MATHESIS

THE MATHEMATICAL FOUNDATIONS OF COMPUTING

"In mathematics, you don't understand things. You just get used to them."

— JOHN VON NEUMANN

Mahdi

LIVING FIRST EDITION · 2025

MATH

The Mathematical Foundations of Computing

"From ancient counting stones to quantum algorithms every data structure tells the story of human ingenuity."

Living First Edition

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MATHESIS:

A Living Architecture of Computing

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Preface

MATHEMATICS IS NOT LEARNEDit is lived. This book emerged not from a plan, but from a necessity I could no longer ignore.

During my work on *Arliz* and *The Art of Algorithmic Analysis*, I confronted an uncomfortable truth: my mathematical foundation was insufficient. Not superficiallyI could manipulate symbols, apply formulas, solve standard problemsbut fundamentally. I lacked the deep, intuitive understanding that transforms mathematics from a tool into a language of thought.

The realization was humbling. Here I was, attempting to write comprehensive treatments of data structures and algorithmic analysis, yet stumbling over concepts that should have been second nature. When working through recurrence relations, I found myself mechanically applying methods without truly grasping why they worked. When analyzing probabilistic algorithms, I could follow the calculations but couldn't see the underlying structure. When dealing with matrix operations in multidimensional arrays, the algebra felt arbitrary rather than inevitable.

This gap became impossible to ignore.

The Decision to Begin Again

I made a choice: to pause my other work and return to the beginning. Not to the beginning of computer science, but to the beginning of mathematical thought itself. If I was to write honestly about computation, I needed to understand the mathematics that makes computation possiblenot as a collection of techniques, but as a coherent intellectual tradition.

I began reading widely. Aristotle's *Organon* for logical foundations. Al-Khwarizmi's *Al-Jabr wa-l-Muqabala* to understand algebra's origins. Ibn Sina's *Al-Shifa* for its systematic treatment of mathematics within broader philosophical context. Euclid's *Elements* to see how axiomatic thinking crystallized geometric intuition. The works of Descartes, Leibniz, Euler, Gausseach revealing how mathematical structures emerged from intellectual necessity.

What struck me most was the continuity. These were not isolated discoveries but conversations across centuries. Khwarizmi built on Greek algebra, which drew from

Babylonian methods. Ibn Sina synthesized Aristotelian logic with Islamic mathematical traditions. European algebraists refined ideas that had traveled from India through Persia. Each generation stood on foundations laid by predecessors, adding new levels of abstraction and generality.

Why This Book Exists

As I studied, I began taking notes. These notes grew into explorations. Those explorations became chapters. Eventually, I realized I was writing a booknot the book I had planned, but the book I needed.

Mathesis is my attempt to understand mathematics as computer scientists and engineers must understand it: not as pure abstraction divorced from application, nor as mere toolbox of techniques, but as living framework for systematic thought. It traces mathematical concepts from their historical origins through their modern formalizations, always asking: Why did this idea emerge? What problem did it solve? How does it connect to computation?

This book completes a trilogy of sorts:

- *Mathesis* provides the mathematical foundations
- *The Art of Algorithmic Analysis* develops analytical techniques
- *Arliz* applies these ideas to concrete data structures

Each stands alone, but together they form a coherent wholea pathway from ancient counting to modern algorithms.

What Makes This Book Different

Most mathematical prerequisites texts for computer science students follow a predictable pattern: rapid surveys of discrete mathematics, linear algebra, probabilitytopics treated as necessary evils, obstacles to overcome before "real" computer science begins. Proofs are minimized, historical context ignored, philosophical motivations unexplored.

This approach fails. It produces students who can manipulate mathematical symbols without understanding what those symbols mean. They can apply algorithms without grasping why those algorithms work. They memorize rather than comprehend.

Mathesis takes a different path. It begins where mathematics began: with humans trying to make sense of quantity, pattern, and structure. It follows the intellectual journey from tally marks on bones to abstract algebraic structures, showing not just

what we discovered but why each discovery was necessary.
Every major concept is developed in three ways:

- **Historical**: How did this idea emerge? What problem motivated it?
- **Mathematical**: What is the precise, formal definition? Why this definition?
- **Computational**: Where does this appear in computer science? How is it used?

