## ART

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# ALGORITHM ANALYSIS

FROM FOUNDATIONS TO PRACTICE

 $\Theta(\log n)$ 

<sup>Ω(n²)</sup> Mahdi

LIVING FIRST EDITION

# Ahlaly

#### ALGORITHMS • ABSTRACTION • ANALYSIS • ART

"From ancient counting stones to quantum algorithms every data structure tells the story of human ingenuity."

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Part I

**Preface** 

#### **Purpose and Scope of This Book**

- 1.1 What This Book Covers
- 1.2 What This Book Does Not Cover
- 1.3 Target Audience: Students, Researchers, and Practitioners

## Why "Precise Analysis" Matters — From Theory to Engineering

- 2.1 The Gap Between Theoretical Complexity and Real-World Performance
- 2.2 Case Studies: When Big-O Isn't Enough
- 2.3 The Role of Constants, Lower-Order Terms, and Hardware

## Mathematical and Algorithmic Prerequisites

- 3.1 Discrete Mathematics
- 3.1.1 Sets, Functions, and Relations
- 3.1.2 Combinatorics: Permutations, Combinations, and Binomial Coefficients
- 3.1.3 Graph Theory Basics
- 3.1.4 Proof Techniques: Induction, Contradiction, and Contrapositive
- 3.2 Elementary Probability Theory
- 3.2.1 Sample Spaces, Events, and Probability Measures
- 3.2.2 Random Variables and Expectations
- 3.2.3 Basic Distributions: Uniform, Bernoulli, Geometric, Binomial
- 3.2.4 Linearity of Expectation
- 3.2.5 Conditional Probability and Independence
- 3.2.6 Variance and Standard Deviation
- 3.2.7 Moment Generating Functions (Brief Introduction)
- 3:3:00-20 Mathematical Analysis

## Structure of the Book: Theorems, Proofs, Examples, and Exercises

- 4.1 How to Read This Book
- 4.2 Notation and Conventions
- 4.3 Types of Exercises: Conceptual, Computational, and Proof-Based
- 4.4 Using Examples Effectively
- 4.5 The Role of Rigor vs. Intuition

## Primary References and Parallel Reading Guide

- 5.1 Classic Textbooks (CLRS, Sedgewick, Kleinberg-Tardos)
- 5.2 Research Papers and Monographs
- 5.3 Online Resources and Lecture Notes
- 5.4 Recommended Reading Order and Study Plans

#### Part II

Foundations of Algorithmic Analysis

#### Introduction to Algorithm Analysis

| 6.1   | What Is Algorithm Analysis?                          |
|-------|--|
| 6.1.1 | Correctness vs. Efficiency                           |
| 6.1.2 | Resource Measures: Time, Space, Energy, I/O          |
| 6.1.3 | The Need for Mathematical Models                     |
| 6.2   | The RAM Model of Computation                         |
| 6.2.1 | Basic Operations and Unit-Cost Assumption            |
| 6.2.2 | Memory Access Model                                  |
| 6.2.3 | Limitations and Extensions of the RAM Model          |
| 6.3   | Measuring Algorithm Performance                      |
| 6.3.1 | Input Size and Problem Instances                     |
| 6.3.2 | Counting Basic Operations                            |
| 6.3.3 | Exact vs. Asymptotic Analysis                        |
| 6.4   | Overview of Complexity Classes                       |
| 6.4.1 | P, NP, NP-Complete, and NP-Hard (Brief Introduction) |
| 6.4.2 | Why We Focus on Polynomial-Time Algorithms           |

#### **Asymptotic Notation**

| <b>7.1</b> | The Need | for Asym | ptotic Analysis |
|------------|----------|----------|-----------------|
|            |          |          |                 |

- 7.1.1 Why Exact Counts Are Often Impractical
- 7.1.2 Growth Rates and Scalability
- 7.2 Big-O Notation (O)
- 7.2.1 Formal Definition
- 7.2.2 Intuition: Upper Bounds
- 7.2.3 Common Functions and Their Growth Rates
- 7.2.4 Examples and Non-Examples
- 7.2.5 Properties of Big-O

**Transitivity** 

Addition and Multiplication Rules

Reflexivity and Asymmetry

#### 7.3 Big-Omega Notation ( $\Omega$ )

- 7.3.1 Formal Definition
- 7.3.2 Intuition: Lower Bounds
- 7.3.3 Examples and Applications
- 7.3.4 Relationship Between O and  $\Omega$

#### 7.4 Big-Theta Notation (⊕)

### **Recurrence Relations and Their Solutions**

| 8.1 Introduction to Recurrence Relation |
|---|
|---|

- 8.1.1 What Are Recurrences?
- 8.1.2 Why They Arise in Algorithm Analysis
- 8.1.3 Examples from Divide-and-Conquer Algorithms

