The Rise of Projective Geometry

Euclid

There is almost nothing known about the personal life of Euclid. This lack of information has lead some to conjecture that he may not actually have existed - “Euclid” was a pseudonym used by some mathematicians in Alexandria (there is a 20th Century analogue of this – Nicholas Bourbaki). According to this view, the few references to Euclid in the ancient Greek works, were probably added by later translators and scribes.

Leaving aside this theory, we date his birth only by internal evidence in the books he wrote. In *The Elements*, we clearly see the theory developed by Eudoxus (c. 370 B.C.) and do not see any of the results of Archimedes (c. 225 B.C.), so he is thought to have lived c. 300 B.C.

Euclid

There was a philospher, Euclid of Meg'ara (a Greek city), who was one of the teachers of Plato, but he lived about a century too early to be the Euclid of geometric fame.

It is not known if Euclid was Greek or an Egyptian who came to the Greek colony of Alexandria. His familiarity with certain subjects implies that he must have spent some time in Athens, at the Academy, but nothing definite is known about this.

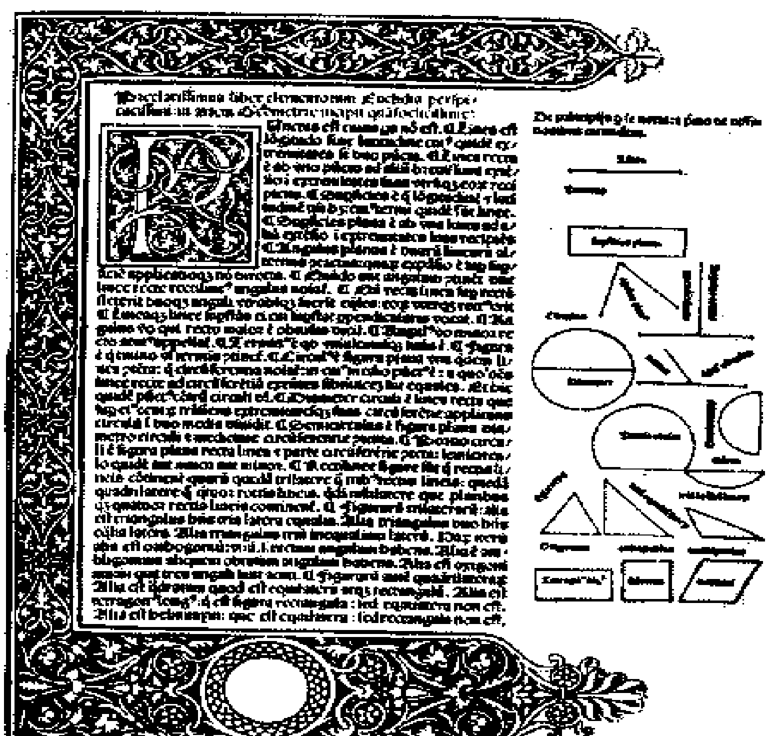
Besides the Elements, he wrote a number of other works. Among them are the *Phænomena*, dealing with the celestial sphere and containing 25 geometric propositions; the *Data*; possibly a treatise on music; and works on optics, porisms, and catoprics. He also wrote a work on the division of figures.

The Elements

Euclid's *Elements* with over 1000 editions is certainly the most influential mathematical work ever published. We do not have any original copies and can not be absolutely certain of the contents of the original. Many of the oldest copies can be traced back to the revision created by Theon of Alexandria (c. 390) since Theon stated that he added a proof that was missing in the copies he used, and the Vatican library has a copy of the *Elements* which does not include this extra proof (and so, must have been copied from an earlier version than Theon's.) Examining the early copies shows how freely the copyists revised and added to the earlier texts.

The Elements

The first printed copy of the Elements appeared in 1482. It was printed in Venice by Erhard Ratdolt and contained a Latin translation due to Johannes Campanus (c. 1260). It is claimed to be the first book to contain figures.



1505

The Elements

Euclid's Postulates

- A straight line can be drawn from any point to any point.
- A finite straight line can be produced continuously in a straight line.
- A circle may be described with any point as center and any distance as radius.
- All right angles are equal to one another.

- If a transversal falls on two lines in such a way that the interior angles on one side of the transversal are less than two right angles, then the lines meet on that side on which the angles are less than two right angles.

The 5^{th} Postulate

Proclus (410-485) wrote a commentary on *The Elements* where he comments on attempted proofs to deduce the fifth postulate from the other four, in particular he notes that Ptolemy had produced a false 'proof'. Proclus then goes on to give a false proof of his own. However he did give the following postulate which is equivalent to the fifth postulate.

Playfair's Axiom:- *Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.*

Although known from the time of Proclus, this became known as *Playfair's Axiom* after John Playfair wrote a famous commentary on Euclid in 1795 in which he proposed replacing Euclid's fifth postulate by this axiom.

The 5th Postulate

Many attempts were made to prove the fifth postulate from the other four, many of them being accepted as proofs for long periods of time until the mistake was found. Invariably the mistake was assuming some 'obvious' property which turned out to be equivalent to the fifth postulate. One such 'proof' was given by Wallis in 1663 when he thought he had deduced the fifth postulate, but he had actually shown it to be equivalent to:-

To each triangle, there exists a similar triangle of arbitrary magnitude.

Saccheri

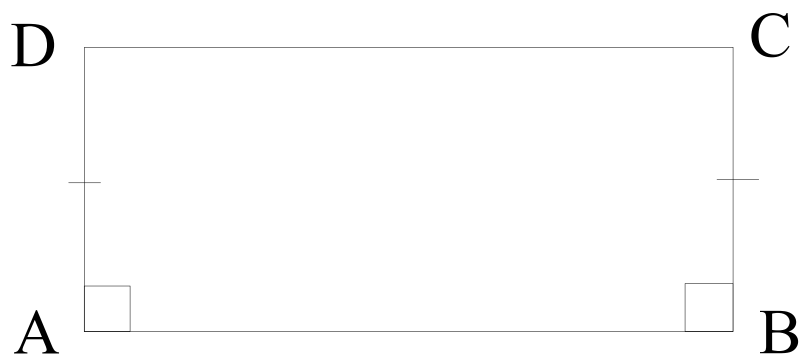
One of the attempted proofs turned out to be more important than most others. It was produced in 1697 by Girolamo Saccheri. The importance of Saccheri's work was that he assumed the fifth postulate false and attempted to derive a contradiction.

