THE MANIFOLD OF COMPLEX STRUCTURES ON A VECTOR SPACE

JOSHUA P. BOWMAN

Contents

Let V be a real vector space of dimension $2n \geq 2$. Here we present some basic tools in the theory of complex structures on V, leading to an explicit proof of Theorem 1.2, whose basic correspondence is well-known but which seems to be something of a "folklore" result.

1. Two definitions

The first, and most common, definition of a complex structure on V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{id}$. This description presents the space $\mathcal{C}(V)$ of complex structures as a closed subset of $\mathrm{Hom}(V)$:

$$\mathcal{C}(V) := \{J \in \mathrm{Hom}(V) \mid J^2 = -\mathrm{id}\}.$$

To better understand this space, we give another definition of complex structures.

Remark 1.1. $\mathbb{C} \otimes V = \mathbb{C} \otimes_{\mathbb{R}} V$ is the complexification of V. The inclusions $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}) \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{C} \otimes V)$ and $\mathrm{Hom}(V) \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{C} \otimes V)$, given by action on the appropriate factor, are injective, and in fact $\mathrm{Hom}(V)$ acts $\mathbb{C}$-linearly on $\mathbb{C} \otimes V$ because its action commutes with multiplication by i. Let $\nu : V \hookrightarrow \mathbb{C} \otimes V$ denote the canonical injection $\nu : v \mapsto 1 \otimes v$. As real vector spaces, we have $\mathbb{C} \otimes V = \nu(V) \oplus i\nu(V)$; these latter summands are, respectively, the positive and negative eigenspaces of complex conjugation on $\mathbb{C} \otimes V$.

$\mathbb{C} \otimes V$ has complex dimension $2n$. Let K be a complex n-dimensional subspace that satisfies $K \cup \overline{K} = \{0\}$, where $\overline{K} = \{\overline{x} \mid x \in K\}$. $\overline{K}$ is also a complex subspace of $\mathbb{C} \otimes V$, and from a dimension count we see that $\mathbb{C} \otimes V = \overline{K} \oplus K$. Since the image of ν is fixed pointwise by complex conjugation, we have $\nu(V) \cap K = \nu(V) \cap \overline{K} = \{0\}$, and therefore the composition of ν with projection onto $\overline{K}$ induces a real-linear isomorphism κ, as in the following diagram (exact in the first row):

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & \mathbb{C} \otimes V \arrow[r] & \overline{K} \arrow[r] & 0 \\
& & V \arrow[u, "\nu"] \arrow[ur, "\kappa"'] & &
\end{tikzcd} \tag{1}$$

Recalling that subspaces of $\mathbb{C} \otimes V$ are points in a Grassmannian manifold, we define

$$\mathcal{U}(V) := \left\{ K \in \mathrm{Gr}_{\mathbb{C}}(n, \mathbb{C} \otimes V) \mid K \cap \overline{K} = \{0\} \right\}.$$

We will also call points of $\mathcal{U}(V)$ complex structures on V.

The main theorem we wish to prove is the following:

Theorem 1.2. *$\mathcal{C}(V)$ and $\mathcal{U}(V)$ are isomorphic complex manifolds of dimension n^2.*

2. $\mathcal{U}(V)$ is open

First, we handle $\mathcal{U}(V)$ separately. Recall that $\mathrm{Gr}_{\mathbb{C}}(n, \mathbb{C} \otimes V)$ is a complex manifold, covered by an atlas of charts taking values in the space of complex-linear maps from a subspace to a complementary subspace. At a point $K \in \mathcal{U}(V)$, the obvious choice of a complementary subspace is $\overline{K}$. Thus the canonical chart on $\mathrm{Gr}_{\mathbb{C}}(n, \mathbb{C} \otimes V)$ centered at K maps into $\mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$. This latter is a complex vector space of dimension n^2, which means $\dim_{\mathbb{C}}(\mathrm{Gr}(n, \mathbb{C} \otimes V)) = n^2$. We use $\mathrm{gr}(A)$ to denote the graph of $A \in \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$.

