# Scattering Theory

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Handwritten notes taken by H. Inoue and N. Tsuzu.

# I) Hilbert space and linear operators.

## I.1) $\mathcal{H}$ and $B(\mathcal{H})$

<u>Def.</u> A Hilbert space $\mathcal{H}$ is a complex vector space, with a scalar (or inner) product $\langle \cdot , \cdot \rangle$, which is complete for the norm $\|f\| = \langle f, f \rangle^{1/2}$, and separable (countable orthogonal basis).

<u>Convention</u>: $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$

$$\forall f, g, h \in \mathcal{H}, \alpha \in \mathbb{C}$$

<u>Examples</u>: $\mathbb{C}^n$ with $\langle \alpha, \beta \rangle = \sum_j \overline{\alpha_j} \beta_j$,

$\ell^2(\mathbb{Z})$ with $\langle a, b \rangle = \sum_{j \in \mathbb{Z}} \overline{a_j} b_j$,

$L^2(\mathbb{R}^n)$ with $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$.

<u>2 topologies on $\mathcal{H}$</u> Consider $\{f_n\}_n \subset \mathcal{H}$

- $\{f_n\}$ <u>converges strongly</u> to $f_\infty \in \mathcal{H}$

$$\| f_n - f_\infty \| \xrightarrow{n\to\infty} 0$$

We write $\operatorname*{s-lim}_{n\to\infty} f_n = f_\infty$.

- $\{f_n\}$ converges weakly to $f_\infty$ if

$$\langle g, f_n - f_\infty \rangle \xrightarrow{n\to\infty} 0 \qquad \forall g \in \mathcal{H}.$$

We write $\operatorname*{w-lim}_{n\to\infty} f_n = f_\infty$.

<u>Lemma</u>: $\operatorname*{s-lim}_{n\to\infty} f_n = f_\infty \iff \begin{cases} \operatorname*{w-lim}_{n\to\infty} f_n = f_\infty, \\ \lim_{n\to\infty} \|f_n\| = \|f_\infty\|. \end{cases}$

(Exercise)

<u>Def</u>: A map $B : \mathcal{H} \to \mathcal{H}$ is a <u>bounded</u> <u>linear map</u> (operator) if $B$ is linear, and $\exists C > 0$ s.t. $\|Bf\| \leq C\|f\|$, $\forall f \in \mathcal{H}$.

We set $\mathcal{B}(\mathcal{H}) = \{$ bounded linear op. $\}$

Also, $\|B\| := \sup_{\substack{f\in\mathcal{H}, \\ \|f\|=1}} \dfrac{\|Bf\|}{\|f\|} = \sup_{\substack{f_i \in \mathcal{M}_i \\ \|f_i\|=1, i=1,2}} |\langle f_1, Bf_2 \rangle|$

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with $\mathcal{M}_i \subset \mathcal{H}$, a dense linear subset.

For any $B \in \mathcal{B}(\mathcal{H})$, we define $B^* \in \mathcal{B}(\mathcal{H})$ by the relation $\langle B^* f, g \rangle = \langle f, Bg \rangle$ (existence based on Riesz lemma).

$B^*$ is called the adjoint of $B$. Then,

$$\left.\begin{array}{ll} \|B^*\| = \|B\|, & (B^*)^* = B, \\ (AB)^* = B^* A^*, & \|B^* B\| = \|B\|^2 \end{array}\right\} \mathcal{B}(\mathcal{H}) \text{ is a } C^*\text{-algebra.}$$

<u>Special operators</u> Let $B \in \mathcal{B}(\mathcal{H})$.

- $B$ is self-adjoint if $B^* = B$,
- $B$ is normal if $B^* B = B B^*$,
- $B$ is a projection if $B^2 = BB = B$,
- $B$ is an orthogonal projection if $B^2 = B = B^*$,
- $B$ is unitary if $BB^* = \mathbb{1}$ and $B^* B = \mathbb{1}$,

where $\mathbb{1}$ is the identity operator, i.e. $\mathbb{1}f = f, \forall f \in \mathcal{H}$.

- $B$ is an isometry if $B^*B = \mathbb{1}$
- $B$ is a partial isometry if $B^*B$ is an orthogonal projection.

<u>3 topologies on $B(\mathcal{H})$</u>

Consider $\{B_n\}_{n\in\mathbb{N}} \subset B(\mathcal{H})$.

- $\{B_n\}$ converges <u>uniformly</u> to $B_\infty \in B(\mathcal{H})$ if $\|B_n - B_\infty\| \xrightarrow{n\to\infty} 0$. We write $\underset{n\to\infty}{u\text{-}\lim} B_n = B_\infty$.

- " " <u>strongly</u> to $B_\infty \in B(\mathcal{H})$ if $\|(B_n - B_\infty)f\| \xrightarrow{n\to\infty} 0$ for $\forall f \in \mathcal{H}$. We write $\underset{n\to\infty}{s\text{-}\lim} B_n = B$.

- $\{B_n\}$ converges <u>weakly</u> to $B_\infty$ if $|\langle g, (B_n - B_\infty) f \rangle| \xrightarrow{n\to\infty} 0$ for $\forall f, g \in \mathcal{H}$. We write $\underset{n\to\infty}{\text{w-lim}}\, B_n = B_\infty$

<u>Polar decomposition</u>

For any $A \in B(\mathcal{H})$, $\exists!$ partial isometry $U$ and a unique <u>positive</u> operator $\underbrace{|A|}_{\text{notation}} \in B(\mathcal{H})$ s.t. $A = U|A|$ and $\ker(A) = \ker U$.

Here, $\ker(A) = \{ f \in \mathcal{H} \mid Af = 0 \}$.

An element $B \in B(\mathcal{H})$ is <u>positive</u> if $\langle f, Bf \rangle \geq 0 \quad \forall f \in \mathcal{H}$. ($\Rightarrow B$ is self-adjoint)

## I. 2) Inside $B(\mathcal{H})$

Let $\{g_j, h_j\}_{j=1}^{N} \subset \mathcal{H}$ and we set ($N \in \mathbb{N}$)

for any $f \in \mathcal{H}$

$$Af := \sum_{j=1}^{N} \underbrace{\langle g_j, f \rangle}_{\in \mathbb{C}} \underbrace{h_j}_{\in \mathcal{H}} \in \mathcal{H}.$$

Check $A \in B(\mathcal{H})$ called a <u>finite rank operator</u>. We set $\mathcal{F}(\mathcal{H}) := \{\text{fin. rank. op.}\}$.

(Ex. What is $A^*$ ?)

<u>Def</u>: $B \in B(\mathcal{H})$ is compact if

$\exists \{A_n\} \subset \mathcal{F}(\mathcal{H})$ s.t. $\underset{n \to \infty}{u\text{-}\lim} A_n = B$.

We set $\mathcal{K}(\mathcal{H}) = \{\text{compact operators}\} \subset B(\mathcal{H})$.

<u>Properties</u>
- $B \in \mathcal{K}(\mathcal{H}) \Rightarrow B^* \in \mathcal{K}(\mathcal{H})$,
- $\mathcal{K}(\mathcal{H})$ is norm closed,
- If $A \in B(\mathcal{H})$, $B \in \mathcal{K}(\mathcal{H}) \Rightarrow AB, BA \in \mathcal{K}(\mathcal{H})$

($\mathcal{K}(\mathcal{H})$ is an <u>ideal</u>)

- If $f_n \xrightarrow[w]{n\to\infty} f_\infty$ and $B \in \mathcal{K}(\mathcal{H})$,

  then $Bf_n \xrightarrow[s]{n\to\infty} Bf_\infty$.

- If $A_n \xrightarrow[s]{n\to\infty} A_\infty$ and $B \in \mathcal{K}(\mathcal{H})$,

  $A_n B \xrightarrow[u]{n\to\infty} A_\infty B$ and $BA_n^* \xrightarrow[u]{n\to\infty} BA_\infty^*$

( A compact. op. improves convergence)

<u>Def:</u> $A \in B(\mathcal{H})$ is <u>Hilbert-Schmidt</u> iff

$\sum_{j\in\mathbb{N}} \|Ae_j\|^2 < \infty$ for $\{e_j\}_{j\in\mathbb{N}}$ an

orthonormal basis. We set

$$B_2 = \{\text{ Hilbert-Schmidt op }\}.$$

<u>Lemma</u> Let $A \in B_2$.

- $\|A\|_{B_2} \equiv \|A\|_{HS} := \sum_{j\in\mathbb{N}} \|Ae_j\|^2$,

  is independent of the choice of the basis.

- $\|A\| \leq \|A\|_{B_2}$,

- $\forall B \in B(\mathcal{H})$, $AB, BA \in \mathcal{B}_2$, and

$$\|AB\|_{B_2} \leq \|B\| \|A\|_{B_2}$$

$\Longrightarrow$ $B_2$ is an ideal.

- $A^* \in \mathcal{B}_2$ and $\|A^*\|_{B_2} = \|A\|_{B_2}$
- $B_2$ is a Hilbert space with scalar product $\langle A, B \rangle := \sum_j \langle Ae_j, Be_j \rangle$.

For $B \in B(\mathcal{H})$, and $B \geq 0$ ($\Leftrightarrow$ $B$ is positive)

$\exists!\, B^{1/2} \in B(\mathcal{H})$, $B^{1/2} \geq 0$ s.t. $B^{1/2} B^{1/2} = B$.

<u>Def</u>: $A \in \mathcal{B}(\mathcal{H})$ is trace class iff $|A|^{1/2}$ is Hilbert-Schmidt. We set $\mathcal{B}_1 = \{\text{trace class op}\}$ and

$$\|A\|_{B_1} := \| |A|^{1/2} \|_{B_2} = \sum_j \| |A|^{1/2} e_j \|^2$$

$$= \sum_j \langle e_j, |A| e_j \rangle$$

indep. of the choice of basis.

<u>Lemma</u> Let $A \in \mathcal{B}_1$.

- $\|A\| \leq \|A\|_{\mathcal{B}_1}$
- If $B \in \mathcal{B}(\mathcal{H})$, then $AB, BA \in \mathcal{B}_1$,

  and $\|AB\|_{\mathcal{B}_1} \leq \|B\| \, \|A\|_{\mathcal{B}_1}$

  ($\rightsquigarrow$ $\mathcal{B}_1$ is an ideal)
- $A^* \in \mathcal{B}_1$
- $A \in \mathcal{B}_1 \iff \exists A_1, A_2 \in \mathcal{B}_2$ s.t. $A = A_1 A_2$

<u>Def:</u> For any $A \in \mathcal{B}_1$, we set

$$\mathrm{tr}(A) := \sum_{j \in \mathbb{N}} \langle e_j, A e_j \rangle$$ ind. of the basis

<u>Lemma</u> $\mathrm{tr}(A_1 A_2) = \mathrm{tr}(A_2 A_1)$

if either $A_1, A_2 \in \mathcal{B}_2$ or

if $A_1 \in \mathcal{B}_1$, $A_2 \in \mathcal{B}(\mathcal{H})$.

Extension to Schatten ideals.

<u>Thm</u> For $j \in \{1,2\}$,

$$\mathcal{F}(\mathcal{H}) \underset{\uparrow}{\subset} B_j \subset \mathcal{K}(\mathcal{H})$$

dense with norm $\|\cdot\|_{B_j}$

dense with $\|\cdot\|$

$\rightsquigarrow$ [Murphy, p64].

## I.3) General linear operators.

<u>Def</u>: A <u>linear operator</u> is a pair

$(A, D(A))$ with $D(A) \subset \mathcal{H}$

operator $\rightarrow$ $\quad$ $\leftarrow$ the domain

a linear subspace of $\mathcal{H}$ and

$A : D(A) \to \mathcal{H}$ a linear map.

$(A, D(A))$ or simply is densely defined if $D(A)$ is dense in $\mathcal{H}$

<u>Remark</u>: any $B \in B(\mathcal{H})$, considered as $(B, \mathcal{H})$ is a linear operator.

We set $\mathrm{Ran}(A) = \{ Af \mid f \in D(A) \}$,
range of A

$\mathrm{Ker}(A) = \{ f \in D(A) \mid Af = 0 \}$.
Kernel of A

⚠ If $(A, D(A))$, $(B, D(B))$ are linear op., then $A+B$ is defined on $D(A) \cap D(B)$ and $AB$ is defined on $\{ f \in D(B) \mid Bf \in D(A) \}$.

<u>Remark</u>: An "operator" $A$ can have several domain.

For example in $L^2(\mathbb{R})$, we can consider

$(-i\partial_x, \underbrace{\mathcal{S}(\mathbb{R})}_{\text{Schwartz}})$, $(-i\partial_x, C_c^\infty(\mathbb{R}))$

or $(-i\partial_x, \underbrace{\mathcal{H}^1(\mathbb{R})}_{\uparrow\ 1^{st}\ \text{Sobolev space}})$

⇝ Some domains are <u>better</u> than other!

⇝ closed op.

Reminder

- $\mathcal{H}$ a Hil. sp, 2 topologies : strong and weak
  $\mathcal{H}$ is complete with them
- $\mathcal{B}(\mathcal{H})$, 3 topologies : uniform, strong, weak
  $B(\mathcal{H})$ is complete with them
- $\mathcal{F}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $B_2$, $B_1$
  finite rank, compact, Hilbert-schmidt, trace class
- $(A, \mathcal{D}(A))$ a linear operator

Several domain for a given A

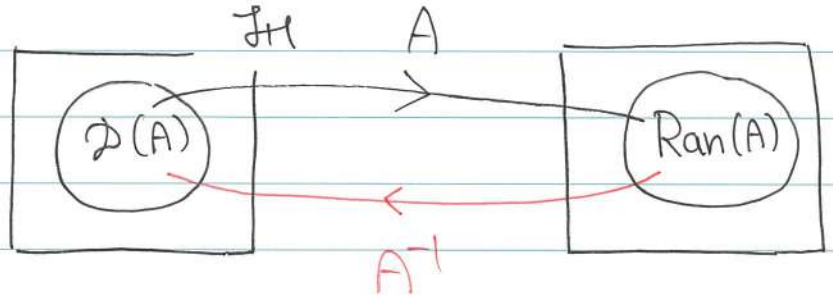
---

Def: $(A, \mathcal{D}(A))$ is closed if $\forall \{f_n\}_n \subset \mathcal{D}(A)$ satisfying $\operatorname{s-lim}_{n\to\infty} f_n = f_\infty \in \mathcal{H}$ and s.t. $\{Af_n\}_n$ is Cauchy, then $f_\infty \in \mathcal{D}(A)$ and $\operatorname{s-lim}_{n\to\infty} Af_n = Af_\infty$.

Any $A \in B(\mathcal{H})$ is closed !

Def: A closed op $(A, \mathcal{D}(A))$ is invertible if $\operatorname{Ker}(A) = \{0\}$. Then, we set

$(A^{-1}, \mathrm{Ran}(A))$, it is closed and one has $AA^{-1} = \mathbb{1}$ on $\mathrm{Ran}(A)$ and $A^{-1}A = \mathbb{1}$ on $\mathcal{D}(A)$.




<u>Thm</u> If $\mathrm{Ran}(A) = \mathcal{H}$ and $(A, \mathcal{D}(A))$ is invertible, then $(A^{-1}, \mathcal{H})$ is bounded. (a boundedly invertible operator)

<u>Lemma</u> (Neumann series)

Let $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| < 1$.

Then, $(1-A)$ is boundedly invertible with $\|(1-A)^{-1}\| \leq (1-\|A\|)^{-1}$

and $(1-A)^{-1} = \sum_{j=0}^{\infty} A^j$

<u>Def</u>: Let $(A, \mathcal{D}(A))$ be closed.

The resolvent set $\rho(A)$ is given by

$$\rho(A) := \{ z \in \mathbb{C} \mid (A-z) \text{ is boundedly invertible} \}$$
$$= \{ z \in \mathbb{C} \mid \mathrm{Ker}(A-z) = \{0\}, \mathrm{Ran}(A-z) = \mathcal{H} \}$$

For $z \in \rho(A)$, the bounded op $(A-z)^{-1}$ is called the resolvent of $A$ at $z$.

We then set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

and call $\sigma(A)$ the spectrum of $A$

<u>Proposition</u> Let $(A, \mathcal{D}(A))$ be closed.

1) $\rho(A)$ is open in $\mathbb{C}$, and $\sigma(A)$ is closed.

2) For $z_1, z_2 \in \rho(A)$, one has

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}$$

1st resolvent equation.

3) $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute

$(AB = BA \Longleftrightarrow A, B \text{ commutes})$

4) The map $\rho(A) \ni z \longmapsto (A-z)^{-1} \in B(\mathcal{H})$

is holomorphic in the uniform topology.

Def: Let $(A, \mathcal{D}(A))$ be closed.

If $f \in \mathcal{D}(A)$ with $f \neq 0$ such that $Af = zf$

for some $z \in \mathbb{C}$, then $f$ is called

an eigenfunction associated with the

eigenvalue $z$. $\dim_{\mathrm{Span}_{\mathbb{C}}} \{ f \in \mathcal{D}(A) \mid Af = zf, f \neq 0 \}$

is called the geometric multiplicity of $z$

Def: For $(A, \mathcal{D}(A))$ densely defined,

its adjoint $(A^*, \mathcal{D}(A^*))$ is defined

by

$$\mathcal{D}(A^*) = \{ f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f, Ag \rangle = \langle f^*, g \rangle \ \forall g \in \mathcal{D}(A) \}.$$

For any $f \in \mathcal{D}(A^*)$, we set $A^* f = f^*$.

<u>Remark</u> (written before def of adjoint)

We set $\sigma_p(A) = \{ \text{eigenvalues of } A \}$

and one has $\sigma_p(A) \subset \sigma(A)$

point spectrum

<u>Question</u> Is $\sigma_p(A)$ always countable?

One observe that

$$\langle A^* f, g \rangle = \langle f, Ag \rangle$$

for $\forall f \in \mathcal{D}(A^*)$ and $g \in \mathcal{D}(A)$.

<u>Notation</u>

If $\mathcal{G} \subset \mathcal{H}$, then $\mathcal{G}^\perp = \{ f \in \mathcal{H} \mid \langle f, g \rangle = 0 \ \forall g \in \mathcal{G} \}$.

<u>Lemma</u>: One has $(A^*, \mathcal{D}(A^*))$ is closed, and $\ker(A^*) = (\mathrm{Ran}(A))^{\perp}$.

<u>Def</u> A densely defined op $(A, \mathcal{D}(A))$ is <u>self-adjoint</u> if

$$(A, \mathcal{D}(A)) = (A^*, \mathcal{D}(A^*)).$$

The equality $\langle Af, g\rangle = \langle f, Ag\rangle$ $\forall f, g \in \mathcal{D}(A)$ does <u>not</u> imply that $A$ is self-adjoint.

If $A$ satisfies this relation, $A$ is called <u>symmetric</u> $\not\Rightarrow$ s.a.

<u>Remark</u>: A self-adjoint op. is closed.

## I. 4) Properties of self-adjoint operators.

**Thm.** If $(A, \mathcal{D}(A))$ is self-adjoint,

then $\sigma(A) \subset \mathbb{R}$.