Girolamo Saccheri entered the Jesuit Order at Genoa in 1685. Five years later he went to Milan where he studied philosophy and theology at the Jesuit College. While in this College he was encouraged to take up mathematics by Tommaso Ceva.

Saccheri was ordained a priest in 1694 at Como and then taught at various Jesuit Colleges through Italy. He taught philosophy at Turin from 1694 to 1697, philosophy and theology at Pavia from 1697 until his death. He also held the chair of mathematics at Pavia from 1699 until his death.

Saccheri

In *Euclides ab Omni Naevo Vindicatus* (1733) [Euclid freed from every flaw] Saccheri did important early work on non-euclidean geometry, although he did not see it as such, rather an attempt to prove the 5th Postulate.



In this figure Saccheri proved that the ***summit angles*** at *D* and *C* were equal. The proof uses properties of congruent triangles which Euclid proved in Propositions 4 and 8 which are proved before the fifth postulate is used. Saccheri considers the cases:

a) *The summit angles are* > 90 *(hypothesis of the obtuse angle).*
b) *The summit angles are* < 90 *(hypothesis of the acute angle).*
c) *The summit angles are* = 90 *(hypothesis of the right angle).*

Saccheri

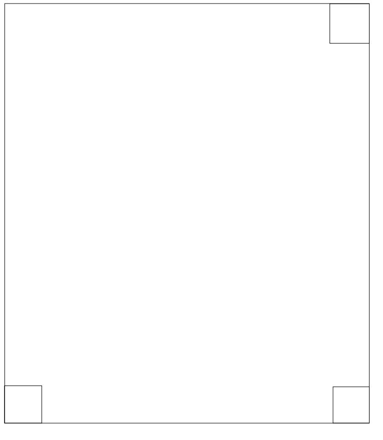
Euclid's fifth postulate is c). Saccheri proved that the hypothesis of the obtuse angle implied the fifth postulate, so obtaining a contradiction. Saccheri then studied the hypothesis of the acute angle and derived many theorems of non-Euclidean geometry without realizing what he was doing. However he eventually 'proved' that the hypothesis of the acute angle led to a contradiction by assuming that there is a 'point at infinity' which lies on a plane. He states, “the hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line.”

According to Heath,

Saccheri ... was the victim of the preconceived notion of his time that the sole possible geometry was the Euclidean, and he presents the curious spectacle of a man laboriously erecting a structure upon new foundations for the very purpose of demolishing it afterwards...

Lambert

In 1766 Lambert wrote *Theorie der Parallellinien* which was a study of the parallel postulate. By assuming that the parallel postulate was false, he followed a similar line to Saccheri, but he did not fall into the trap that Sacceri fell into and investigated the hypothesis of the acute angle without obtaining a contradiction. He managed to deduce a large number of non-euclidean results. He noticed that in this new geometry the sum of the angles of a triangle increases as its area decreases.



Lambert

Of his work on geometry, Folta says:-

Lambert tried to build up geometry from two new principles: measurement and extent, which occurred in his version as definite building blocks of a more general metatheory. Above all, Lambert carefully considered the logical consequences of these axiomatically secure principles. His axioms concerning number can hardly be compared with Euclid's arithmetical axioms; in geometry he goes beyond the previously assumed concept of space, by establishing the properties of incidence. Lambert's physical erudition indicates yet another clear way in which it would be possible to eliminate the traditional myth of three-dimensional geometry through the parallels with the physical dependence of functions. A number of questions that were formulated by Lambert in his metatheory in the second half of the 18*th* *century have not ceased to remain of interest today.*

Legendre

Elementary geometry was by this time engulfed in the problems of the parallel postulate. D'Alembert, in 1767, called it *the scandal of elementary geometry*.

Legendre spent 40 years of his life working on the parallel postulate and the work appears in appendices to various editions of his highly successful geometry book *Eléments de Géométrie*. Published in 1794 it was the leading elementary text on the topic for around 100 years.

In his "*Eléments*" Legendre greatly rearranged and simplified many of the propositions from Euclid's "Elements" to create a more effective textbook. Legendre's work replaced Euclid's "Elements" as a textbook in most of Europe and, in succeeding translations, in the United States and became the prototype of later geometry texts.

Legendre

Legendre proved that Euclid's fifth postulate is equivalent to:-

The sum of the angles of a triangle is equal to two right angles.

Legendre showed, as Saccheri had over 100 years earlier, that the sum of the angles of a triangle cannot be greater than two right angles. This, again like Saccheri, rested on the fact that straight lines were infinite. In trying to show that the angle sum cannot be less than 180 Legendre assumed that through any point in the interior of an angle it is always possible to draw a line which meets both sides of the angle. This turns out to be another equivalent form of the fifth postulate, but Legendre never realized his error himself.

Legendre

Legendre's attempts to prove the parallel postulate

... all failed because he always relied, in the last analysis, on propositions that were "evident" from the Euclidean point of view.

In 1832 (the year Bolyai published his work on non-euclidean geometry) Legendre confirmed his absolute belief in Euclidean space when he wrote:-

It is nevertheless certain that the theorem on the sum of the three angles of the triangle should be considered one of those fundamental truths that are impossible to contest and that are an enduring example of mathematical certitude.

