COMPLEX NUMBERS

A *complex number* is a number of the form $z = a + bi$, with $a, b \in \mathbb{R}$ and $i^2 = -1$. As a set, the complex numbers are written $\mathbb{C}$. Addition and multiplication are defined by

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

(These follow the usual algebraic rules for square roots.) A complex number is represented *geometrically* by an element of $\mathbb{R}^2$: $a + bi \mapsto \binom{a}{b}$. $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$.

Let's investigate the geometric interpretations of adding and multiplying in $\mathbb{C}$.

- Addition is just adding vectors in $\mathbb{R}^2$.
- Multiplication *rotates* and/or *scales*.

Examples. Multiply by i. Multiply by $2i$. Multiply by $1+i$. Multiply by $\sqrt{2}/2 + i\sqrt{2}/2$. Multiply by $1/2 - i\sqrt{3}/2$. Multiply by -1.

Note that both rotating and scaling are linear transformations of $\mathbb{R}^2$.

We know that a point of $\mathbb{R}^2$ can be represented by *polar coordinates*:

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix},$$

where $r = \sqrt{x^2+y^2}$ and θ is the angle (measured counterclockwise) between $\binom{1}{0}$ and $\binom{x}{y}$. Applying this to complex numbers, we have the *polar form*:

$$z = r(\cos\theta + i\sin\theta).$$

r is called the *modulus* (or *absolute value*) of z, and is written $r = |z|$. θ is called an *argument* of z. Of course, θ could have many values, so we usually choose to have it in one of the intervals $[0, 2\pi)$ or $(-\pi, \pi]$.

So we know we can think of complex numbers as real vectors with 2 coordinates. But we've also seen that multiplying by a complex number produces a linear transformation. What are the matrices of these maps?

$$z \mapsto iz \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$z \mapsto \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)z \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

A matrix of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is called a *rotation* matrix.

A matrix of the form $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ is called a *scalar* matrix. Scalar matrices commute with all other matrices.

So we can also think of complex numbers as matrices, the product of a scalar matrix and a rotation matrix (in either order). This preserves the multiplicative structure. Check for yourself that this way of thinking also preserves the additive structure.

Proposition. *Let a be a complex number. Then the map $z \mapsto az$, thought of as a map $\mathbb{R}^2 \to \mathbb{R}^2$, is linear, and is the composition of scaling and rotation.*