Solving Polynomial Equations

Part II

Cubic Equations

Babylonian clay tablets have been found with tables of cubes of numbers. Such tables could be used to solve cubic equations and it has been suggested that they were used for this purpose.[1]

Cubic equations arise in the geometric problem of "duplicating the cube" and these can be solved geometrically by finding the intersection of two conics. Such solutions have been credited to the Greek mathematician Menæchmus (c. 350 B.C.)[2].

Archimedes, in his monumental work *The Sphere and Cylinder,* considers the problem of *cutting a sphere by a plane so that the two segments shall have a given ratio*. This leads to a cubic equation which Eutocius (c. 560) says Archimedes solved by finding the intersection of two conics[3].

Omar Khayyam

Nothing more is known about the cubic equation among the Greeks, but the problem of Archimedes was taken up by the Arabs and Persians of the 9th Century. Much of this work is cited by Omar Khayyam.

Umar ibn Ibrāhīm al Khayyāmī (1048-1131) was a Persian mathematician, astronomer, philosopher and poet (best known in the West for his poetic work the *Rubaiyat*). His major mathematical work, *Treatise on Demonstrations of Problems of al-Jabr and al-Muqābala*, is primarily devoted to the solution of cubic equations. The work is geometric in the Greek tradition, in fact he says that the reader should be familiar with Euclid's *Elements* and *Data* as well as Apollonius' *Conics* since cubic equations can only be solved geometrically using conic sections.

Omar Khayyam

He would have liked to provide algebraic algorithms that solved cubic equations a la al-Khwārizmī, but as he says:

When, however, the object of the problem is an absolute number, neither we, nor any of those who are concerned with algebra, have been able to solve this equation – perhaps others who follow us will be able to fill the gap.[4]

For al-Khayyāmī, as with his predecessors, all numbers were positive and so he classified cubics into 14 cases and analyzed each case in terms of which conic sections were needed for their solutions.

Pacioli

Luca Pacioli (1445-1517) was a Franciscan friar who taught mathematics throughout Italy. He was famous for his teaching, and regretted the low ebb to which teaching had fallen. One reason for this was the lack of available subject material, so he collected mathematical materials for 20 years and in 1494 published one of the earliest mathematical texts, the *Summa de arithmetica, geometrica, proportioni et proportionalita*. This 600 page work was written in Italian (a Tuscan dialect) rather than Latin. It does not contain original material, but because it was comprehensive and printed, it was widely circulated and influential.

In the *Summa*, Pacioli essentially asserts that the solution of the cubic equation is in general impossible.

The Italian Algebraists

Perhaps due to Pacioli's statements, but certainly influenced by them, the Italian algebraists of the 16th Century were very interested in finding an algebraic solution of cubic equations. The story of how this came about is one of the most interesting in the annals of mathematics. Dunham presents this tale fairly well, and I can only add a few embellishments.

Dramatis Personae

The minor characters

Scipio del Ferro (1465 – 1526) – a mathematics professor at the University of Bologna. If it weren't for this incident he would be totally obscure.

Antonio Maria Fior (Florido) – a student of del Ferro; not a particularly noteworthy mathematician.

The Italian Algebraists

Dramatis Personae

The minor characters

Annibale della Nave (1500-1558) – son-in-law of del Ferro and eventual successor to him. He inherited del Ferro's papers.

Zuanne de Tonini da Coi – an off-stage actor in this drama. He was a teacher in Brescia primarily interested in problem solving. He wrote many letters to the antagonists.

Lodovico Ferrari (1522 – 1565) – a very talented mathematician. A poor orphan, he gained employment as a servant in Cardano's household at 14 years old. Realizing that he was bright, Cardano made him his personal secretary, and soon started to teach him mathematics. This brilliant young protégé was Cardano's confidant and colleague before the age of 18.

Tartaglia

Dramatis Personae

The antagonists

Tartaglia (Nicolo Fontana) (1499 – 1557) – In 1512, young Nicolo and his father were in church when the French attacked Brescia. Nicolo's father was killed and he received a sabre wound to his jaw and palate. His mother, finding him in the church with a critical wound, recalled that dogs often licked the wounds of their young, and so, ministered her son. He survived but had a permanent imperfection in his speech. He was called “Tartaglia” meaning “the stammerer” because of this and used that name in his writings. From his will we discover that his brother's name was Fontana. It is said that he always sported a beard to hide his scars.

