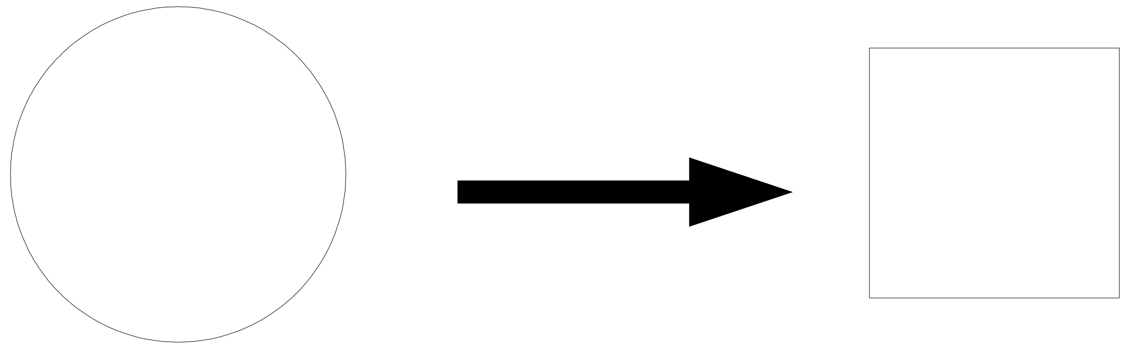
# Squaring the Circle

## A Case Study in the History of Mathematics

# The Problem

Using only a compass and straightedge, construct for any given circle, a square with the same area as the circle.



The general problem of constructing a square with the same area as a given figure is known as the ***Quadrature*** of that figure. So, we seek a ***quadrature of the circle***.

# The Answer

It has been known since 1822 that the quadrature of a circle with straightedge and compass is **impossible**.

*Notes*: First of all we are ***not*** saying that a square of equal area does not exist. If the circle has area A, then a square with side $\sqrt{A}$ clearly has the same area.

Secondly, we are ***not*** saying that a quadrature of a circle is impossible, since it is possible, **but** not under the restriction of using only a straightedge and compass.

# These Notes

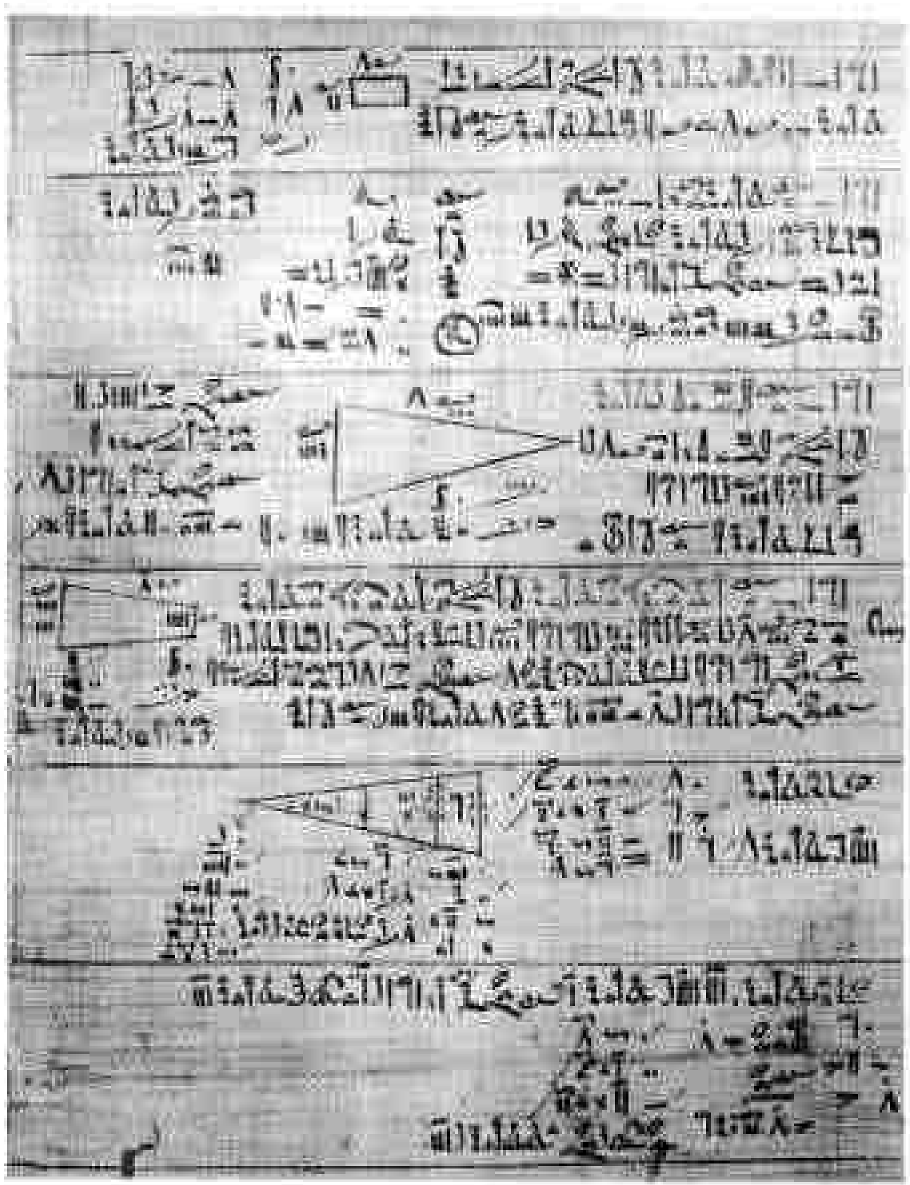
*In this set of notes I will leisurely trace the history of this problem. By "leisurely" I mean that we will take many detours and examine side issues that are of interest (to me at least), both mathematical and historical, as well as sociological and philosophical.*