Part of the proof:

If $z \in \sigma_p(A)$, then $Af = zf$

for some $f \in \mathcal{D}(A)$, $f \neq 0$

and
$$z\|f\|^2 = \langle f, zf\rangle = \langle f, Af\rangle$$
$$= \langle Af, f\rangle = \langle zf, f\rangle$$
$$= \bar{z}\|f\|^2$$

$\Rightarrow z = \bar{z} \iff z \in \mathbb{R}$. $\square$

Remark: If $z_1, z_2 \in \sigma_p(A)$ and

$z_1 \neq z_2$ and $(A, \mathcal{D}(A))$ is self-adjoint,

and if $f_1, f_2 \in \mathcal{H}$, $f_1 \neq 0 \neq f_2$ satisfy

$Af_1 = z_1 f_1$ and $A f_2 = z_2 f_2$ then

$\langle f_1, f_2 \rangle = 0 \iff f_1 \perp f_2$

$\Longrightarrow \sigma_p(A)$ is countable (for self-adjoint

<u>Def</u>: For $(A, \mathcal{D}(A))$ self-adjoint,

We set $\mathcal{H}_p(A) := \overline{\text{subspace spanned by eigenfunctions}}$

closure in $\mathcal{H}$

point subspace

and $\mathcal{H}_c(A) := \mathcal{H} \ominus \mathcal{H}_p(A)$

continuous

orthocomplement of $\mathcal{H}_p(A)$.

<u>Def.</u> A <u>spectral family</u> (or resolution of the identity) is a family of orthogonal projection $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ satisfying

1) $E_\lambda E_\mu = E_{\min\{\lambda, \mu\}}$, non-decreasing property

2) $E_\lambda = E_{\lambda+0} = \underset{\varepsilon \to 0+}{\text{s-lim}}\, E_{\lambda+\varepsilon}$ strongly right continuous

3) $\operatorname{s-lim}_{\lambda \to -\infty} E_\lambda = 0$ and $\operatorname{s-lim}_{\lambda \to \infty} E_\lambda = \mathbb{1}$

normalization

We set $E((a, b]) := E_b - E_a$

and extend it to countable

intersection, union and complement

Borel algebra on $\mathbb{R}$

of intervals $(a, b]$ and get

$$\mathcal{A}_B \ni \upsilon \longmapsto E(\upsilon) \in \{\text{orthogonal projections}\}$$

which is called <u>a spectral measure</u>

Note that we can get the spectral family

from the spectral measure by

$$E_\lambda = E((-\infty, \lambda]).$$

<u>Thm</u> ( Spectral thm )

$\exists$ a bijection betwen the set of self-adjoint operators $(A, \mathcal{D}(A))$ and the set of spectral family (or spectral measure) s.t.

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H} \;\middle|\; \int_{\mathbb{R}} \lambda^2 \langle f, E(d\lambda) f \rangle < \infty \right\}$$

and $Af = \int_{\mathbb{R}} \lambda \, E(d\lambda) f$. Improper Riemann-Stieljes integral.

[Amrein : Hilbert space ... p179-189]

<u>Thm</u> : $\exists$ homomorphism ← *-homomorphism (morphism between $C^*$-algebras)

$$C_b(\mathbb{R}) \ni \varphi \longmapsto \varphi(A) \in B(\mathcal{H})$$

bounded and continuous functions on $\mathbb{R}$

defined by

$$\varphi(A) = \int_{\mathbb{R}} \varphi(\lambda) E(d\lambda)$$

(bounded and continuous functional Calculus)

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and with $\|\varphi(A)\| \leq \|\varphi\|_\infty$.

[ Amrein: Hilbert space §4.2 ].

## II. Scattering theory: time dependent approach.

### II.1) Evolution groups

Thm: $\exists$ bijective relation between self-adjoint operators $(A, \mathcal{D}(A))$ and strongly continuous unitary groups. (Stone's theorem)

A family $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group if

1) $U_t$ is unitary for any $t$,

2) $U_t U_s = U_{t+s} \quad \forall t, s \in \mathbb{R}$,

3) $\operatorname{s-lim}_{t\to 0} U_t = \mathbb{1}$.

Given $\{U_t\}_{t\in\mathbb{R}}$, one has

$$\mathcal{D}(A) := \{ f \in \mathcal{H} \mid \operatorname*{s-lim}_{t\to 0} \frac{1}{t}(U_t - 1) f \text{ exist} \}$$

and then for $f \in \mathcal{D}(A)$, we set

$$Af := \operatorname*{s-lim}_{t\to 0} \frac{i}{t}(U_t - 1) f.$$

Formally, $U_t = \underbrace{e^{-itA} \quad \forall t \in \mathbb{R}}$.

this is well-defined by the functional calculus for $\varphi(\lambda) := e^{-it\lambda}$ but for each fixed $t \in \mathbb{R}$.

$\rightsquigarrow$ Check that $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly conti. unitary group.

## Illustration

$\mathcal{H} = L^2(\mathbb{R}) \ni u \quad , \quad \varphi \in C(\mathbb{R})$

$$[\varphi(X)\, u](x) := \varphi(x)\, u(x) \quad \forall x \in \mathbb{R}$$

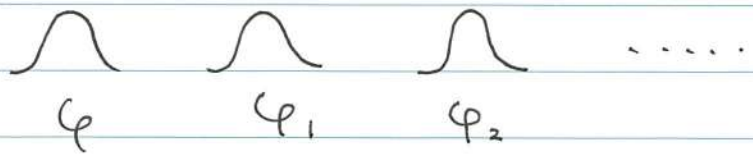
multiplication operator ($\varphi(Q$ or $M_\varphi \ldots$)

- $\varphi(X) \in B(\mathcal{H}) \iff \varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$
  $= C_b(\mathbb{R})$

- If $\varphi \in C_b(\mathbb{R})$, $\varphi(X)^* = \overline{\varphi}(X)$
  (also true if $\varphi$ is not bounded, but the domain has to be specified

- If $\varphi \in C_c(\mathbb{R})$, set $\varphi_n := \varphi(\cdot - n)$ for $n \in \mathbb{N}$,

  then $\varphi_n(X) \xrightarrow[s]{n \to \infty} 0$ but




  $\varphi_n(X) \xrightarrow[u]{n \to \infty}$ does not converge

- $z \in \rho(\varphi(X)) \iff (\varphi(X) - z)^{-1} \in B(\mathcal{H})$

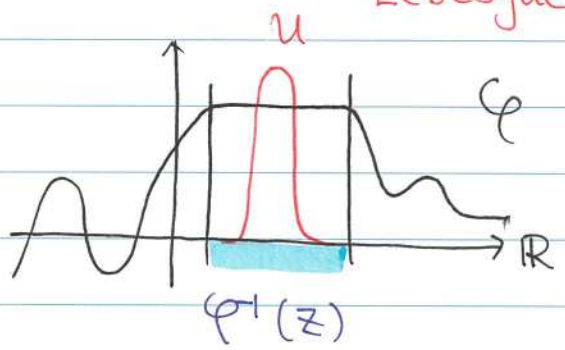
( One has $[(\varphi(X)-z)^{-1}u](x) = (\varphi(x)-z)^{-1}u(x)$ )

$$\Longleftrightarrow z \notin \overline{\varphi(\mathbb{R})} = \overline{\{\varphi(x) \mid x \in \mathbb{R}\}}$$

and $z \in \sigma(\varphi(X)) \Longleftrightarrow z \in \overline{\varphi(\mathbb{R})}$.

- $z \in \sigma_p(\varphi(X)) \Longleftrightarrow |\varphi^{-1}(z)| = |\{x \in \mathbb{R} \mid \varphi(x) = z\}| > 0$

Lebesgue measure




$\varphi(X)u = zu$

Some useful formulas: $A$ is self-adjoint

$$(A-z)^{-1} = i\int_0^\infty e^{izt}\, e^{-itA}\, dt \quad \text{for } \mathrm{Im}(z) > 0,$$

$$(A-z)^{-1} = -i\int_{-\infty}^\infty e^{izt}\, e^{-itA}\, dt \quad \text{for } \mathrm{Im}(z) < 0$$

Converge in the strong topology

<u>Def</u> : $\mathcal{H}_{ac}(A) := \{ f \in \mathcal{H} \mid \langle f, U_t f \rangle \in L^2(\mathbb{R}, dt) \}$

( different from <u>the usual definition</u> but equivalent!)
will be given in the remark later.

<u>Lemma</u> : If $f \in \mathcal{H}_{ac}(A)$, then $\underset{t \to \pm\infty}{w\text{-}\lim} U_t f = 0$

and $f \in \mathcal{H}_c(A)$.

Hint : if $f \in \mathcal{H}_p(A)$ and satisfies $Af = zf$,

then $U_t f = e^{-itA} f = e^{-itz} f$.

For $f \in \mathcal{H}_{ac}(A)$, the function $t \mapsto \langle f, U_t f \rangle$

is uniformly continuous, then $L^2$-condition

implies $\lim_{t \to \infty} \langle f, U_t f \rangle = 0$

<u>Question</u> If $f \in \mathcal{H}_c(A) \ominus \mathcal{H}_{ac}(A) =: \mathcal{H}_{sc}(A)$,

then $\exists \{t_n\} \subset \mathbb{R}$ with $t_n \xrightarrow{n \to \infty} \infty$ s.t.

$\langle f, U_{t_n} f \rangle \to 0$ as $n \to \infty$

$\{t_n\}$ depend on $A$ but not on $f$

In summary, we have set

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A).$$

<u>Remark</u>: The subspaces $\mathcal{H}_p(A)$, $\mathcal{H}_{ac}(A)$, $\mathcal{H}_{sc}(A)$ and also $\mathcal{D}(A)$, $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ are invariant under $\{U_t\}_{t\in\mathbb{R}}$.

Equivalently, the projections $P_p(A)$, $P_{ac}(A)$, $P_{sc}(A)$ and $\underbrace{E_0(A)}_{\mathrm{Ker}(A)}$ commute with $U_t$, $\forall t \in \mathbb{R}$.

<u>Remark</u> (spectral decomposition)

Since $\langle f, U_t f\rangle = \langle f, e^{-itA} f\rangle = \int_{\mathbb{R}} e^{-i\lambda t} \langle f, E(d\lambda) f\rangle$,

relation between the unitary group and the spectral measure.

If we set $\mu_f := \langle f, E(\cdot) f\rangle$

for the measure defined by the spectral measure

- $f \in \mathcal{H}_p(A)$ iff $\mu_f$ is a pure point measure,
- $f \in \mathcal{H}_{ac}(A)$ iff $\mu_f$ is absolutely continuous with respect to the Lebesgue measure,
- $f \in \mathcal{H}_{sc}(A)$ iff $\mu_f$ is singular continuous w.r.to _____.

$\rightsquigarrow$ Possible extensions:

<u>mean ergodic</u> thm + <u>RAGE thm</u>.

## II.2 Wave operators

Consider $(H, \mathcal{D}(H))$ self-adjoint in $\mathcal{H}$.

<u>Question</u> How can describe $e^{-itH}f$ for $|t| \to \infty$.
asymptotic behavior.

<u>Idea 1</u>: Find a "simpler" $(H_0, \mathcal{D}(H_0))$ also self-adjoint s.t.

there exist $f_\pm \in \mathcal{H}$ with

$$\| \underbrace{e^{-itH}}_{\text{unitary}} f - e^{-itH_0} f_\pm \| \xrightarrow{t \to \pm\infty} 0$$

$$\Longleftrightarrow \quad \| e^{-itH} ( f - e^{itH} e^{-itH_0} f_\pm ) \| \xrightarrow{t \to \pm\infty} 0$$

$$\Longleftrightarrow \quad \| f - \underline{e^{itH} e^{-itH_0} f_\pm} \| \xrightarrow{t \to \pm\infty} 0$$

<u>Idea</u> 2 (more complicated)

Find a simpler $(H_0, \mathcal{D}(H_0))$ in another $\mathcal{H}_0$, an operator $\underbrace{J}_{\text{identification operator}} : \mathcal{H}_0 \to \mathcal{H}$ and $f_\pm \in \mathcal{H}_0$ such that

$$\| e^{-itH} f - J e^{-itH_0} f_\pm \| \xrightarrow{t \to \pm\infty} 0$$

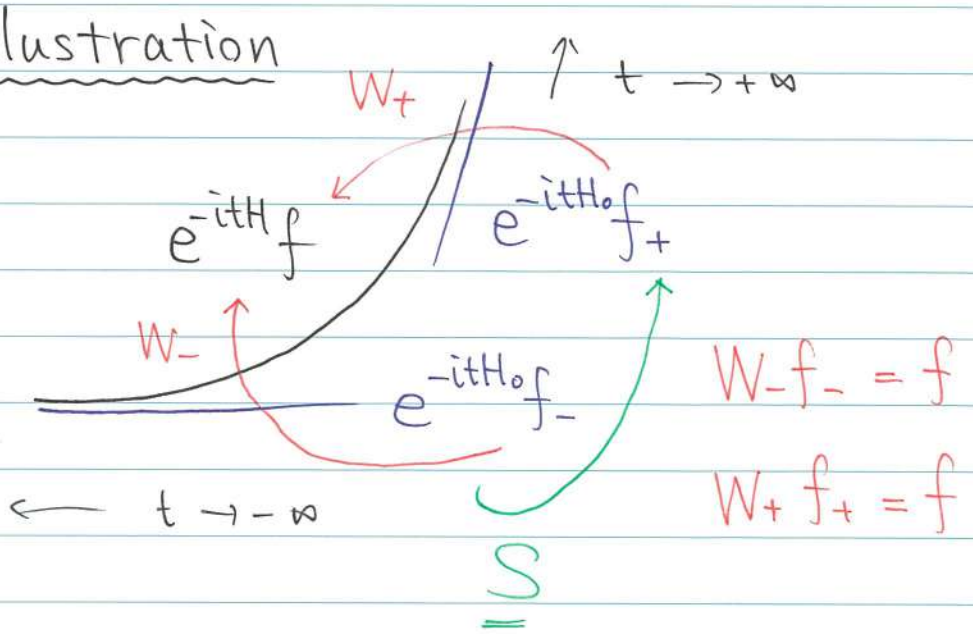
$$\Longleftrightarrow \quad \underbrace{\| f - e^{itH} J e^{-itH_0} f_\pm \| \xrightarrow{t \to \pm\infty} 0}_{\text{not used now.}}$$

<u>Def.</u>: Let $H, H_0$ be 2 self-adjoint operators in $\mathcal{H}$, and let $E$ be an orthogonal projection commuting with $e^{-itH_0}$. We set

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0} E =: W_\pm(H, H_0, E)$$

if it exists and call it the <u>wave operator</u>

<u>Illustration</u>



<u>Remark</u> Usual choice $E = P_{ac}(H_0) \equiv E_{ac}(H_0)$ but other choices are sometimes convenient. If we choose $E = E_{ac}(H_0)$, we simply write $W_\pm(H, H_0)$.

<u>Proposition</u> Suppose that $W_\pm(H, H_0, E)$ exist.

- $W_\pm$ is a partial isometry, and an isometry iff $E = \mathbb{1}$,

intertwining property

- $e^{-isH} W_\pm = W_\pm e^{-isH_0}$,
- $\varphi(H) W_\pm = W_\pm \varphi(H_0) \quad \forall \varphi \in C_b(\mathbb{R})$
- $E^H(\mathcal{V}) W_\pm = W_\pm E^{H_0}(\mathcal{V}) \quad \forall \mathcal{V} \in \mathcal{A}_B$

the spectral measure for $H$ — the spectral meas. for $H_0$

- $H W_\pm = W_\pm H_0$ on $\mathcal{D}(H_0)$

Proof.

1) If $f \perp E\mathcal{H}$, then $W_\pm f = 0$.

If $f \in E\mathcal{H}$, then

$$\|W_\pm f\| = \lim_{t\to\pm\infty} \|e^{itH} e^{-itH_0} \overset{Ef}{\overset{\parallel}{f}}\| = \|f\|.$$

strong limit preserves the norm

2) $e^{-isH} W_\pm f$

$$= \operatorname*{s-lim}_{t\to\pm\infty} \underbrace{e^{-isH} e^{itH}}_{e^{i(t-s)H}} e^{-itH_0} Ef$$

$t' := t-s$

$$= \operatorname*{s-lim}_{t'\to\pm\infty} e^{it'H} e^{-it'H_0} e^{-isH_0} Ef$$

$$= \operatorname*{s-lim}_{t'\to\pm\infty} e^{it'H} e^{-it'H_0} E\, e^{-isH_0} f$$

$$= W_\pm e^{-isH_0} f$$

3) Approximation argument in suitable topology.

4) $W_\pm \frac{i}{t}(e^{-itH_0} - 1) f \xrightarrow[s]{t\to 0} W_\pm H_0 f \quad \forall f \in \mathcal{D}(H_0)$

$\parallel$ intertwining relation

$$\frac{i}{t}(e^{-itH} - 1) \underbrace{W_\pm f}_{\in \mathcal{D}(H) \text{ by existence of limit.}} \xrightarrow[s]{t\to 0} H W_\pm f.$$

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<u>Proposition</u> Consider $H, H_0, E$ and $W_\pm(H, H_0, E)$

and set $F := W_\pm W_\pm^*$ (final range projection)

Exercise
- $F e^{-itH} = e^{-itH} F$
- $W_\pm(H_0, H, F)$ exist, and

$$W_\pm(H_0, H, F) = W_\pm(H, H_0, E)^*$$

- If $E\mathcal{H} \subset \mathcal{H}_{ac}(H_0)$, then $F\mathcal{H} \subset \mathcal{H}_{ac}(H)$.

<u>Proof.</u> If $g \in F\mathcal{H}$, it implies that $\exists f \in \mathcal{H}_{ac}(H_0)$

s.t. $g = W_\pm(H, H_0, E) f$. Then,

$$\langle g, E^H(\cdot) g \rangle = \langle W_\pm f, E^H(\cdot) W_\pm f \rangle$$

any Borel set

intertwining relation

$$= \langle W_\pm f, W_\pm E^{H_0}(\cdot) f \rangle$$

$$= \langle W_\pm^* W_\pm f, E^{H_0}(\cdot) f \rangle$$

initial range projection $= E$

$$= \langle f, E^{H_0}(\cdot) f \rangle,$$

which defines an a.c. measure since $f \in \mathcal{H}_{ac}(H_0)$ $\square$

This computation motivates the following definition.

<u>Def:</u> The wave op $W_\pm(H, H_0, E_{ac}(H_0)) \equiv W_\pm$ are complete if $\operatorname{Ran} W_\pm = \mathcal{H}_{ac}(H)$.

They are asymptotically complete if

$$\operatorname{Ran} W_\pm = \mathcal{H}_{ac}(H) = \mathcal{H}_p(H)^\perp \Bigg\} \Rightarrow \mathcal{H}_{sc}(H) = \{0\}.$$

ortho. complement

<u>Lemma</u> $W_\pm(H, H_0, E_{ac}(H_0))$ are complete iff $W_\pm(H_0, H, E_{ac}(H))$ exist.

