Differential Equations

Carol Ash

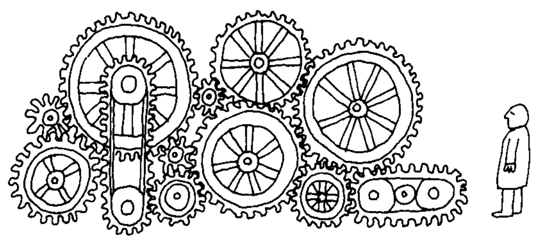


TABLE OF CONTENTS

REFERENCE PAGE FOR EXAMS

convolution integral

$$f(t)*g(t) = \int_{u=-\infty}^{\infty} f(t-u)\ g(u)\ du = \int_{u=-\infty}^{\infty} g(t-u)\ f(u)\ du$$

ANTIDERIVATIVE TABLES

(A) $\int \sin^2 ax\, dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax$

(B) $\int \cos^2 ax\, dx = \frac{x}{2} + \frac{1}{4a} \sin 2ax$

(C) $\int xe^{ax}\, dx = \frac{1}{a} xe^{ax} - \frac{1}{a^2} e^{ax}$

(D) $\int x \cos ax\, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax$

(E) $\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$

(F) $\int \sin mx \cos nx\, dx = \frac{\cos(n-m)x}{2(n-m)} - \frac{\cos(n+m)x}{2(n+m)}$

where $m \neq n$

(G) $\int \sin mx \cos mx\, dx = \frac{\sin^2 mx}{2m}$

(H) $\int \cos mx \cos nx\, dx = \frac{\sin(n-m)x}{2(n-m)} + \frac{\sin(n+m)x}{2(n+m)}$

where $m \neq n$

(I) $\int \sin mx \sin nx\, dx = \frac{\sin(n-m)x}{2(n-m)} - \frac{\sin(n+m)x}{2(n+m)}$

where $m \neq n$

(J) $\int e^{ax} \cos nx\, dx = \frac{e^{ax}(a \cos nx + n \sin nx)}{a^2 + n^2}$

(K) $\int e^{ax} \sin nx\, dx = \frac{e^{ax}(a \sin nx - n \cos nx)}{a^2 + n^2}$

a table of some exact differentials

(22) $\frac{y\,dx - x\,dy}{y^2} = d(\frac{x}{y})$

(23) $\frac{x\,dy - y\,dx}{x^2} = d(\frac{y}{x})$

(24) $\frac{-2x\,dx - 2y\,dy}{(x^2 + y^2)^2} = d(\frac{1}{x^2 + y^2})$

(25) $\frac{x\,dx + y\,dy}{\pm\sqrt{x^2+y^2}} = d(\pm\sqrt{x^2 + y^2})$

(26) $\frac{2x\,dx + 2y\,dy}{x^2 + y^2} = d\,\ell n(x^2 + y^2)$

(27) $\frac{-y\,dx + x\,dy}{x^2 + y^2} = d(\tan^{-1} \frac{y}{x})$

COEFFS FOR FOURIER SINE SERIES

To get $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$ for x in $[0,L]$ choose $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

COEFFS FOR FOURIER COSINE SERIES

To get $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$ for x in $[0,L]$

choose $A_0 = \frac{1}{L} \int_0^L f(x)\, dx$ = average value of f in $[0,L]$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

COEFFS FOR FOURIER FULL SERIES

To get $f(x) = C_0 + \sum_{n=1}^{\infty} \left[C_n \cos \frac{n\pi x}{L/2} + D_n \sin \frac{n\pi x}{L/2} \right]$ for x in $[0,L]$

choose $C_0 = \frac{1}{L} \int_0^L f(x)\, dx$ = average value of f in $[0,L]$

$$C_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L/2}\, dx, \quad D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L/2}\, dx$$

OVER

COEFFS FOR FOURIER BESSSEL SERIES

To get $f(r) = \sum_{n=1}^{\infty} D_n\, J_0\left[\frac{a_n r}{L}\right]$ for r in [0,L], where the a_n's are the zeros of J_0,

$$\text{choose}\quad D_n = \frac{\int_0^L f(r)\, J_0\left[\frac{a_n r}{L}\right] r\, dr}{\int_0^L J_0^2\left[\frac{a_n r}{L}\right] r\, dr}$$

SOME ORDINARY DE WITH VARIABLE COEFFICIENTS THAT TURN UP WHEN YOU SOLVE PDE

$rR'' + R' + r\lambda^2 R = 0$ has solution $R = AJ_0(\lambda r) + BY_0(\lambda r)$

$r^2 R'' + rR' - \lambda^2 R = 0$ has solution $R = Ar^{\lambda} + Br^{-\lambda}$

$r^2 R'' + rR' = 0$ has solution $R = C\, \ell n\, r + D$

INTEGRAL TABLES

(1) $$\frac{2}{L}\int_0^L K \sin\frac{n\pi x}{L}\,dx \quad (\text{where } K \text{ is a constant}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4K}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

(2) $$\frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx = \begin{cases} \frac{-2L}{n\pi} & \text{if } n \text{ is even} \\ \frac{2L}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

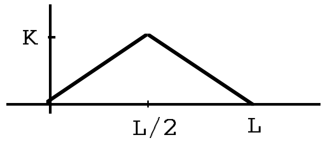
(3) $$\frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4L}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

(4) If $f(x) = \begin{cases} a & \text{for } 0 \le x \le L/2 \\ b & \text{for } L/2 \le x \le L \end{cases}$ where a and b are constants then

(a) $$\frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n=4,8,12, \\ \frac{4(a-b)}{n\pi} & \text{if } n=2,6,10,\ldots \\ \frac{2(a+b)}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

(b) $$\frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \begin{cases} \frac{2(a-b)}{n\pi} & \text{if } n=1,5,9, \\ \frac{-2(a-b)}{n\pi} & \text{if } n=3,7,11,\ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(5) If f(x) looks like this



then

(a) $$\frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8K}{n^2\pi^2} & \text{if } n=1,5,9,\ldots \\ \frac{-8K}{n^2\pi^2} & \text{if } n=3,7,11,\ldots \end{cases}$$

(b) $$\frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \text{ is odd or if } n=4,8,12,\ldots \\ \frac{-16K}{n^2\pi^2} & \text{if } n=2,6,10,\ldots \end{cases}$$

CHAPTER 1 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

SECTION 1.1 SUPERPOSITION

form of a linear differential equation

To illustrate the pattern, here is a typical linear fourth order DE:

(1) $$a_4 y'''' + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = f(x)$$

The coefficients $a_0, \ldots, a_4$ may contain x's or may be constants. They are always real. The function f(x) is called the *forcing function*. The unknown (to be solved for) is the function y(x).

If the forcing function is 0 then the DE is called *homogeneous*.

For example, consider

$$y''' + (\sin x)\, y'' + 2y' + x^2 y = \cos x$$

and

$$y'' + 2y' + x^3 y = 0$$

They are both linear. The first one is third order with forcing function cos x; the second DE is second order homogeneous.

A linear DE does *not* contain terms such as y^2, $y' y''$, e^y, cos y, $1/y'$, etc.

superposition rule

Suppose

y_1 is a solution of $ay'' + by' + cy = f(x)$

y_2 is a solution of $ay'' + by' + cy = g(x)$.

Then

$y_1 + y_2$ is a solution of $ay'' + by' + cy = f(x) + g(x)$

ky_1 is a solution of $ay'' + by' + cy = kf(x)$

For example if

3 sin x is a sol of $ay'' + by' + cy = \cos x$

and

cos 2x is a sol of $ay'' + by' + cy = 12 \cos^2 x$

then

3 sin x + cos 2x is a sol of $ay'' + by' + cy = \cos x + 12 \cos^2 x$

and

4 cos 2x is a sol of $ay'' + by' + cy = 48 \cos^2 x$

proof of the $y_1 + y_2$ part

Assume y_1 solves $ay'' + by' + cy = f(x)$ and y_2 solves $ay'' + by' + cy = g(x)$. This means that

when y_1 is substituted into the LHS of the DE it produces f(x)

when y_2 is substituted into the LHS of the DE it produces g(x).

Substitute $y_1 + y_2$ into $ay'' + by' + cy$:

$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2)$$

$$= a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) \quad \text{(because deriv of sum is sum of derivs)}$$

$$= \underbrace{ay_1'' + by_1' + cy_1}_{f(x)} + \underbrace{ay_2'' + by_2' + cy_2}_{g(x)} \quad \text{(rearrange)}$$

So $y_1 + y_2$ *is* a sol of $ay'' + by' + c = f(x) + g(x)$ QED

linear DE's and physical systems

A linear 2nd order DE has the form

$$ay'' + by' + cy = f(x)$$

If a,b,c are non-negative, the DE can be thought of as describing a "system" where x represents time, f(x) is the input, y(x) is the system's response and a,b,c are "ingredients" of the system, such as mass, resistance etc.

If the coeffs a,b,c are *constants* then the DE corresponds to a *time-invariant* system whose ingredients don't change with time.

The superposition rule for the DE means that in the system, the response to a sum of inputs is the sum of the separate responses, and say tripling an input will triple the response. *Linearity* of a DE (i.e., having the pattern in (1)) corresponds to a system where *superposition* holds.

For example, the DE might describe a block on a spring being acted on by force f(x) at time x and responding with displacement y(x); in this case, a is the block's mass, b is the damping constant associated with the retarding effect of the medium, and c is the spring constant.

Or the DE might describe an LRC network with input voltage f(x) at time x and response y(x) coulombs on the capacitor; in this case,
$a = L$, $b = R$, $c = 1/C$.

general versus particular solutions

A *particular* solution to a DE is any one specific sol. A *general* sol to an n-th order DE is a sol containing n arbitrary constants.

special case of superposition for homog DE

If y_1 and y_2 are particular solutions of

$$ay'' + by' + cy = 0$$

then $y_1 + y_2$, Ay_1, By_2 are also sols, and a general solution is

$$Ay_1 + By_2 .$$

For example, $y = \sin x$ and $y = \cos x$ both have the property that $y'' = -y$ so they are sols to the homog DE $y'' + y = 0$. Other sols are $\sin x + \cos x$, $6 \sin x$, $7 \cos x$, $3 \sin x + 5 \cos x$ etc. and a gen sol is $A \sin x + B \cos x$.

proof

If sol y_1 corresponds to forcing function 0 and y_2 corresponds to forcing function 0, then by the general superposition rule, $Ay_1 + By_1$ corresponds to forcing function $A\cdot 0 + B\cdot 0 = 0$, i.e., to forcing function 0 again.

PROBLEMS FOR SECTION 1.1 (solutions begin in the back)

1. Are the following linear? If so, are they homog?
 (a) $y'' + y' + 7x = 0$
 (b) $xy' + y' + 7y = 0$
 (c) $yy'' + y' + 7y = 0$
 (d) $x^2y''' + y \sin x - \cos x = 0$
 (e) $x^2y''' = y \sin x$
 (f) $x^2y''' = y \sin y$

2. If y_1 and y_2 are sols of $3y'' + 2y' + xy = \cos x$ then what are the following sols of? (a) $y_1 + y_2$ (b) $3y_1$ (c) $y_2 - y_1$

3. If y_1 and y_2 are sols of $3y'' + 2y' + 6y = 0$ what do the following solve?

(a) $y_1 + y_2$ (b) $3y_1$ (c) $y_2 - y_1$

4. Suppose y_1 is a sol to $ay'' + by' + cy = x^2$. Find a sol to

(a) $ay'' + by' + cy = 3x^2$ (b) $3ay'' + 3by' + 3cy = 3x^2$ (c) $3ay'' + 3by' + 3cy = x^2$

5. Check that

x^3 is a solution of $yy' = 3x^5$

and

e^{2x} is a solution of $yy' = 2e^{4x}$

but

$x^3 + e^{2x}$ is *not* a solution of $yy' = 3x^5 + 2e^{4x}$

Doesn't that violate the superposition principle.

APPENDIX SOLVING DIFFERENTIAL EQUATIONS WITH MATHEMATICA

Here's the general solution to y'' + 3y' + 2y = 0.

DSolve[y''[x] + 3y'[x] + 2y[x] ==0,y[x],x]

```
                      x
           C[1] + E  C[2]
  {{y[x] -> --------------}}
                 2 x
                E
```

And here are the general solutions to y'' + 4y' + 13y = 0 and to y'' + 3y' + 3y = 0. Mathematica gives you the general real solution in the first example and the general complex solution in the second example (I don't know why it switched).

DSolve[y''[x] + 4y'[x] + 13y[x] ==0,y[x],x]

```
             C[2] Cos[3 x] - C[1] Sin[3 x]
   {{y[x] -> -----------------------------}}
                          2 x
                         E
```

DSolve[y''[x] + 3y'[x] + 3y[x] ==0,y[x],x]

```
             ((-3 - I Sqrt[3]) x)/2          ((-3 + I Sqrt[3]) x)/2
 {{y[x] -> E                       C[1] + E                       C[2]}}
```

Here's the solution to y'' + 3y' + 2y = 0 with IC y(0) = 1, y'(0) = -3

DSolve[{y''[x] + 3y'[x] + 2y[x] ==0,y[0] == 1, y'[0] == -3}, y[x],x]

```
                 x
           2 - E
  {{y[x] -> ------}}
             2 x
            E
```

Here's the general solution to the nonhomog DE y'' + 3y' + 2y = $6e^{3x}$

DSolve[y''[x] + 3y'[x] + 2y[x] == 6E^(3x), y[x],x]

```
              3 x
           3 E      C[1]   C[2]
  {{y[x] -> ------ + ---- + ----}}
             10      2 x     x
                    E
```

Here's the solution to the DE y'' + 3y' + 2y = $6e^{3x}$ with IC y(0) = 1, y'(0) = 0.

DSolve[{y''[x] + 3y'[x] + 2y[x] == 6E^(3x), y[0] ==1, y'[0]==0}, y[x],x]

```
                   x      5 x
           2 + 5 E  + 3 E
  {{y[x] -> -----------------}}
                      2 x
                 10 E
```

Here's the general solution to y'' + 3y' + 2y = -3 sin 2x

DSolve[y''[x] + 3y'[x] + 2y[x] == -3 Sin[2x], y[x],x]

```
           C[1]   C[2]   9 Cos[2 x]   3 Sin[2 x]
  {{y[x] -> ---- + ---- + ---------- + ----------}}
            2 x     x        20           20
           E       E
```

SECTION 1.2 HOMOGENEOUS LINEAR DE WITH CONSTANT COEFFICIENTS

solution to ay'' + by' + cy = 0 where a,b,c are constant

To solve the DE

$$ay'' + by' + cy = 0$$

where a,b,c are *constants* (this doesn't work if a,b,c have x's in them), first find the roots of the plain equation (called the characteristic equation)

$$am^2 + bm + c = 0$$

If $m = 2, -5$ for instance then the gen sol is $y = Ae^{2x} + Be^{-5x}$.

If $m = 2, 0$ for instance then the gen sol is $y = Ae^{2x} + Be^{0x} = Ae^{2x} + B$.

In general, if $m = m_1, m_2$ (two real numbers, different from one another) then a general solution is

$$y = A\,e^{m_1 x} + B\,e^{m_2 x}$$

Furthermore, it can be shown, but not here, that there are no other solutions to the DE except those of the form $A\,e^{m_1 x} + B\,e^{m_2 x}$.

proof of the rule

Try $y = e^{mx}$ in the DE $ay'' + by' + cy = 0$ to see what m's, if any, make it work:

$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$ (substitute e^{mx} for y in the DE)

$am^2 + bm + c = 0$ (cancel e^{mx})

So the sols m to the characteristic equ $am^2 + bm + c = 0$ determine solutions e^{mx} to the DE.

If $m = 2,-5$ say , then $y = e^{2x}$ and $y = e^{-5x}$ are sols to the DE. By the superposition rule for homog DE, $Ae^{2x} + Be^{-5x}$ is a gen sol. QED

DE with initial conditions

A differential equation plus IC determine a unique solution. To find it, first find a general solution. Then plug in the IC to determine the constants.

(If the DE represents a spring system where f(x) is the input force at time x and y(x) is the position of the block at time x then the IC $y(0) = y_0$ describes the initial position of the block and the IC $y'(0) = y_1$ describes the initial velocity of the block.)

example 1

Find the general solution and any 3 particular solutions to $2y'' + 7y' + 3y = 0$.

solution $2m^2 + 7m + 3 = 0$, $(2m+1)(m+3) = 0$, $m = -\frac{1}{2}, -3$ so a gen sol is

(1) $$y = Ae^{-x/2} + Be^{-3x}$$

Use any values of A and B to get particular solutions:

$$y = \pi e^{-x/2} + 7e^{-3x}$$

$$y = e^{-3x}$$

$$y = e^{-x/2} - e^{-3x} \quad \text{etc.}$$

example 1 continued

Solve $2y'' + 7y' + 3y = 0$ with IC $y(0) = 0$, $y'(0) = 1$. In other words, get the particular solution that satisfies the IC.

Continue with the general solution in (1):

$$y = Ae^{-x/2} + Be^{-3x}$$

Then

$$y' = -\frac{1}{2}Ae^{-x/2} - 3Be^{-3x}.$$

Plug in $y(0) = 0$: $0 = A + B$.

Plug in $y'(0) = 1$: $1 = -\frac{1}{2}A - 3B$.

Solve for A and B:

$$A = \frac{2}{5},\ B = -\frac{2}{5}.$$

Final sol is

$$y = \frac{2}{5}e^{-x/2} - \frac{2}{5}e^{-3x}$$

If a block on a spring has mass 2, the spring constant is 3, the retarding force of the medium is 7, there is no force applied to the block, and initially the block is not displaced but is moving up at velocity 1 then Fig 1 shows the height y of the spring at time x (the block moves up some and then slowly moves back to its equilibrium position at height 0).

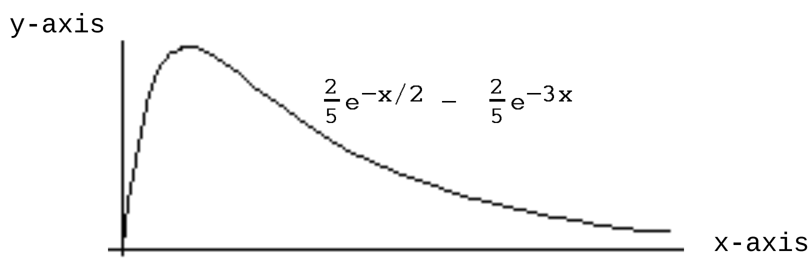


FIG 1

PROBLEMS FOR SECTION 1.2

1. Find a general solution

(a) $y'' + 2y' - 3y = 0$ (b) $y'' + 2y' - 4y = 0$

(c) $4y'' - 25y = 0$ (d) $y'' + 2y' = 0$

2. Solve $y'' - 2y' - 3y = 0$ with IC $y(0) = 0$, $y'(0) = -4$

Honors

3. A student is working on a DE of the form ay" + b' + cy = 0 and gets m = 3,5. According to the rule, she is supposed to conclude that the general solution is $y = Ae^{3x} + Be^{5x}$.

(a) Suppose she says that the general solution is $y = 17Ae^{3x} - \pi Be^{5x}$.
Is she wrong. Explain briefly.

(b) Suppose I plug some IC into the officially sanctioned general solution $y = Ae^{3x} + Be^{5x}$ and get $A = -3$, $B = -16$. so that my final answer to ay" + by' + cy = 0 plus IC is $y = -3e^{3x} - 16e^{5x}$.
If she plugs those same IC into $y = 17Ae^{3x} - \pi Be^{5x}$, what happens.

SECTION 1.3 THE COMPLEX EXPONENTIAL

standard form of a complex number

Expressions of the form a + bi where a and b are real and $i^2 = -1$ are called complex numbers. The real part is a and the imaginary part is b (the imag part is plain b, *not* bi). If the imag part is 0 then the number is real so the complex numbers include the reals as a special case.

If $z = 2 - 3i$ then we write Re $z = 2$ and Im $z = -3$

conjugation

If $z = 2 - 3i$ then the *conjugate* of z, denoted $\overline{z}$, is $2 + 3i$.

In general, if $z = a + bi$ then $\overline{z} = a - bi$.

addition, multiplication, division

Suppose $z = 2 + 3i$ and $w = 7 - 8i$. Then

$$z + w = 9 - 5i$$

$$zw = (2 + 3i)(7 - 8i) = 14 - 24i^2 + 21i - 16i = 38 + 5i$$

$$\frac{z}{w} = \frac{2+3i}{7-8i} = \frac{2+3i}{7-8i}\,\frac{7+8i}{7+8i} = \frac{-10 + 37i}{113} = -\frac{10}{113} + \frac{37}{113}i$$

magnitude and argument (angle)

The number x + iy is pictured as point (x,y) in the plane and is said to have magnitude r and argument (angle) θ as shown in Fig 1. The r and θ are just the polar coordinates corresponding to point (x,y) (always use non-negative r's).

(1)

If x + iy has mag r and argument θ then

$$r = \sqrt{x^2 + y^2}$$

$\tan\theta = y/x$ (but *not* $\theta = \arctan y/x$ ---- see warning coming later)

$x = r\cos\theta$ = Re part

$y = r\sin\theta$ = Im part

$x + iy = r(\cos\theta + i\sin\theta)$ (called the *polar* form of the complex number)

example 1

If $z = 6i$ then by inspection (Fig 2), $r = 6$ and $\theta = 90^\circ$.

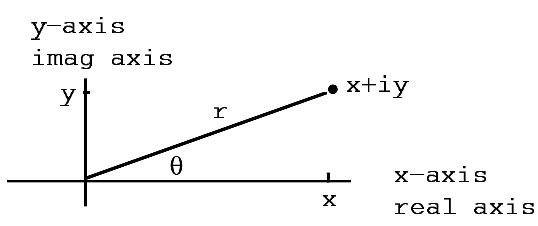


FIG 1

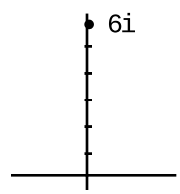


FIG 2

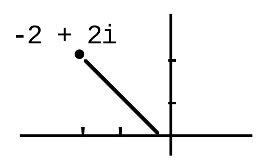


FIG 3

example 2

If $z = -2 + 2i$ (Fig 3) then $r = \sqrt{4 + 4} = \sqrt{8}$, $\theta = 3\pi/4$ by inspection.

arctan y/x versus arctan[x, y]

First of all, let's agree for the sake of this paragraph, to measure angles between $-\pi$ to π. For instance, in Fig 3 we will call the angle $3\pi/4$ (on exams you can also call it $-5\pi/4$ or even $11\pi/4$ or $-26\pi/4$ if you like).

Here is what Mathematica finds for arctan $\frac{2}{-2}$ = arctan(−1) versus arctan[−2, 2].

```
In[1]:=
ArcTan[2/-2]   (*This is arctan(-1)   )
Out[1]=
-Pi
---
 4

In[2]:=
ArcTan[-2, 2]

Out[2]=
3 Pi
----
 4
```

Different answers! Here is what is happening.

There are two angles (between $-\pi$ and π) whose tangent is -1, namely $-\pi/4$ and $3\pi/4$. Arctan(−1) stands for the particular one between $-\pi/2$ and $\pi/2$, namely $-\pi/4$. So arctan $\frac{2}{-2}$ is *not* the angle of $-2 + 2i$.

In general, arctan y/x is the angle between $-\pi/2$ and $\pi/2$ whose tangent is y/x and it is *not* always the angle of $x + iy$.

On the other hand, arctan[x, y] stands for *the* angle (measured between $-\pi$ and π) of $x + iy$ so arctan[−2, 2] did give the angle of $-2 + 2i$ correctly.

In general, if $x + iy$ lies in quadrants I or II then arctan y/x and arctan[x,y] agree and both give you the angle of $x + iy$.

If $x + iy$ lies in quadrants II or III then arctan y/x will *not* be the angle of $x+iy$. You have to add π to arctan y/x to get the right angle.

example 3

If $z = -3 - 2i$ (Fig 4) then

$$r = \sqrt{13}, \quad \tan\theta = \frac{-2}{-3} = \frac{2}{3}.$$

From tables or a calculator, $\tan^{-1}\frac{2}{3} \approx 34^\circ$ so (look at Fig 4) $\theta \approx 34^\circ+180^\circ = 214^\circ$

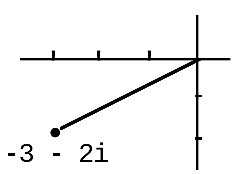


FIG 4

warning

$\tan\theta$ is always y/x.

θ is always arctan[x, y] if your calculator or computer software supports this notation.

But θ is not always arctan $\frac{y}{x}$. So *do not write* $\theta = \arctan\frac{y}{x}$.

example 4

If the number z has mag 2 and angle 27° then

$$\text{Re } z = 2\cos 27^\circ$$
$$\text{Im } z = 2\sin 27^\circ$$
$$z = 2(\cos 27^\circ + i \sin 27^\circ)$$

DeMoivre's law (mag and angle of a product, quotient, power)

If z_1 has mag r_1 and angle θ_1 and z_2 has mag r_2 and angle θ_2 then

$z_1 z_2$ has mag $r_1 r_2$ and angle $\theta_1 + \theta_2$ *(mult mags and add angles)*

z_1/z_2 has mag r_1/r_2 and angle $\theta_1 - \theta_2$ *(divide mags and subtract angles)*

z_1^n has mag r_1^n and angle $n\theta_1$ *(raise mag to n-th power and mult angle by n)*

proof of the $z_1 z_2$ rule

$$z_1 z_2 = r_1(\cos\theta_1 + i \sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \quad \text{by (1)}$$

$$= r_1 r_2 \Big[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \Big]$$

$$= r_1 r_2 \Big[\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2) \Big] \quad \text{by a trig identity}$$

This is of the form $r(\cos\theta + i\sin\theta)$ where $r = r_1 r_2$ and $\theta = \theta_1 + \theta_2$ so $z_1 z_2$ has mag $r_1 r_2$ and angle $\theta_1 + \theta_2$.

example 5

Suppose

z has mag $\sqrt{2}$ and angle 225°,

w has mag $2\sqrt{5}$ and angle -60°,

q has mag 1 and angle -90°.

Then zwq has mag $2\sqrt{10}$ and angle 75°, i.e, $zwq = 2\sqrt{10}\,(\cos 75^\circ + i \sin 75^\circ)$

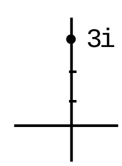
the complex exponential

If r and θ are real and $r \ge 0$ then $re^{i\theta}$ stands for the complex number with mag r and argument θ radians; i.e.,

(2) $$\boxed{re^{i\theta} = r(\cos\theta + i\sin\theta)}$$

For example,

$3e^{\pi i/2}$ has mag 3 and angle 90° so $3e^{\pi i/2} = 3i$



$2e^{3i} = 2(\cos 3 + i \sin 3)$ (meaning 3 radians)

$e^{2ix} = \cos 2x + i \sin 2x$

$e^{-2ix} = \cos(-2x) + i\sin(-2x) = \cos 2x - i \sin 2x$

Here are some special cases of (2).

(3) $$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \quad \text{(mag 1, angle }\theta)}$$

And $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ by trig identities so

(4) $$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta \quad \text{(mag 1, angle }-\theta)}$$

And

$$\boxed{\begin{array}{ll} e^{\pi i} = -1 & \text{(mag 1, angle }\pi) \\ e^{2\pi i} = 1 & \text{(mag 1, angle }2\pi) \\ e^{\pi i/2} = i & \text{(mag 1, angle }\pi/2) \\ e^{-\pi i/2} = -i & \text{(mag 1, angle }-90^{\circ}) \end{array}}$$

Here's the more general definition:

(5) $$\boxed{e^{a+bi} = e^{a}e^{bi} = e^{a}(\cos b + i\sin b) \quad \text{(mag }e^{a}\text{, angle b)}}$$

For example,

$$e^{(2+6i)x} = e^{2x+6ix} = e^{2x}(\cos 6x + i\sin 6x)$$

Here's *why* $re^{i\theta}$ is defined as the complex number with mag r and angle θ.

When two complex numbers are multiplied, DeMoivre's rule says their mags, r_1 and r_2, are multiplied and their angles, θ_1 and θ_2, are added. When the numbers are written in complex exponential form, DeMoivre's rule looks like

$$r_1e^{i\theta_1}\ r_2e^{i\theta_2} = r_1r_2\ e^{i(\theta_1+\theta_2)}$$

and becomes the simple algebra rule "multiply coefficients and add exponents".

In other words, when you use complex exponential notation you don't have to make a special effort to implement DeMoivre's rule; just use familiar rules of exponents and it will be implemented automatically.

footnote

If the angle θ is going to be used as an exponent, why use base e. Why not some other base. Temporarily, you could pick any b and let $b^{i\theta}$ be the complex number with mag 1 and angle θ. But when derivatives are defined (coming up) you would find that the derivative of your b^{ix} turns out to be ib^{ix}. For real functions, only base e acts like this: the derivative of say e^{2x} is $2e^{2x}$ but the derivative of b^{2x} is $2b^{2x}$ *times* $\ln b$. The only way to get the derivative of a complex exponential function to be like the derivative of a real exponential function is to let the complex number with mag 1 and angle x be specifically e^{ix}, not b^{ix} for some other b.

derivative of a complex-valued function of a real variable

To illustrate the definition, if

$$f(x) = x^2 + i\,x^3$$

then

$$f'(x) = 2x + 3ix^2$$

In other words, differentiate the real and imag parts separately. Equivalently, treat i like a constant and differentiate as usual.

In general if $f(x) = u(x) + i\,v(x)$ then $f'(x) = u'(x) + i\,v'(x)$

derivative of the complex exponential

Treat i as a constant and differentiate as usual. For example,

$$D\, e^{2ix} = 2ie^{2ix}$$

$$D\, e^{(2+4i)x} = (2+4i)e^{(2+4i)x}$$

semi proof

I'll show that $D\, e^{2ix} = 2ie^{2ix}$.

By the definition of the derivative of a complex-valued function,

$$D\, e^{2ix} = D(\cos 2x + i \sin 2x) = -2 \sin 2x + 2i \cos 2x.$$

On the other hand,

$$2ie^{2ix} = 2i(\cos 2x + i \sin 2x) = -2 \sin 2x + 2i \cos 2x.$$

So $D\, e^{2ix}$ *is* $2ie^{2ix}$

example 6

$$D\, 5ie^{(2+3i)x} = 5i(2+3i)e^{(2+3i)x} = (-15 + 10i)e^{(2+3i)x}$$

example 7

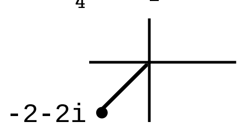
$$D\, x^2 e^{2ix} = x^2 \cdot 2ie^{2ix} + 2xe^{2ix} \quad \text{(product rule)}$$

PROBLEMS FOR SECTION 1.3

1. Find (a) $(2 + 6i)(8-3i)$ (b) $\dfrac{1}{8+3i}$ (c) $\dfrac{2+9i}{4-i}$

2. Find r and θ (a) $-7i$ (b) $4 - 4i\sqrt{3}$ (c) $-4 + 3i$ (d) -7 (e) $10 - 10i$

3. The angle of $-2-2i$ is $-\frac{3}{4}\pi$ by inspection.



(a) What would a computer (or any mathematician) find for $\arctan \frac{-2}{-2}$

(b) What would a computer find for arctan[−2, −2].

4. Use De Moivre's laws to find (a) $(-1 + i)^6$ (b) $(-1 + i)^7$ (c) $(\sqrt{3} + i)^3$

5. Use DeMoivre's law to find the real and imag part.
   (a) $(\sqrt{3} + i)^9$ (b) $\dfrac{8}{(-1+i)^5}$

6. Express in exponential form

(a) $1 - i$ (b) $e^3(\cos 5 + i \sin 5)$ (c) $2(\cos \frac{\pi}{7} - i \sin \frac{\pi}{7})$

7. Sketch roughly (a) e^{6+3i} (b) $e^{\pi i/3}$ (c) $e^{1-\pi i}$ (d) $3e^{\pi i/3}$

8. Find the magnitude and angle (a) $e^{2-3\pi i}$ (b) $2e^{\pi i/4}$
(c) $5ie^{6ix}$

9. Find the real and imag parts
(a) e^{7ix} (b) $5e^{(2-3i)x}$ (c) $(2+3i)e^{5ix}$ (d) $(2+4i)e^{(1-2i)x}$
(e) $e^{3ix} + e^{-3ix}$ (f) $\frac{2}{7-4i} e^{4ix}$

10. Differentiate (a) ie^{4ix} (b) $3e^{(2+4i)x}$ (c) $3e^{4ix} + 5e^{6ix}$

(d) $xe^{\pi ix}$ (e) $(2-i)e^{(3+4i)x}$ (f) $x^3 e^{3ix}$ (g) $ie^{(2-3i)x}$

11. Find $\dfrac{d^2 e^{2ix}}{dx^2}$.

12. Show that if A and B are conjugates then, for any θ, $Ae^{i\theta} + Be^{-i\theta}$ is real.

Honors

13. DeMoivre's rule says that there is a nice connection between the magnitude and angle of each factor and the mag and angle of the product:

If

z_1 has mag r_1 and angle θ_1

and

z_2 has mag r_2 and angle θ_2

then

$z_1 z_2$ has mag $r_1 r_2$ and angle $\theta_1 + \theta_2$.

What is the rule connecting the real and imaginary parts of each factor and the real and imag parts of the product. In other words, fill in the blanks and show how you decided:

If

z_1 has real part x_1 and imag part y_1

and

z_2 has real part x_2 and imag part y_2

then

$z_1 z_2$ has real part ____________________ and imag part __________________ .

SECTION 1.4 COMPLEX SUPERPOSITION

general complex superposition rule

If
$u(x) + iv(x)$ is a sol to $ay'' + by' + cy = f(x) + ig(x)$

then
$u(x)$ is a solution to $ay'' + by' + cy = f(x)$

and
$v(x)$ is a solution to $ay'' + by' + cy = g(x)$

In other words, the real part of the sol goes with the real part of the forcing function and the imag part of the sol goes with the imag part of the forcing function.

For example, if
$y = 2x + ix^2$ is a sol to $ay'' + by' + cy = 6x + i(3x^2 + 2)$
then
$y = 2x$ is a sol to $ay'' + by' + cy = 6x$
and
$y = x^2$ is a sol to $ay'' + by' + cy = 3x^2 + 2$

proof

Suppose $u(x) + iv(x)$ is a sol to $ay'' + by' + cy = f(x) + ig(x)$. This means that

$$a(u + iv)'' + b(u + iv)' + c(u + iv) = f(x) + ig(x)$$

Collect terms to get

$$(*) \quad au'' + bu' + cu + i(av''+ bv' + cv) = f(x) + ig(x)$$

It's a rule of algebra that in an equation like (*), the real part on the LHS must equal the real part on the RHS and the imag part on the LHS must equal the imag part on the RHS. So

$$au'' + bu' + cu = f(x) \quad \text{and} \quad av'' + bv' + cv = g(x)$$

which shows that $u(x)$ is a sol to $ay'' + by' + cy = f(x)$ and $v(x)$ is a solution to $ay'' + by' + cy = g(x)$ QED.

important special case of complex superposition with a complex exponential forcing function

Suppose
$u(x) + iv(x)$ is a sol to $ay'' + by' + cy = 2e^{3ix}$

This means that

$u(x) + iv(x)$ is a sol to $ay'' + by' + cy = 2 \cos 3x + 2i \sin 3x$

Then
$u(x)$ is a sol to $ay'' + by' + cy = 2 \cos 3x$
and
$v(x)$ is a sol to $ay'' + by' + cy = 2 \sin 3x$

special case of complex superposition for homog DE

> Suppose
>
> $u(x) + iv(x)$ is a (complex) sol to the homog equation $ay'' + by' + cy = 0$.
>
> In other words,
>
> $u(x) + iv(x)$ is a sol to the homog equation $ay'' + by + cy = 0 + 0i$.
>
> Then $u(x)$ and $v(x)$ individually are (real) sols to $ay'' + by' + cy = 0$ and
>
> $$Au(x) + Bv(x) \quad \text{(no } i \text{ in here)}$$
>
> is a general real sol.

In other words, *the real and imag parts of a complex homog sol are themselves homog sols (and are real)*. (Remember that the imag part is *real*; it's what is sitting *next to* the i but does *not include* the i.)

For example suppose $3e^{4ix}$ is a sol to $ay'' + by' + cy = 0$.
Then $3 \cos 4x$ (the real part of $3e^{4ix}$) and $3 \sin 4x$ (the imag part of $3e^{4ix}$) are also sols, and $A\, 3\cos 4x + B\, 3\sin 4x$ is a real gen sol.

footnote
3A can be renamed C and 3B can be renamed D so the gen real sol could also be written as $C \cos 4x + D \sin 4x$.

proof
If

$$u(x) + iv(x)$$

is a sol to

$$ay'' + by' + cy = 0 = 0 + 0i$$

then by the general complex superposition rule, $u(x)$ is a sol to

$$ay'' + by' + cy = \text{Re } 0 = 0$$

and $v(x)$ is a sol to

$$ay'' + by' + cy = \text{Im } 0 = 0.$$

So $u(x)$ and $v(x)$ are both homog sols.

PROBLEMS
1. Suppose the equation $ay'' + by' + cy = 4\, e^{3ix}$ has solution

$$y = \frac{4+2i}{i}\, e^{3ix}$$

(This is not a general solution, it's just one little solution.) (Doesn't matter where it came from.)

Find a solution to

(a) $ay'' + by' + cy = 7 \cos 3x$

(b) $ay'' + by' + cy = 7 \cos 3x + 8 \sin 3x$.

SECTION 1.5 HOMOGENEOUS LINEAR DE WITH CONSTANT COEFFICIENTS CONTINUED

solution to ay'' + by' + cy = 0 where a,b,c are constant

To solve the DE

$$ay'' + by' + cy = 0$$

where a,b,c are *constants* (this doesn't work if a,b,c have x's in them), first find the roots of the plain equation (sometimes called the characteristic equation)

$$am^2 + bm + c = 0$$

The solution to the DE depends on the type of roots so there are cases.

case 1 (real unequal roots) (leftover from Section 1.2)

If $m = m_1, m_2$ then a gen sol is $y = Ae^{m_1 x} + Be^{m_2 x}$

case 2 (non-real roots, which, if they occur, must occur in conjugate pairs)

If $m = p \pm qi$ then the gen *complex* sol is

$$y = Ae^{(p+qi)x} + Be^{(p-qi)x}$$

where A and B are arbitrary complex constants;

and the gen *real* sol is

$$y = e^{px}\,(A\cos qx + B\sin qx)$$

where A and B are arbitrary real constants.