# Precursors

It has been written, in many places, that the quadrature problem appears in one of the earliest extant mathematical sources, the Rhind Papyrus (~ 1650 B.C.).

This is not really an accurate statement. If one means by the "quadrature of the circle" simply a quadrature *by any means*, then one is just asking for the determination of the area of a circle. This problem *does* appear in the Rhind Papyrus, but I consider it as just a ***precursor*** to the construction problem we are examining.

# The Rhind Papyrus



The papyrus was found in Thebes (Luxor) in the ruins of a small building near the Ramesseum.[1] It was purchased in 1858 in Egypt by the Scottish Egyptologist A. Henry Rhind and acquired by the British Museum after his death.

The papyrus, written in *hieratic*, the cursive form of hieroglyphics, is a single roll which was originally about 5.4 meters long by 32 cms wide (~18 feet by 13 inches).

# The Rhind Papyrus

However, when the British Museum acquired it, it was shorter and in two pieces, missing the central portion. About 4 years after Rhind made his purchase, the American Egyptologist Edwin Smith bought, in Egypt, what he thought was a medical papyrus. This was given to the New York Historical Society in 1932, where it was discovered that beneath a fraudulent covering lay the missing piece of the Rhind Paprus.

The Society then gave the scroll to the British Museum.[2]

# The Ahmes (Rhind) Papyrus

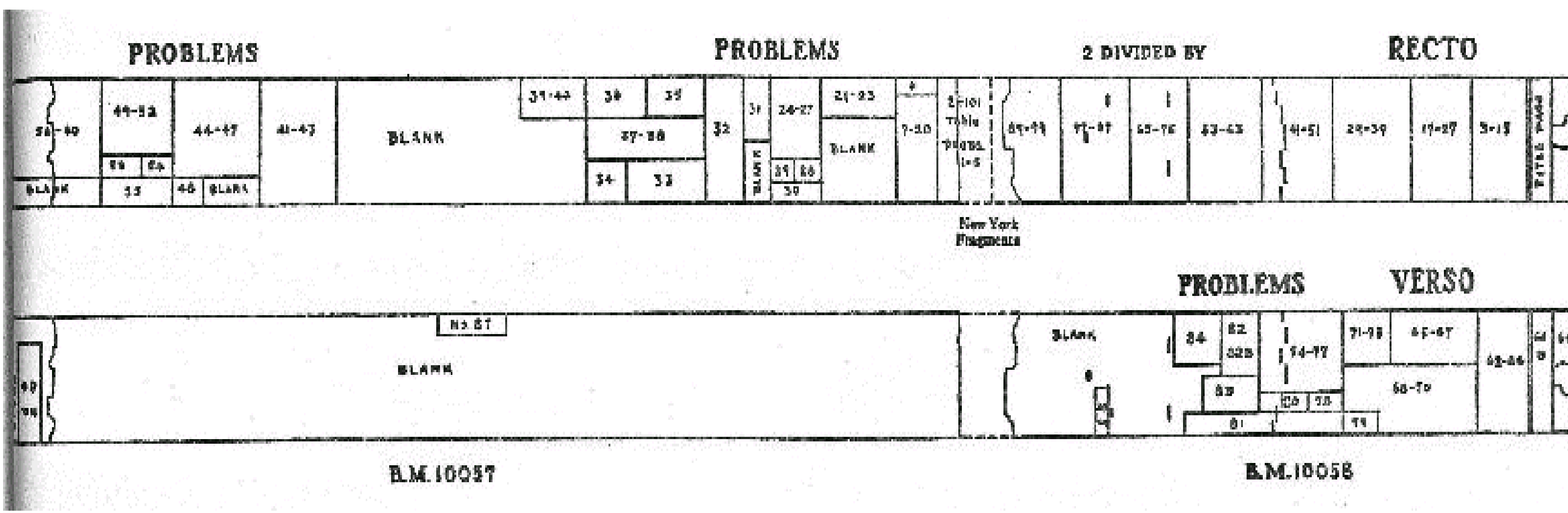
The Rhind Papyrus has been dated to about 1650 B.C. and there is only one older mathematical papyrus, the Moscow Papyrus dated 1850 B.C.

