

# Dealing with the Infinite

# Aristotle

Aristotle introduced an idea which would dominate thinking for two thousand years and is still a persuasive argument to some people today. Aristotle argued against the *actual infinite* and, in its place, he considered the *potential infinite*. His idea was that we can never conceive of the natural numbers as a whole. However they are potentially infinite in the sense that given any finite collection we can always find a larger finite collection.

# Aristotle

Aristotle discussed this in Chapters 4-8 of Book III of *Physics* where he claimed that denying that the actual infinite exists and allowing only the potential infinite would be no hardship to mathematicians:-

*Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untransversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.*

# Aristotle's Influence

This point of view was dominant until the 19<sup>th</sup> Century. Although it was mildly challenged by John Wallis's infinite series, the limit process of the Newton-Liebniz Calculus, and the development of mathematical induction, it remained in full force.

Gauss, in a letter to Schumacher in 1831, argued against the actual infinite:-

*I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a facon de parler, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction.*

# Aristotle's Influence

It was essentially the work of one man, Georg Cantor, which broke this strangle hold of Aristotle.

Cantor argued that Aristotle was making a distinction which was only in his use of words:-

*... in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.*

This then is his story.

# Georg Ferdinand Ludwig Philipp Cantor

Georg Cantor's father, Georg Waldemar Cantor, was a successful merchant, working as a wholesaling agent in St Petersburg, then later as a broker in the St Petersburg Stock Exchange. Georg Waldemar Cantor was born in Denmark and he was a devoutly religious man with a deep love of culture and the arts. Georg's mother, Maria Anna Böhm, was Russian and very musical. Certainly Georg inherited considerable musical and artistic talents from his parents being an outstanding violinist. Georg was brought up a Lutheran, this being the religion of his father, while Georg's mother was a Roman Catholic.

After early education at home from a private tutor, Cantor attended primary school in St Petersburg, then in 1856 when he was eleven years old the family moved to Germany.

# Georg Cantor



Cantor's father had poor health and the move to Germany was to find a warmer climate than the harsh winters of St Petersburg. At first they lived in Wiesbaden, where Cantor attended the Gymnasium, then they moved to Frankfurt. Cantor studied at the Realschule in Darmstadt where he lived as a boarder. He graduated in 1860 with an outstanding report, which mentioned in particular his exceptional skills in mathematics, in particular trigonometry. After attending the Höhere Gewerbeschule in Darmstadt from 1860 he entered the Polytechnic of Zurich in 1862. The reason Cantor's father chose to send him to the Höheren Gewerbeschule was that he wanted Cantor to become:-

*... a shining star in the engineering firmament.*

# Georg Cantor



However, in 1862 Cantor had sought his father's permission to study mathematics at university and he was overjoyed when eventually his father consented. His studies at Zurich, however, were cut short by the death of his father in June 1863. Cantor moved to the University of Berlin where he became friends with Hermann Schwarz who was a fellow student. Cantor attended lectures by Weierstrass, Kummer and Kronecker. He spent the summer term of 1866 at the University of Göttingen, returning to Berlin to complete his dissertation on number theory *De aequationibus secundi gradus indeterminatis* in 1867.



# Georg Cantor



While at Berlin Cantor became much involved with the Mathematical Society being president of the Society during 1864-65. He was also part of a small group of young mathematicians who met weekly in a wine house. After receiving his doctorate in 1867, Cantor taught at a girl's school in Berlin. Then, in 1868, he joined the Schellbach Seminar for mathematics teachers. During this time he worked on his habilitation and, immediately after being appointed to Halle in 1869, he presented his thesis, again on number theory, and received his habilitation.

# Georg Cantor



At Halle the direction of Cantor's research turned away from number theory and towards analysis. This was due to Heine, one of his senior colleagues at Halle, who challenged Cantor to prove the open problem on the uniqueness of representation of a function as a trigonometric series (Fourier series). This was a difficult problem which had been unsuccessfully attacked by many mathematicians, including Heine himself as well as Dirichlet, Lipschitz and Riemann.

# Fourier



Jean Baptiste Fourier (1768 – 1830) was a soldier during the French Revolution and under Napoleon, an assistant lecturer at the École Polytechnique, and a mathematician/scientist who worked on explaining the properties of heat. His fame was cinched by a series of mathematical ideas that were derived by unintentionally making a number of errors that would lead him to formulating ideas and a theorem which would take mathematicians over 150 years to correctly justify. Fourier, in his work on describing the mathematics behind the theory of heat, had concluded that *any function* or graph could be described by a series of trigonometric functions, now called a Fourier series. When Fourier developed his formula for coefficients in the trigonometric series, he was unaware that Euler had already developed it (correctly). But unlike Fourier, Euler believed it only applied to a small class of functions.