Cardano

Dramatis Personae

The antagonists

Girolamo Cardano (1501 – 1576) – “Cardano was a man of remarkable contrasts. He was an astrologer, yet a serious student of philosophy, a gambler and yet a first class algebraist, a physicist of accurate habits of observation and yet a man whose statements were extremely unreliable, a physician and yet the father and defender of a murderer, at one time a professor in the University of Bologna and at another time an inmate of an almshouse, a victim of blind superstition and yet the rector of the College of Physicians at Milan, a heretic who ventured to publish the horoscope of Christ, and yet a recipient of a pension from the Pope, always a man of extremes, always a man of genius, always a man devoid of principle.” - Smith, *History of Mathematics*.

The Setting

Academic life in the 16th Century was quite different from what it is today. Positions were controlled by wealthy donors and they would only support academics who had established reputations. To obtain such a reputation, one had to engage in public “debates”. For mathematicians these debates consisted of each party presenting a list of problems to their opponent. Whoever could solve the most problems (from both lists) in a given time was declared the winner by a judge. Besides the prestige, prizes or wagers were often involved.

In this system, it was to an individual's advantage to keep their methods and knowledge secret, especially if they had a desire to advance themselves.

The Play

Recall that Omar Khayyam had classified 14 types of cubic equations which had positive coefficients and a positive root. Around 1506 (according to Tartaglia) or 1514-5 (according to Cardano) Scipione del Ferro found a method for solving the form $x^3 + ax = b$. (Historians disagree on whether or not this is the only form he knew how to solve or even whether he discovered it himself or found it in earlier Arabic writings.) Del Ferro did not publish this result, but did write it out in detail in a notebook. He imparted this knowledge to his student Florido. [The rather dramatic story that he imparted this information on his deathbed is often told, he died in 1526. Also, there are statements about his only informing Florido, but other Bolognese mathematicians were aware of the solution ... whether they got it from del Ferro or from his notebook is unclear.]

The Play

Tartaglia was a very bright self-educated mathematician who was making a living by tutoring and teaching in Venice and the surrounding area. He participated in several debates to gain a reputation in order to advance himself. In 1530, Da Coi wrote to Tartaglia and posed two problems, namely

$$x^3 + 3x^2 = 5$$

and

$$x^3 + 6x^2 + 8x = 1000,$$

neither of which he could solve. Tartaglia worked on these problems and by 1535 had found a solution for the general form of

$$x^3 + ax^2 = c.$$

He made known the fact that he could solve these cubics, probably hoping to entice someone into challenging him. Later that year Florido took the bate and challenged Tartaglia to a debate.

The Play

Florido's list consisted of 30 problems, all of them involving cubics whose solutions could be obtained from del Ferro's method (the *cubo e cosa* case). Tartaglia's list included some cubics of the *cubo e quadrato* case that he could solve, but also more general problems from other areas. The wager in this debate was to be 30 lavish banquets provided by the loser.

Tartaglia worked frantically on Florido's problems, and just in the nick of time found his own solution for the *cubo e cosa* case. He was able to solve all of Florido's problems in two hours, while Florido could handle only a few of Tartaglia's problems. With this resounding success, Tartaglia's prestige soared and he magnanimously relieved Florido of his obligation under the wager. (Nothing else is known about Florido.)

Enter Cardano

Da Coi urged Tartaglia to publish his solutions, but Tartaglia declined to do so. Cardano, who was very keen to see the solutions, also wrote to Tartaglia and was similarly rebuffed. In 1539 however Cardano offered Tartaglia an introduction to the Marchese del Vasto and an opportunity for Tartaglia to exploit his own work in artillery science. This was too much for Tartaglia to by pass, so he agreed to come to Milan. However, when he arrived the Marchese was out of town and Cardano set to work to find out the secrets of his guest. With great reluctance, Tartaglia gave his solution of the *cubo e cosa* case to Cardano, but only after making Cardano swear to never reveal the secret:

I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them.