<u>Proof.</u> 1) Assume that $W_\pm(H, H_0, E_{ac}(H_0))$ are complete $\Rightarrow \forall g \in \mathcal{H}_{ac}(H)\ \exists f \in \mathcal{H}_{ac}(H_0)$ s.t.

$$g = W_\pm(H, H_0, E_{ac}(H_0)) f$$

$$\Longleftrightarrow \quad 0 = \lim_{t \to \pm\infty} \| e^{itH} e^{-itH_0} f - g \|$$

unitarity

$$= \lim_{t \to \pm\infty} \| f - e^{itH_0} e^{-itH} g \|$$

$$\Longrightarrow \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_0} e^{-itH} E_{ac}(H) \text{ exist}$$

$$\equiv W(H_0, H, E_{ac}(H)).$$

2) If $W_\pm(H_0, H, E_{ac}(H))$ exists

$\Longrightarrow$ $\forall g \in \mathcal{H}_{ac}(H)$ one has

$$f := W_\pm(H_0, H, E_{ac}(H))\, g \in \mathcal{H}_{ac}(H_0)$$

proved before.

$$\Longleftrightarrow 0 = \lim_{t\to\pm\infty} \| f - e^{itH_0} e^{-itH} g \|$$

$$= \lim_{t\to\pm\infty} \| e^{itH} e^{-itH_0} f - g \|$$

$$\Rightarrow g = W_\pm(H, H_0, E_{ac}(H_0)) f$$

$$\Rightarrow \operatorname{Ran}\left(W_\pm(H, H_0, E_{ac}(H_0))\right) = \mathcal{H}_{ac}(H). \quad \square$$

Thus, if $W_\pm(H, H_0, E_{ac}(H_0))$ are complete, these operators define a unitary map from $\mathcal{H}_{ac}(H_0)$ to $\mathcal{H}_{ac}(H)$, and $H|_{\mathcal{H}_{ac}(H)}$ and $H_0|_{\mathcal{H}_{ac}(H_0)}$ are unitarily equivalent.

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$$\left( \begin{array}{l} \text{In } \mathcal{H} = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{sc}(H) \oplus \mathcal{H}_p(H), \\ \text{the op } H \text{ takes the form} \\ \qquad H = H\big|_{\mathcal{H}_{ac}(H)} \oplus H\big|_{\mathcal{H}_{sc}(H)} \oplus H\big|_{\mathcal{H}_p(H)}. \end{array} \right)$$

<u>Remark</u> For the asymptotic completeness, we need a stronger argument.

<u>Question</u> How can we prove completeness or asymptotic completeness?

## II.3 <u>A few criteria.</u>

<u>Lemma</u>. Let $\{W_t\}_{t\in\mathbb{R}} \subset B(\mathcal{H})$ and $f \in \mathcal{H}$ s.t. $\{W_t f\}_{t\in\mathbb{R}} \subset \mathcal{H}$ is strongly $C^1$ and s.t.

$$\left( t \longmapsto \| (W_t f)' \| \right) \in L^1((1,\infty))$$

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Then, $\{W_t f\}_{t\in\mathbb{R}}$ is strongly convergent as $t \longrightarrow +\infty$.

<u>Proof.</u> One has

$$\|W_{t_1} f - W_{t_2} f\|$$

$$= \left\| \int_{t_1}^{t_2} (W_s f)' \, ds \right\|$$

$$\leq \int_{t_1}^{t_2} \|(W_s f)'\| \, ds \xrightarrow{t_1, t_2 \to \infty} 0$$

since the function is $L^1$.

$\Longrightarrow \{W_t f\}_{t\in\mathbb{R}}$ is Cauchy.

$\Longrightarrow \{W_t f\}_{t\in\mathbb{R}}$ is convergent since $\mathcal{H}$ is complete in the strong topology $\square$.

The argument is similar for $t \longrightarrow -\infty$.

<u>Proposition</u> <u>Cook's Criterion</u>

Suppose $\mathcal{D}(H_0) = \mathcal{D}(H)$ and that

$$\left(t \mapsto \| (H-H_0) e^{-itH_0} f \|\right) \in L^1((1,\infty))$$

for $f$ in a dense subset of $E$ ~~fH~~

contained in $\mathcal{D}(H_0)$. Then,

$$W_+ (H, H_0, E) \text{ exist.}$$

<u>Key observation</u> for the proof:

Consider $W_t := e^{itH} e^{-itH_0}$ and observe that

$$\left\| \frac{d}{dt} W_t f \right\| = \| e^{itH} (iH - iH_0) e^{-itH_0} f \|$$

$$= \| (H - H_0) e^{-itH_0} f \|$$

We have used that $\mathcal{D}(H) = \mathcal{D}(H_0)$ and

$$e^{-itH_0} \mathcal{D}(H_0) = \mathcal{D}(H_0).$$

$\Rightarrow$ by the previous lemma.

$\{W_t f\}_{t \in \mathbb{R}}$ is a strongly convergent sequence

Finally, a denisty argument. //

For completeness, there are 2 different approaches: the trace class approach and the smooth approach.

(Kato, T. has played an important role for both approaches)

<u>Thm</u> (developed by Kato, Rosenblum and Pearson)

([Kato] Thm X.4.4, p 542).

Let $H_1, H_2$ be arbitrary self-adjoint op. such that $H_1 - H_2 \in B_1(\mathcal{H})$

trace class.

Then, $W_\pm(H_2, H_1, E_{ac}(H_1))$ and $W_\pm(H_1, H_2, E_{ac}(H_2))$ exist. $\Rightarrow$ We have Completeness.

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<u>Remark</u> The condition $H_1 - H_2 \in B_1(\mathcal{H})$ is very strong, and thus not really applicable. In the next statement, this condition is weakened and the result is stronger.

Let us introduce

$$\phi : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \{\lambda_1, \cdots, \lambda_n\} \subset \mathbb{R}$$

s.t. $\phi \in C^1((\lambda_j, \lambda_{j+1}))$ for $j = 0, \cdots, n$

with $\lambda_0 := -\infty$ and $\lambda_{n+1} = +\infty$,

$\phi'$ is [redacted] locally of bounded variation ← taken from book of Kato.

$$\underline{\underline{\phi'(\lambda)}} > 0 \quad \text{on} \quad (\lambda_j, \lambda_{j+1})$$

$\phi'$ : abs conti.

<u>Thm</u> (Kato, Thm X.4.12)

Let $(H_1 - z)^{-1} - (H_2 - z)^{-1} \in B_1(\mathcal{H})$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, and let $\phi$ be as above. Then,

$$W_\pm(\phi(H_j), \phi(H_k), E_{ac}(H_k))$$

exist for $j, k \in \{1, 2\}$, $j \neq k$, they are complete, and are independent of $\phi$. In particular, $W_\pm(H_j, H_k, E_{ac}(H_k))$ exist and are complete.

This property is called the invariance principle.

The proof takes 17 pages in Kato's book. The main ingredients are

- baby lemma (later)

- If $A \in B_1$ and self-adjoint,

then $A = \sum_{j=1}^{\infty} c_j |f_j\rangle\langle f_j|$

with $\|f_j\| = 1$ and $\langle f_j, f_k \rangle = 0$ if $j \neq k$

and $\sum_{j=1}^{\infty} |c_j| < \infty$.

- the following technical lemma: $\forall \psi \in L^2(\mathbb{R})$

$$\int_0^\infty \left| \text{l.i.m} \int_{-\infty}^{\infty} e^{-i\lambda t} e^{-is\phi(\lambda)} \psi(\lambda)\, d\lambda \right|^2 dt \xrightarrow{s \to \infty} 0$$

$$\Longleftrightarrow \quad \operatorname*{s-lim}_{s \to \infty} \chi_{(0,\infty)}(X)\, \mathcal{F}\, e^{-is\phi(X)} = 0$$

<u>Lemma</u> Let $\mathcal{H} = L^2(\mathbb{R})$, $H_1 = X$ and

$$\left( [Xf](x) = x f(x) \quad \forall f \in \mathcal{H} \right).$$

$$H_2 := H_1 + c|f\rangle\langle f| \quad , \quad c \in \mathbb{R}, \ f \in \mathcal{H}.$$

take $\|f\| = 1$.

Then, $W_\pm(H_2, H_1, \underbrace{E_{ac}(H_1)}_{= \mathbb{1}})$ exist.

<u>proof</u>: 1) Observe that for $u \in \mathcal{H}$

$$\|(H_2 - H_1) e^{-iH_1 t} u\| = \| c|f\rangle\langle f| e^{-itX} u\|$$

$$= |c| \, |\langle f, e^{-itX} u\rangle| \, \|f\|$$

$$= |c| \left| \int_{\mathbb{R}} e^{-itx} \overline{f(x)} \, u(x) \, dx \right|.$$

If $f, u \in \mathcal{S}(\mathbb{R})$ (Schwartz space), then

$$\left( t \longmapsto \int_{\mathbb{R}} e^{-itx} \overline{f(x)} \, u(x) \, dx \right) \in \mathcal{S}(\mathbb{R})$$

and then it is in $L^1(\mathbb{R})$. Then,

by Cook's criterion, the wave op exist.

2) Estimate on the rate of convergence.

Still for $u, f \in \mathcal{S}(\mathbb{R})$,

$$W(t) := e^{itH_2} e^{-itH_1} \xrightarrow[s]{t \to +\infty} W_+ .$$

One has

$$\| (W_+ - W(t)) u \|^2$$

$$= \underbrace{\| W_+ u \|^2}_{\| u \|^2} + \| W(t) u \|^2 - \langle W_+ u, W(t) u \rangle - \langle W(t) u, W_+ u \rangle$$

$$= 2 \| W_+ u \|^2 - 2 \operatorname{Re} \langle W(t) u, W_+ u \rangle$$

$$= -2 \operatorname{Re} \langle (W(t) - W_+) u, W_+ u \rangle$$

$$= 2 \operatorname{Re} \left\langle \int_t^\infty \left( \frac{d}{ds} e^{+isH_2} e^{-isH_1} u \right) ds, W_+ u \right\rangle$$

Since $u \in \mathcal{D}(H_1) = \mathcal{D}(H_2)$

$$= -2 \operatorname{Im} \left\langle \int_t^\infty \left( e^{isH_2} \cdot c |f\rangle\langle f| e^{-isH_1} u \right) ds, W_+ u \right\rangle$$

$$= -2c \operatorname{Im} \int_t^\infty \langle e^{isH_1} W_+^* |f\rangle\langle f| e^{-isH_1} u, u \rangle \, ds$$

intertwining relation for $W_+^*$

$$\leq 2|c| \int_t^\infty |\langle e^{isH_1} W_+^* f, u\rangle \langle f, e^{-isH_1} u\rangle| \, ds$$

$$\leq 2|c| \underbrace{\left( \int_t^\infty |\langle e^{isH_1} W_+^* f, u\rangle|^2 ds \right)}_{\circledast}^{1/2}$$

$$\times \underbrace{\left( \int_t^\infty |\langle f, e^{-isH_1} u\rangle|^2 ds \right)^{1/2}}_{\circledast}$$

One has

$$\circledast \leq 2\pi \, \| \mathcal{F}(\bar{u} W_+^* f) \|^2$$

$$= 2\pi \, \| \bar{u} \, W_+^* f \|^2$$

$$\leq 2\pi \, \|u\|_{L^\infty}^2 \, \|W_+^* f\|$$

$$\leq 2\pi \, \|u\|_{L^\infty}^2 \, \|f\|^2 .$$

$$\circledast \leq 2|c| \sqrt{2\pi} \, \|u\|_{L^\infty} \left( \int_t^\infty |\langle e^{-isH_1} u, f\rangle|^2 ds \right)^{1/2} .$$

Then,

$\pm W_+$

$$\|(W(t) - W(t'))u\|$$

$$\leq |c|^{1/2} (8\pi)^{1/4} \|u\|_{L^\infty}^{1/2} \left( \left[ \int_t^\infty |\langle e^{-isH_1} u, f\rangle|^2 ds \right]^{1/4} + \left[ \int_{t'}^\infty |\langle e^{-isH_1} u, f\rangle|^2 ds \right]^{1/4} \right)$$

$\ast\ast$

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"Valid" for $u \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$.

3) By approximation, show that (**) is indeed valid for $u \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$. (Kato. p539).

4) Since the integrals on the r.h.s. of (**) exist for any $f \in \mathcal{H}$, $\|f\| = 1$ and $u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the limit $\operatorname{s-lim}_{t \to \pm\infty} W(t)$ exist on a dense subset of $\mathcal{H}$ $\Longrightarrow$ $W_\pm$ exist by Cook's criterion.

$\square$

<u>Second approach</u>: smooth approach.

<u>Def</u>: Let $H, H_0$ be self-adj. op in $\mathcal{H}$, with spectral measure $E^{H_0}$, $E^H$ respectively. Let $J \subset \mathbb{R}$ be an open set. The <u>local wave operators</u> $W_\pm(H, H_0, E^{H_0}(J))$ are <u>complete</u> $\mathrm{Ran}\left(W_\pm(H, H_0, E^{H_0}(J))\right)$ coincide with $E^H(J)\mathcal{H}$.

By the same proof as before, $W_\pm(H, H_0, E^{H_0}(J))$ are complete iff $W_\pm(H_0, H, E^H(J))$ exist.

<u>Def.</u> Let $H = H^*$ and let $G : \mathcal{D}(H) \to \mathcal{G}$ be linear and <u>continuous</u>, with $\mathcal{G}$ a new Hil. space. $G$ is locally $H$-smooth on an open interval $J \subset \mathbb{R}$ if $\forall K \subset J$ compact, $\exists C_K < \infty$ s.t.

$$\int_{\mathbb{R}} \| G e^{itH} f \|_{\mathcal{H}}^2 dt \leq C_K \|f\|^2$$

for $\forall f \in \mathcal{H}$ with $\chi_K(H) f \equiv E^H(K) f = f$.

<u>Remark</u>: Here $\mathcal{D}(H)$ is endowed with the graph norm, i.e. $\|f\|_{\mathcal{D}(H)}^2 = \|f\|^2 + \|Hf\|^2$ $(\forall f \in \mathcal{D}(H))$.

<u>Thm</u> Let $H_0 = H_0^*$, $H = H^*$ with spectral measures $E^{H_0}$, $E^H$ resp. let $J \subset \mathbb{R}$ be an open interval and assume that

$$\langle Hg, f \rangle - \langle g, H_0 f \rangle = \langle Gg, G_0 f \rangle$$

$$\forall g \in \mathcal{D}(H),\ f \in \mathcal{D}(H_0)$$

with $G_0$ (resp. $G$) locally $H_0$-smooth (resp. $H$-smooth) on $J$.

Then $W_\pm(H, H_0, E^{H_0}(\mathcal{J}))$ and

$W_\pm(H_0, H, E^{H}(\mathcal{J}))$ exist.

$\rightsquigarrow$ completeness. ✓

<u>Proof.</u>

($1^{st}$ part): Assume $f = \chi_K(H_0) f$ and $\in \mathcal{D}(H_0)$

$g = \chi_K(H) g \in \mathcal{D}(H)$. One has

$$\left| \langle g, (W(t_2) - W(t_1)) f \rangle \right|$$

$$= \left| \int_{t_1}^{t_2} \langle g, \frac{d}{ds} e^{isH} e^{-isH_0} f \rangle ds \right|$$

$$= \int_{t_1}^{t_2} \left| \langle -iH e^{-isH} g, e^{-isH_0} f \rangle + \langle e^{-isH} g, \overset{-iH_0}{\vee} e^{-iH_0 s} f \rangle \right| ds$$

$$= \int_{t_1}^{t_2} \left| \langle H e^{-isH} g, e^{-isH_0} f \rangle - \langle e^{-isH} g, H_0 e^{-isH_0} f \rangle \right| ds$$

$$= \int_{t_1}^{t_2} \left| \langle G e^{-isH} g, G_0 e^{-isH_0} f \rangle \right| ds$$

$$\leq \int_{t_1}^{t_2} \| G e^{-isH} g \| \, \| G_0 e^{-isH_0} f \| ds$$

$$\leq \left(\int_{t_1}^{t_2} \|G e^{-isH} g\|^2 ds\right)^{1/2} \left(\int_{t_1}^{t_2} \|G_0 e^{-isH_0} f\|^2 ds\right)^{1/2}$$

$$\leq C_K^{1/2} \|g\| \left(\int_{t_1}^{t_2} \|G_0 e^{-isH_0} f\|^2 ds\right)^{1/2}$$

We infer by taking a sup on $g \in \mathcal{H}_P$, $\|g\|=1$ s.t. $E^H(K)g=g$,

$$\operatorname*{s-lim}_{t\to\pm\infty} E^H(J)\, e^{itH} e^{-itH_0} E^{H_0}(J) \quad \text{exist}$$

to be replaced by $\mathbb{1}$. ← to be done in 1 or 2 weeks

<u>Remark</u> The existence of wave op. $\operatorname*{s-lim}_{t\to\pm\infty} E^{H_0}(J) e^{itH_0} e^{-itH} E^H(J)$ is proved similarly from ⊛, because the assumption is symmetric between $H_0$ and $H$.

## II.4 Scattering Operator

If $W_\pm(H, H_0, E_{ac}(H_0))$ are complete, we set

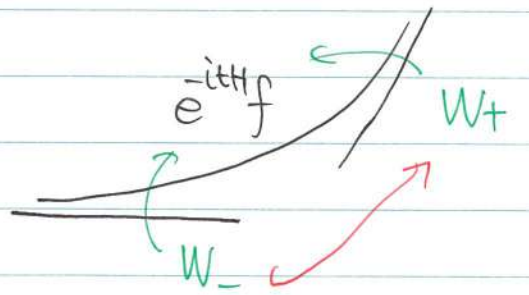
$$S := W_+(H, H_0, E_{ac}(H_0))^* W_-(H, H_0, E_{ac}(H_0))$$

and is called the scattering operator.

This operator is unitary from $\mathcal{H}_{ac}(H_0)$ to itself (because of completeness), and

$$[S, e^{-itH_0}] = 0$$

(by the intertwining relation)



# III) Scattering theory : time independent approach.

## III.1 Local H-smoothness.

Let $\mathcal{H}, \mathcal{G}$ be 2 Hilbert spaces, let $H = H^*$ and $J \subset \mathbb{R}$, open int. (finite!!) Let $G : \mathcal{D}(H) \to \mathcal{G}$

Then, $G_J := G E^H(J) \in \mathcal{B}(\mathcal{H}, \mathcal{G})$

<u>Lemma</u> The following are equivalent (and equal)

1) $$\gamma_1^2 := \frac{1}{2\pi} \sup_{\substack{f \in \mathcal{H} \\ \|f\|=1}} \int_{-\infty}^{\infty} \| G_J e^{-itH} f \|^2 dt < \infty,$$

2) $$\gamma_2^2 := \frac{1}{(2\pi)^2} \sup_{\substack{\|f\|=1, \\ \varepsilon > 0}} \int_{-\infty}^{\infty} \left\{ \| G_J R(\lambda + i\varepsilon) f \|^2 + \| G_J R(\lambda - i\varepsilon) f \|^2 \right\} d\lambda < \infty$$

with $R(z) = (H - z)^{-1}$.

3) $$\gamma_3^2 := \sup_{\|f\|=1, \varepsilon > 0} \int_{-\infty}^{\infty} \| G_J \delta_\varepsilon (H - \lambda) f \|^2 d\lambda < \infty$$

with $$\delta_\varepsilon(H - \lambda) = \frac{1}{2\pi i} \left( R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon) \right) = \frac{\varepsilon}{\pi} R(\lambda + i\varepsilon) R(\lambda - i\varepsilon)$$

$$= \frac{1}{\pi} \operatorname{Im} R(\lambda + i\varepsilon).$$

4) $\gamma_4^2 := \sup_{\substack{\lambda \in \mathbb{R} \\ \varepsilon > 0}} \| G_J R(\lambda + i\varepsilon) G_J^* \| < \infty,$

5) $\gamma_5^2 := \sup_{\substack{\Lambda \subset \mathbb{R} \\ (\text{interval})}} \frac{1}{|\Lambda|} \| G_J E^H(\Lambda) \|^2 < \infty.$

<u>Proof.</u> (We write $G$ for $G_J$).