The goal is not merely competence but *mathematical maturity*the ability to think mathematically, to see structure where others see complexity, to recognize patterns that transcend specific contexts.

Acknowledgment

This book owes debts to thinkers separated by millennia: to Aristotle for showing that thought itself can be systematized; to Al-Khwarizmi for demonstrating that symbolic manipulation can solve problems; to Ibn Sina for integrating mathematics into comprehensive philosophical systems; to Descartes for making geometry algebraic; to Leibniz for dreaming of universal mathematical language; to Turing for showing that mathematics could be mechanized.
More immediately, I thank the readers of my other books whose questions and insights helped me understand what I had missed. Your engagement made me a better writer and thinker.

Final Thoughts

Mathematics is hard. It should be hardwe are training our minds to think in ways that don't come naturally, to see abstractions that don't exist in physical world, to follow chains of reasoning that extend far beyond immediate intuition.
But mathematics is also beautiful. When you finally understand a proof, when a pattern suddenly becomes clear, when disparate concepts unite into coherent theorythose moments justify every frustration that preceded them.
This book is my attempt to share both the difficulty and the beauty. To show not just mathematical results but the intellectual journey that produced them. To help you develop not just mathematical knowledge but mathematical intuition.

Welcome to **Mathesis**. Let us begin at the beginning.

Mahdi
2025

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I would like to express my gratitude to everyone who supported me during the creation of this book. Special thanks to the open-source community for their invaluable resources and to all those who reviewed early drafts and provided feedback.

Introduction

THIS BOOK is structured as an intellectual journeya carefully designed progression through the landscape of mathematical thought that has shaped computational science. Each part represents not merely a collection of related topics, but a distinct phase in humanity's mathematical understanding, building systematically toward the comprehensive foundation needed for modern computer science and engineering.

The Architecture of Mathematical Knowledge

Mathematics is not a linear sequence of facts to be memorized. It is a vast, interconnected web of ideas, where each concept illuminates and is illuminated by countless others. This book's structure reflects that reality. We begin with originsthe cognitive and historical roots of mathematical thinkingand progressively build toward the sophisticated abstractions that enable modern computation.

The journey follows a natural arc:

Parts I-VI: Historical and Foundational Development
We trace mathematics from its primordial origins through ancient civilizations to the Renaissance mathematical revolution. These parts are not merely historicalthey reveal *why* mathematical concepts emerged in particular forms, *what problems* motivated their development, and *how* each innovation prepared the ground for subsequent advances.

Parts VII-XII: The Analytical Revolution
From calculus through measure theory and functional analysis, we explore the mathematics of continuity, change, and infinite processes. These parts develop the analytical machinery essential for understanding algorithms, complexity, and computational systems.

Parts XIII-XVII: Abstract Structures and Modern Mathematics
Probability theory, combinatorics, computational mathematics, category theory, and twentieth-century synthesis reveal mathematics' power through abstraction. Here

we see how general frameworks unify diverse phenomena and enable systematic reasoning.

Parts XVIII-XXIV: Applied and Specialized Mathematics
The connection between mathematics and physics, contemporary frontiers, and specialized applications to electrical engineering, robotics, artificial intelligence, computer vision, natural language processing, quantum computing, and deep learning demonstrate how abstract mathematics becomes practical power.

Three Dimensions of Understanding

Throughout this journey, we maintain three interwoven perspectives:

1. Historical Development
Understanding *how* mathematical ideas emerged reveals *why* they take particular forms. When you see Babylonian mathematicians wrestling with positional notation, or Greek geometers discovering incommensurability, or Islamic scholars systematizing algebra, you understand these concepts' essential nature in ways that pure formal definition cannot convey.

Mathematics did not spring fully formed from abstract contemplation. It emerged from necessityfrom practical problems requiring systematic solution, from intellectual puzzles demanding resolution, from the human drive to understand pattern and structure. Each major mathematical development represents humanity solving a problem, confronting a paradox, or discovering an unexpected connection.