Gauss

The first person to really come to understand the problem of the parallels was Carl Friedrich Gauss (1777- 1855) the dominant mathematical figure of his time, and undoubtedly one of the greatest mathematicians of all time. His meditations on the subject can be traced through letters written to colleagues over a period of three decades. He began work on the fifth postulate in 1792 while only 15 years old, at first attempting to prove the parallel postulate from the other four. By 1813 he had made little progress and wrote:

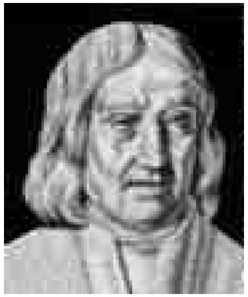
In the theory of parallels we are even now not further than Euclid. This is a shameful part of mathematics...

Gauss

However by 1817 Gauss had become convinced that the fifth postulate was independent of the other four postulates. In a letter to Wilhelm Olbers he wrote:

I keep coming closer to the conviction that the necessary truth of our geometry cannot be proved, at least by the human intellect for the human intellect. Perhaps in another life we shall arrive at other insights into the nature of space which at present we can not reach.

He began to work out the consequences of a geometry in which more than one line can be drawn through a given point parallel to a given line. Perhaps most surprisingly of all Gauss never published this work but kept it a secret. At this time thinking was dominated by Immanuel Kant (1724-1804) who had stated that Euclidean geometry is *the inevitable necessity of thought* and Gauss disliked controversy.



Bolyai

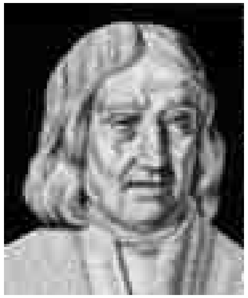
Gauss discussed the theory of parallels with his friend, the Hungarian mathematician Farkas (Wolfgang) Bolyai (1775-1856) who made several false proofs of the parallel postulate. Farkas Bolyai taught his son, János Bolyai, mathematics but, despite advising his son *not to waste one hour's time on that problem* of the fifth postulate, János Bolyai did work on the problem.

In 1823 Bolyai wrote to his father saying *I have discovered things so wonderful that I was astounded ... out of nothing I have created a strange new world*. However it took Bolyai a further two years before it was all written down and he published his *strange new world* as a 24 page appendix to his father's book *Tentamen juventutem studiosam in elementa metheseos* [Essay for studious youths on the elements of mathematics], although just to confuse future generations the appendix was published before the book itself.

Bolyai

Gauss, after reading the 24 pages, described János Bolyai in these words while writing to a friend: *I regard this young geometer Bolyai as a genius of the first order* . However in some sense Bolyai only assumed that the new geometry was possible. He then followed the consequences in a not too dissimilar fashion from those who had chosen to assume the fifth postulate was false and seek a contradiction. However the real breakthrough was the belief that the new geometry was possible. Gauss, however impressed he sounded in the above quote with Bolyai, rather devastated Bolyai by telling him that he (Gauss) had discovered all this earlier but had not published. Although this must undoubtedly be true, it detracts in no way from Bolyai's incredible breakthrough.

Bolyai



Gauss wrote to Farkas:

If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. ... On the other hand it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner.

Bolyai

János could not escape the feeling that Gauss had learned of his discoveries from the elder Bolyai prior to the publication of the Appendix and that Gauss was trying to claim priority unjustly. He later came to realize that his suspicions had been unfounded. It was many years before the work of János Bolyai gained the recognition it deserved. Even Gauss, who heartily approved of this work, hesitated to endorse it in print. Indeed, in talking about his own work in 1829, Gauss says:

It may take very long before I make public my investigations on this issue; in fact, this may not happen in my lifetime for I fear the "clamor of the Boeotians."

(Boeotia was a province of ancient Greece whose inhabitants were known for their dullness and ignorance.)

János published nothing further.

Lobachevsky

Nor is Bolyai's work diminished because Lobachevsky published a work on non-Euclidean geometry in 1829.

Nicolai Ivanovitch Lobachevsky (1793-1856) studied at the University of Kasan (Russia), where he became Professor and then Rector in 1827. About 1815 he began work on the theory of parallels, attempting to prove Euclid's 5th Postulate, ala Legendre. By 1823 he had given up this approach and turned to the development of an alternate geometry. He called this Imaginary Geometry and later Pangeometry. He presented his ideas in 1826 to the mathematics and physics department of the University of Kasan in a lecture entitled *A succinct exposition of the principles of geometry with a rigorous demonstration of the theorem of parallels*. The manuscript for this lecture was lost.

Lobachevsky

However, in 1829, he published a second memoir in Russian in the Kazan Messenger a local university publication, called *On the Principles of Geometry*. This was the earliest published account of non-Euclidean geometry. Lobachevsky's attempt to reach a wider audience had failed when his paper was rejected by Ostrogradski.

Throughout his life he wrote about and revised his theory. In 1837 an article in French appeared in *Crelle's* Journal. In 1840, a book *Geometric Researches on the Theory of Parallels* was published in German. It was from this volume that Gauss learned about Lobachevsky's work, and he then informed Frakas and János.

The nature of real truth of course cannot but be one and the same in Maros-Vásárhely as in Kamschatka and on the Moon, or, to be brief, anywhere in the world; and what one finite, sensible being discovers, can also not impossibly be discovered by another.

- J. Bolyai

Lobachevsky

In Lobachevsky's 1840 booklet he explains clearly how his non-Euclidean geometry works.

All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes - into cutting and non-cutting. The boundary lines of the one and the other class of those lines will be called parallel to the given line.

Hence Lobachevsky has replaced the fifth postulate of Euclid by:- Lobachevsky's Parallel Postulate. *There exist two lines parallel to a given line through a given point not on the line.*

Lobachevsky went on to develop many trigonometric identities for triangles which held in this geometry, showing that as the triangle became small the identities tended to the usual trigonometric identities.

Riemann

G.F. Bernhard Riemann (1826-1866) seems to have been a good, but not outstanding, pupil who worked hard at the classical subjects such as Hebrew and theology. He showed a particular interest in mathematics and the director of his Gymnasium allowed Bernhard to study mathematics texts from his own library. On one occasion he lent Bernhard Legendre's book on the theory of numbers and Bernhard read the 900 page book in six days.