# Fixing it up

In 1823, Cauchy attempted to deal with the convergence of Fourier series in general terms. His techniques however were not powerful enough to deal with all cases, even when the question of convergence was incontestable, and in some instances were based on unsound arguments. In 1829, Dirichlet, who was unsatisfied with Cauchy's work, took up the convergence question, anxious to establish the theory of Fourier series with greater rigor and clarity. His results show that convergence is obtained if the function is continuous, and more generally if additional conditions were satisfied at a finite number of singularities. Riemann, in his *Habilitationsschrift* (1854), takes up the question where Dirichlet left off, and extends the number of allowed singularities to an infinite number, provided they form a set of measure zero.

# Cantor's Early Work



Another question that arises concerning Fourier series is that of uniqueness – given an arbitrary function, under what conditions is there only one Fourier series which represents that function?

In an 1870 paper Heine gave some conditions and noted that even the greatest minds, including Dirichlet, Lipschitz, and Riemann, had been unable to solve the uniqueness problem.

With this challenge so neatly articulated by Heine in mind, Cantor set out to show that there was only one way in which such a development could occur, assuming that one existed. He was also determined to do away with as many restrictive assumptions as he could, but this made the problem considerably more difficult.

# Cantor's Early Work



Cantor solved the problem proving uniqueness of the representation by April 1870.

*If a function of a real variable  $f(x)$  is given by a trigonometric series convergent for every value of  $x$ , then there is no other series of the same form which likewise converges for every value of  $x$  and represents the function  $f(x)$ .*

In the proof Cantor uses the requirement that zero be represented by a convergent trigonometric series at each point of the domain. He published further papers between 1870 and 1872 dealing with trigonometric series which extend the theorem by loosening the restrictions. For instance, in a note of 1871, he shows that the theorem remains valid even if there are a **finite** number of points in the domain at which either the convergence of the series or the representation of zero does not hold. In the 1872 paper he further

# Cantor's Early Work



extends this to an **infinite** number of points, provided they are distributed in a certain way (based on an idea due to Hankel). This basic idea could be repeated, giving an infinite number of an infinite number of points ... but Cantor had to find a better way to express which sets of points the exclusion could be applied to. This led him to examine the structure of the real (especially irrational) numbers. He was unhappy with the current state of the theory of irrational numbers which he felt presupposed the existence of the numbers it was trying to define, and so, he developed a theory of irrational numbers without this defect. With his new series definition of irrational numbers, it was an easy matter to describe the exceptional points in his uniqueness theorem.

# Cantor's Early Work

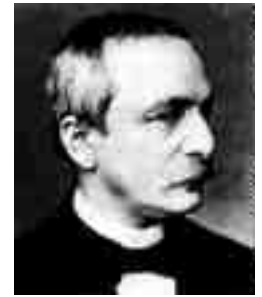


Kronecker, at the University of Berlin, suggested a simplification in Cantor's original proof of the uniqueness theorem, which Cantor appreciated and incorporated several times in later versions.

It is worth noting that Kronecker's interest shows that the two were still on good terms during Cantor's first years at Halle. But as Cantor began to produce remarkable extensions of his uniqueness theorem, Kronecker became increasingly uneasy. As Cantor began to press further, developing ideas which Kronecker thoroughly opposed, Kronecker became Cantor's earliest and most determined critic.



# Kronecker (1832-1891)



**Leopold Kronecker's** parents were well off, his father, Isidor Kronecker, being a successful business man while his mother was Johanna Prausnitzer who also came from a wealthy family. The families were Jewish, the religion that Kronecker kept until a year before his death when he became a convert to Christianity.

Kronecker was taught mathematics at Liegnitz Gymnasium by Kummer, and it was due to Kummer that Kronecker became interested in mathematics. Kummer immediately recognised Kronecker's talent for mathematics and he took him well beyond what would be expected at school, encouraging him to undertake research.

# Kronecker



Kronecker became a student at Berlin University in 1841 and there he studied under Dirichlet and Steiner. After spending the summer of 1843 at the University of Bonn, which he went to because of his interest in astronomy rather than mathematics, he then went to the University of Breslau (following Kummer) for the winter semester of 1843-44. Back in Berlin he worked on his doctoral thesis on algebraic number theory under Dirichlet's supervision. The thesis, *On complex units* was submitted on 30 July 1845 and he took the necessary oral examination on 14 August. Dirichlet commented on the thesis saying that in it Kronecker showed:-

*... unusual penetration, great assiduity, and an exact knowledge of the present state of higher mathematics.*

# Kronecker



However, just as it looked as if he would embark on an academic career, Kronecker left Berlin to deal with family affairs. He helped to manage the banking business of his mother's brother and, in 1848, he married the daughter of this uncle, Fanny Prausnitzer. He also managed a family estate but still found the time to continue working on mathematics, although he did this entirely for his own enjoyment.

Certainly Kronecker did not need to take on paid employment since he was by now a wealthy man. His enjoyment of mathematics meant, however, that when circumstances changed in 1855 and he no longer needed to live on the estate outside Liegnitz, he returned to Berlin. He did not wish a university post, rather he wanted to take part in the mathematical life of the university and undertake research interacting with the other mathematicians.