Enter Cardano

Furthermore, Tartaglia only gave his solution in cryptic verse (according to Tartaglia he later provided more details, but Cardano does not mention this):

Quando chel cubo con le cose appresso
When the cube is brought closer to the things
Se aggualia à qualche numero discreto
It becomes equal to some distinct number
Trouan dui altri differenti in esso. ...
Two others are found, whose difference it is. ...

Cardano keeps his word and there is no mention of the cubic solution in his arithmetic book which appeared soon thereafter. He even sent Tartaglia a copy to verify this. (According to Cardano, Tartaglia's reaction was to nit-pick at trifles.)

Enter Ferrari

Cardano may not have kept his word with respect to his secretary/pupil/colleague Ferrari. Together they worked out reductions of all the other forms to the *cubo e cose* case. Da Coi (whom Tartaglia once called a "devil" - he must have had an unpleasant personality) wrote to Cardano in 1540 and posed a problem:

Divide 10 into three parts such that they shall be in continued proportion and that the product of the first two shall be 6.

(This leads to the equation $x^4 + 6x^2 + 36 = 60x$.)

Da Coi thought the equation could not be solved, but Cardano thought otherwise although he could not do it himself. He gave the problem to Ferrari who solved it, but whose solution involved solving a cubic equation, so publishing it would run counter to Cardano's oath.

Ars Magna

Now chaffing under the restriction of his oath, with many new results to publish, Cardano and Ferrari, in 1543 on a trip to Firenze (Florence) stopped in Bologna and visited Hannibal Nave (Annibale di Nave del Ferro), del Ferro's son-in-law who had succeeded him at the University of Bologna and taken his name. Nave showed them del Ferro's notebook which contained the *cubo e cose* solution.

Cardano now rationalized that he could publish his own work using del Ferro's solution without breaking his oath to Tartaglia. In 1545 Cardano published his masterpiece, *Artis Magnae, sive de regvlis algebraicis, liber vnvs* in Nürnberg, Germany. The heart of this book is the complete solution (all cases, each given separate chapters) of the cubic equation and Ferrari's solution of the general quartic (***biquadratic***) equation (20 cases identified, with a few examples worked out).

Ars Magna

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit, eſt in ordine Decimus.

Habes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſ ſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Ne-que ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aūt librum ideo ſeorſim edere placuit, ut hoc abſtruſiſsimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.

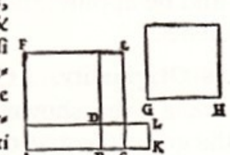
relinquitur prima 6 m: ℞ 30 6/7, hæ autem quantitates proportionales ſunt, & quadratum ſecundæ eſt æquale duplo producti ſecundæ in primam, cum quadruplo primæ, ut proponebatur.

De cubo & rebus æqualibus numero. Cap. XI.

Scipio Ferreus Bononienſis iam annis ab hinc triginta ferme capitulum hoc inuenit, tradidit uero Anthonio Mariæ Florido Veneto, qui cũ in certamen cũ Nicolao Tartalea Brixellenſe aliquando ueniſſet, occaſionem dedit, ut Nicolaus inuenerit & ipſe, qui cum nobis rogantibus tradidiſſet, ſuppreſſa demonſtratione, freti hoc auxilio, demonſtrationem quæſiuimus, eamq́ in modos, quod difficillimum fuit, redactam ſic ſubiecimus.

DEMONSTRATIO.