At various times, one wishes to study the points in the complement of $\mathcal{U}(V)$. The next two lemmas will be useful to this end.

Lemma 2.1. *If $L \subset \mathbb{C} \otimes V$ is a complex subspace and $\dim_{\mathbb{C}}(L \cap \overline{L}) = p$, then $L \cap \overline{L}$ contains a real p-dimensional subspace that is fixed pointwise by complex conjugation.*

Proof. Set $L' = \{x + \overline{x} \mid x \in L \cap \overline{L}\}$. This is clearly a real subspace that is fixed pointwise by complex conjugation. We see moreover that $iL' = \{x - \overline{x} \mid x \in L \cap \overline{L}\}$, because $i(x + \overline{x}) = ix - \overline{ix}$ and $L \cap \overline{L}$ is invariant under m_i, and therefore $L \cap \overline{L} = L' \oplus iL'$ as real vector spaces. Because $\dim_{\mathbb{R}} L' = \dim_{\mathbb{R}} iL'$, we conclude both are equal to p. □

Lemma 2.2. *Let $K \in \mathcal{U}(V)$. For all $A \in \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$, there exists a nonzero $x \in K$ such that $Ax = \overline{x}$ if and only if $\mathrm{gr}(A) \cap \overline{\mathrm{gr}(A)} \neq \{0\}$.*

Proof. If $x \neq 0$ satisfies $Ax = \overline{x}$, then $x + \overline{x}$ is nonzero and is fixed by complex conjugation, hence it lies in $\mathrm{gr}(A) \cap \overline{\mathrm{gr}(A)}$. Conversely, if $\mathrm{gr}(A) \cap \overline{\mathrm{gr}(A)} \neq \{0\}$, then Lemma 2.1 implies that there exists a nonzero $x \in K$ such that $\overline{x + Ax} = x + Ax$. Because $x \in K$, by definition $\overline{x} \in \overline{K}$. But $\mathbb{C} \otimes V$ is the direct sum of K and $\overline{K}$, and therefore since $Ax \in \overline{K}$ and the two sums above are equal, we must have $Ax = \overline{x}$. □

Now we prove

Lemma 2.3. *$\mathcal{U}(V)$ is an open subset of $\mathrm{Gr}_{\mathbb{C}}(n, \mathbb{C} \otimes V)$.*

Proof. Let $K \in \mathcal{U}(V)$. For any $A \in \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$, let $\overline{A} \in \mathrm{Hom}_{\mathbb{R}}(K)$ denote the composition of A with complex conjugation. If A is such that $\mathrm{gr}(A) \cap \overline{\mathrm{gr}(A)} \neq \{0\}$, then Lemma 2.2 implies that $\overline{A}$ has an eigenvalue of 1. Eigenvalues vary continuously with linear maps, and because K is sent to $0 \in \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$ under the canonical chart at K, there exists an open neighborhood around K contained in $\mathcal{U}(V)$. □

Because an open subset of a complex manifold is itself a complex manifold, we have shown that $\mathcal{U}(V)$ is a complex manifold of dimension n^2. (Here one benefit of using our second definition of a complex structure becomes readily apparent.)

3. Passing between $\mathcal{C}(V)$ and $\mathcal{U}(V)$

In this section we introduce functions $j : \mathcal{U}(V) \to \mathcal{C}(V)$ and $k : \mathcal{C}(V) \to \mathcal{U}(V)$, which will allow us to move fluidly between the two spaces. First, we prove an elementary lemma that will unify some arguments in this section. It generalizes the standard method of splitting a real-linear function on $\mathbb{C}^n$ into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts (for this analogy, see especially the application of this result in Lemma 3.4.)

Lemma 3.1. *Let W be a real even-dimensional vector space. Let $J_1, J_2 \in \mathrm{Hom}(W)$ be complex structures on W, and set $W' = \ker(J_1 - J_2)$, $W'' = \ker(J_1 + J_2)$. Then W' and W'' are invariant subspaces for J_1 and J_2. If J_1 and J_2 commute, then $W = W' \oplus W''$.*

Proof. Observe that $(J_1 - J_2)J_1 = J_2(J_2 - J_1)$; because J_2 is invertible, $\ker(J_1 - J_2) = \ker -J_2(J_1 - J_2)$, and therefore W' is invariant under J_1. Similarly, W'' is invariant under J_1, and W' and W'' are invariant under J_2.