In the spring of 1846 Riemann enrolled at the University of Göttingen. His father had encouraged him to study theology and so he entered the theology faculty. However he attended some mathematics lectures and asked his father if he could transfer to the faculty of philosophy so that he could study mathematics.

Riemann

Riemann was always very close to his family and he would never have changed courses without his father's permission. This was granted, however, and Riemann then took courses in mathematics from Moritz Stern and Gauss.

It may be thought that Riemann was in just the right place to study mathematics at Göttingen, but at this time the University of Göttingen was a rather poor place for mathematics. Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognised Riemann's genius. Stern, however, certainly did realise that he had a remarkable student and later described Riemann at this time saying that he:-

... already sang like a canary.

Riemann

Riemann moved from Göttingen to Berlin University in the spring of 1847. The main person to influence Riemann at this time, was Dirichlet.

Riemann was bound to Dirichlet by the strong inner sympathy of a like mode of thought. Dirichlet loved to make things clear to himself in an intuitive substrate; along with this he would give acute, logical analyses of foundational questions and would avoid long computations as much as possible. His manner suited Riemann, who adopted it and worked according to Dirichlet's methods.

- F. Klein

Riemann

Riemann's work always was based on intuitive reasoning which fell a little below the rigour required to make the conclusions watertight. However, the brilliant ideas which his works contain are so much clearer because his work is not overly filled with lengthy computations. It was during his time at the University of Berlin that Riemann worked out his general theory of complex variables that formed the basis of some of his most important work.

Riemann

In 1849 he returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851.

Riemann's thesis studied the theory of complex variables and, in particular, what we now call Riemann surfaces.

In proving some of the results in his thesis Riemann used a variational principle which he was later to call the ***Dirichlet Principle*** since he had learnt it from Dirichlet's lectures in Berlin. The Dirichlet Principle did not originate with Dirichlet, however, as Gauss, Green and Thomson had all made use if it. Riemann's thesis, one of the most remarkable pieces of original work to appear in a doctoral thesis, was examined on 16 December 1851. In his report on the thesis Gauss described Riemann as having:-

... a gloriously fertile originality.

Riemann

On Gauss's recommendation Riemann was appointed to a post in Göttingen and he worked for his Habilitation, the degree which would allow him to become a lecturer. He spent thirty months working on his Habilitation dissertation which was on the representability of functions by trigonometric series. He gave the conditions of a function to have an integral, what we now call the condition of Riemann integrability. In the second part of the dissertation he examined the problem which he described in these words:-

While preceding papers have shown that if a function possesses such and such a property, then it can be represented by a Fourier series, we pose the reverse question: if a function can be represented by a trigonometric series, what can one say about its behaviour.

Riemann

To complete his Habilitation Riemann had to give a lecture. He prepared three lectures, two on electricity and one on geometry. Gauss had to choose one of the three for Riemann to deliver and, against Riemann's expectations, Gauss chose the lecture on geometry. Riemann's lecture *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the hypotheses that lie at the foundations of geometry), delivered on 10 June 1854, became a classic of mathematics.

There were two parts to Riemann's lecture. In the first part he posed the problem of how to define an n-dimensional space and ended up giving a definition of what today we call a Riemannian space. Freudenthal writes

It possesses shortest lines, now called geodesics, which resemble ordinary straight lines. In fact, at first approximation in a geodesic coordinate system such a metric is flat Euclidean, in the same way that a curved surface up to higher-order terms looks like its tangent plane. Beings living on the surface may discover the curvature of their world and compute it at any point as a consequence of observed deviations from Pythagoras' theorem.

Riemann

The second part of Riemann's lecture posed deep questions about the relationship of geometry to the world we live in. He asked what the dimension of real space was and what geometry described real space. The lecture was too far ahead of its time to be appreciated by most scientists of that time. Monastyrsky writes :-

Among Riemann's audience, only Gauss was able to appreciate the depth of Riemann's thoughts. ... The lecture exceeded all his expectations and greatly surprised him. Returning to the faculty meeting, he spoke with the greatest praise and rare enthusiasm to Wilhelm Weber about the depth of the thoughts that Riemann had presented.

It was not fully understood until sixty years later. Freudenthal writes:-

The general theory of relativity splendidly justified his work. In the mathematical apparatus developed from Riemann's address, Einstein found the frame to fit his physical ideas, his cosmology, and cosmogony: and the spirit of Riemann's address was just what physics needed: the metric structure determined by data.

Riemann

This lecture was not published until 1868, two years after Riemann's death but was to have a profound influence on the development of a wealth of different geometries. Riemann briefly discussed a 'spherical' geometry in which every line through a point P not on a line AB meets the line AB. In this geometry no parallels are possible.

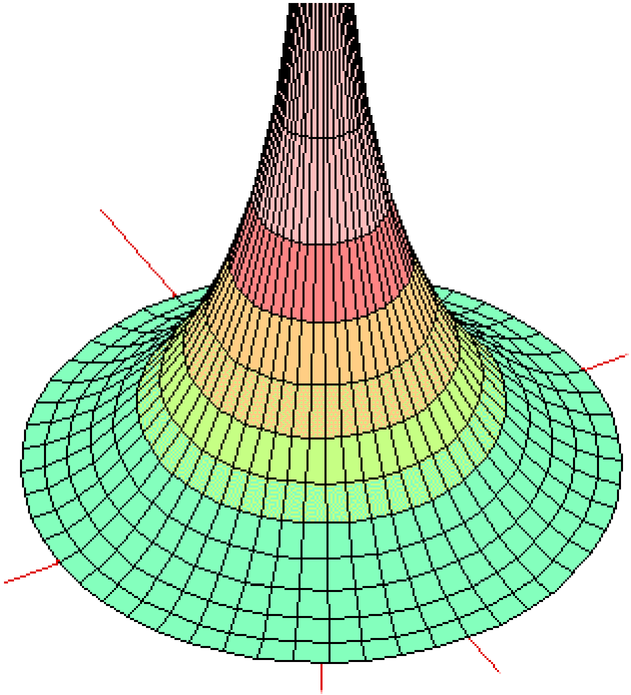
Consistency

It is important to realise that neither Bolyai's nor Lobachevsky's description of their new geometry had been proved to be consistent [free from any logical contradictions]. In fact it was no different from Euclidean geometry in this respect although the many centuries of work with Euclidean geometry was sufficient to convince mathematicians that no contradiction would ever appear within it.