The papyrus was written by an Egyptian scribe A'h-mosè, commonly called Ahmes by modern writers. It appears to be a copy of an older work.

> *" Accurate reckoning of entering into things, knowledge of existing things all, mysteries, secrets all. Now was copied book this in year 33, month four of the inundation season, [under the majesty] of the king of [Upper and] Lower Egypt, 'A-user-Rê'* [15th Dynasty reign of the Hyksos Pharaoh, Apepi I]*, endowed with life, in likeness to writings of old made in the time of the king of Upper [and Lower] Egypt, Ne-ma'et-Rê'* [Amenemhet III (1842 - 1797 B.C.)]. *Lo the scribe A'h-mosè writes copy this."* [3]

# The Ahmes (Rhind) Papyrus

The scroll consists of 87 problems (with solutions) and is a rich primary source of ancient Egyptian mathematics, describing the Egyptian methods of multiplying and dividing, the Egyptian use of unit fractions, their employment of false position, and many applications of mathematics to practical problems.



# The Ahmes (Rhind) Papyrus

Our concern is with problem 50 which reads:

***A circular field has diameter 9 khet. What is its area?***

(A ***khet*** is a length measurement of about 50 meters.)

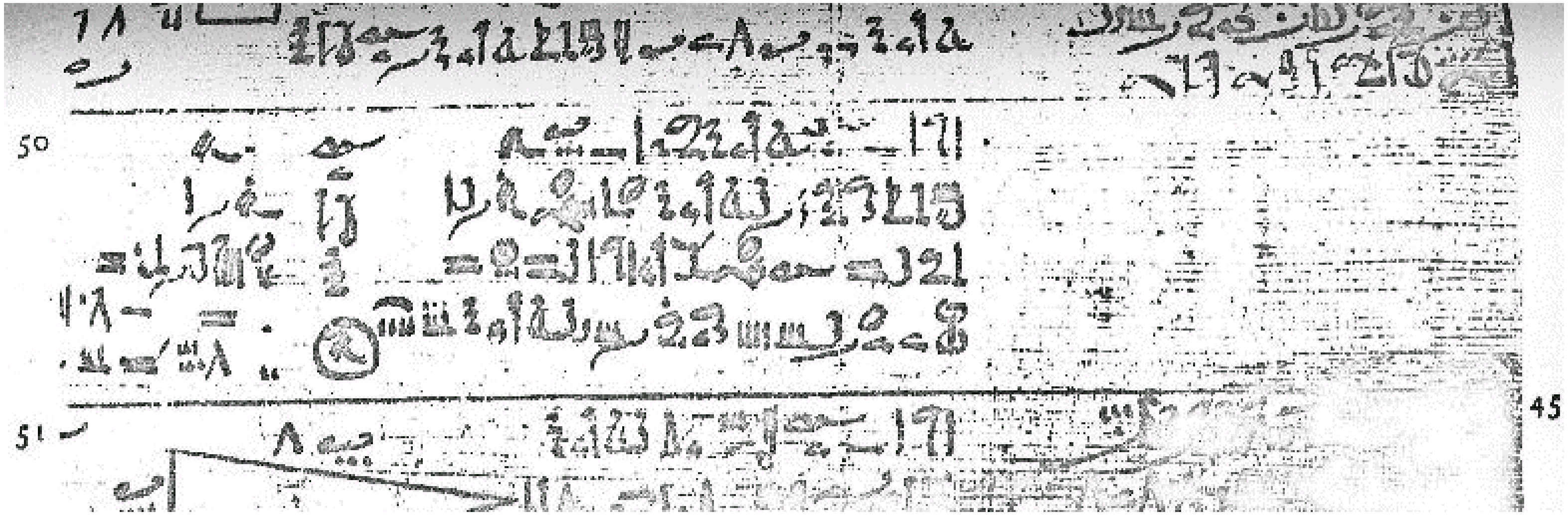
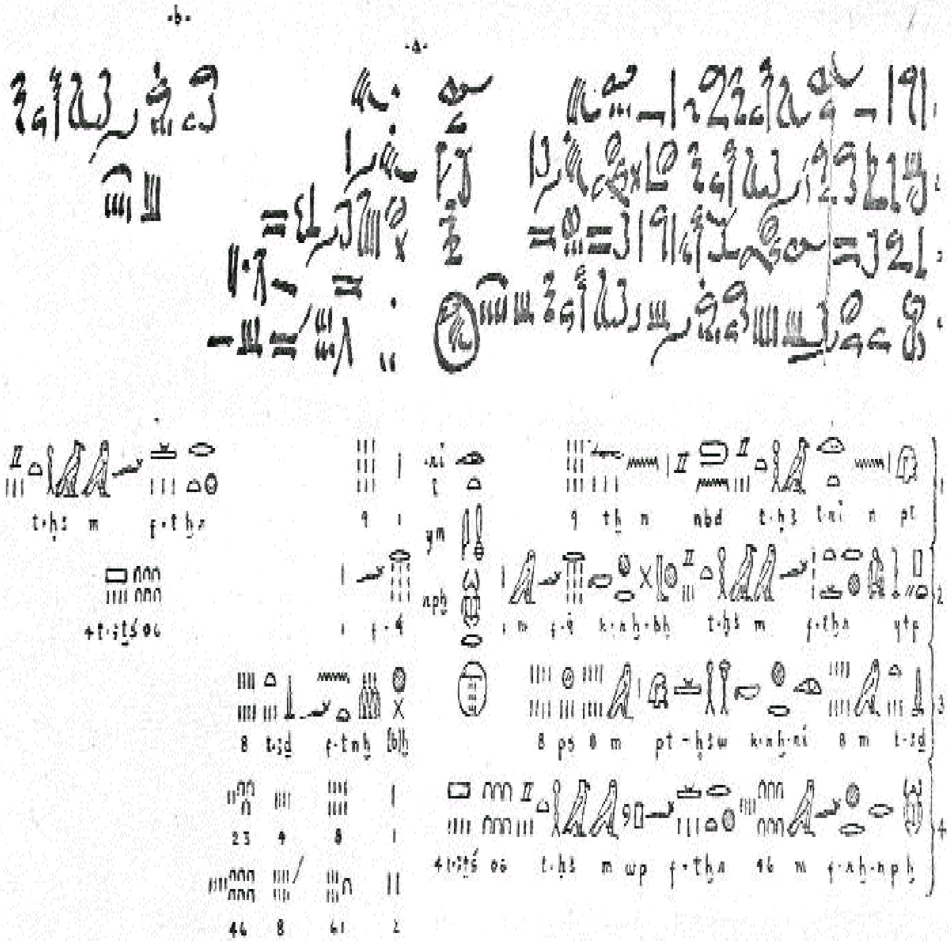


Photo of Problem 50

