Euler's identity

In Exercise 1.5.10, you showed that

$$\mathbf{A} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \implies e^{\mathbf{A}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta. \end{bmatrix}$$

Since $\mathbf{A}$ is the matrix of the complex number $i\theta$, this computation implies *Euler's identity:*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using the properties of exponents, we can write $e^z = e^{x+iy} = e^x e^{iy}$. By Euler's identity, we see that the imaginary part of z determines the polar angle of e^z, while the real part of z determines the distance from e^z to the origin. In particular, if a is a real number, $e^{ia} \in S^1$.

A fun particular case: $e^{i\pi} = -1$, IOW $\boxed{e^{i\pi} + 1 = 0}$—this equation unifies five important, and until now apparently unrelated, constants!

Something more serious: *Euler's identity is the only trigonometric formula you will ever need to remember.* For example, we have as an immediate consequence *de Moivre's formula:*

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also get the trig formulas for sums of angles:

$$\begin{aligned} \cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta); \end{aligned}$$

now equate real and imaginary parts.

Derivatives

We've begun exploring the meaning of the derivative as the "best linear approximation." It's useful at this point to see what this means for our friends the linear functions.

Proposition. The derivative of a linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ at every point $\mathbf{x} \in \mathbb{R}^n$ is $DT(\mathbf{x}) = T$.

It's important to note that this result does say that the derivative of a linear function is "constant," in some sense—namely, that you get the *same* map at every point. But since, at each point, the derivative is a linear map $\mathbb{R}^n \to \mathbb{R}^m$, it is not a constant function.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$. Recalling the definition of the derivative, we have that $DT(\mathbf{x}) = L$ is the unique linear function such that

$$\lim_{\mathbf{h}\to 0} \frac{T(\mathbf{x}+\mathbf{h}) - T(\mathbf{x}) - L(\mathbf{h})}{|\mathbf{h}|} = \mathbf{0},$$

if it exists. But because T is linear, the numerator of the above expression simplifies to $T(\mathbf{h}) - L(\mathbf{h})$, which if we take $L = T$ is identically zero. So $DT(\mathbf{x}) : \mathbf{h} \mapsto T(\mathbf{h})$. □

Special case of yesterday's last example: $\mathbf{A} \mapsto \mathbf{A}^2$. We know that, for real numbers, the derivative of $f(x) = x^2$ at a is $2a$ (i.e., the map $h \mapsto 2ah$). Is it true that $\mathbf{H} \mapsto 2\mathbf{A}\mathbf{H}$ is the derivative of $g : \mathbf{A} \mapsto \mathbf{A}^2$?

Directional derivative:

$$\begin{aligned} D_{\mathbf{H}}g(\mathbf{A}) &= \lim_{t\to 0} \frac{(\mathbf{A}+t\mathbf{H})^2 - \mathbf{A}^2}{t} \\ &= \lim_{t\to 0} \frac{\mathbf{A}^2 + t\mathbf{A}\mathbf{H} + t\mathbf{H}\mathbf{A} + t^2\mathbf{H} - \mathbf{A}^2}{t} \\ &= \lim_{t\to 0} \left(\frac{t}{t}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) + \frac{t^2}{t} \right) = \mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A} \end{aligned}$$

We haven't quite shown that g is differentiable at $\mathbf{A}$; you'll do this in class tomorrow. We'll assume for now that it is. What the above tells us, then, is that

$$[Dg(\mathbf{A})]\mathbf{H} = \mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}.$$