MATHESIS

THE MATHEMATICAL FOUNDATIONS OF COMPUTING

"In mathematics, you don't understand things. You just get used to them."

— JOHN VON NEUMANN

Mahdi

LIVING FIRST EDITION · 2025

MATH

The Mathematical Foundations of Computing

"From ancient counting stones to quantum algorithms every data structure tells the story of human ingenuity."

Living First Edition

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A Living Architecture of Computing

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Preface

MATHEMATICS IS NOT LEARNEDit is lived. This book emerged not from a plan, but from a necessity I could no longer ignore.

During my work on *Arliz* and *The Art of Algorithmic Analysis*, I confronted an uncomfortable truth: my mathematical foundation was insufficient. Not superficiallyI could manipulate symbols, apply formulas, solve standard problemsbut fundamentally. I lacked the deep, intuitive understanding that transforms mathematics from a tool into a language of thought.

The realization was humbling. Here I was, attempting to write comprehensive treatments of data structures and algorithmic analysis, yet stumbling over concepts that should have been second nature. When working through recurrence relations, I found myself mechanically applying methods without truly grasping why they worked. When analyzing probabilistic algorithms, I could follow the calculations but couldn't see the underlying structure. When dealing with matrix operations in multidimensional arrays, the algebra felt arbitrary rather than inevitable.

This gap became impossible to ignore.

The Decision to Begin Again

I made a choice: to pause my other work and return to the beginning. Not to the beginning of computer science, but to the beginning of mathematical thought itself. If I was to write honestly about computation, I needed to understand the mathematics that makes computation possiblenot as a collection of techniques, but as a coherent intellectual tradition.

I began reading widely. Aristotle's *Organon* for logical foundations. Al-Khwarizmi's *Al-Jabr wa-l-Muqabala* to understand algebra's origins. Ibn Sina's *Al-Shifa* for its systematic treatment of mathematics within broader philosophical context. Euclid's *Elements* to see how axiomatic thinking crystallized geometric intuition. The works of Descartes, Leibniz, Euler, Gausseach revealing how mathematical structures emerged from intellectual necessity.

What struck me most was the continuity. These were not isolated discoveries but conversations across centuries. Khwarizmi built on Greek algebra, which drew from

Babylonian methods. Ibn Sina synthesized Aristotelian logic with Islamic mathematical traditions. European algebraists refined ideas that had traveled from India through Persia. Each generation stood on foundations laid by predecessors, adding new levels of abstraction and generality.

Why This Book Exists

As I studied, I began taking notes. These notes grew into explorations. Those explorations became chapters. Eventually, I realized I was writing a booknot the book I had planned, but the book I needed.

Mathesis is my attempt to understand mathematics as computer scientists and engineers must understand it: not as pure abstraction divorced from application, nor as mere toolbox of techniques, but as living framework for systematic thought. It traces mathematical concepts from their historical origins through their modern formalizations, always asking: Why did this idea emerge? What problem did it solve? How does it connect to computation?

This book completes a trilogy of sorts:

- *Mathesis* provides the mathematical foundations
- *The Art of Algorithmic Analysis* develops analytical techniques
- *Arliz* applies these ideas to concrete data structures

Each stands alone, but together they form a coherent wholea pathway from ancient counting to modern algorithms.

What Makes This Book Different

Most mathematical prerequisites texts for computer science students follow a predictable pattern: rapid surveys of discrete mathematics, linear algebra, probabilitytopics treated as necessary evils, obstacles to overcome before "real" computer science begins. Proofs are minimized, historical context ignored, philosophical motivations unexplored.

This approach fails. It produces students who can manipulate mathematical symbols without understanding what those symbols mean. They can apply algorithms without grasping why those algorithms work. They memorize rather than comprehend.