# The Ahmes (Rhind) Papyrus



## Problem 50

tp n ir·t ꜣḥ·t dbn n ḫt·w[1] 9 pty rḫt · f m ꜣḥ·t
*Example of making a field round of* khet 9. *What is the amount of it in area?*

ḥb·ḫr · k 9̇ · f m 1 ḏꜣ·t m 8 ir·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
*Take away thou* 1/9 *of it, namely,* 1; *the remainder is* : 8. *Make thou the multiplication* : 8 *times* 8; *becomes it* : 64;

rḫt · f pw m ꜣḥ·t 60[2] sṯꜣ·t 4
*the amount of it, this is, in area,* 60 setat 4.

ir·t my ḫpr
*The doing as it occurs:*

| | | | |
|---|---|---|---|
| 1 | 9 | 1 | 8 |
| 9̇ · f *of it* | 1. | 2 | 16 |
| | | 4 | 32 |
| | | \8 | 64 |

ḥ[b] ḫnt · f ḏꜣ·t 8
*Take away from it; the remainder is* 8.

rḫt · f m ꜣḥ·t 60[2] sṯꜣ·t 4
*The amount of it in area:* 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

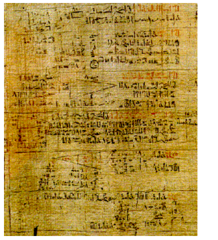
[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

# The Ahmes (Rhind) Papyrus

This problem (and solution) as well as all the other problems is phrased in terms of specific numbers. Does this mean that the problems are meant to be illustrative of general methods, or are all problems with different parameters considered different?

Examination of other problems indicates that a definite algorithm was being used since the same technique is applied to different parameters.