# Kronecker



Of course since Kronecker did not hold a university appointment, he did not lecture at this time but was remarkably active in research publishing a large number of works in quick succession. These were on number theory, elliptic functions and algebra, but, more importantly, he explored the interconnections between these topics. Kummer proposed Kronecker for election to the Berlin Academy in 1860, and the proposal was seconded by Borchardt and Weierstrass. On 23 January 1861 Kronecker was elected to the Academy and this had a surprising benefit.

Members of the Berlin Academy had a right to lecture at Berlin University. Although Kronecker was not employed by the University, or any other organization for that matter, Kummer suggested that Kronecker exercise his right to lecture at the University and this he did beginning in October 1862.

# Kronecker



For the best students his lectures were demanding but stimulating. However, he was not a popular teacher with the average students :-

*Kronecker did not attract great numbers of students. Only a few of his auditors were able to follow the flights of his thought, and only a few persevered until the end of the semester.*

Berlin was attractive to Kronecker, so much so that when he was offered the chair of mathematics in Göttingen in 1868, he declined.

# Kronecker



Kronecker's primary contributions were in the theory of equations and higher algebra, with his major contributions in elliptic functions, the theory of algebraic equations, and the theory of algebraic numbers. However the topics he studied were restricted by the fact that he believed in the reduction of all mathematics to arguments involving only the integers and a finite number of steps. Kronecker is well known for his remark:-

*God created the integers, all else is the work of man.*

Kronecker believed that mathematics should deal only with finite numbers and with a finite number of operations. He was the first to doubt the significance of non-constructive existence proofs. It appears that, from the early 1870s, Kronecker was opposed to the use of irrational numbers, upper and lower limits, and the Bolzano-Weierstrass theorem, because of their non-constructive nature. Another consequence of his philosophy of mathematics was that to Kronecker transcendental numbers could not exist.

# Kronecker



In 1870 Heine published a paper *On trigonometric series* in *Crelle's Journal*, but Kronecker had tried to persuade Heine to withdraw the paper. Again in 1877 Kronecker tried to prevent publication of Cantor's work in *Crelle's Journal*, not because of any personal feelings against Cantor (which has been suggested by some biographers of Cantor) but rather because Kronecker believed that Cantor's paper was meaningless, since it proved results about mathematical objects which Kronecker believed did not exist. Kronecker was on the editorial staff of *Crelle's Journal* which is why he had a particularly strong influence on what was published in that journal. After Borchardt died in 1880, Kronecker took over control of *Crelle's Journal* as the editor and his influence on which papers would be published increased.

# Kronecker



Although Kronecker's view of mathematics was well known to his colleagues throughout the 1870s and 1880s, it was not until 1886 that he made these views public. In that year he argued against the theory of irrational numbers used by Dedekind, Cantor and Heine giving the arguments by which he opposed:-

*... the introduction of various concepts by the help of which it has frequently been attempted in recent times (but first by Heine) to conceive and establish the "irrationals" in general. Even the concept of an infinite series, for example one which increases according to definite powers of variables, is in my opinion only permissible with the reservation that in every special case, on the basis of the arithmetic laws of constructing terms (or coefficients), ... certain assumptions must be shown to hold which are applicable to the series like finite expressions, and which thus make the extension beyond the concept of a finite series really unnecessary.*



# Kronecker



Lindemann had proved that  $\pi$  is transcendental in 1882, and in a lecture given in 1886 Kronecker complimented Lindemann on a beautiful proof but, he claimed, one that proved nothing since transcendental numbers did not exist. So Kronecker was consistent in his arguments and his beliefs, but many mathematicians, proud of their hard earned results, felt that Kronecker was attempting to change the course of mathematics and write their line of research out of future developments. Kronecker explained his programme based on studying only mathematical objects which could be constructed with a finite number of operations from the integers in *Über den Zahlbegriff* in 1887.

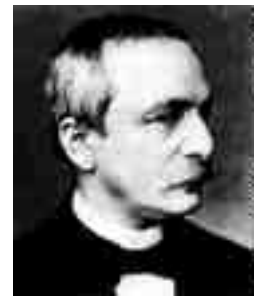
# Kronecker



Another feature of Kronecker's personality was that he tended to fall out personally with those who he disagreed with mathematically. Of course, given his belief that only finitely constructible mathematical objects existed, he was completely opposed to Cantor's developing ideas in set theory. Not only Dedekind, Heine and Cantor's mathematics was unacceptable to this way of thinking, and Weierstrass also came to feel that Kronecker was trying to convince the next generation of mathematicians that Weierstrass's work on analysis was of no value.

Kronecker had no official position at Berlin until Kummer retired in 1883 when he was appointed to the chair. But by 1888 Weierstrass felt that he could no longer work with Kronecker in Berlin and decided to go to Switzerland, but then, realizing that Kronecker would be in a strong position to influence the choice of his successor, he decided to remain in Berlin.