Mathesis takes a different path. It begins where mathematics began: with humans trying to make sense of quantity, pattern, and structure. It follows the intellectual journey from tally marks on bones to abstract algebraic structures, showing not just

what we discovered but why each discovery was necessary.
Every major concept is developed in three ways:

- **Historical**: How did this idea emerge? What problem motivated it?
- **Mathematical**: What is the precise, formal definition? Why this definition?
- **Computational**: Where does this appear in computer science? How is it used?

The goal is not merely competence but *mathematical maturity*the ability to think mathematically, to see structure where others see complexity, to recognize patterns that transcend specific contexts.

Acknowledgment

This book owes debts to thinkers separated by millennia: to Aristotle for showing that thought itself can be systematized; to Al-Khwarizmi for demonstrating that symbolic manipulation can solve problems; to Ibn Sina for integrating mathematics into comprehensive philosophical systems; to Descartes for making geometry algebraic; to Leibniz for dreaming of universal mathematical language; to Turing for showing that mathematics could be mechanized.
More immediately, I thank the readers of my other books whose questions and insights helped me understand what I had missed. Your engagement made me a better writer and thinker.

Final Thoughts

Mathematics is hard. It should be hardwe are training our minds to think in ways that don't come naturally, to see abstractions that don't exist in physical world, to follow chains of reasoning that extend far beyond immediate intuition.
But mathematics is also beautiful. When you finally understand a proof, when a pattern suddenly becomes clear, when disparate concepts unite into coherent theorythose moments justify every frustration that preceded them.
This book is my attempt to share both the difficulty and the beauty. To show not just mathematical results but the intellectual journey that produced them. To help you develop not just mathematical knowledge but mathematical intuition.

Welcome to **Mathesis**. Let us begin at the beginning.

Mahdi
2025

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I would like to express my gratitude to everyone who supported me during the creation of this book. Special thanks to the open-source community for their invaluable resources and to all those who reviewed early drafts and provided feedback.

Introduction

THIS BOOK is structured as an intellectual journeya carefully designed progression through the landscape of mathematical thought that has shaped computational science. Each part represents not merely a collection of related topics, but a distinct phase in humanity's mathematical understanding, building systematically toward the comprehensive foundation needed for modern computer science and engineering.

The Architecture of Mathematical Knowledge

Mathematics is not a linear sequence of facts to be memorized. It is a vast, interconnected web of ideas, where each concept illuminates and is illuminated by countless others. This book's structure reflects that reality. We begin with originsthe cognitive and historical roots of mathematical thinkingand progressively build toward the sophisticated abstractions that enable modern computation.

The journey follows a natural arc:

Parts I-VI: Historical and Foundational Development
We trace mathematics from its primordial origins through ancient civilizations to the Renaissance mathematical revolution. These parts are not merely historicalthey reveal *why* mathematical concepts emerged in particular forms, *what problems* motivated their development, and *how* each innovation prepared the ground for subsequent advances.

Parts VII-XII: The Analytical Revolution
From calculus through measure theory and functional analysis, we explore the mathematics of continuity, change, and infinite processes. These parts develop the analytical machinery essential for understanding algorithms, complexity, and computational systems.

Parts XIII-XVII: Abstract Structures and Modern Mathematics
Probability theory, combinatorics, computational mathematics, category theory, and twentieth-century synthesis reveal mathematics' power through abstraction. Here

we see how general frameworks unify diverse phenomena and enable systematic reasoning.

Parts XVIII-XXIV: Applied and Specialized Mathematics
The connection between mathematics and physics, contemporary frontiers, and specialized applications to electrical engineering, robotics, artificial intelligence, computer vision, natural language processing, quantum computing, and deep learning demonstrate how abstract mathematics becomes practical power.

Three Dimensions of Understanding

Throughout this journey, we maintain three interwoven perspectives:

1. Historical Development
Understanding *how* mathematical ideas emerged reveals *why* they take particular forms. When you see Babylonian mathematicians wrestling with positional notation, or Greek geometers discovering incommensurability, or Islamic scholars systematizing algebra, you understand these concepts' essential nature in ways that pure formal definition cannot convey.