For homework problems and exams it is always intended that you give *real* solutions unless specifically stated otherwise.

case 3 (repeated roots)

If $m = m_1, m_1$ then $y = Ae^{m_1 x} + B\,\underline{\underline{x}}\,e^{m_1 x}$ (step up by x).

example 1

If $m = 3 \pm 4i$ then the gen complex sol is

$$y = Ae^{(3+4i)x} + Be^{(3-4i)x}$$

and the gen real sol is

$$y = e^{3x}\,(P\cos 4x + Q\sin 4x)$$

If $m = \pm 5i$ (think of m as $0 \pm 5i$) then the gen real sol is

$$y = e^{0x}(A\cos 5x + B\sin 5x) = A\cos 5x + B\sin 5x$$

If $m = 2,2$ then the gen sol is $y = Ae^{2x} + B\underline{\underline{x}}e^{2x}$

proof of the rules in cases 2 and 3

Try $y = e^{mx}$ in the DE $ay'' + by' + cy = 0$ to see what m's, if any, make it work:

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \qquad \text{(substitute } e^{mx} \text{ for y in the DE)}$$

$$am^2 + bm + c = 0 \qquad \text{(cancel } e^{mx}\text{)}$$

So the sols m to the characteristic equ $am^2 + bm + c = 0$ determine solutions e^{mx} to the DE.

case 2

Suppose $m = 3 \pm 4i$. Then $y = e^{(3+4i)x}$ and $y = e^{(3-4i)x}$ are sols to the DE.

By the superposition rule for homog DE, $y = Ae^{(3+4i)x} + Be^{(3-4i)x}$ is a general (complex) solution, which proves part of case 2.
Now I'll try to get a gen *real* sol.

Consider the complex sol

$$e^{(3+4i)x} = e^{3x}(\cos 4x + i\sin 4x).$$

By the complex superposition rule for homog DE, its real part $e^{3x}\cos 4x$ and its imag part $e^{3x}\sin 4x$ are also homog sols. So a general (real) sol is

$$A\,e^{3x}\cos 4x + B\,e^{3x}\sin 4x,$$

i.e.,

$$e^{3x}(A\cos 4x + B\sin 4x).$$

This proves the rest of case 2.

Why didn't I consider the other complex sol $e^{(3-4i)x}$? It has real part $e^{3x}\cos 4x$ and imag part $-e^{3x}\sin 4x$. These are also real sols but they don't contribute anything new; they are already included in the gen sol $e^{3x}(A\cos 4x + B\sin 4x)$.

case 3

Suppose the characteristic equation has repeated roots $m = a,a$.
Then the characteristic equation was (or simplified to)

$$(m-a)(m-a) = 0$$

$$m^2 - 2am + a^2 = 0$$

and the DE was (or simplified to)

$$(*) \qquad y'' - 2ay' + a^2y = 0.$$

We know from case 1 that e^{ax} is a solution to the DE. What about xe^{ax} (stepped up).
Test it: If $y = xe^{ax}$ then

$$y' = ax\,e^{ax} + e^{ax}$$

$$y'' = a^2x\,e^{ax} + 2ae^{ax}$$

Substitute into the left side of the DE in (*) to get

$$a^2x\,e^{ax} + 2ae^{ax} - 2a(ax\,e^{ax} + e^{ax}) + a^2xe^{ax}$$

which is ZERO. Meaning that xe^{ax} is a sol to the DE in (*)..

And $Ae^{ax} + Bxe^{ax}$ is a general solution.

example 2

Find a general (real) solution to $y'' - 2y' + 5y = 0$

solution The characteristc equation is

$$m^2 - 2m + 5 = 0,$$

so

$$m = 1 \pm 2i$$

and

$$y_{gen} = e^x(A\cos 2x + B\sin 2x)$$

warning

If $m = 1 \pm 2i$ then a gen real homog sol is

$$y = e^x(A\cos 2x + B\sin 2x)$$ *without* an i.

Try not to confuse this with the ordinary algebraic fact that

$$e^{(1+2i)x} = e^x(\cos 2x + i\sin 2x)$$ *with* an i.

warning

If $m = 1 \pm 2i$ then the gen homo sol is

$$e^{\boxed{x}}(A\cos 2x + B\sin 2x) \quad \text{RIGHT}$$

not

$$e^{\boxed{1}}(A\cos 2x + B\sin 2x) \quad \text{WRONG}$$

example 3

If $y'' + 4y = 0$ then $m^2 + 4 = 0$, $m = \pm 2i$, and

$$y_{gen} = A\cos 2x + B\sin 2x$$

warning

The characteristic equ for $y'' + 4y = 0$ is $m^2 + \boxed{4} = 0$, *not* $m^2 + \boxed{4m} = 0$.

higher and lower order DE

If a 5th-order linear homog DE with constant coeffs has a characteristic equ with roots m = 2,2,2,2,5 then a gen sol is

$$y = Ae^{2x} + Bxe^{2x} + Cx^2e^{2x} + Dx^3e^{2x} + Ee^{5x}$$

(keep stepping up by x for repeated m's)

If a 6th-order linear homog DE with constant coeffs has an aux equ with roots $m = 2 \pm 3i,\ 2 \pm 3i,\ 0,\ 7$ then a gen sol is

$$y = e^{2x}(A \cos 3x + B \sin 3x) + xe^{2x}(C \cos 3x + D \sin 3x) + E + Fe^{7x}$$

If a first-order DE has m = 4 then a gen solution is $y = Ae^{4x}$

DE with initial conditions

A differential equation plus IC determine a unique solution. To find it, first find a general solution. Then plug in the IC to determine the constants.

(If the DE represents a spring system where f(x) is the input force at time x and y(x) is the position of the block at time x then the IC $y(0) = y_0$ describes the initial position of the block and the IC $y'(0) = y_1$ describes the initial velocity of the block.)

example 4

Solve $2y'' + 5y' - 3y = 0$ with IC $y(0) = 0$, $y'(0) = 1$.

solution $2m^2 + 5m - 3 = 0$, $(2m-1)(m+3) = 0$, $m = \frac{1}{2}, -3$ so a gen sol is

$$y = Ae^{x/2} + Be^{-3x}$$

Then $y' = \frac{1}{2}Ae^{x/2} - 3Be^{-3x}$

Plug in $y(0) = 0$: $0 = A + B$.
Plug in $y'(0) = 1$: $1 = \frac{1}{2}A - 3B$.
Solve for A and B: $A = \frac{2}{7}$, $B = -\frac{2}{7}$.
Final sol is

$$y = \frac{2}{7}e^{x/2} - \frac{2}{7}e^{-3x}$$

example 5

Solve $y'' - 6y' + 25y = 0$ with IC $y(0) = 1$, $y'(0) = 11$.

solution $m^2 - 6m + 25 = 0$, $m = 3 \pm 4i$ so the gen real sol is

(1) $$y = e^{3x}(C \cos 4x + D \sin 4x)$$ (where C and D are arbitrary real constants)

The IC $y(0) = 1$ makes $C = 1$. We have

$$y' = e^{3x}(-4 \sin 4x + 4D \cos 4x) + 3e^{3x}(\cos 4x + D \sin 4x)$$

so to get $y'(0) = 11$ you need $11 = 4D + 3$, $D = 2$. Final answer is

$$y = e^{3x}(\cos 4x + 2 \sin 4x)$$

It's also possible to use the general complex sol

(2) $\quad y = Ae^{(3+4i)x} + Be^{(3-4i)x}$ (where A and B are arbitrary complex constants)

and plug the IC in there. To get IC $y(0)=1$ you need

$$A + B = 1.$$

We have

$$y' = (3+4i)Ae^{(3+4i)x} + (3-4i)Be^{(3-4i)x}$$

so to get $y'(0) = 11$ you need

$$(3+4i)A + (3-4i)B = 11.$$

Solve the two equations in A and B to get $A = \frac{1}{2} - i$, $B = \frac{1}{2} + i$. Final sol is

$$y = (\tfrac{1}{2} - i)e^{(3+4i)x} + (\tfrac{1}{2} + i)e^{(3-4i)x}$$

$$= (\tfrac{1}{2} - i)e^{3x}(\cos 4x + i \sin 4x) + (\tfrac{1}{2} + i)e^{3x}(\cos[-4x] + i \sin[-4x]$$

$$= e^{3x}(\cos 4x + 2 \sin 4x)$$ as before (the i's cancel out)

in case you're still wondering where (1) came from

The general complex solution in (2) has an i in it.
The general real solution in (1) doesn't have an i in it.
Where did (1) come from and where did the i go?

When m comes out to be $3 \pm 4i$ you get the two specific *complex* solutions

$$e^{(3+4i)x} \quad \text{and} \quad e^{(3-4ix)}$$

You can use them immediately to build the *general complex* solution in (2).
You can also use them to get *real* solutions, like this. First,

(3) $\quad e^{(3+4i)x}$ (*with* an i) $= e^{3x}(\cos 4x + i \sin 4x)$ (*with* an i)

(4) $\quad e^{(3-4i)x}$ (*with* an i) $= e^{3x}(\cos 4x - i \sin 4x)$ (*with* an i)

Second, by the complex superposition principle for homogeneous DE in §1.4, if (3) produces 0 when it's subsituted into $y'' - 6y' + 25y$ then its real part

(5) $\quad e^{3x} \cos 4x$ (*without* an i)

and its imag part

(6) $\quad e^{3x} \sin 4x$ (*without* an i)

both produce 0 also.

(Similarly, the real part of (4) and the imag part of (4) also produce 0 but they aren't different enough from (5) and (6) to be of any use.)

Then by the ordinary superposition principle for homog DE, since (5) and (6) are homog solutions, the general real solution is

$$y = Ce^{3x} \cos 4x + De^{3x} \sin 4x$$ (*without* an i),

where C and D are arbitrary real constants. That's where (1) comes from.

The general complex solution in (2) and the general real solution in (1) are *not* equal. The gen real sol produces only solutions with no i's in them and the gen complex sol can produce solutions with and without i's. The general complex solution includes all the real solutions as a special case. In particular, if A and B in (2) are conjugates, then the i's cancel out and (2) will be real. And when you plug (real) IC into (2), the i's cancel out and you get a real solution.

PROBLEMS FOR SECTION 1.5

1. Write the general real solution given the following roots of the characteristic equation
(a) $m = -3 \pm 5i$ (b) $m = \pm 2i$ (c) $m = 3 \pm 4i$

2. Find a general real sol.
(a) $y'' + \pi^2 y = 0$ (b) $y'' - \pi^2 y = 0$ (c) $y'' + 2y' + 4y = 0$

3. Find a gen real sol (a) $y'' + 4y' + 5y = 0$ (b) $y'' + 4y = 0$

4. If $y(x)$ is the position at time x of a particle on a number line and k is a positive constant then $y'' = -k^2 y$ describes *simple harmonic motion* (where the force on the particle is proportional to its position and is directed toward the origin). Find the gen solution.

5. Solve $y''' + y'' + 4y' + 4y = 0$ with IC $y(0) = 0$, $y'(0) = -1$, $y''(0) = 5$
(The m's are -1, $\pm 2i$.)

6. Write the general real sol given the following roots of the characteristic equ
(a) $-3, -3, -3, \pm 5, \pm 4i, -2 \pm 3i, -2 \pm 3i, -2 \pm 3i, 0$
(b) 0,0,0,3
(c) $2 \pm \sqrt{5}$, $\pm i$, $\pm i$

(d) $2 \pm i\sqrt{5}$, 0, 3

(e) $\pm i$, $\pm 2i$, 1

7. Solve $y''' + 3y'' = 0$ with conditions $y(0) = 0$, $y'(0) = 2$, $y'(\infty) = 1$

8. Go backwards and invent a DE with the given gen sol if possible
(a) $Ae^{2x} + Be^{3x}$ (b) $A + Bx + Ce^{x}$ (c) $Ae^{2x} + Bxe^{2x}$
(d) $e^{2x}(A \cos 3x + B \sin 3x)$

9. Find a gen sol to $y' + 2y = 0$.

10. Look at the DE

$$ay'' + by' + cy = 0 \text{ with IC } y(0) = y_0,\ y'(0) = y_1$$

where a,b and c are *positive* (so that the DE corresponds to a real life system with damping). Show that the solution is transient (i.e., $y(x) \to 0$ as $x \to \infty$) (a) with a physical argument
(b) with a mathematical argument
For the mathematical version, there are 3 possibilities for the solution depending on whether the m's are non-real, real and unequal, real and equal. Just do it for the case that the m's are non-real.

11. Consider $ay'' + by' + cy = 0$ with IC $y(0) = 0$, $y'(0) = 0$.
(a) Predict the solution easily using a physical argument in the case that a,b,c are not negative so that the DE corresponds to a real life system.
(b) Solve it with the methods of this section (whether or not a,b,c are negative). There are 3 cases, m's are non-real, real and unequal, real and equal. Just do it for the case that the m's are real and unequal.

Honors

12. Look at #12(b) in §1.3 where the problem is to show that if a,b,c are positive then the solution to $ay'' + by' + cy = 0$ with some IC is transient.
The problem said to just do it in the case that the m's are non-real.

Now do it in the case that the two m's are the same (repeated root case).
And point out where you used (if at all) the hypothesis that a,b,c, are positive.

13. Suppose a spring has fixed mass $M > 0$, and fixed spring constant $k > 0$.
It will be immersed in a medium with damping constant $c \geq 0$.
You will get to choose c.
The spring is initially displaced and/or moving at time $t = 0$ (i.e., there are some IC).
No other force is applied to the spring.

Then the displacement $y(t)$ at time t satisfies the DE

$$My''(t) + cy'(t) + ky(t) = 0 \quad \text{plus IC.}$$

(a) For what c's will the solution be overdamped (Fig A).
(b) For what c's will the solution be damped (Fig B).
(c) For what c's will the solution be undamped (Fig C).

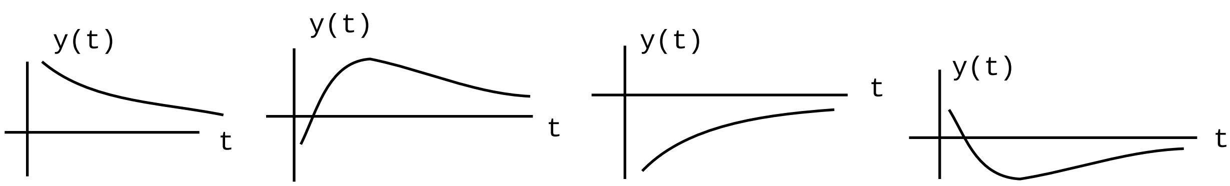


FIG A overdamped responses

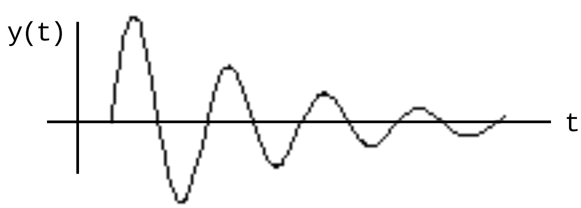


Fig B damped response

FIG C undamped response

14. (a) Let $y(x) = \dfrac{v(x)}{x}$. Use the quotient rule to find $y'(x)$ and $y''(x)$ (in terms of $v(x)$ and its derivatives).

For example, if $y = x^3\, v(x)$ then

$$y' = x^3\, v'(x) + 3x^2 v(x) \text{ by the product rule}$$

and

$$y'' = x^3\, v''(x) + 3x^2\, v'(x) + 3x^2\, v'(x) + 6x\, v(x).$$

You do the same thing with $y = \frac{v(x)}{x}$ using the quotient rule (or any rule you like as long as it comes out right).

(b) The second order differential equation

$$xy'' + 2y' + 9xy = 0$$

has *variable* coefficients so Chapter 1 doesn't apply. But get a general solution anyway by making the substitution $y = \frac{v(x)}{x}$ to get a new equ in $v(x)$ instead of $y(x)$.

Then solve the new differential equation for $v(x)$, substitute back, and get the general solution to the original differential equation.

15. Go back to #11 and do part (b) in the remaining cases
case 2 The m's are real and equal
case 3 The m's are non-real

16. In Section 1.3, one of the problems said to show that if A and B are conjugates then $Ae^{i\theta} + Be^{i\theta}$ is real.
What does this have to do with general complex solutions versus general real solutions.
solution
It shows that the general complex solution does include *some* real solutions. It remains to be shown that it includes *all* real solutions.
For that you need the following:
Given any C and D, it is possible to find A and B such that
$Ae^{i\theta} + Be^{i\theta} = C \cos\theta + D \sin\theta$.

SECTION 1.5A TRIG REVIEW

Think of x as time.

graph of 5 cos x (Fig 1)

Period = 2π seconds per cycle

Frequency = $\frac{1}{2\pi}$ cycles per sec

Angular frequency = 1 cycle per 2π seconds

Amplitude = 5

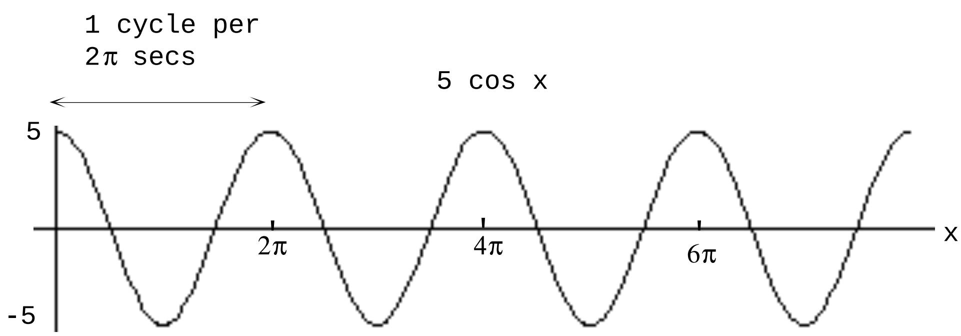


FIG 1

graph of 5 cos 3x (Fig 2)

period = $\frac{2\pi}{3}$ seconds per cycle

Frequency = $\frac{3}{2\pi}$ cycles per sec

Angular frequency = 3 cycles per 2π seconds

Amplitude = 5

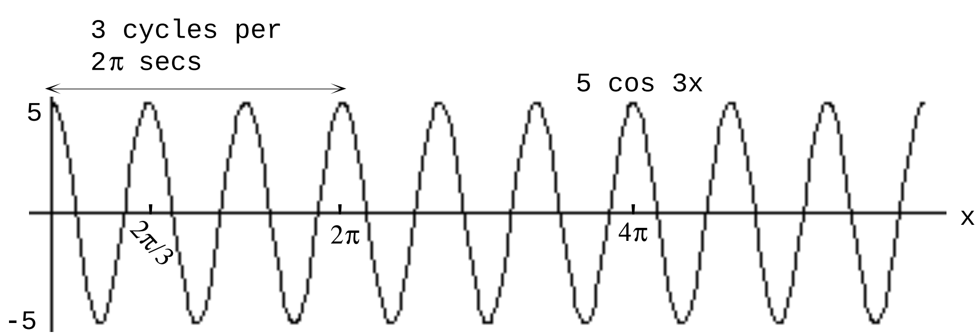


FIG 2

graph of 5 cos(3x – π/4) (Fig 3)

Period = $\frac{2\pi}{3}$ seconds per cycle

Angular frequency = 3 cycles per 2π seconds

Phase angle = $\pi/4$

To get the graph, shift (translate) the graph of 5 cos 3x to the right by $\frac{\pi/4}{3}$, i.e. by $\frac{\pi}{12}$

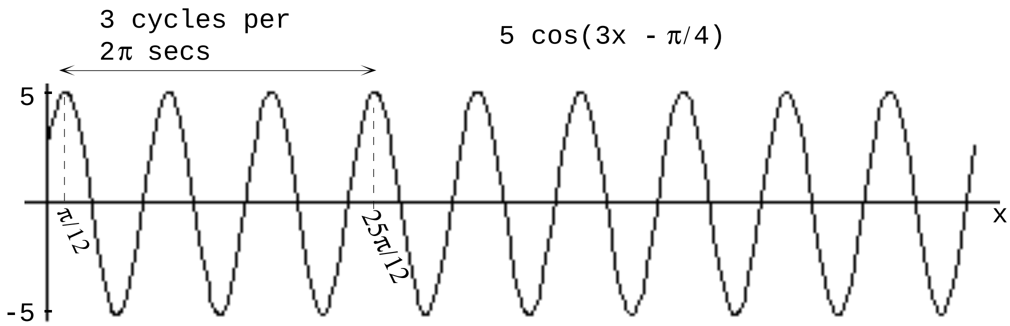


FIG 3

question
In general, to get the graph of $y = f(x - a)$, translate the graph of $y = f(x)$ to the right by a.
So to get the graph of $y = \cos(3x - \pi/4)$ why don't we shift cos 3x to the right by $\pi/4$
answer
To get the amount of translation you have to rewrite $\cos(3x - \pi/4)$ as $\cos[\,3(x - \pi/12)\,]$. Then you have $x - \pi/12$ sitting where x used to be and the amount of translation is $\pi/12$.
Or just plot some points to see.

combining a sine and cosine with the same frequency
By a trig identity,

$$r\cos(bx - \theta) = r(\cos bx \cos\theta + \sin bx \sin\theta)$$

$$(1) \qquad = \underbrace{r\cos\theta}_{\text{call this A}} \cos bx + \underbrace{r\sin\theta}_{\text{call this B}} \sin bx$$

Now read (1) from right to left and note that A,B and r,θ are related the way rectangular and polar coords are related.

(2)

$$A\cos bx + B\sin bx = r\cos(bx - \theta)$$

where

$$r = \sqrt{A^2 + B^2}$$
$$\theta = \arctan[A,B] \quad \text{(the angle in Fig 4)}$$

In particular, if a cosine and sine have the same frequency and have respective amplitudes A and B then their sum is another harmonic oscillation with the same frequency, with amplitude $\sqrt{A^2+B^2}$ and with phase angle arctan[A, B].

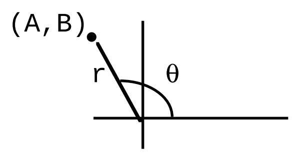


FIG 4

For example,

$$e^{3x}(C\cos 4x + D\sin 4x)$$

with arbitrary constants C and D can be written as

$$re^{3x}\cos(4x - \theta)$$

with arbitrary constants r and θ.

For example,

$$e^{3x}(\cos 4x + 2\sin 4x)$$

can be written as

$$\sqrt{5}\, e^{3x}\cos(4x - \theta_o)$$

where θ_o is the specific angle in Fig 5, namely arctan[1, 2].

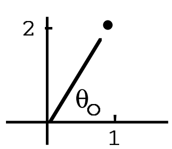


FIG 5

warning

In Fig 5, because the point (1,2) is in quadrant I, θ_o can be called arctan $\frac{2}{1}$ as well as actan[1,2].

But in general, the phase angle θ in Fig 4 should be called arctan[A,B] not arctan B/A. The two are *not* necessarily the same (see Section 1.3) and only arctan[A,B] is always right.

SECTION 1.6 NON-HOMOGENEOUS LINEAR DE WITH CONSTANT COEFFICIENTS

the general non-homog solution

Consider the non-homog equation

$$(*) \qquad ay'' + by' + cy = f(x)$$

Let y_h be the *general homog solution*, i.e., the solution to $ay'' + by' + c = 0$ which you learned to find in the last section.

Let y_p be a *particular solution* to the given nonhomog solution in (*), i.e., a solution to (*) containing no arbitrary constants (this section and the next will show you how to find one).

A general solution to the given non-homog DE is $y_h + y_p$.

This general solution contains two arbitrary constants (in the y_h part) which you'll need in order to satisfy the initial conditions.

proof

y_h is a solution of $ay'' + by' + cy = 0$.

y_p is a solution of $ay'' + by' + cy = f(x)$.

By superposition,

$y_h + y_p$ is a solution of $ay'' + by' + cy = 0 + f(x) = f(x)$.

So $y_h + y_p$ is a solution of the given DE $ay'' + by' + cy = f(x)$. And it is a general solution because it contains two arbitrary constants (in the y_h term).

finding a particular solution

I want to find a particular solution, to be called y_p, for

$$ay'' + by' + cy = f(x).$$

There are several cases.

(1) f is constant.
Suppose $f(x) = 6$ for all x. Try $y_p = A$.
Substitute the trial y_p into the DE to determine A.

(2) f is a polynomial.
Suppose $f(x) = 7x^3 + 2x$ (a cubic). Try $y_p = Ax^3 + Bx^2 + Cx + D$ (a cubic not missing any terms even though f(x) is missing a few).

Substitute the trial y_p into the DE to determine A,B,C,D.

Similarly if $f(x) = 3x^2 + 4x + 1$ (quadratic) then try $y_p = Ax^2 + Bx + C$

And so on for any poly f.

(3) f is exponential.
If $f(x) = 9e^{3x}$ try $y_p = Ae^{3x}$.
Substitute the trial y_p into the DE to determine A.

(4) f is sine or cosine.
Suppose $f(x) = 2\sin 5x$ or $f(x) = 2\cos 5x$.

method 1 Try $y_p = A\cos 5x + B\sin 5x$.
Substitute the trial y_p into the DE to determine A and B.

method 2 Switch to the new problem

$$ay'' + by' + cy = 2e^{5ix}$$

Try $y_p = Ae^{5ix}$.
Substitute into the switched DE to determine A.

Then to find y_p for the original DE:

> Take the *real* part of the switched y_p if f(x) is a *cosine.*
> Take the *imag* part if f(x) is a *sine.*

This works because of complex superposition: When you get a y_p for

$$ay'' + by' + cy = 2e^{5ix}$$

you are really getting a particular sol to

$$ay'' + by' + cy = 2\cos 5x + 2i\sin 5x.$$

So
the real part of your y_p is a particular sol to $ay'' + by' + cy = 2\cos 5x$,
the imag part if your y_p is a particular sol to $ay'' + by' + cy = 2\sin 5x$.

Exams will probably insist that you use the complex exponential method. But it's also important to know that the real y_p has the form $A\cos 5x + B\sin 5x$.

example 1

Find a general solution to $y''' + 4y' = 3e^{2x}$.

solution First solve $y''' + 4y' = 0$ to get y_h:

$$m^3 + 4m = 0, \quad m(m^2 + 4) = 0, \quad m = 0, \pm 2i$$

so

$$y_h = A + B\cos 2x + C\sin 2x$$

Then try $y_p = De^{2x}$. We have

$$y_p' = 2De^{2x}, \qquad y_p'' = 4De^{2x}, \qquad y_p''' = 8De^{2x}$$

Substitute into the DE to determine D. You need

$$8De^{2x} + 4\cdot 2De^{2x} = 3e^{2x}$$

$$16De^{2x} = 3e^{2x}$$

To make this true for all x, equate coeffs of e^{2x}: $16D = 3$, $D = \frac{3}{16}$.

So $y_p = \frac{3}{16}e^{2x}$.

Finally, $y_{gen} = y_h + y_p = A + B\cos 2x + C\sin 2x + \frac{3}{16}e^{2x}$.

example 2

Find a general solution to $y'' + 2y' + 5y = 5x^2 + 2$.

solution First find y_h.

$$m^2 + 2m + 5 = 0, \quad m = -1 \pm 2i, \quad y_h = e^{-x}(A \cos 2x + B \sin 2x)$$

Then try

$$y_p = Cx^2 + Dx + E.$$

We have $y_p' = 2Cx + D$ and $y_p'' = 2C$. Substitute into the DE:

$$2C + 2(2Cx + D) + 5(Cx^2 + Dx + E) = 5x^2 + 2$$

$$(**) \qquad 5Cx^2 + (4C + 5D)x + 2C + 2D + 5E = 5x^2 + 2$$

To make this true for all x, equate corresponding coefficients.

Equate x^2 coeffs: $5C = 5$
Equate x coeffs: $4C + 5D = 0$
Equate constant terms: $2C + 2D + 5E = 2$

So $C = 1$, $D = -\frac{4}{5}$, $E = \frac{8}{25}$ and $y_p = x^2 - \frac{4}{5}x + \frac{8}{25}$.

Finally,

$$y_{gen} = y_h + y_p = e^{-x}(A \cos 2x + B \sin 2x) + x^2 - \frac{4}{5}x + \frac{8}{25}$$

warning

If the forcing function is say $5x^2$, don't try plain Ax^2 as the trial y_p; instead try $y_p = Ax^2 + Bx + C$. In general, even if a polynomial forcing function is missing some lower degree terms, you should *not* leave out any terms in your trial y_p.

example 3

Find a gen sol to $y'' + 4y' + 2y = 3 \sin 2x$.

solution $m^2 + 4m + 2 = 0, \quad m = \dfrac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$

$$y_h = Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x}$$

To get y_p, switch to

$$y'' + 4y' + 2y = 3e^{2ix}.$$

and try

$$\text{switched } y_p = Ce^{2ix}.$$

Then

$$y_p' = 2i\,Ce^{2ix}, \qquad y_p'' = 4i^2\,Ce^{2ix} = -4Ce^{2ix}.$$

Substitute into the DE to determine C:

$$-4Ce^{2ix} + 8i\,Ce^{2ix} + 2Ce^{2ix} = 3e^{2ix}$$

$$C(-2 + 8i)e^{2ix} = 3e^{2ix}$$

So you need

$$C(-2 + 8i) = 3$$

$$C = \frac{3}{-2+8i} = \frac{3}{-2+8i} \cdot \frac{-2 - 8i}{-2 - 8i} = \frac{-3 - 12i}{34}$$

So

(5) switched $y_p = \frac{-3 - 12i}{34} e^{2ix} = \frac{-3 - 12i}{34} (\cos 2x + i \sin 2x)$

To get y_p for the original SINE forcing function (which is the imag part of the switched forcing function $3e^{2ix}$) take the IMAG part of (5):

$$y_p = -\frac{12}{34} \cos 2x - \frac{3}{34} \sin 2x$$

> **warning**
> y_p is *not* $-\frac{12}{34} \boxed{i} \cos 2x - \frac{3}{34} \boxed{i} \sin 2x$. The imag part of (5) is what is sitting *next* to the i and *does not include the i itself.*

Then

$$y_{gen} = \underbrace{Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x}}_{\text{transient}} \underbrace{- \frac{12}{34} \cos 2x - \frac{3}{34} \sin 2x}_{\text{steady state solution}}$$

steady state solution

If transient terms (terms which $\to 0$ as $x \to \infty$) are ignored and the remaining solution is periodic then that remaining solution is called the *steady state solution* (it's the periodic behavior you see "eventually").

The DE in example 3 has steady state solution $-\frac{12}{34} \cos 2x - \frac{3}{34} \sin 2x$.
By (2) in Section 1.5A, the steady state solution can be written as

$$\sqrt{\left(-\frac{12}{34}\right)^2 + \left(-\frac{3}{34}\right)^2} \cos (2x - \theta),$$

i.e., as

$$\frac{1}{34} \sqrt{153} \cos(2x - \theta)$$

where $\theta = \arctan[\, -\frac{12}{34}, -\frac{3}{34}\,] = \arctan[-4, -1]$.

The steady state response is harmonic with amplitude $\frac{1}{34} \sqrt{153}$ and angular frequency 2 cycles per 2π seconds (compared with the input which was harmonic with amplitude 3 and the same angular frequency).

warning

The switched equation $y'' + 4y' + 2y = 3e^{2ix}$ is not the *same* as the old equation $y'' + 4y' + 2y = 3e^{2ix}. = 3 \sin 2x$. But they are related because taking the imag part of a solution to the switched equation produces a solution to the old equation.

example 3 continued

Find a general solution to $y'' + 4y' + 2y = 3\cos 2x$.

solution This time, since the forcing function is a *cosine*, take the *real* part of (5) to get y_p. Final answer is

$$y_{gen} = Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x} - \frac{3}{34}\cos 2x + \frac{12}{34}\sin 2x$$

another method for example 3

Here's a way to get y_p for $y'' + 4y' + 2y = 3\sin 2x$ *without* using the complex exponential (*but exams will probably insist that you use the complex exponential method).*

Try

$$y_p = C\cos 2x + D\sin 2x$$

Then

$$y_p' = -2C\sin 2x + 2D\cos 2x$$

$$y_p'' = -4C\cos 2x - 4D\sin 2x$$

Substitute into the DE and choose C and D to make it work. You need

$$-4C\cos 2x - 4D\sin 2x + 4(-2C\sin 2x + 2D\cos 2x) + 2(C\cos 2x + D\sin 2x) = 3\sin 2x$$

Equate the coeffs of $\sin 2x$ on each side: $\quad -4D - 8C + 2D = 3$

Equate the coeffs of $\cos 2x$ on each side: $\quad -4C + 8D + 2C = 0$

Solve to get $\quad C = -\frac{12}{34}, \quad D = -\frac{3}{34}$

$$y_p = -\frac{12}{34}\cos 2x - \frac{3}{34}\sin 2x$$

as before

solving non-homog DE with initial conditions

A 2nd order DE has infinitely many solutions; in fact it has a general solution with two arbitrary constants.

A DE with IC has exactly one solution. To find it, *first* find the general solution $y_h + y_p$ and *then* plug in the IC to determine the constants in the gen sol.

example 4

Solve $y'' + 4y = 3x^2$ with IC $y(0) = 0$, $y'(0) = 0$.

solution First get y_h. We have $m^2 + 4 = 0$, $\quad m = \pm 2i$,

$$y_h = P\cos 2x + Q\sin 2x$$

Try

$$y_p = Ax^2 + Bx + C.$$

Then

$$y_p' = 2Ax + B, \qquad y_p'' = 2A.$$

Substitute into the DE :

$$4Ax^2 + 4Bx + 2A + 4C = 3x^2$$

Equate corresponding coeffs:

$$4A = 3, \qquad 4B = 0, \qquad 2A + 4C = 0.$$

So

$$A = \frac{3}{4}, \quad B = 0, \quad C = -\frac{3}{8},$$

$$y_p = \frac{3}{4}x^2 - \frac{3}{8}$$

and a general sol is

$$y = P\cos 2x + Q\sin 2x + \frac{3}{4}x^2 - \frac{3}{8}.$$

To get the IC $y(0) = 0$ you need

$$0 = P - \frac{3}{8}, \quad P = \frac{3}{8}.$$

To get the IC $y'(0) = 0$, first find

$$y' = -2P\sin 2x + 2Q\cos 2x + \frac{3}{2}x$$

and then plug in $x = 0$, $y' = 0$:

$$0 = 2Q, \quad Q = 0.$$

The final answer is

$$y = \frac{3}{8}\cos 2x + \frac{3}{4}x^2 - \frac{3}{8}$$

warning Once the constants P and Q are determined, the solution is no longer called *general*. It's now the particular solution satisfying the IC.

warning

(1) When you solve a non-homog DE with IC, determine the various constants at the appropriate stage.

First find y_h (containing arbitrary constants).

Then find y_p (the *trial* y_p contains constants but they must be immediately determined to get the *genuine* y_p).

Then $y_{gen} = y_h + y_p$ (contains arbitrary constants in the y_h part).

Finally, use the IC to determine the constants in y_{gen}.

Don't use the IC on y_h alone at the beginning of the problem.

(2) If you follow the correct procedures to determine the constants, they should come out to be just that, namely *constants*. You will look silly if you conclude that $B = 3x$ or $B = 6x^2$ if B is supposed to be a *constant* (i.e., no x's in it).

forcing functions which change formulas

Look at

$$y'' + y = f(x) \text{ with IC } y(0) = 1,\ y'(0) = 0$$

where

$$f(x) = \begin{cases} 5e^{2x} & \text{if } 0 \le x \le \pi/2 \\ 2 & \text{if } x \ge \pi/2 \end{cases} \quad \text{(FIG 1)}$$

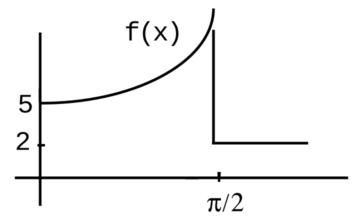


FIG 1

I'll solve and furthermore make the solution and its derivative continuous.
First find a gen sol to $y'' + y = 5e^{2x}$. We have

$$m^2 + 1 = 0,\ m = \pm i, \quad y_h = A\cos x + B\sin x.$$

Try $y_p = Pe^{2x}$. Substitute into the DE to get

$$4Pe^{2x} + Pe^{2x} = 5e^{2x}.$$

$$5Pe^{2x} = 5e^{2x}$$
$$5P = 5, \quad P = 1.$$

So

$$y_{gen} = A\cos x + B\sin x + e^{2x} \quad \text{for } 0 \le x \le \pi/2$$

Then find a gen sol to $y'' + y = 2$.

Still have $y_h = C\cos x + D\sin x$ as above *but using different constants.*

Try $y_p = Q$. Substitute into the DE to get

$$0 + Q = 2, \quad Q = 2.$$

So

$$y_{gen} = C\cos x + D\sin x + 2 \quad \text{for } x \ge \pi/2$$

In other words,

$$y = \begin{cases} A\cos x + B\sin x + e^{2x} & \text{for } 0 \le x \le \pi/2 \\ C\cos x + D\sin x + 2 & \text{for } x \ge \pi/2 \end{cases}$$

warning
If the constants are A,B in the $x \le \pi/2$ part, use *different* constants C,D in the $x \ge \pi/2$ part

Plug in the IC using the *first* piece of y_{gen}, the part which holds when $x = 0$.
To get $y(0) = 1$ you need $A + 1 = 1$, $A = 0$.
We have $y' = B\cos x + 2e^{2x}$ so to get $y'(0) = 0$ you need $B + 2 = 0$, $B = -2$. So

$$y = \begin{cases} -2\sin x + e^{2x} & \text{for } 0 \le x \le \pi/2 \\ C\cos x + D\sin x + 2 & \text{for } x \ge \pi/2 \end{cases}$$

You still have constants available for the choosing and you should take the opportunity to make the solution continuous. To do this, make the two pieces agree at $x = \frac{1}{2}\pi$. When $x = \frac{1}{2}\pi$, the first piece is $-2 + e^{\pi}$ and the second piece is $D + 2$. So make

$$-2 + e^{\pi} = D + 2$$
$$D = -4 + e^{\pi}$$

Choose the remaining constant to make y' continuous. So far,

$$y' = \begin{cases} -2 \cos x + 2e^{2x} & \text{if } x \le \pi/2 \\ -C \sin x + D \cos x & \text{if } x \ge \pi/2 \end{cases}$$

Make the two pieces agree at $\pi/2$. To do this you need

$$2e^{\pi} = -C$$
$$C = -2e^{\pi}.$$

The final sol is

$$(6) \qquad y = \begin{cases} -2 \sin x + e^{2x} & \text{if } x \le \pi/2 \\ -2e^{\pi} \cos x + (-4 + e^{\pi}) \sin x + 2 & \text{if } x \ge \pi/2 \end{cases}$$

warning

The IC get plugged into that part of the sol which holds when x = 0 so in this example they determine A and B only. They have nothing to do with C and D which are determined by continuity requirements.