Recall that if $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

and $\check{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda t} g(t)\, dt$, then

$$\int_{\mathbb{R}} |g(t)|^2 dt = \|g\|^2 = \|\check{g}\|^2 = \int |\check{g}(\lambda)|^2 d\lambda$$

If $g \in L^1(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{H})$ then

it is still true (replace $|\cdot| \rightsquigarrow \|\cdot\|$ norm in $\mathcal{H}$).

One has

$$\int_{-\infty}^{\infty} \| G e^{-itH} f \|^2 dt$$

$$= \sup_{\varepsilon > 0} \left\{ \int_{-\infty}^{0} e^{2\varepsilon t} \| G e^{-itH} f \|^2 dt + \int_{0}^{\infty} e^{-2\varepsilon t} \| G e^{-itH} f \|^2 dt \right\}$$

$$= \sup_{\varepsilon>0} \left\{ \int_{-\infty}^{0} \| G e^{\varepsilon t} e^{-itH} f \|^2 dt + \int_{0}^{\infty} \| G e^{-\varepsilon t} e^{-itH} f \|^2 dt \right\}$$

$$= \sup_{\varepsilon>0} \left\{ \int_{\mathbb{R}} \| \underbrace{\chi_-(t) G e^{\varepsilon t} e^{-it} f}_{=: g_-} \|^2 dt + \int_{\mathbb{R}} \| \underbrace{\chi_+(t) G e^{-\varepsilon t} e^{-itH} f}_{=: g_+} \|^2 dt \right\}$$

$$= \sup \left\{ \int_{\mathbb{R}} \| \check{g}_-(\lambda) \|^2 d\lambda + \int_{\mathbb{R}} \| \check{g}_+(\lambda) \|^2 d\lambda \right\} \quad (*)$$

with $\check{g}_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\lambda t} G e^{\varepsilon t} e^{-itH} f \, dt$ $\quad (\varepsilon = \bar{z}(-i\varepsilon))$

$$= \frac{1}{\sqrt{2\pi}} i G (H - \lambda + i\varepsilon)^{-1} f$$

$$\check{g}_+(\lambda) = \frac{1}{\sqrt{2\pi}} (-i) G (H - \lambda - i\varepsilon)^{-1} f$$

Then, $(*)$

$$= \frac{1}{2\pi} \sup_{\varepsilon>0} \left\{ \int_{\mathbb{R}} \left\{ \| G(H-\lambda+i\varepsilon)^{-1} f \|^2 + \| G(H-\lambda-i\varepsilon)^{-1} f \|^2 \right\} d\lambda \right\}$$

By taking sup on $f \in \mathcal{H}$, $\|f\| = 1$, we get $\gamma_1 = \gamma_2$.

One has

$$\int_{\mathbb{R}} \left\| G \frac{1}{2\pi i} \left( R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon) \right) f \right\|^2 d\lambda$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \Big\{ \| G R(\lambda+i\varepsilon) f \|^2 + \| G R(\lambda-i\varepsilon) f \|^2$$

$$+ \underbrace{\langle G R(\lambda+i\varepsilon) f, G R(\lambda-i\varepsilon) f \rangle}_{=0}$$

$$+ \langle G R(\lambda-i\varepsilon) f, G R(\lambda+i\varepsilon) f \rangle \Big\} d\lambda$$

since their Fourier transform ($g_+$ or $g_-$) have disjoint support ($\langle g, h \rangle = \langle \mathcal{F}g, \mathcal{F}h \rangle$).

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \left( \| G R(\lambda+i\varepsilon) f \|^2 + \| G R(\lambda-i\varepsilon) f \|^2 \right) d\lambda$$

$$\rightsquigarrow \quad \gamma_2 = \gamma_3 .$$

One has

$$\int_{\mathbb{R}} \| G \delta_\varepsilon (H-\lambda) f \|^2 d\lambda$$

$$\leq \sup_{\lambda \in \mathbb{R}} \| G \delta_\varepsilon (H-\lambda)^{1/2} \| \int_{\mathbb{R}} \| \delta_\varepsilon (H-\lambda)^{1/2} f \|^2 d\lambda$$

$$= \sup_{\lambda} \| G \delta_\varepsilon (H-\lambda) G^* \| \cdot \| f \|^2$$

since $\|B\|^2 = \|BB^*\|$ for any $B \in B(\mathcal{H})$.

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Since

$$\int_{\mathbb{R}} \| \delta_\varepsilon (H-\lambda)^{1/2} f \|^2 d\lambda = \int_{\mathbb{R}} \langle f, \delta_\varepsilon (H-\lambda) f \rangle d\lambda$$

and $\int_{\mathbb{R}} \delta_\varepsilon (x)\, dx = 1$ for any $\varepsilon > 0$.

Thus, we get $\gamma_3 \leq \gamma_4$. Then we show that $\gamma_4 < \underbrace{\gamma_5 < \gamma_3}$, we need the functional calculus + Stone formula. □

<u>Remark.</u>

1) In 4) we have

$$\sup_{\lambda \in \mathbb{R}, \varepsilon > 0} \| G_{\mathcal{J}} \delta_\varepsilon (H-\lambda) G_{\mathcal{J}}^* \|$$

$$= \sup_{\lambda \in \mathcal{J}, \varepsilon > 0} \| G_{\mathcal{J}} \delta_\varepsilon (H-\lambda) G_{\mathcal{J}}^* \|$$

one has to prove something.

2) As already seen, an operator $G : \mathcal{D}(H) \to \mathcal{G}$ satisfying one of these conditions

is called H-smooth on $J$.

<u>Question</u> How can one find locally H-smooth op?

$\rightsquigarrow$ Commutator method or Mourre theory provide a framework for this

<u>Example</u> If $H = -\Delta$ in $L^2(\mathbb{R}^d)$, then $\langle X \rangle^{-\frac{1}{2}-\varepsilon}$ is loc. H.smooth on any interval of $\mathbb{R}^*$, for any $\varepsilon > 0$.

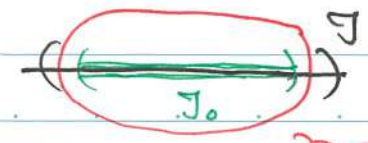
End of proof of the thm of last week:

One has to show that

$$\operatorname*{s-lim}_{t\to\infty} E^H(\mathbb{R}\setminus J)\, e^{itH} e^{-itH_0} E^{H_0}(J) = 0$$

Commute Commute.

Since we can show the statement on a dense set, let us consider $J_0 \subset J$, with strict inclusion.




$\gamma \subset \mathbb{C}$

and we consider a path $\gamma$ circling $\mathcal{J}_0$ in $\mathbb{C}$. Then, one has

$$2\pi i E^{H_0}(\mathcal{J}) = -\int_\gamma R_0(z) E^{H_0}(\mathcal{J}_0)\, dz$$

and $\displaystyle\int_\gamma E^H(\underline{\mathbb{R}\setminus\mathcal{J}})\, R(z)\, dz = 0$

since $\mathcal{J}_0 \cap \mathbb{R}\setminus\mathcal{J}_0 = \phi$

$$\Rightarrow \quad 2\pi i\, E^H(\mathbb{R}\setminus\mathcal{J})\, E^{H_0}(\mathcal{J}_0)$$

$$= \int_\gamma E^H(\mathbb{R}\setminus\mathcal{J}) \big(\underbrace{R(z) - R_0(z)}\big) E^{H_0}(\mathcal{J})\, dz. \quad (*)$$

One has

$$\langle R(\bar{z}) g, f\rangle - \langle g, R_0(z) f\rangle$$

$$= \langle R(\bar{z}) g, (H_0 - z) R_0(z) f\rangle$$

$$- \langle (H - \bar{z}) R(\bar{z}) g, R_0(z) f\rangle$$

$$= \langle R(\bar{z}) g, H_0 R_0(z) f\rangle$$

$$- \langle H R(\bar{z}) g, R_0(z) f\rangle$$

$$= - \langle G R(\bar{z}) g, \; G_0 R_0(z) f \rangle$$

$$\Rightarrow \langle g, \; \underline{(R(z) - R_0(z))} f \rangle$$

$$= - \langle g, \; \underline{(G R(\bar{z}))^* G_0 R_0(z) f} \rangle$$

$$\circledast = - E^H(\mathbb{R} \setminus J) \int_\gamma \underbrace{(G R(\bar{z}))^*}\, \underbrace{G_0 R_0(z) E^{H_0}(J_0)}\, dz$$

Let us set $\varphi(t) := \underbrace{G_0 R_0(z)} e^{-itH_0} E^{H_0}(J_0) f$.

is also $H_0$-smooth on $J$

One has (since $G_0 R_0(z)$ is $H_0$-smooth on $J$)

$$\int_1^\infty \| \varphi(t) \|^2 \, dt < \infty .$$

Also $\| \varphi'(t) \|$ is bounded

$$\Longrightarrow \| \varphi'(t) \| \in o(1) \quad \text{as } t \to \infty$$

$$\Longrightarrow E^H(\mathbb{R} \setminus J) (G R(\bar{z}))^* G_0 R_0(z) e^{-itH_0} E^{H_0}(J_0) \xrightarrow[\text{strongly}]{t \to \infty} 0$$

By taking care of the dependence on $z$ for $z \in \gamma$, one gets that

$$E^{H}(\mathbb{R} \setminus \mathcal{J})\, e^{-itH_0} E^{H_0}(\mathcal{J}_0) \xrightarrow[\text{strongly}]{t \to \infty} 0$$

$e^{itH}$

$\#$

## III.2 Some resolvent equation

Question: Can one get $H$-smoothness from $H_0$-smoothness, by a perturbative argument?

For any self-adjoint op. $(H, D(H))$, we can consider $D(H)$ with a new scalar product (and norm)

$$\langle f, g \rangle_{D(H)} := \langle f, g \rangle_{\mathcal{H}} + \langle Hf, Hg \rangle_{\mathcal{H}} \quad \forall f, g \in D(H)$$

$$\|f\|_{D(H)} = \sqrt{\langle f, f \rangle_{D(H)}} = \sqrt{\|f\|^2 + \|Hf\|^2}$$

$\rightsquigarrow$ It makes $D(H)$ a new Hilbert space.

We call $\|\cdot\|_{D(H)}$ the graph norm of $H$

We shall often consider the situation

$$\langle Hf, g \rangle - \langle f, H_0 g \rangle = \langle Gf, G_0 g \rangle$$

$$\forall f \in D(H) \quad \forall g \in D(H_0) \quad (*)$$

with $G_0 \in B(D(H_0), \mathcal{Y})$, $G \in B(D(H), \mathcal{Y})$

<u>Remark</u> Since $\|(H+i)f\|^2 = \langle (H+i)f, (H+i)g \rangle =$

$$= \|Hf\|^2 + \|f\|^2 + \underbrace{\langle Hf, if \rangle + \langle if, Hf \rangle}_{=0} = \|f\|^2_{D(H)}$$

Then $G \in B(D(H), \mathcal{Y})$

$$\Longleftrightarrow \|Gf\|^2_{\mathcal{Y}} \leq c\|f\|^2_{D(H)} = c\|(H+i)f\|^2$$

$$\forall f \in D(H)$$

Set $f = (H+i)^{-1} g$ for some $g \in \mathcal{H}$

$$\Longleftrightarrow \|G(H+i)^{-1} g\|^2_{\mathcal{Y}} \leq c\|g\|^2_{\mathcal{H}}$$

$$\Longleftrightarrow G(H+i)^{-1} \in B(\mathcal{H}, \mathcal{Y})$$

$$\Longleftrightarrow G(H-z)^{-1} \in B(\mathcal{H}, \mathcal{Y}) \text{ for any } z \in \rho(H)$$

Now, from ⊛, one has for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\langle \underbrace{R(\bar{z})}_{(H-\bar{z})^{-1}} f, g \rangle - \langle f, \underbrace{R_0(z)}_{(H_0-z)^{-1}} g \rangle$$

$$= -\langle GR(\bar{z}) f, G_0 R_0(z) g \rangle \qquad ◎$$

$$= -\langle f, (GR(\bar{z}))^* G_0 R_0(z) g \rangle$$

$$\Rightarrow R(z) - R_0(z) = -\left(G R(\bar{z})\right)^* \left(G_0 R_0(z)\right)$$

<u>Question</u> Can we write

$$R(z) - R_0(z) = -\underbrace{R(z)}_{\in B(D(H)^*,\, \mathcal{H})} \underbrace{G^*}_{\in B(\mathcal{Y},\, D(H)^*)} G_0 R_0(z) \quad ?$$

dual space

Without loss of too much generality, we can consider $G_0 \in B(\mathcal{H}, \mathcal{Y})$ and $G \in B(\mathcal{H}, \mathcal{Y})$

We will assume this in the future

From now, we consider

$$R(z) - R_0(z) = -R(z)\, G^* G_0 R_0(z)$$

multiply by $G_0$ and $G^*$

$$\Rightarrow G_0 R(z) G^* - G_0 R_0(z) G^* = -\left(G_0 R(z) G^*\right)\left(G_0 R_0(z) G^*\right)$$

$$\Rightarrow \underline{\left(1 - G_0 R(z) G^*\right)\left(1 + G_0 R_0(z) G^*\right) = 1}$$

Suppose that $G^* G_0$ is self-adjoint

$$\Rightarrow G^* G_0 = \left(G^* G_0\right)^* = G_0^* G$$

From ⓪ with everything on the l.h.s, we get

$$R(z) - R_0(z) = -R_0(z)\, G_0^* G\, R(z)$$

$$= -R_0(z)\, G^* G_0\, R(z)$$

multiply by $G_0$ and $G^*$

$$\Longrightarrow G_0 R(z) G^* - G_0 R_0(z) G^* = -(G_0 R_0(z) G^*)(G_0 R(z) G^*)$$

$$\Longrightarrow \underline{(1 + G_0 R_0(z) G^*)(1 - G_0 R(z) G^*) = 1}$$

Thus, we get $(1 - G_0 R(z) G^*)$ and $(1 + G_0 R_0(z) G^*)$ are boundedly invertible, and inverse of each others.

Then we have

$$R(z) = R_0(z) - \overbrace{R(z)}^{R_0(z) - R_0(z) G^* G_0 R(z)}\, G^* G_0 R_0(z) \qquad \square$$

$$= R_0(z) - R_0(z)\, G^* G_0 R_0(z) + R_0(z)\, G^* G_0 R(z)\, G^* G_0 R_0(z)$$

$$= R_0(z) - R_0(z) G^* (1 - G_0 R(z) G^*)\, G_0 R_0(z)$$

$$= R_0(z) - R_0(z)\, G^* (1 + G_0 R_0(z) G^*)^{-1} G_0 R_0(z)$$

does not contain $R(z)$!

<u>Remark</u> This equality is valid for any $z \in \rho(H) \cap \rho(H_0) \subset \mathbb{C} \setminus \mathbb{R}$

Let us still set

$$B_0(z) := -G_0 R_0(z) G^*$$

$$B(z) := G R(z) G^*$$

Then one has from $\boxed{\square}$

$$\underbrace{G R(z) G^*}_{B(z)} = G R_0(z) G^* - \underbrace{G R(z) G^*}_{B(z)} G_0 R_0(z) G^*$$

$$\Rightarrow B(z)\left(1 + G_0 R_0(z) G^*\right) = G R_0(z) G^*$$

$$\Rightarrow \underline{B(z)} = G R_0(z) G^* \left(1 + G_0 R_0(z) G^*\right)^{-1}$$

$$= \underline{G R_0(z) G^* \left(1 - B_0(z)\right)^{-1}}$$

## III.3 Local existence and completeness of $W_\pm$

In the notation is introduced in the previous section

One has :

<u>Thm 1</u>. Let $J$ be an open interval and suppose that

1) $G_0$ is $H_0$-smooth on $J$

2) $GR_0(z)G^*$ is uniformly bounded for $\mathrm{Re}(z) \in J$ and $\mathrm{Im}(z) \neq 0$

3) $\sup_{\mathrm{Re}(z) \in J} \| G_0 R_0(z) G^* \| < 1$ (*)

Then $W_\pm(H, H_0, E^{H_0}(J))$ exist and are complete

($\Leftrightarrow$ existence of $W_\pm(H_0, H, E^H(J))$)

<u>Remark</u> (*) implies some smallness condition on the pertubation $G^* G_0$

<u>proof</u>

From the relation $B(z) = GR_0(z)G^*(1 - B_0(z))^{-1}$

together with the assumption 2) and 3)

We get that

$$\sup_{\substack{\mathrm{Re}(z) \in J \\ \mathrm{Im}(z) \neq 0}} \| B(z) \| = \sup_{\substack{\mathrm{Re}(z) \in J \\ \mathrm{Im}(z) \neq 0}} \| GR(z) G^* \| < \infty$$

$\Longrightarrow$ $G$ is $H$-smooth on $J$

With the assumption 1), we can use the main thm. of Section II.3 and get the result $\square$

<u>Remark</u>: We have seen that $G$ is $H$-smooth on $J$

if $\sup_{\substack{\lambda \in J \\ \varepsilon \neq 0}} \| G_J \operatorname{Im} R(\lambda + i\varepsilon) G_J^* \| < \infty$

but this is still equivalent to

$$\sup_{\substack{\lambda \in J \\ \varepsilon \neq 0}} \| G \operatorname{Im} R(\lambda + i\varepsilon) G^* \| < \infty$$

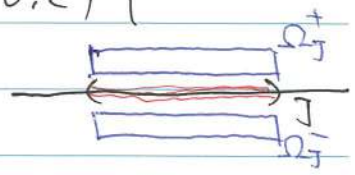
In order to avoid the $3^{rd}$ condition,

we can look more carefully at $(1 - B_0(z))^{-1}$

We need a preliminary result on holomorphic function.

<u>Thm</u> Let $J \subset \mathbb{R}$ an open interval and set

$$\Omega_J^{\pm} := \{ z \in \mathbb{C} \mid \operatorname{Re}(z) \in J, \ \pm \operatorname{Im} z \in (0, \varepsilon) \}$$




Consider the map $\Omega_J^{\pm} \ni z \mapsto A(z) \in B(\mathcal{Y})$ s.t.

1) it is holomorphic

2) $A^{\ell}(z) \in K(\mathcal{H})$ for some $\ell \in \mathbb{N}$ and all $z \in \Omega_J^{\pm}$

3) $\exists z_0 \in \Omega_J^{\pm}$ s.t. $(1 - A(z_0))^{-1} \in B(\mathcal{H})$

4) the map admits a continuous extension to $J \subset \mathbb{C}$.

Then, set $N := \{\lambda \in \bar{J} \mid 1 \in \sigma_p(A(\lambda \pm i0))\}$,

This set has Lebesgue measure $0$ and is closed.