Mathematics did not spring fully formed from abstract contemplation. It emerged from necessityfrom practical problems requiring systematic solution, from intellectual puzzles demanding resolution, from the human drive to understand pattern and structure. Each major mathematical development represents humanity solving a problem, confronting a paradox, or discovering an unexpected connection.

2. Formal Mathematical Structure
History provides intuition, but mathematics demands precision. Each concept receives rigorous formal treatment: definitions, theorems, proofs, examples, counterexamples. We develop mathematical maturitythe ability to think precisely, reason systematically, and construct valid arguments.

Formal mathematics is not pedantry. It is the discipline that distinguishes reliable reasoning from wishful thinking, valid inference from plausible error. When you understand *why* definitions must be precise, *how* theorems connect to definitions, and *what* proofs actually accomplish, mathematics transforms from mysterious ritual into comprehensible structure.

3. Computational Application
Mathematics for computer scientists and engineers must connect to computation. Throughout, we emphasize: Where does this concept appear in algorithms? How does this theorem enable practical computation? Why does this abstraction matter for software systems?

This computational perspective is not separate from "pure" mathematicsit reveals mathematics' essential character. Computation is systematic symbol manipulation following precise rules. Mathematics is systematic reasoning about structure and pattern. They are intimately connected.

Navigation Strategies

This book supports multiple reading paths:

The Complete Journey

Work through systematically from Part I to Part XXIV. This provides the fullest understanding and reveals how mathematical ideas build on one another. Recommended for students building comprehensive foundations.

The Reference Approach

Use the book as a reference when specific mathematical understanding is needed. Each part is relatively self-contained, with clear prerequisites noted. The extensive index and cross-references enable targeted consultation.

The Curious Explorer

Follow your interests. Skip parts that don't immediately engage you. Return when ready. Mathematics rewards patienceconfusion often precedes understanding. Some concepts require mental maturation; return later and they suddenly make sense.

Prerequisites and Preparation

This book assumes:

- **Mathematical maturity equivalent to first-year university mathematics**
- **Comfort with algebraic manipulation and basic proof techniques**
- **Willingness to work through difficult material systematically**
- **Patience with abstraction and formal reasoning**

If you find early parts too easy, skip ahead. If later parts seem too difficult, return to earlier materialmathematical understanding develops through repeated engagement from different perspectives.

The Living Nature of This Work

Like all my books, *Mathesis* evolves continuously. As I discover better explanations, identify errors, or recognize new connections, the book improves. Your engagement-through corrections, suggestions, and questionscontributes to this evolution.

Mathematics itself is not static. New theorems are proved, old proofs simplified, unexpected connections discovered. A book about mathematics should reflect this dynamic reality.

A Word of Encouragement

The journey ahead is challenging. Mathematics demands sustained mental effort, tolerance for confusion, and persistence through difficulty. But the rewards justify the struggle:

- **Intellectual power**: Mathematical thinking enables systematic problem-solving across domains
- **Deep understanding**: Surface-level knowledge becomes genuine comprehension
- **Professional capability**: Mathematical maturity distinguishes good practitioners from exceptional ones
- **Aesthetic pleasure**: Mathematics possesses profound beautypatterns, elegance, surprising connections

When concepts seem opaque, persist. When proofs seem impenetrable, work through them line by line. When exercises seem impossible, struggle with them. Mathematical understanding arrives not in sudden revelation but through patient, sustained engagement.

Every mathematicianfrom ancient Babylonian scribes to modern research leadershas experienced the frustration you will feel. Every significant mathematical insight in history required someone to persist through confusion toward clarity. You walk a path trodden by countless others; you will arrive.

Begin

Twenty-four parts await. Each reveals another dimension of mathematical thought. Each builds the foundation for computational understanding. Each represents humanity's long conversation with quantity, pattern, and structure.

Welcome to **Mathesis**. The journey begins with a simple question: How did humans learn to count?