Beltrami

The first person to put the Bolyai-Lobachevsky non-Euclidean geometry on the same footing as Euclidean geometry was Eugenio Beltrami (1835-1900).

His 1868 paper *Essay on an interpretation of non-euclidean geometry* gives a concrete realisation of the non-euclidean geometry of Lobachevsky and Bolyai and connects it with Riemann's geometry. The concrete realization uses the ***pseudosphere***, a surface generated by the revolution of a tractrix about its asymptote.



The top half of a pseudo-sphere

Beltrami

Beltrami studied at Pavia from 1853 to 1856, and there he was taught by Brioschi who had been appointed as professor of applied mathematics at the University of Pavia the year before Beltrami began his studies. Beltrami would have liked to continue his mathematical studies but he was suffering financial hardship so in 1856 he had to stop his studies and take up a job. He was employed as the secretary to a railway engineer and this job took him first to Verona and then to Milan.

At Milan Beltrami began to work hard at his mathematical studies again and in 1862 he published his first paper. He was appointed to the University of Bologna in 1862 as a visiting professor of algebra and analytic geometry. After two years in Bologna, Beltrami accepted the chair of geodesy at the University of Pisa, which he held from 1864 to 1866. At Pisa he became friendly with Betti. In 1866 he returned to Bologna where he was appointed professor of rational mechanics.

Beltrami

When the Kingdom of Italy was established in 1861 Turin was the capital. In 1870 Italian troops entered Rome. The city had been held by the Pope with support from the French, but after Napoleon III was defeated and abdicated, French support to hold Rome evaporated. A new University of Rome was set up in the new Italian capital and Beltrami was appointed to the chair of rational mechanics there in 1873. After three years in Rome, Beltrami moved to Pavia to take up the chair of mathematical physics there. However, Beltrami returned to Rome in 1891 and spent his last years teaching there.

Finally we should mention an important contribution by Beltrami to the history of mathematics. This appears in a 1889 publication in which Beltrami brought to the attention of the mathematical world Saccheri's 1733 study of the parallel postulate. He compared Saccheri's results with the non-euclidean geometry of Lobachevski and Bolyai.

Beltrami

Beltrami in this 1868 paper did not set out to prove the consistency of non-Euclidean geometry or the independence of the Euclidean parallel postulate. What he suggested was that Bolyai and Lobachevsky had not really introduced new concepts at all but had described the theory of geodesics on surfaces of negative curvature. Beltrami wrote in this paper:-

We have tried to find a real foundation to this doctrine, instead of having to admit for it the necessity of a new order of entities and concepts.

Beltrami

Hoüel translated both Lobachevsky's and Beltrami's work into French in 1870 and he noted how Beltrami's paper proved the independence of the Euclid's parallel postulate.

The 1868 paper should have appeared sooner but it was delayed in its publication because Cremona was not entirely happy that it was not based on a circular argument. Cremona worried that euclidean geometry was being used to describe non-euclidean geometry and he saw a possible logical difficulty in this. Cremona was wrong, but his worries caused Beltrami to put his work on one side for a while but the work of Riemann convinced Beltrami that his methods were sound.

Klein

In fact Beltrami's model was incomplete but it certainly gave a final decision on the fifth postulate of Euclid since the model provided a setting in which Euclid's first four postulates held but the fifth did not hold. It reduced the problem of consistency of the axioms of non-Euclidean geometry to that of the consistency of the axioms of Euclidean geometry.

Beltrami's work on a model of Bolyai - Lobachevsky's non-Euclidean geometry was completed by Klein in 1871. Klein went further than this and gave models of other non-Euclidean geometries such as Riemann's spherical geometry. Klein's work was based on a notion of distance defined by Cayley in 1859 when he proposed a generalized definition for distance.

Many Geometries

Klein showed that there are three basically different types of geometry. In the Bolyai - Lobachevsky type of geometry, straight lines have two infinitely distant points. In the Riemann type of spherical geometry, lines have no (or more precisely two imaginary) infinitely distant points. Euclidean geometry is a limiting case between the two where for each line there are two coincident infinitely distant points.

Poincaré

Henri Poincaré (1854-1912) entered the École Polytechnique in 1873, graduating in 1875. He was well ahead of all the other students in mathematics but, perhaps not surprisingly given his poor coordination, performed no better than average in physical exercise and in art. Music was another of his interests but, although he enjoyed listening to it, his attempts to learn the piano while he was at the École Polytechnique were not successful. His memory was remarkable and he retained much from all the texts he read but not in the manner of learning by rote, rather by linking the ideas he was assimilating particularly in a visual way. His ability to visualise what he heard proved particularly useful when he attended lectures since his eyesight was so poor that he could not see the symbols properly that his lecturers were writing on the blackboard.

After graduating from the École Polytechnique, Poincaré continued his studies at the École des Mines.

Poincaré

After completing his studies at the École des Mines Poincaré spent a short while as a mining engineer at Vesoul while completing his doctoral work. As a student of Charles Hermite, Poincaré received his doctorate in mathematics from the University of Paris in 1879. His thesis was on differential equations and the examiners were somewhat critical of the work. They praised the results near the beginning of the work but then reported that the :-

... remainder of the thesis is a little confused and shows that the author was still unable to express his ideas in a clear and simple manner. Nevertheless, considering the great difficulty of the subject and the talent demonstrated, the faculty recommends that M Poincaré be granted the degree of Doctor with all privileges.