Sit igitur exempli cauſa cubus G H & ſexcuplum lateris G H æquale 20, & ponam duos cubos A E & C L, quorum differentia ſit 20, ita quod productum A C lateris, in C K latus, ſit 2, tertia ſcilicet numeri rerum pars, & abſcindam C B, æqualem C K, dico, quod ſi ita fuerit, lineam A B reſiduum, eſſe æqualem G H, & ideo rei æſtimationem, nam de G H iam ſupponebatur, quod ita eſſet, perficiam igitur per modum primi ſuppoſiti 6ᵗ capituli huius libri, corpora D A, D C, D E, D F, ut per D C intelligamus cubum B C, per D F cubum A B, per D A triplum C B in quadratum A B, per D E triplum A B in quadratũ B C. quia igitur ex A C in C K fit 2, ex A C in C K ter fiet 6 numerus rerum, igitur ex A B in triplum A C in C K fiunt 6 res A B, ſeu ſexcuplum A B, quare triplum producti ex A B, B C, A C, eſt ſexcuplum A B, at uero differentia cubi A C, à cubo C K, & exiſtenti à cubo B C ei æq̃le ex ſuppoſito, eſt 20, & ex ſuppoſito primo 6ᵗ capituli, eſt aggregatum corporum D A, D E, D F, tria igitur hæc corpora ſunt 20, poſita uero B C m: cubus A B, æqualis eſt cubo A C, & triplo A C in quadratum C B, & cubo B C m: & triplo B C in quadratum A C m: per demonſtrata illic, differentia autem tripli B C in quadratum A C, à triplo A C in quadratum B C eſt productum A B, B C, A C, quare cum hoc, ut demonſtratum eſt, æquale ſit ſexcuplo A B, igitur addito ſexcuplo A B, ad id quod fit ex A C in quadratum B C ter, fiet triplum B C in quadratum A C, cum igitur B C ſit m: iam oſtenſum eſt, quod productum C B in



CARDAN'S SOLUTION OF THE CUBIC

First page of the solution as given in the first edition of Cardan's *Ars* Nürnberg, 1545. The solution was slightly expanded in the second Basel, 1570

in quadratum A C ter, eſt m: & reliquum quod ei æquatur eſt p: igitur triplum C B in q̃dratum A B, & triplum A C in q̃dratũ C B, & ſexcuplũ A B nihil faciunt. Tanta igitur eſt differentia, ex cõmuni animi ſententia, ipſius cubi A C, à cubo B C, quantum eſt quod cõflatur ex cubo A C, & triplo A C in quadratum C B, & triplo C B in quadratum A C m: & cubo B C m: & ſexcuplo A B, hoc igitur eſt 20, quia differentia cubi A C, à cubo C B, fuit 20, quare per ſecundum ſuppoſitum 6ᵗ capituli, poſita B C m: cubus A B æquabitur cubo A C, & triplo A C in quadratum B C, & cubo B C m: & triplo B C in quadratum A C m: cubus igitur A B, cum ſexcuplo A B, per communem animi ſententiam, cum æquetur cubo A C & triplo A C in quadratum C B, & triplo C B in quadratum A B m: & cubo C B m: & ſexcuplo A B, quæ iam æquatur 20, ut probatum eſt, æquabuntur etiam 20, cum igitur cubus A B & ſexcuplum A B æquentur 20, & cubus G H, cum ſexcuplo G H æquentur 20, erit ex communi animi ſententia, & ex dictis, in 35ᵃ pᵗ & 31ᵃ undecimi elementorum, G H æqualis A B, igitur G H eſt differentia A C & C B, ſunt autem A C & C B, uel A C & C K, numeri ſeu liniæ continentes ſuperficiem, æqualem tertiæ parti numeri rerum, quarum cubi differunt in numero æquationis, quare habebimus regulam.

REGVLA.

Deducito tertiam partem numeri rerum ad cubum, cui addes quadratum dimidij numeri æquationis, & totius accipe radicem, ſcilicet quadratam, quam ſeminabis, uniq́ dimidium numeri quod iam in ſe duxeras, adijcies, ab altera dimidium idem minues, habebisq́ Binomium cum ſua Apotome, inde detracta ℞ cubica Apotomæ ex ℞ cubica ſui Binomij, reſiduũ quod ex hoc relinquitur, eſt rei æſtimatio.