#### 8.2 The Substitution Method

- 8.2.1 Guessing the Solution
- 8.2.2 Proving by Induction
- 8.2.3 Examples: Mergesort, Binary Search
- 8.2.4 Strengthening the Inductive Hypothesis

#### 8.3 The Recursion-Tree Method

- 8.3.1 Visualizing the Recurrence
- 8.3.2 Summing Over Levels
- 8.3.3 Examples and Illustrations
- 8.3.4 Limitations and When to Use

#### 8.4 The Master Theorem

8.4.1 Statement of the Master Theorem (Standard Form)

#### Best-Case, Worst-Case, and Average-Case Analysis

| 9.1 | <b>Defining</b> | Input | Classes |
|-----|-----------------|-------|---------|
|-----|-----------------|-------|---------|

- 9.1.1 What Constitutes an "Input"?
- 9.1.2 Problem Instances and Instance Distributions
- 9.2 Best-Case Analysis
- 9.2.1 Definition and Purpose
- 9.2.2 Examples: Insertion Sort, Linear Search
- 9.2.3 When Best-Case Matters (and When It Doesn't)
- 9.3 Worst-Case Analysis
- 9.3.1 Definition and Motivation
- 9.3.2 Guarantees and Robustness
- 9.3.3 Examples: Quicksort, Searching in Unsorted Arrays
- 9.3.4 Lower Bounds and Optimality
- 9.4 Average-Case Analysis
- 9.4.1 Definition: Expected Running Time
- 9.4.2 Assumptions About Input Distributions
- 9.4.3 Probabilistic Models: Uniform, Gaussian, etc.
- 7.4.5 I Tobabilistic Widacis. Offitolini, Gaussian, etc.

**Examples: Quicksort, Hashing, Skip Lists** 

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| 10.1.2            | Indicator Random Variables                     |               |
| 10.1.3            | Linearity of Expectation                       |               |
| 10.2              | <b>Expected Running Time</b>                   |               |
| 10.2.1            | Formal Definition                              |               |
| 10.2.2            | Computing Expectations via Indicator Variables |               |
| 10.2.3            | Examples: Hiring Problem, Randomized Quicksort |               |
| 10.3              | Probabilistic Bounds                           |               |
| 10.3.1            | Markov's Inequality                            |               |
| 10.3.2            | Chebyshev's Inequality                         |               |
| 10.3.3            | Chernoff Bounds                                |               |
| 10.3.4            | Applications to Load Balancing and Hashing     |               |
| 10.4              | Randomized Algorithms                          |               |
| 10.4.1            | Randomized Quicksort (Detailed Analysis)       |               |
| 10.4.2            | Randomized Selection (Quickselect)             |               |
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10.4.4 Bloom Filters and Probabilistic Data Structures

# Part III Advanced Analysis Techniques

#### **Amortized Analysis**

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| 11.1.1 | Motivation: Why Average Per-Operation Cost? |
| 11.1.2 | Amortized vs. Average-Case Analysis         |
| 11.1.3 | When to Use Amortized Analysis              |
| 11.2   | Aggregate Analysis                          |
| 11.2.1 | Definition and Methodology                  |
| 11.2.2 | Example: Dynamic Array (Vector) Resizing    |
| 11.2.3 | <b>Example: Binary Counter Increment</b>    |
| 11.2.4 | Example: Stack with Multipop                |
| 11.3   | The Accounting Method                       |
| 11.3.1 | Conceptual Framework: Credits and Debits    |
| 11.3.2 | <b>Defining Amortized Costs</b>             |
| 11.3.3 | Example: Dynamic Array via Accounting       |
| 11.3.4 | <b>Example: Splay Trees (Introduction)</b>  |
| 11.3.5 | <b>Ensuring Non-Negative Credit Balance</b> |
| 11.4   | The Potential Method                        |

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#### **Space Complexity Analysis**

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|---------------|---|
| 12.1.1        | Why Space Matters                               |
| 12.1.2        | Types of Memory: Stack, Heap, Static            |
| 12.1.3        | In-Place vs. Out-of-Place Algorithms            |
| 12.2          | Measuring Space Usage                           |
| 12.2.1        | Auxiliary Space vs. Total Space                 |
| 12.2.2        | Recursive Call Stack Depth                      |
| 12.2.3        | Implicit vs. Explicit Data Structures           |
| 12.3          | <b>Examples of Space Complexity Analysis</b>    |
| 12.3.1        | Iterative Algorithms: Loops and Arrays          |
| 12.3.2        | Recursive Algorithms: Mergesort, Quicksort      |
| 12.3.3        | Dynamic Programming: Memoization vs. Tabulation |
| 12.3.4        | Graph Algorithms: BFS, DFS, Shortest Paths      |
| 12.4          | Space-Time Tradeoffs                            |
| 12.4.1        | Caching and Memoization                         |
| 12.4.2        | Lookup Tables and Precomputation                |
| FrsPEdAn 3202 | Compression and Succinct Data Structures        |