In general, for a second order differential equation you'll have enough arbitrary constants available to make y and y' continuous. For a first-order DE you'll only have enough constants to make y continuous. For a third-order DE you'll have enough constants to make y, y' and y" continuous.

example 6 continued

Continue from (6) and find a neat description of the steady state response.

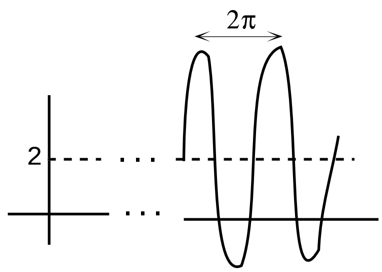
solution The steady state response is the periodic function

$$2e^{\pi} \cos x + (-4 + e^{\pi})\sin x + 2.$$

By (1) from Section 1.5A it can be rewritten as

$$\sqrt{(-2e^{\pi})^2 + (-4 + e^{\pi})^2}\ \cos(x - \theta_0) + 2$$

where $\theta_o = \arctan[2e^{\pi}, -4+e^{\pi}]$. So the steady state solution (Fig 2) is harmonic oscillation (above and below 2) with period 2π, angular frequency 1 (cycle per 2π sec) and amplitude $\sqrt{4e^{2\pi} + (-4 + e^{\pi})^2}$.



steady state response

FIG 2

warning

The steady state solution in (6) has nothing to do with $-2 \sin x + e^{2x}$ since that part of the solution holds only when $x \le \pi/2$; the steady state solution is supposed to be the periodic response (if any) that holds for *large* x.

superposition rule for initial conditions

Suppose

$$ay'' + by' + cy = f(x) \text{ with IC } y(0) = 5,\ y'(0) = 6 \text{ has solution } y_1(x)$$

and

$$ay'' + by' + cy = g(x) \text{ with IC } y(0) = 7,\ y'(0) = 8 \text{ has solution } y_2(x)$$

Then $y_1 + y_2$ is a solution to

$$ay'' + by' + cy = f(x) + g(x)$$ [this much holds by plain superposition from §1.1]

with

IC $y(0) = 5+7 = 12$, $y'(0) = 6+8 = 14$ [this is the new idea]

In particular, suppose $y_1(x)$ is the solution to

$$ay'' + by' + cy = f(x) \text{ with IC } y(0) = 0\ ,\ y'(0) = 0$$

and $y_2(x)$ is the solution to

$$ay'' + by' + cy = 0 \text{ with IC } y(0) = 5\ ,\ y'(0) = 6$$

Then $y_1 + y_1$ is a solution to

$$ay'' + by' + cy = f(x) \text{ with IC } y(0) = 5,\ y'(0) = 6$$

proof

If $y_1(0) = 5$ and $y_2(0) = 7$ then $(y_1 + y_2)(0) = 5 + 7 = 12$

If $y_1'(0) = 6$ and $y_2'(0) = 8$ then $(y_1 + y_1)'(0) = (y_1' + y_2')(0) = 6 + 8 = 14$

mathematical catechism (you should know the answers to these questions)

Question What does it mean to say that a solution $y(x)$ is transient.
Answer It means that $y(x) \to 0$ as $x \to \infty$.

Question What does it mean to say that the steady state solution is $q(x)$.
Answer It means that $q(x)$ is periodic, and the solution to the differential equation was $q(x)$ + transient terms.

PROBLEMS FOR SECTION 1.6

1. Find a general real sol (a) $y'' + 3y' - 5y = 4e^{2x}$ (b) $y' + 2y = e^{3x}$

2. Find a general real sol (a) $y'' + 9y = -162x^2$ (b) $y'' - 4y = 2$

3. Solve (a) $y'' + 3y' + 2y = 2 - 4x$ with IC $y(0) = 0$, $y'(0) = 0$
(b) $y'' + y = 1$ with IC $y(0) = 0$, $y'(0) = 2$

4. Find a gen real sol.
(a) $y'' + y' + y = 73 \sin 3x$
(b) $y'' + y' + y = 73 \cos 3x$
(c) $y'' + y' + y = 5 \sin 3x + 4 \cos 3x$

5. Solve and then identify the steady state sol if there is one.

$y'' + 4y' + 5y = 8 \sin x$ with IC $y(0) = 0$, $y'(0) = 0$

6. Suppose the gen sol to a DE is $y = A \cos 2x + B \sin 2x + x^2 - 5$

(a) Choose IC so that the final sol is $y = \cos 2x + 6 \sin 2x + x^2 - 5$
(b) Find the DE

7. For $y'' - 3y' + 2y = 2x$ you should try $y_p = Ax + B$. What happens if you ignore all my warnings and try $y_p = Ax$

8. Suppose $(5 + 3i)e^{2ix}$ is a particular solution to

$$ay'' + by' + cy = e^{2ix} \qquad \text{(a,b,c are real constants)}$$

Find a particular solution to $ay'' + by' + cy = 5 \cos 2x + 7 \sin 2x$

9. Solve and get as much continuity as possible. And find the steady state solution if there is one.

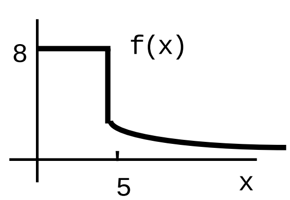
(a) $y'' + y = \begin{cases} x & \text{if } 0 \le x \le \pi \\ \pi e^{\pi - x} & \text{if } x \ge \pi \end{cases}$ with IC $y(0) = 0$, $y'(0) = 1$

(b) $y'' - 2y' - 3y = f(x)$ with IC $y(0) = 8$, $y'(0) = 0$

where $f(x) = \begin{cases} -12 & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x \ge 2 \end{cases}$

(c) $y' + 2y = f(x)$ with IC $y(0) = 0$ where $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \ge 1 \end{cases}$

(d) $y' + 4y = f(x)$ with IC $y(0) = 1$ where $f(x) = \begin{cases} 8 & \text{if } 0 \le x \le 5 \\ 6e^{-2x} & \text{if } x \ge 5 \end{cases}$



10. If y_1 solves

$$3y'' + 2y' + y = 0 \text{ with IC } y(2) = 5,\ y'(2) = 7$$

and y_2 solves

$$3y'' + 2y' + y = \cos x \text{ with IC } y(2) = 3,\ y'(2) = 4$$

what does $y_1 + y_2$ solve.

11. Your friend solved

$$y'' + 3y' - 4y = f(x) \text{ with IC } y(0) = 2,\ y'(0) = 1$$

and got solution $y_1(x)$.

Your class was assigned the *same* DE but with IC $y(0) = 1$, $y'(0) = 4$.

Take advantage of her y_1 to find your solution (i.e., solve your problem in terms of y_1).

honors

12.(a) Let z be an arbitrary complex number and let $\overline{z}$ stand for the conjugate of z. Show that the real part of z is $\frac{1}{2}(z + \overline{z})$ and the imag part of z is $-\frac{1}{2}\, i(z - \overline{z})$.

(b) Find the general solution to $y'' + 4y = -6 \cos 3x$.

(c) (very interesting) Here is Mathematica (version 2) doing part (b):

```
In[44]:=
DSolve[y''[x] + 4y[x] == -6 Cos[3x],y[x],x]

Out[44]=
                                                 3 (Cos[3 x] - I Sin[3 x])
{{y[x] -> C[2] Cos[2 x] - C[1] Sin[2 x] + ------------------------- +
                                                          5

     3 (Cos[3 x] + I Sin[3 x])
     -------------------------}}
                 5
```

The answer looks funny. Use part (a) to explain what the Mathematica routine seems to be doing.

13. In example 2, y_p came out to be $x^2 - \frac{4}{5}x + \frac{8}{25}$ and

$$y_{gen} = e^{-x}(A \cos 2x + B \sin 2x) + x^2 - \frac{4}{5}x + \frac{8}{25}$$

Find (by inspection), a dozen other particular solutions.

14. (follow up to #13) In HW #4, I said that another particular solution in example 2 in Section 1.4 is $8e^{-x} \cos 2x + x^2 - \frac{4}{5}x + \frac{8}{25}$.
Suppose you found the general solution in this example by using y_h and this new particular solution.
How does it compare with the general solution you get using y_h and the "first born" particular solution $x^2 - \frac{4}{5}x + \frac{8}{25}$.
Do you get a valid general solution using the new particular solution.
Explain a little.

SECTION 1.7 NON-HOMOGENEOUS DE CONTINUED (STEPPING UP)

stepping up y_p

Suppose you want a particular solution for $ay'' + by' + cy = f(x)$. There are exceptions to the rules in (1)–(4) of the preceding section.

(1') Suppose $f(x) = 6$. Ordinarily you try $y_p = A$.

But if A is already a homog sol (i.e., one of the m's is 0) then $y_p = A$ can't be made to satisfy the equation $ay'' + by' + cy = 6$ since it already satisfies the equation $ay'' + by' + cy = 0$ no matter what A is. Try $y_p = Ax$ instead.

If A and x are both homog sols (i.e., m = 0,0) try $y_p = Ax^2$ (step up more).

If A, x, x^2 are all homog sols (i.e., m = 0,0,0) try $y_p = Ax^3$ etc.

(2') Suppose $f(x) = 6x^2 + 3$. Ordinarily you try $y_p = Ax^2 + Bx + C$.

But if C is a homog sol (which happens if one of the m's is 0) try

$$y_p = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx \quad \text{(step up)}$$

If C and x are both homog sols (i.e., if m = 0,0) try

$$y_p = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2 \quad \text{(step up more)}$$

If C, x, x^2 are all homog sols (which happens if m = 0,0,0) try

$$y_p = x^3(Ax^2 + Bx + C) = Ax^5 + Bx^4 + Cx^3 \quad \text{etc.}$$

(3') If $f(x) = 9e^{3x}$ you ordinarily try $y_p = Ae^{3x}$.

But if e^{3x} is a homog sol (i.e., one of the m's is 3) try $y_p = Axe^{3x}$ (step up).

If e^{3x} and xe^{3x} are both homog sols (i.e., m = 3,3) try $y_p = Ax^2e^{3x}$ (step up more).

(4') Suppose f(x) is 2 sin 5x or 2 cos 5x.

If you don't use the complex exponential method then ordinarily you would try $y_p = A \cos 5x + B \sin 5x$.

But if cos 5x and sin 5x are homog sols (which happens when $m = \pm 5i$) then try

$$y_p = x(A \cos 5x + B \sin 5x).$$

If cos 5x, sin 5x, x cos 5x and x sin 5x are all homog sols (i.e., if $m = \pm 5i, \pm 5i$) then try

$$y_p = x^2(A \cos 5x + B \sin 5x).$$

(But don't step up if m is say $2 \pm 5i$ or $6 \pm 5i$; only if m is plain $\pm 5i$)

If you use the complex exponential method and switch to the problem

$$ay'' + by' + cy = 2e^{5ix}$$

then ordinarily you would try $y_p = Ae^{5ix}$. But if e^{5ix} is a homog sol (i.e., if $m = \pm 5i$) try $y_p = Axe^{5ix}$ instead.

If e^{5ix} and xe^{5ix} are both homog sols (i.e., if m = ±5i, ±5i) then try $y_p = Ax^2e^{5ix}$.

(But don't step up if m is say 2 ±5i or 6 ± 5i; step up only if m is plain ±5i.)

example 1
Find a general solution to $y'' + y = \sin x$

First find y_h. We have $m^2 + 1 = 0$, $m = \pm i$,

$$y_h = A \cos x + B \sin x.$$

To get y_p, switch to

$$y'' + y = e^{ix}.$$

Ordinarily you would try $y_p = Ce^{ix}$ but e^{ix} is a homog sol (since one of the m's is i) so instead try

$$y_p = Cxe^{ix}$$

Then

$$y_p' = iCxe^{ix} + Ce^{ix} \quad \text{and} \quad y_p'' = -Cxe^{ix} + 2iCe^{ix}$$

Substitute into $y'' + y = e^{ix}$ to get

$$2i\,Ce^{2ix} = e^{ix}$$

$$2iC = 1, \qquad C = \frac{1}{2i} = -\frac{1}{2}i$$

So for the switched DE,

(**) $$y_p = -\frac{1}{2}i\,xe^{ix} = -\frac{1}{2}i\,x\,(\cos x + i \sin x)$$

The original forcing function is a *sine* so take the *imag* part to get the y_p for the original problem:

$$y_p = -\frac{1}{2}x \cos x$$

warning The imag part is *not* $-\frac{1}{2}\boxed{i}\,x \cos x$.
The imag part is what is *next* to the i, just $-\frac{1}{2}x \cos x$

The final answer is

$$y_{gen} = y_h + y_p = A \cos x + B \sin x - \frac{1}{2}x \cos x$$

example 1 again
Here's a way to get y_p *without* using the complex exponential (*but exams will probably insist that you use the complex exponential method).* Try

$$y_p = x(C \cos x + D \sin x) \quad \text{(stepped up because } \cos x \text{ and } \sin x \text{ are homog sols)}$$

Then

$$y_p' = x(-C \sin x + D \cos x) + C \cos x + D \sin x$$
$$y_p'' = x(-C \cos x - D \sin x) + -C \sin x + D \cos x + -C \sin x + D \cos x$$

Substitute into the DE and choose C and D to make it work. You need

$$x(-C\cos x - D\sin x) + -C\sin x + D\cos x + -C\sin x + D\cos x$$
$$+ x(C\cos x + D\sin x) = \sin x$$

The $x\cos x$ and $x\sin x$ terms drop out on the left side, which agrees with the right side.
Equate coeffs of $\sin x$ on each side $\quad -C - C = 1$, $C = -\frac{1}{2}$.
Equate coeffs of $\cos x$ on each side $\quad D + D = 0$, $D = 0$.

So $y_p = -\frac{1}{2}x\cos x$ as before.

PROBLEMS FOR SECTION 1.7

1. Find a general sol to $y'' - y' - 2y = 6e^{-x}$.

2. The DE $y'' - 3y' + 2y = 6e^{2x}$ has $m = 2,1$ so e^{2x} is a homog sol. So you should step up and try $y_p = Axe^{2x}$. What happens if you forget to step up (you dope) and try $y_p = Ae^{2x}$.

3. Given the following forcing functions and roots of the characteristic equation, what y_p would you try

	forcing function	roots of characteristic equation
(a)	$6e^{3x}$	$m = 2,6$
(b)	$6e^{3x}$	$m = 3,6$
(c)	$6e^{3x}$	$m = 3,3,3$

4. Find a general solution to $y''' - y' = x$.

5. Find a general solution to $y'' = 3x^2$

 (a) using all the fancy stuff in this section
 (b) in a more sensible fashion using ordinary calculus

6. Find a real gen sol (a) $y'' + 9y = 4\cos 3x$ (b) $y'' + 4y = 6\sin 2x$.

7. Given the following forcing functions and roots of the characteristic equation, what y_p would you try

	forcing function	roots of characteristic equation
(a)	$x^4 + 2x$	$m = -1,2$
(b)	$x^3 + 2$	$m = 0,0,0,0,0,3$
(c)	$5\sin 2x$	$m = 3 \pm 2i$
(d)	$2\cos 4x$	$m = \pm 4i, \pm 4i$

8. Suppose you were about to try the following particular solutions to a second order DE. What m's would make you change your mind and step up instead. To what?

(a) Try $y_p = Ae^{-4x}$

(b) Try $y_p = Ax^2 + Bx + C$

(c) Try $y_p = Ae^{2ix}$

9. Look at the equation $y'' + 2y' = x + 4$

(a) Ordinarily you would try $y_p = Ax + B$ but $m = 0, -2$ so B is a homog sol so you should step up. What happens if you don't.

(b) Find a general solution using the methods of this section.
(c) Find a general solution again by first antidifferentiating (once) on both sides.

HONORS

10. (What happens when you try too much or try too little)

(a) I was trying to find a particular solution for a 3rd order DE with forcing function $7e^{-5x}$. Two of the me's were 5 and 5 and the third m was not 5 so I followed the rules and tried $y_p = Ax^2e^{-5x}$ (stepped up). When I determined A, I got A = 17.

Suppose you don't follow the rules (you dope) and you try $y_p = (Bx^2 + Cx + D)e^{-5x}$. What happens when you try to find B, C, D.

Be as specific as you can even through you don't know the actual differential equation. It is not good enough to say "you get into trouble" or "it works anyway". I want to know more precisely what you run into.

(b) I was looking for a particular solution for a 3rd order DE with forcing function $5x^2$. I followed the rules and tried $y_p = Ax^2 + Bx + C$ (after first checking that this was not a instance where I should step up). When I determined A,B,C, I got A = 13, B = −2, C = 3.

Suppose you don't follow the rules and try $y_p = Dx^2$. What happens when you try to find D.

SECTION 1.8 NON-HOMOGENEOUS DE CONTINUED (SUMS AND PRODUCTS)

a particular sol for a sum forcing function

To get y_p for say

$$ay'' + by' + cy = 3x^2 + 6e^{5x},$$

first find y_{p1} for

$$ay'' + by' + cy = 3x^2$$

and y_{p2} for

$$ay'' + by' + cy = 6e^{5x}.$$

Then, by superposition,

$$y_p = y_{p1} + y_{p2}$$

Equivalently, try as your y_p a sum of what you would have tried separately for $3x^2$ and $6e^{5x}$, namely

$$y_p = Ax^2 + Bx + C + De^{5x}$$

The usual stepping up rules apply when you try y_{p1} and y_{p2}.

For instance, if m = 5,7 then for y_{p1} above you should try $Ax^2 + Bx + C$ (no stepping up) and for y_{p2} you should try Dxe^{5x} (step up), or all in one shot try

$$y_p = Ax^2 + Bx + C + Dxe^{5x}.$$

example 1

Find a gen sol to $y''' + 4y' = e^x + \sin x$

solution First find y_h.

$m^3 + 4m = 0$, $m(m^2 + 4) = 0$, $m = 0, \pm 2i$, $y_h = A + B\cos 2x + C\sin 2x$.

Now you want y_p.

First get y_{p1} for $y''' + 4y' = e^x$. Try

$$y_{p1} = De^x.$$

Substitute to get

$$De^x + 4De^x = e^x$$

$$5De^x = e^x$$

$$5D = 1,\ D = \frac{1}{5},$$

$$y_{p1} = \frac{1}{5}e^x$$

Then find y_{p2} for $y''' + 4y' = \sin x$. Switch to

$$y''' + 4y' = e^{ix}$$

For the switched DE try

$$y_{p2} = Fe^{ix}.$$

Substitute into the switched DE to get

$$-iFe^{ix} + 4iFe^{ix} = e^{ix}$$

$$3iFe^{ix} = e^{ix}$$

$$3iF = 1, \quad F = \frac{1}{3i} = -\frac{1}{3}i$$

$$\text{switched } y_{p2} = -\frac{1}{3}i\ e^{ix} = -\frac{1}{3}i(\cos x + i \sin x)$$

Take the imag part to get the unswitched y_{p2}:

$$y_{p2} = -\frac{1}{3}\cos x.$$

Finally, for the original DE,

$$y_p = \frac{1}{5}e^x - \frac{1}{3}\cos x$$

$$y_{gen} = y_h + y_p = A + B\cos 2x + \frac{1}{5}e^x - \frac{1}{3}\cos x$$

a particular solution for a product forcing function

Look at $ay'' + by' + cy = f(x)$.

(1) If $f(x) = (6x^2 + 4)\,e^{5x}$, try $y_p = (Ax^2 + Bx + C)e^{5x}$

As usual, plug the trial y_p into the DE to determine A, B, C.

(2) Suppose $f(x) = 6x^2 \sin 2x$ (similarly for $6x^2 \cos 2x$).

method 1 (complex version) Switch to the equation $ay'' + by' + cy = 6x^2 e^{2ix}$ and try $y_p = (Ax^2 + Bx + C)e^{2ix}$. After you determine the constants A,B,C take the imag part (take real part if it was a cosine forcing function).

method 2 (real version) Try $y_p = (Ax^2 + Bx + C)\sin 2x + (Dx^2 + Ex + F)\cos 2x$.

(3) Suppose $f(x) = 5e^{-3x}\sin 4x$ (similarly for $f(x) = 5e^{-3x}\cos 4x$)

method 1 (complex version) Switch to the equation $ay'' + by' + cy = 5e^{-(3+4i)x}$. Try $y_p = Ae^{(-3+4i)x}$ and after you find the constant A, take the imag part (take real part if it was a cosine forcing function).

method 2 (real version) Try $y_p = e^{-3x}(A\cos 4x + B\sin 4x)$.

stepping up when there is a product forcing function

(1) To get y_p for $ay'' + by' + cy = (6x^2 + 4)\,e^{5x}$, watch out if one (or both) of the m's is 5.

Ordinarily you would try

$$y_p = (Ax^2 + Bx + C)e^{5x} = Ax^2e^{5x} + Bxe^{5x} + Ce^{5x}.$$

But if say m=5,17 so that e^{5x} is a homog sol, try

$$y_p = x(Ax^2 + Bx + C)e^{5x} = (Ax^3 + Bx^2 + Cx)e^{5x}.$$

(Step up the *whole* product, not just the Ce^{5x} term.)

And if m = 5,5 so that e^{5x} and xe^{5x} are both homog sols, step up to
$y_p = x^2(Ax^2 + Bx + C)e^{5x} = (Ax^4 + Bx^3 + Cx^2)e^{5x}$.

Then, as usual, plug the trial y_p into the DE to determine A, B, C.

(2) Suppose you want y_p for $ay'' + by' + cy = 6x^2 \sin 2x$. Watch out if $m = \pm 2i$.

complex version

Ordinarily you would switch to the equation $ay'' + by' + cy = 6x^2 e^{2ix}$ and try

$$y_p = (Ax^2 + Bx + C)e^{2ix} = Ax^2 e^{2ix} + Bxe^{2ix} + Ce^{2ix}.$$

But if $m = \pm 2i$ so that e^{2ix} is a homog sol, step up to

$$y_p = x(Ax^2 + Bx + C)e^{2ix} = (Ax^3 + Bx^2 + Cx)e^{2ix}.$$

Plug the trial y_p into the DE to determine A, B, C and take the imag part to get y_p for the original DE.

Similarly for $ay'' + by' + cy = 6x^2 \cos 2x$, except take the real part ultimately.

real version

Ordinarily you would try

$$y_p = (Ax^2 + Bx + C)\sin 2x + (Dx^2 + Ex + F)\cos 2x.$$
$$= Ax^2 \sin 2x + Bx \sin 2x + C \sin 2x + Dx^2 \cos 2x + Ex \cos 2x + F \cos 2x.$$

But if $m = \pm 2i$ so that $\sin 2x$ and $\cos 2x$ are homog sols step up to

$$y_p = x(Ax^2 + Bx + C)\sin 2x + x(Dx^2 + Ex + F)\cos 2x$$
$$= (Ax^3 + Bx^2 + Cx)\sin 2x + (Dx^3 + Ex^2 + Fx)\cos 2x$$

Plug the trial y_p into the DE to determine A, B, C, D, E, F and take the imag part to get y_p for the original DE.

Similarly for $ay'' + by' + cy = 6x^2 \cos 2x$, except take the real part ultimately.

(3) Suppose you want y_p for $ay''+ by' + cy = 7e^{-3x}\cos 5x$. Watch out if $m = \pm 3i$.

complex version

Ordinarily you would switch to the equation

$$y'' + by' + cy= 7e^{(-3+5i)x}$$

and try

$$y_p = Ae^{(-3+5i)x}.$$

But if $m = -3\pm 5i$ so that $e^{(-3+5i)x}$ and $e^{(-3-5i)x}$ are homog sols, step up to $y_p = Axe^{(-3+5i)x}$.

(Note that if $m = -3$ so that e^{-3x} is a homog solution, you should *not* step up.)

Then plug the trial y_p into the DE to determine A and take the imag part to get y_p for the original DE.

Similarly for $ay'' + by' + cy = 7e^{-3x} \sin 5x$, except take the real part ultimately.

real version

Ordinarily you would try

$$y_p = e^{-3x}(A \cos 5x + B \sin 5x)$$
$$= Ae^{-3x} \cos 5x + Be^{-3x} \sin 5x.$$

But if $m = -3 \pm 5i$ so that $e^{-3x} \cos 5x$ and $e^{-3x} \sin 5x$ are homog sols, step up and try $y_p = xe^{-3x}(A \cos 5x + B \sin 5x)$.

Plug the trial y_p into the DE to determine A and B.

example 2

Find a general sol to $y'' + 2y' + 2y = e^{3x} \sin 2x$.

solution Begin with $m^2 + 2m + 2 = 0$, $m = -1 \pm i$.

$$y_h = e^{-x}(C \cos x + D \sin x).$$

To get y_p switch to the DE

$$y'' + 2y' + 2y = e^{(3+2i)x}$$

and try

$$y_p = Ae^{(3+2i)x}.$$

Then

$$y_p' = A(3 + 2i) e^{(3+2i)x},$$

$$y_p'' = A(3 + 2i)^2 e^{(3+2i)x} = A(5 + 12i) e^{(3+2i)x}$$

Substitute into the switched DE:

$$A(5 + 12i)e^{(3+2i)x} + 2A(3 + 2i) e^{(3+2i)x} + 2Ae^{(3+2i)x} = e^{(3+2i)x}$$
$$A(13 + 16i)e^{(3+2i)x} = e^{(3+2i)x}$$
$$(13 + 16i)A = 1, \qquad A = \frac{1}{13 + 16i} = \frac{13 - 16i}{425}$$

So

$$\text{switched } y_p = \frac{13 - 16i}{425} e^{(3+2i)x} = \frac{13 - 16i}{425} e^{3x}(\cos 2x + i \sin 2x)$$

This is a particular solution to

$$y'' + 2y' + y = e^{(3+2i)x}$$

Since the original forcing function was a sine, the imag part of $e^{(3+2i)x}$, take the imag part of y_p to get

$$\text{original } y_p = -\frac{16}{425} e^{3x} \cos 2x + \frac{13}{425} e^{3x} \sin 2x.$$

Then for the original DE,

$$y_{gen} = y_h + y_p = e^{-x}(C \cos x + D \sin x) - \frac{16}{425} e^{3x} \cos 2x + \frac{13}{425} e^{3x} \sin 2x.$$

footnote

In example 2, to avoid using the complex exponential you can try

$$y_p = e^{3x}(A \cos 2x + B \sin 2x).$$

You would end up with $A = -\frac{16}{425}$, $B = \frac{13}{425}$.

example 3

Find the form of the particular solution to $y'' + y = e^{-2x} + 3xe^{-2x}$

solution Find the homog sol to see if it's necessary to step up y_p: $m^2 = -1$, $m = \pm i$,

$$y_h = A \cos x + B \sin x$$

No stepping up necessary. Think of the the forcing function as the product $(3x+1)e^{-2x}$ and try

$$(*) \qquad y_p = (Px + Q)e^{-2x}$$

If you think of the forcing function as a sum then you would try

$$y_p = Ce^{-2x} + (Dx + E)e^{-2x}$$

which simplifies to (*) because you can combine $Ce^{-2x} + Ee^{-2x}$ into $(C+E)e^{-2x}$ and replace C+E by Q.

differential equations not included in this chapter

type 1 Linear DE with *variable* coeffs such as $\boxed{x}y'' + \boxed{x^2}y' + 6y = 7x^2$

Superposition rules still hold but the idea of solving a characteristic equ to get m's for y_h doesn't apply anymore. It is *not* correct to try to solve

$$xm^2 + x^2m + 6 = 0$$

for m and use $Ae^{m_1 x} + Ae^{m_2 x}$

Furthermore, trying y_p of some standard form doesn't necessarily work when the coeffs are variable.

type 2 Linear DE with constant but *non-real* coefficients
The complex superposition rule doesn't hold in this case.

type 3 *Non*-linear DE such as $\boxed{y''y} + y = 5$ and $\boxed{(y'')^2} + y' + 6y = 5x$

In this case, superposition doesn't hold. Even if you could get y_h and y_p (which you can't), the gen sol would not be $y_h + y_p$

PROBLEMS FOR SECTION 1.8

1. Find a gen sol (a) $y'' + 9y = 5e^{x} + 3x$ (b) $y'' - 4y = e^{2x} + 2$

2. Describe how you would find a particular solution for
(a) $y'' + 2y' + 10y = 6\cos 3x + 7\sin 3x$
(b) $y'' + 2y' + 10y = 6\cos 3x + 7\sin 4x$.

3. Find a gen real sol (a) $y'' + 4y' + 3y = 2e^{2x}\cos 4x$

(b) $y'' + 4y' + 3y = 5e^{2x}\sin 4x$

(c) $y'' - 3y' + y = 3e^{x}\sin x$ (d) $y'' - y = xe^{x}$

4. Given the following forcing functions and roots of the characteristic equ. What would you try for y_p.

forcing function	roots of characteristic equation
(a) $x^2 e^{2x}$	$m = 1, -1$
(b) $x^2 e^{2x}$	$m = 2, 2$
(c) $e^{3x}\cos 4x$	$m = \pm 4i, \pm 4i$ (4th order DE)
(d) $e^{3x}\sin 4x$	$m = 3 \pm 4i$
(e) $e^{3x}\cos 4x$	$m = 2 \pm 4i$
(f) $e^{x}(x^2 + 1)$	$m = 1, 2$
(g) $x^2\sin x$	$m = 0, 0, 0, 3$ (4th order DE)
(h) $e^{x}(x^2 + 1)$	$m = 0, 2$

5. What particular solution would you try.

(a) $y'' + 6y' + 2y = 2e^{2x} - x^2 e^{2x}$

(b) $y'' - 4y' + 4y = 2e^{2x} - x^2 e^{2x}$

6. (a) Find a particular sol to $2y'' + 2y = 3x\cos x$ using the complex exponential.
(b) Find the particular sol again without using the complex exponential.

REVIEW PROBLEMS FOR CHAPTER 1

1. Solve $y'' - y = xe^x$ with IC $y(0) = 1$, $y'(0) = 0$

2. Find a general real solution (a) $y'' + 2y' + y = 3 \cos 2x$
(b) $y'' + 2y' + y = 6 \sin 2x$

3. Solve $y'' + 6y' + 10y = f(x)$ with IC $y(0) = 1$, $y'(0) = 2$ where

$$f(x) = \begin{cases} 50x & \text{if } 0 \le x \le \pi \\ 10 & \text{if } x \ge \pi \end{cases}$$

4. Suppose that $5e^{-2x} + 3 \sin x$ and $6e^{-x} + 3 \sin x$ are solutions to a second order linear differential equation. Find the equation.

5. Suppose a solution to $y'' + 3y' - 4y = f(x)$ is $y = 3x^2$.

Find a dozen other particular solutions.

6. Suppose y_1 is a solution to

$$y'' + 3y' - 4y = f(x) \text{ with IC } y(0) = 2,\ y'(0) = 3$$

Look at $y_1 + 4x^2 + 1$ (I just made up $4x^2 + 1$ at random and tacked it onto y_1).
What DE plus IC does $y_1 + 4x^2 + 1$ solve.

7. Are these linear? If so, are they homogeneous?

(a) $y' = y$ (b) $y' = x$ (c) $y'' = y$ (d) $y''y = x$ (e) $xy'' = y$

8. As usual, let a,b,c be real constants,.
Are these statements true sometimes, always or never.

(a) If y_1 and y_2 are solutions of $ay'' + by' + cy = f(x)$ then $Ay_1 + By_2$ is also a solution for any A and B.

(b) If y_1 and y_2 are solutions of $ay'' + by' + cy = f(x)$ then $y_1 - y_2$ is a solution of $ay'' + by' + cy = 0$.

9. Find a general solution (a) $y' = - y$ (b) $y'' = x + y$

10. The velocity $v(t)$ of a falling object with mass m satisfies the DE

$$mv' = mg - cv$$

where g and c as well as m are constants.

(The object experiences a downward force mg due to gravity and a retarding force cv proportional to its velocity due to air resistance. Their sum, the total force, is mv' by Newton's law that force = mass × acceleration.)

Find $v(t)$ if the initial velocity is 0. Then find the "limiting" velocity $v(\infty)$.

11. Look at the equation

$$y'' + 2y' = x + 4 \qquad (1)$$

Then $m = 0, -2$ and

$$y_h = C + De^{-2x}$$

Ordinarily you would try

$$y_p = Ax + B.$$

But since B is a homog solution you should step up. What happens if you don't, i.e., what happens when you try $Ax + B$.

CHAPTER 2 THE IMPULSE RESPONSE

SECTION 2.1 THE UNIT IMPULSE AND THE IMPULSE RESPONSE

This chapter is about systems in which inputs f(t) and outputs y(t) are related by a DE of the form

$$ay'' + by' + cy = f(t)$$

where a,b,c, are constants.
So the systems satisfy superposition and are time-invariant meaning that the ingredients such as mass, resistance etc. do not change with time.

the delta function

Fig 1 shows the function $\delta(t)$, called the *unit impulse* at time 0. It is thought of as an "infinite" force applied for a "split second" at time t = 0, producing an impulse (area under the curve) of 1 (Fig 1).

The function $\delta(t-t_0)$ is a unit impulse occurring at time t_0 (Fig 2).

The function $7\delta(t)$ is an impulse of "size" (i.e., area) 7 at time t = 0 (Fig 3).

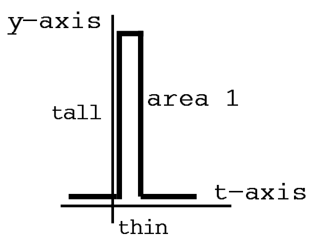


FIG 1 $\delta(t)$

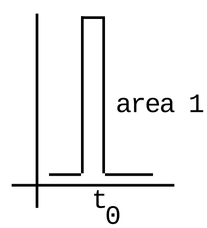


FIG 2 $\delta(t-t_0)$

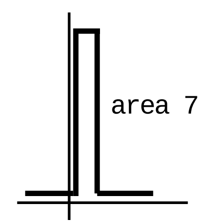


FIG 3 $7\delta(t)$

A delta function is sometimes drawn as a vertical arrow f with height equal to the "area enclosed". Fig 4 shows the arrow representation of $7\delta(t - t_0)$. Fig 5 shows the

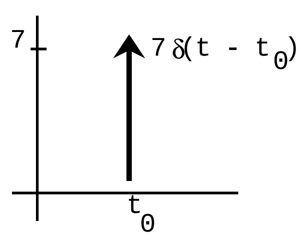


FIG 4

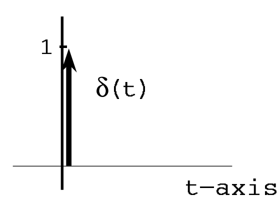


FIG 5

the impulse response h(t)

The *impulse response* of a system is its response to the input $\delta(t)$ when the system is initially at rest. The impulse response is usually denoted h(t). Sometimes it's called Green's function.
In other words, if the input to an initially-at-rest system is $\delta(t)$ then the output is named h(t).

finding the impulse response

Suppose inputs $f(t)$ and outputs $y(t)$ are related by the DE

$$ay'' + by' + cy = f(t)$$

By definition, the system's impulse response $h(t)$ is the solution to

$$ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0.$$

To find the impulse response, solve the new problem

$$ay'' + by' + cy = \boxed{0} \quad \text{with IC } y(0) = 0\ ,\ y'(0) = \boxed{1/a}$$

Here's a physical interpretation of the rule. You have given the block at the end of a resting spring a quick hard kick at time 0. Now that the kick is over, it is still time 0 and no more kick is coming in, the block is not yet displaced but the quick kick gave it initial velocity $1/a$.

quickie pseudo proof

Consider the differential equation

(1) $\quad ay'' + by' + cy = \delta(t)$ with IC $y(0) = 0,\ y'(0) = 0$

Think of $y(t)$ as the displacement of a block on a spring at time t. You can do this if a,b,c aren't negative. The boxed rule still holds if one of the coeffs is negative but this proof would be no good in that case.

Physicists say that

(2) $\quad$ impulse = change in momentum

where

(3) $\quad$ momentum = mass $\times$ velocity

The mass of the block is the coefficient a (this is stated, but without explanation unfortunately, on page 2 of Section 1.2).

The velocity of the block when the impulse hits is 0 because the IC in (1) is $y'(0) = 0$.

The impulse imparted to the block is 1 because the forcing function in (1) is the unit impulse function $\delta(t)$.

So (2) becomes

$$1 = \text{change in } \ a \times \text{velocity}$$

But a is a fixed constant so

$$1 = a \times \text{change in velocity}$$

and

$$\text{change in velocity} = 1/a$$

Look at what happens right after the spike part of the input $\delta(t)$ has acted on the block. The block's velocity goes up by $1/a$, the block hasn't changed position yet, it's still time 0 (practically) and for the rest of time the input in (1) is the 0 part of the delta function. So to get the response,i.e., the solution to (1), solve

$$ay'' + by' + cy = 0 \text{ with IC } y(0) = 0,\ y'(0) = 0 + 1/a = 1/a$$

example 1

Given a system where the input f(t) and response y(t) are related by

$$2y'' + 8y' + 6y = f(t)$$

Find the system's impulse response.

solution The problem says to solve

$$2y'' + 8y' + 6y = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

To do it, switch to the problem

$$2y'' + 8y' + 6y = 0 \text{ with IC } y(0) = 0,\ y'(0) = 1/2.$$

Solve $2m^2 + 8m + 6 = 0$, $m = -1, -3$. So $y = Ae^{-t} + Be^{-3t}$.

To satisfy IC y(0) = 0 you need

$$A + B = 0$$

We have $y' = -Ae^{-t} - 3Be^{-3t}$ so to satisfy IC $y'(0) = 1/2$ you need

$$-A - 3B = 1/2$$

So A = 1/4, B = − 1/4 and

(1) $$h(t) = \frac{1}{4} e^{-t} - \frac{1}{4} e^{-3t} \text{ for } t \ge 0$$

Note that h(t) is always 0 for $t \le 0$ since there is no response until the impulse hits at time t = 0

In other words, for this system, if the input is the delta function in Fig 5 then the response is the function h(t) in Fig 6.