> *"In mathematics, you don't understand things. You just get used to them."*
>
> — John von Neumann

"Pure mathematics is, in its way, the poetry of logical ideas."

— Albert Einstein

"Mathematics is the language in which God has written the universe."

— Galileo Galilei

Part I

Origins of Mathematical Thought

MATHEMATICS DID NOT emerge fully formed from human minds. It was forged through millennia of necessity, observation, and intellectual struggle. Before symbols existed, before numbers had names, our ancestors confronted the fundamental challenge: how to comprehend and communicate quantity, pattern, and structure.

This part traces mathematics from its primordial originswhen humanity first distinguished "one" from "many"through the revolutionary abstractions that made systematic thought possible. We examine not merely what ancient peoples calculated, but how they reasoned, what cognitive leaps enabled mathematical thinking, and why certain cultures developed particular mathematical frameworks.

What Makes This Different:

- ***Cognitive Foundations:*** *We explore the neurological and psychological basis for mathematical intuition*
- ***Archaeological Evidence:*** *Physical artifacts reveal how abstract concepts became material reality*
- ***Cultural Contexts:*** *Mathematical systems emerged from specific human needs and worldviews*
- ***Conceptual Evolution:*** *We trace how simple counting became sophisticated abstraction*

> *"The numbers are a match for the transcendent world, and the transcendent world is a match for the numbers."*
>
> — ARISTOTLE, METAPHYSICS

Chapter 1

The Primordial Urge to Count and Order

Chapter 2

Cognitive Foundations of Number Sense

Chapter 3

Archaeological Evidence of Early Quantification

Chapter 4

Body Counting and Finger Mathematics

Chapter 5

Tally Systems and External Memory

Chapter 6

The Neolithic Revolution and Administrative Mathematics

Chapter 7

Token Systems and Proto-Writing

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The Birth of Symbolic Representation

Part II

Ancient Number Systems and Positional Notation

WITH SETTLED CIVILIZATIONS *came new mathematical demands. Agricultural surplus required accounting; astronomical observation demanded precision; architecture necessitated geometric sophistication. The ancient world responded with remarkably diverse mathematical systems, each reflecting the unique needs and insights of its culture.*

This part examines the major mathematical traditions of antiquity: Mesopotamian sexagesimal notation, Egyptian hieroglyphic numbers and unit fractions, the revolutionary Chinese rod calculus and matrix methods, and the sophisticated Indian numeral system that would transform world mathematics. We explore not merely their computational techniques, but the conceptual frameworks that made such techniques possible.

What Makes This Different:

- ***Comparative Analysis:*** *We examine why different cultures developed distinct mathematical approaches*
- ***Positional Revolution:*** *The conceptual leap from concrete to abstract representation*
- ***Computational Practice:*** *How ancient peoples actually performed calculations*
- ***Cultural Transmission:*** *The paths by which mathematical knowledge spread across civilizations*

> *"I have found a very great number of exceedingly beautiful theorems."*
>
> — ARCHIMEDES, AS REPORTED BY PLUTARCH

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Part III

Greek Mathematical Philosophy

The Greeks transformed mathematics from a computational tool into a philosophical discipline. They asked not merely "how to calculate?" but "why is this true?" Their demand for logical proof, their development of axiomatic systems, and their conception of mathematics asthe study of eternal, perfect forms fundamentally altered human intellectual history.

This part explores Greek mathematical philosophy from the Pythagoreans' mystical number theory through Euclid's systematic geometry to Archimedes' sophisticated methods of exhaustion. We examine how Greek philosophical commitments shaped mathematical practice, how logical rigor emerged as a mathematical virtue, and how Greek achievements influenced all subsequent mathematical development.