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#### Cache-Aware and I/O Complexity

| 13.1    | Introduction to the Memory Hierarchy                             |
|---------|--|
| 13.1.1  | Registers, Cache (L1, L2, L3), RAM, Disk                         |
| 13.1.2  | Latency and Bandwidth Characteristics                            |
| 13.1.3  | Why Algorithm Design Must Consider Memory                        |
| 13.2    | The External Memory Model (I/O Model)                            |
| 13.2.1  | Parameters: $N$ (data size), $M$ (memory size), $B$ (block size) |
| 13.2.2  | I/O Complexity: Counting Block Transfers                         |
| 13.2.3  | Comparison with RAM Model  |
| 13.3    | I/O-Efficient Algorithms   |
| 13.3.1  | Scanning and Sorting   |
| Externa | l Merge Sort   |

13.3.2 Matrix Operations

I/O Complexity:  $O((N/B)\log_{M/B}(N/B))$ 

**Matrix Transposition** 

Matrix Multiplication

13.3.3 Graph Algorithms

I/O-Efficient BFS and DFS

**Minimum Spanning Tree** 

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## Cache-Aware Scheduling and Analysis for Multicores

| Introduction to Multicore and Parallel Computing                |
|---|
| Shared vs. Distributed Memory                                   |
| Parallel Models: PRAM, Fork-Join, Work-Stealing                 |
| Performance Metrics: Work, Span, Parallelism                    |
| Cache Coherence and Consistency                                 |
| MESI and MOESI Protocols  |
| False Sharing in Multicore Systems                              |
| Impact on Algorithm Design                                      |
| Cache-Aware Parallel Algorithms                                 |
| Parallel Sorting with Cache Awareness                           |
| Parallel Matrix Multiplication (Strassen, Coppersmith-Winograd) |
|   |

14.4.1 WCET Analysis in Cache-Aware Contexts

**Real-Time and Embedded Systems** 

14.3.3 Load Balancing and Task Granularity

# Part IV Lower Bounds and Optimality

15.5 Exercises

# Lower Bounds for Comparison-Based Algorithms

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| 15.1.1                             | Modeling Algorithms as Decision Trees   |
| 15.1.2                             | Height of Decision Trees and Worst-Case Complexity  |
| 15.2                               | <b>Sorting Lower Bound</b>  |
| 15.2.1                             | Information-Theoretic Argument  |
| 15.2.2                             | $\Omega(n \log n)$ Lower Bound for Comparison Sorting   |
| 15.2.3                             | Implications and Optimal Algorithms   |
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| 15.3                               | <b>Selection and Searching Lower Bounds</b>   |
|                                    | Selection and Searching Lower Bounds Finding the Minimum: $\Omega(n)$   |
| 15.3.1                             |   |
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| 15.3.1<br>15.3.2<br>15.3.3         | Finding the Minimum: $\Omega(n)$<br>Finding Median: Adversary Arguments   |
| 15.3.1<br>15.3.2<br>15.3.3<br>15.4 | Finding the Minimum: $\Omega(n)$<br>Finding Median: Adversary Arguments<br>Searching in Sorted Arrays: $\Omega(\log n)$ |

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# Algebraic and Non-Comparison Lower Bounds

| 16.1   | Algebraic Decision Trees                |
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| 16.1.1 | <b>Extending Beyond Comparisons</b>     |
| 16.1.2 | <b>Element Distinctness Lower Bound</b> |
| 16.2   | <b>Communication Complexity</b>         |
| 16.2.1 | Models and Definitions                  |
| 16.2.2 | Applications to Data Structures         |
| 16.3   | Cell-Probe Model                        |
| 16.3.1 | <b>Lower Bounds for Data Structures</b> |
| 16.3.2 | Dynamic vs. Static Data Structures      |
| 16.4   | Exercises                               |

### Part V

**Specialized Topics and Applications** 

# Analysis of Specific Algorithm Paradigms

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| 17.1.2      | Examples: Mergesort, Quicksort, Strassen's Algorithm |
| 17.1.3      | Optimality and Lower Bounds                          |
| 17.2        | Greedy Algorithms                                    |
| 17.2.1      | Correctness via Exchange Arguments                   |
| 17.2.2      | Matroid Theory (Brief Introduction)                  |
| 17.2.3      | Examples: Huffman Coding, Kruskal's MST              |
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