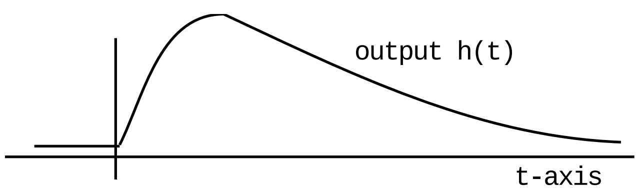


FIG 6

warning If you leave out "for $t \ge 0$" in (1) then you are suggesting that the impulse response is $\frac{1}{4} e^{-t} - \frac{1}{4} e^{-3t}$ for *all* t (Fig 7) which is *wrong*.

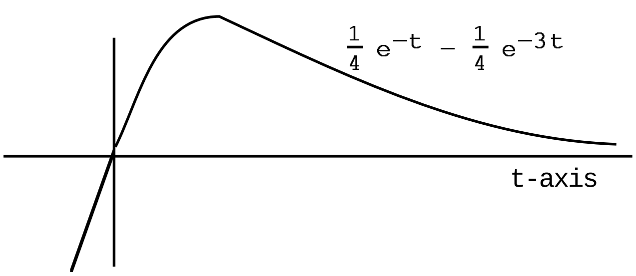


FIG 6A

response to a non-unit impulse

If the response to δ(t) is h(t) then, by superposition, the response to the impulse 7δ(t) is 7h(t).

example 1 continued

The solution to

$$2y'' + 8y' + 6y = 5\delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

is

$$y = 5h(t) = 5\left(\frac{1}{4} e^{-t} - \frac{1}{4} e^{-3t} \right) \quad \text{for } t \ge 0$$

response to a delayed impulse

Suppose a system is initially at rest.
And, as usual, suppose its response at time t to input $\delta(t)$ is $h(t)$. Let $t_0 \ge 0$.
Then the system's response at time t to $\delta(t-t_0)$, a unit impulse at time $t = t_0$, is $h(t-t_0)$.

In other words, if the impulse is delayed then the response is delayed.

footnote

This is not as obvious as it seems. It hold only because the system is initially at rest and is time invariant so *nothing happens between time t=0 and $t=t_o$* (no block moves, no current flows, no ice melts, no birds sing); the system remains suspended in time and therefore responds to the delayed impulse in the same way in which it would have responded to the original impulse.

example 1 continued

The solution to

$$2y'' + 8y' + 6y = \delta(t-4) \quad \text{with zero IC}$$

is

$$y = h(t-4) = \begin{cases} 0 & \text{for } t \le 4 \\ \frac{1}{4} e^{-(t-4)} - \frac{1}{4} e^{-3(t-4)} & \text{for } t \ge 4 \end{cases}$$

warning

The solution is *not* simply $y = \frac{1}{4} e^{-(t-4)} - \frac{1}{4} e^{-3(t-4)}$.
The sol has this formula *only* for $t \ge 4$.
The solution is 0 until time t=4 since the impulse hasn't hit yet.

Fig 7 shows the input $\delta(t-4)$ and Fig 8 shows the response $h(t-4)$.

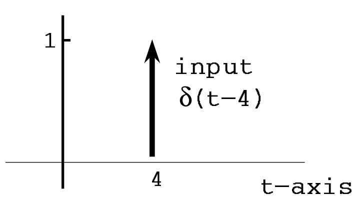


FIG 7

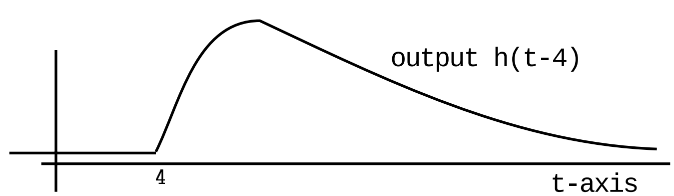


FIG 8

PROBLEMS FOR SECTION 2.1

1. Find the impulse response of a system whose input $f(t)$ and output $y(t)$ are related by

(a) $2y'' + 2y = f(t)$ (b) $2y'' - y' - y = f(t)$

2. (a) Solve $y'' + 4y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$.

(b) Solve $y'' + 4y = 6\delta(t-2)$ with IC $y(0) = 0$, $y'(0) = 0$.

3. Let $h(t) = 1/t^2$, $t \geq 0$, be the impulse response of a system. If the system is initially at rest, find the response of the system at time 3 to

(a) a unit impulse at time 0
(b) an impulse of size 6 at time 0
(c) a unit impulse at time 2

4. Go back to example 1. The impulse response is in (1) and its graph is in Fig 6. Where does the peak occur; i.e., when does the response stop growing and start dying out.

HONORS

5. Suppose that for a certain physical system, inputs $f(t)$ and outputs $y(t)$ are related by

$$ay'' + by' + cy = f(t).$$

Your roommate found the impulse response of the system, $h(t)$, i.e., the solution to

$$ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 0,\ y'(0) = 0$$

You were asked to find the response of the system at time t to the input $\delta(t)$ when the system is *not* initially at rest; namely, with $y(0) = 4$, $y'(0) = 5$.

You are going to take advantage of your roommate's answer, plus superposition, to come up with your answer. In particular, fill in the following blank:

The response at time t to the input $\delta(t)$ when the IC are $y(0) = 4$, $y'(0) = 5$

is $h(t)$ [that I stole from my roommate] + ______________________

Don't actually try to compute what goes in the blank (because you don't have a, b, c). Just explain briefly what you would do to get the blank and why.

SECTION 2.2 GETTING READY TO CONVOLVE

integrating a "multi-piece" function

$$\text{If}\quad g(x) = \begin{cases} x^3 & \text{for } 1 \le x \le 2 \\ 4 & \text{for } 2 \le x \le 5 \\ 0 & \text{otherwise} \quad \text{(Fig 1)} \end{cases}$$

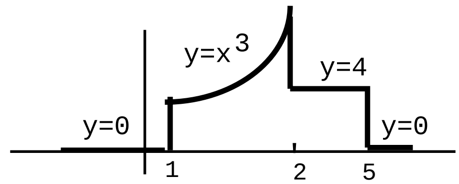


FIG 1

then

$$\int_{-\infty}^{\infty} g(x)\,dx = \underbrace{\int_{-\infty}^{1} 0\,dx}_{0} + \int_1^2 x^3\,dx + \int_2^5 4\,dx + \underbrace{\int_5^{\infty} 0\,dx}_{0} = \int_1^2 x^3\,dx + \int_2^5 4\,dx$$

In general, to integrate a multi-piece function, integrate the pieces and add, ignoring the intervals where the function is 0

vertical lines in a graph

The graph of $g(x)$ in Fig 1 is ambiguous because of the vertical segments at $x=1,2,5$. In other words, you can't find $g(1)$, $g(2)$, $g(5)$ from the diagram. For our purposes, it doesn't matter.

integrating a product of multi-piece functions

$$\text{If}\quad f(x) = \begin{cases} e^x & \text{for } 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} x^3 & \text{if } 1 \le x \le 4 \\ 7 & \text{if } 4 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

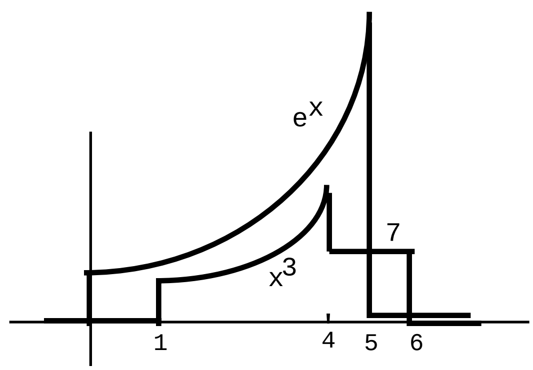


FIG 2

then (look at at the picture)

$$f(x)g(x) = \begin{cases} x^3 e^x & \text{for } 1 \le x \le 4 \\ 7e^x & \text{for } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_1^4 x^3 e^x\,dx + \int_4^5 7e^x\,dx$$

the graph of y = g(t–x) in an x,y coord system

Consider say $y = x^3$. The graph of $y = (2-x)^3$ can be found by first translating left 2 and then reflecting in the y-axis. Fig 3 shows the several steps.

begin | intermediate step, translate left 2 | end, reflect in y-axis

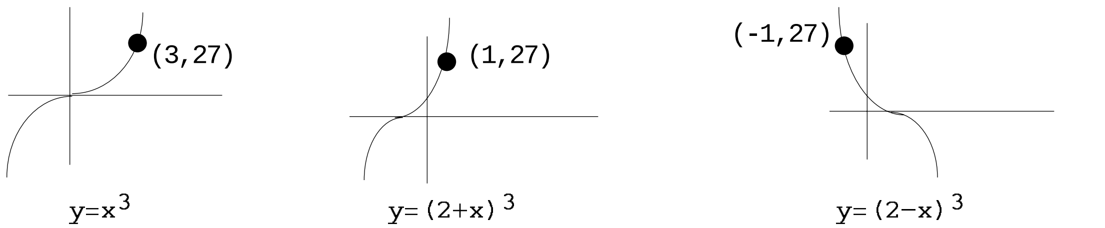


FIG 3 Going from $y = x^3$ to $y = (2-x)^3$

> In general, to sketch the graph of $g(t-x)$ in an x,y coord system, translate the graph of $y = g(x)$ *left* by t and then reflect in the y-axis.
>
> Alternatively, you can reflect in the y-axis *first* and then translate *right* by t.

example 1 Let

$$g(x) = \begin{cases} xe^x & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \quad \text{(Fig 4)} \end{cases}$$

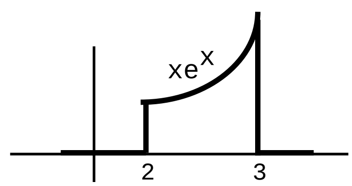
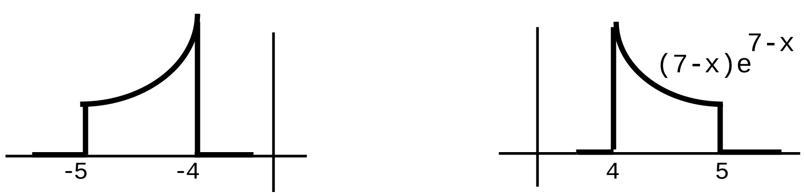


FIG 4 graph of g(x)

Fig 5 shows the graph of $g(7-x)$.



intermediate step | graph of $g(7-x)$

FIG 5

The graph shows that the nonzero piece lies between x = 4 and x = 5. The formula for the nonzero piece in the new graph is found by replacing x by $7-x$ so all in all

$$g(7-x) = \begin{cases} (7-x)e^{7-x} & \text{for } 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

More generally, Fig 6 shows the graph of $g(t-x)$ in an x,y coordinate system (the y-axis is not drawn because where it is depends on the size of t):

$$g(t-x) = \begin{cases} (t-x)e^{t-x} & \text{for } t-3 \le x \le t-2 \\ 0 & \text{otherwise} \end{cases}$$

intermediate step   graph of g(t−x)

FIG 6

drawing the graph of f(x) + g(x)

To get the graph of say x + sin x, you can draw the graph of x and the graph of sin x (Fig 7) and add (painstakingly) their heights (Fig 7).

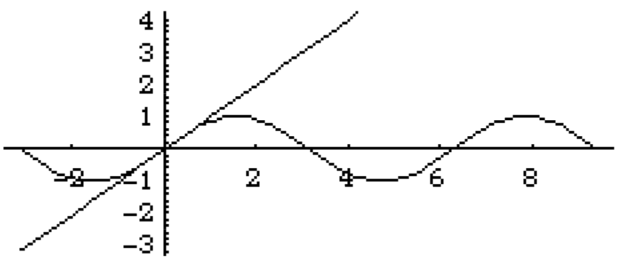


FIG 7

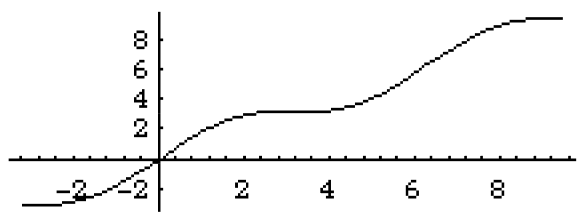


FIG 8

example 2

Figs 9, 10, 11 show the graphs of f(x), g(x) and k(x).

The graph of f(x) + g(x) + k(x) is in Fig 12.

The curve in Fig 12 was obtained by adding *heights* from Figs 9, 10, 11. Don't refer to it as "adding areas".

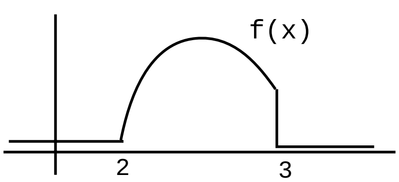


FIG 9

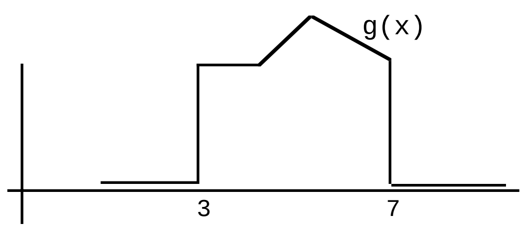


FIG 10

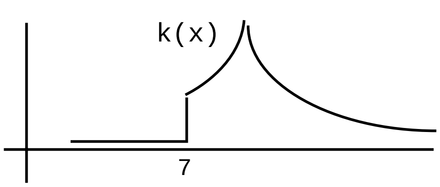


FIG 11

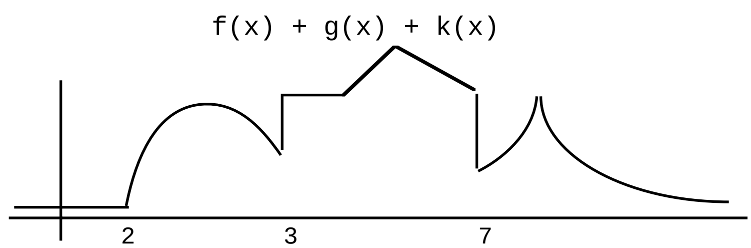


FIG 12

SECTION 2.3 CONVOLUTION

This chapter is about systems in which inputs f(t) and outputs y(t) are related by a DE of the form

$$ay'' + by' + cy = f(t) \text{ where } a,b,c, \text{ are constants.}$$

In other words, the systems satisfy superposition and are time-invariant (the ingredients such as mass, resistance etc. do not change with time).

the convolution of two functions

Given functions f(t) and g(t), the two integrals

$$(1) \qquad \int_{u=-\infty}^{\infty} f(u)\, g(t-u)\, du \quad \text{and} \quad \int_{u=-\infty}^{\infty} f(t-u)\, g(u)\, du$$

can be shown to be equal (that's a theorem) (proved in problem 9).

Each is referred to as the *convolution* of f(t) and g (t) (that's a definition), and each is denoted by $f(t)*g(t)$ or equivalently by $g(t)*f(t)$.

In each integral, u is the dummy variable of integration and t is "carried along" so that the convolution of f(t) and g(t) is another function of t.

The two formulas in (1) are given on the reference page you will get with exams.

finding the response of an initially-at-rest system to input f(t) given its impulse response h(t)

If a system is initially at rest, its impulse response h(t) determines its response to all other inputs as follows.

Let h(t) be a system's impulse response.
If the system is initially at rest then its response y(t) to input f(t) is given by

$$(2) \quad y(t) = h(t)*f(t) = \int_{u=-\infty}^{\infty} f(u)\; h(t-u)\; du = \int_{u=-\infty}^{\infty} f(t-u)\; h(u)\; du$$

In other words, the *output of an initially-at-rest system corresponding to a particular input can be found by convolving the input with the system's impulse response*.

In other other words, if h(t) is the impulse response, namely the output of the initially-at-rest system when the input is $\delta(t)$, then the convolution $h(t)*f(t)$ is the output of the initially-at-rest system when the input is f(t).

proof of (2)

Think of the input f(t) (Fig 1) as a sum of impulses (i.e., the sum of the impulse heights in Fig 1 is the f(t) height).

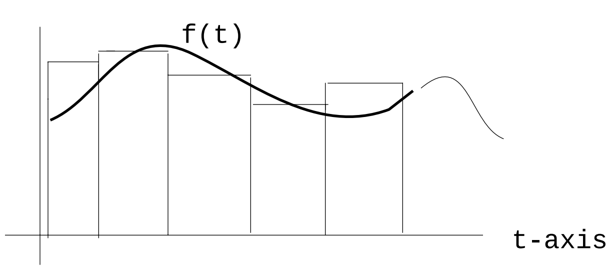


FIG 1

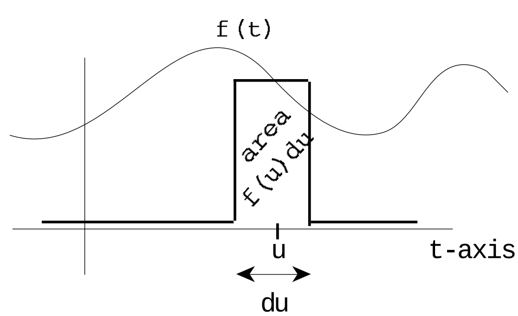


FIG 2

Remember that an impulse of say "size" 5 with spike at time 7 is named $5\delta(t-7)$. Now look at a typical impulse (Fig 2) in this sum occurring at time u with width du. Its height is f(u) so its area ("size") is f(u) du and its name is

(3) $\quad f(u)\ du\ \delta(t-u)$

By definition, the response to the impulse $\delta(t)$ is h(t). So (Section 2.1), the response to the (delayed, non-unit) impulse in (3) is

(4) $\quad f(u)\ du\ h(t-u)$

footnote This wouldn't work if the system were not initially at rest. As pointed out in Section 2.1, the response to a delayed impulse would not necessarily just be the delayed impulse response unless the system is time invariant and initially at rest.

By superposition, the response at time t to *the sum of all the impulses in Fig 1* (i.e., to f itself), when the system is initially at rest, is the sum of the responses in (4).

footnote This wouldn't work if the system were not initially at rest. If you add responses which satisfy say IC y(0) = 4 then the sum satisfies IC y(0) = 4 + 4 + 4 + $\cdots$. But in this case, the responses in (4) all satisfy initial conditions like y(0) = 0 and therefore so does the sum.

The summing of the responses in (4) is done with an integral. So the response y(t) at time t to input f(t) is

(5) $$y(t) = \int_{-\infty}^{\infty} f(u)\ du\ h(t-u) = f(t)*h(t) \qquad \text{QED}$$

footnote (very subtle) The sum (integral) in (5) could actually start adding from u=0 instead of $u=-\infty$ since the input f(t) starts at time 0. And the sum could stop at u=t instead of $u=\infty$, since at time t, you can only get the response from impulses that occur before time t.
But it doesn't hurt to start out using the limits $u=-\infty$ to $u=\infty$. When we actually calculate the integral you will use the fact that f(u) = 0 if $u \le 0$ and h(t−u) = 0 if $u \ge t$ and this will automatically change the limits to u=0 to u=t. Don't worry about it now.

example 1
Given a system with impulse response h(t) in Fig 4. Find the response y(t) of the initially-at-rest system to the input f(t) in Fig 5.

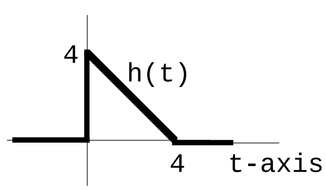


FIG 3

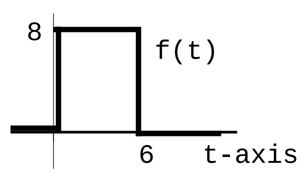


FIG 4

I'll use the first of the two convolution integrals in (2):

$$y(t) = h(t)*f(t) = \int_{u=-\infty}^{\infty} h(t-u) f(u)\ du$$

The problem starts with two functions of *t* namely the impulse response h(t) and the input f(t) in Figs 3 and 4. And the answer will be a function of *t*, namely the response y(t). But the *working* letter for the convolution integral is u; you are integrating f(u) times h(t−u) with respect to u and carrying t along as if it were a constant.

Since h(t−u) and f(u) both change formulas, the best way to keep track of them, and ultimately their product, is to draw their graphs in a u,y coord system.

The graph of f(u) in a u,y coord system looks the same as the f graph in Fig 4 but with the horizontal axis named u instead of t (Fig 5).

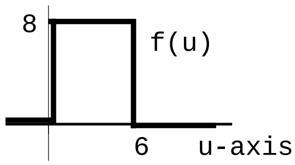
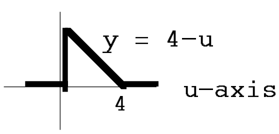


FIG 5

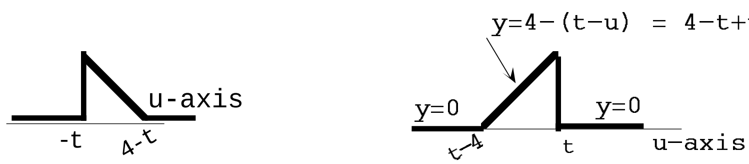
The graph of h(u) in a u,y coord system looks the same as the h graph in Fig 3 but with a horizontal axes named u instead of t (Fig 6). Translate left by t and then reflect in the y-axis to get the graph of h(t−u). Fig 7 shows the two steps.

In Fig 6, the slanted line has equation y = 4−u. Replace u by t−u to get the equ of the slanted part in the graph of h(t−u) in Fig 7, namely y = 4 − (t−u).



graph of h(u)

FIG 6



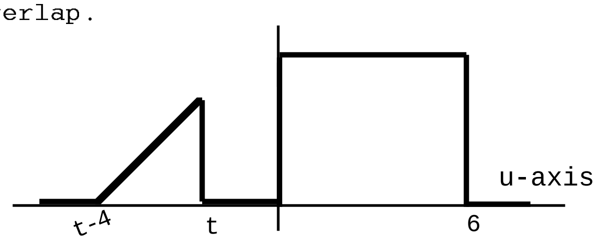
intermediate step

graph of h(t−u)

FIG 7 Getting the graph of y = h(t−u)

The function h(t−u) is either 4 − t + u or 0 and the function f(u) is either 8 or 0 so the product is either 8(4 − t + u) or 0. But *when* it is 8(4 − t+ u) and *when* it is 0 depends on the "constant" t. The best way to keep track is with more graphs and you need cases to accommodate all the possibilities, i.e., all the ways in which the h(t−u) and f(u) graphs can overlap.

case 1 t ≤ 0

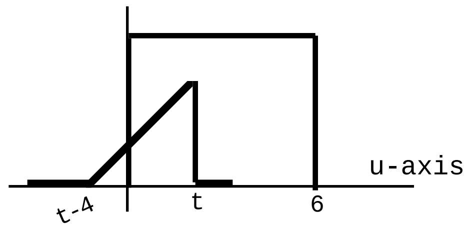


From the diagram you can see that in this case, one or another of f(u) and h(t−u) is always 0 so their product is always 0. So

$$y(t) = \int_{-\infty}^{\infty} h(t-u)f(u)\,du = \int_{-\infty}^{\infty} 0\,du = 0$$

case 2 $t-4 \le 0$ and $t \ge 0$,

i.e., $0 \le t \le 4$



From the diagram you can see that outside the interval [0,t] at least one of h(t−u) and f(u) is 0 so their product is 0 and does not contribute to the convolution. integral. Inside the interval their product is $8(4-t+u)$. So

$$y(t) = \int_{-\infty}^{\infty} h(t-u)\, f(u)\, du \qquad = \int_{-\infty}^{\infty} 8(4-t+u)\, du$$

$$= \left[8(4-t)u + 4u^2\right]\Big|_{u=0}^{u=t} \qquad \text{(antidiff w.r.t. u and treat t as a constant)}$$

$$= 8(4-t)t + 4t^2$$
$$= -4t^2 + 32t$$

warning

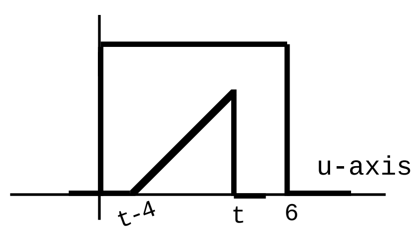
The name of the case is $0 \le t \le 4$ but the integral is *not* $\int_0^4$; it's $\int_{u=0}^{u=t}$.

To get $0 \le t \le 4$ as the title of the case, decide what will make the triangle overlap the box as shown in the diagram: the left end of the triangle must be to the left of 0 but the right end must be past 0. (The right end must also not be past 6 but that's taken care of already by saying the left end hasn't passed 0.) This means $t-4 \le 0$ and $t \ge 0$ which is $0 \le t \le 4$

To get $\int_{u=0}^{u=t}$ look at the picture and decide where the product is nonzero: To the left of u=0, the box is 0. To the right of u=t the triangle turns 0. The product is nonzero for $0 \le u \le t$ so $\int_{u=-\infty}^{\infty}$ turned into $\int_{u=0}^{t}$.

case 3 $t-4 \ge 0$ and $t \le 6$

i.e., $4 \le t \le 6$



From the diagram you can see that outside the interval [t−4,t] the product h(t−u) f(u) is 0 (because outside the interval one or another of the factors is 0). And inside the interval, h(t−u)f(u) is 8(4−t+u). So

$$y(t) = \int_{u=-\infty}^{u=\infty} h(t-u)f(u)\, du$$

$$= \int_{u=t-4}^{u=t} 8(4-t+u)\, du$$

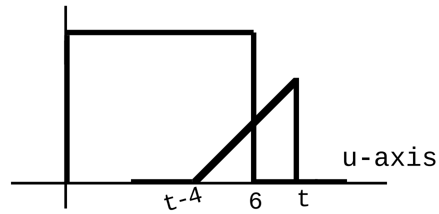
$$= 8(4-t)u + 4u^2 \Big|_{u=t-4}^{u=t}$$

$$= 8(4-t)t + 4t^2 - \left(8(4-t)(t-4) + 4(t-4)^2\right)$$
$$= 64$$

warning
The name of the case is $4 \le t \le 6$ but the integral is $\int_{u=t-4}^{u=t}$ not $\int_4^6$

case 4 $t - 4 \le 6$ and $t \ge 6$

i.e., $6 \le t \le 10$



The product $h(t-u)f(u)$ is 0 except for the interval $[t-4,6]$. So

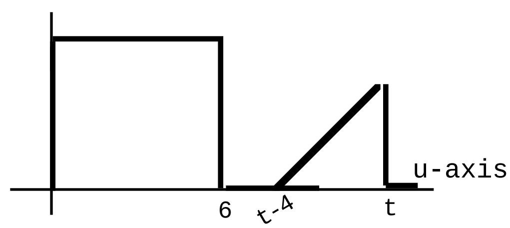
$$y(t) = \int_{u=-\infty}^{u=\infty} h(t-u)d(u)\, du$$

$$= \int_{u=t-4}^{u=6} 8(4 - t + u)\, du$$

$$= 8(4 - t)u + 4u^2 \Big|_{u=t-4}^{u=6} = -4t^2 - 80t + 400$$

case 5 $t - 4 \ge 6$,

i.e., $t \ge 10$



One or another of $f(u)$ and $h(t-u)$ is always 0 so their product is always 0 . So

$$y(t) = \int_{u=-\infty}^{u=\infty} 0\, du = 0$$

All in all the response is

$$(5) \quad y = \begin{cases} 0 & \text{if } t \le 0 \\ -4t^2 + 32t & \text{if } 0 < t < 4 \\ 64 & \text{if } 4 < t < 6 \\ 4t^2 - 80t + 400 & \text{if } 6 \le t \le 10 \\ 0 & \text{if } t \ge 10 \end{cases}$$

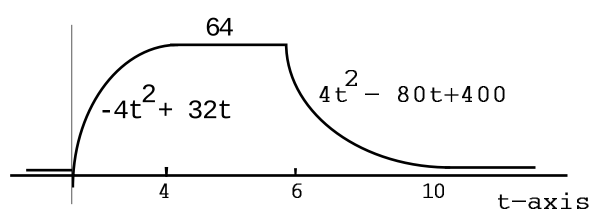


FIG 8

What good was all this? Now you know that if the input $\delta(t)$ produces the output $h(t)$ in Fig 3 then the input $f(t)$ in Fig 4 produces the output in Fig 8.

warning

1. The cases must include all values of t. The cases cannot jump from $3 \le t \le 5$ to $6 \le t \le 9$ omitting $5 \le t \le 6$. If one case is $3 \le t \le 5$ then the next case must pick up from there with $5 \le t \le$ ...

And the cases must not overlap. You can't have one case named $2 \le t \le 3$ and another case named $t \ge 2$.

2. In example 1, don't write $h(u) = 4 - u$ since $h(u)$ is $4 - u$ only for $0 \le u \le 4$. For other u's, $h(u)$ is 0.

example 2

Suppose the impulse response of a system is

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases} \qquad \text{(Fig 9)}$$

If the system is initially at rest, find its response to input

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \ge 0 \end{cases} \qquad \text{(Fig 10)}$$

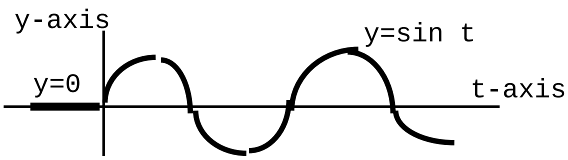


FIG 9 graph of h(t)

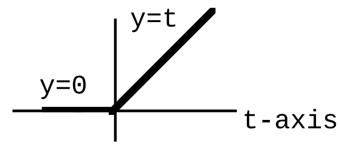


FIG 10 graph of f(t)

solution The response is h(t)*f(t). I'll use the second convolution integral in (2):

$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)\,f(t-u)\,du$$

Fig 11 shows the graph of h(u) in a u,y coord system.

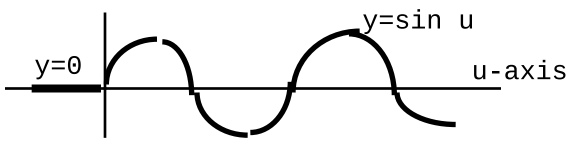
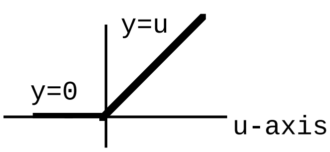


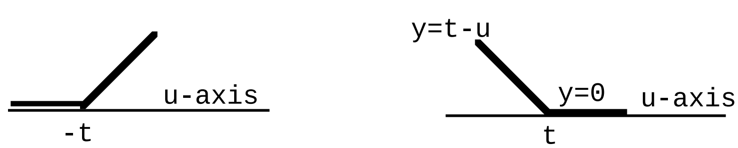
FIG 11 graph of h(u)

Fig 12 shows the graph of f(u) in a u,y coord system.
Fig 13 translates the f({u) graph left by t and then reflects in the y-axis to get the graph of f(t-u)



graph of f(u)

FIG 12



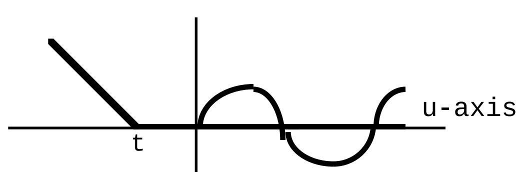
intermediate step

graph of f(t-u)

FIG 13

case 1 $t \le 0$

One or another of h(u) and f(t-u) is always 0 so

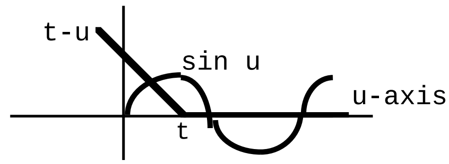


$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)\,f(t-u)\,du = \int_{u=-\infty}^{u=\infty} 0\,du = 0$$

warning Do not write $h(t)*f(t) = \int_{u=-\infty}^{\infty} \sin u \cdot (t-u)\,du$ in case 1.

When $t \le 0$, the product h(u)f(t-u) is *not* $\sin u \cdot (t-u)$. It is *zero*, which is why the integral is 0.

case 2 $t \geq 0$



The product $h(u)f(t-u)$ is nonzero only in interval $[0,t]$ (to the left of that interval $h(u)$ is 0 and to the right of that interval $f(t-u)$ is 0. So

$$h(t)*f(t) = \int_{u=-\infty}^{u=\infty} h(u)f(t-u)\,du$$

$$= \int_{u=0}^{t} \sin u \cdot (t-u)\,du$$

warning
Be careful with letters.
The problem starts with $h(t) = \sin t$ but in the convolution integral you have to use $h(\boxed{u})$ which is $\sin \boxed{u}$.
It makes a difference.

$$= t\int_{u=0}^{t} \sin u\,du - \int_{u=0}^{t} u \sin u\,du$$

$$= \Big[-t \cos u - u \cos u + \sin u \Big]\Big|_{u=0}^{t} \qquad \text{(ref page antideriv tables (E))}$$

$$= t - \sin t$$

So all in all

$$h(t)*f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t - \sin t & \text{if } t \geq 0 \end{cases}$$

special case

If $f(t)$ and $h(t)$ are each 0 until time t and then each maintains a single formula apiece for $t \geq 0$ as in example 2 then their convolution requires only the trivial case $t \leq 0$ (where the answer is 0) and the significant case $t \geq 0$ where the convolution integral has limits $\int_{u=0}^{t}$

warning

But if either $f(t)$ or $h(t)$ changes formula for $t \geq 0$ then you are not in the special case and the one case $t \geq 0$ is not enough, as in example 1.

which convolution integral to use

The result in (2) offers a choice of two convolution integrals. They produce the same answer so either one can be used.

In example 1, $f(t)$ is "simpler" than $h(t)$ so it would have been better to use the version with $f(t-u)h(u)$ so that the simpler of the two graph is the one that gets flipped. The example used the other version to give extra flipping practice.

In example 2, one version uses the integrand $(t-u) \sin u$ and the other version uses the integrand $u \sin(t-u)$. Pick the version for which the antidifferentiation is easier, namely $(t-u) \sin u$.

mathematical catechism (you should know the answers to these questions)

Question 1 What does it mean to say that $h(t)$ is the impulse response of a (linear time-invariant) system.

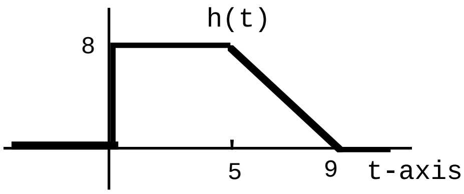
Answer 1 It means that if the input into the system when it is initially at rest is $\delta(t)$ then the response of the system at time t is $h(t)$.

Question 2 If h(t) is the impulse response of a system then what is the significance of the convolution $h(t)*f(t)$.
Answer 2 It's the response of the system at time t if it is initially at rest and then gets input f(t).

PROBLEMS FOR SECTION 2.3

1. Draw the graph of h(t−u) in a u,y coord system and find the equations of the various pieces.

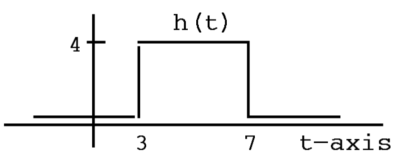
(a)

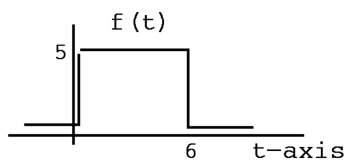


(b)

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$$

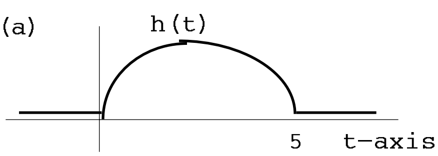
2. The diagram shows the impulse response h(t) of a system. Find the response of the system to input f(t) if the system is initially at rest. And sketch the response when you get it.
(The solution flips h.)

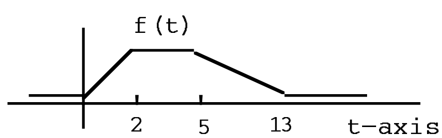




3. Suppose a system has impulse response h(t). If the system is initially at rest, when does its response to the input f(t) die out (if ever).

(a)





(b) $h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$ and f(t) is the same as in part (a)

4. Repeat example 1 but flip the f(u) this time instead of h(u).

5. Suppose a system is initially at rest.

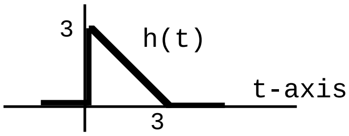
Suppose $f(t)*g(t) = c(t)$.

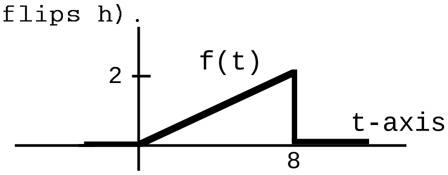
Fill in the blanks. (There are two possible answers. Give them both.

If the system has impulse response _________________ then input _____________

produces output __________________ .

6. The diagram shows the impulse response h(t) of a system, and a function f(t). The problem is to find the response y(t) of the system to input f(t) if the system is initially at rest (the solution flips h).



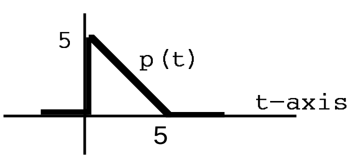


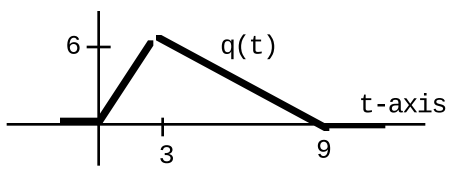
7. (a) Look at a system in which inputs f(t) and outputs y(t) are related by $2y'' + 8y' + 6y = f(t)$.
Let h(t) be the system's impulse response.
(You don't have to find h(t). Give the answers in terms of h(t).)

(a) Suppose the system is initially a rest.
What is the response of the system to input f(t).
(b) Suppose the system is *not* initially at rest.
In particular suppose $y(0) = 7$, $y'(0) = 9$.
What is the response of the system to input f(t).

8. (a) Given the functions p(t) and q(t) in the diagram. Find their convolution but stop before computing the integrals. Just set them up. (There are lots of cases.)
(b) Suppose p(t) is the impulse response of a system. What does this mean physically.

(c) If p(t) is the impulse response of a system, what does the convolution that you computed in part (a) represent physically.





9. The last footnote on page 2 of this section tried to show why the second convolution integral in (1) should give the same result as the first. Show again that the two integrals in (1) are equal using ordinary substitution from calculus.