# Kronecker



Kronecker was of very small stature and extremely self-conscious about his height. An example of how Kronecker reacted occurred in 1885 when Schwarz sent him a greeting which included the sentence:-

*He who does not honour the Smaller, is not worthy of the Greater.*

Here Schwarz was joking about the small man Kronecker and the large man Weierstrass. Kronecker did not see the funny side of the comment, however, and never had any further dealings with Schwarz (who was Weierstrass's student and Kummer's son-in-law). Others however displayed more tact and, for example, Helmholtz who was a professor in Berlin from 1871, managed to stay on good terms with Kronecker.

# Kronecker



We should not think that Kronecker's views of mathematics were totally eccentric. Although it was true that most mathematicians of his day would not agree with those views, and indeed most mathematicians today would not agree with them, they were not put aside. Kronecker's ideas were further developed by Poincaré and Brouwer, who placed particular emphasis upon intuition. Intuitionism stresses that mathematics has priority over logic, the objects of mathematics are constructed and operated upon in the mind by the mathematician, and it is impossible to define the properties of mathematical objects simply by establishing a number of axioms.

# Back to Cantor



Cantor was promoted to Extraordinary Professor at Halle in 1872 and in that year he began corresponding with Dedekind who he would meet personally while on holiday in Interlaken, Switzerland in 1874.

Dedekind published his definition of the real numbers by "Dedekind cuts" also in 1872 and in this paper Dedekind refers to Cantor's 1872 paper, which Cantor had sent him, in which he defined irrational numbers in terms of convergent sequences of rational numbers.

Much of what we know about Cantor's thinking at this period comes from the frequent correspondence between these two.

# Dedekind (1831-1916)



**Richard Dedekind's** father was a professor at the Collegium Carolinum in Brunswick. His mother was the daughter of a professor who also worked at the Collegium Carolinum. Richard was the youngest of four children and never married. He was to live with one of his sisters, who also remained unmarried, for most of his adult life.

He entered the University of Göttingen in the spring of 1850 with a solid grounding in mathematics. Göttingen was a rather disappointing place to study mathematics at this time, and it had not yet become the vigorous research center that it turned into soon afterwards. Mathematics was directed by M A Stern and G Ulrich. Gauss also taught courses in mathematics, but mostly at an elementary level.

# Dedekind



The first course to really make Dedekind enthusiastic was, rather surprisingly, a course on experimental physics taught by Weber. More likely it was Weber who inspired Dedekind rather than the topic of the course.

In the autumn term of 1850, Dedekind attended his first course given by Gauss. It was a course on least squares and :-

*... fifty years later Dedekind remembered the lectures as the most beautiful he had ever heard, writing that he had followed Gauss with constantly increasing interest and that he could not forget the experience.*

He received his doctorate from Göttingen in 1852 and he was to be the last pupil of Gauss. However he was not well trained in advanced mathematics and fully realized the deficiencies in his mathematical education.

# Dedekind



At this time Berlin was the place where courses were given on the latest mathematical developments but Dedekind had not been able to learn such material at Göttingen. By this time Riemann was also at Göttingen and he too found that the mathematical education was aimed at students who were intending to become secondary school teachers, not those with the very top abilities who would go on to research careers. Dedekind therefore spent the two years following the award of his doctorate learning the latest mathematical developments and working for his habilitation.

In 1854 both Riemann and Dedekind were awarded their habilitation degrees within a few weeks of each other. Dedekind was then qualified as a university teacher and he began teaching at Göttingen giving courses on probability and geometry.



# Dedekind



Gauss died in 1855 and Dirichlet was appointed to fill the vacant chair at Göttingen. Dedekind attended courses by Dirichlet on the theory of numbers, on potential theory, on definite integrals, and on partial differential equations. They soon became close friends and the relationship was in many ways the making of Dedekind, whose mathematical interests took a new lease of life with the discussions between the two. Bachmann, who was a student in Göttingen at this time :- *... recalled in later years that he only knew Dedekind by sight because Dedekind always arrived and left with Dirichlet and was completely eclipsed by him.*

Dedekind wrote in a letter in July 1856 :- *What is most useful to me is the almost daily association with Dirichlet, with whom I am for the first time beginning to learn properly; he is always completely amiable towards me, and he tells me without beating about the bush what gaps I need to fill and at the same time he gives me the instructions and the means to do it. I thank him already for infinitely many things, and no doubt there will be many more.*

# Dedekind



While at Göttingen, Dedekind applied for J L Raabe's chair at the Polytechnikum in Zürich. Dirichlet supported his application writing that Dedekind was 'an exceptional pedagogue'. In the spring of 1858 the Swiss councillor who made appointments came to Göttingen and Dedekind was quickly chosen for the post. Dedekind was appointed to the Polytechnikum in Zürich and began teaching there in the autumn of 1858.