Exemplum. cubus & 6 poſitiones, æquantur 20, ducito 2, tertiam partem 6, ad cubum, fit 8, duc 10 dimidium numeri in ſe, fit 100, iunge 100 & 8, fit 108, accipe radicem quæ eſt ℞ 108, & eam geminabis, alteri addes 10, dimidium numeri, ab altero minues tantundem, habebis Binomiũ ℞ 108 p: 10, & Apotomen ℞ 108 m: 10, horum accipe ℞ᵃˢ cubᵃˢ & minue illam quę eſt Apotomæ, ab ea quæ eſt Binomij, habebis rei æſtimationem, ℞ v: cub: ℞ 108 p: 10 m: ℞ v: cubica ℞ 108 m: 10.

cub⁹ p: 6 reb⁹ æq̃lis 20
2 20
8 —— 10
108
℞ 108 p: 10
℞ 108 m: 10
℞ v: cu. ℞ 108 p: 10
m: ℞ v: cu. ℞ 108 m: 10

Aliud, cubus p: 3 rebus æquetur 10, duc 1, tertiam partem 3, ad cubum, fit 1, duc 5, dimidium 10, ad quadratum, fit 25, iunge 25 & 1,

H 2 fiunt

CARDAN'S SOLUTION OF THE CUBIC

Continuation of the solution as given on page 462. For the meaning of the symbols see page 428

The Fallout

Naturally, Tartaglia was furious with Cardano when the Ars Magna was published. There followed a barrage of public letters, full of invective, charges and counter-charges between them. To regain his prestige Tartaglia demanded a debate with Cardano. Cardano refused but offered a debate with Ferrari. This was at first rejected by Tartaglia who wanted revenge against Cardano, but eventually Tartaglia was offered a position which he desired, and the conditions required that a debate be held, so he acceded. The debate between Tartaglia and Ferrari was held in Milan in 1548 (with much fanfare, this was the superbowl of debates). After one day Tartaglia realized that he was outmatched by Ferrari, so he slunk away at night, forfeiting the debate to the young Ferrari.

Postscript

Tartaglia eventually wrote the best treatise on arithmetic that appeared in Italy in his century (*General Trattato di nvmeri, et misvre*, in two parts 1556 – 1560).

At eighteen, Ferrari was glad to sever all relations with Cardano and began teaching by himself in Milan. He was very successful and did so well in debates that he attracted the attention of the Cardinal of Mantua through whom he secured a position that brought him abundant means. He became a professor at Bologna, but died after one year there, probably poisoned by his sister.

Cardano's life had many ups and downs, chronicled in his autobiography. It is said that he predicted the date of his own death, and ensured that prediction by committing suicide.

The Cubic Equation

The treatment of the cubic in the Ars Magna will strike the modern reader as redundant and torturous due to the several cases considered which we now know are equivalent. The treatment is not abstract, all problems are carried out with specific numbers and even though the final result is an algebraic formula, the proofs are all geometric.

I will forgo the historical treatment – deciphering what Cardano wrote – and present the modern derivation of this formula. I do this since it appears that this result has fallen out of the modern curriculum, so even this recent version has taken on an historical hue.

The Cubic Equation

This derivation is a variant of one due to Viète (1615) although he did not permit his symbols to take both positive and negative values. It was the Dutch mathematician Johann Hudde (c. 1658) who firmly recognized this possibility.

Starting with the general cubic equation

$$\mathbf{x^3 + ax^2 + bx + c = 0},$$

we make the substitution $x = y - 1/3\, a$. After simplifying, we see that the equation has been transformed to

$$y^3 + py + q = 0, \quad (\textit{the cubo e cosa form})$$

where

$$p = b - \frac{a^2}{3} \quad \textit{and} \quad q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

The Cubic Equation

We can now substitute $y = u + v$, introducing two variables in place of one to give us an additional degree of freedom. Simplification gives us

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Using our extra degree of freedom, we now set $p + 3uv = 0$, and need to solve two equations:

$$u^3 + v^3 = -q \quad \text{and}$$

$$uv = -1/3\ p, \text{ or equivalently } u^3v^3 = -1/27\ p^3\ .$$

Using the relationship between the roots and coefficients of a quadratic equation, we see that the quantities u^3 and v^3 must be roots of the quadratic equation,

$$z^2 + qz - p^3/27 = 0.$$

Now, by using the quadratic formula and taking cube roots we obtain the Cardano-Tartaglia formula:

The Cubic Equation

$$y = u + v = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

As we know, there are three roots to a cubic (two of which may be complex). Once we have found one the other two can be obtained from the original equation by removing the factor corresponding to this root and solving the resultant quadratic.

Cardano did consider the number of roots of his equations, and counted the negative (***fictitious***) roots, although they were not considered as solutions.