2. Formal Mathematical Structure
History provides intuition, but mathematics demands precision. Each concept receives rigorous formal treatment: definitions, theorems, proofs, examples, counterexamples. We develop mathematical maturitythe ability to think precisely, reason systematically, and construct valid arguments.

Formal mathematics is not pedantry. It is the discipline that distinguishes reliable reasoning from wishful thinking, valid inference from plausible error. When you understand *why* definitions must be precise, *how* theorems connect to definitions, and *what* proofs actually accomplish, mathematics transforms from mysterious ritual into comprehensible structure.

3. Computational Application
Mathematics for computer scientists and engineers must connect to computation. Throughout, we emphasize: Where does this concept appear in algorithms? How does this theorem enable practical computation? Why does this abstraction matter for software systems?

This computational perspective is not separate from "pure" mathematicsit reveals mathematics' essential character. Computation is systematic symbol manipulation following precise rules. Mathematics is systematic reasoning about structure and pattern. They are intimately connected.

Navigation Strategies

This book supports multiple reading paths:

The Complete Journey
Work through systematically from Part I to Part XXIV. This provides the fullest understanding and reveals how mathematical ideas build on one another. Recommended for students building comprehensive foundations.

The Reference Approach
Use the book as a reference when specific mathematical understanding is needed. Each part is relatively self-contained, with clear prerequisites noted. The extensive index and cross-references enable targeted consultation.

The Curious Explorer
Follow your interests. Skip parts that don't immediately engage you. Return when ready. Mathematics rewards patienceconfusion often precedes understanding. Some concepts require mental maturation; return later and they suddenly make sense.

Prerequisites and Preparation

This book assumes:

- **Mathematical maturity equivalent to first-year university mathematics**
- **Comfort with algebraic manipulation and basic proof techniques**
- **Willingness to work through difficult material systematically**
- **Patience with abstraction and formal reasoning**

If you find early parts too easy, skip ahead. If later parts seem too difficult, return to earlier materialmathematical understanding develops through repeated engagement from different perspectives.

The Living Nature of This Work

Like all my books, *Mathesis* evolves continuously. As I discover better explanations, identify errors, or recognize new connections, the book improves. Your engagement-through corrections, suggestions, and questionscontributes to this evolution.

Mathematics itself is not static. New theorems are proved, old proofs simplified, unexpected connections discovered. A book about mathematics should reflect this dynamic reality.

A Word of Encouragement

The journey ahead is challenging. Mathematics demands sustained mental effort, tolerance for confusion, and persistence through difficulty. But the rewards justify the struggle:

- **Intellectual power**: Mathematical thinking enables systematic problem-solving across domains
- **Deep understanding**: Surface-level knowledge becomes genuine comprehension
- **Professional capability**: Mathematical maturity distinguishes good practitioners from exceptional ones
- **Aesthetic pleasure**: Mathematics possesses profound beautypatterns, elegance, surprising connections

When concepts seem opaque, persist. When proofs seem impenetrable, work through them line by line. When exercises seem impossible, struggle with them. Mathematical understanding arrives not in sudden revelation but through patient, sustained engagement.

Every mathematicianfrom ancient Babylonian scribes to modern research leadershas experienced the frustration you will feel. Every significant mathematical insight in history required someone to persist through confusion toward clarity. You walk a path trodden by countless others; you will arrive.

Begin

Twenty-four parts await. Each reveals another dimension of mathematical thought. Each builds the foundation for computational understanding. Each represents humanity's long conversation with quantity, pattern, and structure.

Welcome to **Mathesis**. The journey begins with a simple question: How did humans learn to count?

"In mathematics, you don't understand things. You just get used to them."

— John von Neumann

"Pure mathematics is, in its way, the poetry of logical ideas."

— Albert Einstein

"Mathematics is the language in which God has written the universe."