Poincaré

Immediately after receiving his doctorate, Poincaré was appointed to teach mathematical analysis at the University of Caen. Reports of his teaching at Caen were not wholly complimentary, referring to his sometimes disorganised lecturing style. He was to remain there for only two years before being appointed to a chair in the Faculty of Science in Paris in 1881. In 1886 Poincaré was nominated for the chair of mathematical physics and probability at the Sorbonne. The intervention and the support of Hermite was to ensure that Poincaré was appointed to the chair and he also was appointed to a chair at the École Polytechnique. In his lecture courses to students in Paris :-

... changing his lectures every year, he would review optics, electricity, the equilibrium of fluid masses, the mathematics of electricity, astronomy, thermodynamics, light, and probability.

Poincaré held these chairs in Paris until his death at the early age of 58.

Poincaré

We should say a little about his way of thinking and working. He is considered as one of the great geniuses of all time and there are two very significant sources which study his thought processes. One is a lecture which Poincaré gave to l'Institute Général Psychologique in Paris in 1908 entitled *Mathematical invention* in which he looked at his own thought processes which led to his major mathematical discoveries. The other is a book by Toulouse who was the director of the Psychology Laboratory of l'École des Hautes Études in Paris. Although published in 1910 the book recounts conversations with Poincaré and tests on him which Toulouse carried out in 1897.

Poincaré

Toulouse explains that Poincaré kept very precise working hours. He undertook mathematical research for four hours a day, between 10 am and noon then again from 5 pm to 7 pm. He would read articles in journals later in the evening. An interesting aspect of Poincaré's work is that he tended to develop his results from first principles. For many mathematicians there is a building process with more and more being built on top of the previous work. This was not the way that Poincaré worked and not only his research, but also his lectures and books, were all developed carefully from basics. Perhaps most remarkable of all is the description by Toulouse of how Poincaré went about writing a paper. Poincaré:-

... *does not make an overall plan when he writes a paper. He will normally start without knowing where it will end. ... Starting is usually easy. Then the work seems to lead him on without him making a wilful effort. At that stage it is difficult to distract him. When he searches, he often writes a formula automatically to awaken some association of ideas. If beginning is painful, Poincaré does not persist but abandons the work.*

Poincaré

Toulouse then goes on to describe how Poincaré expected the crucial ideas to come to him when he stopped concentrating on the problem:-

Poincaré proceeds by sudden blows, taking up and abandoning a subject. During intervals he assumes ... that his unconscious continues the work of reflection. Stopping the work is difficult if there is not a sufficiently strong distraction, especially when he judges that it is not complete ... For this reason Poincaré never does any important work in the evening in order not to trouble his sleep.

As Miller notes:-

Incredibly, he could work through page after page of detailed calculations, be it of the most abstract mathematical sort or pure number calculations, as he often did in physics, hardly ever crossing anything out.

Poincaré

We should note that, despite his great influence on the mathematics of his time, Poincaré never founded his own school since he did not have any students. Although his contemporaries used his results they seldom used his techniques.

Poincaré achieved the highest honours for his contributions of true genius. He was elected to the Académie des Sciences in 1887 and in 1906 was elected President of the Academy. The breadth of his research led to him being the only member elected to every one of the five sections of the Academy, namely the geometry, mechanics, physics, geography and navigation sections. In 1908 he was elected to the Académie Francaise and was elected director in the year of his death. He won numerous prizes, medals and awards.Poincaré was only 58 years of age when he died :-

M Henri Poincaré, although the majority of his friends were unaware of it, recently underwent an operation in a nursing home. He seemed to have made a good recovery, and was about to drive out for the first time this morning. He died suddenly while dressing.

Poincaré

Finally we look at Poincaré's contributions to the philosophy of mathematics and science. The first point to make is the way that Poincaré saw logic and intuition as playing a part in mathematical discovery. He wrote in *Mathematical definitions in education* (1904):-

It is by logic we prove, it is by intuition that we invent.

In a later article Poincaré emphasised the point again in the following way:-

Logic, therefore, remains barren unless fertilised by intuition.

McLarty gives examples to show that Poincaré did not take the trouble to be rigorous. The success of his approach to mathematics lay in his passionate intuition. However intuition for Poincaré was not something he used when he could not find a logical proof. Rather he believed that formal arguments may reveal the mistakes of intuition and logical argument is the only means to confirm insights. Poincaré believed that formal proof alone cannot lead to knowledge. This will only follow from mathematical reasoning containing content and not just formal argument.

Poincaré

Now it is reasonable to ask what Poincaré meant by "intuition". This is not straightforward, since he saw it as something rather different in his work in physics to his work in mathematics. In physics he saw intuition as encapsulating mathematically what his senses told him of the world. But to explain what "intuition" was in mathematics, Poincaré fell back on saying it was the part which did not follow by logic:-

... to make geometry ... something other than pure logic is necessary. To describe this "something" we have no word other than intuition.

The same point is made again by Poincaré when he wrote a review of Hilbert's *Foundations of geometry* (1902):-

The logical point of view alone appears to interest [Hilbert]. *Being given a sequence of propositions, he finds that all follow logically from the first. With the foundations of this first proposition, with its psychological origin, he does not concern himself.*

Poincaré

Poincaré believed that one could choose either euclidean or non-euclidean geometry as the geometry of physical space. He believed that because the two geometries were topologically equivalent then one could translate properties of one to the other, so neither is correct or false. For this reason he argued that euclidean geometry would always be preferred by physicists. This, however, has not proved to be correct and experimental evidence now shows clearly that physical space is not euclidean.

Fano

Gino Fano (1871-1952) studied at the University of Turin which he entered in 1888. His studies there were directed by Corrado Segre and he was also influenced by Castelnuovo. Corrado Segre had been appointed to the chair of higher geometry in Turin the year that Fano entered the University of Turin. This was an exciting place for research in geometry and it is not surprising that Fano was led to specialise in this area.