In fact it was while he was thinking how to teach differential and integral calculus, the first time that he had taught the topic, that the idea of a Dedekind cut came to him. He recounts that the idea came to him on 24 November 1858. His idea was that every real number  $r$  divides the rational numbers into two subsets, namely those greater than  $r$  and those less than  $r$ . Dedekind's brilliant idea was to represent the real numbers by such divisions of the rationals.

# Dedekind



Dedekind and Riemann travelled together to Berlin in September 1859 on the occasion of Riemann's election to the Berlin Academy of Sciences. In Berlin, Dedekind met Weierstrass, Kummer, Borchardt and Kronecker.

The Collegium Carolinum in Brunswick had been upgraded to the Brunswick Polytechnikum by the 1860s, and Dedekind was appointed to the Polytechnikum in 1862. With this appointment he returned to his home town and even to his old educational establishment where his father had been one of the senior administrators for many years. Dedekind remained there for the rest of his life, retiring on 1 April 1894. After he retired, Dedekind continued to teach the occasional course and remained in good health in his long retirement.

# Dedekind



Dedekind made a number of highly significant contributions to mathematics and his work would change the style of mathematics into what is familiar to us today. One remarkable piece of work was his redefinition of irrational numbers in terms of Dedekind cuts which, as we mentioned above, first came to him as early as 1858. He published this in *Stetigkeit und Irrationale Zahlen* in 1872. In it he wrote:-

*Now, in each case when there is a cut  $(A_1, A_2)$  which is not produced by any rational number, then we create a new, irrational number  $a$ , which we regard as completely defined by this cut; we will say that this number  $a$  corresponds to this cut, or that it produces this cut.*

As well as his analysis of the nature of number, his work on mathematical induction, including the definition of finite and infinite sets, and his work in number theory, particularly in algebraic number fields, is of major importance.

# Dedekind



Dedekind loved to take his holidays in Switzerland, the Austrian Tyrol or the Black Forest in southern Germany. On one such holiday in 1874 he met Cantor while staying in the beautiful city of Interlaken and the two discussed set theory. Dedekind was sympathetic to Cantor's set theory as is illustrated by this quote from *Was sind und was sollen die Zahlen* (1888) regarding determining whether a given element belongs to a given set :-

*In what way the determination comes about, or whether we know a way to decide it, is a matter of no consequence in what follows. The general laws that are to be developed do not depend on this at all.*

In this quote Dedekind is arguing against Kronecker's objections to the infinite and, therefore, is agreeing with Cantor's views.

# Dedekind



Dedekind's brilliance consisted not only of the theorems and concepts that he studied but, because of his ability to formulate and express his ideas so clearly, he introduced a new style of mathematics that been a major influence on mathematicians ever since. As Edwards writes :-

*Dedekind's legacy ... consisted not only of important theorems, examples, and concepts, but a whole style of mathematics that has been an inspiration to each succeeding generation.*

# Cantor

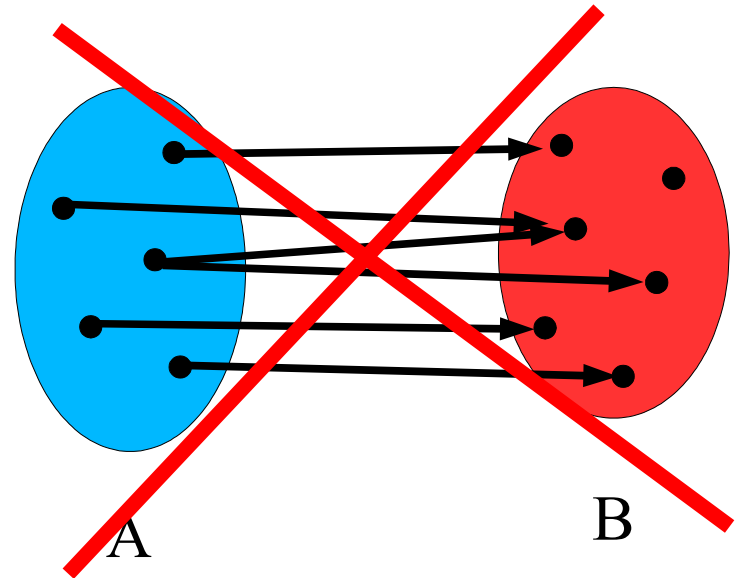
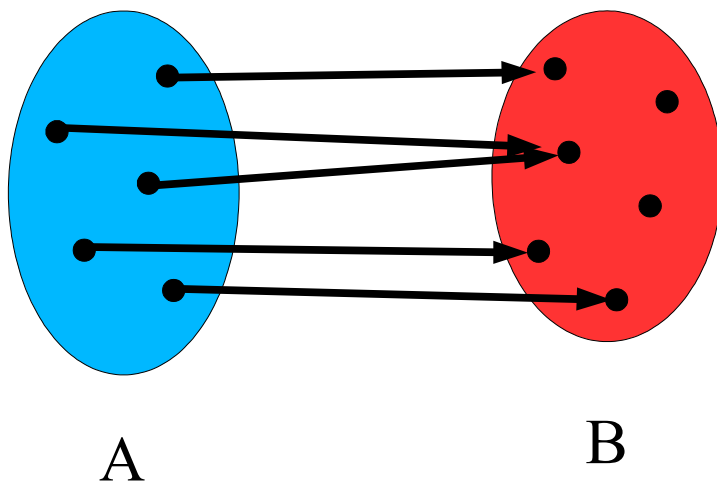


