Limits of rational functions

Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2 y}{x^4 + y^2}.$$

We want to consider the limit of this function at $\binom{0}{0}$: does it exist? What is it, if it does exist? Let's consider first approaching the origin along a line. If the line is $x = 0$, then the function is constantly zero along this line, and so it has a limit along this line. Any other line has the equation $y = mx$, so we make this substitution. We get:

$$\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{x^2(mx)}{x^4 + (mx)^2} = \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{mx^3}{x^4 + m^2x^2} = \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{mx}{x^2 + m^2} = 0.$$

The last equality follows immediately if $m = 0$, and if $m \neq 0$, then the limitand is continuous in x at the origin, so we can just substitute.

But what happens along a different path, say $y = mx^2$? Then the limit becomes

$$\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{x^2(mx^2)}{x^4 + (mx^2)^2} = \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{mx^4}{x^4 + m^2x^4} = \lim_{\binom{x}{y}\to\binom{0}{0}} \frac{m}{1 + m^2},$$

which *depends on m*. For example, if $m = 0$, then it's zero, while if $m = 1$, it's $1/2$.

So, even though f has a limit along every line approaching the origin, and the limits are all the same along these lines, f itself *does not* have a limit at the origin. A limit of a function of 2 (or more) variables must be the same regardless of the method of approach.

By similar means, you should be able to show the following proposition:

Proposition. *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^d y}{x^{2d} + y^2}.$$

Then the limit of f at $\binom{0}{0}$ along any path $y = p(x)$, where p is a polynomial of degree less than d, is zero. However, the limit of f along paths of the form $y = mx^d$ varies with m. Hence, the limit of f at $\binom{0}{0}$ does not exist.

There is a kind of limit that occurs often enough that it's useful to know immediately that the limit is zero. As you saw in recitation this week, I struggled to come up with a viable statement of this result. This is one of those times **you should understand the technique of the proof more than knowing exactly what the theorem is,** and I expect to see you use the kind of inequalities that appear in the proof.

Proposition. *Suppose $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial in n variables, where every term has degree greater than d. Then*

$$\lim_{|\mathbf{x}|\to 0} \frac{|p(\mathbf{x})|}{|\mathbf{x}|^d} = 0.$$

Proof. The basic idea, and one you should internalize and recognize immediately, is that the length of any component of a vector is less than or equal to the length of the whole vector. That is,

$$|x_i| \leq \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2} \quad \text{for all } 1 \leq i \leq n.$$

By the triangle inequality, the absolute value of a polynomial is less than or equal to the sum of the absolute values of its terms. Therefore we can just show that the proposition is true for a monomial $a{x_1}^{i_1} \dots {x_n}^{i_n}$, and it generalizes immediately to any polynomial satisfying the given conditions. Our assumption is $d' = i_1 + \cdots + i_n > d$. Then, by the above observation,

$$|a{x_1}^{i_1} \dots {x_n}^{i_n}| \leq |a| \left(\sqrt{{x_1}^2 + \cdots + {x_n}^2} \right)^{d'} = |a||\mathbf{x}|^{d'}.$$

Thus the limitand is bounded above at all points by

$$\frac{|a||\mathbf{x}|^{d'}}{|\mathbf{x}|^d} = |a||\mathbf{x}|^{d'-d}.$$

Because we have assumed $d' > d$, this expression certainly approaches zero as $|\mathbf{x}| \to 0$. Therefore the original limit is zero. $\square$