In addition, the map $z \longmapsto (1 - A(z))^{-1}$ is meromorphic on $\Omega_J^{\pm}$ and is continuous in norm as $z$ approaches a point in $\bar{J} \setminus N$. [Yaf, p46-48]

<u>Thm 2.</u> Let $\mathcal{J}$ be an open interval in $\mathbb{R}$

Suppose that :

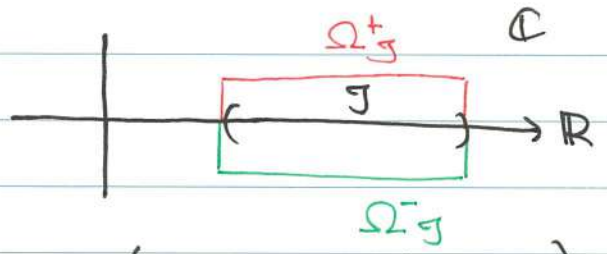
1) $G_0$ is $H_0$-smooth on $\mathcal{J}$,

2) $(G_0 R_0(z) G^*)^l \in \mathcal{K}(\mathcal{G})$ for some $l \in \mathbb{N}$,

3) The map $z \longmapsto G_0 R_0(z) G^*$

and $z \longmapsto G R_0(z) G^*$ are holomorphic

in $\Omega^{\pm}_{\mathcal{J}}$ and <u>extend continuously to</u> $\mathcal{J}$




Then $W_{\pm}(H, H_0, E^{H_0}_{ac}(\mathcal{J}))$ exist and

are complete $\left( \Longleftrightarrow W_{\pm}(H_0, H, E^{H}_{ac}(\mathcal{J})) \text{ exist} \right)$

<u>Remark</u> : There is no smallness condition in the assumptions, but the result is only on $\mathcal{H}_{ac}(H_0)$ or $\mathcal{H}_{ac}(H)$.

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<u>Proof.</u>

Recall that $\mathcal{N}$ was introduced in the abstract thm and $\mathcal{N}$ is closed and of measure 0 (in $\mathcal{J}$). It implies that $\exists \{\mathcal{J}_n\}_{n\in\mathbb{N}}$ subintervals in $\mathcal{J}$, $\mathcal{J}_n$ open, with

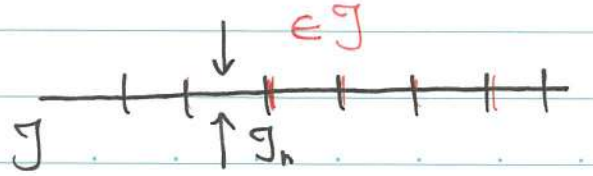
$$\mathcal{J}\setminus\mathcal{N} = \bigsqcup_n \mathcal{J}_n \quad \text{and} \quad |\mathcal{J}| = \left|\bigsqcup_n \mathcal{J}_n\right| .$$

We shall now use the abstract thm ⊛ with $A(z) = -B_0(z) = G_0 R_0(z) G^*$.

As a consequence of thm ⊛, the map

$$z \longmapsto \left(1 - G_0 R_0(z) G^*\right)^{-1}$$

is norm continuous when $z$ approaches any point in $\mathcal{J}_n$

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$\Longrightarrow \quad z \longmapsto (1 - G_0 R_0(z) G^*)^{-1}$ is

locally norm continuous on $\mathcal{J}_n$

$\Longrightarrow \quad z \longmapsto GR(z)G^* = GR(z)G^*(1 - G_0 R_0(z) G^*)^{-1}$

is locally norm continuous on $\mathcal{J}_n$

$\Longrightarrow$ $G$ is locally $H$-smooth on $\mathcal{J}_n$

Since $G_0$ is also locally $H_0$-smooth on $\mathcal{J}$, it implies that $\underline{W_\pm(H, H_0, E^{H_0}(\mathcal{J}_n))}$ exist and are complete.

Since $N = \mathcal{J} \setminus \bigsqcup_n \mathcal{J}_n$ is of Lebesgue measure 0 one has $E^H_{ac}(\mathcal{J}) = E^H_{ac}(\bigsqcup_n \mathcal{J}_n) = \bigoplus_n E^H_{ac}(\mathcal{J}_n)$

is not true for $E^H_{sc}$ or $E^H_p$.

and the same for $H_0$ instead of $H$.

One infers that $W_\pm(H, H_0, E^{H_0}_{ac}(\mathcal{J}))$ exist and are complete.

A few more information for thm 1 and thm 2.

<u>Proposition</u> Suppose that assumptions of thm 1 are satisfied, and suppose that $\mathrm{Ker}(G) = \{0\}$. Then the spectrum of $H$ on $\mathcal{J}$ is purely a.c.

<u>Proof</u>

In the proof of thm 1, we get that $G$ is $H$-smooth on $\mathcal{J}$. It follows from the condition 5 on $H$-smoothness that

$$\langle G^* f, E^H(\Lambda) G^* f \rangle \leq \gamma_5^2 |\Lambda| \, \|f\|$$

for any interval $\underline{\underline{\Lambda \subset \mathcal{J}}}$.

$$\Longrightarrow \quad G^* f \in \mathcal{H}_{ac}(H) \cap E^H(\mathcal{J}) \mathcal{H}$$

$$\Longleftrightarrow \quad \mathrm{Ran}(G^*) \subset \mathcal{H}_{ac}(H) \cap E^H(\mathcal{J})\mathcal{H}$$

$$\Longleftrightarrow \quad \mathrm{Ran}\left(GE^H(\mathcal{J})\right) \subset \mathcal{H}_{ac}(H)$$

Recall that $\mathcal{H} = \underbrace{\mathrm{Ran}(G_{\mathcal{J}}^*)}_{\subset \mathcal{H}_{ac}(H)} \oplus \underbrace{\mathrm{Ker}(G_{\mathcal{J}})}_{\supset \mathcal{H}_c(H) \,:=\, \mathcal{H}_{sc}(H) \oplus \mathcal{H}_p(H)}$

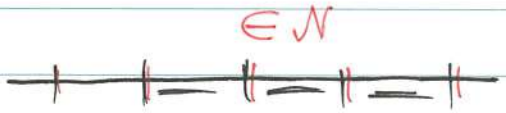
We infer $\mathcal{H}_c(H) \subset \underbrace{\mathrm{Ker}(G_{\mathcal{J}})}$

$$= \mathrm{Ker}\left(GE^H(\mathcal{J})\right) = \mathrm{Ker}(G)\big|_{E^H(\mathcal{J})\mathcal{H}}$$

$$= \{0\},$$

Thus we get $\mathcal{H}_c(H) \cap E^H(\mathcal{J})\mathcal{H} = \{0\}$.

⚠ The proof is valid in $E^H(\mathcal{J})\mathcal{H}$. $\square$




(localize to subintervals)

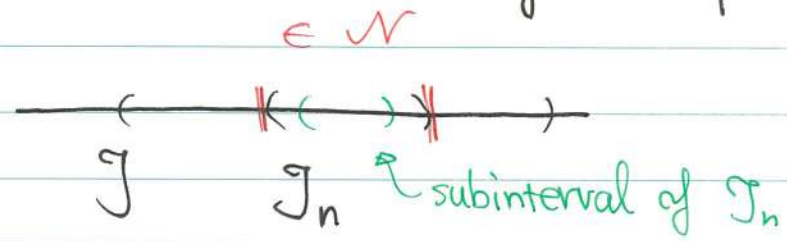
<u>Proposition</u> Suppose that assumptions of thm 2 are satisfied, and recall that

$$\mathcal{N} = \{\lambda \in \overline{\mathcal{J}} \mid 1 \in \sigma(B_0(\lambda \pm i0))\}$$

Then, if $\ker(G) = \{0\}$ then $H$ is purely a.c. on $\mathcal{J} \setminus \mathcal{N}$.

<u>Idea of the proof</u>:

A local version of the previous proof.




in any strict subinterval of $\mathcal{J}_n$.

<u>What about $\mathcal{N}$?</u>

With stronger assumptions one gets that $\mathcal{N} = \sigma_p(H) \cap \mathcal{J}$, and the multiplicities of the eigenvalue $\lambda \in \sigma_p(H)$ and 1 in $\sigma(B_0(\lambda \pm i0))$ coincide.

$B_0(z) := G_0 R_0(z) G^*$

# IV Scattering theory : stationary expressions.

## IV.1 Preliminaries

We always consider $H = H^*$, $H_0 = H_0^*$ in $\mathcal{H}$.

The wave operators have been defined with strong limits but we could use other topologies. For example,

$$W_\pm^{\text{weak}}(H, H_0) = \underset{t \to \pm\infty}{w\text{-}\lim} E_{ac}(H) e^{itH} e^{-itH_0} E_{ac}(H_0)$$

such that $W_\pm^{\text{weak}}(H, H_0)^* = W_\pm^{\text{weak}}(H_0, H)$.

but then these operators are not always isometries.

Consider $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ s.t.

$$\int_0^\infty \omega(t)\, dt = 1$$

and

$$\mathcal{F}\Big( \mathbb{R} \ni x \longmapsto \overset{\psi(x)}{\overset{\shortparallel}{e^{-x}\omega(e^{-x})}} \Big)$$

Fourier transform

$\Big( \Longleftrightarrow \{ \psi(\cdot + \alpha) \mid \alpha \in \mathbb{R} \}$ is dense in $L^1(\mathbb{R}) \Big)$

has no real zero.

Then we set $\omega_\varepsilon(t) := \varepsilon\,\omega(\varepsilon t)$.

$W_\pm^{\mathrm{Abel}}(H, H_0) \in B(\mathcal{H})$ are defined by the following relation (if it exist)

$$\lim_{\varepsilon \searrow 0} \int_0^\infty \omega_\varepsilon(t) \left\| e^{itH} e^{-itH_0} E_{ac}^{H_0} f - W_+^{\mathrm{Abel}}(H, H_0) f \right\|^p dt = 0$$

for all $f \in \mathcal{H}$.

⚠ Tauberian theorem: $W_+^{\mathrm{Abel}}(H, H_0)$ does not depend on $\omega$ and on $p \geq 1$.

[Yaf p76, BW Chap 6 p 97].

We could also consider $W_\pm^{\text{weak Abel}}(H, H_0)$ given by

$$w\text{-}\lim_{\varepsilon \searrow 0} \int_0^\infty \omega_\varepsilon(t)\, E_{ac}(H)\, e^{itH} E_{ac}(H_0)\, dt \quad (*)$$

⚠ These formulations are weaker than the original one, which means that the existence of these operators does NOT imply the existence of $W_\pm(H, H_0)$.

Let us consider $f_0 \in \mathcal{H}_{ac}(H_0)$ and $f \in \mathcal{H}_{ac}(H)$ and choose $\omega(t) = 2e^{-2t}$

$$\langle f, \circledast f_0 \rangle$$

$$= \lim_{\varepsilon \searrow 0} \int_0^\infty 2\varepsilon e^{-2\varepsilon t} \langle e^{-itH} f, e^{-itH_0} f_0 \rangle dt$$

$$= \lim_{\varepsilon \searrow 0} 2\varepsilon \int_{-\infty}^\infty \langle \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH} f, \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH_0} f_0 \rangle dt$$

$$= \lim_{\varepsilon \searrow 0} \frac{2\varepsilon}{2\pi} \int_{\mathbb{R}} \langle (t \mapsto \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH} f)(\lambda), (t \mapsto \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH_0} f_0)(\lambda) \rangle d\lambda$$

inverse Fourier

$$= \underline{\lim_{\varepsilon \searrow 0}} \frac{\varepsilon}{\pi} \underline{\int_{\mathbb{R}} \langle (H-\lambda-i\varepsilon)^{-1} f, (H_0-\lambda-i\varepsilon)^{-1} f_0 \rangle \, d\lambda}$$

<u>Lemma</u>

$$\langle W_{\pm}^{\text{Abel weak}} f_0 , f \rangle$$

$$:= \lim_{\varepsilon \searrow 0} \int_0^{\pm\infty} 2\varepsilon e^{\mp 2\varepsilon t} \langle e^{-itH_0} E_{ac}^{H_0} f_0 , e^{-itH} E_{ac}^{H} f \rangle dt$$

$$= \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \langle (H_0 - \lambda \mp i\varepsilon)^{-1} E_{ac}^{H_0} f_0 ,$$

$$(H - \lambda \mp i\varepsilon)^{-1} E_{ac}^{H} f \rangle dt$$

<u>Def.</u> Let $\mathcal{L}_0, \mathcal{L}$ be 2 linear dense subspace of $\mathcal{H}$ and suppose that for any $f_0 \in \mathcal{L}_0$, $f \in \mathcal{L}$ the limit

$$\lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \langle (H_0 - \lambda \mp i\varepsilon)^{-1} f_0 , (H - \lambda \mp i\varepsilon)^{-1} f \rangle =: a_{\pm}(\lambda, f_0, f)$$

exist for a.e. $\lambda$

can depend on $f_0$ and $f$

Then the stationary wave operators are defined by

Then, the <u>stationary wave operators</u> are defined by

$$\langle \mathcal{U}_{\pm} f_0, f \rangle := \int_{\mathbb{R}} a_{\pm}(\lambda, f_0, f)\, d\lambda .$$

<u>Thm</u> : [Yaf, Thm 4 p160]

If $\mathcal{U}_{\pm}$ exist, then $W_{\pm}^{\text{Abel weak}}$ exist as well and are equal.

<u>Remark</u>

Since the existence of $W_{\pm}^{\text{Abel weak}}$ is a weaker requirement, their existence can be ensured under weaker condition

$\rightsquigarrow$ locall smoothness in the weak sense.

## IV.2 Smoothness in the weak sense

We introduce a condition on $G : \mathcal{H} \to \mathcal{G}$ but a version for $G : \mathcal{D}(H) \to \mathcal{G}$ also exists.

**Proposition** Let $\mathcal{J}$ be a Borel set and $G : \mathcal{H} \to \mathcal{G}$. Then, the following conditions are equivalent

1) $\| G \, \mathrm{Im}(R(\lambda + i\varepsilon)) G^* \| \leq c(\lambda)$ for a.e. $\lambda \in \mathcal{J}$ — does not depend on $\varepsilon \in (0,$

simple

not simple

2) $\varepsilon^{1/2} \| G R(\lambda + i\varepsilon) \| \leq d(\lambda)$ for a.e. $\lambda \in \mathcal{J}$

3) $\underset{\varepsilon \searrow 0}{\text{w-lim}} \; G \, \mathrm{Im}(R(\lambda + i\varepsilon)) G^*$ exist for a.e. $\lambda \in \mathcal{J}$

Spectral theorem

4) $\underset{\eta \to 0}{\text{w-lim}} \; \frac{1}{2\eta} G \left( E^H_{\lambda+\eta} - E^H_{\lambda-\eta} \right) G^*$ exist

spectral family

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for a.e. $\lambda \in \mathcal{J}$.

5) $\| G(\underbrace{E^H(\lambda-\eta, \lambda+\eta)}_{\text{spectral measure}}) G^* \| \leq C'(\lambda)\eta$

for a.e. $\lambda \in \mathcal{J}$

6) $\| G\, E^H((\lambda-\eta, \lambda+\eta)) \| \leq C''(\lambda)\eta^{1/2}$

for a.e. $\lambda \in \mathcal{J}$.

$$\boxed{\begin{aligned} &E^H((a, b]) \\ &= E_b^H - E_a^H \end{aligned}}$$

The proof involves some measure theory argument.

<u>Remark</u>

If one of these conditions is satisfied then $G$ is said to be <u>H-smooth on $\mathcal{J}$ in the weak sense</u>.

<u>Remark</u> From 1) one observes that if $G$ is $H$-smooth on $\mathcal{J}$, then it is $H$-smooth on $\mathcal{J}$ in the weak sense.

The following lemma shows the relation between the imaginary part of the resolvent of $H$ on the real axis, and the derivative of the spectral measure.

<u>Lemma</u> Let $G_1, G_2 \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ be $H$-smooth operators on $\mathcal{J}$ in the weak sense. Then, for a. e. $\lambda \in \mathcal{J}$, one has

do NOT depend on "$f_0$" or "$f$" !!

$$\text{w-}\lim_{\varepsilon \searrow 0} \frac{1}{\pi} G_1 \operatorname{Im}(R(\lambda + i\varepsilon)) G_2^* \qquad \left( [\text{Yaf}, \text{p } 155] \right.$$

$$= \text{w-}\lim_{\eta \searrow 0} \frac{1}{2\eta} G_1 \left( E^H_{\lambda-\eta} - E^H_{\lambda+\eta} \right) G_2^*$$

because it is a symmetric difference

compare with the first definition!

We still give a result for which a.e. $\lambda$ depends on $f$:

<u>Lemma</u>. Let $G$ be $H$-smooth on $J$ in the weak sense, and let $f \in \mathcal{H}$. Then the following exist and are equal for a.e. $\lambda \in J$

can depend on $f$

$$\underset{\varepsilon \searrow 0}{\text{w-lim}} \frac{1}{\pi} G \, \mathrm{Im}\left(R(\lambda + i\varepsilon)\right) f = \underset{\eta \to 0}{\text{w-lim}} G \left(E^H_{\lambda-\eta} - E^H_{\lambda+\eta}\right) f.$$

<u>Remark</u> (back to $H$-smooth (not in the weak sense))

From the condition (2) of $H$-smoothness

$$\sup_{\|f\|=1,\, \varepsilon>0} \int_{-\infty}^{\infty} \left\{ \|G_J R(\lambda + i\varepsilon) f\|^2 + \|G_J R(\lambda - i\varepsilon) f\|^2 \right\} d\lambda < \infty$$

$\rightsquigarrow$ Hardy class.

$$\Longrightarrow \underset{\varepsilon \searrow 0}{\text{s-lim}} G_J R(\lambda \pm i\varepsilon) f \quad \text{exist for a.e. } \lambda \in J.$$

Paley-Winer theorem

depends on $f$.

## IV.3 Stationary expression

**Thm.** Let $H, H_0$ be self-adjoint in $\mathcal{H}$ s.t. $H - H_0 = G^* G_0$ with $G_0$ $H_0$-smooth on $\mathcal{J}$ and $G$ $H$-smooth on $\mathcal{J}$ in the weak sense.

Then, the stationary wave operator exist on $\mathcal{J}$, namely for $f \in E_{ac}^{H}(\mathcal{J})\mathcal{H}$ and $f_0 \in E_{ac}^{H_0}(\mathcal{J})\mathcal{H}$,

$$a_\pm(\lambda, f_0, f) = \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \langle (H_0 - \lambda \mp i\varepsilon)^{-1} f_0, (H - \lambda \mp i\varepsilon)^{-1} f \rangle$$

exist for a.e. $\lambda \in \mathcal{J}$, and

$$\langle \mathcal{U}_\pm f_0, f \rangle = \int_{\mathbb{R}} a_\pm(\lambda, f, f_0)\, d\lambda .$$

Also $W_\pm(H, H_0, E_{ac}^{H_0}(\mathcal{J}))$ exist and are equal to $\mathcal{U}_\pm$.

If $G$ is also $H$-smooth on $\mathcal{J}$,

then the completeness also holds.