What Makes This Different:

- ***Philosophical Integration:*** *Mathematics as inseparable from metaphysics and epistemology*
- ***Proof Culture:*** *The emergence of demonstration as mathematical necessity*
- ***Geometric Focus:*** *Why Greeks privileged geometric over arithmetic reasoning*
- ***Logical Foundations:*** *Aristotelian logic as framework for mathematical thought*

> *"There is no royal road to geometry."*
>
> — Euclid to Ptolemy I

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Greek Mechanical Mathematics and Computation

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Indian and Islamic Mathematical Synthesis

While Europe struggled through its Dark Ages, mathematical brilliance flourished elsewhere. Indian mathematicians developed the decimal place-value system and conceived of zero as number-revolutionary insights that transformed human capacity for calculation. Islamic scholars preserved, synthesized, and extended Greek and Indian mathematics, creating algebra as a systematic discipline and developing sophisticated astronomical and geometric methods.

This part examines these transformative contributions: the philosophical and practical implications of zero, the development of positional decimal notation, al-Khwarizmi's systematization of algebra, and the geometric innovations of Persian and Arab mathematicians. We explore how these advances emerged from specific intellectual contexts and how they spread to reshape global mathematics.

What Makes This Different:

- ***Conceptual Revolution:*** *How zero changed mathematical possibility*
- ***Algebraic Thinking:*** *The emergence of symbolic manipulation as mathematical method*
- ***Cultural Synthesis:*** *How Islamic scholars unified diverse mathematical traditions*
- ***Computational Efficiency:*** *Practical mathematical methods for complex calculations*

> *"Al-jabr is the restoration and balancing of broken parts."*
>
> — Muhammad ibn Musa al-Khwarizmi

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The House of Wisdom and Knowledge Transmission

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Medieval European Mathematics

Medieval Europe received Greek and Islamic mathematics through translation, gradually absorbing and extending these traditions. The rise of universities, the development of systematic educational curricula, and the needs of commerce and architecture drove mathematical innovation. Though often dismissed as a period of stagnation, the medieval era laid crucial institutional and intellectual foundations for the Renaissance explosion of mathematical creativity.

This part examines how European scholars engaged with inherited mathematical traditions, how monastic and university education systematized mathematical knowledge, and how practical needsnavigation, commerce, architecturedrove theoretical advances. We explore the slow but crucial development of mathematical notation and the gradual shift toward algebraic thinking.

What Makes This Different:

- ***Institutional Context:*** *How universities shaped mathematical development*
- ***Translation Movement:*** *The transmission of Greek and Arabic texts to Latin Europe*
- ***Practical Mathematics:*** *Commercial arithmetic and its theoretical implications*
- ***Notational Evolution:*** *The gradual development of symbolic mathematical language*

> *"In omni doctrina et scientia delectabili et utili, quam nullus ignorare debet..."*
>
> — Leonardo Fibonacci, Liber Abaci

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The Renaissance Mathematical Revolution

The Renaissance unleashed mathematical creativity of unprecedented scope. The development of symbolic algebra transformed mathematics from geometric and rhetorical reasoning into symbolic manipulation. The invention of analytic geometry unified algebra and geometry, revealing deep connections between equations and curves. The solution of cubic and quartic equations demonstrated that systematic algebraic methods could solve problems that had resisted Greek geometry.

This part traces these revolutionary developments: Viète's symbolic algebra, Cardano's solution methods, Descartes' analytical geometry, and the broader cultural and intellectual context that made such innovations possible. We examine how new notational systems enabled new mathematical thought, and how Renaissance mathematics prepared the ground for the calculus revolution.

What Makes This Different:

- ***Symbolic Revolution:*** *How notation changed what could be thought*
- ***Algebraic-Geometric Unity:*** *The emergence of coordinate systems and analytical methods*
- ***Solution Systematization:*** *General methods replacing case-by-case geometric arguments*
- ***Cultural Context:*** *How Renaissance humanism and artisanal practice influenced mathematics*

> *"Ars magna, the great art, is the art of solving equations of the third and fourth degree."*
>
> — Gerolamo Cardano

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