

# Quadratic forms

## THE GAUSS–JORDAN ELIMINATION COMPLETES THE SQUARE

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## Definition of quadratic forms

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A quadratic form is a special multi-variable function.

### Definition

Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix. Assume that  $a_{k,j}$  denotes the appropriate element of that matrix. The function  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

$$Q(x_1, x_2, \dots, x_n) = \sum_{k=1}^n \sum_{j=1}^n a_{k,j} x_k x_j$$

is called  $n$  variables quadratic function.

If  $n = 1$  then  $Q(x_1) = ax_1^2$ .

If  $n = 2$  then

$$Q(x_1, x_2) = a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{2,1}x_2x_1 + a_{2,2}x_2^2.$$

Compute the case when  $n = 3$ .

## Theorem

*If  $A \in \mathbb{R}^{n \times n}$  is a symmetric matrix then the quadratic form determined by  $A$  is*

$$Q(x) = x \cdot Ax$$

*for every  $x \in \mathbb{R}^n$ .*

## Example

Write the symmetric matrix of quadratic form given before. Write the quadratic form of a symmetric matrix given before.

## Theorem

*The quadratic form determined by  $A + B$  is the same as the sum of quadratic forms determined by  $A$  and  $B$  respectively. Similarly, the quadratic form determined by  $\alpha A$  is the same as the form determined by  $A$  multiplied by  $\alpha$ .*

## Dyadic decomposition

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## Dyadic decomposition

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Dyad and completing the square

The dyad is a special symmetric matrix representing the perfect squares.

### Definition

A symmetric matrix  $A \in \mathbb{R}^{n \times n}$  is called *dyad*, if there exists a vector  $a \in \mathbb{R}^{n \times 1}$  for which  $A = a \cdot a^T$

### Theorem

*A quadratic form is a complete square if and only if it is determined by a dyad. If  $A = a \cdot a^T$  where  $a = (a_1, a_2, \dots, a_n)$ , then  $Q(x_1, x_2, \dots, x_n) = (a_1x_1 + a_2x_2 + \dots + a_nx_n)^2$ , here  $Q$  is the quadratic form determined by the dyad  $A$ .*



## Dyadic decomposition

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High school method for a few variable only

## Theorem

*Every symmetric matrix is a linear combination of dyads. Thus every quadratic form is a linear combination of complete squares.*

## Example

Completing the square, rewrite the following quadratic forms as a linear combination of perfect squares.

$$Q(x_1, x_2) = x_1^2 + 4x_1x_2 - 5x_2^2,$$

$$Q(x_1, x_2) = 4x_1^2 + 6x_1x_2 + 9x_2^2,$$

$$Q(x_1, x_2) = 5x_1^2 - 6x_1x_2 + x_2^2,$$

$$Q(x_1, x_2, x_3) = 5x_1^2 - 6x_1x_2 + x_1x_3,$$

# Dyadic decomposition

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Professional method

## Dyadic decomposition using Gauss–Jordan elimination

Every  $n$ -variable quadratic form can be decreased by a multiply of a dyad such that the remainder quadratic form has one variable less:

### Theorem

*Consider a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ . Assume, that  $a_{1,1} \neq 0$  and denote  $d = \frac{1}{a_{1,1}}a_1$ , where  $a_1$  is the first column of matrix  $A$ . Then*

$$R = A - a_{1,1}d \cdot d^T$$

*is a symmetric matrix, where the first row (and column) is the zero vector.*

### Proof.

The sum of symmetric matrices is a symmetric matrix. If  $\delta_k$  denotes the  $k$ -th coordinate of  $d$ , then  $r_{k,j} = a_{k,j} - a_{1,1}\delta_k\delta_j$ . If  $k = 1$  we obtain

$$r_{1,j} = a_{1,j} - a_{1,1} \cdot 1 \cdot \frac{a_{1,j}}{a_{1,1}} = 0.$$



The  $k, j$  term of the matrix of the remainder quadratic form is

$$r_{k,j} = a_{k,j} - a_{1,1}\delta_k\delta_j = a_{k,j} - a_{1,1}\frac{a_{k,1}}{a_{1,1}}\delta_j = a_{k,j} - a_{k,1}\delta_j.$$

If  $k > 1$  and  $j > 1$ , then the above formula just computes the new coordinates of the  $j$ -th column after the first column has entered to the base at the first position. Thus we proved the following theorem.

### Theorem

*Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix and assume that  $a_{1,1} \neq 0$ . After separating the first complete square, as in the theorem above, the matrix of the remainder quadratic form is the bottom right  $n - 1 \times n - 1$  matrix of the transformation table, when  $a_{1,1}$  is the pivot term.*

# Problems

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Write the following quadratic forms as a linear combination of complete squares.

1.  $Q(x_2, x_3) = 13x_2^2 - 4x_2x_3 + 8x_3^2,$

2.  $Q(x_1, x_2, x_3) = 5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3,$

3.  $Q(z_1, z_2, z_3) = 2z_1^2 + \frac{3}{2}z_3^2 + 2z_1z_2 - 4z_1z_3 + 2z_2z_3,$

4.  $Q(x_1, x_2, x_3, x_4) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 4x_1x_2 + 2x_1x_4 + 2x_2x_3 - 4x_3x_4$

## Definiteness of a quadratic form

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# Definiteness

Classify a quadratic form with respect to the range!

## Definition

Let  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  be a nonzero quadratic form. It is called

**positive definite**, if  $Q(x) > 0$  for every  $x \in \mathbb{R}^n, x \neq 0$ ;

**positive semidefinite**, if  $Q(x) \geq 0$  for every  $x \in \mathbb{R}^n$ , but there exists a  $z \in \mathbb{R}^n$  such that  $Q(z) = 0$ ;

**negative definite**, if  $Q(x) < 0$  for every  $x \in \mathbb{R}^n, x \neq 0$ ;

**negative semidefinite**, if  $Q(x) \leq 0$  for every  $x \in \mathbb{R}^n$ , but there exists a  $z \in \mathbb{R}^n$  such that  $Q(z) = 0$ ;

**indefinite**, if there exist two vectors  $x, y \in \mathbb{R}^n$  for which  $Q(x) > 0$  but  $Q(y) < 0$ .

# Definiteness of a quadratic form

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## Summary

## Definiteness after completing the square

It is not hard to decide the definiteness of a quadratic form if it is given with its dyad decomposition. Let  $Q$  be an  $n$ -variable non-zero quadratic form.

If the dyad decomposition

- includes positive and negative pivot terms then  $Q$  is an *indefinite* quadratic form.

Now assume the dyad decomposition includes exactly  $n$  complete squares. If

- all pivot terms are positive numbers then  $Q$  is *positive definite*;
- all pivot terms are negative numbers then  $Q$  is *negative definite*.

## Definiteness after completing the square ...

If the number of dyads is less than  $n$ , the same characterization holds true but the quadratic form is semidefinite only:

Thus if

- all pivot terms are positive numbers then  $Q$  is *positive semidefinite*;
- all pivot terms are negative numbers then  $Q$  is *negative semidefinite*.

Thank you for your attention!



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