Key argument in the proof:

recall 1) $\delta_\varepsilon(H-\lambda) = \frac{1}{2\pi i}\left(R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon)\right)$

$$= \frac{\varepsilon}{\pi} R(\lambda+i\varepsilon)R(\lambda-i\varepsilon) = \frac{1}{\pi}\operatorname{Im} R(\lambda+i\varepsilon)$$

2) $\langle R_0(z)f_0, f\rangle = \langle f_0, R(\bar{z})f\rangle$

$$+ \langle G_0 R_0(z) f_0, G R(\bar{z}) f\rangle$$

For $z = \lambda \pm i\varepsilon$, one has

$$\frac{\varepsilon}{\pi}\langle R_0(z)f_0, R(z)f\rangle$$

$$= \frac{\varepsilon}{\pi}\left[\langle f_0, R(\bar{z})R(z)f\rangle + \langle G_0 R_0(z) f, G R(\bar{z}) R(z) f\rangle\right]$$

$$= \langle f_0, \delta_\varepsilon(H-\lambda)f\rangle + \frac{1}{\pi}\langle \underbrace{G_0 R_0(\lambda+i\varepsilon)f_0}, \underbrace{G\operatorname{Im}(R(\lambda+i\varepsilon))f}\rangle$$

also converges as $\varepsilon \searrow 0$ for a.e. $\lambda \in \mathcal{J}$

strongly converges as $\varepsilon \to 0$ (by the remark)

weakly converges as $\varepsilon \searrow 0$ (by previous lemma for a.e. $\lambda \in \mathcal{J}$)

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## IV. 4 Direct integrals [Baum-Woll, p67 - 68]

Let $\{\mathcal{H}(\lambda)\}_{\lambda\in\mathbb{R}}$ be a family of Hilbert spaces and let $m$ be a Borel measure on $\mathbb{R}$.

We set

$$\mathcal{H} := \int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda)\, m(d\lambda)$$

as the Hilbert space of all vector valued functions $\lambda \longmapsto f(\lambda) \in \mathcal{H}(\lambda)$, which are measurable and square integrable with respect to the measure $m$.

The scalar product is for $f, g \in \mathcal{H}$

$$\langle f, g\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle f(\lambda), g(\lambda)\rangle_{\mathcal{H}(\lambda)}\, m(d\lambda)$$

$\Longrightarrow$ square integrable means

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \|f(\lambda)\|_{\mathcal{H}(\lambda)}^2 m(d\lambda) < \infty .$$

<u>Remark</u> : If $\mathcal{H}(\lambda) = \mathbb{C}$ for each $\lambda$,

then $\mathcal{H} = L^2(\mathbb{R}, m(d\lambda))$.

If $\mathcal{H}(\lambda) = \mathbb{C}^n$, then $\mathcal{H} = L^2(\mathbb{R}, m(d\lambda); \mathbb{C}^n)$

$$= \underbrace{\left(L^2(\mathbb{R}, m(d\lambda))\right)^n}_{n\text{-copies}} .$$

[ <u>June 28</u> (Thursday)

Seminar on Scattering theory

6.30 — 8.00 ]

<u>Remark</u>. One version of the spectral theorem states that given a self-adjoint op. $H$ in $\mathcal{H}$,

$\exists\ \mathcal{F} : \mathcal{H} \longrightarrow \int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda)\, m(d\lambda)$

$\mathcal{F}$ unitary, such that

$$[\mathcal{F}Hf](\lambda) = \lambda [\mathcal{F}f](\lambda) \quad \forall\ f \in D(H)$$

$$\Longleftrightarrow \mathcal{F}H\mathcal{F}^* = X$$ multiplication by the variable

Also for any $\varphi \in L^\infty(\mathbb{R})$, then

$$\mathcal{F}\varphi(H)\mathcal{F}^* = \varphi(X)$$ ← op. of multiplication by $\varphi$

$$= M_\varphi$$

In paticular $\mathcal{F}E^H(J)\mathcal{F}^* = \chi_J(X)$ ← multiplication op. by this characteristic function.

⚠ The Hilbert sp. $\mathcal{H} = \int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda)\, m(d\lambda)$ and $\mathcal{F}$ are not unique. We call $(\mathcal{H}, \mathcal{F})$ a spectral representation of $H$.

<u>Lem.</u> Let $B \in B(\mathcal{H})$ s.t. $B$ "commutes" with $H$.

it means $Be^{-itH} = e^{-itH}B \quad \forall\ t \in \mathbb{R}$.

$\Longleftrightarrow B(H-z)^{-1} = (H-z)^{-1}B \quad \forall\ z \in \mathbb{C}\setminus\mathbb{R}$

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Then, $\exists \{b(\lambda)\}_{\lambda\in\mathbb{R}}$ with $b(\lambda) \in B(\mathcal{H}(\lambda))$ for $m$ a.e. $\lambda$

s.t. $[\mathcal{F}Bf](\lambda) = b(\lambda)[\mathcal{F}f](\lambda)$

$\left( \Longleftrightarrow \mathcal{F}B\mathcal{F}^* = b(X) \right.$ operator valued multiplication op. or direct integral operator

Also $\|B\| = m\text{-}\operatorname{esssup}_{\lambda\in\mathbb{R}} \|b(\lambda)\|_{\mathcal{H}(\lambda)}$

$= \inf\{\lambda\in\mathbb{R} \mid m(b^{-1}((\lambda,\infty))) = 0\}$

In scattering theory, we consider only $m_{ac}$ and the restriction $\mathcal{H}_{ac}(H) \xrightarrow{\mathcal{F}} \int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda) \underbrace{m_{ac}(d\lambda)}_{\eta(\lambda)\,d\lambda}$

which is still unitary. (*)

<u>Remark</u> Since $m_{ac}$ is absolutely continuous with respect to the Lebesgue measure, we can change $\mathcal{F}$ and get a unitary map

$[\tilde{\mathcal{F}}f](\lambda) := \sqrt{\eta(\lambda)}[\mathcal{F}f](\lambda)$

$$\mathcal{H}_{ac}(H) \xrightarrow{\tilde{\mathcal{F}}} \int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda)\,d\lambda$$ ← Lebesgue measure

With the notation one has

$$\langle \underbrace{E^H_{ac}(J)}_{E^H(J)E_{ac}(H)} f, g \rangle_{\mathcal{H}} = \langle \mathcal{F}E^H_{ac}(J)f, \mathcal{F}g \rangle$$

$$= \int_J \langle [\mathcal{F}f](\lambda), [\mathcal{F}g](\lambda) \rangle_{\mathcal{H}(\lambda)}$$

$$=: \int_J \langle \tilde{f}(\lambda), \tilde{g}(\lambda) \rangle_{\mathcal{H}(\lambda)} \quad \text{with } \tilde{f} = \mathcal{F}f \quad \cdots (1)$$

<u>Question</u> If $B \in \mathcal{B}(\mathcal{H})$, what about

$E^H_{ac} B E^H_{ac}$ in $\int_{\mathbb{R}}^{\oplus} \mathcal{H}(\lambda)\, d\lambda$ ?

<u>Def.</u> An operator $B \in \mathcal{B}(\mathcal{H})$ is called a integral operator in the spectral representation of $H|_{\mathcal{H}^H_{ac} \equiv \mathcal{H}_{ac}(H)}$ if for almost every $(\mu, \nu) \in \sigma_{ac}(H) \times \sigma_{ac}(H)$ there exists $b(\mu, \nu) : \mathcal{H}(\nu) \longrightarrow \mathcal{H}(\mu)$ s.t.

$\forall f, g \in \mathcal{H}_{ac}(H)$

$$\langle Bf, g \rangle_{\mathcal{H}} = \iint_{\hat{\sigma}\ \hat{\sigma}} \langle b(\mu, \nu) \tilde{f}(\nu), \tilde{g}(\mu) \rangle_{\mathcal{H}(\mu)}\, d\nu\, d\mu$$

with $\hat{\sigma} = \sigma_{ac}(H)$, if the integrant is measurable with respect to $d\nu\, d\mu$ on $\hat{\sigma} \times \hat{\sigma}$ and if the integral converges.

We call $\{b(\mu, \nu)\}_{\mu, \nu \in \hat{\sigma}}$ the kernel of $B$ in the spectral rep. of $H$.

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<u>Remark</u> This notation can be extended to unbounded $B$ if we impose suitable condition on $f$ and $g$, or if we consider $\underbrace{\varphi(H) B \psi(H)}$ with $\varphi, \psi \in C_c(\mathbb{R})$ and assume that is bounded.

Compactly supported functions on $\mathbb{R}$

<u>Remark</u> Often, this definition of integral operator is too general since the kernel $b(\mu,\nu)$ is defined only for a.e. $(\mu,\nu) \in \hat{\sigma} \times \hat{\sigma}$

In paticular $b(\lambda,\lambda)$ is not well-defined!

We need more regularity.

The notion of $H$-smooth op. in the weak sense is again going to play a role

Recall that if $G : \mathcal{H} \to \mathcal{G}$ is $H$-smooth is the weak then for a.e. $\lambda$ one has

$$\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, G\, \delta_\varepsilon(H-\lambda)\, G^* = \underset{\eta \to 0}{w\text{-}\lim}\, \frac{1}{2\eta}\, G\,(E_{\lambda+\eta} - E_{\lambda-\eta})\, G^*$$

$\square$

the complement is measure zero.

We call $\Lambda_1$ the subset of $\mathbb{R}$ of full measure where (Π) holds, and call the limit $\Delta(\lambda, G) \in B(\mathcal{G})$

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{G}$ and let $\Lambda_2 \subset \mathbb{R}$ be the subset of $\lambda \in \mathbb{R}$ s.t.

$[\mathcal{F}G^*\varphi_j](\lambda)$ is well-defined, $\forall j \in \mathbb{N}$

It follows that $\Lambda_2$ is of full measure and $[\mathcal{F}G^*\varphi_j](\lambda)$ is well-defined for any $\lambda \in \Lambda_2$

Let us set $\mathcal{D} = \mathrm{Span}(\{\varphi_j\}_{j\in\mathbb{N}}) \underset{\text{dense}}{\subset} \mathcal{G}$

$\Rightarrow \underbrace{[\mathcal{F}G^*\varphi](\lambda)}_{\in \mathcal{H}(\lambda)}$ is well-defined $\forall \varphi \in \mathcal{D}$ and $\lambda \in \Lambda_2$

We set $Z(\lambda, G) : \mathcal{D} \longrightarrow \mathcal{H}(\lambda)$

$$Z(\lambda, G)\varphi := [\mathcal{F}G^*\varphi](\lambda), \quad \forall \lambda \in \Lambda_2$$

From the equality (Π) and (1) one has for any $\varphi, \psi \in \mathcal{G}\mathcal{H}_{ac}(H)$

$$\lim_{\varepsilon \searrow 0} \langle \delta_\varepsilon(H-\lambda) G^*\varphi, G^*\psi \rangle$$

$$= \lim_{\eta \to 0} \langle \frac{1}{2\eta} G(E_{\lambda+\eta} - E_{\lambda-\eta}) G^*\varphi, \psi \rangle$$

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$$= \langle \tfrac{d}{d\lambda} G E_\lambda G^* \varphi, \psi \rangle$$

$$= \langle \widetilde{G\varphi}(\lambda), \widetilde{G\psi}(\lambda) \rangle_{\mathcal{H}(\lambda)} \quad ①$$

$$= \langle [\mathcal{F}G\varphi](\lambda), [\mathcal{F}G\psi](\lambda) \rangle$$

$$= \langle Z(\lambda, G)\varphi, Z(\lambda, G)\psi \rangle \quad \text{for } \lambda \in \Lambda_1 \cap \Lambda_2 = \Lambda$$

$$\Rightarrow \underset{\substack{\cap \\ B(\mathcal{G})}}{\Delta(\lambda, G)} = Z(\lambda, G)^* Z(\lambda, G) \quad \forall \lambda \in \Lambda$$

$$\Rightarrow Z(\lambda, G) \in B(\mathcal{G}, \mathcal{H}(\lambda)) \quad \forall \lambda \in \Lambda$$

We set $Z(\lambda, G) = 0$ for $\lambda \in \Lambda^c$

$$\Rightarrow Z(\lambda, G) \in B(\mathcal{G}, \mathcal{H}(\lambda)) \quad \forall \lambda \in \hat{\sigma} \subset \mathbb{R}$$

<u>Def.</u> Let $B \in B(\mathcal{H})$ and consider

$A :=$ in the

weak

Then

$E^H_{ac}$

$a$

for any $\mu, \nu \in \hat{\sigma}$.

Definition not correct

<u>Lemma.</u>

If $A = G^* B G$ with $G$ $H$-smooth in the weak sense and $B \in \mathcal{B}(\mathcal{G})$, then the kernel

$$a(\mu,\nu) := Z(\mu, G) B Z(\nu, G)^*$$

is an integral kernel for $E^H_{ac} A E^H_{ac}$ in the spectral representation of $H|_{\mathcal{H}_{ac}(H)}$.

<u>Proof.</u>

Observe first that for any $\varphi \in D$ (introduced before) and $f \in \mathcal{H}$, one has for a.e. $\lambda \in \hat{\sigma}$:

$$\frac{d}{d\lambda} \langle G E^H_\lambda f, \varphi \rangle_{\mathcal{G}} = \frac{d}{d\lambda} \langle E_\lambda f, \underbrace{G^* \varphi}_{\in \mathcal{H}_{ac}(H)} \rangle_{\mathcal{H}}$$

$$= \frac{d}{d\lambda} \langle \mathcal{F} E_\lambda f, \mathcal{F} G^* \varphi \rangle_{\mathcal{H}}$$

$$= \frac{d}{d\lambda} \int_{\hat{\sigma}} \langle \underbrace{[\mathcal{F} E^H_\lambda f]}_{= \chi_{(-\infty,\lambda]}(\mu) \cdot [\mathcal{F} f](\mu)}(\mu), [\mathcal{F} G^* \varphi](\mu) \rangle_{\mathcal{H}(\mu)} d\lambda$$

$$= \frac{d}{d\lambda} \int_{\hat{\sigma} \cap (-\infty, \lambda]} \langle \tilde{f}(\mu), [\mathcal{F} G^* \varphi](\mu) \rangle_{\mathcal{H}(\mu)} \, d\mu$$

$$= \langle \tilde{f}(\lambda), \underbrace{[\mathcal{F} G^* \varphi](\lambda)}_{\text{well-defined for all } \lambda.} \rangle_{\mathcal{H}(\lambda)}$$

for a.e. $\lambda \in \hat{\sigma}$

(depending on $f$)

$$= \langle \tilde{f}(\lambda), Z(\lambda, G) \varphi \rangle_{\mathcal{H}(\lambda)}$$

$$= \langle Z(\lambda, G)^* \tilde{f}(\lambda), \varphi \rangle_{\mathcal{G}}$$

Thus

$$\frac{d}{d\lambda} G E_\lambda f = Z(\lambda, G)^* \tilde{f}(\lambda) \quad \text{for a.e. } \lambda.$$

Thus, $f, g \in \mathcal{H}_{ac}(H)$, one has

$$\langle A f, g \rangle_{\mathcal{H}} = \langle G^* B G f, g \rangle_{\mathcal{H}}$$

$$\int_{\hat{\sigma}} \frac{d}{d\mu} E_\mu \, d\mu = \mathbb{1}_{ac}(H)$$

$$= \int_{\hat{\sigma}} \frac{d}{d\mu} \langle E_\mu G^* B G f, g \rangle \, d\mu$$

$$= \int_{\hat{\sigma}} \langle BGf, Z(\mu, G)^* \tilde{g}(\mu) \rangle d\mu$$

$$= \int_{\hat{\sigma}} \left[ \int_{\hat{\sigma}} \frac{d}{d\nu} \langle BGE_\nu f, Z(\mu, G)^* \tilde{g}(\mu) \rangle d\nu \right] d\mu$$

$$= \int_{\hat{\sigma}} \left[ \int_{\hat{\sigma}} \langle BZ(\nu, G)^* \hat{f}(\nu), Z(\mu, G)^* \tilde{g}(\mu) \rangle d\nu \right] d\mu$$

$$= \int_{\hat{\sigma}} \left[ \int_{\hat{\sigma}} \langle a(\mu, \nu) \tilde{f}(\nu), \tilde{g}(\mu) \rangle d\nu \right] d\mu$$

$\square$.

## IV.5 Scattering Operator and matrix

Recall that $S = \underbrace{W_+^* W_-}_{\text{time-dependent approach}} = \underbrace{\mathcal{U}_+^* \mathcal{U}_-}_{\text{from the stationary approach}}$

when $W_\pm = \mathcal{U}_\pm$.

Therefore, for $f, g \in \mathcal{H}_{ac}(H_0)$, one has

$$\langle \mathcal{U}_+^* \mathcal{U}_- f, g \rangle = \langle \mathcal{U}_- f, \mathcal{U}_+ g \rangle$$

$$= \int \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \langle R_0(\lambda - i\varepsilon) f, \underbrace{R(\lambda - i\varepsilon)}_{\text{resolvent equation}} \mathcal{U}_+ g \rangle d\lambda$$

$$= \int \lim_{\varepsilon \searrow 0} \Big\{ \underline{\langle f, \delta_\varepsilon(H_0-\lambda) U_+ g \rangle}$$

$$+ \langle G_0 R(\lambda - i\varepsilon) f, G\, \delta_\varepsilon(H-\lambda) U_+ g \rangle \Big\} d\lambda$$

$$= \int \lim_{\varepsilon \searrow 0} \Big\{ \frac{\varepsilon}{\pi} \langle \underbrace{R(\lambda+i\varepsilon)}_{\text{resolvent eq.}} f, R_0(\lambda+i\varepsilon) g \rangle$$

$$+ \frac{\varepsilon}{\pi} \langle R(\lambda+i\varepsilon) G^* G_0 R_0(\lambda-i\varepsilon) f, R_0(\lambda+i\varepsilon) g \rangle \Big\} d\lambda$$

$$= \int \lim_{\varepsilon \searrow 0} \Big\{ \langle \delta_\varepsilon(H_0-\lambda) f, g \rangle - \frac{\varepsilon}{\pi} \langle R(\lambda+i\varepsilon) V R_0(\lambda+i\varepsilon) f, R_0(\lambda+i\varepsilon) g \rangle$$

$$+ \frac{\varepsilon}{\pi} \langle R(\lambda+i\varepsilon) V R_0(\lambda-i\varepsilon) f, R_0(\lambda+i\varepsilon) g \rangle \Big\} d\lambda$$

$$= \int \lim_{\varepsilon \searrow 0} \Big\{ \langle \delta_\varepsilon(H_0-\lambda) f, g \rangle - \langle \underbrace{R(\lambda+i\varepsilon)}_{\text{resolvent eq}} V \delta_\varepsilon(H_0-\lambda) f, R_0(\lambda+i\varepsilon) g \rangle \Big\} d\lambda$$

$$= \int \lim_{\varepsilon \searrow 0} \Big\{ \langle \delta_\varepsilon(H_0-\lambda) f, g \rangle - 2\pi i \langle (V - VR(\lambda+i\varepsilon)V) \delta_\varepsilon(H_0-\lambda) f,$$

$$\delta_\varepsilon(H_0-\lambda) g \rangle \Big\} d\lambda$$

It means

$$\langle Sf, g \rangle = \langle f, g \rangle - 2\pi i \int \lim_{\varepsilon \to 0} \langle (V - VR(\lambda+i\varepsilon)V) \delta_\varepsilon(H_0-\lambda) f,$$

$$\delta_\varepsilon(H_0-\lambda) g \rangle d\lambda$$

Since $S$ commutes with $H_0$, it can be represented in the spectral representation of $H_0$, it means

$$\langle E_{ac}^{H} S E_{ac}^{H} f, g \rangle$$

$$= \int_{\hat{\sigma}} \langle S(\lambda) \tilde{f}(\lambda), \tilde{g}(\lambda) \rangle d\lambda$$

but what is $S(\lambda)$ ?

called the scattering matrix.

$$S(\lambda) : \mathcal{H}(\lambda) \longrightarrow \mathcal{H}(\lambda) \text{ for a.e. } \lambda.$$

For giving an expression for $S(\lambda)$, assume that $\underset{\varepsilon \searrow 0}{w\text{-}\lim}\, GR(\lambda + i\varepsilon) G^* \equiv GR(\lambda + i0) G^*$ exists

(this is a stronger requirement than H-smoothness in the weak sense or in the initial sense)

Then, $\underset{\varepsilon\to 0}{w\text{-}\lim}\, V R(\lambda+i\varepsilon) V = G_0^* B(\lambda+i0) G_0$

for a.e. $\lambda \in \hat{\sigma}$.