— Galileo Galilei

Part I

Logic and the Foundations of Reasoning

BEFORE MATHEMATICS, *there was logic. Before we can reason about numbers, structures, or algorithms, we must understand reasoning itself. This part develops the formal systems of propositional and predicate logic, proof theory, and the philosophical foundations that make mathematics possible.*

What Makes This Different:

- ***Philosophical Depth:*** *From Aristotelian syllogisms to modern proof assistants*
- ***Complete Formalization:*** *Natural deduction, sequent calculus, resolution*
- ***Computational Connection:*** *Logic as the foundation of programming languages*
- ***Metamathematical Results:*** *Completeness, soundness, decidability*

> *"Logic is the beginning of wisdom, not the end."*
>
> — SPOCK (AND ARISTOTLE, ESSENTIALLY)

Chapter 1

Propositional Logic and the Calculus of Reasoning

Chapter 2

Predicate Logic and Quantificational Reasoning

Chapter 3

Modal Logic: Necessity, Possibility, and Temporal Reasoning

Chapter 4

Intuitionistic Logic and Constructive Mathematics

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The Curry-Howard Correspondence: Proofs as Programs

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Metalogic: Completeness, Soundness, and Decidability

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Gödel's Incompleteness Theorems: The Limits of Formal Systems

Part II

Set Theory: The Language of Mathematical Objects

SETS ARE *the atoms of mathematical discourse. Every mathematical objectnumbers, functions, spaces, categoriesis ultimately constructed from sets. This part develops axiomatic set theory from ZFC, explores the paradoxes that necessitate axiomatization, and examines the philosophical implications of mathematical existence.*

What Makes This Different:

- ***Axiomatic Rigor:*** *Full development of Zermelo-Fraenkel set theory*
- ***Philosophical Context:*** *What does mathematical existence mean?*
- ***Computational Relevance:*** *Sets as data structures, type theory*
- ***Foundations of Infinity:*** *Cantor's paradise and its implications*

> *"No one shall expel us from the Paradise that Cantor has created."*
>
> — DAVID HILBERT

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Naive Set Theory and Its Paradoxes

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Axiomatic Set Theory: ZFC

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Relations, Functions, and Mappings

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Cardinality and the Arithmetic of Infinity

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Ordinal Numbers and Transfinite Induction

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Constructible Universe and Forcing

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Alternative Foundations: Category Theory and HoTT

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Algebraic Structures: Symmetry and Abstraction

ALGEBRA IS *the study of structure. By abstracting the essential properties of mathematical operationsassociativity, commutativity, identity, inverseswe discover profound unifying patterns. This part develops abstract algebra from groups through categories, revealing how seemingly disparate areas of mathematics share deep structural unity.*

What Makes This Different:

- ***Structural Thinking:*** *Focus on morphisms and universal properties*
- ***Computational Algebra:*** *Algorithms for algebraic computation*
- ***Cryptographic Applications:*** *Group theory in RSA, elliptic curves*
- ***Categorical Perspective:*** *Algebra as the study of categories with structure*

> *"Algebra is the offer made by the devil to the mathematician... All you need to do is give me your soul: give up geometry."*
>
> — MICHAEL ATIYAH

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Groups: Symmetry and Transformation

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Number Theory: The Integers and Their Mysteries

NUMBER THEORY*once dismissed as the purest of pure mathematicsnow underpins modern cryptography, pseudorandom generation, and algorithmic complexity. This part develops both classical and computational number theory, from Euclid's algorithm to elliptic curves.*

What Makes This Different:

- ***Computational Focus:*** *Complexity analysis of every algorithm*
- ***Cryptographic Applications:*** *RSA, Diffie-Hellman, ECC in depth*
- ***Analytic Methods:*** *Connection to complex analysis and the Riemann hypothesis*
- ***Algorithmic Number Theory:*** *Primality, factorization, discrete logarithm*

> *"Mathematics is the queen of sciences, and number theory is the queen of mathematics."*
>
> — CARL FRIEDRICH GAUSS

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Divisibility and the Fundamental Theorem of Arithmetic

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Modular Arithmetic and Congruences

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The Euclidean Algorithm and Its Complexity

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DISCRETE MATHEMATICS is the native language of computer science. Unlike continuous mathematics, we deal with countable, finite, or denumerable structures: graphs, permutations, recursive sequences. This part develops the combinatorial and graph-theoretic foundations essential for algorithm design and analysis.