To split W into the sum $W' \oplus W''$, we want to find a projection $P : W \to W'$ such that $\mathrm{id} - P$ is a projection $W \to W''$. That is, we want to solve the system

$$\begin{cases} (J_1 - J_2)P = 0 \\ (J_1 + J_2)(\mathrm{id} - P) = 0 \\ P^2 = P \end{cases}$$

for P. The first equation gives $J_1 P = J_2 P$. Substituting into the second equation, we get $J_1 + J_2 - 2J_1 P = 0$, from which

$$P = \frac{1}{2}(\mathrm{id} - J_1 J_2) \qquad \text{and} \qquad \mathrm{id} - P = \frac{1}{2}(\mathrm{id} + J_1 J_2).$$

The third equation is then satisfied if and only if $(J_1 J_2)^2 = \mathrm{id}$, i.e., $J_1 J_2 = J_2 J_1$. □

We use m_z to denote multiplication by $z \in \mathbb{C}$. Suppose $K \in \mathcal{U}(V)$ is given, and define

$$j(K) := \kappa^{-1} \circ m_i \circ \kappa$$

where κ is as in (1). Then $j(K) \in \mathrm{Hom}(V)$ and $(j(K))^2 = \kappa^{-1} \circ m_i \circ m_i \circ \kappa = \kappa^{-1} \circ m_{-1} \circ \kappa = -\mathrm{id}$. Therefore j is a map $\mathcal{U}(V) \to \mathcal{C}(V)$.

Now suppose we are given $J \in \mathcal{C}(V)$, and let $k(J)$ be the real $2n$-dimensional subspace

$$k(J) := \{1 \otimes v + i \otimes Jv \mid v \in V\} \subset \mathbb{C} \otimes V.$$

Note that $\overline{k(J)} = \{1 \otimes v - i \otimes Jv \mid v \in V\} = k(-J)$.

Lemma 3.2. *For all $J \in \mathcal{C}(V)$, $k(J)$ and $\overline{k(J)}$ are the complex subspaces of $\mathbb{C} \otimes V$ defined by the respective equations $ix = -Jx$ and $ix = Jx$, and $\mathbb{C} \otimes V = k(J) \oplus \overline{k(J)}$.*

Proof. Because $-J = J^{-1}$, for every $x = 1 \otimes v + i \otimes Jv \in k(J)$, we have

$$ix = i(1 \otimes v + i \otimes Jv) = -1 \otimes Jv + i \otimes (-J)Jv = -J(1 \otimes v + i \otimes Jv) = -Jx;$$
$$i\overline{x} = i(1 \otimes v - i \otimes Jv) = 1 \otimes Jv + i \otimes (-J)Jv = J(1 \otimes v - i \otimes Jv) = J\overline{x},$$

which shows that $k(J)$ and $\overline{k(J)}$ satisfy the equations, as claimed. Now observe that both J and m_i are complex structures on $\mathbb{C} \otimes V$, and by Remark 1.1 they commute. The first equation describes the kernel of $m_i + J$, and the second describes the kernel of $m_i - J$.

Therefore the result follows from Lemma 3.1 (in particular, the equations describe complex subspaces because their solution sets are invariant under m_i). $\square$

As a corollary to Lemma 3.2, we see that k is a map $\mathcal{C}(V) \to \mathcal{U}(V)$.

