ and $m \geq n$, and so by (2) above a map $\varphi : 2^\omega \to \omega^\omega$ is defined by

$$\varphi(x) = \bigcup_{n\in\omega} (t^{p_n}_{x\restriction n})^0.$$

Similarly, (b) and (2) above guarantee that for each $k \in \omega$ a map $\gamma_k : 2^\omega \to \omega^\omega$ is defined by

$$\gamma_k(x) = \bigcup_{n>k} g^{p_n}_{x\restriction n-k-1,k}.$$

Now by (3) above we have

$$(\varphi(s_k{}^\frown i^\frown z), \varphi(s_k{}^\frown(1-i)^\frown z), \gamma_k(z)) \in [T]$$

for $z \in 2^\omega$, which shows that $\varphi(s_k{}^\frown i^\frown z)\, \mathcal{G}\, \varphi(s_k{}^\frown(1-i)^\frown z)$. Thus φ is a homomorphism, and is continuous by Exercise 9 below.

Case 2: $\prec$ is wellfounded, in which case there is a Borel ω-colouring of $\mathcal{G}$.

For each $p \in \mathbb{P}$, let E_p be the set of all pairs $((f_u)_{u\in 2^n}, (\gamma_{v,k})_{v\in 2^{n-k-1},k<n})$ such that

(1) $f_u \in \omega^\omega \times \omega^\omega$ and $\gamma_{v,k} \in \omega^\omega$;
(2) $t^p_u \subseteq f_u$ and $g_{v,k} \subseteq \gamma_{v,k}$.
(3) For $k < n$, $v \in 2^{n-k-1}$ and $i \in \{0, 1\}$,

$$(f^0_{s_k{}^\frown i^\frown v}, f^0_{s_k{}^\frown(1-i)^\frown v}, \gamma_{v,k}) \in [T].$$

It can be shown that E_p is a closed set in the appropriate product space, see Exercise 10 below. We let

$$D_p = \{x \in \omega^\omega : (\exists((f_u)_{u\in 2^n}, (\gamma_{v,k})_{v\in 2^{n-k-1},k<n}) \in E_p)\ x = f^0_{s_n}\},$$

which is then analytic (see Exercise 11 below).

Claim: For all $p \in \mathbb{P}$, the set $D_p \setminus \bigcup_{q\prec p} D_q$ is $\mathcal{G}$-discrete.

Proof of Claim. Suppose not, and let $x, x' \in D_p \setminus \bigcup_{q\prec p} D_q$ such that $x\,\mathcal{G}\,x'$. Let $((f_u)_{u\in 2^n}, (\gamma_{v,k})_{v\in 2^{n-k-1},k<n})$ and $((f'_u)_{u\in 2^n}, (\gamma'_{v,k})_{v\in 2^{n-k-1},k<n})$ witness that $x, x' \in D_p$. Let $\gamma \in \omega^\omega$ be such that $(x, x', \gamma) \in [T]$. Define $q \in \mathbb{P}$ with $\mathrm{ht}(q) = n + 1$ as follows:

(i) $t^q_{u^\frown 0} = f_u \restriction (n + 1)$;

(ii) $t^q_{u^\frown 1} = f'_u \restriction (n+1)$;
(iii) for $k < n$, $v \in 2^{n-k-1}$ and $i \in 2$, $g^q_{v^\frown 0,k} = \gamma_{v,k} \restriction (n+1)$ and $g^q_{v^\frown 1,k} = \gamma'_{v,k} \restriction (n+1)$;
(iv) $g^q_{\emptyset,n} = \gamma \restriction (n+1)$.

Then $q \in \mathbb{P}$, $q \prec p$, and $x, x' \in D_q$, contradicting the choice of x, x'. Claim.$\dashv$

One particular consequence of the previous claim is that if $p \in \mathbb{P}$ is terminal, i.e., if p is $\prec$ minimal and so has no extensions, then D_p is an analytic $\mathcal{G}$-discrete set. By Lemma 1 there is a $\mathcal{G}$-discrete Borel set $B_p \supseteq D_p$.

Now, for a possibly non-terminal p, suppose that for each $q \prec p$ we have already defined a $\mathcal{G}$-discrete Borel set B_q such that $D_q \subseteq B_q \cup \bigcup_{r \prec q} B_r$. Then
$$D_p \setminus \bigcup_{q \prec p} B_q \subseteq D_p \setminus \bigcup_{q \prec p} D_q,$$
so, using the claim above, $D_p \setminus \bigcup_{q \prec p} B_q$ is $\mathcal{G}$-discrete. Moreover, $D_p \setminus \bigcup_{q \prec p} B_q$ is analytic (since we are removing a countable union of Borel sets from D_p). By Lemma 1 we can then find $B_p \supseteq D_p \setminus \bigcup_{q \prec p} B_q$ which is $\mathcal{G}$-discrete.