In 1873 Cantor proved the rational numbers *countable*, i.e. they may be placed in one-one correspondence with the natural numbers. He also showed that the *algebraic numbers*, i.e. the numbers which are roots of polynomial equations with integer coefficients, were countable. However his attempts to decide whether the real numbers were countable proved harder. He had proved that the real numbers were not countable by December 1873 and published this in a paper in 1874. It is in this paper that the idea of a one-one correspondence appears for the first time, but it is only implicit in this work.

A *transcendental number* is an irrational number that is not a root of any polynomial equation with integer coefficients. Liouville established in 1851 that transcendental numbers exist. Twenty years later, in this 1874 work, Cantor showed that in a certain sense 'almost all' numbers are transcendental by proving that the real numbers were not countable while he had proved that the algebraic numbers were countable.

# A little mathematical background

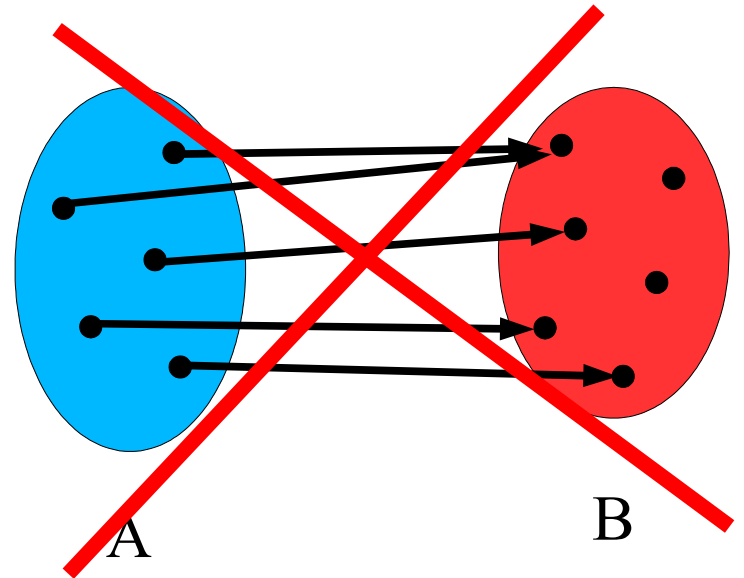
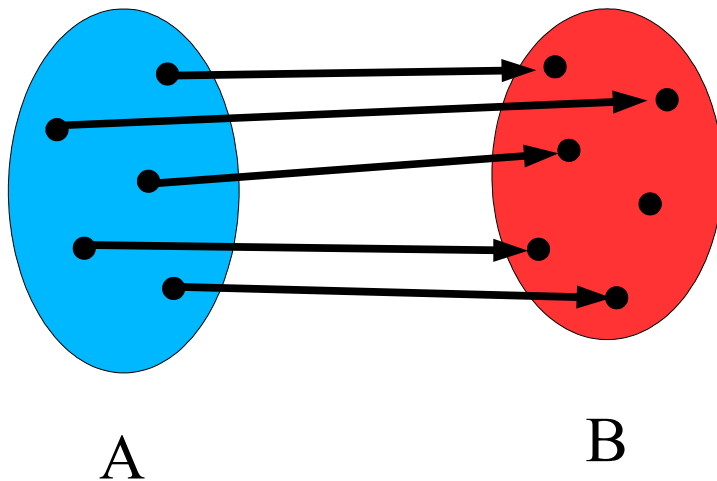
A **function**  $f$  from a set  $A$  (the *domain*) to a set  $B$  (the *codomain*) is a set of ordered pairs  $(a,b)$  with the property that every element of the domain is the first coordinate of one and only one ordered pair. In the diagram below we indicate the ordered pairs by arrows starting in set  $A$  and ending in set  $B$  (the function is the collection of all arrows).





