THE ART OF ALGORITHM ANALYSIS

FROM FOUNDATIONS TO PRACTICE

Mahdi

LIVING FIRST EDITION

Analy

ALGORITHMS • ABSTRACTION • ANALYSIS • ART

*"From ancient counting stones to quantum algorithms—
every data structure tells the story of human ingenuity."*

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Updated October 15, 2025

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THE ART OF ALGORITHMIC ANALYSIS: ALGORITHMIC COST ANALYSIS AND ASYMPTOTIC REASONING

A Living Architecture of Computing

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Preferred Citation Format:
Mahdi. (2025). *The Art of Algorithmic Analysis*. Retrieved from
https://github.com/m-mdy-m/algorithms-data-structures/tree/main/books/books

Version Control: This is a living document. Check the repository for the most current version and revision history.

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TECHNICAL SPECIFICATIONS

Typeset with: LaTeX using Charter and Palatino font families
Graphics: TikZ and custom illustrations
Standards: Follows academic publishing conventions
Encoding: UTF-8 with full Unicode support
Format: Available in PDF, and LaTeX source formats

License last updated: October 15, 2025

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Acknowledgments

I would like to express my gratitude to everyone who supported me during the creation of this book. Special thanks to the open-source community for their invaluable resources and to all those who reviewed early drafts and provided feedback.

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