In other words, we have recursively defined $\mathcal{G}$-discrete Borel sets B_p for all $p \in \mathbb{P}$ such that $D_p \subseteq B_p \cup \bigcup_{q \prec p} B_q$. Since $D_\phi = \omega^\omega$, where ϕ denotes the maximal element of $\mathbb{P}$, this allows us to define a Borel ω-colouring as follows: Let $(p_i)_{i \in \omega}$ enumerate $\mathbb{P}$ (which is easily seen to be a countable set), and define $c : \omega^\omega \to \omega$ by $c(x) = i$ iff i is least such that $x \in B_{p_i}$. Then c is Borel (see Exercise 12 below) and is a colouring since $c^{-1}(\{i\}) \subseteq B_{p_i}$. $\square$

Exercise 7. Prove that KST holds for all Polish X (and not just ω^ω).

Hint: Use Theorem 7.9 in Kechris' book [2] to move the problem from X to ω^ω so that KST in ω^ω applies.

Exercise 8. Prove that $\mathrm{dom}(\mathcal{G})$ is analytic when $\mathcal{G}$ is an analytic graph on a Polish space.

Exercise 9. Show that the map φ defined in Case 1 in the proof of KST is continuous.

Exercise 10. Show that E_p, defined in Case 2 in the proof of KST, is closed. (First make sure you understand which product space E_p is a subset of. Hint: The space is a finite product of 2^n copies of $\omega^\omega \times \omega^\omega$ and for each $k < n$ a copy of $(\omega^\omega)^{2^{n-k-1}}$.)

Exercise 11. Use the previous exercise to show that D_p, as defined in Case 2 of the proof of KST, is analytic.

Exercise 12. Show that $c^{-1}(\{i\})$ is Borel for all i, where c is the colouring defined in case 2 of the proof of KST, and conclude that c is a Borel function. (Hint: Induction on i should do it.)

Exercise 13. Let $\mathcal{G}$ and $\mathcal{H}$ be graphs on sets X and Y, respectively. A *strong homomorphism* (or *reduction*) of $\mathcal{G}$ to $\mathcal{H}$ is a function $h : X \to Y$ such that

$$(\forall x, x' \in X)\ x\,\mathcal{G}\,x' \iff h(x)\ \mathcal{H}\ h(x').$$

(Notice the if and only if.) Show, by giving an example, that the $\mathcal{G}_0$ dichotomy is *false* if we replace "homomorphism" by "strong homomorphism".

Side remark: An injective strong homomorphism is usually called an *embedding.*

3. Silver's theorem and other applications

Our first application is a purely classical proof of Silver's theorem, as was promised at the beginning of this note:

Theorem 1 (Silver's dichotomy theorem, 1980)**.** *Let E be a co-analytic equivalence relation on a Polish space X. Then exactly one of the following hold:*

(I) *E has countably many equivalence classes.*
(II) *There is a Cantor set $C \subseteq X$ which meets each E equivalence class in at most one point.*

We shall give a proof using Mycielski's theorem, which you proved in your mandatory home-exercise:

Theorem 3 (Mycielski's Theorem)**.** *If E is an equivalence relation on a Polish space X and E is meager as a subset of X^2, then E has perfectly many equivalence classes, in the sense that there is a continuous injection $f\colon 2^\omega \to X$ such that $x \neq x' \Rightarrow f(x) \not{E}\, f(x')$ for any $x, x' \in 2^\omega$.*

Proof of Silver's dichotomy theorem. Define a graph on X by $x\,\mathcal{G}\,x'$ iff $x \not{E} x'$. If $c\colon X \to \omega$ is a Borel coloring of $\mathcal{G}$, then cearly

$$c'([x]_E) = n \iff c(x) = \min\{k \mid (\exists x' \in [x]_e) c(x') = k\}$$

gives a well-defined function from $\{[x]_E \mid x \in X\}$ to ω, which is injective because E is transitive. Thus E has only countably many equivalence classes.

By theorem 2, it suffices to show that there can be no continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$; so towards a contradiction, assume $\varphi\colon 2^\omega \to X$ is such a homomorphism.