# Ahmes Papyrus: Problem 50



Ahmes' solution is:

*Take away thou 1/9 of it, namely 1; the remainder is 8. Make thou the multiplication 8 times 8; becomes it 64; the amount of it, this is, in area 64 setat.*

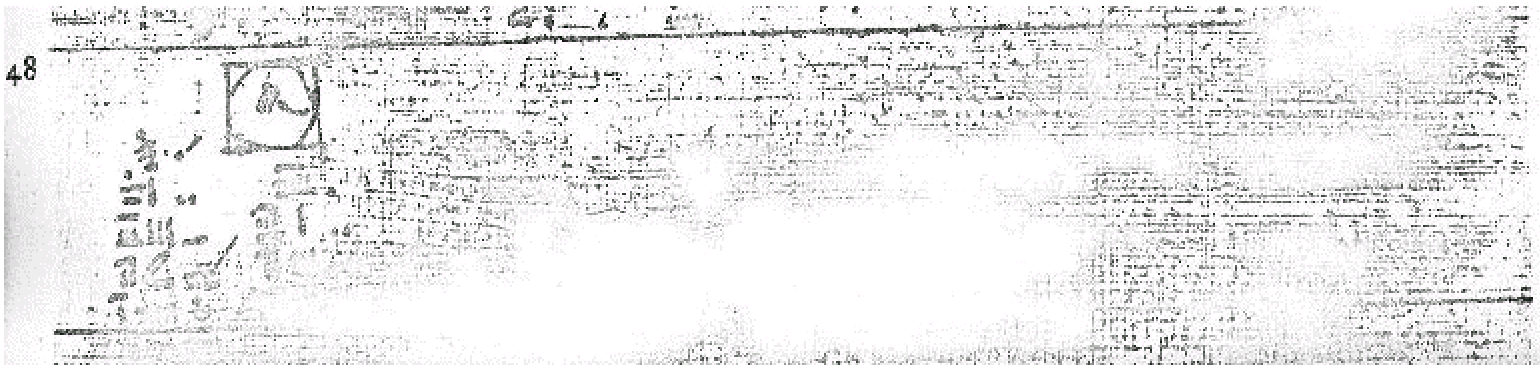
If we take this solution as a general formula, then in modern notation we obtain the "formula" for the area A of a circle of diameter d as:

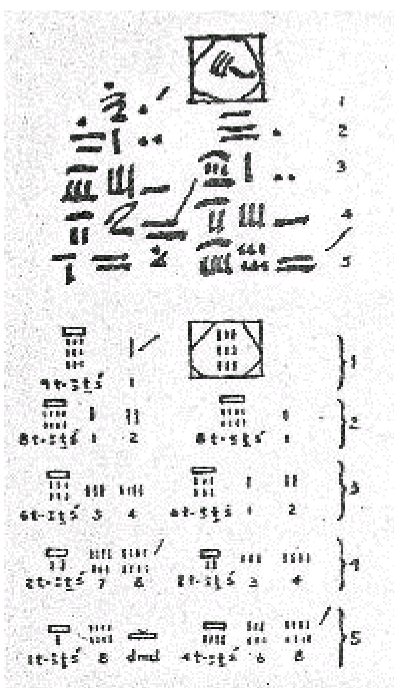
$$A = \left(d - \frac{d}{9}\right)^2 = \frac{64}{81} d^2 .$$

From this we deduce (since $A = \pi d^2/4$) that the ancient Egyptians implicitly used the value

$$\pi = 256/81 = (4/3)^4 = 3.\overline{160493827}\ldots$$

# Ahmes Papyrus: Problem 48



*Problem 48*

| | | | |
|---|---|---|---|
| 1 | šṯꜣ-t 8 setat | \1 | šṯꜣ-t 9[2] setat |
| 2 | 1 " 6[1] | 2 | 1 " 8 |
| 4 | 3 " 2 | 4 | 3 " 6[1] |
| \8 | 6 " 4. | \8 | 7 " 2 |
| | | dmd *Total* | 8 " 1. |

This problem compares the areas of a circle of radius 9 and the circumscribing square.

The number before the word šṯꜣ-t denotes the number of times ten *setat*. Thus the second line of the first table represents 16 *setat*, the third line 32 *setat*, and so on. See volume 1, page 33. The writing of multiples of ten *setat* in this way is explained by the fact that ten *setat* is equivalent to the old Egyptian unit called a "thousand-of-land" equal to a thousand cubit-strips or cubits-of-land (see volume 1, page 33). Griffith and Peet consider these numbers as representing so many thousands-of-land (Griffith, volume 16, page 236; see also volume 14, pages 410–415. Peet, page 25 and under Problems 48–55).

