... The Sequel

a.k.a. Gödel's Girdle

Formal Systems

A ***Formal System*** for a mathematical theory consists of:

1. A complete list of the symbols to be used.
2. Rules of syntax
   The rules that determine properly formed statements.
3. Axioms
   Those statements whose truth is assumed.
4. Production Rules
   The legal steps in proofs. The laws of logic.

Principia Mathematica

Russell and Whitehead's monumental *Principia Mathematica* of 1910 is an attempt to recast set theory as a formal system.

$*54{\cdot}42.\quad \vdash :: \alpha \in 2 . \supset :. \beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv . \beta \in \iota \text{“} \alpha$

Dem.

$\vdash . *54{\cdot}4 . \quad \supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y . \supset :.$

$\qquad \beta \subset \alpha . \exists ! \beta . \equiv : \beta = \Lambda . \vee . \beta = \iota\text{‘}x . \vee . \beta = \iota\text{‘}y . \vee . \beta = \alpha : \exists ! \beta :$

$[*24{\cdot}53{\cdot}56 . *51{\cdot}161] \qquad \equiv : \beta = \iota\text{‘}x . \vee . \beta = \iota\text{‘}y . \vee . \beta = \alpha \qquad (1)$

$\vdash . *54{\cdot}25 . \mathrm{Transp} . *52{\cdot}22 . \supset \vdash : x \neq y . \supset . \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}x . \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}y :$

$[*13{\cdot}12] \quad \supset \vdash : \alpha = \iota\text{‘}x \cup \iota\text{‘}y . x \neq y . \supset . \alpha \neq \iota\text{‘}x . \alpha \neq \iota\text{‘}y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y . x \neq y . \supset :.$

$\qquad \beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv : \beta = \iota\text{‘}x . \vee . \beta = \iota\text{‘}y :$

$[*51{\cdot}235] \qquad \equiv : (\exists z) . z \in \alpha . \beta = \iota\text{‘}z :$

$[*37{\cdot}6] \qquad \equiv : \beta \in \iota \text{“} \alpha \qquad (3)$

$\vdash . (3) . *11{\cdot}11{\cdot}35 . *54{\cdot}101 . \supset \vdash . \mathrm{Prop}$

$*54{\cdot}43.\quad \vdash :. \alpha , \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\qquad \vdash :. (\exists x , y) . \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \mathrm{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that 1 + 1 = 2.

After 362 pages, the proposition that 1 + 1 = 2 is established.

Consistency

The goal of using formal systems is to ensure consistency ...
it never occurs that a proof exists for a statement and also for the negation of that statement.

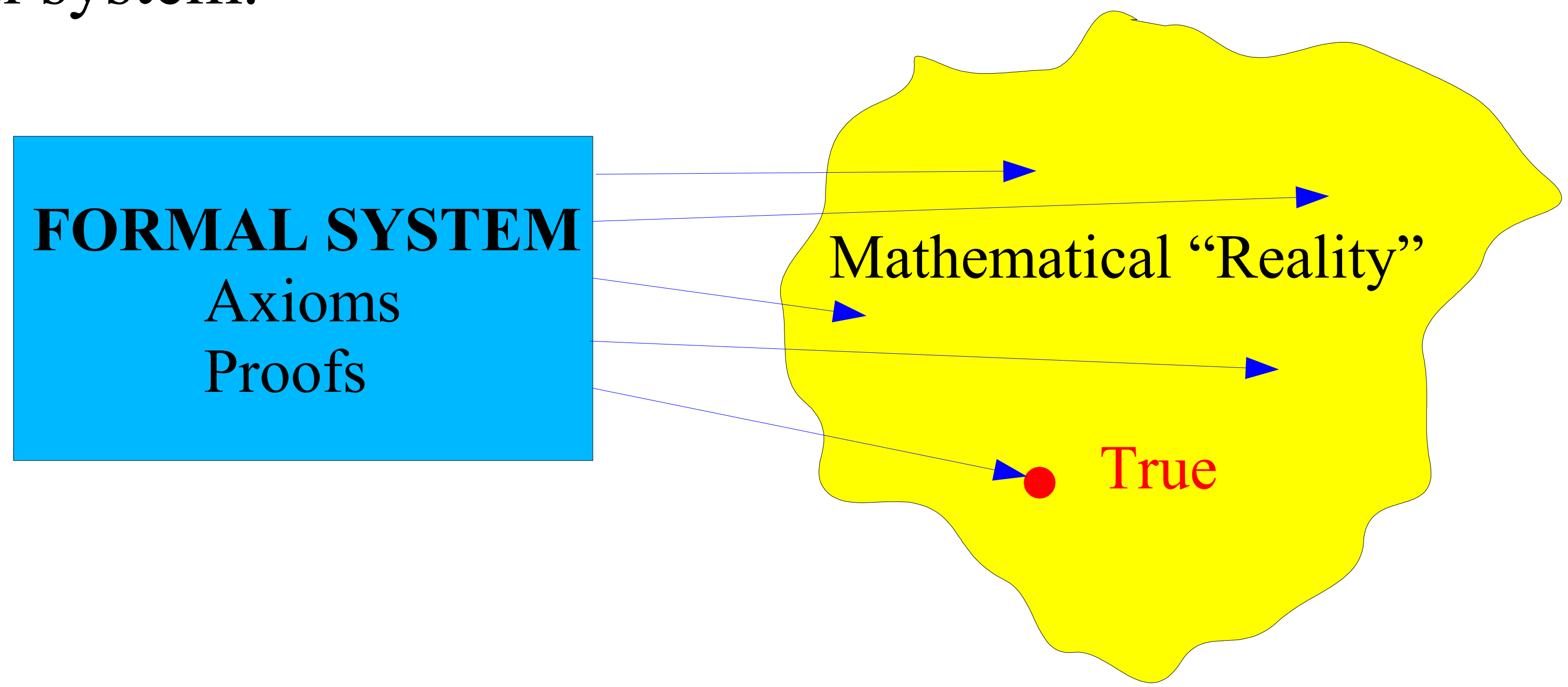
If a system is not consistent then all statements in that system are true!