Define an equivalence relation on 2^ω by $x\ F\ x' \iff \varphi(x)\ E\ \varphi(x')$, i.e. $F = (\varphi \times \varphi)^{-1}(E)$. Note that F is co-analytic, for each $x \in 2^\omega$, $F|_x = [x]_F$ and that this set is $\Pi^1_1(2^\omega)$ (since taking the intersection with the closed set $\{x\} \times 2^\omega$ preserves being co-analytic) and hence has the Baire property.

Assume that there is some $x \in 2^\omega$ such that $[x]_F$ is not meager in X. Then we can find a nonempty basic open N_s such that $N_s \subseteq^* [x]_F$. As in lemma **??**, we can find $x, x' \in N_s$ such that $x \mathcal{G}_0 x'$, and thus we must have $\varphi(x) \not\!E\, \varphi(x')$, contradicting that $x\, F\, x'$.

But since every vertical section $F|_x = [x]_F$ is meager, by Kuratowski-Ulam (see [2, 8.41]), F is meager as a subset of $(2^\omega)^2$. Apply Mycielski's theorem for F, obtaining $f\colon 2^\omega \to 2^\omega$ s.t. $x = x' \iff f(x)\, F\, f(x')$; let $C = (\varphi \circ f)(2^\omega)$. Note that for $x, x' \in 2^\omega$, $x = x' \iff f(x)\, F\, f(x') \iff x\, E\, x'$; in particular $\varphi \circ f$ is injective and hence C is a Cantor set (a homeomorphic image of 2^ω). □

Note that the proof uses only classical methods, i.e. we don't have to use effective descriptive set theory. If you want, compare this to the proof given in [1, ?].

The second application is a uniformization theorem. There are many uniformization theorems, two of which I will discuss below (one without proof, as an example to help you get better acquainted with this type of theorem).

Let X, Y be sets and $A \subseteq X \times Y$ a binary relation. We say that a relation F *uniformizes* A if and only if for any $x \in X$,

$$\big[(\exists y \in Y)(x, y) \in A\big] \iff \big[(\exists! y \in Y)(x, y) \in F\big].$$

In other words, we can view F as a function $F\colon \operatorname{dom}(A) \to Y$, where $\operatorname{dom}(A)$ denotes the set of all $x \in X$ such that the vertical section $A|_x = \{y \in Y \mid (x, y) \in A\}$ is non-empty; and for each x in that set, $F(x) \in A|_x$ (See also [2, 18.A], there's a very informative picture).

Since this is no different from saying that F is a *choice function* for the family

$$\prod_{x \in \operatorname{dom}(A)} A|_x,$$

the axiom of choice tells us that every relation $A \subseteq X \times Y$ can be uniformized by *some* F; the question is more interesting when we require F to be 'nice' in some way, or 'effective' in some way. Uniformization theorems tell us that we can obtain such a 'nice' F, if A itself isn't 'too awful':

Theorem 4 (Novikov-Kondô)**.** *If X, Y are Polish spaces and $A \subseteq X \times Y$ is co-analytic, there is $F \subseteq X \times Y$ co-analytic which uniformizes A.*

The proof of this theorem is beyond the scope of our course (see e.g. [2, 36.12]).

While Novikov-Kondô uniformization (also called Π^1_1-uniformization) is powerful and easy to state, in many applications we want F to be of lower complexity than Π^1_1. Unfortunately, Borel relations can *not* in general be uniformed by Borel relations; but there are cases in which A *can* be uniformized by a Borel relation. You may want to take a look at chapter 18 in [2], from which we prove the following uniformization theorem.

Theorem 5 (Lusin-Novikov)**.** *Let X, Y be Polish (or just standard Borel) and let $A \subseteq X \times Y$ be Borel. If every section $A|_x$ is countable, then A has a Borel uniformization and* $\mathrm{proj}_X(A)$ *is Borel. Moreover, A can be written as $\bigcup_{i\in\omega} f_i$, where each f_i is Borel and the graph of a function $f_i\colon \mathrm{dom}(f_i) \to Y$ (i.e. for $x \in X$, there is* at most one *$y \in Y$ such that $(x, y) \in f_i$).*

Our proof (using the $\mathcal{G}_0$-dichotomy) is much less involved than the one in Kechris (see[2, 18.10]).

We need the equivalence relation E_0 on 2^ω, given by xE_0y precisely if x and y are different only at finitely many places, i.e.

$$(\exists n)(\forall m \geq n)\; x(m) = y(m).$$

The proof we are about to give uses the following exercise:

Exercise 14. For any $x, y \in 2^\omega$, xE_0y if and only if there is a finite sequence $x_0, \ldots, x_n$ of elements of 2^ω such that $x_0 = x$, $x_n = y$, and for each $k < n$, $x_k\; \mathcal{G}_0\; x_{k+1}$. In other words, E_0 is the unique equivalence relation whose classes are exactly the connected components of $\mathcal{G}_0$ (the 'transitivization' of $\mathcal{G}_0$).