Lemma 3.3. *j and k are inverses of each other.*

Proof. Let $J \in \mathcal{C}(V)$. Then for all $v \in V$,

$$\nu(v) = 1 \otimes v = \frac{1}{2}(1 \otimes v + i \otimes Jv) + \frac{1}{2}(1 \otimes v - i \otimes Jv) \qquad \in k(J) \oplus \overline{k(J)},$$

and so $\kappa(v) = \frac{1}{2}(1 \otimes v - i \otimes Jv)$. Thus we find, in applying $j(k(J))$ to $v \in V$,

$$\begin{aligned} j(k(J))v &= (\kappa^{-1} \circ m_i \circ \kappa)v \\ &= \frac{1}{2}(\kappa^{-1} \circ m_i)(1 \otimes v - i \otimes Jv) \\ &= \frac{1}{2}\kappa^{-1}(1 \otimes Jv - i \otimes J(Jv)) = Jv, \end{aligned}$$

which implies $j \circ k = \mathrm{id} : \mathcal{C}(V) \to \mathcal{C}(V)$.

Let $K \in \mathcal{U}(V)$. Let $x = 1 \otimes u + i \otimes v$ be a nonzero element of K. Because $K \cap \overline{K} = \{0\}$, x is the unique element of K such that $\nu(u) = \frac{1}{2}x + \frac{1}{2}\overline{x}$, from which we conclude $\kappa(u) = \frac{1}{2}\overline{x}$. Similarly, there exists a unique $w \in V$ such that $\kappa(j(K)u) = \frac{1}{2}(1 \otimes j(K)u + i \otimes w)$. By the definition of $j(K)$, $\kappa \circ j(K) = m_i \circ \kappa$, and therefore

$$1 \otimes j(K)u + i \otimes w = i(1 \otimes u - i \otimes v) = 1 \otimes v + i \otimes u.$$

From the direct sum splitting $\mathbb{C} \otimes V = \nu(V) \oplus i\nu(V)$, we see that $v = j(K)u$. Hence $K = k(j(K))$, which implies $k \circ j = \mathrm{id} : \mathcal{U}(V) \to \mathcal{U}(V)$. $\square$

Given $J \in \mathcal{C}(V)$, we will call $A \in \mathrm{Hom}(V)$ J-linear (respectively, J-antilinear) if it satisfies $AJ = JA$ (respectively, $AJ = -JA$). We will denote the space of J-linear functions by $\mathrm{Hom}_J(V)$, and the space of J-antilinear functions by $\mathrm{Hom}_{\overline{J}}(V)$.

Lemma 3.4. *$\mathrm{Hom}(V)$ is the direct sum of $\mathrm{Hom}_J(V)$ and $\mathrm{Hom}_{\overline{J}}(V)$.*

Proof. Let $\mathscr{L}_J$ (respectively, $\mathscr{R}_J$) denote left-multiplication (respectively, right-multiplication) by J in $\mathrm{Hom}(V)$. Then $\mathscr{L}_J$ and $\mathscr{R}_J$ are complex structures on $\mathrm{Hom}(V)$ that commute, and

$$\mathrm{Hom}_J(V) = \ker(\mathscr{L}_J - \mathscr{R}_J), \qquad \text{and} \qquad \mathrm{Hom}_{\overline{J}}(V) = \ker(\mathscr{L}_J + \mathscr{R}_J).$$

Hence the result follows from Lemma 3.1. $\square$

In Remark 1.1, we observed that $\mathrm{Hom}(\mathbb{C})$ and $\mathrm{Hom}(V)$ also act on $\mathbb{C} \otimes V$. We now show, given $K \in \mathcal{U}(V)$, how $\mathrm{Hom}_{j(K)}(V)$ and $\mathrm{Hom}_{\overline{j(K)}}(V)$ relate to $\mathrm{Hom}_{\mathbb{C}}(K)$, $\mathrm{Hom}_{\mathbb{C}}(\overline{K})$, and $\mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$.

Lemma 3.5. *Let $J \in \mathcal{C}(V)$, and set $K = k(J)$. Then we have the following equalities:*

$$\mathrm{Hom}_{\mathbb{C}}(K) = \mathrm{Hom}_{\mathbb{C}}(\overline{K}) = \mathrm{Hom}_J(V)$$

$$\mathrm{Hom}_{\mathbb{C}}(K, \overline{K}) = \mathrm{Hom}_{\overline{J}}(V)$$

when elements of the latter sets are restricted to the appropriate subspaces.