[1] The numeral sign here which resembles the ordinary sign for 60 is probably a special sign used in writing both 6 *setat* and 6 *hekat*. See Introduction.

[2] The numeral 9 here is a special sign used in writing both 9 *setat* and 9 *hekat*. See Problems 53 and 84.

# Ahmes Papyrus: Problem 48

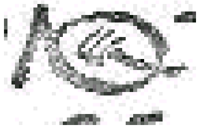
Problem 48 is the only problem in the papyrus which does not have a statement. It consists only of a diagram and a calculation of $8^2$ and $9^2$.

Chace's interpretation is "This problem compares the areas of a circle of radius[sic] 9 and the circumscribing square."

Gillings[4] challenges this interpretation, pointing out how well the circles of problems 50 and 41 are drawn.

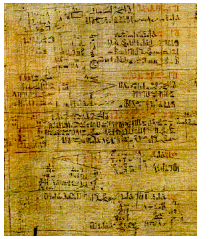
Prob. 48

Prob. 50



Prob. 41

# Ahmes Papyrus: Problem 48



Gillings' interpretation of this problem is that Ahmes is providing the justification of the rule for finding the area of a circle.

He proceeds to show how the rule could be obtained by examining the areas of an octagon (which would be “close” to the area of a circle) and the square which circumscribes it.

In his discussion he makes the “tongue in cheek” comment that Ahmes should be considered the first circle squarer![5] Others have seemingly missed the point that Gillings was joking, and have in all seriousness claimed this *honor* for Ahmes.

# Earliest Greek Study

To talk about the quadrature problem with straightedge and compass, one must turn to the ancient Greeks; for these were the tools of the Greek geometers.

The first Greek known to be connected to the problem is **Anaxagoras** (c. 499 – c. 427 B.C.). Although his chief work was in philosophy, where his prime postulate was "reason rules the world," he was interested in mathematics and wrote on the quadrature of the circle and perspective. Plutarch (c.46 - c.120) reports that he did this mathematical work while he was in jail (for being a Persian sympathizer)[6]. Only fragments of his work are extant, and it is unclear what his contribution actually is.

# Lunes of Hippocrates

**Hippocrates of Chios** (c. 440 B.C.) was a contemporary of Anaxagoras and was described by Aristotle as being skilled in geometry but otherwise stupid and weak[7]. [Not to be confused with Hippocrates of Cos who also lived around this time on an island not far from the island of Chios and is considered to be the "father of medicine"; originator of the Hippocratic Oath.]

Hippocrates of Chios is mentioned by ancient writers as the first to arrange the propositions of geometry in a scientific fashion and as having published the secrets of Pythagoras in the field of geometry. Proclus (c. 460) credits him with the *method of reduction* ... reducing a problem to a simpler one, solving the simpler problem and then reversing the process.

# Lunes of Hippocrates

Hippocrates enters our story because he provided the first example of a quadrature of a curvilinear figure (one whose sides are not line segments). He worked with certain ***lunes*** (crescents) formed by two circular arcs.

This work is also historically important, since it is the first *known* “proof”. We don't have Hippocrates' original words, rather Simplicius' summary (530 A.D.) of Eudemus' account (335 B.C.) in his now lost *History of Geometry,* of Hippocrates' proof (440 B.C.)[8].

# Lunes of Hippocrates

Hippocrates' proof uses three preliminary results:

1. The Pythagorean Theorem ($a^2 + b^2 = c^2$)
2. An angle subtended by a semicircle is a right angle.
3. The ratio of the areas of two circles is the same as the ratio of the squares of their diameters. (Euclid XII.2)

The first two of these were well known to geometers of Hippocrates' time. Eudemos (again via Simplicus) credits Hippocrates with the third result, but Archimedes (c. 225 B.C.) implies that the result is due to Eudoxus (408 – c. 355 B.C.)[9]. This puts the credit for the proof in doubt and current thinking is that Hippocrates probably didn't have a rigorous proof.

# Euclid XII 2

Before we examine Hippocrates' lunes, let's consider this result on the areas of circles.

We examine the result as given in Euclid's (c. 300 B.C.) masterpiece, *The Elements*. It is clear that this work is, at least in part, a compilation of earlier Greek work. The second proposition in book XII (out of 13) is:

*Circles are to one another as the squares on the diameters*.

Notice how the proposition is phrased in geometrical terms – not the "squares *of* the diameters" an algebraic operation, but the "squares *on* the diameters" referring to the geometric square with side equal to a diameter. The ancient Greeks had only a rudimentary algebraic notation and relied almost exclusively on geometric ideas in their writing and thinking.