We also need the following lemma:

Lemma 3. *Any function continuous function $f\colon 2^\omega \to X$, where X is Polish, such that $xE_0y \Rightarrow f(x) = f(y)$ must be constant. (In fact it suffices that X is T_0, i.e. has a basis which separates points.)*

Proof. Let $x \in \mathrm{ran}(f)$ be arbitrary and O be an open neighborhood of x. Then $A = f^{-1}(O)$ is open; also A is E_0 *invariant*, meaning if $z \in A$ and zE_0z' then $z' \in A$. But then $A = 2^\omega$, for A must contain some basic open neighborhood N_s, where $s \in 2^{<\omega}$; and for every $z \in 2^\omega$ we can find $z' \in N_s$ such that zE_0z'. Since O was arbitrary, $\mathrm{ran}(f) = \{x\}$. □

Now we are ready to proof Lusin-Novikov uniformisation.

Proof. Below, we shall prove the 'moreover' statement, from which the theorem follows: for each $i \in \omega$, $\mathrm{dom}(f_i) = \mathrm{proj}_X(f_i)$ is the injective image of a Borel set and hence itself Borel, by theorem 15.1 in [2]. Thus, we can let

$$f'_i = f_i \setminus \Big[\bigcup_{k<i} \big(\mathrm{dom}(f_k) \times Y\big)\Big];$$

clearly, the Borel relation $F = \bigcup_{i\in\omega} f_i'$ is a function when restricted to its domain, which equals $\mathrm{dom}(A)$, and therefore F uniformizes A. Again using [2, 15.1], $\mathrm{proj}_X(A)$ is Borel.

It remains to prove the 'moreover' statement. Define a graph $\mathcal{G}$ on $X \times Y$ by $(x, y)\,\mathcal{G}\,(x', y')$ iff $x = x'$, $y \neq y'$ and both $(x, y), (x', y') \in A$. If this graph admits a Borel ω-coloring $c\colon X \times Y \to \omega$, let $f_i = c^{-1}(\{i\}) \cap A$ for $i \in \omega$. Clearly, each[4] f_i is Borel and if $(x, y), (x', y') \in f_i$ and $(x, y) \neq (x', y')$, then $x \neq x'$, so f_i can be seen as (the graph of) a function $f_i\colon \mathrm{dom}(f_i) \to Y$.

By theorem 2, it thus suffices to show that there can be no continuous homomorhism from $\mathcal{G}_0$ to $\mathcal{G}$. Towards a contradiction, assume $\varphi\colon 2^\omega \to X \times Y$ is such a homomorphism. Note that we have $xE_0y \Rightarrow \varphi(x)\mathcal{G}\varphi(y)$; this is by exercise 14. In particular, $xE_0y \Rightarrow \varphi_X(x) = \varphi_X(y)$ By lemma 3, φ_X is constantly equal to some $x_0 \in X$.

Note that like every vertical section of $\mathcal{G}$, $\mathcal{G}|_{x_0} = \mathcal{G} \cap \{x_0\} \times Y$ *does* admit a Borel ω-coloring: let $\{a_k\colon k \in \omega\}$ be an injective enumeration of $A|_{x_0}$ and define a Borel c coloring of $\mathcal{G}|_{x_0}$ by $h(y) = 0$ for $y \notin A|_{x_0}$, and otherwise $h(y) = k + 1$ for the unique k such that $y = a_k$. But this is a contradiction, since $\varphi \circ h$ is a Borel ω-coloring of $\mathcal{G}_0$. □

In lecture, I started with an alternative proof of this theorem, not using the fact about E_0, which is much more complicated and shows much less (it doesn't allow us to see that φ_X is in fact constant). It may serve as an illustration of how the right tools make a proof much more informative and understandable. It does tell us, though, that an open graph either has a Borel ω-coloring or else contains a perfect complete set (or clique, i.e. a set of pairwise connected points). Compare this with the following exercise:

Exercise 15. Any closed graph has a Borel ω-coloring.

(Hint: cover the diagonal by basic open neighborhood U such that $U^2 \cap \mathcal{G} = \emptyset$).

Alternative ending of the proof. Start as in the previous proof of the 'moreover' statement. Again, we show that there can be no continuous homomorhism from $\mathcal{G}_0$ to $\mathcal{G}$, so towards a contradiction, assume $\varphi\colon 2^\omega \to X \times Y$ is such a homomorphism. This time we will find 'by hand' a Cantor set contained in a single vertical section of A.