In 1892 Fano graduated from Turin and then, in 1893, he went to Göttingen to undertake research and of course to study under Felix Klein. Twenty years earlier, in 1872, Klein had produced his synthesis of geometry as the study of the properties of a space that are invariant under a given group of transformations, known as the *Erlanger Programm* (1872). The Erlanger Programm gave the unified approach to geometry that is now the standard accepted view. Corrado Segre corresponded regularly with Klein and in this way Fano had been brought to Klein's attention. In fact this had led to Fano translating the Erlanger Programm into Italian while he was an undergraduate at Turin.

Fano

Fano became Castelnuovo's assistant in Rome in 1894, a position he held for four years. Following this assistantship, Fano went to Messina in the extreme northeastern Sicily where he worked from 1899 to 1901. He had left that city well before an earthquake struck Messina on 28 December 1908, almost totally destroying the city and killing 78000. By this time Fano was far away in Turin where he had been appointed as professor at the University in 1901. In 1911 Fano married Rosetta Cassin and their two children, both of the sons, became professors in the United States.

He taught at Turin from 1901 until 1938 when he was deprived of his chair by the Fascist Regime. After this Fano went to Switzerland where he taught Italian students at an international camp near Lausanne. After the end of World War II Fano, who was seventy-four years old by this time, continued to travel and lecture on mathematics. In particular he visited the United States where he lectured and he also lectured in his native Italy during the remaining six years of his life.

Fano

Fano's work was mainly on projective and algebraic geometry. Fano was a pioneer in finite geometry and one of the first people to try to set geometry on an abstract footing. Before Hilbert was to make such abstract statements Fano said:-

As a basis for our study we assume an arbitrary collection of entities of an arbitrary nature, entities which, for brevity, we shall call points, but this is quite independent of their nature.

Hilbert

David Hilbert (1862- 1943) attended the gymnasium in his home town of Königsberg. After graduating from the gymnasium, he entered the University of Königsberg. There he went on to study under Lindemann for his doctorate which he received in 1885 for a thesis entitled *Über invariante Eigenschaften specieller binärer Formen, insbesondere der Kugelfunctionen*. One of Hilbert's friends there was Minkowski, who was also a doctoral student at Königsberg, and they were to strongly influence each others mathematical progress.

In 1884 Hurwitz was appointed to the University of Königsberg and quickly became friends with Hilbert, a friendship which was another important factor in Hilbert's mathematical development. Hilbert was a member of staff at Königsberg from 1886 to 1895, being a Privatdozent until 1892, then as Extraordinary Professor for one year before being appointed a full professor in 1893.

Hilbert

In 1892 Schwarz moved from Göttingen to Berlin to occupy Weierstrass's chair and Klein wanted to offer Hilbert the vacant Göttingen chair. However Klein failed to persuade his colleagues and Heinrich Weber was appointed to the chair. Klein was probably not too unhappy when Weber moved to a chair at Strasbourg three years later since on this occasion he was successful in his aim of appointing Hilbert. So, in 1895, Hilbert was appointed to the chair of mathematics at the University of Göttingen, where he continued to teach for the rest of his career.

Hilbert's eminent position in the world of mathematics after 1900 meant that other institutions would have liked to tempt him to leave Göttingen and, in 1902, the University of Berlin offered Hilbert Fuch' chair. Hilbert turned down the Berlin chair, but only after he had used the offer to bargain with Göttingen and persuade them to set up a new chair to bring his friend Minkowski to Göttingen.

Hilbert

Hilbert submitted a paper proving the finite basis theorem to *Mathematische Annalen*. However Gordan was the expert on invariant theory for *Mathematische Annalen* and he found Hilbert's revolutionary approach difficult to appreciate. He refereed the paper and sent his comments to Klein:-

The problem lies not with the form ... but rather much deeper. Hilbert has scorned to present his thoughts following formal rules, he thinks it suffices that no one contradict his proof ... he is content to think that the importance and correctness of his propositions suffice. ... for a comprehensive work for the Annalen this is insufficient.

However, Hilbert had learnt through his friend Hurwitz about Gordan's letter to Klein and Hilbert wrote himself to Klein in forceful terms:-

... I am not prepared to alter or delete anything, and regarding this paper, I say with all modesty, that this is my last word so long as no definite and irrefutable objection against my reasoning is raised.

Hilbert

At the time Klein received these two letters from Hilbert and Gordan, Hilbert was an assistant lecturer while Gordan was the recognised leading world expert on invariant theory and also a close friend of Klein's. However Klein recognised the importance of Hilbert's work and assured him that it would appear in the *Annalen* without any changes whatsoever, as indeed it did.

Hilbert expanded on his methods in a later paper, again submitted to the *Mathematische Annalen* and Klein, after reading the manuscript, wrote to Hilbert saying:-

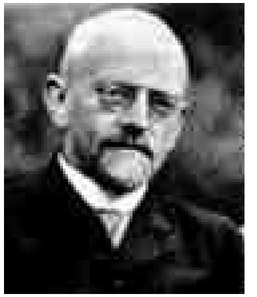
I do not doubt that this is the most important work on general algebra that the Annalen has ever published.

Hilbert

Hilbert's work in geometry had the greatest influence in that area after Euclid. A systematic study of the axioms of Euclidean geometry led Hilbert to propose 21 such axioms and he analysed their significance. He published *Grundlagen der Geometrie* in 1899 putting geometry in a formal axiomatic setting. The book continued to appear in new editions and was a major influence in promoting the axiomatic approach to mathematics which has been one of the major characteristics of the subject throughout the 20th century.

Hilbert's famous 23 Paris problems challenged (and still today challenge) mathematicians to solve fundamental questions. Hilbert's famous speech *The Problems of Mathematics* was delivered to the Second International Congress of Mathematicians in Paris in 1900. It was a speech full of optimism for mathematics in the coming century and he felt that open problems were the sign of vitality in the subject:-

Hilbert



The great importance of definite problems for the progress of mathematical science in general ... is undeniable. ... [for] as long as a branch of knowledge supplies a surplus of such problems, it maintains its vitality. ... every mathematician certainly shares ..the conviction that every mathematical problem is necessarily capable of strict resolution ... we hear within ourselves the constant cry: There is the problem, seek the solution. You can find it through pure thought...