What Makes This Different:

- ***Algorithmic Emphasis:*** *Every result connects to computation*
- ***Generating Functions:*** *Systematic enumeration techniques*
- ***Graph Algorithms:*** *From Euler to modern network science*
- ***Ramsey Theory:*** *The mathematics of inevitable structure*

> *"Combinatorics is an honest subject. No adèles, no sigma-algebras. You count balls in a box, and you either have the right number or you haven't."*
>
> — GIAN-CARLO ROTA

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Generating Functions and Recurrence Relations

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Spectral Graph Theory

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Mathematical Analysis: Limits, Continuity, and Infinity

A NALYSIS IS *the mathematics of the infinite and the infinitesimal. From calculus to measure theory, we study limits, continuity, convergencethe machinery needed to reason about algorithms' asymptotic behavior. This part builds rigorous foundations for continuous mathematics and its discrete approximations.*

What Makes This Different:

- ***Asymptotic Focus:*** *Everything connects to algorithm analysis*
- ***Rigorous ε-δ:*** *No hand-waving about limits*
- ***Lebesgue Integration:*** *Modern measure-theoretic approach*
- ***Functional Analysis:*** *Infinite-dimensional perspectives*

> *"In mathematics, you don't understand things. You just get used to them."*
>
> — JOHN VON NEUMANN

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PROBABILITY THEORY *provides the mathematical framework for reasoning under uncertainty. From randomized algorithms to machine learning, probabilistic thinking permeates modern computation. This part develops probability from measure-theoretic foundations to concentration inequalities and stochastic processes.*

What Makes This Different:

- ***Measure-Theoretic Rigor:*** *Probability as a branch of analysis*
- ***Concentration Bounds:*** *Chernoff, Hoeffding, Azuma inequalities*
- ***Randomized Algorithms:*** *Probabilistic method and derandomization*
- ***Stochastic Processes:*** *Markov chains, martingales, Brownian motion*

> *"The theory of probability is at bottom nothing but common sense reduced to calculus."*
>
> — Pierre-Simon Laplace

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TOPOLOGY STUDIES properties preserved under continuous deformation. While initially abstract, topological thinking appears throughout computer science: fixed-point theorems in semantics, computational topology in data analysis, homotopy type theory in foundations.

What Makes This Different:

- ***Computational Topology:*** *Persistent homology and TDA*
- ***Homotopy Type Theory:*** *Topological foundations for CS*
- ***Fixed-Point Theorems:*** *Applications to program semantics*
- ***Metric Spaces:*** *Foundations for analysis and optimization*

> *"Topology is the mathematics of continuity."*
>
> — HENRI POINCARÉ

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Formal Languages and Automata Theory

COMPUTATION IS *symbolic manipulation following formal rules. This part develops the theory of formal languages, automata, and computabilitythe mathematical foundations of what computers can and cannot do.*

What Makes This Different:

- ***Philosophical Depth:*** *What is computation?*
- ***Chomsky Hierarchy:*** *The structure of syntactic complexity*
- ***Decidability:*** *The limits of algorithmic solvability*
- ***Complexity Theory:*** *P vs NP and beyond*

> *"Computer science is no more about computers than astronomy is about telescopes."*
>
> — EDSGER DIJKSTRA

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Mathematical Logic and Foundations

WHAT ARE *the foundations of mathematics itself? This part examines the deepest questions: Can mathematics be reduced to logic? Are there mathematical truths beyond proof? What is the relationship between syntax and semantics?*

What Makes This Different:

- ***Metamathematics:*** *Mathematics studying itself*
- ***Model Theory:*** *Structures and their theories*
- ***Proof Theory:*** *The mathematics of proofs*
- ***Philosophical Implications:*** *What can we know?*

> *"In mathematics, the art of asking questions is more valuable than solving problems."*
>
> — GEORG CANTOR

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