10. If the impulse response of a system is

$$h(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$$

find the response, when initially at rest, to

(a) input $f(t) = \begin{cases} 6 & \text{if } 0 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$

(b) input $\delta(t)$

11. Find $f(t) * g(t)$ if

(a) $g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$ and $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \ge 0 \end{cases}$

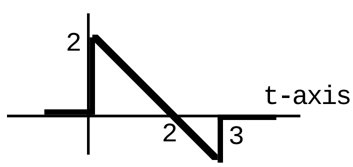
(b) $g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \cos t & \text{if } t \ge 0 \end{cases}$ and $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \sin t & \text{if } t \ge 0 \end{cases}$

Use the identity $\sin x \cos y = \dfrac{\sin(x+y) + \sin(x-y)}{2}$.

12. Find $f(t)*f(t)$ if

(a) $f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{t} & \text{if } t \ge 0 \end{cases}$ (b) $f(t) = \begin{cases} a & \text{if } -b \le x \le b \\ 0 & \text{otherwise} \end{cases}$

13. The diagram shows the output of a system when the input is $\delta(t)$ and the system is initially at rest.



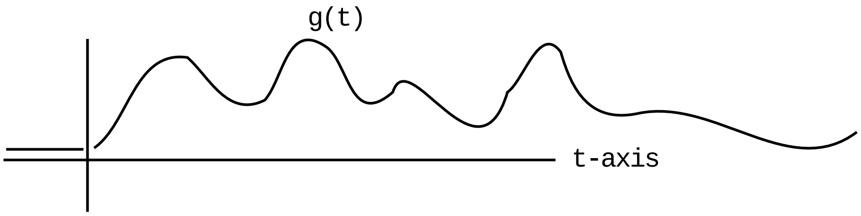
Find the response of the system to the input

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$$

when the system is initially at rest and find the steady state response.

HONORS

14. Start with an arbitrary function $g(t)$ for $t \geq 0$.



Find the convolution $\delta(t) * g(t)$

You can't actually compute the convolution integral the way you do an ordinary integral because $\delta(t)$ is not an ordinary function.
You'll have to do it some other way.

Suggestion You can do it by inspection by thinking about what $\delta(t)*g(t)$ represents physically for some hypothetical system.
And explain how you got your answer.

15. Suppose the impulse response of a system is $\delta(t)$. In other words, if the input is $\delta(t)$ then the output is also $\delta(t)$.

Now use input $f(t)$. Explain (clearly, logically, grammatically, briefly) why the system's output is $f(t)$.

In other words, if the system is a copy–cat when you put in $\delta(t)$, explain why it is a copy–cat when you put in anything else.

summary These two results are equivalent.
(1) $\delta(t)*f(t) = f(t)$
(2) If the impulse response is $\delta(t)$ then the initially–at–rest system is a copy–cat.

(2) is an immediate corollary of (1): If the impulse response is $\delta(t)$ then the response to $f(t)$ is $\delta(t)*f(t)$ which is $f(t)$, by (1).
(1) follows from (2): Since $h(t) = \delta(t)$, the response to $f(t)$ is $\delta(t)*f(t)$. So $\delta(t)*f(t) = f(t)$.
My second method for proving (1) essentially proves (2) first and then gets back to (1).

16. (this was an exam problem)
The inputs $f(t)$ and outputs $y(t)$ of a system are related by

$$2y'' + 8y = f(t)$$

(a) Find the impulse response of the system.
(b) Solve

$$2y'' + 8y = \frac{t^5 \tan t}{1 + t^2} \quad \text{with IC } y(0) = 2,\ y'(0) = 3.$$

But assume you have a computer that can do any convolution so that your answer is allowed to contain unevaluated convolutions, e.g., your answer could look like

$$\frac{(e^t \sin t) * \cos t + t^4}{t^2 + t^3 * t^4} .$$

In other words, your answer must be a specific function of t but it can contain unevaluated convolutions.

REVIEW PROBLEMS FOR CHAPTER 2

1. A system's input $f(t)$ and output $y(t)$ are related by $4y'' + y = f(t)$.

(a) Find the impulse response.
(b) Find the response of the system to $\delta(t-11)$ with IC $y(0) = 0$, $y'(0) = 0$.

2. Let

$$h(t) = \begin{cases} -2t + 6 & \text{if } 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the response of an initially-at-rest system to input $f(t)$ if the system has impulse response $h(t)$.

CHAPTER 3 LINEAR RCURRENCE RELATIONS (DIFFERENCE EQUATIONS)

SECTION 3.1 INTRODUCTION

examples of recurrence relations (rr) with initial conditions (IC)

Suppose $y_1, y_2, y_3, \ldots, y_n, \ldots$ is a sequence of numbers beginning with

(1) $$y_1 = 1, \quad y_2 = -1$$

and satisfying the condition

(2) $$y_n = 5y_{n-1} - 6y_{n-2}$$

or equivalently

(2') $$y_{n+2} = 5y_{n+1} - 6y_n$$

The two equivalent versions in (2) and (2') say that

any term in the sequence = 5 × preceding term − 6 × pre-preceding term.

Then

(3)
$$\begin{aligned} y_3 &= 5y_2 - 6y_2 = 5(-1) - 6(1) = -11 \\ y_4 &= 5y_3 - 6y_2 = 5(-11) - 6(-1) = -49 \\ y_5 &= 5y_4 - 6y_3 = 5(-40) - 6(-11) = -179 \\ &\text{etc.} \end{aligned}$$

The conditions in (1) are called initial conditions (IC) and the equation in (2) is called a recurrence relation (rr) or a difference equation (ΔE).

Given a rr with IC, the sequence is determined and you can write as many successive terms as you like. The aim of the topic is to find a formula for the n-th term y_n. This process is called solving the rr. For example we will soon show that the solution to the rr in (2) with the IC in (1) is $y_n = -3^n + 2 \cdot 2^n$.

linear recurrence relations with constant coefficients

A rr of the form

(5) $$ay_{n+2} + by_{n+1} + cy_n = f_n$$

is called a linear second order rr with constant coefficients . The function f_n is called the *forcing function*. The unknown (to be solved for) is y_n, the n-th term of the sequence.

If f_n is 0 then the rr is called *homogeneous*.

The rr in (5) is called *second* order because it takes *two* IC to get the sequence started, i.e., because (5) describes how to get a term of the sequence from the *two* preceding terms.

For example,

$$3y_n + y_{n-4} = 0$$

is a homog 4−th order linear rr; it says that

any term = $-\frac{1}{3}$ times the pre-pre-pre-preceding term.

The rr

$$y_{n+10} = y_{n+5} - y_{n+4}$$

can be rewritten as

$$y_{n+6} = y_{n+1} - y_n$$

and is 6-th order (not 10-th order).

Until IC are specified, a rr has many solutions. A *general solution* to an n-th order rr is a solution containing n arbitrary constants (to ultimately be determined by n IC).

example 1
Show that

$$y_n = -3n^2 - n - 2$$

is a solution to the (nonhomog) rr

$$y_{n+2} - y_{n+1} - 6y_n = 18n^2 + 2$$

Substitute the supposed solution into the lefthand side of the rr to see if it works.

$$\text{LHS} = \underbrace{-3(n+2)^2 - (n+2) - 2}_{y_{n+2}} - \underbrace{[-3(n+1)^2 - (n+1) - 2]}_{y_{n+1}} - 6\underbrace{[-3n^2 - n - 2]}_{y_n}$$

$$= -3(n^2 + 4n + 4) - n - 2 - 2 - [-3(n^2 + 2n + 1) - n - 1 - 2] - 6[-3n^2 - n - 2]$$

$$= 18n^2 + 2$$

YEs, that does equal the righthand side so $-3n^2 - n - 2$ *is* a solution

example 2
Let

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2,$$

i.e, S_n is the sum of the first n squares. Suppose you want a formula for S_n. You know that

$$S_{n+1} = \underbrace{1^2 + 2^2 + \ldots + n^2}_{S_n} + (n+1)^2$$

so

$$S_{n+1} = S_n + (n+1)^2 \tag{6}$$

The (nonhomog) rr in (6) together with the IC $S_1 = 1$ determines S_n and later in the chapter you"ll be able to solve the rr and find the formula for S_n

superposition rule

If u_n is a solution of $ay_{n+2} + by_{n+1} + cy_n = f_n$

and v_n is a solution of $ay_{n+2} + by_{n+1} + cy_n = g_n$

then

$u_n + v_n$ is a solution of $ay_{n+2} + by_{n+1} + cy_n = f_n + g_n$

ku_n is a solution of $ay_{n+2} + by_{n+1} + cy_n = kf_n$

proof of the u + v rule

Assume that when substituted into $ay_{n+2} + by_{n+1} + cy_n$, u_n produces f_n and v_n produces g_n. Substitute $u_n + v_n$ to see what happens:

$$a(u_{n+2} + v_{n+2}) + b(u_{n+1} + v_{n+1}) + c(u_n + v_n)$$

$$= \underbrace{au_{n+2} + bu_{n+1} + cu_n}_{f_n \text{ by hypothesis}} + \underbrace{av_{n+2} + bv_{n+1} + cv_n}_{g_n \text{ by hypothesis}} = f_n + g_n \quad \text{QED}$$

special case of superposition for homog recurrence relations

A constant multiple of a solution to a homog rr is also a sol.
The sum of sols to a homog rr is also a solution.

In particular if u_n and v_n are sols to

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

then a general solution is

$$Au_n + Bv_n$$

linear recurrence relations and physical systems

A linear rr such as $ay_{n+2} + by_{n+1} + cy_n = f_n$ often describes some physical system with inputs and outputs appearing at discrete time intervals (e.g., every minute): n represents time, f_n is an input at time n and y_n is the system's response. The constants a,b,c are "ingredients" of the system, such as mass, resistance etc. A rr with constant coeffs corresponds to a time invariant system whose ingredients don't change with time.

The superposition rule for the rr means that the response to a sum of inputs is the sum of the separate responses and tripling an input for instance will triple the response. Linear rr's corresponds to systems where physical superposition holds.

complex superposition

If

$$ay_{n+2} + by_{n+1} + cy_n = f_n + ig_n$$

has sol

$$u_n + iv_n$$

then u_n is a solution of $ay_{n+2} + by_{n+1} + cy_n = f_n$

and v_n is a solution of $ay_{n+2} + by_{n+1} + cy_n = g_n$

In other words, the real part of the sol goes with the real part of the forcing function and the imag part of the sol goes with the imag part of the forcing function.

proof

Suppose $u_n + iv_n$ is a sol to $ay_{n+2} + by_{n+1} + cy_n = f_n + ig_n$ Then

$$a(u_{n+2} + iv_{n+2}) + b(u_{n+1} + iv_{n+1}) + c(u_n + iv_n) = f_n + ig_n$$

Collect terms:

$$au_{n+2} + bu_{n+1} + cu_n + i(av_{n+2} + bv_{n+1} + cv_n) = f_n + ig_n$$

The left side can't equal the right side unless the real parts are equal and the imag parts are equal. So

$$au_{n+2} + bu_{n+1} + cu_n = f_n \quad \text{and} \quad av_{n+2} + bv_{n+1} + cv_n = g_n$$

So

u_n is a sol to $ay_{n+2} + by_{n+1} + cy_n = f_n$

and

v_n is a sol to $ay_{n+2} + by_{n+1} + cy_n = g_n$ QED

special case of complex superposition for homog recurrence relations

If

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

has complex sol

$$u_n + iv_n$$

then u_n and v_n individually are real sols.

In other words, the real and imag parts of a complex homog sol are real homog sols.

PROBLEMS FOR SECTION 3.1

1. The Fibonacci sequence is defined by the recurrence relation

$$y_{n+2} = y_{n+1} + y_n \quad \text{with IC} \quad y_0 = 0,\ y_1 = 1$$

Find y_2, y_3, y_4, y_5

2. Write the recurrence relation for the sum S_n of the first n integers. In other words, if $S_n = 1 + 2 + 3 + \ldots + n$, write a rr (plus IC) satisfied by S_n.

3. Find the order of the rr $6y_{n+7} - 2y_{n+5} + 3y_{n+4} = 0$; i.e., how many IC do you need to get started.

4. Suppose $y_{n+2} - 2y_n = n^3$ and $y_1 = 2$, $y_2 = -3$. Find y_3, y_4, y_5.

5. Substitute to show that $y_n = n2^n$ satisfies the ΔE $y_{n+2} - 4y_{n+1} + 4y_n = 0$

6. If u_n and v_n are sols of $3y_{n+4} + 5y_{n+1} - 2y_n = \sin \pi n$ then what are the following sols of.

(a) $u_n + v_n$ (b) $3u_n$ (c) $u_n - v_n$

7. If u_n and v_n are solutions to $ay_{n+2} + by_{n+1} + cy_n = 0$, what are the following solutions of. (a) $u_n + v_n$ (b) $6u_n$ (c) $u_n - v_n$

8. Rewrite the recurrence relation $S_{n+1} = S_n + (n+1)^2$ (from example 2) so that it involves

(a) S_n and S_{n-1} instead of S_{n+1} and S_n

(b) S_{n+6} and S_{n+5} instead of S_{n+1} and S_n

SECTION 3.2 HOMOGENEOUS RECURRENCE RELATIONS

finding the general sol to a second order homog rr

To solve

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

or equivalently to solve

$$ay_n + by_{n-1} + cy_{n-2} = 0$$

first find the roots of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

The solution to the rr depends on the type of roots so there are cases.

case 1 (real unequal roots)

If $\lambda = \lambda_1, \lambda_2$ then $y_n = A\lambda_1^n + B\lambda_2^n$

case 2 (repeated real roots)

If $\lambda = \lambda_1, \lambda_1$ then $y_n = A\lambda_1^n + B\underline{\underline{n}}\lambda_1^n$ (step up by n)

case 3 (non-real roots, which can only occur in conjugate pairs)

If $\lambda = a \pm bi$, then the gen *complex* solution is $y_n = A\lambda_1^n + B\lambda_2^n$.
To get the general *real* sol, find the mag r and angle θ of either root, say of $a + bi$. Then a general *real* sol is

$$y_n = r^n(A\cos n\theta + B\sin n\theta)$$

For homework problems and exams it is always intended that you give *real* solutions unless specifically stated otherwise.

example 1

If $\lambda = -2, 5$ then $y_n = A(-2)^n + B\,5^n$.

If $\lambda = 2, 1$ then $y_n = A2^n + B\,1^n = A2^n + B$.

Remember that $1^n = 1$

If $\lambda = 2, 2$ then $y_n = A2^n + Bn2^n$.

Suppose $\lambda = 3 \pm 3i$. The number $3 + 3i$ has magnitude $3\sqrt{2}$ and angle $\pi/4$ and a general (real) sol is

$$y_n = (3\sqrt{2})^n\left(C\cos\frac{n\pi}{4} + D\sin\frac{n\pi}{4}\right)$$

semiproof

To get solutions to $ay_{n+2} + by_{n+1} + cy_n = 0$ try $y_n = \lambda^n$ to see what values of λ, if any, make it work. We have

$$a\lambda^{n+2} + b\lambda^{n+1} + c\lambda^n = 0 \quad \text{(substitute)}$$

$$a\lambda^2 + b\lambda + c = 0 \quad \text{(divide by } \lambda^n\text{)}$$

So the sols λ to the characteristic equ $a\lambda^2 + b\lambda + c = 0$ determine solutions λ^n to the homog rr.

case 1 Suppose $\lambda = -2,5$. Then $(-2)^n$ and 5^n are sols. By superposition, $A(-2)^n + B\,5^n$ is a general sol.

case 2 Suppose $\lambda = 2,2$. Then 2^n is a solution. It can be proved (but it takes a while) that another sol is $n2^n$. Then by superposition, a gen sol is $A2^n + Bn2^n$.

case 3 Suppose $\lambda = 3 \pm 3i$. Then as in the other cases, $(3 + 3i)^n$ and $(3 - 3i)^n$ are (complex) sols and $A(3 + 3i)^n + B(3 - 3i)^n$ is a gen (complex) solution. To get *real* sols, use the complex superposition principle and take the real and imag parts of the complex solutions $(3 + 3i)^n$ and $(3 - 3i)^n$. First write the complex solutions so that their real and imag parts are evident.

$3 + 3i$ has mag $3\sqrt{2}$ and angle $\frac{\pi}{4}$

$3 - 3i$ has mag $3\sqrt{2}$ and angle $-\frac{\pi}{4}$

By DeMoivre's rule (Section 1.3, page 3),

$(3 + 3i)^n$ has mag $(3\sqrt{2})^n$ and angle $\frac{n\pi}{4}$

$(3 - 3i)^n$ has mag $(3\sqrt{2})^n$ and angle $-\frac{n\pi}{4}$

So

$$(3 + 3i)^n = (3\sqrt{2})^n \left(\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}\right)$$

$$(3 - 3i)^n = (3\sqrt{2})^n \left(\cos\frac{n\pi}{4} - i \sin\frac{n\pi}{4}\right)$$

The real parts are both $(3\sqrt{2})^n \cos\frac{n\pi}{4}$ and the imag parts are $\pm\, (3\sqrt{2})^n \sin\frac{n\pi}{4}$. This gives three real solutions but only two "independent" sols, namely

$$(3\sqrt{2})^n \cos\frac{n\pi}{4} \quad \text{and} \quad (3\sqrt{2})^n \sin\frac{n\pi}{4}$$

By superposition for homog rr, a general (real) sol is

$$C\,(3\sqrt{2})^n \cos\frac{n\pi}{4} + D\,(3\sqrt{2})^n \sin\frac{n\pi}{4} \qquad \text{QED}$$

example 2

Find a general solution to $y_{n+2} + 3y_{n+1} + 2y_n = 0$.

We have $\lambda^2 + 3\lambda + 2 = 0$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda = -2, -1$$

So

$$y_n = A(-2^n) + B(-1)^n$$

example 3

Find a gen solution to $y_{n+2} + 4y_n = 0$.

The characteristic equ is $\lambda^2 + 4 = 0$ so $\lambda = \pm 2i$.

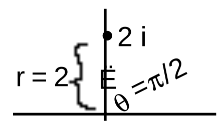


FIG 1

The number 2i has mag 2 and angle $\pi/2$ (Fig 1) so

$$y_n = 2^n \left(A \cos \frac{n\pi}{2} + B \sin \frac{n\pi}{2}\right)$$

warning

1. If $\lambda = \pm 2i$ then the gen real homog sol is $2^n \left(A \cos \frac{n\pi}{2} + B \sin \frac{n\pi}{2}\right)$ WITHOUT an i in it.

Don't confuse this with the fact that the polar form of $(2i)^n$ is $2^n\left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right)$ WITH an i.

2. If $\lambda = -2$ then the solution includes $(-2)^n$. Don't mistakenly write this as -2^n which actually means $-(2^n)$.

3. The characteristic equ for $y_{n+2} + 4y_n = 0$ is $\lambda^2 + \boxed{4} = 0$, *not* $\lambda^2 + \boxed{4\lambda} = 0$.

example 4

Find the general solution to a homog rr if $\lambda = -1 \pm i\sqrt{3}$

The number $-1 + i\sqrt{3}$ (Fig 2) has

$$r = \sqrt{1 + 3} = 2 \text{ and } \theta = \arctan[-1,\sqrt{3}] = \frac{2\pi}{3}$$

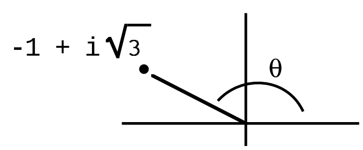


FIG 2

So

$$y_n = 2^n \left(A \cos \frac{2n\pi}{3} + B \sin \frac{2n\pi}{3}\right)$$

warning

In example 4, θ is not $\arctan \frac{\sqrt{3}}{-1}$ which is $-\frac{\pi}{3}$.

See arctan[x,y] versus arctan y/x in Section 1.3.

finding the gen solution to a homog rr of any order

If a 3rd order homog rr has a characteristic equ with roots

$$\lambda = 2, 3, -4$$

then the general sol is

$$y_n = A2^n + B3^n + C(-4)^n$$

If a 5th order homog rr has a characteristic equ with roots

$$\lambda = 2,\ 2,\ 2,\ 2,\ 5$$

then a gen sol is

$$y_n = A2^n + Bn2^n + Cn^2\,2^n + Dn^3\,2^n + E5^n \quad \text{(keep stepping up by n)}$$

If a first order homog rr has $\lambda = 4$ then a gen sol is $y_n = A4^n$

Suppose a 4-th order homog rr has

$$\lambda = \pm 3i,\ \pm 3i$$

The number 3i has $r = 3$, $\theta = \frac{\pi}{2}$ so

$$y_n = 3^n\left(A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2}\right) + n3^n\left(C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2}\right) \quad \text{(step up by n)}$$

solving a homog rr with IC

First find a general solution. Then plug in the IC to determine the constants.

For example consider

$$y_{n+2} - 5y_{n+1} + 6y_n = 0 \quad \text{with IC } y_1 = 1,\ y_2 = -1$$

We have

$$\lambda^2 - 5\lambda + 6 = 0, \quad (\lambda-3)(\lambda-2) = 0, \quad \lambda = 3,2$$

So a gen sol is

$$y_n = A3^n + B2^n$$

Plug in $y_1 = 1$ (i.e., set $n = 1$, $y_n = 1$) to get

$$1 = 3A + 2B$$

Plug in $y_2 = -1$ (i.e., set $n = 2$, $y_n = -1$) to get

$$-1 = 9A + 4B$$

Solve the system of two equations in A and B to get $A = -1$, $B = 2$. Then the final sol is

$$y_n = -3^n + 2\cdot 2^n$$

warning

If you are solving a homog differential equation and $m = -2 \pm 2i$ then

$$y = e^{-2x}\,(A\cos 2x + B\sin 2x)$$

But if you are solving a homog recurrence relation and $m = -2 \pm 2i$ then $r = \sqrt{8}$, $\theta = \frac{3\pi}{4}$ and

$$y_n = (\sqrt{8})^n\left(A\cos\frac{3n\pi}{4} + B\sin\frac{3n\pi}{4}\right)$$

If $m = -2,-4$ and it's a differential equation then $y = Ae^{-2x} + Be^{-4x}$ but if it's a recurrence relation then $y_n = A(-2)^n + B(-4)^n$

Don't mix up the two types of problems

PROBLEMS FOR SECTION 3.2

1. Find a general solution.

(a) $y_{n+2} - 3y_{n+1} - 10y_n = 0$ (b) $y_{n+2} + 3y_{n+1} - 4y_n = 0$

(c) $2y_{n+2} + 2y_{n+1} - y_n = 0$ (d) $y_n + 3y_{n-1} - 4y_{n-2} = 0$

2. Solve $y_{n+2} + 2y_{n+1} - 15y_n = 0$ with IC $y_0 = 0$, $y_1 = 1$.

3. Given $y_{n+2} - y_{n+1} - 6y_n = 0$ with $y_0 = 1$, $y_1 = 0$.

(a) Before doing any solving, find y_3.
(b) Now solve and find a formula for y_n.
(c) Use the formula from part (b) to find y_3 again, as a check.

4. Find a general (real) sol.

(a) $y_{n+2} + 2y_{n+1} + 2y_n = 0$ (b) $y_{n+2} + y_{n+1} + y_n = 0$

5. If $y_0 = 0$, $y_1 = 2$ and $y_{n+2} + 4y_{n+1} + 8y_n = 0$ find y_{102}.

6. If the auxiliary equation of a homog linear rr with constant coeffs has the following roots, find a general (real) solution.
(a) $-3,4,4,\ -\sqrt{3} \pm i$ (b) $1,\ \pm 2,\ 3,\ \pm 2i,\ \pm 2i$

7. The Fibonacci sequence begins with $y_0 = 0$, $y_1 = 1$ and from then on each term is the sum of the two preceding terms. Find a formula for y_n.

8. Suppose a sequence begins with 2,5 and then each term is the average of the two preceding terms.
(a) Find the fifth term by working your way out to it.
(b) Find a formula for the n-th term.
(c) Find the fifth term again using the formula from part (b).

9. If the characteristic equation of a homog rr has the following roots, find a general sol
(a) $-3,4,4$ (b) $5,5,5,5,2$ (c) $1,1,1,6,-7$

10. Find a general sol to $y_{n+2} + 6y_{n+1} + 9y_n = 0$

11. Go backwards and find a rr with the general sol $y_n = A + Bn + C2^n$.

12. Solve $y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} = 0$ with $y_1 = 0$, $y_2 = 1$, $y_3 = 0$.

13. Solve by inspection and then solve again (overkill) with the methods of this section.

(a) $y_{n+1} - y_n = 0$ with $y_1 = 4$

(b) $ay_{n+2} + by_{n+1} + cy_n = 0$ with $y_o = 0$, $y_1 = 0$

14. Suppose $y_1 = 5$, $y_2 = 7$ and thereafter each term is the average of the two surrounding terms.

(a) Write out some terms and see if you can find a formula for y_n by guessing.

(b) Find a formula for y_n by solving a rr.

SECTION 3.3 NONHOMOGENEOUS RECURRENCE RELATIONS

finding the general solution to a nonhomog rr

Look at

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

Let h_n be the *general homog solution* (i.e., the sol to $ay_{n+2} + by_{n+1} + cy_n = 0$). This is found with the method of the preceding section.

Let p_n be any *particular nonhomog sol* (i.e., sol, with no constants, to the given nonhomog rr). This section will show you how to do this.

Then

$$\boxed{y_n = h_n + p_n \text{ is a } \textit{general} \text{ nonhomog sol}}$$

The same idea works for a nonhomog rr of any order.

proof

$h_n + p_n$ is a solution by superposition: h_n produces 0 and p_n produces f_n so the sum produces $0 + f_n$. Furthermore $h_n + p_n$ is a *general* solution because it contains the necessary arbitrary constants in the h_n part.

finding a particular nonhomog solution

Consider

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

I want to find a particular solution, denoted p_n. There are several cases.

case 1 f_n is a constant

Suppose $f_n = 6$. Try $p_n = A$

Substitute the trial p_n into the rr to determine A

case 2 f_n is a polynomial

Suppose $f_n = 7n^3 + 2n$ (a cubic). Try $p_n = An^3 + Bn^2 + Cn + D$ (a cubic not missing any terms even though f_n was missing a few)

Substitute the trial p_n into the rr to determine A,B,C,D.

Similarly if $f_n = 3n^2 + 4n + 1$ (quadratic) then try $p_n = An^2 + Bn + C$

etc.

case 3 f_n is an exponential

Suppose $f_n = 9 \cdot 2^n$. Then try $p_n = A2^n$

Substitute the trial p_n into the rr to determine A *case 4* f_n is a sine or cosine

Suppose f_n is $4 \sin \pi n$ or $4 \cos \pi n$.

method 1 Switch to

$$ay_{n+2} + by_{n+1} + cy_n = 4e^{\pi in}$$

Try $p_n = Ae^{\pi in}$ and substitute into the switched rr to determine A.

Take the imag part if $f_n = 4 \sin \pi n$. Take the real part if $f_n = 4 \cos \pi n$

method 2 Don't switch at all and try $p_n = A \cos \pi n + B \sin \pi n$.

example 1

Find a general solution to $y_{n+2} - 5y_{n+1} + 6y_n = 4^n$.

First find h_n:

$$\lambda^2 - 5\lambda + 6 = 0, \ (\lambda-2)(\lambda-3) = 0, \ \lambda = 2,3,$$

$$h_n = B\,2^n + C\,3^n$$

Then try

$$p_n = A\,4^n$$

Substitute into the rr to see what value of A will make it work. You need

$$A\,4^{n+2} - 5A\,4^{n+1} + 6A\,4^n = 4^n$$

Rewrite the left side to display all the 4^n terms:

$$A\,4^2\,4^n - 5A\cdot 4\cdot 4^n + 6A\;4^n = 4^n$$

$$16A\,4^n - 20A\,4^n + 6A\,4^n = 4^n$$

$$2A\,4^n = 4^n$$

You need

$$2A = 1, \quad A = \frac{1}{2}\ .$$

So

$$p_n = \frac{1}{2}\,4^n$$

The general solution is

$$y_n = h_n + p_n \ = B\,2^n + C\,3^n + \frac{1}{2}\,4^n$$

solving a nonhomog rr with IC

First find a general solution (i.e., $h_n + p_n$) and THEN plug in the IC to determine the contants in the gen solution.

example 2

Solve $y_{n+2} - 5y_{n+1} + 6y_n = n$ with IC $y_0 = 1$, $y_1 = 2$

solution First get the homogeneous solution:

$$\lambda^2 - 5\lambda + 6 = 0,\ \lambda = 2,3, \quad h_n = A\,2^n + B\,3^n$$

Now try

$$p_n = Cn + D$$

Then

$$p_{n+1} = C(n+1) + D$$

$$p_{n+2} = C(n+2) + D$$

Substitute into the rr to get

$$C(n+2) + D - 5\big(C(n+1) + D\big) + 6(Cn + D) = n$$
$$2Cn + 2D - 3C = n$$

Now match the coeffs of corresponding terms on each side.
The n coeffs must be equal so $2C = 1$
The constant terms must be equal so $2D - 3C = 0$
So $C = \frac{1}{2}$, $D = \frac{3}{4}$,

$$p_n = \frac{1}{2}n + \frac{3}{4}$$

The general sol is

$$y_n = h_n + p_n = A\,2^n + B\,3^n + \frac{1}{2}n + \frac{3}{4}$$

To get $y_0 = 1$ you need

$$1 = A + B + \frac{3}{4}$$

To get $y_1 = 2$ you need

$$2 = 2A + 3B + \frac{1}{2} + \frac{3}{4}$$

Solve:

$$A = 0, \quad B = \frac{1}{4}$$

Final answer is

$$y_n = \frac{1}{4}3^n + \frac{1}{2}n + \frac{3}{4}$$

warning

1.If the forcing function is n or 5n or $-6n$ try $p_n = An + B$, not just n. Similarly if the forcing function is $3n^2$ or $n^2 + 3$ or $9n^2 + n$, try $p_n = An^2 + Bn + C$, a quadratic *not* missing any terms.

2. Determine the various constants at the appropriate stage. For a nonhomog rr with IC, first find h_n (containing constants) . Then find p_n (the trial p_n contains constants but they must be immediately determined to get the genuine p_n). The general solution is $y_n = h_n + p_n$ (contains constants via the h_n part). Use the IC to determine the constants in the gen sol. *Don't use the IC on h_n alone in the middle of the problem* .

example 3

Solve $y_{n+2} + 2y_{n+1} - 3y_n = 10 \sin \frac{n\pi}{2}$ with IC $y_0 = 2$, $y_1 = 9$

We have $\lambda^2 + 2\lambda - 3 = 0$, $\lambda = -3, 1$, $h_n = P(-3)^n + Q$

method 1 for p_n

Switch to

$$y_{n+2} + 2y_{n+1} - 3y_n = 10\, e^{\frac{1}{2}n\pi i}$$

and try

$$p_n = D\, e^{\frac{1}{2}n\pi i}$$

Substitute into the switched rr:

$$De^{\frac{1}{2}(n+2)\pi i} + 2D\, e^{\frac{1}{2}(n+1)\pi i} - 3D\, e^{\frac{1}{2}n\pi i} = 10\, e^{\frac{1}{2}n\pi i}$$

Rewrite the exponentials:

$$D\, \underbrace{e^{\pi i}}_{-1}\, e^{\frac{1}{2}n\pi i} + 2D\, \underbrace{e^{\frac{1}{2}\pi i}}_{i}\, e^{\frac{1}{2}n\pi i} - 3D\, e^{\frac{1}{2}n\pi i} = 10\, e^{\frac{1}{2}n\pi i}$$

Collect terms and find D:

$$(-4 + 2i)D = 10, \qquad D = \frac{10}{-4 + 2i} = -2 - i$$

So for the switched rr,

$$p_n = (-2 - i)\, e^{\frac{1}{2}n\pi i} = (-2-i)(\cos \tfrac{1}{2}n\pi + i \sin \tfrac{1}{2}n\pi)$$

Take the imag part to get the particular sol for the original rr

$$p_n = -\cos \tfrac{1}{2}n\pi - 2 \sin \tfrac{1}{2}n\pi$$

method 2 for p_n

Try

$$p_n = A \sin \frac{n\pi}{2} + B \cos \frac{n\pi}{2}$$

Substitute into the rr:

$$A \sin \tfrac{1}{2}\pi(n+2) + B \cos \tfrac{1}{2}\pi(n+2) + 2\left[A \sin \tfrac{1}{2}\pi(n+1) + B \cos \tfrac{1}{2}\pi(n+1) \right]$$

$$- 3\left[A \sin \tfrac{1}{2}\pi n + B \cos \tfrac{1}{2}\pi n \right] = 10 \sin \tfrac{1}{2}n\pi$$

Expand the sines and cosines:

$$A\left[\sin\frac{n\pi}{2}\underbrace{\cos\pi}_{-1}+\cos\frac{n\pi}{2}\underbrace{\sin\pi}_{0}\right]+B\left[\cos\frac{n\pi}{2}\underbrace{\cos\pi}_{-1}-\sin\frac{n\pi}{2}\underbrace{\sin\pi}_{0}\right]$$

$$+\,2A\left[\sin\frac{n\pi}{2}\underbrace{\cos\frac{\pi}{2}}_{0}+\cos\frac{n\pi}{2}\underbrace{\sin\frac{\pi}{2}}_{1}\right]$$

$$+\,2B\left[\cos\frac{n\pi}{2}\underbrace{\cos\frac{\pi}{2}}_{0}-\sin\frac{n\pi}{2}\underbrace{\sin\frac{\pi}{2}}_{1}\right]$$

$$-\,3A\sin\frac{n\pi}{2}-3B\cos\frac{n\pi}{2}$$

$$=\ 10\sin\tfrac{1}{2}n\pi$$

Collect terms:

$$(2A-4B)\cos\frac{n\pi}{2}+(-4A-2B)\sin\frac{n\pi}{2}=10\sin\tfrac{1}{2}n\pi$$

Match the coefficients and solve for A and B:

$$2A-4B=0,\quad -4A-2B=10,\quad A=-2,\quad B=-1,$$

$$p_n=-2\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}$$

Finally

$$y_n=h_n+p_n=P(-3)^n+Q-2\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}$$

To get $y_0=2$ you need $2=P+Q-1$

To get $y_1=9$ you need $9=-3P+Q-2$

So $P=-2$, $Q=5$. Answer is

$$y_n=-2(-3)^n+5\ -2\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}$$

stepping up p_n

Look at

$$ay_{n+2}+by_{n+1}+cy_n=f_n$$

There are some exceptions to the rules about what to try for p_n

case 1 Suppose $f_n=6$. Ordinarily you try $p_n=A$.
But if A is already a homog sol (i.e., if one of the λ's is 1) try $p_n=An$ (step up).
If A and n are both homog sols (which happens when $\lambda=1,1$) try $p_n=An^2$
If A,n,n^2 are all homog sols (i.e., $\lambda=1,1,1$) try $p_n=An^3$ etc.

case 2 Suppose $f_n = 6n^2 + 3$. Ordinarily you try $p_n = An^2 + Bn + C$.
But if C is a homog sol (i.e, one of the λ's is 1) then try

$$p_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn.$$

If C and n are both homog sols (i.e., λ = 1,1) try

$$p_n = n^2(An^2 + Bn + C) = An^4 + Bn^3 + Cn^2$$

case 3 Suppose $f_n = 9 \cdot 2^n$. Ordinarily you try $p_n = A\,2^n$.

But if 2^n is a homog sol (i.e., one of the λ's is 2) try $p_n = An\,2^n$.
If 2^n and $n\,2^n$ are both homog sols (λ = 2,2) try $p_n = An^2\,2^n$.

case 4 Suppose $f_n = 4\cos\frac{\pi}{3}n$ (similarly for $f_n = 4\sin\frac{\pi}{3}n$).
If you're not using the complex exponential then ordinarily you would try

$$p_n = A\cos\frac{\pi}{3}n + B\sin\frac{\pi}{3}n$$

If you use the complex exponential method and switch to

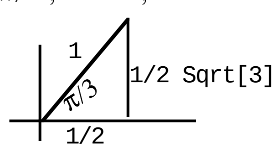
$$ay_{n+2} + y_{n+1} + cy_n = 4e^{n\pi i/3}$$

then ordinarily you would try

$$p_n = Ae^{n\pi i/3} \quad \text{(and eventually take the imag part)}$$

If $\cos\frac{\pi}{3}n$ and $\sin\frac{\pi}{3}n$ are homog sols

footnote This happens if one of the λ's is non-real with mag 1 and angle π/3, i.e., if the λ's are $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$.



try

$$p_n = n(A\cos\frac{\pi}{3}n + B\sin\frac{\pi}{3}n) \quad \text{for the real method}$$

or

$$p_n = Ane^{n\pi i/3} \quad \text{for the complex method.}$$

example 4

Solve $y_{n+2} - 4y_{n+1} + 4y_n = 6 \cdot 2^n$ with $y_0 = 0$, $y_1 = 0$.

Start with $\lambda^2 - 4\lambda + 4 = 0$, $\lambda = 2,2$, $h_n = A\,2^n + Bn\,2^n$

Ordinarily with forcing function $6 \cdot 2^n$ you try $p_n = C\,2^n$. But since 2^n and $n\,2^n$ are homog sols, step up to

$$p_n = Cn^2\,2^n$$

Then

$$p_{n+1} = C(n+1)^2\, 2^{n+1} = C(n^2 + 2n + 1)\ 2\cdot 2^n = 2C(n^2 + 2n + 1)\ 2^n$$

$$p_{n+2} = C(n+2)^2\ 2^{n+2} = C(n^2 + 4n + 4)\ 2^2\ 2^n = 4C(n^2 + 4n + 4)\ 2^n$$

Substitute into the rr:

$$4C(n^2 + 4n + 4)2^n \ - \ 4\cdot 2C(n^2 + 2n + 1)2^n + \ 4\cdot Cn^2\ 2^n \ = 6\cdot 2^n$$

Collect terms and equate coeffs:

$$8C\ 2^n = 6\cdot 2^n, \quad C = \frac{3}{4}$$

So

$$p_n = \frac{3}{4}\, n^2\ 2^n$$

and a gen sol is

$$y_n = A\ 2^n + Bn\ 2^n + \frac{3}{4}\, n^2\ 2^n$$

To get $y_0 = 0$ and $y_1 = 0$ you need $0 = A,\ 0 = 2A + 2B + \frac{3}{2}, \quad A = 0,\ B = -\frac{3}{4}$.

Final answer is

$$y_n = -\frac{3}{4}\ n\, 2^n + \frac{3}{4}\, n^2\, 2^n$$

particular solution for a sum forcing function

Look at

$$ay_{n+2} + by_{n+1} + cy_n = n^2 + 3^n$$

Find p_n's separately for

$$ay_{n+2} + by_{n+1} + cy_n = n^2$$

and

$$ay_{n+2} + by_{n+1} + cy_n = 3^n$$

and add them (because of the superposition rule).