# Euclid XII 2

Today we would state the result as:

*The ratio of the areas of two circles equals the ratio of the squares of their diameters*.

This phrasing underlines the more algebraic way in which we view such problems. One needs to be careful in studying ancient mathematics not to dismiss the difficulties that were overcome by the ancients because they appear simple to us. This simplicity is a result of a viewpoint that took thousands of years to develop.

Euclid's proof is an example of the *method of exhaustion*, a technique used several times in Euclid and according to Archimedes, perfected by the mathematician Eudoxus. Simply put, the idea is to “exhaust” the area of a given circle by inscribing in it polygons of increasingly many sides. This is combined with a double *reductio ad absurdum,* that is, he proves A = B by showing that A < B and B < A both lead to contradictions.

# Euclid XII 2

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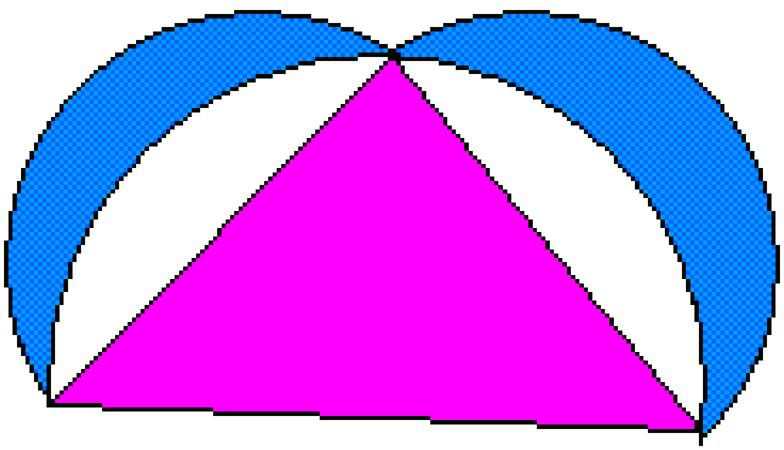
Euclid XII 2

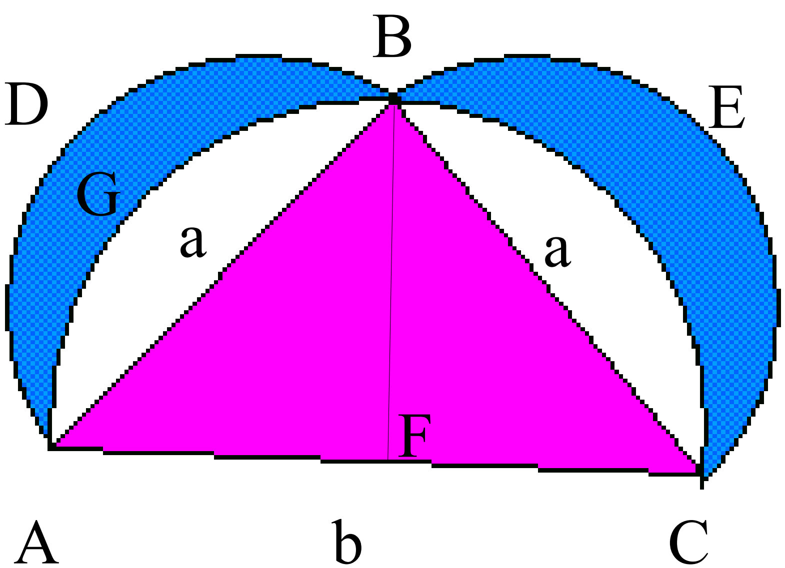
# Quadrature of Hippocrate's Lunes

*Given an isosceles right triangle, the area of the lunes determined by the semicircle on the hypotenuse and the semicircles on the sides of the triangle is equal to the area of the triangle.*



The theorem remains true for any right triangle, but Hippocrates does not seem to be aware of this.

# Quadrature of Hippocrate's Lunes




Since ΔABC is an isoceles right triangle, by Pythagoras' Theorem

$$b^2 = a^2 + a^2 = 2a^2$$

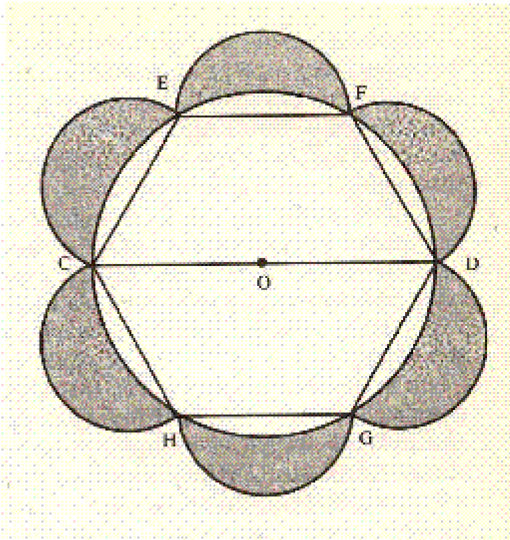
By Euclid XII 2, the area of semicircle ADB is to the area of semicircle ABC as ½$a^2$ is to ½$b^2$, i.e., 1 to 2.