Hilbert's problems included the continuum hypothesis, the well ordering of the reals, Goldbach's conjecture, the transcendence of powers of algebraic numbers, the Riemann hypothesis, the extension of Dirichlet's principle and many more. Many of the problems were solved during the last century, and each time one of the problems was solved it was a major event for mathematics.

Hilbert's Axioms

David Hilbert (1862 - 1943)

Undefined terms: *Point, Line, Plane, Between, Congruent, On.*

21 Axioms needed for Euclidean Geometry.

III. In a plane α, there can be drawn through any point A, lying outside of a straight line *a*, one and only one straight line that does not intersect the line *a*.

Birkhoff

George D. Birkhoff (1884-1944) entered the University of Chicago in 1902, spending a year there, then moved to Harvard University where he studied from 1903 to 1905. While at Harvard he submitted the results he had obtained with Vandiver to the *Annals of Mathematics* in 1904 and this joint number theory paper became his first publication. During these two years at Harvard the teacher who influenced him most was Bôcher who taught him algebra and classical analysis. Birkhoff was awarded his A.B. by Harvard in 1905 and his A.M. in 1906.

Birkhoff returned to the University of Chicago in 1905 to study for his doctorate. His research concentrated on asymptotic expansions, boundary value problems, and Sturm-Liouville type problems but his thesis advisor Eliakim Moore appears to have been a less influential guide to Birkhoff than was Poincaré. Birkhoff read Poincaré's works on differential equations and celestial mechanics and he learnt more, and was more strongly influenced in the direction his research was taking, by Poincaré than from his supervisor. Archibald writes:-

Birkhoff

During his two years at Harvard, one in college and one in the graduate school, Bôcher and Osgood were in their prime; but during the next two years [*at Chicago*] *he worked under Eliakim Moore, Bolza and Maschke, in the most inspiring mathematical center in the United States at that time. His dissertation was done quite independently however.*

The doctoral thesis which Birkhoff submitted was entitled *Asymptotic Properties of Certain Ordinary Differential Equations with Applications to Boundary Value and Expansion Problems* and it led to the award of his Ph.D. in 1907. It was an important thesis, not just for the results which it contained but also for the fact that the work had natural important extensions. Birkhoff himself developed the ideas further in the following years, as did two of his students, Rudolph Langer and Marshall Stone. For example Birkhoff and Langer published an important extension in 1923. Birkhoff's work on linear differential equations, difference equations and the generalised Riemann problem mostly all arose from the basis he laid in his thesis.

Birkhoff

In 1923 the American Mathematical Society made the first award of the Bôcher Memorial Prize to Birkhoff for his memoir, *Dynamical systems with two degrees of freedom* which he had published in the *Transactions of the American Mathematical Society* in 1917. He had a long association with the AMS being Vice-President in 1919, Colloquium lecturer in 1920 when he lectured on Dynamical Systems, he edited the *Transactions of the American Mathematical Society* from 1921 to 1924 and was President from 1925 to 1926.

Perhaps this high level of involvement with the AMS already suggest that Birkhoff worked tirelessly to advance mathematics in America. In fact Veblen described his efforts in this direct as "a sort of religious devotion". He realised, of course, that advancing mathematics in America meant having close contacts with advances in the rest of the world. His close contacts with mathematicians in Europe made him a natural person to write a report for the International Education Board on the state of mathematics in Europe.

Birkhoff

Birkhoff's most famous result is the “ergodic theorem” of 1931-32. But it is not only the ergodic theorem that made Birkhoff the most famous mathematician in America in his day. He had already achieved this distinction in most mathematicians eyes many years earlier when he proved Poincaré's Last Geometric Theorem, a special case of the 3-body problem, in 1913. Poincaré had stated his theorem in *Sur un théorème de géométrie* in 1912 but could only give a proof in certain special cases. Birkhoff's proof in 1913 was :-

.. one of the most exciting mathematical events of the era.

Birkhoff

The foundations of relativity and quantum mechanics were also topics which Birkhoff studied. Jointly with R E Langer, he published the monograph *Relativity and Modern Physics* in 1923. Near the end of his life he published a more speculative work, combining his ideas on philosophy and science, in *Electricity as a Fluid* in 1938. He also did important work on the four colour theorem. He developed a mathematical theory of aesthetics which he applied to art, music and poetry. Before writing *Aesthetic measure* he spent a year travelling round the world studying art, music and poetry and various countries.

He has told us that the formal structure of western music, the riddle of melody, began to interest him in undergraduate days; somewhat intense consideration of the mathematical elements here involved led him to apply his theory also to aesthetic objects such as polygons, tilings, vases, and even poetry.

Birkhoff

There was, however, a negative side to Birkhoff's character which we should comment on. Einstein said:-

G D Birkhoff is one of the world's great anti-Semites.

In a similar vein Chandler Davis, in a personal communication, writes:-

G D Birkhoff was an early teacher of mine, and his son Garrett was my (much appreciated) thesis supervisor. G D (but not Garrett) was consistently anti-Semitic, as shown in correspondence over the years; see R Phillips' article and S Mac Lane coming to Birkhoff's defence. He systematically kept Jews out of his department, but apparently relented late in life and favoured appointing ONE by the 1940s. He also helped some Jewish refugees find jobs NOT at Harvard in the 1930s, while acting generally to hinder their entry. Though his record is mixed and some were more implacably anti-Semitic than he was, his actions in this regard are important because of his very great influence. However, it does not seem to be true (as rumoured credibly at the time) that he opposed the appointment of Oscar Zariski to his department. As I mentioned, Garrett was not anti-Semitic at all.

Birkhoff's Axioms

G. D. Birkhoff (1884 - 1944)

I. **The points of any line can be put into one-to-one correspondence with the real numbers.**

II. Two distinct points determine a unique line.

III. **The half-lines through any point can be put into one-to-one correspondence with the real numbers (mod 2π).**

IV. There exist similar, but not congruent, triangles.