Equivalently, try $p_n = An^2 + Bn + C + D\,3^n$

particular solution for some product forcing functions

(1) If $f_n = 6n^2\cdot 3^n$ try $p_n = (An^2 + Bn + C)3^n$

If 3^n is a homog sol step up to $p_n = n(An^2 + Bn + C)3^n$

If 3^n and $n3^n$ are both homog sols step up to $p_n = n^2(An^2 + Bn + C)3^n$ etc.

(2) Suppose $f_n = n^2 \sin n\pi$ (similarly for $n^2 \cos n\pi$) .

One method is to try

$$p_n = (An^2 + Bn + C)(D\cos n\pi + E\sin n\pi)$$

$$= (Fn^2 + Gn + H)\cos n\pi + (Pn^2 + Qn + R)\sin n\pi$$

But if $\sin n\pi$ and $\cos n\pi$ are homog sols then step up and try

$$p_n = n(Fn^2 + Gn + H)\cos n\pi + n(Pn^2 + Qn + R)\sin n\pi$$

The second method is to try

$$p_n = (An^2 + Bn + C)e^{\pi i n}$$

and take the imag part. But if $\sin n\pi$ and $\cos n\pi$ are homog sols then step up and try

$$p_n = n(An^2 + Bn + C)e^{\pi i n}$$

(3) Suppose $f_n = 2^n \sin\frac{n\pi}{2}$ (similarly for $2^n \cos\frac{n\pi}{2}$)

One method is to try

$$p_n = 2^n\left(A\sin\frac{n\pi}{2} + B\cos\frac{n\pi}{2}\right)$$

But if $2^n \sin\frac{n\pi}{2}$ and $2^n\cos\frac{n\pi}{2}$ are homog sols then step up and try

$$p_n = n2^n\left(A\sin\frac{n\pi}{2} + B\cos\frac{n\pi}{2}\right)$$

Another method is to try

$$p_n = A\,2^n\, e^{\frac{1}{2}n\pi i}$$

and take the imag part. But if $2^n\sin\frac{n\pi}{2}$ and $2^n\cos\frac{n\pi}{2}$ are homog sols then step up and try

$$p_n = A\,n2^n\, e^{\frac{1}{2}n\pi i}$$

recurrence relations not included in this chapter

1. Linear rr's with *variable* coeffs such as $\boxed{n^3}\, y_{n+2} + \boxed{n}\, y_{n+1} + 6y_n = 8n^2$.

Superposition rules still hold but the idea of solving a characteristic equ to get h_n doesn't apply anymore. It is *not* correct to solve the "characteristic equ" $n^3\lambda^2 + n\lambda + 6 = 0$ for λ and use $h_n = A\lambda_1{}^n + B\lambda_2{}^n$

Furthermore, trying p_n of a certain standard form doesn't necessarily work when the coeffs are variable.

2. Linear rr's with constant but *non-real* coefficients
The complex superposition rule doesn't hold in this case.

3. *Non*-linear rr's such as $\boxed{y_{n+2}\, y_n}$ + y_n = 5 or $\boxed{y_{n+2}^2}$ + $y_{n+1} - y_n$ = 2n.

In this case, superposition doesn't hold. Even if you could get h_n and p_n (which you can't), the gen sol would not be $y_h + y_p$

The methods of this chapter are only for $ay_{n+2} + by_{n+1} + cy_n = f_n$ (plus similar equations of higher or lower order) where a,b,c are real constants.

PROBLEMS FOR SECTION 3.3

1. Given $y_n - 2y_{n-1} = 6n$ with $y_1 = 2$

(a) Find y_4 recursively by first finding y_2 and y_3

(b) Find a formula for y_n

(c) Use the formula from (b) to find y_4 again as a check

(d) Rewrite the equation so that it involves y_{n+1} and y_n instead of y_n and y_{n-1}

2. Solve (a) $y_{n+2} - y_{n+1} - 2y_n = 1$ with IC $y_1 = 1$, $y_2 = 3$

(b) $y_{n+2} + 2y_{n+1} - 15y_n = 6n + 10$ with IC $y_0 = 1$, $y_1 = -\frac{1}{2}$

3. Find a general sol to $y_{n+2} - 3y_{n+1} + y_n = 10\cdot 4^n$

4. Solve $y_{n+2} - y_{n+1} - 6y_n = 18n^2 + 2$ with $y_0 = -1$, $y_1 = 0$

5. (a) Find a particular solution to $y_{n+2} - 2y_n = 5\cos n\pi$

(b) Oops, $\cos n\pi$ is 1 if n is even and 0 if n is odd so it equals $(-1)^n$.
So the forcing function in part (a) is $5(-1)^n$.
Find p_n in part (a) again from this new point of view.

(c) Find a particular $y_{n+1} - 2y_n = 10\sin\frac{n\pi}{2}$

6. Given the following forcing functions and roots of the characteristic equ.
What p_n would you try

	forcing function f_n	λ's
(a)	$n^4 + 2n$	$\pm i$
(b)	$n^4 + 2$	1,1,1,1,3
(c)	$6\cdot 2^n$	2, 6
(d)	$6\cdot 2^n$	3, 6
(e)	3^n	3, 3
(f)	$5\cos\frac{n\pi}{2}$	$\pm i$
(g)	$5\cos\frac{n\pi}{2}$	$\pm 2i$

7. Solve $2y_{n+1} - y_n = (\frac{1}{2})^n$ with $y_1 = 2$

8. Solve $y_{n+2} - 2y_{n+1} + y_n = 1$ with $y_0 = 1$, $y_1 = \frac{1}{2}$

9. Let S_n be the sum of the first n squares, i.e.,

$$S_n = 1^2 + \ {}^2 + \ldots + n^2$$

Find a formula for S_n by writing a recurrence relation plus IC and solving it.

10. (a) For $y_{n+2} - 3y_{n+1} + 2y_n = 6 \cdot 2^n$ you have $h_n = A2^n + B$ so for p_n you should step up and try $p_n = Cn\,2^n$. What happens if you forget to step up and try $p_n = A\,2^n$

(b) For $2y_{n+2} + 3y_{n+1} + 4y_n = 18n$ you should try $p_n = An \underline{\underline{+ B}}$. What happens if you violate warning 1 and try $p_n = An$

11. Solve $y_{n+1} + 2y_n = 3 + 4^n$ with $y_0 = 2$

12. Find a general real sol to $y_{n+4} - 16y_n = n + 3^n$

13. Solve $y_{n+2} - y_{n+1} + y_n = 2^n$ with IC $y_0 = 1$, $y_1 = 3$

14. Solve $y_{n+2} - 3y_{n+1} + 2y_n = 8n \cdot 3^n$ with IC $y_0 = -16$, $y_1 = -40$

REVIEW PROBLEMS FOR CHAPTER 3

1. Find a general solution to $y_{n+2} - 9y_n = 56n^2$.

2. Find a gen sol to $y_{n+2} - 2y_{n+1} + 4y_n = 0$.

3. Solve $2y_{n+1} + 4y_n = 6 \cdot 7^n$ with IC $y_1 = 5$.

4. Find a general solution to $y_{n+2} + 5y_{n+1} - y_n = 6$.

5. Find a formula for the sum of the first n integers .

(Let $S_n = 1 + 2 + \ldots + n$, find a rr and IC for S_n and solve.)

6. Find a gen sol $y_{n+2} - 9y_n = 5 \cdot 3^n$.

7. Suppose you want to solve the rr

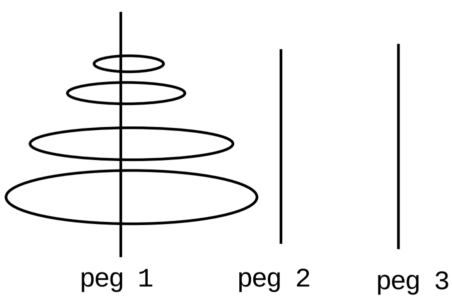
$$y_{n+2} - 2y_{n+1} = 0.$$

Following the rules you would solve $\lambda^2 - 2\lambda = 0$ and get $\lambda = 0,2$ so the gen sol is

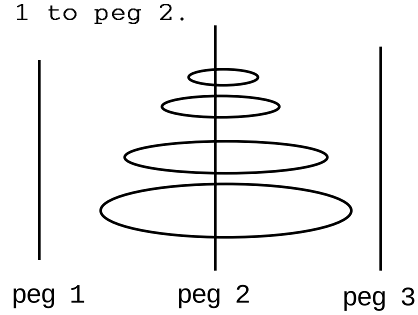
$$y_n = A\,0^n + B2^n = B2^n$$

And you suddenly lost one of your two constants (which you would need if you were going to satisfy two IC). You've always led a good clean life. How could something like this happen to you and what are you going to do about it.

8. (The tower of Hanoi) The game begins with n rings in increasing size on peg 1. The idea is to transfer them all to peg 2 but never place a larger ring on top of a smaller ring at any stage of the game. Rings may be moved temporarily to peg 3 (the storage peg) as they eventually go from peg 1 to peg 2.



GAME BEGINS



GAME ENDS

The problem is to find the minimum number of moves it takes.

Let y_n be the min number of moves required in a game with n rings; i.e., y_n is the min number of moves it takes to transfer n rings from a first peg to a second peg when you have a third peg available for storage.

(a) Write a recurrence relation for y_n and find IC.

(b) Solve the rr from part (a).

CHAPTER 4 SOME FIRST ORDER DIFFERENTIAL EQUATIONS

SECTION 4.1 LINEAR FIRST ORDER WITH NOT NECESSARILY CONSTANT COEFFICIENTS

solution to y' + P(x)y = Q(x)

A typical first order linear DE has the form

$$ay' + by = f(x).$$

If a and b are constants then the methods of the preceding sections work for y_h and y_p and the gen solution is $y_h + y_p$

There is another method which not only works in the case of constant a and b but also works if they are not constant.

> To solve
>
> $$ay' + by = f(x)$$
>
> divide by a to get the form
>
> $$y' + P(x)y = Q(x)$$
>
> Find $\int P(x)\,dx$ and let
>
> $$I = e^{\int P(x)\,dx}$$
>
> Then the solution for y is given by
>
> (1) $\quad Iy = \int IQ\,dx$
>
> Add an arbitrary constant when you do this integral
>
> To finish up, solve for y by dividing by I

Don't bother inserting an arbitrary constant when you find $\int P(x)\,dx$ (if you do it will only cancel out later anyway -- see problem #3(a)). But do put one in when you find $\int IQ\,dx$. Otherwise you won't get a *general* solution.

proof

Take the equation

$$y' + P(x)y = Q(x)$$

and multply on both sides by an as-of-yet undetermined function I(x), called an integrating factor:

(*) $\quad I(x)y' + I(x)P(x)y = I(x)Q(x).$

The lefthand side $I(x)y' + I(x)P(x)y$ would be the derivative of the product $I(x)y(x)$ if you had

(**) $\quad I(x)P(x) = I'(x)$

So if I(x) is chosen to satisfy (**) then the DE in (*) becomes

$$(Iy)' = IQ$$

and its solution is

$$Iy = \int IQ\,dx$$

To finish up you need a function I(x) satisfying (**), i.e., you need a function whose derivative equals the original function times P(x). One such function is

$$I(x) = e^{\int P(x)\,dx};$$

This works because its derivative is $e^{\int P(x)\,dx}$ *times* the derivative of $\int P(x)\,dx$, i.e., its derivative is the original function times P(x).

example 1

To find a general solution to

(2) $\quad xy' - 3y = x^5$ (first order linear, *variable* coeffs)

rearrange to get

(3) $\quad y' - \frac{3}{x}\,y = x^4$

Then

(4) $\quad P(x) = -\frac{3}{x}, \quad \int P(x)\,dx = -3 \ln x$

$$I = e^{\int P(x)\,dx} = e^{-3 \ln x} = e^{\ln x^{-3}}$$ **review** $\ln x^a = a \ln x$

$$= x^{-3}$$ **review** $e^{\ln a} = a$

Now use (1):

(5) $\quad \frac{y}{x^3} = \int \frac{x^4}{x^3}\,dx = \int x\,dx = \frac{1}{2}x^2 + K$

> **warning**
> 1. Q(x) is x^4 from line (3), not x^5 from line (2).
> 2. The arbitrary constant must be inserted *here*. It's wrong to leave it out completely and it's wrong it casually insert it at some later stage. Do it *now*.

Final solution is

$$y = \frac{1}{2}x^5 + Kx^3$$

warning

Make sure that the coeff of y' is 1 before going into the P,Q routine.

example 1 continued

I'll check that the solution is correct.

Let $y = \frac{1}{2}x^5 + Kx^3$. Find $xy' - 3y$ to see if it comes out to be x^5:

$$xy' - 3y = x\left(\frac{5}{2}x^4 + 3Kx^2\right) - 3\left(\frac{1}{2}x^5 + Kx^3\right)$$
$$= \frac{5}{2}x^5 + 3Kx^3 - \frac{3}{2}x^5 - 3Kx^3$$
$$= x^5 \quad \text{QED}$$

warning about mathematical style

To do this check in example 1, i.e., to show that $y = \frac{1}{2}x^5 + Kx^3$ satisfies $xy' - 3y = x^5$, it is neither good style nor good logic to write like this.

Don't write like this

$$xy' - 3y = x^5$$
$$x\left(\frac{5}{2}x^4 + 3Kx^2\right) - 3\left(\frac{1}{2}x^5 + Kx^3\right) = x^5$$
$$\frac{5}{2}x^5 + 3Kx^3 - \frac{3}{2}x^5 - 3Kx^3 = x^5$$
$$x^5 = x^5 \text{ TRUE!}$$

Don't write like this

First of all, it is silly keep repeating the x^5's on the righthand side; the essence of the argument is in the lefthand sides where $xy' - 3y$ turned into x^5.
And any "proof" in mathematics that *begins* with what you want to prove and *ends* with something TRUE, like $x^5 = x^5$ (or B = B or 0 = 0) is at best badly written and at worst incorrect and *drives me crazy*.

With this "method" I can prove that 3 = 4:

3 = 4
4 = 3
7 = 7 (add)
TRUE!!!

So conclude that 3 = 4 ??????

What you *should* do to check that $xy' - 3y$ equals x^5 is work on one of them until it turns into the other or work on each one *separately* until they turn into the same thing. Don't write $xy' - 3y = x^5$ as the *first* line of your proof. It should be your *last* line, as in my example 1 continued.

example 2

Find a gen solution to

$$y' + 2y = e^{3x} \quad \text{(first order linear, } \textit{constant} \text{ coeffs)}$$

method 1 You can use the methods of Chapter 1 since the DE has constant coeffs.

$m + 2 = 0$, $m = -2$

$y_h = Ae^{-2x}$.

Try

$y_p = Be^{3x}$

Substitute into the DE to get

$$3Be^{3x} + 2Be^{3x} = e^{3x}$$

Equate coeffs of e^{3x}: $5B = 1$, $B = 1/5$

So $y_{gen} = y_h + y_p = Ae^{-2x} + \frac{1}{5}e^{3x}$

method 2 You can also use the method of this section:

$$P(x) = 2,\ Q(x) = e^{3x},\ \int P(x)\,dx = 2x,\ I = e^{2x}.$$

By (1),

$$ye^{2x} = \int e^{5x}\,dx = \frac{1}{5}e^{5x} + K$$

$$y = Ke^{-2x} + \frac{1}{5}e^{3x}$$

example 3

The DE

$$xy'' + y' = 0$$

is second order when the unknown is the function y but if you consider the unknown to be y' then it is first order. Rearrange to get

$$(y')' + \frac{1}{x}(y') = 0.$$

So

$$\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln x, \quad I = e^{\ln x} = x$$

and by (1),

$$xy' = \int x\cdot 0\,dx = K,$$

$$y' = \frac{K}{x}.$$

Finally, antidifferentiate to get

$$y = K \ln x + C$$

PROBLEMS FOR SECTION 4.1

1. Solve

(a) $(x + 2)^2 y' + 4(x + 2)y = -6$

(b) $y' = x - 4xy$

(c) $2y' + 2y = e^{2x}$

(d) $y' - y \cot x = \csc x$ For reference: $\int \cot x\,dx = \ln \sin x$

2. The DE $xy'' + y' = 4x$ is second order with variable coeffs but if you consider y' to be the unknown then it is linear first order.
Solve it for y' first and then find y.

3. (a) Solve problem #1(b) again and this time insert an arbitrary constant when you find $\int P(x)\,dx$ (you're entitled). What happens?

(b) What happens if you solve problem #1(b) and you leave out the arbitrary constant when you find $\int IQ\,dx$

4. Solve $y' = ky$ two ways (k is a fixed constant).

5. Solve $y' + y = e^{-x}$ with IC $y(-1) = 3$.

6. (a) Why is it not quite legal to say that if K is an arbitrary constant then e^K is just another arbitrary constant and can be renamed C.
(b) Is it OK to turn $\ln K$ into a new arbitrary constant called C.

7. Solve $xy' + 2y = x^2 + 1$.

8. Let

$$f(x) = \begin{cases} x & \text{for } x \le 3 \\ 0 & \text{for } x \ge 3 \end{cases}$$

Solve $y' - \frac{1}{x}y = f(x)$ with condition $y(1) = 2$ and make the solution continuous.

Honors

9. Let $y(t)$ be the fish population in a lake at time t. If the fish are left alone then the population grows at a rate proportional to the size of the population, i.e., $y'(t) = ry(t)$ where r is a positive constant.

If the fish are harvested at the constant rate of h fish per unit of time (where h is a positive constant) then the differential equation becomes

$$y'(t) = ry(t) - h$$

Suppose there are N fish initially.

(a) Find $y(t)$.
(b) Let $N = 40$, $r = 2$ specifically. For what values of h does the lake get fished out.
(c) Continue from part (b) with $N = 40$ and $r = 2$. One of those values of h for which the lake gets fished out is $h = 100$. For this value of h, *when* does the lake get fished out.

SECTION 4.2 SEPARABLE DIFFERENTIAL EQUATIONS

the algebra of arbitrary constants

If A and B are arbitrary constants then so are A + B, 3A, A−B, AB etc. and may be re−named C_1, C_2, C_3 etc.

separating variables

One way to solve

(1) $$y' = \frac{x}{y^2}$$

is to rewrite the equation as

(2) $$y^2(x)\ y'(x) = x$$

and antidifferentiate on both sides with respect to x to get

(3) $$\int y^2(x)\ y'(x)\ dx = \int x\ dx$$

(4) $$\frac{1}{3} y^3(x) = \frac{1}{2} x^2 + C$$

(An antideriv of $y^2(x)y'(x)$ is $\frac{1}{3} y^3(x)$. Differentiate it back, using the chain rule, to see.)

The procedure in (1)−(4) is usually written in the following more convenient style:

(1A) $$\frac{dy}{dx} = \frac{x}{y^2}$$

(2A) $$y^2\ dy = x\ dx$$

(3A) $$\int y^2\ dy = \int x\ dx$$

(4A) $$\frac{1}{3} y^3 = \frac{1}{2} x^2 + C$$

In (4A) there's an arbitrary constant on one side only because if you put in two constants you get

$$\frac{1}{3} y^3 + A = \frac{1}{2} x^2 + B$$

which reduces to (4A) anyway when you let C = B−A.

So far the solution y has been found *implicitly* in (4'A). The *explicit* solution is

(5) $$y = \sqrt[3]{\frac{3}{2} x^2 + 3C}$$

or equivalently

(6) $$y = \sqrt[3]{\frac{3}{2} x^2 + D}$$

To check the solution in (6) find y' and x/y^2 to see if they are equal:

$$y' = \frac{1}{3}\left(\frac{3}{2} x^2 + D\right)^{-2/3} \cdot 3x$$

$$\frac{x}{y^2} = \frac{x}{\left(\frac{3}{2} x^2 + D\right)^{2/3}}$$

They came out equal so the solution checks out.

In general:

> If it's possible to separate variables so that the DE has the form
>
> x-stuff dx = y-stuff dy
>
> (as in (2')) then the DE is called separable and is solved by antidiffing on both sides and inserting an arbitrary constant on one side.

Only first order DE, that is, equations involving y' but not y'', y''' etc., can be separated.

The separation process usually leads to an *implicit* solution for y. If it is feasible to solve for y *explicitly*, do it, but otherwise settle for an implicit solution.

The solution will contain one arbitrary constant and is called the general solution. If you are given some condition then the constant can be determined to get the specific solution satisfying the DE plus condition.

warning

The variables must be separated before this method of integrating w.r.t. y on one side and w.r.t. x on the other side can be used. The DE

$$y' = \frac{2x}{x + 3y^2}$$

can be written as

$$(x + 3y^2)\ dy = 2x\ dx$$

but there is no way to continue and separate the variables. The DE can't be solved by the method in this section.

Here's a WRONG way to try to solve it. Write the DE as

(7) $\displaystyle\int (x + 3y^2)\ dy = \int 2x\ dx$ dangerous to have x's on the dy side or y's on the dx side

(8) $xy + y^3 = x^2$ WRONG

It's wrong to go from (7) to (8) because in (7), y is a function of x, dy is $y'(x)dx$, and the left side of (7) is an abbreviation for $\int \left[x + 3y^2(x)\right]\ y'(x)\ dx$. It does *not* equal $xy + y^3$ because $xy+y^3$ differentiates back to $xy' + y + 3y^2y'$, *not* to $x+3y^2$.

warning

Don't wait until the end of the problem to insert an arbitrary constant. At line (4A) don't write

$$\frac{1}{3}\ y^3 = \frac{1}{2}\ x^2,$$

then solve for y to get

$$y = \sqrt[3]{\frac{3}{2}\ x^2}$$

and *then* (too late) insert the arbitrary constant to get

$$y = \sqrt[3]{\frac{3}{2}\ x^2} + C \qquad \text{WRONG WRONG}$$

The constant must be inserted *at the antidifferentiation step*, not later.

antiderivative for 1/x

The usual choice is

(9) $$\int \frac{1}{x}\,dx = \ln x + C$$

but it is also true that

(10) $$\boxed{\int \frac{1}{x}\,dx = \ln Kx}$$

because

$$D \ln Kx = \frac{1}{Kx} \cdot K = \frac{1}{x}$$

Here's another way to see (10):

$\ln x + C = \ln x + \ln K$ (rewrite the arbitrary constant)

$= \ln Kx$ (log algebra)

I think the version in (10) is often more useful than (9).

example 1

Use separation to find the general solution to $w'(t) = 2 - \frac{1}{5}w(t)$.

solution

$$\frac{dw}{dt} = 2 - \frac{1}{5}w$$

$$\frac{dw}{2 - \frac{1}{5}w} = dt$$

Now antidiff and use either (9) or (10).

version 1 (better)

$-5 \ln K(2 - \frac{1}{5}w) = t$ (antidiff and use (10) to insert the constant)

$\ln K(2 - \frac{1}{5}w) = -\frac{1}{5}t$ (Divide by -5)

$K(2 - \frac{1}{5}w) = e^{-t/5}$ (take exp)

$2 - \frac{1}{5}w = Ae^{-t/5}$ (divide by K and let 1/K be renamed A)

$w = 10 - Be^{-t/5}$ (solve for w and let 5A be renamed B)

version 2

$-5 \ln(2 - \frac{1}{5}w) = t + K$ (antidiff and use (9) to insert the constant)

(11) $\ln(2 - \frac{1}{5}w) = -\frac{1}{5}t + C$ (let $-\frac{1}{5}K$ be renamed C)

$2 - \frac{1}{5}w = e^{-t/5 + C}$ (take exp on both sides)

$2 - \frac{1}{5}w = e^{-t/5}\, e^{C}$ (rule of exponents)

$2 - \frac{1}{5}w = Be^{-t/5}$ (let e^C be renamed B)

$w = 10 - De^{-t/5}$ (solve for w and let 5B be renamed D)

warning

When you take exp on both sides of (11) it's *wrong* to get

$$(2 - \tfrac{1}{5}w) = e^{-t/5} \quad \text{PLUS} \quad e^C \quad \text{WRONG} \quad \text{WRONG}$$

which turns into

$$(2 - \tfrac{1}{5}w) = e^{-t/5} + A$$

You *should* have

$$2 - \tfrac{1}{5}w = e^{-t/5} + C \qquad \text{RIGHT}$$

which turns into

$$2 - \tfrac{1}{5}w = e^{-t/5} \text{ TIMES } e^C = Ae^{-t/5}$$

If you use version 1 you won't run the risk of this mistake.

example 2

Solve $y'(t) = -\frac{1}{3}y(t)$ with IC $y(0) = 150$

solution

$$\frac{dy}{dt} = -\frac{1}{3}y$$

$$\frac{dy}{y} = -\frac{1}{3}dt$$

$$\ell n\ Ky = -\frac{1}{3}t$$

$$Ky = e^{-t/3}$$

$$y = \frac{1}{K}e^{-t/3}$$

(12) $$y = Ce^{-t/3}$$

To determine the specific solution satisfying the IC, set $y = 150$, $t = 0$ in (8) to get

$$150 = Ce^0, \qquad C = 150.$$

The final answer is $y= 150\ e^{-t/3}$

example 3

Find a general solution to $y' = 4xy + 3x$

solution

$$\frac{dy}{dx} = x(4y + 3)$$

$$\frac{dy}{4y + 3} = x\,dx$$

$$\frac{1}{4}\ \ell n\ K(4y + 3) = \frac{1}{2}x^2$$

$$\ell n\ K(4y + 3) = 2x^2 \quad \text{(by (10))}$$

$$K(4y + 3) = e^{2x^2}$$

$$4y + 3 = Ae^{2x^2}$$

$$y = \frac{Ae^{2x^2} - 3}{4}$$

warning

It is true that if A and B are arbitrary constants then A+B = C (i.e., A+B is just another arbitrary constant). And 1/A = C. And $e^A = C$. But it is not true that $Ae^{2x^2} = C$. An arbitrary constant can't swallow x-stuff.

orthogonal families

The equation

$$x^2 + 3y^2 = K,$$

where K is an arbitrary constant, describes a family of ellipses. I'll find the family of curves orthogonal to the ellipse family.

step 1 Go backwards from the family of ellipses to the differential equation for the family: Differentiate w.r.t. x on both sides of the equation and remember to treat y as y(x).

$$2x + 6yy' = 0$$

$$y' = -\frac{x}{3y} \tag{13}$$

For each point (x,y), the differential equation in (13) give sthe slope of the curve in the family that passes through that point. For example, the ellipse in the family that passes through point (7,2) has slope −7/6 at that point.

step 2 Find the differential equation for the orthogonal family.

The slopes on the orthogonal family should be the negative reciprocals of the slopes on the original family. So the orthogonal family satisfies the DE

$$y' = \frac{3y}{x}\ . \tag{14}$$

step 3 Solve the DE in (14).

$$\frac{dy}{y} = 3\ \frac{dx}{x}$$

$$\ell n\ KY = 3\ \ell n\, x$$

$$\ell n\ Ky = \ell n\, x^3$$

$$Ky = x^3$$

$$y = Ax^3$$

This is the equation of the orthogonal family. Fig 1 shows some of the members in each family.

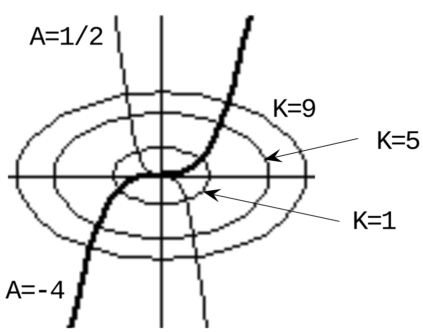


FIG 1

PROBLEMS FOR SECTION 4.2

1. Find a general solution if the equation is separable.

(a) $y' = -x \sec y$ (b) $dx + x^3 y\, dy = 0$ (c) $x^2 + y^4 \dfrac{dy}{dx} = 0$

(d) $y' = \dfrac{y}{2x + 3}$ (e) $x^2\, dy = e^y\, dx$ (f) $y' = \dfrac{5x + 3}{y}$

(g) $y' = \dfrac{y}{x + y}$ (h) $y' = \dfrac{1}{xy + x}$

2. Take your solutions to #1(e) and (f) and check that they really satisfy the differential equations.

3. Find the particular solution satisfying the given condition.

(a) $y' = xy$ with $y(1) = 3$

(b) $yy' + 5x = 3$ with $y(2) = 4$

(c) $y' \dfrac{e^y}{x} = 3$ with $y(0) = 2$

(d) $y' = y^4 \cos x$ with $y(0) = 2$

4. The DE $w'(t) = 2 - \frac{1}{5} w(t)$ in example 1 is not only separable but also first order linear. Solve it again.

5. If y is implicitly given by $xy^2 = y + 7$, find y explicitly.

6. You have some radioactive stuff which is decaying at a rate proportional the amount there, where the constant of proportionality is 10. In particular, if $y(x)$ is the amount of stuff at time x then $y' = -10y$.
(a) If you start with G grams, at what time will you have only G/2 grams left.
(b) If you would like your initial G grams to decay to G/2 grams by time 3, you should start with new radioactive stuff with what constant of proportionality instead of 10.

7. Find the orthogonal family and draw a picture.
(a) $xy = K$
(b) $y = Ax^2$
Suggestion: Before you differentiate w.r.t. x on both sides of the equation of the family, isolate the arbitrary constant so that it will differentiate away.

SECTION 4.3 EXACT DIFFERENTIAL EQUATIONS

the differential of a function

(1-dim version) Suppose $y = f(x)$ and x changes by dx producing a corresponding change in y. The differential of f is defined by

$$dy = f'(x)dx.$$

It was shown in calculus that the differential approximates the change in y.

(2-dim version) Suppose $z = f(x,y)$ and x changes by dx, y changes by dy producing a corresponding change in z. The differential of f is defined by

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{1}$$

It was shown in calculus that the differential approximates the corresponding change in z.

Mathematicians use the notation Δz for the change in z and use dz for the differential in (1) which approximates the change in z. Outside of pure mathematics the distinction between (1) and Δz is blurred and often both are referred to as dz.

example 1

Let $z = x^2y^3$. Then

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = 2xy^3\,dx + 3x^2y^2\,dy$$

meaning that if x and y change by dx and dy respectively there is a corresponding change in z given approximately by $2xy^3\,dx + 3x^2y^2\,dy$

example 2

To find $d(3q^2)$ use the one−dimensional differential formula $dy = f'(x)\,dx$ to get $d(3q^2) = 6q\,dq$.

sum, product, quotient and chain rules for differentials

Let u and v be functions of one or more variables. Then

$$d(u + v) = d(u) + d(v) \tag{2}$$

$$d(uv) = u\,d(v) + v\,d(u) \tag{3}$$

$$d\left(\frac{u}{v}\right) = \frac{v\,d(u) - u\,d(v)}{v^2} \tag{4}$$

$$d[f(u)] = f'(u)d(u) \tag{5}$$

For example, by (5),

$$d(\ell n\ u) = \frac{1}{u}d(u),$$

$$d(\sin u) = \cos u\ d(u).$$

A differential can always be found directly using (1) but sometimes (2)−(5) are more convenient. For example, by (1),

$$d\ \ell n(2x + 3y) = \frac{\partial \ell n(2x+3y)}{\partial x}\,dx + \frac{\partial \ell n(2x+3y)}{\partial y}\,dy$$

$$= \frac{2}{2x + 3y}\,dx + \frac{3}{2x + 3y}\,dy.$$

But also

$$d\ \ell n(2x + 3y) = \frac{1}{2x + 3y}\, d(2x + 3y) \quad \text{(by (5))}$$

$$= \frac{1}{2x + 3y}(2\ dx + 3\ dy) \quad \text{(by (2))}$$

$$= \frac{2\ dx + 3\ dy}{2x + 3y}$$

For example, By (1),

$$d\ \sin x^2y^3 = \frac{\partial \sin(x^2y^3)}{\partial x}\, dx + \frac{\partial \sin(x^2y^3)}{\partial y}\, dy$$

$$= 2xy^3 \cos x^2y^3\ dx + 3x^2y^2 \cos x^2y^3\ dy$$

And also

$$d\ \sin x^2y^3 = \cos x^2y^3\ d(x^2y^3) \quad \text{by (5)}$$

$$= \cos x^2y^3\ (x^2\ 3y^2)\ dy + 2xy^3\ dx) \quad \text{by (3).}$$

exact differentials

Example 1 began with a function $f(x,y) = x^2y^3$ and found $df = 2xy^3\,dx + 3x^2y^2\,dy$. To identify and solve exact differential equations you have to consider the opposite problem: given the differential expression $2xy^3\,dx + 3x^2y^2\,dy$, find a function $f(x,y)$ with that differential. In general, an expression of the form

(6) $$p(x,y)\ dx + q(x,y)\ dy$$

is called a *differential form*. It is possible (in fact, likely) that (6) simply is not df for any f.

> If there does exist a function $f(x,y)$ such that
>
> (7) $$df = p(x,y)\ dx + q(x,y)\ dy$$
>
> then the differential $p\ dx + q\ dy$ is called *exact*.

> In other words $p\ dx + q\ dy$ is exact if there is an $f(x,y)$ such that
>
> (8) $$\frac{\partial f}{\partial x} = p(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = q(x,y)$$

For example, consider the differential

(9) $$\underbrace{(3x^2y^2 + 2y^3 + x)}_{p}\ dx + \underbrace{(2x^3y + 6xy^2 + \cos y + 7)}_{q}\ dy$$

The problem is to find $f(x,y)$, if possible, so that (7) and (8) hold. Begin by antidifferentiating p with respect to x:

(10) $$\text{tentative } f = x^3y^2 + 2xy^3 + \tfrac{1}{2}x^2$$

The derivative w.r.t. y of this tentative answer is

(11) $2x^3y + 6xy^2$.

Compare this with q in (9). Since (11) is missing the terms cos y + 7, fix up (10) by adding sin y + 7y to get

$$\text{better } f = x^3y^2 + 2xy^3 + \frac{1}{2}x^2 + \sin y + 7y.$$

Now it has the correct partial w.r.t. y. Note that fixing up the answer like this does not change its partial derivative w.r.t x since *the additional terms do not contain the variable x*. So the final answer, including the standard arbitrary constant, is

$$f(x,y) = x^3y^2 + 2xy^3 + \frac{1}{2}x^2 + \sin y + 7y + C$$

You can check the answer by finding its partials to see that you do get p and q.

example 3

Let

(12) $p = 3x^2y^2 + 2y^3$ and $q = 2x^3y + 6xy^2 + 8xy^3$.

Try, but find it impossible, to obtain an f such that df = p dx + q dy. In other words, show that p dx + q dy is *not* exact.

solution Antidifferentiate p to get

(13) $x^3y^2 + 2xy^3$

Differentiate this tentative answer w.r.t. y to get

(14) $2x^3y + 6xy^2$

and compare it with q. The term $8xy^3$ is missing from (14) and can be produced only if you expand (13) to $x^3y^2 + 2xy^3 + 2xy^4$. But the extra term $2xy^4$ *contains the variable x* so when you differentiate the expanded tentative answer w.r.t. x you no longer get the desired p. So it is not possible to find a function f with partials p and q: the differential p dx + dy is not exact.

a criteria for exactness

Given p dx + q dy, one way to decide if there exists an f such that df = p dx + q dy holds is to simply try to find it as in the preceding examples. It is also possible to get a test for determining *in advance* if an f exists. Then the antidifferentiating process for *finding* f need be used only when the criterion guarantees the *existence* of f. I'll find the criterion and then use it in examples.

If (7) holds then

$$\frac{\partial f}{\partial x} = p \quad \text{and} \quad \frac{\partial f}{\partial y} = q$$

so

$$\frac{\partial q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

and so $$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

In more advanced courses, the converse can be proved: if $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ then (7) holds.

So here's the criterion:

(15)	If $\frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y}$ then $p\,dx + q\,dy$ is not exact
(16)	If $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ then $p\,dx + q\,dy$ is exact

example 3 repeated

To see if

$$(3x^2y^2 + 2y^3)\,dx + (2x^3y + 6xy^2 + 8xy^3)\,dy$$

is exact, find

$$\frac{\partial q}{\partial x} = 6x^2y + 6y^2 + 8y^3, \quad \frac{\partial p}{\partial y} = 6x^2y + 6y^2.$$

The two are not identical so, by (15), the differential is not exact.

exact differential equations

Consider the equation

$$y' = \frac{2x - y^3}{3xy^2}$$

Then

$$\frac{dy}{dx} = \frac{2x - y^3}{3xy^2},$$

$$3xy^2\,dy = (2x - y^3)\,dx.$$

The equation is not separable so try a second approach. Write the equation as

$$\underbrace{(y^3 - 2x)}_{p}\,dx + \underbrace{3xy^2}_{q}\,dy = 0 \tag{17}$$

Since

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \qquad \text{(both are } 3y^2\text{)}$$

the left side of (17) is an exact differential df. To find f, antidifferentiate p w.r.t. x to get the terms

$$xy^3 - x^2.$$

The derivative w.r.t. y of this tentative f is $3xy^2$, precisely q, so the tentative f is final: the differential equation can be written as

$$d(xy^3 - x^2) = 0$$

Since the differential is 0, if x changes by dx and y changes by dy, the function $xy^3 - x^2$ itself does not change. Therefore it is a constant function.

In general, *f(x,y) is constant if and only if df = 0*, analogous to the 1-dim rule that f'(x) is constant if and only if f'(x) = 0. So

$$xy^3 - x^2 = K$$

where K is an arbitrary constant, and this describes an *implicit* solution to the original diff equ. The *explicit* solution is found by solving for y to get

$$y = \sqrt[3]{\frac{x^2 + K}{x}}$$

Here's the overall idea.

(18) Consider the differential equation

$$p\, dx + q\, dy = 0 \quad \text{where } \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

The left side of the DE is an exact differential, the equation is called exact, and there is a function $f(x,y)$ such that the equation can be written as $df = 0$. The solution $y(x)$ to the differential equation is given implicitly by

$$f(x,y) = K\ .$$

An explicit solution is found by solving the implicit solution for y, if possible.