For $r \in 2^\omega$, denote by $\varphi(r)_X$ and $\varphi(r)_Y$ the components of $\varphi(r) \in X \times Y$. We shall construct a map $j\colon 2^{<\omega} \to 2^{<\omega}$, defining $j(s)$ by induction on the length of $s \in 2^{<\omega}$, together with a family $\{z_s \mid s \in 2^{<\omega}\}$. Let $j(\emptyset) = \emptyset$. Fix

[4] Note some f_i might be empty; this doesn't seem to hurt. If you want you can throw the empty ones away and re-enumerate, since we're not claiming any 'effectiveness' of the enumeration.

some arbitrary $z_\emptyset \in 2^\omega$, with $s_0 \subseteq z_\emptyset$ for simplicity. For $i \in \{0,1\}$, let $z_{<i>}$ be such that $z_{<i>} \supseteq s_0{}^\frown i$ and $z_{<i>}$ agrees with $z_\emptyset$ for $k \geq \text{length}(s_0)+1$. Thus $z_{<0>} \,\mathcal{G}_0\, z_{<1>}$, so $\varphi(z_{<0>}), \varphi(z_{<1>}) \in A$, $\varphi(z_{<0>})_Y \neq \varphi(z_{<1>})$ and letting $x_0 = \varphi(z_\emptyset)$, $\varphi(z_{<0>})_X = \varphi(z_{<1>})_X = x_0$. We can find $t_0, t_1 \in 2^{<\omega}$ such that for $i \in \{0,1\}$, $s_0{}^\frown i \subseteq t_i \subseteq z_{<i>}$ and for $(r, r') \in N_{t_0} \times N_{t_1}$ we have $\varphi(r)_Y \neq \varphi(r')_Y$. Let $j(<0>) = t_0$ and $j(<1>) = t_1$.

For the general case of the induction, assume $j(s)$ has been defined, say $\text{length}(j(s)) = n$.

Write z^* for the sequence $k \mapsto z_\emptyset(n+k)$; note we can assume by induction that $\varphi(j(s){}^\frown z^*)_X = x_0$. Let $m > n$ be least satisfying $s_m \subseteq j(s){}^\frown z^*$ and proceed analogously to the first step: write z' for the sequence $k \mapsto z_\emptyset(m+1+k)$ and for $i \in \{0,1\}$, define $z_{s^\frown i}$ to be $s_m{}^\frown i^\frown z'$. Then we have $z_{s^\frown 0} \,\mathcal{G}_0\, z_{s^\frown 1}$, so again $\varphi(z_{s^\frown 0})_X = \varphi(z_{s^\frown 1})_X = x_0$, $\varphi(z_{s^\frown 0})_Y \neq \varphi(z_{s^\frown 1})_Y$ and $\varphi(z_{s^\frown 0}), \varphi(z_{s^\frown 0}) \in A$. Again, we can find $t_0, t_1 \in 2^{<\omega}$ such that for $i \in \{0,1\}$, $s_m{}^\frown i \subseteq t_i \subseteq z_{s^\frown i}$ and for all $(r, r') \in N_{t_0} \times N_{t_1}$ we have $\varphi(r)_Y \neq \varphi(r')_Y$. Again we let $j(s^\frown 0) = t_0$ and $j(s^\frown 1) = t_1$.

Finally, we let $j^*\colon 2^\omega \to X \times Y$ be given by

$$j^*(r) = \bigcup_{n \in \omega} j(r \restriction n).$$

Clearly j^* is continuous, whence for $h = \varphi \circ j^*$ we have that $h\colon 2^\omega \to X \times Y$ is continuous with respect to τ and hence also with respect to the product topology. It is clear by construction that j^* is injective: given $r_0 \neq r_1$, let s be their longest common initial segment. We made sure that for all $(r, r') \in N_{j(s^\frown 0)} \times N_{j(s^\frown 1)}$ we have $\varphi(r)_Y \neq \varphi(r')$; so in particular $\varphi(r_0)_Y \neq \varphi(r_1)_Y$.

We finish the proof by showing that $h(2^\omega) \subseteq A_{x_0}$. By construction, for each $n \in \omega$ we have $j^*(r), z_{r \restriction n} \in N_{j(r \restriction n)}$. Thus, $\lim_n z_{r \restriction n} = j^*(r)$ and as for each $n \in \omega$, $\varphi(z_{r \restriction n})_X \in A_{x_0}$ which is closed in τ while φ is continuous w.r.t. τ, we have $h(r) = \varphi(j^*(r)) \in A_{x_0}$. □

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