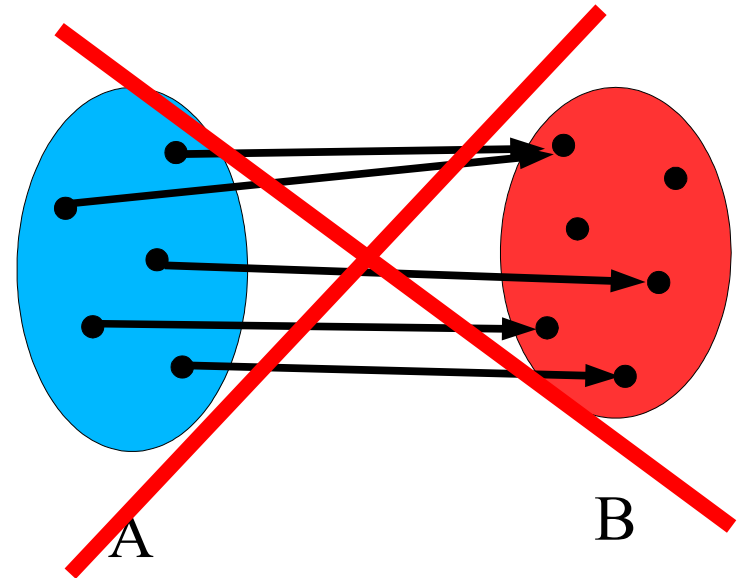
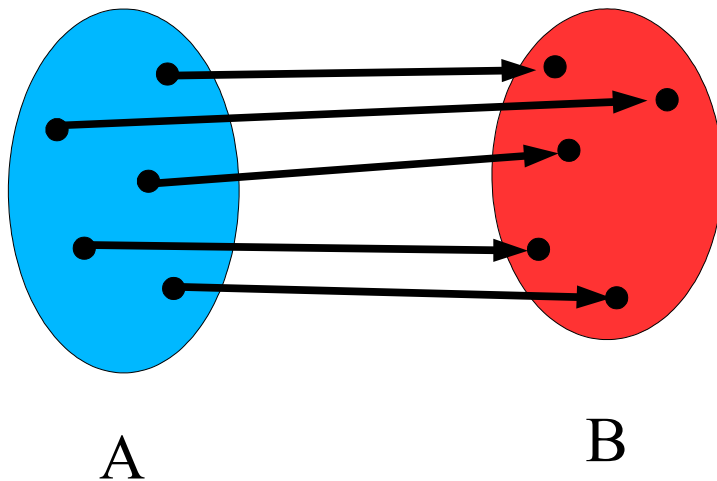
# A little mathematical background

A function  $f$  from a set  $A$  (the *domain*) to a set  $B$  (the *codomain*) is *one-to-one* (an *injection*) if no two elements of the domain go to the same element of the codomain.



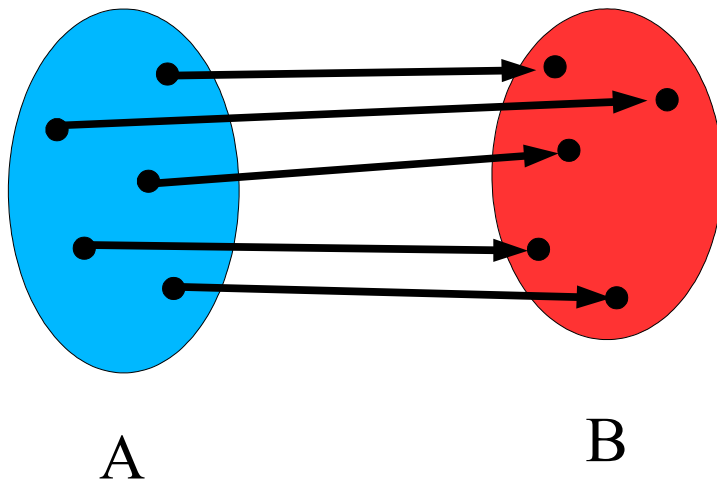
# A little mathematical background

A function  $f$  from a set  $A$  (the *domain*) to a set  $B$  (the *codomain*) is *onto* (a *surjection*) if every element of the codomain is the endpoint of an arrow (is the second coordinate of at least one ordered pair).



# A little mathematical background

A function  $f$  from a set  $A$  (the *domain*) to a set  $B$  (the *codomain*) which is both one-to-one and onto is called a ***one-to-one correspondence*** (a **bijection**). Note that the composition of two bijections is also a bijection.



# Equipotence

If there exists a bijection between two sets  $A$  and  $B$ , we say that  $A$  and  $B$  are *equivalent*. Cantor would have said that the two sets have the same **power** (or potency – the German word **Mächtigkeit**, can be translated either way) or that the sets were **equipotent**.

Equivalent finite sets clearly have the same number of elements, but for infinite sets this concept (**having the same number of elements or having the same size**) is not really clear and was the subject of Cantor's work ... so he wished to use a different word to describe the relation which did not carry the same baggage as the word “size”.

# Equipotence

**Theorem:** The set of natural numbers is equivalent to the set of even natural numbers.

*Pf:* We need to produce a bijection between these sets. One very simple one is given by  $f(n) = 2n$ . That is:

$$1 \leftrightarrow 2$$

$$2 \leftrightarrow 4$$

$$3 \leftrightarrow 6$$

$$4 \leftrightarrow 8 \quad \text{etc.}$$

This function is clearly one-to-one and onto the even integers, so it is a bijection.