If this holds, then any H-smooth op in the weak sense is also $H_0$-smooth in the weak sense. Indeed,

$$\varepsilon^{1/2} \| G R_0(\lambda+i\varepsilon) \|$$

$$\leq \varepsilon^{1/2} \| G R(\lambda+i\varepsilon) \| + \varepsilon^{1/2} \| G R(\lambda+i\varepsilon) G^* G_0 R_0(\lambda+i\varepsilon) \|$$

$$\leq \varepsilon^{1/2} \underbrace{\| G R(\lambda+i\varepsilon) \|}_{G \text{ is H-smooth}} + \underbrace{\| G R(\lambda+i\varepsilon) G^* \|}_{\text{bounded}} \varepsilon^{1/2} \underbrace{\| G_0 R_0(\lambda+i\varepsilon) \|}_{G_0 \text{ is } H_0\text{-smoo}}$$

$$\leq \exists C(\lambda).$$

With this we get that

$$\langle (V - VR(\lambda+i\varepsilon)V) \delta_\varepsilon(H_0-\lambda) f, \delta_\varepsilon(H_0-\lambda) g \rangle$$

$$= \langle G_0 \delta_\varepsilon(H_0-\lambda) f, G \delta_\varepsilon(H_0-\lambda) g \rangle$$

$$- \langle B(\lambda+i\varepsilon) G_0 \delta_\varepsilon(H_0-\lambda) f, G_0 \delta_\varepsilon(H-\lambda) g \rangle$$

⊛

This one has

with additional assumption.

$$\lim_{\varepsilon\to 0} \circledast = \langle Z(\lambda, G_0)^* \tilde{f}(\lambda), Z(\lambda, G)^* \tilde{g}(\lambda)\rangle$$
$$- \langle B(\lambda+i0) Z(\lambda, G_0)^* \tilde{f}(\lambda), Z(\lambda, G_0)^* \tilde{g}(\lambda)\rangle$$

By collecting all this one gets

<u>Thm</u> Let $G$ be $H$-smooth on $\mathcal{J}$, let $G_0$ be $H_0$-smooth on $\mathcal{J}$, and assume that $\lim_{\varepsilon\to 0} G R(\lambda+i\varepsilon) G^*$ exists on $\mathcal{J}$.

Then, for a.e. $\lambda$,

$$S(\lambda) = \mathbb{1} - 2\pi i \left[ Z(\lambda, G) Z(\lambda, G_0)^* - Z(\lambda, G_0) B(\lambda+i0) Z(\lambda, G_0)^* \right]$$

<u>Remark</u> With slightly stronger assumptions one gets :

$$S(\lambda) = \mathbb{1} - 2\pi i Z(\lambda, G) \left(1 - B_0(\lambda+i0)\right)^{-1} Z(\lambda, G_0)^*$$

with $B_0(z) = - G_0 R_0(z) G^*$

for a.e. $\lambda$

# V Schrödinger operators.

We illustrate previous concepts, but also use additional structures.

## V.1 The Laplace operator on $\mathbb{R}^d$.

Consider $\mathcal{H} = L^2(\mathbb{R}^d)$ and the Laplace operator

$$H_0 = -\Delta = -\sum_{j=1}^{d} \partial_j^2$$

and $D(H_0) = \mathcal{H}^{\underline{\underline{2}}}(\mathbb{R}^d)$ ← Sobolev space of order $\underline{\underline{2}}$

We set $F : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ for the Fourier transform with

$$[Ff](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ which extends to a unitary operator.

Check that $FH_0F^* = X^2$ multiplication op

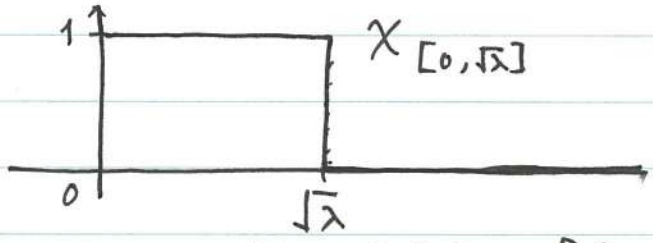
by the function $\mathbb{R}^d \ni x \longmapsto X^2 = \sum_{j=1}^{d} x_j^2 \in \mathbb{R}$.

Then $\mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R}^d) = F\,\mathcal{D}(X^2)$

with $\mathcal{D}(X^2) = \{ f \in L^2(\mathbb{R}^d) \mid x \mapsto x^2 f(x) \in L^2(\mathbb{R}^d) \}$

<u>Exercise</u>

Check that $E_\lambda^{H_0} = F\,\chi_{[0,\sqrt{\lambda}]}(|X|)\,F^*$




and that $\mathcal{H}_{ac}(H_0) = \mathcal{H}$

$$\Longrightarrow \mathcal{H}_{sc}(H_0) = \mathcal{H}_p(H_0) = \{0\}.$$

Consider
$$\begin{aligned} \mathcal{H} &= \int_{\mathbb{R}_+}^{\oplus} L^2(\mathbb{S}^{d-1})\, d\lambda \\ &= L^2(\mathbb{R}_+, d\lambda\,;\, L^2(\mathbb{S}^{d-1})) \\ &\cong L^2(\mathbb{R}_+, d\lambda) \otimes L^2(\mathbb{S}^{d-1}). \end{aligned}$$

Then we set

$$\mathcal{F}_0 : \mathcal{H} \longrightarrow \mathcal{H}$$

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$$[\mathcal{F}_0 f](\underset{\in \mathbb{R}_+}{\lambda}, \underset{\in \mathbb{S}^{d-1}}{\omega}) \equiv [[\mathcal{F}_0 f](\lambda)](\omega)$$

$$:= 2^{-\frac{1}{2}} \lambda^{(d-2)/4} [Ff](\lambda^{1/2}\omega).$$

<u>Check</u> $[\mathcal{F}_0 H_0 f](\lambda, \omega) = \lambda [\mathcal{F}_0 f](\lambda, \omega)$

$\Longleftrightarrow \mathcal{F}_0 H_0 \mathcal{F}_0^* = X$ multiplication by the function $\mathbb{R}_+ \ni \lambda \longmapsto \lambda \in \mathbb{R}$

It means that $(\mathcal{H}, \mathcal{F}_0)$ determines a spectral representation of $H_0$.

Recall that $F\varphi(H_0)F^* = \varphi(X^2)$ $\forall \varphi \in L^\infty(\mathbb{R})$.

$\Longleftrightarrow \varphi(H_0) = F^*\varphi(X^2)F$

$$\Longrightarrow [(H_0 - z)^{-1} f](x)$$

$$= [F^*(X^2 - z)Ff](x)$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} [(X^2 - z)^{-1} Ff](\xi) d\xi$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (\xi^2 - z)^{-1} [Ff](\xi) d\xi$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (\xi^2 - z)^{-1} \left[ \int_{\mathbb{R}^d} e^{-ix'\cdot\xi} f(x')\, dx' \right] d\xi$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} e^{i\xi\cdot(x-x')} (\xi^2 - z)^{-1} dx \right] f(x')\, dx'$$

$$= \int_{\mathbb{R}^d} R_0(z; x, x') f(x')\, dx' \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d)$$

with $R_0(z; x, x') = \frac{1}{(2\pi)^d} \int e^{i(x-x')\cdot\xi} (\xi^2 - z)^{-1} d\xi$.

is called the Green function of $H_0$

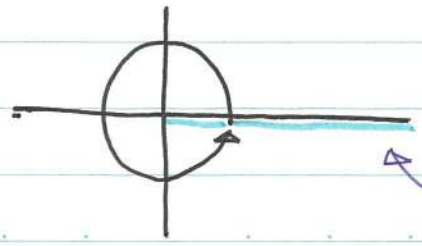
By computing this function in the sense of distributions, we get

$$R_0(z; x, x') = i\, 4^{-1} (2\pi)^{-\nu} z^{\nu/2} |x-x'|^{-\nu} H_\nu^{(1)}(z^{1/2}|x-x'|)$$

with $\nu = \frac{d-2}{2}$

Hankel function

with $\arg(z) \in [0, 2\pi)$



cut in the complex plane

Note that for $d=1$

$$R_0(z;x,x') = i2^{-1}(2\pi)^{-1/2}\exp(iz^{1/2}|x-x'|)$$

and for $d=3$

$$R_0(z;x,x') = (4\pi|x-x'|)^{-1}\exp(iz^{1/2}|x-x'|).$$

Observe that for $\lambda>0$ and $\varepsilon>0$,

one has $(\lambda+i\varepsilon)^{\nu/2} = \lambda^{\nu/2}(1+i\frac{\varepsilon}{\lambda})^{\nu/2} \longrightarrow \lambda^{\nu/2}$ as $\varepsilon\to 0$

and $(\lambda-i\varepsilon)^{\nu/2} = \lambda^{\nu/2}(\underbrace{1-i\frac{\varepsilon}{\lambda}}_{\approx e^{2\pi i}})^{\nu/2} \longrightarrow \lambda^{\nu/2}e^{i\pi\nu}$.

In particular for $\nu=1$

$$(\lambda+i\varepsilon)^{1/2} \longrightarrow \lambda^{1/2},$$

$$(\lambda-i\varepsilon)^{1/2} \longrightarrow -\lambda^{1/2}.$$

By using the relation

Bessel function

$$H_\nu^{(1)}(x) - e^{i\pi\nu}H_\nu^{(1)}(-x) = 2I_\nu(x),$$

One has

$$\frac{1}{2\pi i}\Big(R_0(\lambda+i\varepsilon;x,x')-R_0(\lambda-i\varepsilon;x,x')\Big)$$

$$\xrightarrow{\varepsilon\to 0}\frac{1}{2\pi i}\, i\,4^{-1}(2\pi)^{-\nu}|x-x'|^{-\nu}\lambda^{\nu/2}\Big\{H_\nu^{(1)}(\lambda^{1/2}|x-x'|)-e^{i\pi\nu}H_\nu^{(1)}(-\lambda^{1/2}|x-x'|)\Big\}$$

$$= 2\,I_\nu(\lambda^{1/2}|x-x'|)$$

$$=\frac{1}{2}(2\pi)^{-\nu-1}\lambda^{\nu/2}|x-x'|^{\nu}\,I_\nu(\lambda^{1/2}|x-x'|)\ .\ (*)$$

⚠ We have computed the limit $\varepsilon\searrow 0$ of $\mathrm{Im}\,R_0(\lambda+i\varepsilon;x,x')$ for fixed $x$ and $x'$.

It does <u>not</u> correspond to the convergence of $G\,\mathrm{Im}\,(H_0-\lambda-i\varepsilon)^{-1}G^*$ as $\varepsilon\searrow 0$ for suitable $G$.

However, if the latter limit exists, then $(*)$ will appear somewhere in its expression.

The unitary group associated with $H_0$ is also explicitly known :

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$$\left[e^{-itH_0}f\right](x) = (4\pi i t)^{-d/2}\int_{\mathbb{R}^d}\exp\left(\frac{i}{4t}|x-x'|^2\right)f(x')dx'$$

with $(4\pi i t)^{-d/2} = |4\pi t|^{-d/2}\, e^{-i\frac{\pi d}{4}}\,\mathrm{sgn}(t)$

sign functions.

Let us provide one more definition which has no counterpart in the abstract framework:

$$\Gamma_0(\lambda) : \mathcal{S}(\mathbb{R}^d) \longrightarrow L^2(\mathbb{S}^{d-1})$$

$$[\Gamma_0(\lambda)f](\omega) = [\mathcal{F}_0 f](\lambda, \omega).$$

$\Gamma_0(\lambda)$ extends then to a continuous operator $\mathcal{H}^\alpha(\mathbb{R}^d)$ to $L^2(\mathbb{S}^{d-1})$ for $\alpha > \frac{1}{2}$

Sobolev space of order $\alpha$

$$= F\{f \in L^2(\mathbb{R}^d) \mid |x|^\alpha f \in L^2(\mathbb{R}^d)\}$$

$\Gamma_0(\lambda)$ is called a trace operator or a restriction operator cf. Yaf II p71–75.

The relation with the abstract theory:

Since $\langle X \rangle^{-\alpha} = \left\{ (1+\sum_{j=1}^{d} x_j^2)^{1/2} \right\}^{-\alpha} \in \mathcal{B}(\mathcal{H})$

is locally $H_0$-smooth on $(0,\infty)$ for

$\alpha > \frac{1}{2}$, $Z(\lambda, \langle X \rangle^{-\alpha})$ is well-defined

and one has

$$Z(\lambda, \langle X \rangle^{-\alpha}) = \Gamma_0(\lambda) \langle X \rangle^{-\alpha}$$

$$: \mathcal{H} \to L^2(\mathbb{S}^{d-1}) \text{ bounded.}$$

## V.2 The Schrödinger operator.

In $\mathcal{H} = L^2(\mathbb{R}^d)$, we shall consider

$H = H_0 + V$ with $V$ a perturbation

of $H_0$. If $V \in \mathcal{B}(\mathcal{H})$, then

$\mathcal{D}(H) = \mathcal{D}(H_0)$. If $V$ is "small"

with respect to $H_0$, then we still have

$\mathcal{D}(H) = \mathcal{D}(H_0)$.

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But if $V$ is not "small", then $\mathcal{D}(H) \neq \mathcal{D}(H_0)$ and one has to work for giving a meaning to $H$.

$\rightsquigarrow$ perturbation theory either in the sense of operators or in the sense of forms.

(look at $(\varphi, \psi) \mapsto \langle \varphi, H\psi \rangle$
$= \langle H^{1/2}\varphi, H^{1/2}\psi \rangle$
if $H \geq 0$)

From now on, we consider $V$ a multiplication operator by a function $V \in L^\infty(\mathbb{R}^d)$,

with $V(x) \in \mathbb{R}$ $\forall x \in \mathbb{R}^d$ and $V(x) \xrightarrow{|x| \to \infty} 0$

<u>Remark</u>: We can study much more general $V$.

not bounded, not real and not vanishing at infinity

<u>Lem.</u> If $V \in L^\infty(\mathbb{R}^d)$, $V$ real and $V(x) \xrightarrow{|x| \to \infty} 0$,

then $H = H_0 + V$ is self-adjoint on $\mathcal{H}^2(\mathbb{R}^d)$,

and $\sigma(H) = [0, \infty) \cup \{\lambda_j\}_{j \in N}$

with $\lambda_j < 0$ and $\lambda_j \xrightarrow[j \to \infty]{} 0$ when $N = \mathbb{N}$.

<u>proof</u> By the resolvent equation

$$R(z) = R_0(z) - \underbrace{R(z) \underbrace{V R_0(z)}_{\in K(\mathcal{H})}}_{\in K(\mathcal{H})} \quad \text{for any } z \in \mathbb{C} \setminus \mathbb{R}$$

$$\Rightarrow \sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty)$$

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<u>Remark</u>. The discrete spectrum of $H$, denoted by $\sigma_d(H)$, consists in the set of isolated eigenvalues of finite multiplicity.

no spectrum around the eigenvalue

The essential spectrum of $H$ is $\sigma_{ess}(H) = \sigma(H) \setminus \sigma_d(H)$

The remaining part of the spectrum of $H$ is then discrete, and cannot accumulate at $-\infty$ since $H = H_0 + V$ is bounded from below by $-\|V\|$

Observe that $0$ is the only possible accumulation point of the eigenvalues, since otherwise $\lambda_\infty < 0$ would be in the essential spectrum, which is not true since $\sigma_{ess}(H) = [0, \infty)$ $\square$

<u>Remark</u> We could also have some positive eigenvalues.

## <u>V.3</u> Wave and scattering operators

<u>Def.</u> A function $V \in L^\infty(\mathbb{R}^d)$, $V$ real, is <u>short-range</u> if $|V(x)| \leq C \langle x \rangle^{-1-\varepsilon}$ for some $\varepsilon > 0$.

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$V$ is called a <u>short-range potential</u>

Consider the following important statement:

multiplication op.

<u>Thm.</u> For any $\alpha > \frac{1}{2}$, the operator $\langle X \rangle^{-\alpha}$ is locally $H_0$-smooth on $(0,\infty)$ and it is also locally $H$-smooth on $(0,\infty)$, when $H = H_0 + V$ with $V$ short-range

<u>Remark</u>. This theorem can be proved with Mourre theory, or the method of conjugate operator.

<u>Thm.</u> If $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is short-range, then $W_\pm(H_0 + V, H_0)$ exist and asymptotically complete.