(19) More generally, if a differential equation can be written as

$$df = dg$$

(rather than as $df = 0$) then its solution is given implicitly by

$$f(x,y) = g(x,y) + K$$

example 4

Find the particular solution to

$$y' = \frac{x^2 - y}{x}$$

satisfying the condition $y(3) = 1$.

solution The equation is

$$\frac{dy}{dx} = \frac{x^2 - y}{x}$$

$$\underbrace{(x^2 - y)}_{p}\, dx \quad \underbrace{-\, x}_{q}\, dy = 0$$

Since $\partial q/\partial x$ and $\partial p/\partial y$ both equal -1, the equation is exact. In particular, it can be written as

$$d\left(\tfrac{1}{3}x^3 - xy\right) = 0$$

so the solution is given implicitly by

(20) $$\frac{1}{3}x^3 - xy = K$$

and explicitly by

(21) $$y = \frac{1}{3}x^2 - \frac{K}{x}$$

To find K, substitute $x=3$, $y=1$ in either (20) or (21). Using (20) which is more convenient, you have $9 - 3 = K$, $K = 6$, so the solution is

$$y = \frac{1}{3}x^2 - \frac{6}{x}\ .$$

warning

After you find $\frac{1}{3}x^3 - xy$ in example 4, *you must know what to do with it*.

Here are some *non-solutions*:

$$\frac{1}{3}x^3 - xy \qquad \textit{not the solution}$$

$$\frac{1}{3}x^3 - xy = 0 \qquad \textit{not the solution}$$

$$f(x,y) = \frac{1}{3}x^3 - xy \qquad \textit{not the solution}$$

$$y = \frac{1}{3}x^3 - xy \qquad \textit{not the solution}$$

The solution is the function $y(x)$ defined implicitly by the equation

$$\frac{1}{3}x^3 - xy = K \qquad \textit{implicit solution}$$

and explicitly by

$$y = \frac{1}{2}x^2 - \frac{K}{x} \qquad \textit{explicit solution}$$

warning

In example 4 here are some *wrong* ways to identify p and q:

$$\underbrace{(x^2 - y)}_{p}\ dx = \underbrace{x}_{q}\ dy \qquad \textit{wrong}$$

$$\underbrace{(x^2 - y)}_{p}\ dx - \underbrace{x}_{q}\ dy \qquad \textit{wrong}$$

The *correct* identification is

$$\underbrace{(x^2 - y)}_{p}\ dx\ \underbrace{-\ x}_{q}\ dy \qquad \textit{right} \qquad (\text{q gets a } \textit{minus} \text{ sign})$$

warning

Here's another wrong way to do example 4:

$$x^2\,dy = (x^2 - y)\,dx \qquad (\text{so far, so good})$$

$$\int x^2\,dy = \int (x^2 - y)\,dx \qquad \text{Hmmmmm!}$$

$$x^2y = \frac{1}{3}x^3 - xy + K \qquad \text{WRONG WRONG}$$

This was an attempt to separate variables (Section 4.1) but they *aren't separated* (there is an x on the dyside and a y on the dx side).

a brief table of exact differentials (this is on the reference page)

(22) $$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$$

(23) $$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

(24) $$\frac{-2x\,dx - 2y\,dy}{(x^2 + y^2)^2} = d\left(\frac{1}{x^2 + y^2}\right)$$

(25) $$\frac{x\,dx + y\,dy}{\pm\sqrt{x^2+y^2}} = d\left(\pm\sqrt{x^2 + y^2}\right)$$

(26) $$\frac{2x\,dx + 2y\,dy}{x^2 + y^2} = d\,\ln(x^2 + y^2)$$

(27) $$\frac{-y\,dx + x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

integrating factors

Look at the equation

$$y\,dx - x\,dy = y^3\,dy.$$

The right side is an exact differential, namely $d(\frac{1}{4}y^4)$, but the left side is not exact since $p(x,y) = y$, $q(x,y) = -x$, $\frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y}$. But compare the left side with (22) to see that it can be made exact if you multiply by $1/y^2$. So multiply on both sides to get

$$\frac{y\,dx - x\,dy}{y^2} = y\,dy$$

The left side is now the exact differential in (22) and fortunately the right side is still exact. The equation can be written as

$$d\left(\frac{x}{y}\right) = d\left(\frac{1}{2}y^2\right)$$

By (19), the implicit solution is

$$\frac{x}{y} = \frac{1}{2}y^2 + K$$

It is not convenient to solve for y and get the explicit solution so I'll settle for the implicit version.

A factor, $1/y^2$ in this case, which changes a differential equation from non-exact to exact is called an *integrating factor*. A table of exact differentials like (22)-(27) can serve as goals.

PROBLEMS FOR SECTION 4.3

1. Check formulas (22)-(27) by finding the differential indicated on the righthand side to see if you get the lefthand side.

2. Suppose a point has polar coords r, θ and rectangular coords x, y. If r changes by dr and θ changes by $d\theta$, find dx and dy.

3. Decide if the expression is an exact differential df and if so, find f.

(a) $2xy\,dx + y\,dy$ (b) $(x^3 + 3x^2y)\,dx + (x^3 + y^3)\,dy$

(c) $\dfrac{y}{x^2}\,dx + \left(5 - \dfrac{1}{x}\right)dy$

4. Find q so that $xy^3\,dx + q\,dy$ is exact

5. Solve the DE if it is exact. Find the explicit solution whenever possible.

(a) $(6x^2 + y^2)\,dx + (2xy + 3y^2)\,dy = 0$ (b) $(3x^2 + y)\,dx + x\,dy = 0$

(c) $y' = \dfrac{x - y\cos x}{y + \sin x}$ (d) $y' = e^{xy}$

(e) $(2r\cos\theta - 1)\,dr = r^2\sin\theta\,d\theta$ (f) $(x + y)\,dx + (x^2 + y^2)\,dy = 0$

(g) $\cos x \cos y\,dx - \sin x \sin y\,dy = x^3\,dx$

(h) $(ye^{-x} - \sin x)\,dx = (e^{-x} + 2y)\,dy$

6. Check that your answer to #5(h) really does satisfy the DE.

7. Solve.

(a) $2xy\,dx + (x^2 + y)\,dy = 0$ with $y(1) = 4$

(b) $2\sin(2x + 3y)\,dx + 3\sin(2x + 3y)\,dy = 0$ with $y(0) = \pi/2$

(c) $\dfrac{1}{x + y}\,dx + \dfrac{1}{x + y}\,dy = dx$ with $y=1$ when $x = 0$

8. The equation $(x^2 + 2)\,dx + 3y\,dy = 0$ is both exact and separable. Solve it twice.

9. Find an integrating factor and then solve.

(a) $(x^2 + y^2)\,dx = x\,dy - y\,dx$ (b) $y\,dx - x\,dy = y^2\,dx$

(c) $\sqrt{x^2 + y^2}\,dy = x\,dx + y\,dy$ (d) $y' = \dfrac{x}{x^2 + y^2 - y}$

(e) $x\,dy - y\,dx = 2x^3\,dx + 2x^2y\,dy$

SECTION 4.4 DIRECTION FIELDS

the direction field of a first order DE

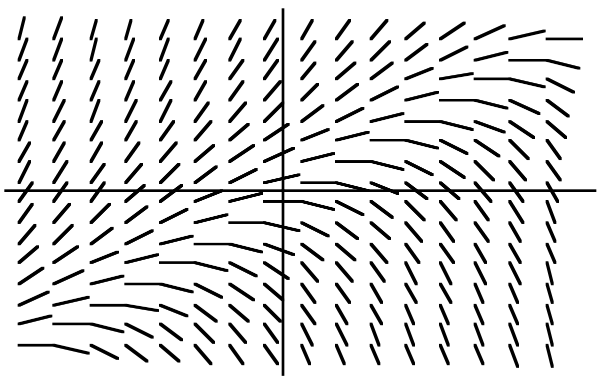
Look at the differential equation

(1) $$y' = y - x$$

The direction field of the DE consists of a lot of little line segments such that the segment at point (x,y) has slope y−x. For example, at point (4,1) draw a small line segment with slope 3.

Here's a Mathematica program which sketches the direction field. .

```
directionField[f_,{x_,x1_,x2_},{y_,y1_,y2_},d_, s_] :=
      Table[Graphics[Line[{{x,y}, {x,y} +
                s{1,f}/Sqrt[1 + f^2]}]],{x,x1,x2,d},{y,y1,y2,d}]
(*The parameter s is the length of each segment and d is the
spacing, both horizontal and vertical, of the grid*)

MyField = directionField[y - x, {x,-3,3},{y,-3,3},.4,.4];
Show[MyField, Axes->True, Ticks->None];
```



Here's the connection between the direction field and the solutions to the DE. The general solution to the DE is a family of curves in the plane such that the slope at the point (x,y) on any curve in the family is y−x.

The DE is first order linear so here's one way to solve it:

$$y' - y = -x$$

$$P(x) = -1, \quad Q(x) = -x$$

$$I = e^{\int P(x)\,dx} = e^{-x}$$

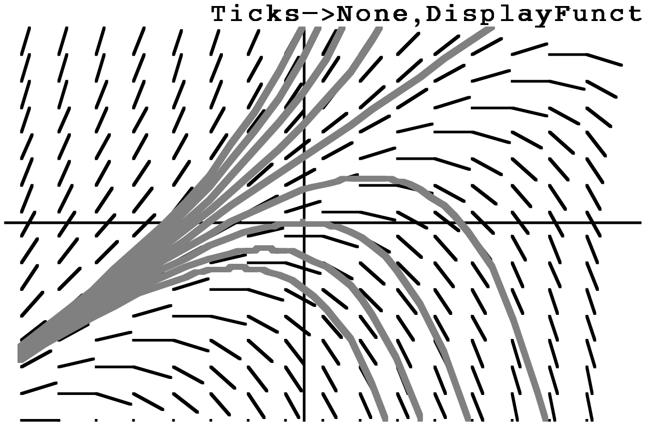
$$e^{-x}\, y = \int IQ = \int -xe^{-x}\,dx = xe^{-x} + e^{-x} + K \quad \text{(tables)}$$

The general solution is

(2) $$y = x + 1 + Ke^{x}$$

Here's the graph of some of the solutions (namely the ones where K = −2, −1.5, −1, −.5, 0, .5, 1, 1.5, 2) superimposed on top of the direction field.

```
SomeSols = Plot[Release[Table[k E^x + x + 1,{k,-2,2,.5}]],{x,-3,3},
        PlotStyle->{{GrayLevel[.5],Thickness[.01]}},
        PlotRange->{-3,3},DisplayFunction->Identity];

Show[{MyField,SomeSols}, Axes->True,
           Ticks->None,DisplayFunction->$DisplayFunction];
```



If you solve (1) with the IC y(2) = 1 you will get the particular solution that passes through the point (2,1). Substitute the condition into (2) to get

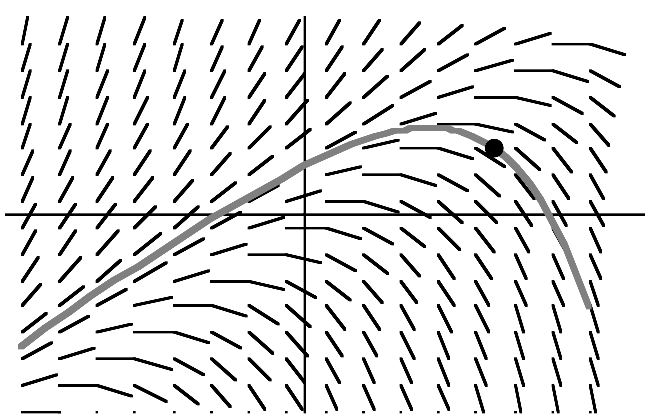
$$1 = 2 + 1 + Ke^{2}$$
$$K = -2e^{-2}$$

So the solution satisfying the IC is

$$y = x + 1 \; -2e^{-2}\, e^{x}$$

Here's a picture of the solution along with the direction field.

```
MySol = Plot[ x + 1 - 2 E^(-2) E^x ,{x,-3,3},
      PlotStyle->{{GrayLevel[.5],Thickness[.01]}},
      DisplayFunction->Identity]

MyPoint = Graphics[{PointSize[.03],Point[{2,1}]}]

Show[{MyField,MySol, MyPoint}, Axes->True, PlotRange->{-3,3},
      Ticks->None,DisplayFunction->$DisplayFunction]
```



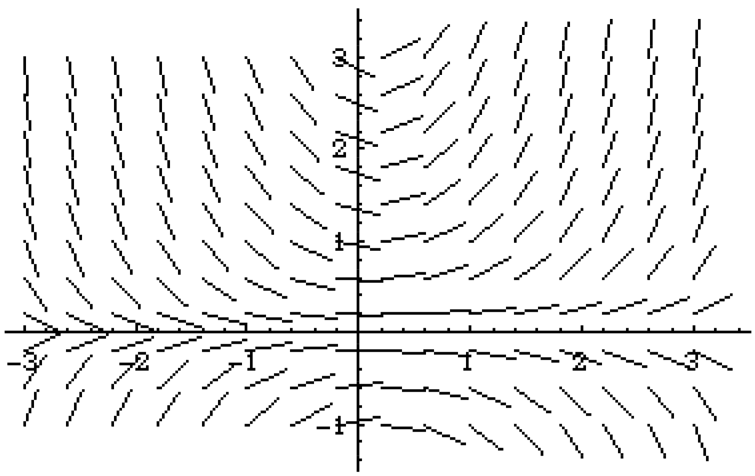
In general, you can get a rough sketch of the solution corresponding to IC $y(x_0) = y_0$ by sketching the "path" in the direction field that goes through point (x_0,y_0).

PROBLEMS FOR SECTION 4.4

1. Mathematica can draw nice direction fields but you can sketch them by hand too. Try doing it for these equations and then solve the equation and see if the solutions go with the direction field.

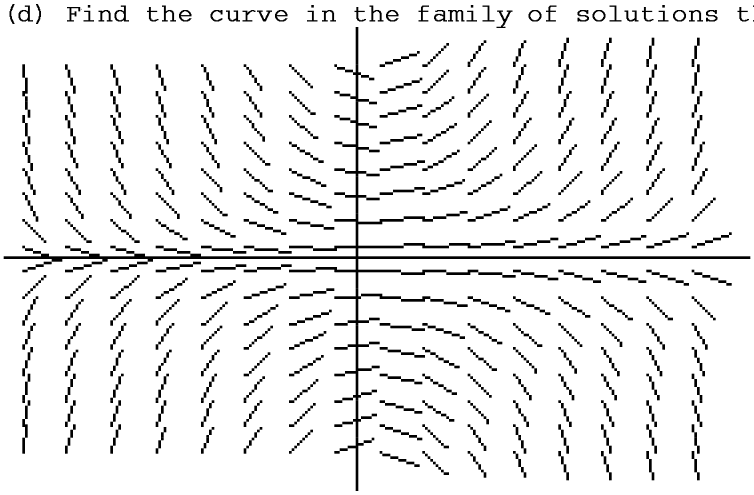
(a) $y' = x/y$ (b) $y' = y/x$

2. Here's the direction field of a first order DE. Sketch the solution satisfying the IC $y(0) = 1$.



3. Look at the DE $y' = xy$
The diagram shows its direction field.

(a) What are all those little segments supposed to be?
(b) Solve the differential equation.
(c) Plot some of the solutions.
(d) Find the curve in the family of solutions through point (4,3).



Direction field for $y' = xy$

REVIEW PROBLEMS FOR CHAPTER 4

1. Solve in as many ways as possible, for practice (using this chapter and/or earlier chapters)

(a) $(x^2 + 2)\,dx + 3y\,dy = 0$

(b) $y' = -y$

(c) $y' = \dfrac{2x - y}{x}$ with $y(1) = 2$

(d) $y'' = y$

(e) $y'' = 3y' + 12$

(f) $y' = e^{x+y}$

(g) $xy'' - y' = 1$ with conditions $y(1) = 2$, $y'(1) = 3$

2. Back in the review problems for Chapters 1,2 there was a problem about the velocity $v(t)$ of a falling object with mass m,

$$mv' = mg - cv \quad \text{where } g, c\ m \text{ are constants,}$$

It was solved by treating the equation as linear first order with constant coefficients. Try it again with the methods of this chapter.

3. Sketch the direction field of the DE $y' = x + y$ and then solve the equation and see if the solutions go with your direction field.

For reference: $\displaystyle\int xe^{ax}\,dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$.

4. Find the family orthogonal to $y = Ax^3$ and draw a picture.

5. Show that any separable DE can be rearranged to be exact (you have to be general here) but not vice versa (you must find a specific counterexample here).

REFERENCE PAGE FOR EXAM 2

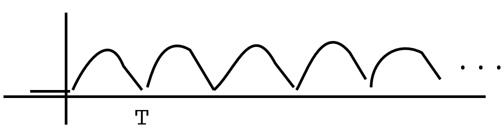
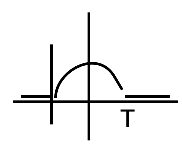
DEFINITION OF THE TRANSFORM $F(s) = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$

TRANSFORMS OF DERIVATIVES $f'(t) \leftrightarrow sF(s) - f(0)$,

$$f''(t) \leftrightarrow s^2 F(s) - sf(0) - f'(0)$$

TRANSFORM OF A CONVOLUTION $f(t)*g(t) \leftrightarrow F(s)G(s)$

TRANSFORM TABLE

$u(t)$	$\dfrac{1}{s}$
$r(t)$	$\dfrac{1}{s^2}$
$t^n\ u(t)$	$\dfrac{n!}{s^{n+1}}$
$\sin at\ u(t)$	$\dfrac{a}{s^2+a^2}$
$\cos at\ u(t)$	$\dfrac{s}{s^2+a^2}$
$e^{at}\ u(t)$	$\dfrac{1}{s-a}$
$\delta(t)$	1
	$\dfrac{1}{1-e^{-Ts}}$ × transform of 

SHIFTING RULES

$f(t-a)u(t-a) \leftrightarrow e^{-as} F(s)$

$e^{at} f(t) \leftrightarrow F(s-a)$

INVERSE TRANSFORMS

It's understood that these inverse tranforms are good for any values of a,b,c (including non-real values) as long as you don't end up dividing by 0.

(1)	1	$\delta(t)$
(2)	$\dfrac{1}{s-a}$	$e^{at}\ u(t)$
(3)	$\dfrac{1}{s}$	$u(t)$
(4)	$\dfrac{1}{s^2}$	$t\ u(t)$
(5)	$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}\ u(t)$
(6)	$\dfrac{s}{s^2+a^2}$	$\cos at\ u(t)$

(7)	$\frac{1}{s^2 + a^2}$	$\frac{1}{a} \sin at \, u(t)$
(8)	$\frac{s}{(s^2 + a^2)^2}$	$\frac{1}{2a} t \sin at \, u(t)$
(9)	$\frac{1}{(s^2 + a^2)^2}$	$\frac{1}{2a^3} (\sin at - at \cos at) \, u(t)$
(10)	$\frac{1}{s(s^2 + a^2)}$	$\frac{1}{a^2} (1 - \cos at) \, u(t)$
(11)	$\frac{1}{s^2(s^2 + a^2)}$	$\frac{1}{a^3} (at - \sin at) \, u(t)$
(12)	$\frac{1}{s^3(s^2 + a^2)}$	$\left(\frac{1}{2a^2} t^2 + \frac{1}{a^4} \cos at - \frac{1}{a^4} \right) u(t)$
(13)	$\frac{s^2}{(s^2 + a^2)^2}$	$\frac{1}{2a} (\sin at + at \cos at) \, u(t)$
(14)	$\frac{1}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{1}{b^2 - a^2} \left(\frac{1}{a} \sin at - \frac{1}{b} \sin bt \right) u(t)$
(15)	$\frac{s}{(s^2 + a^2)(s^2 + b^2)}$	$\frac{1}{b^2 - a^2} (\cos at - \cos bt) \, u(t)$
(16)	$\frac{1}{s^2(s-a)}$	$\left(\frac{1}{a^2} e^{at} - \frac{t}{a} - \frac{1}{a^2} \right) u(t)$
(17)	$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b} (e^{at} - e^{bt}) \, u(t)$
(18)	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{a-b} (ae^{at} - be^{bt}) \, u(t)$
(19)	$\frac{1}{s^2 - a^2}$	$\frac{1}{a} \sinh at \, u(t)$ (special case of (17))
(20)	$\frac{s}{s^2 - a^2}$	$\cosh at \, u(t)$ (special case of (18))
(21)	$\frac{s}{(s-a)^2}$	$(at + 1) \, e^{at} \, u(t)$
(22)	$\frac{1}{(s-a)(s-b)(s-c)}$	$\left[\frac{e^{at}}{(a-b)(a-c)} + \frac{e^{bt}}{(b-a)(b-c)} + \frac{e^{ct}}{(c-a)(c-b)} \right] u(t)$
(23)	$\frac{s}{(s-a)(s-b)(s-c)}$	$\left[\frac{ae^{at}}{(a-b)(a-c)} + \frac{be^{bt}}{(b-a)(b-c)} + \frac{ce^{ct}}{(c-a)(c-b)} \right] u(t)$
(24)	$\frac{s^2}{(s-a)(s-b)(s-c)}$	$\left[\frac{a^2 \, e^{at}}{(a-b)(a-c)} + \frac{b^2 \, e^{bt}}{(a-b)(c-b)} + \frac{c^2 \, e^{ct}}{(a-c)(b-c)} \right] u(t)$
(25)	$\frac{1}{(s-a)(s^2 + b^2)}$	$\frac{1}{a^2 + b^2} \left[e^{at} - \cos bt - \frac{a}{b} \sin bt \right] u(t)$
(26)	$\frac{1}{(s-a)^2(s-b)}$	$\left[\frac{-e^{at}}{(a-b)^2} + \frac{e^{bt}}{(a-b)^2} + \frac{te^{at}}{a-b} \right] u(t)$

CHAPTER 5 THE (ONE-SIDED) LAPLACE TRANSFORM

SECTION 5.1 INTRODUCTION

the unit step function u(t)

The function u(t) is defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Fig 1 shows the function u(t) and some variations.

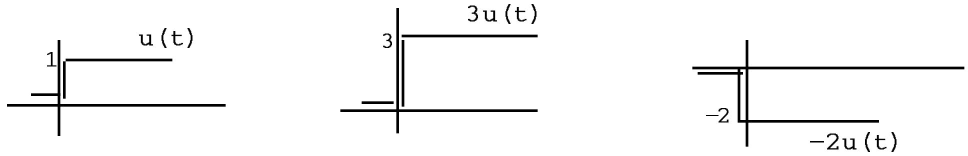


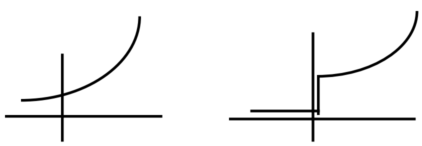
FIG 1

the function f(t)u(t)

If f(t) is an arbitrary function then

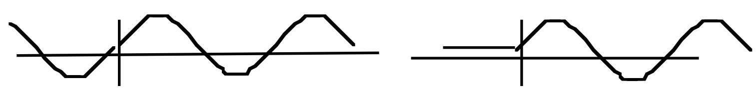
$$f(t)\,u(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t > 0 \end{cases}$$

In other words, multiplying a function f(t) by u(t) kills f(t) until time t=0 and thereafter leaves it alone. Fig 2 shows the function e^t versus $e^t\,u(t)$ and Fig 3 shows $\sin t$ versus $\sin t\; u(t)$.



e^t $\quad$ $e^t\; u(t)$

FIG 2



sin t $\quad$ sin t u(t)

FIG 3

All functions in this chapter are intended to be 0 until time t = 0 since they are pictured as inputs and outputs of a system which is initialized at t = 0. So the chapter is concerned with functions such as $\sin t\; u(t)$ and $e^t u(t)$ rather than with plain $\sin t$ and e^t.

the unit ramp r(t)

The function tu(t) is called the unit ramp and denoted r(t). In other words,

$$r(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \ge 0 \end{cases}$$

Fig 4 shows the functions r(t), 3r(t) and −2r(t).

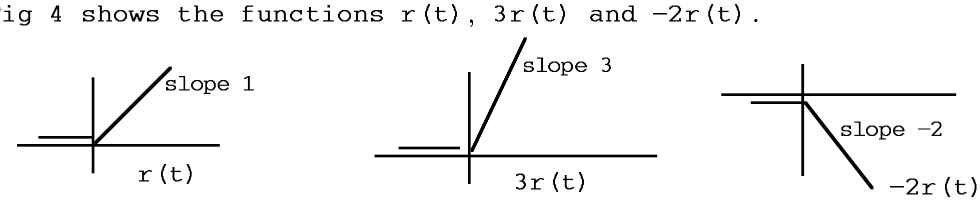


FIG 4

the functions $f(t)\delta(t)$ and $f(t)\delta(t-a)$

Multiplying $\delta(t-3)$ by $f(t)$ leaves the zero heights on the $\delta(t-3)$ graph unchanged and multiplies the impulse at $t = 3$ by $f(3)$ so that the area becomes $f(3)$ instead of 1. In fact, $f(t)\delta(t-3)$ simplifies to $f(3)\delta(t-3)$.

> In general, $f(t)\delta(t-a)$ is the same as $f(a)\delta(t-a)$, an impulse of area $f(a)$ occurring at time $t = a$ (Fig 5).
>
> In particular, $f(t)\delta(t)$ is the same as $f(0)\delta(t)$, an impulse of area $f(0)$ occurring at $t=0$ (Fig 6)

For example, $t^2\,\delta(t-4)$ is the same as $16\delta(t-4)$, an impulse of area 16 at $t=4$.
For example, $t\delta(t)$ is the zero function (carrying zero area). It isn't an impulse anymore.

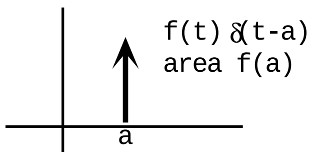


FIG 5

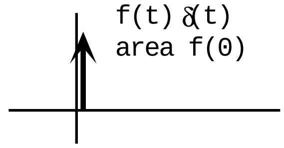


FIG 6

the sifting property of the delta function

From the box above, the area under the graph of $f(t)\delta(t-a)$ is $f(a)$ and it is all concentrated at $t=a$. So:

(1) $$\boxed{\int_a^{\text{above } a} f(t)\delta(t-a)\,dt = f(a)}$$

In particular

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$\int_{0}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$$

And if a is *not* in the interval of integration then $\int_{\text{interval}} f(t)\delta(t-a)\,dt = 0$.

For example,

$$\int_0^{\pi} \delta(t - \tfrac{1}{2}\pi)\,\sin t\,dt = \sin \tfrac{1}{2}\pi = 1$$

$$\int_{\pi}^{2\pi} \delta(t - \tfrac{1}{2}\pi)\,\sin t\,dt = 0$$

($\pi/2$ is not in the interval of integration)

$$\int_{t=-\infty}^{\infty} \frac{t^3+3}{t^4+7}\,\delta(t)\,dt = \frac{0^3+3}{0^4+7} = \frac{3}{7}$$

definition of the (one-sided) Laplace transform and inverse transform

Start with a function f(t) which is 0 for $t \le 0$. Then

(2) $$\boxed{F(s) = \int_{t=0}^{t=\infty} e^{-st} f(t)\, dt}$$

We're only considering $s > 0$ because if $s \le 0$, the integral probably doesn't even converge. In fact, for some f(t) the transform will exist only for say $s \ge 5$ or $s \ge 2$ or $s \ge \pi$. Don't worry about it.

The live variable in (2) is s, and the dummy variable of integration is t.

F(s) is called the Laplace transform of f(t)
f(t) is called the inverse transform of F(s).
You can write

$$f(t) \leftrightarrow F(s)$$

$$\mathcal{L} f(t) = F(s)$$

$$\mathcal{L}^{-1} F(s) = f(t)$$

some basic transform pairs

f(t)	F(s)	
u(t)	$\frac{1}{s}$	
r(t)	$\frac{1}{s^2}$	
$t^n\, u(t)$	$\frac{n!}{s^{n+1}}$	
cos at u(t)	$\frac{s}{s^2 + a^2}$	
sin at u(t)	$\frac{a}{s^2 + a^2}$	
$e^{at}\, u(t)$	$\frac{1}{s - a}$	for $s > a$
$\delta(t)$	1	

proof for u(t)

If f(t) = u(t) then

$$F(s) = \int_0^\infty u(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_{t=0}^{\infty} = 0 + \frac{1}{s} = \frac{1}{s}$$

provided $s > 0$ so that $\lim_{t \to \infty} e^{-st} = 0$

proof for r(t)

If f(t) = r(t) = t u(t) then

$$F(s) = \int_0^\infty t\, e^{-st}\, dt = \left(-\frac{1}{s} t\, e^{-st} - \frac{1}{s^2} e^{-st}\right)\Big|_{t=0}^{\infty}$$ antideriv is on the ref page for exam 1

$$= \underbrace{-\frac{1}{s} \frac{t}{e^{st}}\Big|_{t=\infty}}_{0} \underbrace{- \frac{1}{s^2} e^{-st}\Big|_{t=\infty}}_{0} \underbrace{+ \frac{1}{s} t\, e^{-st}\Big|_{t=0}}_{0} \underbrace{+ \frac{1}{s^2} e^{-st}\Big|_{t=0}}_{1/s^2} = \frac{1}{s^2}$$

proof for sin at u(t)

If $f(t) = \sin at\ u(t)$ then

$$F(s) = \int_{t=0}^{\infty} e^{-st} \sin at\, dt = \left. \frac{e^{-st}\,(-s \sin at - a \cos at)}{s^2 + a^2} \right|_{t=0}^{\infty} = \frac{a}{s^2 + a^2}$$

proof for δ(t)

$$\pounds\, \delta(t) = \int_0^{\infty} e^{-st}\, \delta(t)\, dt = e^{-s \cdot 0} \quad \text{(sifting property)} \quad = 1$$

linearity property of the transform

$$\pounds\,[f(t) + g(t)] = \pounds\, f(t) + \pounds\, g(t)$$

$$\pounds\,[af(t)] = a\, \pounds\, f(t)$$

In other words,

(3) $f(t) + g(t) \leftrightarrow F(s) + G(s)$

(4) $a\,f(t) \leftrightarrow a\,F(s)$

proof of (3)

$$\pounds\,[f(t) + g(t)] = \int_0^{\infty} [f(t) + g(t)]\, e^{-st}\, dt$$

$$= \int_0^{\infty} f(t)\, e^{-st}\, dt + \int_0^{\infty} g(t)\, e^{-st}\, dt$$

$$= \pounds\, f(t) + \pounds\, g(t)$$

corollary of linearity

$$(at^2 + bt + c)\,u(t) \leftrightarrow a\,\frac{2}{s^3} + b\,\frac{1}{s^2} + c\,\frac{1}{s}$$

example 1

If $f(t) = (4 + 3t^2)\,u(t)$ then

$$F(s) = \frac{4}{s} + 3\,\frac{2}{s^3} = \frac{4}{s} + \frac{6}{s^3}$$

example 2

$$6r(t) \leftrightarrow \frac{6}{s^2}$$

$$-3u(t) \leftrightarrow \frac{-3}{s}$$

warning

Suppose

$$f(t) = \sin 3t \; u(t)$$

To indicate that the transform is $\dfrac{3}{s^2 + 9}$ either write

$$\sin 3t \; u(t) \;\leftrightarrow\; \frac{3}{s^2+9} \qquad \text{OK}$$

or write

$$\mathcal{L}\, \sin 3t \; u(t) = \frac{3}{s^2+9} \qquad \text{OK}$$

or write

$$F(s) = \frac{3}{s^2+9} \qquad \text{OK}$$

but *don't* write

$$\sin 3t \; u(t) = \frac{3}{s^2+9} \qquad \text{WRONG WRONG WRONG}$$

review

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

PROBLEMS FOR SECTION 5.1

1. Find the transform

(a) $t^5 u(t)$ (b) $t^3 u(t)$ (c) $e^{3t} u(t)$ (d) $e^{-4t} u(t)$ (e) $\sin 4t \; u(t)$ (f) $\cos 5t \; u(t)$

2. Derive the transform formula for $e^{-at} u(t)$

3. Find the transform

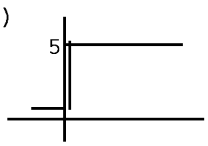
(a) $\cosh t \; u(t)$

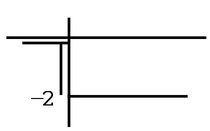
(b) $\sin^2 4t \; u(t)$ (Use the identity $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$)

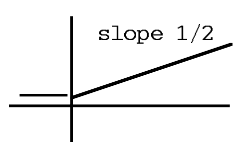
(c) $\cos(at + b)u(t)$ (Use the identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$)

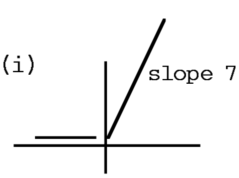
(d) $(8t^2 + 2t - 3)\; u(t)$

(e) $(2e^{3t} - \sin \pi t)\; u(t)$

(f)


(g)


(h)


(i)


(j) $e^{3t+4}\; u(t)$ (use a little algebra first)

(k) $6\; \delta(t)$

(ℓ) $(4t^3 - 3t^2 + 5t + 2)\; u(t)$

4. Find

(a) $\int_0^{2\pi} \delta(t - \frac{\pi}{4}) \cos t \, dt$ (b) $\int_0^{2\pi} t^3 \, \delta(t-7) \, dt$

(c) $\int_0^{2\pi} t^3 \, \delta(t-6) \, dt$ (d) $\int_{-\infty}^{\infty} \delta(t) \cos t \, dt$

(e) $\int_{-\infty}^{\infty} 6\delta(t) \, dt$ (f) $\int_0^{\infty} 6\delta(t) \, dt$

(g) $\int_{-\infty}^{\infty} t^2 \, \delta(t) \, dt$ (h) $\int_0^{\infty} e^t \, \delta(t) \, dt$

(i) $\int_{-\infty}^{\infty} t^2 \, \delta(t-2) \, dt$ (j) $\int_0^{\infty} e^t \, \delta(t-2) \, dt$

5. Find by inspection (a) $\int_0^{\infty} e^{-st} \; t^4 \, dt$ (b) $\int_0^{\infty} e^{-su} \; u^4 \, du$

(c) $\int_0^{\infty} e^{-wt} \, t^4 \, dt$ (d) $\int_0^{\infty} t^4 \, e^{3s^2 t} \; t^4 \, dt$

HONORS

6. (a) Use integration by parts to express the transform of $t^n u(t)$ in terms of the transform of $t^{n-1} u(t)$.

(b) I already derived the transform of $tu(t)$, i.e., of $r(t)$. Use the result in (a) to find the transform of $t^2 u(t)$, $t^3 u(t)$, $t^4 u(t)$ and $t^n u(t)$.

SECTION 5.2 FINDING TRANSFORMS

the graph of f(t–a)u(t–a)

The graph of $f(t-a)u(t-a)$ is found by translating the graph of $f(t)u(t)$ to the right by a units (Figs 1–4), i.e., by delaying the signal.

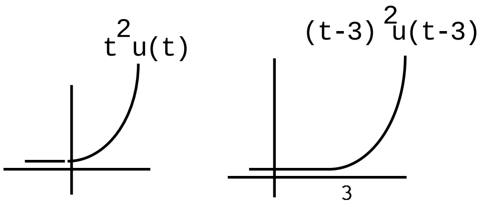


FIG 1

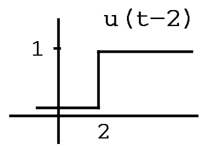


FIG 2

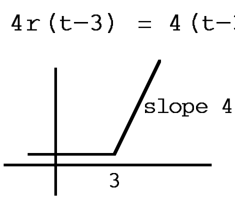


FIG 3

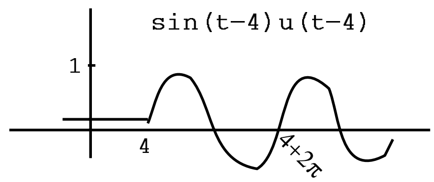


FIG 4

t-shifting rule

Let a be a positive number. Delaying $f(t)u(t)$ until time a will multiply the transform by e^{-as}. In other words, if $f(t)u(t) \leftrightarrow F(s)$ then

$$\boxed{f(t-a)u(t-a) \leftrightarrow e^{-as} F(s)}$$

For example (Fig 1),

$$t^2u(t) \leftrightarrow \frac{2}{s^3}$$

so

$$(t-3)^2\, u(t-3) \leftrightarrow \frac{2e^{-3s}}{s^3}$$

Similarly, for the delayed signals in Figs 2–4,

$$u(t-2) \leftrightarrow \frac{e^{-2s}}{s}$$

$$4r(t-3) \leftrightarrow \frac{4e^{-3s}}{s^2}$$

$$\sin(t-4)u(t-4) \leftrightarrow \frac{e^{-4s}}{s^2+1}$$

proof of the t-shifting rule

$$\pounds f(t-a)u(t-a) = \int_0^\infty f(t-a)u(t-a)\, e^{-st}\, dt = \int_a^\infty f(t-a)\, e^{-st}\, dt.$$

Now let $v = t-a$, $dv = dt$. When $t = a$, $v = 0$. When $t = \infty$, $v = \infty$. So

$$£ f(t-a)u(t-a) = \int_0^\infty f(v)\ e^{-s(v+a)}\ dv$$

$$= e^{-as} \underbrace{\int_0^\infty f(v)\ e^{-sv}\ dv}_{\text{This is the integral for } F(s) \text{ (but with dummy variable } v \text{ instead of } t)} = e^{-as}\ F(s)$$

adding and subtracting ramps

First note that if you add two lines with slopes m_1 and m_2 you get a line with slope $m_1 + m_2$ because

$$m_1x + b_1 + m_2x + b_2 = (m_1 + m_2)x + b_1+b_2$$

Figs 5-7 shows how to use this to combine ramps graphically.