# Countable

A set which is equivalent to the set of natural numbers is said to be *denumerable*.

A set which is either *denumerable* or *finite* is called *countable*.

A set which is not countable is called *uncountable*.

Are there any uncountable sets?

Cantor was the first to even ask such a question – and answer it. The real numbers are an uncountable set, in fact, just the real numbers in the open interval  $(0,1)$  form an uncountable set. This remarkable theorem has an equally remarkable proof.

# Cantor's Diagonalization Argument

We will consider the real numbers in the interval  $(0,1)$  written as decimals. There are numbers which have two decimal representations, such as  $0.5000\dots$  and  $0.49999\dots$  and we shall agree to write these in the first form (without the repeating 9's).

Suppose that there exists a bijection from the natural numbers onto this set of reals. We will derive a contradiction. Using the “hypothetical” bijection, we can write the real number that 1 is associated to first, and the real number that 2 is associated with directly under this, and so on. We will have created a list (infinite) which has all the real numbers in the interval in it because the function is onto. This list might look like :

# Cantor's Diagonalization Argument

1  $\leftrightarrow$  0. 1 2 4 6 0 9 8 3 7 5 9 9 7 6 ...  
2  $\leftrightarrow$  0. 3 4 7 9 3 3 2 1 0 0 0 3 9 2 ...  
3  $\leftrightarrow$  0. 5 5 5 5 5 5 8 9 9 9 2 2 1 0 ...  
4  $\leftrightarrow$  0. 8 9 2 0 0 1 3 4 4 4 4 6 8 9 ...  
⋮

Now we can construct a real number in the interval which can not appear on the list ... thus contradicting the assumption that they are all present. This number is constructed by going down the diagonal of the list, and picking a digit other than 0, 9 or the digit on the diagonal.

$X = 0 . \text{ } \textcircled{3} \textcircled{7} \textcircled{6} \textcircled{2} \dots$



# Real Algebraic Numbers

An *algebraic number* is a real number which is a solution of a polynomial equation with integer coefficients. A real number which is not algebraic is called *transcendental*.

The rational numbers are all algebraic, since the rational number  $r/s$  ( $r, s$  integers, and  $s \neq 0$ ) satisfies the equation  $sx - r = 0$ . But there are also irrational numbers which are algebraic, for instance,

$\sqrt[3]{5}$  satisfies the equation  $x^3 - 5 = 0$ , and so is algebraic.

Cantor finds a bijection from the set of natural numbers to the set of all algebraic numbers, thus showing that this set is denumerable, and hence countable. Since the reals are uncountable and the algebraic reals are countable ... there must be lots of transcendentals!!

# Real Algebraic Numbers

Cantor reasons as follows: Every algebraic number satisfies a polynomial of smallest degree with integer coefficients which can not be factored (is **irreducible**) and has a positive leading coefficient, with no integer other than 1 dividing all the coefficients. Also, any polynomial of degree  $n$  has at most  $n$  solutions.

If there is a way to list out all of these polynomials and their roots (i.e., assign to each root of any such polynomial a unique integer – its position in the list) then the set will be shown to be denumerable.

Cantor defines the **height** of a polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n, \text{ to be}$$
$$H = n - 1 + |a_0| + |a_1| + |a_2| + \dots + |a_n|.$$

# Real Algebraic Numbers

Thus, the height of  $5x - 2$  is 7 and the height of  $x^3 + 5$  is also 7, but the height of  $x^2 - 10x + 12$  is 23. Because the height is larger than the degree and any of the coefficients, there can only be a finite number of polynomials with integer coefficients of a given height.

Since each of these has only a finite number of solutions, we get only a finite number of algebraic numbers associated with polynomials of a given height. These we arrange in increasing order, and put them all on a list arranged by increasing height.

This would go like this: There only irreducible polynomial of height 1 is  $x$ , so our list will start with 0. For height 2, we can only get  $x + 1$  and  $x - 1$ , so the algebraic numbers at this height are -1 and 1. For height 3, any irreducible quadratic must have a non-zero constant, so the only possibility is  $x^2 + 1$  which has no real root. The linear ones are  $x + 2$ ,  $x - 2$ ,  $2x + 1$  and  $2x - 1$ . Thus, for height 3 we get the algebraic numbers -2,  $-\frac{1}{2}$ ,  $\frac{1}{2}$ , 2 in order. And so, on ...

# 1874

Cantor titled his paper, “*On a Property of the Collection of All Real Algebraic Numbers*” and it was published in the *Journal für die reine und angewandte Mathematik* (a.k.a. Crelle's Journal).

The most significant theorem in the paper is clearly the uncountability of the reals. Why then is there no mention of this result in the title of the paper?

Cantor chose his title carefully ... it doesn't hint at the very provocative theorem contained in it, one that Kronecker would heartiedly disapprove of. So Cantor picked a title that would likely get by the censors who didn't look at the paper too carefully!!