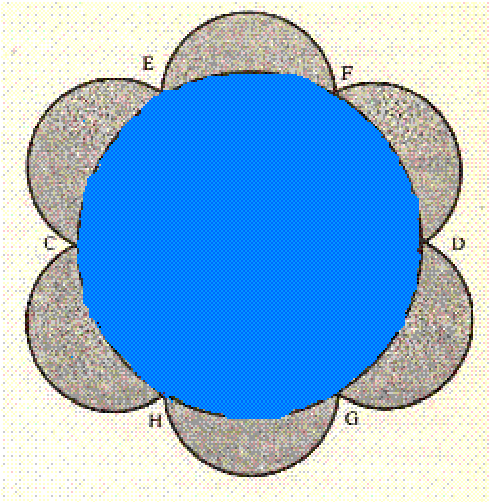
Drop the perpendicular from B to AC. This line divides the semicircle ABC into two equal sectors and also divides ΔABC into two congruent triangles. So, the area of semicircle ADB = area of sector AFB. If x is the area of the region AGB, then

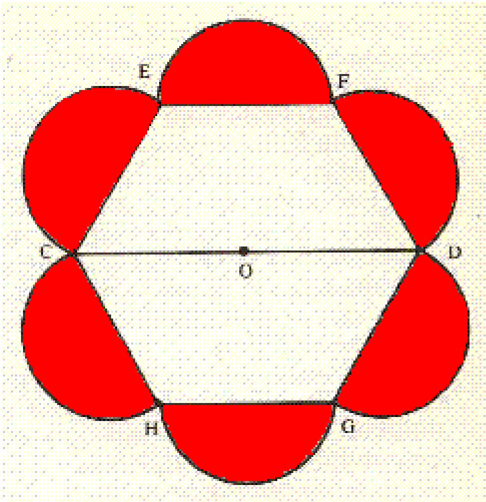
area of lune ADB + x = area of ΔAFB + x.

So the area of the lune ADB = ½ area of ΔABC.

# Lunes of Hippocrates




Consider a regular hexagon inscribed in a circle whose side is ½ of the diameter of the circle. The figure above can be thought of in two ways: circle + 6 lunes = hexagon + 6 semicircles .
Since the diameter of the red semicircles is ½ the diameter of the blue circle, circle = 4 circles = 8 semicircles. So, we get

1 circle = 2 semicircles = hexagon – 6 lunes.

So if we can square these lunes, we can square a circle ....
... but these are **not** Hippocrates' lunes!

# A Literary Aside

Arguments such as this may have engendered a hope that with enough work a quadrature of the circle could be accomplished.

The desire to find such a quadrature must have been well known to the general Greek populace, and not just the small set of mathematicians, for we find it referenced in one of Aristophanes famous comedies.

Not only is the problem known, but in order to achieve the comic effect, it must have been known as a fruitless endeavor.

# Aristophanes

In the ***Birds***, performed in 414 BC, a new city has to be founded from scratch. The main character, Peisthetaerus, is visited by various people who offer their services.[11]

*Enter*

**METON**: I come amongst you ...

**PEISTHETAERUS**: Some new misery this! Come to do what? What's your scheme's form and outline? What's your design? What buskin's on your foot?

**METON**: I come to land-survey this Air of yours, and mete it out by acres.

**PEISTHETAERUS**: Heaven and earth! Whoever are you?

**METON**: Whoever am I? I'm Meton, known throughout Hellas and Colonnus.

**PEISTHETAERUS**: Aye, and what are these?

# Aristophanes

**METON**: They're rods for Air-surveying. I'll just explain. The Air's, in outline, like one vast extinguisher; so then, observe, applying here my flexible rod, and fixing my compass there, - you understand?
**PEISTHETAERUS**: I don't.
**METON**: With the straight rod I measure out, that so the circle may be squared; and in the centre a market-place; and streets be leading to it straight to the very centre; just as from a star, though circular, straight rays flash out in all directions.
**PEISTHETAERUS**: Why, the man's a Thales!