This follows from the tautology $P \Rightarrow (\sim P \Rightarrow Q)$

If P and ~P are both true, then Q must be true ....
but Q is arbitrary – every statement is true!!

Completeness

A formal system (axiomatization) for a mathematical subject is ***complete*** if every true statement in the subject has a proof in the formal system.



You must be a Platonist to understand this definition.

The Richard Paradox

Due to Jules Richard (1905).
Consider any property that a natural number can have written as an English statement. Order these statements by length, and lexicographically when they have the same length. This produces a well ordered list of properties of the natural numbers, and sets up an association between the natural numbers and the properties of natural numbers (i.e., a property is associated with its position on this list.)

#	property
⋮	⋮
13	is a prime
.	.
15	is a square
⋮	⋮

The Richard Paradox

A number is ***Richardian*** iff it does not have the property associated with it.

#	property
⋮	⋮
13	is a prime
.	.
15	is a square
⋮	⋮
n	is Richardian
⋮	⋮

In our example, 15 is Richardian, but 13 is not.

What about n?

Gödel Numbering

Every symbol, statement and proof of the formal system gets its own unique Gödel number.

Each symbol of the system is assigned an arbitrary natural number (in our example, we will work with 10 symbols):

Gödel Number	Symbol	Meaning
1	$\sim$	not
2	$\vee$	or
3	$\Rightarrow$	implication (if .. then..)
4	$\exists$	there exists
5	$=$	equals
6	0	zero
7	s	immediate successor
8	(	punctuation mark
9	)	punctuation mark
10	,	punctuation mark

Gödel Numbering

Statements are assigned numbers as in the following example:

Statement	$0 = 0$
Symbol numbers	6 5 6
Statement number	$2^6 \times 3^5 \times 5^6$
	$= 243{,}000{,}000$

Proofs are similarly assigned numbers:

$$\begin{aligned} \text{statement}_1 &= n_1 \\ \text{statement}_2 &= n_2 \\ &\vdots \\ \text{statement}_k &= n_k \end{aligned}$$

Proof number $= 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times \ldots \times p_k^{\,n_k}$

Gödel Numbering

Key idea!!!

The meta-mathematical statement :

statement y is the last statement in proof x

is an ARITHMETICAL relationship between x and y, namely

$$x = 2^{n_1} \times 3^{n_2} \times 5^{n_3} \times \ldots \times p_k^{\,y}$$

Thus, this meta-mathematical statement can be formally expressed within the system (provided the system is complex enough to permit statements about numbers.)

Some Mechanics

Let us abbreviate this arithmetical relationship by

Dem (x,y)

and note that for any x and y this is a statement in the formal system, and so, it will have its own Gödel number.

On the other hand, speaking meta-mathematically, this arithmetic statement can be interpreted as saying that the statement with Gödel number y is proved by the proof with Gödel number x (since the last statement of a proof is the statement that you are trying to prove.)

Some Mechanics

The Gödel substitution function is:

sub (a, y, b)

and is the Gödel number of the statement you obtain by:-

in the statement with Gödel number a, replace each occurrence of the variable with Gödel number y by the number b.

The Gödel Statement

Suppose that the variable y has Gödel number 13 and consider the arithmetic statement with Gödel number n :-

$$n : (\forall x) \sim \mathrm{Dem}(x, \mathrm{sub}(y,13,y)).$$

If we replace y by n we get the *Gödel statement*

$$G : (\forall x) \sim \mathrm{Dem}(x, \mathrm{sub}(n,13,n))$$

i.e., $G : (\forall x) \sim \mathrm{Dem}(x, G).$

Meta-mathematically this statement says that it itself does not have a proof!

G: There is no proof of G

If there is a proof of G, then G has no proof. (G $\Rightarrow$ ~G)

If there is no proof of G, then this proves G. (~G $\Rightarrow$ G)

Now, if the formal system is *consistent* it can not lead to this contradiction ... so the statement G can neither be proved nor disproved, i.e., G is an ***undecidable*** statement in the system.

(*For you Platonists out there only*) The formal system must be *incomplete* since G is obviously a true statement that has no proof.

G: There is no proof of G

Gödel takes this one step further :-

$A : (\forall x) \sim (x \wedge \sim x)$ is the statement that the formal system is consistent.

Gödel proves that $A \Rightarrow G$ (if A is provable then so is G)

and since G is undecidable this means that

A can not be proved in the formal system!!

- A formal system complex enough to talk about numbers can not prove its own consistency.

Aftermath

But all is not lost ...
there are meta-mathematical proofs that arithmetic is consistent.

Gerhard Gentzen supplied one in 1936, but it is not finitistic – in fact it uses transfinite induction – so it would not satisfy the requirements of Hilbert's program.