<u>proof</u>. Consider $G_0 := \langle X \rangle^{-\alpha}$ with $\alpha$ satisfying $\alpha > \frac{1}{2}$ and $2\alpha < 1 + \varepsilon$. (from the short-range condition)

Set $G := (V \langle X \rangle^{2\alpha}) \langle X \rangle^{-\alpha}$

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Then $G_0 G^* = G_0 G = V = H - H_0$

and $V \langle X \rangle^{2\alpha} \in B(\mathcal{H})$

By the previous theorem, $G_0$ is locally $H_0$-smooth on $(0, \infty)$ and $G$ is locally $H$-smooth on $(0, \infty)$

It then follows from the $3^{rd}$ theorem in Section II.3 that for any open interval $J$ with $\bar{J} \subset (0, \infty)$ one has the existence of $W_\pm(H, H_0, E^{H_0}(J))$ and $W_\pm(H_0, H, E^{H}(J))$

The existence means that $W_\pm(H, H_0, E^{H_0}(J))$ is complete, it means $\mathrm{Ran}(W_\pm(H, H_0, E^{H_0}(J))) = E^{H}(J)\mathcal{H}$

Since $E^{H_0}(\mathbb{R}_+) = \mathbb{1}$ one infers that

$W_\pm(H, H_0)$ exist with $\mathrm{Ran}(W_\pm(H, H_0)) = E^{H}(\mathbb{R}_+)\mathcal{H}$

$\Longrightarrow$ Completeness $(\Leftrightarrow \mathrm{Ran}(W_\pm(H, H_0)) = \mathcal{H}_{ac}(H))$

since $(\mathbb{1} - E^{H}(\mathbb{R}_+))\mathcal{H}$ consists only on the vector space generated by the discrete spectrum.

$$\Rightarrow (\mathbb{1} - E^H(\mathbb{R}_+))\mathcal{H} \subset \mathcal{H}_p(H)$$

Thus, $\mathcal{H}_{sc}(H)$ is empty, and this corresponds to the asymptotic completeness. $\square$

<u>Prop.</u> If $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is short-range, then $S := W_+^* W_-$ exists, can be diagonalized in the spectral representation of $H_0$, and one has $\mathcal{F}_0 S \mathcal{F}_0^* = \int_{\mathbb{R}_+}^{\oplus} S(\lambda)\, d\lambda$ with

$$S(\lambda) = 1 - 2\pi i\, P_0(\lambda)\left(V - VR(\lambda + i0)V\right) P_0(\lambda)^*$$

from $L^2(S^{d-1})$ to $L^2(S^{d-1})$

<u>Remark</u>. the proof consists in checking the assumption of the last theorem of Section IV.5 and by using the equality $Z(\lambda, G) = P_0(\lambda) G$

## <u>V.4</u> Radiation condition

Recall that

$$R_0(z, x, x') = i 4^{-1} (2\pi)^{-\nu} z^{\nu/2} |x - x'| H_\nu^{(1)}\left(z^{1/2} |x - x'|\right)$$

with $\nu = \frac{d-2}{2}$

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By considering $|z^{1/2}|x-x'|| \longrightarrow \infty$

we get

$$R_0(z,x,x') = \frac{1}{2}\exp\left(-(d-3)\frac{\pi i}{4}\right)(2\pi)^{-\frac{(d-1)}{2}} z^{\frac{(d-3)}{4}} \cdot$$

$$\cdot |x-x'|^{\frac{(d-1)}{2}} \exp\left(iz^{1/2}|x-x'|\right)\left(1+O(|z|^{-1/2}|x-x'|^{-1/2}\right)$$

and for fixed $x'$ inside a compact set, and

for $x \to \infty$, one has $|x-x'| = |x| - \hat{x}\cdot x' + O(|x|^{-1})$

with $\hat{x} = \frac{x}{|x|}$

and $e^{iz^{1/2}|x-x'|} = e^{iz^{1/2}|x|}\, e^{-iz^{1/2}\hat{x}\cdot x'}\left(1+O(|x|^{-1})\right)$

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<u>Lem.</u> For any $\lambda > 0$ and $f \in L^2_c(\mathbb{R}^d)$ one has

$$[R_0(\lambda \pm i0) f](x) = \pi^{1/2} \lambda^{-1/4} [\mathcal{F}_0(\lambda) f](\pm \hat{x}) W^{\pm}(|x|, \lambda) + O(|x|^{-\frac{d+1}{2}})$$

as $|x| \longrightarrow \infty$.

with $W^{\pm}(r, \lambda) = r^{-(d-1)/2} \exp(\pm i \lambda^{1/2} r \mp i\pi(d-3)/4)$

outgoing / incoming spherical waves

$\varepsilon \to 0$. for + sign. $\lambda^{(d-3)/4}$
for − sign $\lambda^{(d-3)/4} e^{2\pi i \frac{d-3}{4}}$

"proof"

$$[R(\lambda \pm i\varepsilon) f](x) \cong \int_{\mathbb{R}^d} \frac{1}{2}(2\pi)^{-(d-1)/2} (\lambda \pm i\varepsilon)^{(d-3)/4} \cdot |x|^{-(d-1)/2} \cdot$$

$$\cdot e^{-i\frac{\pi}{4}(d-3)} e^{i(\lambda \pm i\varepsilon)^{1/2}|x|} \cdot e^{-i(\lambda \pm i\varepsilon)^{1/2} \hat{x} \cdot x'} f(x')\, dx' + \text{remainder}$$

$$\xrightarrow{\varepsilon \searrow 0} W^{\pm}(|x|, \lambda) \int_{\mathbb{R}^d} \frac{1}{2}(2\pi)^{-(d-1)/2} \lambda^{(d-3)/4} \cdot$$

$$\cdot e^{\mp i \lambda^{1/2} \hat{x} \cdot x'} f(x')\, dx' + \text{remainder} \qquad \square$$

Observe that

$$\partial_r W^{\pm}(r, \lambda) = \pm i \lambda^{1/2} W^{\pm}(r, \lambda) + O(r^{-\frac{d+1}{2}})$$

$$\Longleftrightarrow \partial_r W^{\pm}(r, \lambda) \mp i \lambda^{1/2} W^{\pm}(r, \lambda) = O(r^{-\frac{d+1}{2}})$$

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<u>Def.</u> For fixed $\lambda > 0$, a function $U : \mathbb{R}^d \to \mathbb{C}$ satisfies <u>the outgoing / incoming radiation condition</u> at infinity

if $\liminf_{r \to \infty} \int_{|x|=r} \left| [\partial_r U](x) \pm i\lambda^{1/2} U(x) \right|^2 r^{d-1} d\omega = 0.$

<u>Remark</u> This assumption is slightly weaker than

$$[\partial_r U](x) \pm i\lambda^{1/2} U(x) = o\left(|x|^{-(d-1)/2}\right)$$

or $\partial_r U(x) \pm i\lambda^{1/2} U(x) \in \mathcal{H}_{-\frac{1}{2}}(\mathbb{R}^d)$ weighted Hilbert sp. with weight $\langle x \rangle^{1/2}$

$$\mathcal{H}_\alpha(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d) \mid \langle x \rangle^\alpha f \in L^2(\mathbb{R}^d) \}$$

for $\alpha \geq 0$.

For $\alpha < 0$ $\mathcal{H}_\alpha(\mathbb{R}^d) = \{ f \in L^2_{loc}(\mathbb{R}^d) \mid \langle x \rangle^\alpha f \in L^2(\mathbb{R}^d) \}$.

Before the main statement of the section, we need to introduce 2 new spaces ! Observe that for $\alpha > \beta > \frac{1}{2}$

Agmon-Hörmander space

$$\mathcal{H}_\alpha(\mathbb{R}^d) \subset \mathcal{H}_\beta(\mathbb{R}^d) \subset \underbrace{B}_{\substack{\| \\ FB^{1/2}_{2,1}}} \subset \mathcal{H}_{1/2}(\mathbb{R}^d) \subset \overset{L^2(\mathbb{R}^d)}{\overset{\|}{\mathcal{H}}} \subset \mathcal{H}_{-1/2}(\mathbb{R}^d) \subset B^* \subset \mathcal{H}_\beta(\mathbb{R}^d) \subset \mathcal{H}_{-\alpha}(\mathbb{R}^d)$$

Fourier transform Besov space

With these spaces, some previous results can be extended:

1) $\Gamma_0(\lambda) : \mathcal{H}_\alpha(\mathbb{R}^d) \longrightarrow L^2(S^{d-1})$, bounded for any $\alpha > 1/2$

$\rightsquigarrow$ $\Gamma_0(\lambda) : B \longrightarrow L^2(S^{d-1})$ is uniformly bounded in $\lambda$ on compact subset of $(0,\infty)$, and strongly continuous in $\lambda$

2) For $H = H_0 + V$ with $V$ short range

$$\sup_{\substack{\lambda \in J \subset (0,\infty) \\ \text{compact} \\ \varepsilon > 0}} \| \langle X \rangle^{-\alpha} R(\lambda \pm i\varepsilon) \langle X \rangle^{-\alpha} \| < \infty \quad \text{for } \alpha > \tfrac{1}{2}$$

$$\Longleftrightarrow \sup_{\substack{\lambda \in J \\ \varepsilon > 0}} \| R(\lambda \pm i\varepsilon) \|_{\mathcal{H}_\alpha(\mathbb{R}^d) \to \mathcal{H}_{-\alpha}(\mathbb{R}^d)} < \infty$$

$$\rightsquigarrow \sup_{\substack{\lambda \in J \\ \varepsilon > 0}} \| R(\lambda \pm i\varepsilon) \|_{B \to B^*} < \infty$$

3) For $V$ short range,

$$\Gamma_\pm(\lambda) := \Gamma_0(\lambda)(1 - VR(\lambda \pm i0)) : \mathcal{H}_\alpha(\mathbb{R}^d) \to L^2(S^{d-1})$$
$$: B \to L^2(S^{d-1})$$

and $\Gamma_\pm(\lambda)^* : B^* \longrightarrow L^2(S^{d-1})$ are also bounded.

(and uniformly bounded on compact subset of $(0,\infty)$)

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Main statement of the Section :

<u>Thm.</u> Let $V : \mathbb{R}^d \to \mathbb{R}$ be short range,

let $\lambda > 0$ and $f \in \mathcal{B}$.

Set $U(\lambda \pm i0) := R(\lambda \pm i0) f \in \mathcal{B}^*$.

Then, $U(\lambda \pm i0)$ is the only solution in $\mathcal{H}^2_{loc}(\mathbb{R}^d)$

of the equation $(-\Delta + V - \lambda) U = f$

which satisfies the radiation condition.

## <u>V.5</u> Homogeneous Schrödinger equation

We shall analyze the solution of

$(-\Delta + V)\psi = \lambda \psi$ ... ⊛ for $\lambda > 0$ and $V$ short range

<u>Prop.</u> For any $b \in L^2(S^{d-1})$, $\Gamma_\pm(\lambda)^* b$ satisfies ⊛

<u>Proof</u>

$$(-\Delta + V - \lambda)\Gamma_\pm(\lambda)^* b = (-\Delta + V - \lambda)(1 - R(\lambda \mp i0)V)\Gamma_0(\lambda)^* b$$

$$= (-\Delta + V - \lambda - \underbrace{(-\Delta + V - \lambda) R(\lambda \mp i0)}_{\mathbb{1}} V)\Gamma_0(\lambda)^* b$$

$$= \underbrace{(-\Delta - \lambda)\Gamma_0(\lambda)^* b}_{0}$$

$= 0.$ since $\Gamma_0(\lambda)(-\Delta-\lambda)g = 0$ for suitable $g$

$\square$

We can then deduce the main result ;

<u>Prop.</u> For any $b \in L^2(S^{d-1})$ one has

$$[\Gamma_-(\lambda)^* b](x) = -i 2^{-1} \lambda^{-1/4} \pi^{-1/2} \{ [S(\lambda)b](\hat{x}) W^+(|x|,\lambda) - b(-\hat{x}) W^-(|x|,\lambda) \} + o(|x|^{-(d-1)/2})$$

$$[\Gamma_+(\lambda)^* b](x) = -i 2^{-1} \lambda^{-1/4} \pi^{-1/2} \{ b(\hat{x}) W^+(|x|,\lambda) - [S(\lambda)^* b](-\hat{x}) W^-(|x|,\lambda) \} + o(|x|^{-(d-1)/2}$$

as $|x| \longrightarrow \infty$

<u>Remark</u>

Either these expression define $S(\lambda)$ and $S(\lambda)^*$, or one observe that the asymptotic expansions of $\Gamma_\pm(\lambda)^*$ involve expressions corresponding to the stationary expression of $S(\lambda)$

In the formar case, one can observe that

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$$\|S(\lambda)b\|_{L^2(S^{d-1})} = \|b\|_{L^2(S^{d-1})}$$

and $\|S(\lambda)^* b\|_{L^2(S^{d-1})} = \|b\|_{L^2(S^{d-1})}$

$\Longrightarrow$ $S(\lambda)$ is unitary

⚠ It is important to realize that the 2 approaches are fundamentally different!

1) We start with evolution group $\rightsquigarrow$ wave operators $\rightsquigarrow$ scattering operator $\rightsquigarrow$ scattering matrix.

2) We look at solutions of $(-\Delta + V - \lambda)\psi = 0$. for $\psi$ satisfying the radiation condition, and the asymptotic expansion leads to $S(\lambda)$ directly

By further analysis of $\Gamma_\pm(\lambda)$ one gets that this operator from $\mathcal{B}$ to $L^2(S^{d-1})$ is surjective.

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Our next aim is to recover the wave operators from $\{\Gamma_\pm(\lambda)\}_{\lambda>0}$.

For any $f \in \mathcal{B}$, we set

$$[\mathcal{F}_\pm f](\lambda) := \Gamma_\pm(\lambda) f \in L^2(S^{d-1})$$

<u>Thm.</u> Let $V$ be short range.

The operators $\mathcal{F}_\pm$ extends continuously to a map from $\mathcal{H}$ to $\underbrace{\int_{[0,\infty)}^{\oplus} L^2(S^{d-1})\, d\lambda}$

$$= L^2(\mathbb{R}_+, L^2(S^{d-1})) = \mathcal{K}$$

and satisfies $\mathcal{F}_\pm \mathcal{F}_\pm^* = \mathbb{1}_{\mathcal{K}}$ and $\mathcal{F}_\pm^* \mathcal{F}_\pm = E_{ac}(H)$

In addition, $\mathcal{F}_\pm$ diagonalizes $H$, which means

$$[\mathcal{F}_\pm H f](\lambda) = \lambda [\mathcal{F}_\pm f](\lambda) \quad \forall f \in D(H).$$

<u>proof of</u>

It is sufficient to check that

$$\langle (\mathcal{F}_\pm H - X \mathcal{F}_\pm) f, g \rangle_{\mathcal{H}} = 0 \quad \forall g \in C_c(\mathbb{R}_+, L^2(S^{d-1}))$$
$$\forall f \in D(H)$$

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$$= \int_0^\infty \langle [(\mathcal{F}_\pm H - X\mathcal{F}_\pm) f](\lambda), g(\lambda) \rangle_{L^2(S^{d-1})} \, d\lambda$$

$$= \int_0^\infty \langle \Gamma_\pm(\lambda) H f - \lambda \Gamma_\pm(\lambda) f, g(\lambda) \rangle_{L^2(S^{d-1})} \, d\lambda$$

$$= \int_0^\infty \langle (H-\lambda) f, \Gamma_\pm(\lambda)^* g(\lambda) \rangle_{\mathcal{H}} \, d\lambda$$

$$= \int_0^\infty \langle f, \underbrace{(H-\lambda)\Gamma_\pm(\lambda)^*}_{=0} g(\lambda) \rangle_{\mathcal{H}} \, d\lambda$$

$$= 0.$$

The other parts of the proof are based on stationary expressions. See Chapter IV $\square$

Let me show that $W_\pm(H, H_0) = \mathcal{F}_\pm^* \mathcal{F}_0$

Indeed, one has for $f, f_0 \in B$,

$$\lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \langle (H_0 - \lambda \mp \varepsilon)^{-1} f_0, (H - \lambda \mp \varepsilon)^{-1} f \rangle_{\mathcal{H}}$$

resolvent equation

$$= \lim_{\varepsilon \searrow 0} \langle \delta_\varepsilon (H_0 - \lambda) f_0, (1 - VR(\lambda \pm i\varepsilon)) f \rangle_{\mathcal{H}}$$

$$= \langle \Gamma_0(\lambda)^* \Gamma_0(\lambda) f_0, (1 - VR(\lambda \pm i0)) f \rangle_{\mathcal{H}}$$

$$= \langle \Gamma_0(\lambda) f_0, \Gamma_0(\lambda)(1 - VR(\lambda \pm i0)) f \rangle_{L^2(S^{d-1})}$$

$$= \langle \Gamma_0(\lambda) f_0, \Gamma_\pm(\lambda) f \rangle_{L^2(S^{d-1})}$$

Since for short-range potential, the wave operators exist and equal to the stationary expression one has

$$\langle W_\pm(H, H_0) f_0, f \rangle_{\mathcal{H}}$$
$$= \int_0^\infty \langle \Gamma_0(\lambda) f_0, \Gamma_\pm(\lambda) f \rangle_{L^2(S^{d-1})} d\lambda$$
$$= \int_0^\infty \langle [\mathcal{F}_0 f_0](\lambda), [\mathcal{F}_\pm f](\lambda) \rangle_{L^2(S^{d-1})} d\lambda$$
$$= \langle \mathcal{F}_0 f_0, \mathcal{F}_\pm f \rangle_{\mathcal{H}}$$
$$= \langle \mathcal{F}_\pm^* \mathcal{F}_0 f_0, f \rangle_{\mathcal{H}} \qquad \square$$

## V.6 Wave functions

When $V = 0$, solution of $-\Delta \psi = \lambda \psi$ are given by $\psi_0(x, \omega, \lambda) = e^{i\lambda^{1/2} x \cdot \omega}$ for fixed $\lambda > 0$ and $\omega \in S^{d-1}$.

We shall look to solutions of $H\psi = \lambda\psi$ which behave like $\psi_0(\cdot, \omega, \lambda)$ at infinity

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We shall consider

$$|V(x)| \le C\langle x\rangle^{-\rho} \quad \rho > 1$$

<u>Def.</u> We call <u>wave function</u> $\psi$ for $\lambda > 0$ and $\omega \in S^{d-1}$ a solution of $H\psi = \lambda\psi$ s.t.

$$\psi = \psi_0(\cdot, \omega, \lambda) - \theta_\pm(\cdot, \omega, \lambda)$$

with $\theta_\pm(\cdot, \omega, \lambda)$ satisfying the radiation condition.

<u>Prop.</u> If $\rho > \frac{d+1}{2}$ then

We need $V$ short range so that $V\psi_0 \in \mathcal{H}$

$$\psi(x) := \psi_0(x, \omega, \lambda) - R(\lambda \mp i0)V\psi_0(\cdot, \omega, \lambda) \quad (**)$$

is a wave function

Alternatively, wave function can be defined as the solution of the so-called <u>Lippmann-Schwinger</u> equation $\psi_\pm = \psi_0(\cdot, \omega, \lambda) - R_0(\lambda \mp i0)V\psi_\pm$ $(*)$

<u>Lem.</u> If $\rho > \frac{d+1}{2}$, $\lambda > 0$ and $\omega \in S^{d-1}$ are fixed, then $(*)$ has a unique solution which coincides with $(**)$

Based on the L.S. equation, one can get:

<u>Thm</u> For $\rho > \frac{d+1}{2}$ and as $|x| \longrightarrow \infty$

$$\psi_\pm(x) = e^{i\lambda^{1/2} x\cdot\omega} + a^\pm(\hat{x},\omega,\lambda)\,\omega^\pm(|x|,\lambda) + o_{a.u.}\left(|x|^{-(d-1)/2}\right)$$

with $a^\pm(\underset{S^{d-1}}{\underset{\cap}{\varphi}},\omega,\lambda) = -\pi^{1/2}\lambda^{-1/4}\left[\Gamma_0(\lambda)V\psi_\pm\right](\mp\varphi)$

$\rightsquigarrow$ related to Born approximation

<u>Def.</u> The coefficient $a^-(\varphi,\omega,\lambda)$ is called the <u>scattering amplitude</u> at energy $\lambda$ for the direction $\omega$ of the incident plane wave and the direction $\varphi$ of observation

<u>Thm.</u> For $\rho > \frac{d+1}{2}$ the operator

$$\frac{1}{2\pi i}\left(S(\lambda)-1\right) : L^2(S^{d-1}) \longrightarrow L^2(S^{d-1})$$

is an integral operator with kernel

$$-(2\pi)^{-(d+1)/2}\,\lambda^{(d-1)/4}\,a^-(\omega,\omega',\lambda) \quad (=: k(\omega,\omega',\lambda))$$

It means for $g \in L^2(S^{d-1})$

$$\left[\frac{1}{2\pi i}\left(S(\lambda)-1\right)g\right](\omega) = \int_{S^{d-1}} k(\omega,\omega',\lambda)\,g(\omega')\,d\omega'$$

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## Conclusion

We have seen several approaches for scattering theory, namely

- The time dependent approach
- " independent "
  $\rightsquigarrow$ stationary expression
- Spherical wave approach, with radiation condition
- Lippmann - Schwinger approach

for Schrödinger op. on $\mathbb{R}^d$

They all lead to similar results but under slightly different assumption, and in a more or less abstract framework

Next step : 1) inverse problem
(find $V$ from the scattering data.)
$\rightsquigarrow$ closer to applications

2) geometric scattering theory or scattering theory on manifolds.