Proof. Let $A \in \mathrm{Hom}_J(V)$. Then K and $\overline{K}$ are invariant subspaces of A, because

$$A(1 \otimes v \pm i \otimes Jv) = 1 \otimes Av \pm i \otimes AJv = 1 \otimes Av \pm i \otimes J(Av) \qquad \text{for all } v \in V.$$

A similar computation show that if $A \in \mathrm{Hom}_{\overline{J}}(V)$, then A sends K to $\overline{K}$ (as well as $\overline{K}$ to K). Thus we have the inclusions $\mathrm{Hom}_J(V) \subset \mathrm{Hom}_{\mathbb{C}}(K)$ and $\mathrm{Hom}_{\overline{J}}(V) \subset \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$.

By Remark 1.1, these inclusions are injective, and so $\dim \mathrm{Hom}_J(V) \leq \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(K)$ and $\dim \mathrm{Hom}_{\overline{J}}(V) \leq \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(K, \overline{K})$. Observe that, because K and $\overline{K}$ have complex dimension n, $\dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(K) + \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(K, \overline{K}) = 2n^2 + 2n^2 = 4n^2 = \dim \mathrm{Hom}(V)$. Thus observation and Lemma 3.4 together imply that the inclusions must be surjections, which proves the equalities. □

By composing k with the canonical charts at points of $\mathcal{U}(V)$, we get canonical charts on $\mathcal{C}(V)$. The next lemma gives explicit formulas for these charts and their inverses, as well as domains on which they are defined.

Lemma 3.6. *Let $J_0 \in \mathcal{C}(V)$, and set $K_0 = k(J_0)$. If $J \in \mathcal{C}(V)$ is such $J_0 J$ does not have 1 as an eigenvalue, then $k(J)$ is the graph in $K_0 \oplus \overline{K_0} = \mathbb{C} \otimes V$ of*

$$A = (\mathrm{id} + J_0 J)(\mathrm{id} - J_0 J)^{-1} \qquad \in \mathrm{Hom}_{\overline{J_0}}(V) = \mathrm{Hom}_{\mathbb{C}}(K_0, \overline{K_0}).$$

Conversely, if $A \in \mathrm{Hom}_{\overline{J_0}}(V)$ does not have 1 as an eigenvalue, then

$$j(\mathrm{gr}(A)) = J_0(\mathrm{id} + A)^{-1}(\mathrm{id} - A) \qquad \in \mathcal{C}(V).$$

Proof. By definition, $k(J) = \{1 \otimes w + i \otimes Jw \mid w \in V\}$. Since $k(J) \subset K_0 \oplus \overline{K_0}$, for every $w \in V$ there exist unique u and v in V such that

$$1 \otimes w + i \otimes Jw = (1 \otimes u + i \otimes J_0 u) + (1 \otimes v - i \otimes J_0 v).$$

This leads to the equations

$$\begin{cases} w = u + v \\ Jw = J_0(u - v) \end{cases}.$$

Multiplying the first equation by J_0, and then adding and subtracting the second equation from the result, we obtain the system

$$\begin{cases} (J_0 + J)w = 2J_0 u \\ (J_0 - J)w = 2J_0 v \end{cases}.$$

Solving for v in terms of u, we find

$$v = (\mathrm{id} + J_0 J)(\mathrm{id} - J_0 J)^{-1} u,$$

which proves the first result.

Suppose $A \in \mathrm{Hom}_{\overline{J}}(V)$ has an eigenvector v with eigenvalue -1. Then Jv is an eigenvector of A with eigenvalue 1, since $AJv = -JAv = Jv$. Therefore if A does not have 1 as an eigenvalue, neither does it have -1 as an eigenvalue, and $A + \mathrm{id}$ is invertible. The second result now follows from the first by solving the first equation for J. □

Lemma 3.7. *j and k are continuous.*

Proof. Immediate from the local forms in Lemma 3.6. □

4. The complex structure on $\mathcal{C}(V)$

From the results of the previous section, we find

Lemma 4.1. *$\mathcal{C}(V)$ is a smooth submanifold of* $\mathrm{Hom}(V)$*.*

Proof. Lemmas 3.3 and 3.7 imply that $\mathcal{C}(V)$ is homeomorphic to the topological manifold $\mathcal{U}(V)$. It is also an algebraic variety because it is defined by a polynomial equation. Therefore it is a smooth manifold. $\square$

Lemma 4.2. *The tangent space to $\mathcal{C}(V)$ at a point J is $T_J\mathcal{C}(V) = \mathrm{Hom}_{\overline{J}}(V)$.*

Proof. Because $\mathcal{C}(V)$ is the zero set of the function $f : A \mapsto A^2 + \mathrm{id}$, the tangent space to $\mathcal{C}(V)$ at J is the kernel of df_J. This yields the condition that $H \in T_J\mathcal{C}(V)$ if and only if $JH + HJ = 0$, which is the defining equation of $\mathrm{Hom}_{\overline{J}}(V)$. $\square$

Define $\mathscr{J}$ on $T\mathcal{C}(V)$ by

$$\mathscr{J}|_{T_J\mathcal{C}(V)} = \mathscr{L}_J,$$

where $\mathscr{L}_J$ is left-multiplication by J, as in the proof of Lemma 3.4. We will show that $\mathscr{J}$ is an almost complex structure on $\mathcal{C}(V)$, and that it coincides with the pullback of the complex structure on $\mathcal{U}(V)$ by k, which implies that $\mathscr{J}$ is in fact integrable, making $\mathcal{C}(V)$ into a complex manifold isomorphic to $\mathcal{U}(V)$. This will complete the proof of Theorem 1.2.

Lemma 4.3. *$\mathscr{J}$ is an almost complex structure on $\mathcal{C}(V)$.*

Proof. By Lemma 3.1 and the proof of Lemma 3.4, $\mathscr{L}_J$ is a complex structure on $\mathrm{Hom}(V)$, and $\mathrm{Hom}_{\overline{J}}(V)$ is an invariant subspace of $\mathscr{L}_J$. Therefore by Lemma 4.2, $\mathscr{J}$ maps $T\mathcal{C}(V)$ to itself fiberwise, and the result follows. $\square$

Lemma 4.4. *$\mathscr{J}$ coincides with the pullback of the complex structure m_i on $\mathcal{U}(V)$ by k.*

Proof. For convenience, rewrite the chart on $\mathcal{C}(V)$ at J_0 given in Lemma 3.6 as $k_0(J) = 2(\mathrm{id} - J_0J)^{-1} - \mathrm{id}$. Let $H \in \mathrm{Hom}_{\overline{J_0}}(V)$. We compute $[Dk_0(J_0)]H$, the directional derivative of k at J_0 in the direction of H:

$$\begin{aligned}
[Dk_0(J_0)]H &= \lim_{t\to 0} \frac{1}{t}\big(k_0(J_0 + tH) - k_0(J_0)\big) \\
&= \lim_{t\to 0} \frac{1}{t}\left(2(\mathrm{id} - J_0(J_0 + tH))^{-1} - 2(\mathrm{id} - {J_0}^2)^{-1}\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left((\mathrm{id} - \frac{1}{2}tJ_0J)^{-1} - \mathrm{id}\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(-\mathrm{id} + \mathrm{id} + \frac{1}{2}tJ_0H + o(t)\right) = \frac{1}{2}J_0H.
\end{aligned}$$

Thus $Dk_0(J_0)$ is the linear map $H \mapsto \frac{1}{2}J_0H$, which gives an identification of $\mathrm{Hom}_{\overline{J_0}}(V)$ with itself as the tangent space at $J_0 \in \mathcal{C}(V)$ and at $k(J_0) \in \mathcal{U}(V)$, respectively. By Lemma 3.2, $\mathscr{L}_{J_0} = m_i$ is also the complex structure on $\mathrm{Hom}_{\overline{J_0}}(V)$ as the tangent space to $\mathcal{U}(V)$ at J_0. The above computation implies that $\mathscr{L}_{J_0}[Dk_0(J_0)]H = [Dk_0(J_0)]\mathscr{L}_{J_0}H$, which shows that $\mathscr{L}_{J_0}$ is the pullback of m_i by k. $\square$