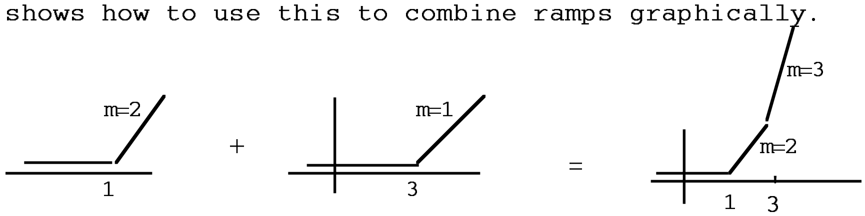


FIG 5

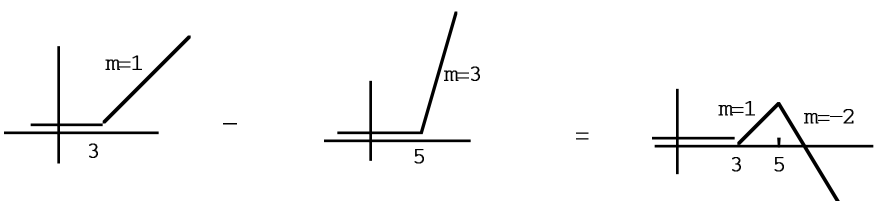


FIG 6

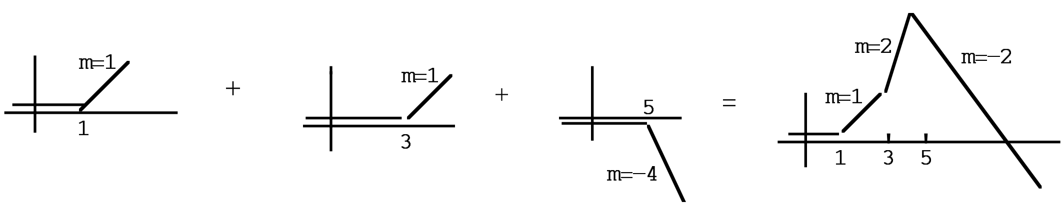


FIG 7

finding transforms by decomposing into ramps and steps

Look at the function in Fig 8.
Fig 9 shows that

Fig 8 = $4r(t) - 5r(t-1) + r(t-5)$.

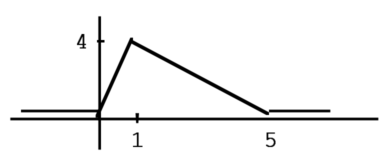


FIG 8

So the transform of Fig 8 is

$$\frac{4}{s^2} - \frac{5e^{-s}}{s^2} + \frac{e^{-5s}}{s^2}$$

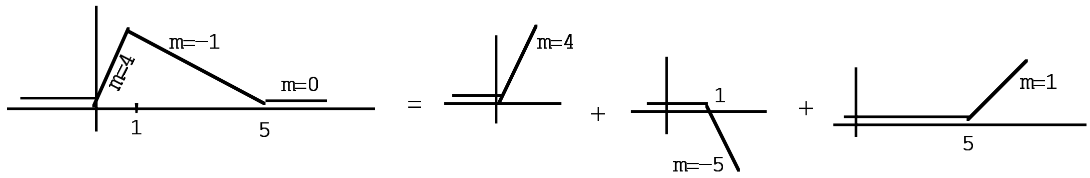


FIG 9

warning

In Fig 9, if you stop with the first two terms in the decomp, i.e., if you stop with

$$4r(t) - 5r(t-1)$$

then you have the function in Fig 10 instead of the desired Fig 8. Make sure you have enough terms in your decomposition to get what you want.

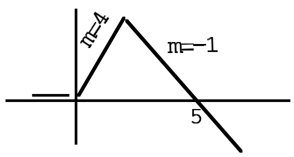


FIG 10

example 1

The function in Fig 11 can be written as r(t) − r(t−3) (Fig 12). So the transform of Fig 11 is

$$\frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$

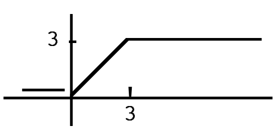


FIG 11

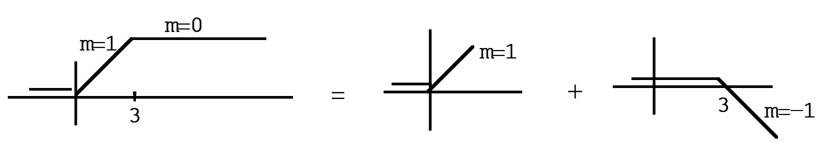


FIG 12

transform of a jumpy f(t)

I'll find the transform of the function in Fig 13 which jumps from 8 down to 0.

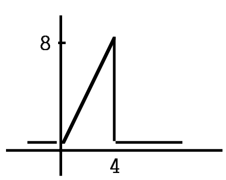
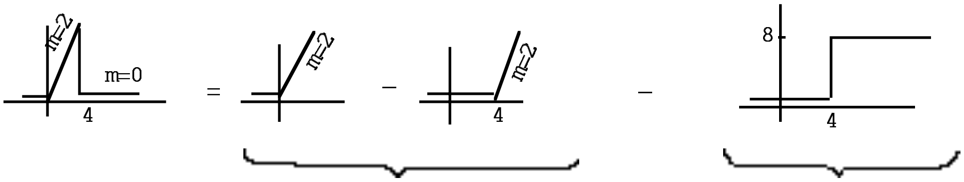


FIG 13

Fig 14 shows the decomposition of Fig 13.



so far the decomp goes up to height 8 and levels off there

subtract to make it jump down to height 0

FIG 14

So

$$\text{Fig 13} = 2r(t) - 2r(t-4) - 8u(t-4)$$

and the transform of Fig 13 is

$$\frac{2}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{8e^{-4s}}{s}$$

describing a pulse

Fig 15 shows that

$$u(t-4) - u(t-7) = \begin{cases} 1 & \text{if } 4 \le t \le 7 \\ 0 & \text{otherwise} \end{cases}$$

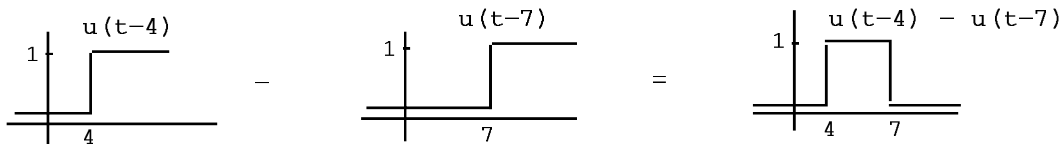


FIG 15

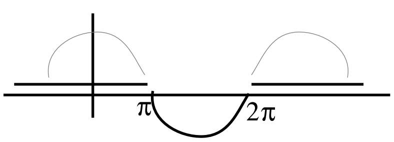
So

$$f(t)\Big[u(t-4) - u(t-7)\Big] = \begin{cases} f(t) & \text{if } 4 \le t \le 7 \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

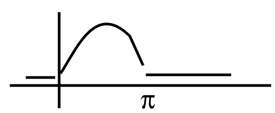
$$f(t)\Big[u(t) - u(t-7)\Big] = \begin{cases} f(t) & \text{if } 0 \le t \le 7 \\ 0 & \text{otherwise} \end{cases}$$

Figs 16 and 17 show some sine pulses.



$\sin t\Big[u(t-\pi) - u(t-2\pi)\Big]$

FIG 16



$\sin t\Big[u(t) - u(t-\pi)\Big]$

FIG 17

using algebraic maneuvers and t-shifting to find the transform of a pulse

Let

$$f(t) = \begin{cases} e^{2t} & \text{if } 0 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

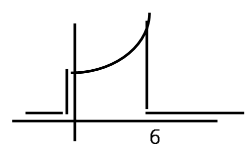


FIG 18

Then

$$f(t) = e^{2t}\,[\,u(t) - u(t-6)\,]$$

$$= e^{2t}\,u(t) - e^{2t}\,u(t-6)$$

You can't use the t-shifting rule on $e^{2t}\,u(t-6)$ because the exponent has plain t in it, not t−6. But you can get it into a more useful form as follows:

$$f(t) = e^{2t}\,u(t) - \underbrace{e^{2(t-6)+12}}_{\text{TRICK}}\,u(t-6) = e^{2t}\,u(t) - \underbrace{e^{12}}_{\text{constant}}\;\underbrace{e^{2(t-6)}\,u(t-6)}_{\text{now can t-shift}}$$

So

$$F(s) = \frac{1}{s-2} - e^{12}\ \frac{e^{-6s}}{s-2} = \frac{1}{s-2}\ (1 - e^{12-6s})$$

example 3

Find the transform of the pulse f(t) in Fig 19

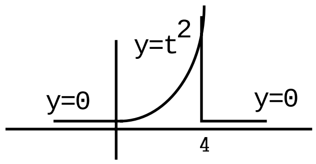


FIG 19

We have

$$f(t) = t^2 \Big[u(t) - u(t-4)\Big] = \underbrace{t^2\, u(t)}_{\text{t-shifting}} - \ t^2\, u(t-4) \quad \text{not ready for}$$

$$= t^2\, u(t) - \underbrace{\Big[(t-4) + 4\Big]^2}_{\text{TRICK}}\ u(t-4)$$

$$= t^2\, u(t) - \underbrace{\Big[(t-4)^2 + 8(t-4) + 16\Big]\ u(t-4)}_{\text{ready for t-shifting}}$$

So

$$F(s) = \frac{2}{s^3} - e^{-4s}\left[\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right]$$

warning

The t-shifting rule does not apply directly to $t^2\ u(t-4)$; the transform is *not* $\frac{2e^{-4s}}{s^3}$ because $\frac{2e^{-4s}}{s^3}$ goes with $(t-4)^2\ u(t-4)$ and not with $t^2\ u(t-4)$. The t-shifting rule applies in example 3 *after* you use algebra to get

$$\Big[(t-4)^2 + 8(t-4) + 16\Big]\ u(t-4)$$

example 4

Find the transform of the sine pulse in Fig 20.

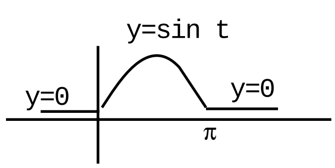


FIG 20

method 1 Fig 21 shows the pulse decomposed into

$$\sin t\ u(t) + \sin(t-\pi)\ u(t-\pi)$$

So the transform is

$$\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

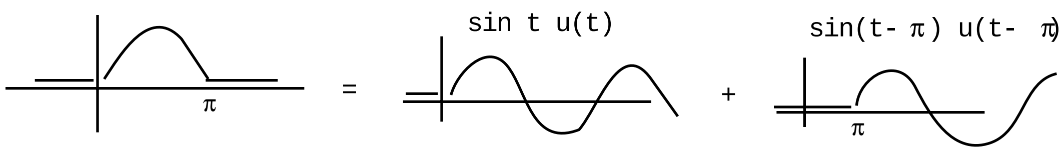


FIG 21

method 2

$$\text{Fig 16} = \sin t \left[u(t) - u(t-\pi) \right]$$

$$= \sin t\; u(t) - \sin t\; u(t-\pi)$$

$$= \sin t\; u(t) - \sin \underbrace{\left[(t-\pi) + \pi \right]}_{\text{TRICK}} u(t-\pi)$$

$$= \sin t\; u(t) + \sin(t-\pi)\; u(t-\pi) \quad \text{by the identity } \sin(x+\pi) = -\sin x$$

So the transform of the function in Fig 20 is

$$\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

example 5

Find the transform of the pulse in Fig 22

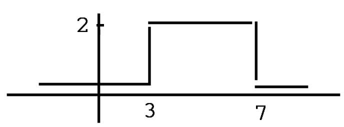


FIG 22

The pulse is $2u(t-3) - 2u(t-7)$ so its transform is $\frac{2}{s}\left[e^{-3s} - e^{-7s} \right]$.

review of geometric series

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r} \quad \text{provided that } -1 < r < 1$$

transform of a periodic function

Suppose $f(t)u(t)$ has period T for $t \geq 0$ (Fig 23).

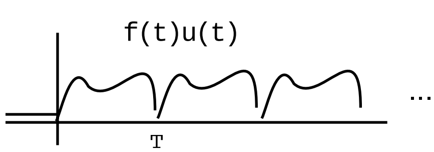


FIG 23

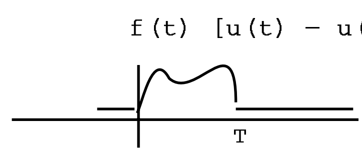


FIG 24

To find its transform, first find the transform of

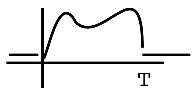
$$f(t) \left[u(t) - u(t-T) \right] \quad \text{(Fig 24)},$$

the signal consisting of the first period followed by zero. Then multiply by

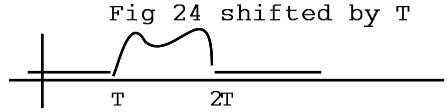
$$\frac{1}{1-e^{-Ts}}$$

proof

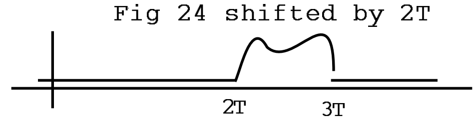
Fig 24

f(t)u(t) = 

\+ Fig 24 shifted by T



\+ Fig 24 shifted by 2T



\+ . . .

If you let G(s) denote the transform of Fig 24 then

$$f(t)u(t) \leftrightarrow G(s) + e^{-Ts}G(s) + e^{-2Ts}G(s) + e^{-3Ts}G(s) + \ldots$$

$$= G(s)\left[1 + e^{-Ts} + (e^{-Ts})^2 + (e^{-Ts})^3 + \ldots\right]$$

The series in the brackets is geometric with

$$a = 1,\ r = e^{-Ts},$$

Since T and s are positive, e^{-TS} is between 0 and 1 so the series converges to $\dfrac{1}{1-e^{-Ts}}$

and

$$f(t)u(t) \leftrightarrow \text{transform of Fig 24} \times \frac{1}{1-e^{-Ts}}$$

example 6

The function in Fig 25 has period 2a for $t \ge 0$. So

$$\text{transform of Fig 25} = \text{transform of Fig 26} \times \frac{1}{1-e^{-2as}}$$

Fig 27 shows the decomposition of Fig 26:

$$\text{Fig 26} = 5u(t) - 10u(t-a) + 5u(t-2a)$$

So

$$\text{transform of Fig 26} = \frac{5}{s} - \frac{10e^{-as}}{s} + \frac{5e^{-2as}}{s}$$

and

$$\text{Fig 25} \leftrightarrow \frac{5 - 10e^{-as} + 5^{-2as}}{s(1 - e^{-2as})}$$

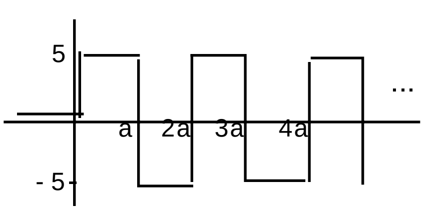


FIG 25

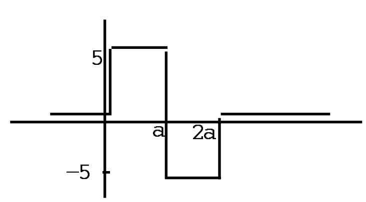


FIG 26

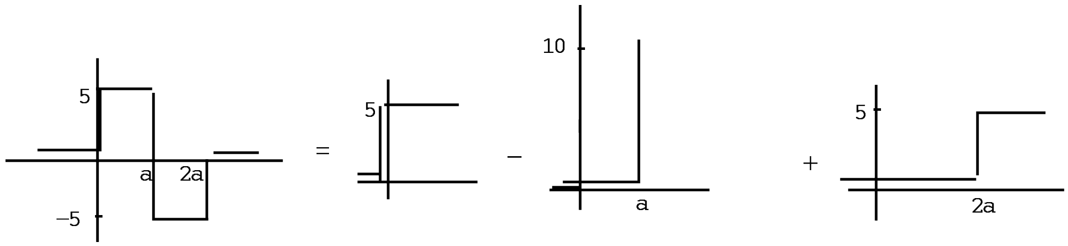
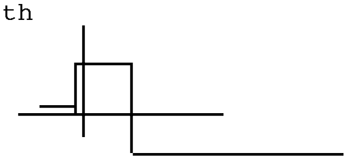


FIG 27

warning Don't forget the last term in the decomposition in Fig 27. If you leave it out you will end up with



when you wanted to get Fig 26.

s-shifting rule

$$e^{at}\, f(t) \quad \leftrightarrow \quad F(s-a) \qquad \text{for } s > a$$

In other words, multiplying a function by e^{at} shifts the tranform to the right by a.

proof of the s-shifting rule

$$(*) \qquad \mathcal{L}\, e^{at}\, f(t) \;=\; \int_0^\infty e^{at}\, f(t)\, e^{-st}\, dt \;=\; \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$$

The integral on the righthand side is the same as the integral for the Laplace transform F(s) but with s−a instead of s. So the integral is F(s−a).

footnote The proof of the s-shifting rule doesn't work unless s > a so that s−a > 0 because the original definition of the Laplace transform as $\int_0^\infty f(t)\, e^{-st}\, dt$ required s > 0. And in the last integral in (*), s−a is playing the role of the s in the definition.
So the s-shifting rule really holds only for s > a.

example 7

To find the transform of $e^{-3t} \sin t\; u(t)$, find the transform of $\sin t\; u(t)$ and then shift:

$$e^{-3t} \sin t\; u(t) \quad \leftrightarrow \quad \frac{1}{(s+3)^2 + 1}$$

example 8

Tables list the transform pair

$$\cosh at\; u(t) \leftrightarrow \frac{s}{s^2 - a^2}$$

So, by the s-shifting rule,

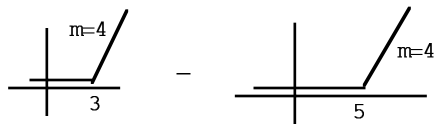
$$e^{3t} \cosh at\; u(t) \leftrightarrow \frac{s-3}{(s-3)^2 - a^2}$$

PROBLEMS FOR SECTION 5.2

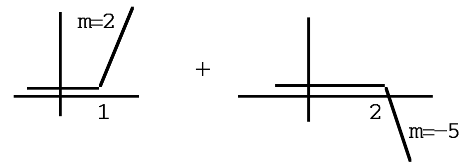
1. Sketch the graph
(a) $9-t^2$ (b) $(9-t^2)\ u(t)$ (c) $(9-t^2)u(t-3)$ (d) $(9-t^2)\ u(t-5)$

2. Draw a picture of the indicated sum

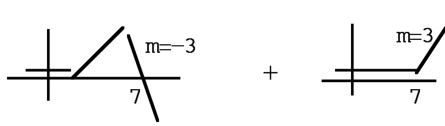
(a)



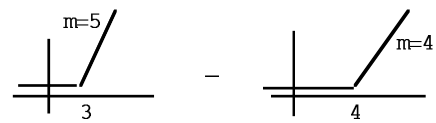
(b)



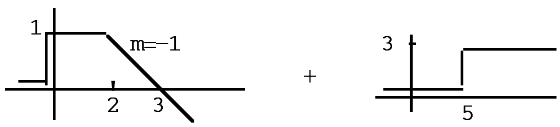
(c)



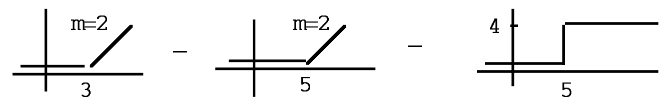
(d)



(e)

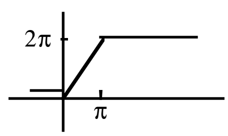


(f)

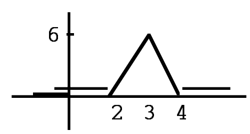


3. Find the transform

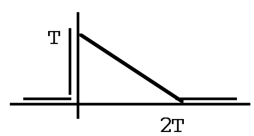
(a)



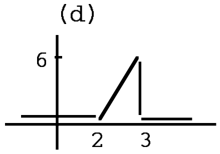
(b)



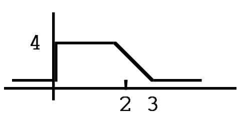
(c)



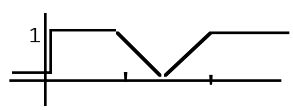
(d)



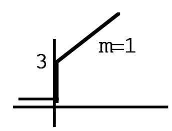
(e)



(f)

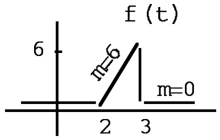


(g)



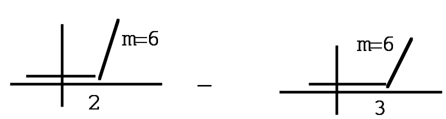
4. Suppose you want the transform of the function f(t) in the lefthand diagram. What have you done wrong if you decompose f(t) into $6r(t-2) - 6r(t-3)$ (see the righthand diagram) and conclude that the transform is $\dfrac{6e^{-2s}}{s^2} - \dfrac{6e^{-3s}}{s^2}$.

f(t)



Problem 4 f(t)

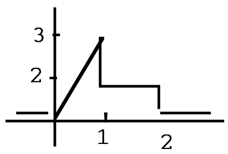
6r(t−2) 6r(t−3)



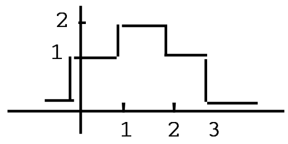
Problem 4 would-be decomposition

5. Find the transform

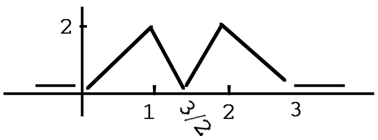
(a)



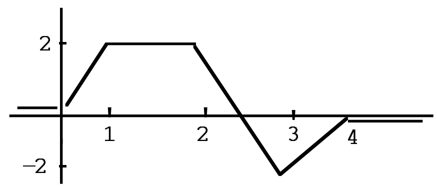
(b)



(c)



(d)



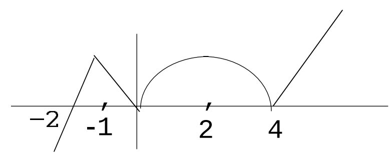
6. Sketch the graph (a) $9-t^2$ (b) $(9 - t^2)\left[\, u(t) - u(t-3) \,\right]$

(c) $(9 - t^2)\left[\, u(t-2) - u(t-4) \,\right]$

7. Given the graph of f(t) in the diagram.
Sketch the graph of

(a) $f(t)\, u(t)$ (b) $f(t)\, u(t-2)$

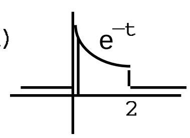
(c) $f(t-2)\, u(t-2)$ (d) $f(t)\left[u(t) - u(t-4)\right]$



Problem 7

8. Find the transform

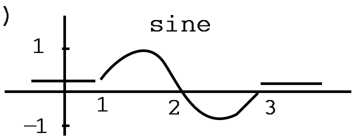
(a)



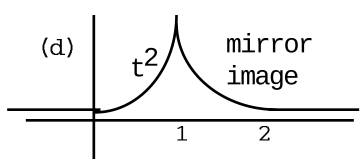
(b)

$$f(t) = \begin{cases} t^3 & \text{if } 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(c)



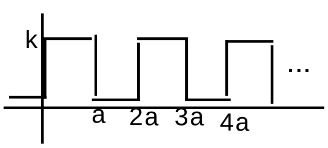
(d)



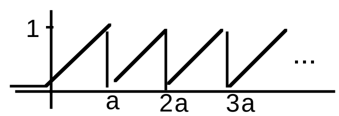
9. Find the transform of $tu(t-5)$.

10. Find the transforms of the following functions which are periodic on $[0,\infty)$

(a)

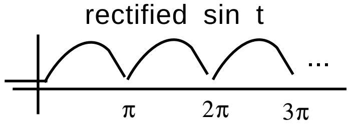


(b)



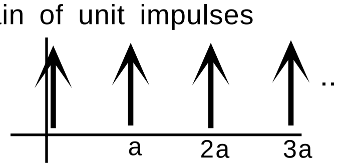
rectified sin t

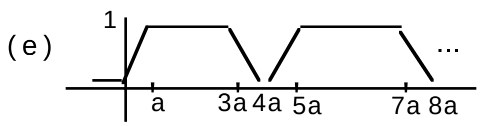
(c)

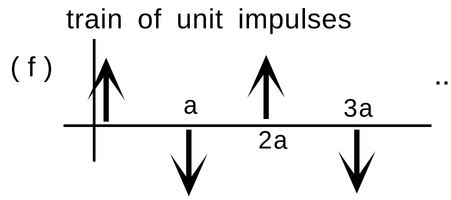


train of unit impulses

(d)





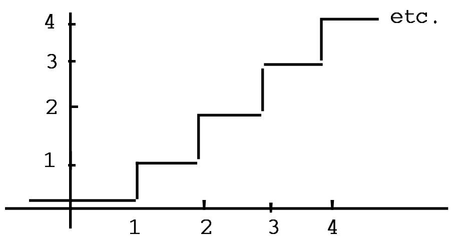


11.(a) Show that the transform of the staircase in the diagram is

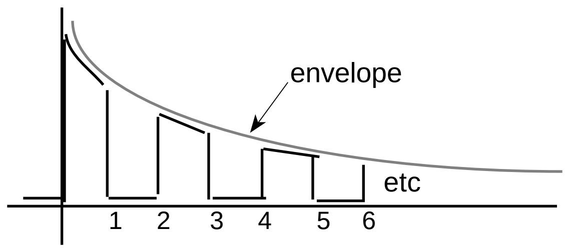
$$\frac{1}{s(e^s - 1)}$$

(b) Show that the transform of the indicated function with envelope e^{-2t} is

$$\frac{1}{s + 2} \qquad \frac{1}{1 + e^{-(s + 2)}}$$



Problem 11(a)



Problem 11(b)

12. Find the transform
    (a) $e^{3t} \cos 4t \; u(t)$ (b) $t^2 \, e^{-4t} \, u(t)$

13. Find $\int_{t=5}^{\infty} e^{-st} (t-5)^7 \, dt$ by inspection (don't do any integration).

SECTION 5.3 FINDING INVERSE TRANFORMS

using linearity in reverse

If the inverses of F(s) and G(s) are in the table of inverse transforms, you can find the inverse of 3F(s) and F(s) + G(s) using the linearity rule in reverse.

For example the tables list

$$\frac{1}{s^2 + a^2} \leftrightarrow \frac{1}{a} \sin at \; u(t)$$

and

$$\frac{s}{s^2 + a^2} \leftrightarrow \cos at \; u(t)$$

so

$$\pounds^{-1}\left[\frac{2}{s^2 + 5} + \frac{6s}{s^2 + 3} \right] = \frac{2}{\sqrt{5}} \sin \sqrt{5}\, t \; u(t) + 6 \cos \sqrt{3}\, t \; u(t)$$

using the t-shifting rule in reverse

You used the rule

$$f(t-a)u(t-a) \leftrightarrow e^{-as} F(s)$$

to find the transform of the delayed signal $f(t-a)u(t-a)$. It can also be used from right to left to find inverse transforms:

> To find the inverse transform of $e^{-as} F(s)$, invert F(s) (remember to put in the u(t)) and then delay until time a.

For example, the tables list

$$\frac{1}{s^4} \leftrightarrow \frac{t^3}{3\,!} u(t)$$

so

$$(1) \qquad \frac{e^{-5s}}{s^4} \leftrightarrow \frac{(t-5)^3}{3\,!} u(t-5) \qquad \text{(Fig 1)}$$

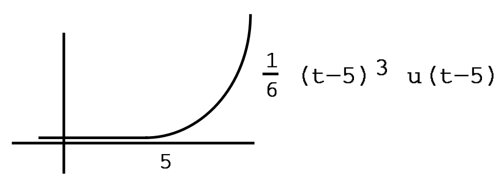


FIG 1

warning

If you t-shift $\frac{t^3}{3\,!} u(t)$ to find the inverse transform of $\frac{e^{-5s}}{s^4}$ make sure you t-shift *all* of it:

WRONG $\quad \frac{e^{-5s}}{s^4} \leftrightarrow \frac{t^3}{3\,!} u(t-5) \quad$ WRONG

WRONG $\quad \frac{e^{-5s}}{s^4} \leftrightarrow \frac{(t-5)^3}{3\,!} u(t) \quad$ WRONG

WRONG $\quad \frac{e^{-5s}}{s^4} \leftrightarrow \frac{(t-5)^3}{3\,!} \quad$ WRONG

RIGHT $\quad \frac{e^{-5s}}{s^4} \leftrightarrow \frac{(t-5)^3}{3\,!} u(t-5) \quad$ RIGHT

using the s-shifting rule in reverse

You can use the rule

$$e^{at} f(t)u(t) \leftrightarrow F(s-a)$$

to find the transform of a signal multiplied by an exponential. It's more useful read from right to left: shifting the transform to the right by a corresponds to multiplying the original function by e^{at}

For example, you know that

$$\mathcal{L}^{-1}\frac{s}{s^2+7} = \cos\sqrt{7}\,t\;u(t)$$

so, by the s-shifting rule,

$$\mathcal{L}^{-1}\frac{s-5}{(s-5)^2+7} = e^{5t}\cos\sqrt{7}\,t\;u(t)$$

inverting simple fractions where the denominator contains linear factors

Tables list

$$\frac{2}{s-4} \leftrightarrow 2e^{-4t}$$

So

(2) $$\frac{5}{3s+7} = \frac{5}{3}\,\frac{1}{s+\frac{7}{3}} \quad \text{(rearrange)} \quad \leftrightarrow \frac{5}{3}\,e^{-7t/3}\,u(t)$$

Tables list

$$\frac{1}{s^{10}} \leftrightarrow \frac{t^9}{9!}\,u(t)$$

so, by s-shifting,

(3) $$\frac{1}{(s+4)^{10}} \leftrightarrow \frac{t^9}{9!}\,e^{-4t}\,u(t)$$

completing the square to invert some fractions with nonfactorable quadratic denominators

(4) $$\frac{1}{s^2-3s+4} = \frac{1}{s^2-3s+\frac{9}{4}+4-\frac{9}{4}} = \frac{1}{(s-\frac{3}{2})^2+\frac{7}{4}} \quad \text{(complete the square)}$$

$$\leftrightarrow \frac{2}{\sqrt{7}}\sin\frac{2}{\sqrt{7}}t\quad e^{-3t/2} \quad \text{(basic formula plus s-shifting)}$$

(5) $$\frac{7s}{s^2-6s+14} = \frac{7s}{(s-3)^2+5} \quad \text{complete the square}$$

But the fraction isn't ready to be inverted yet because the denominator is s-shifted but the numerator isn't. So rearrange the numerator to match the shifted denom:

$$\begin{aligned}\frac{7s}{s^2-6s+14} &= \frac{7(s-3)+21}{(s-3)^2+5}\\ &= \frac{7(s-3)}{(s-3)^2+5} + \frac{21}{(s-3)^2+5}\end{aligned}$$

$$\leftrightarrow 7\cos\sqrt{5}\,t\;e^{3t}\,u(t) + \frac{21}{\sqrt{5}}\sin\sqrt{5}\,t\;e^{3t}\,u(t)$$

warning

To invert

$$\frac{7s}{(s-3)^2 + 5}$$

don't forget that the numerator *must be rearranged to match the shifted denominator*. If they *don't* match (the denom has s−3 but the numerator has s) the fraction can't be inverted using s−shifting.

On the other hand, to invert

$$\frac{e^{-s}}{(s-3)^2 + 5} \tag{6}$$

don't try to turn e^{-s} into $e^{-(s-3)}$. Treat the e^{-s} in (6) as the signal to first invert the fraction

$$\frac{1}{(s-3)^2 + 5}$$

getting

$$\frac{1}{\sqrt{5}}\, e^{-3t} \sin\sqrt{5}\, t\ u(t) \qquad \text{(s−shfting rule)}$$

and then t−shift to get

$$\frac{e^{-s}}{(s-3)^2 + 5} \quad \leftrightarrow \quad \frac{1}{\sqrt{5}}\, e^{-3(t-1)} \sin\sqrt{5}\,(t-1)\ u(t-1)$$

how to invert fractions where the denominator is cubic or worse

Suppose F(s) is of the form

$$\frac{\text{polynomial}}{\text{another poly of higher degree}}$$

I put many of these in the tables so look there first.

If you get one that's not in the tables then one way to find an inverse transform is to decompose F(s) (ugh) into a sum of simpler partial fractions (you learned how to do that in calculus---see handout on decomposition) and then invert the pieces. That's how many of the formulas in the reference table were derived in the first place.

If you have access to Mathematica you can take transforms and inverse transforms of many functions directly.

example 1

$$\frac{1}{s^2 - 5} \leftrightarrow \frac{1}{\sqrt{5}} \sinh\sqrt{5}\, t\ u(t) \qquad \text{(tables (19))}$$

$$\frac{1}{(s+4)^2 - 5} \leftrightarrow \frac{1}{\sqrt{5}}\, e^{-4t} \sinh\sqrt{5}\, t\ u(t) \qquad \text{(s−shifting rule)}$$

$$\frac{e^{-2s}}{s^2 - 5} \leftrightarrow \frac{1}{\sqrt{5}} \sinh\sqrt{5}\,(t-2)\ u(t-2) \qquad \text{(t−shifting rule)}$$

$$\frac{e^{-2s}}{(s+4)^2 - 5} \leftrightarrow e^{-4(t-2)} \sinh\sqrt{5}\,(t-2)\ u(t-2) \qquad \text{(s−shifting and t−shifting)}$$

review of factoring quadratics

$$ax^2 + bx + c = a\left[x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]$$

example 2

$$\frac{1}{s^2 - 2s - 2} = \frac{1}{\left[s-(1+\sqrt{3})\right]\ \left[s-(1-\sqrt{3})\right]}$$

$$\leftrightarrow \quad \frac{1}{2\sqrt{3}} \left[e^{(1+\sqrt{3})t} - e^{(1-\sqrt{3})t}\right] u(t) \qquad \text{(tables (17))}$$

example 3

Let

$$F(s) = \frac{1}{s(s^2 - 2s + 5)}$$

The second factor in the denominator is nonfactorable. The decomposition is

$$\frac{1}{s(s^2 - 2s + 5)} = \frac{1/5}{s} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5}$$

The first term inverts to $\frac{1}{5}\, u(t)$.

The second term is like (5) above:

$$(*) \quad \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5} = \frac{-\frac{1}{5}s + \frac{2}{5}}{(s-1)^2 + 4} \qquad \text{(complete square)}$$

$$= -\frac{1}{5}\ \frac{(s-1) - 1}{(s-1)^2 + 4} \qquad \text{(rearrange numerator to match the shift in the denom)}$$

$$= -\frac{1}{5}\left[\frac{s-1}{(s-1)^2 + 4} - \frac{1}{(s-1)^2 + 4}\right]$$

$$\leftrightarrow -\frac{1}{5}\, e^t \cos 2t\, u(t) + \frac{1}{5}\,\frac{1}{2}\, e^t \sin 2t\, u(t) \qquad \text{(s-shifting)}$$

So

$$\frac{1}{s(s^2 - 2s + 5)} \leftrightarrow \left[\frac{1}{5} - \frac{1}{5}\, e^t \cos 2t + \frac{1}{10}\, e^t \sin 2t\right] u(t)$$

example 4

Let

$$F(s) = \frac{s^2 + 2s - 2}{s(s+2)(s-4)}$$

method 1 for inverse transforming

Split $F(s)$ into three fractions and use (18),(17) and (22) in the tables.

$$F(s) = \frac{s^2}{s(s+2)(s-4)} + \frac{2s}{s(s+2)(s-4)} - \frac{2}{s(s+2)(s-4)}$$

$$= \frac{s}{(s+2)(s-4)} + \frac{2}{(s+2)(s-4)} - \frac{2}{s(s+2)(s-4)}$$

$$\leftrightarrow -\frac{1}{6}(-2e^{-2t} - 4e^{4t})u(t) - \frac{2}{6}(e^{-2t} - e^{4t})u(t) - 2(-\frac{1}{8} + \frac{1}{12}e^{-2t} + \frac{1}{24}e^{4t})u(t)$$

$$= \left[\frac{1}{4} - \frac{1}{6}e^{-2t} + \frac{11}{12}e^{4t}\right] u(t)$$

method 2 for inverse transforming

F(s) decomposes into

$$\frac{1/4}{s} - \frac{1/6}{s+2} + \frac{11/12}{s-4}$$

So

$$f(t) = \left[\frac{1}{4} - \frac{1}{6}e^{-2t} + \frac{11}{12}e^{4t}\right] u(t)$$

PROBLEMS FOR SECTION 5.3

1. Find the inverse transform (a) $\frac{5}{s^2}$ (b) $\frac{5}{s^4}$ (c) $\frac{5}{s-3}$ (d) $\frac{4}{s^2+5}$

(e) $\frac{1}{3-s}$ (f) $\frac{4s}{s^2+5}$

2. Find the inverse transform and draw its graph

(a) $\frac{e^{-2s}}{s}$ (b) $\frac{e^{-2s}}{s^2+3}$ (c) $\frac{e^{-2s}}{s+3}$ (d) e^{-2s}

3. Find the inverse transform

(a) $\frac{1}{(s-3)^3}$ (b) $\frac{1}{(s+2)^4}$ (c) $\frac{1}{(s-5)^2}$ (d) $\frac{1}{s+6}$ (e) $\frac{e^{-3s}}{(s+6)^8}$

4. (a) $\frac{2}{3s+4}$ (b) $\frac{1}{2s+1}$ (c) $\frac{3s}{2s^2+5}$

(d) $\frac{1}{s^2+5}$ (e) $\frac{1}{(s+4)^2+5}$ (f) $\frac{e^{-2s}}{s^2+5}$ (g) $\frac{e^{-2s}}{(s+4)^2+5}$

(h) $\frac{1}{s^2+2s}$ (i) $\frac{1}{s^2+2s+1}$ (j) $\frac{s+1}{s^2-3s+3}$

5. (a) Derive the inverse transform formulas that you'll find in the tables for these fractions by decomposing into simpler fractions.

(i) $\frac{1}{s^2-a^2}$ (ii) $\frac{s}{(s-a)^2}$

(b) The tables are missing the inverse transform of $\frac{1}{(s-a)^2}$ What is it?

6. Find the inverse transform of $\frac{s}{s+1}$ (first use long division)

7. For this problem just find the *form* of the partial fraction decomposition and use it to find the *form* of the inverse transform, all without actually computing the constants involved in the decomposition.

(a) $\frac{1}{s^3(s-2)}$ (b) $\frac{s}{(s-1)(s+2)^4}$

8. Find the inverse transform. If a partial fraction decomposition is necessary, just find the form of the answer without actually computing the constants involved in the decomposition. (a) $\dfrac{s^2 + 6s + 5}{s(s^2 + 4s + 5)}$ (b) $\dfrac{s + 1}{s^2(s^2 + 1)}$

(c) $\dfrac{1}{s^2 + 4s + 7}$ (d) $\dfrac{s}{s^2 + 3s + 3}$ (e) $\dfrac{s + 3}{s^2 + 2}$

(f) $\dfrac{s + 4}{2s^2 + 4s + 5}$ (g) $\dfrac{1}{(s+4)^2}$ (h) $\dfrac{s}{(s+4)^2}$ (i) $\dfrac{1}{s^2 + 4}$ (j) $\dfrac{s}{s^2 + 4}$

(k) $\dfrac{10 - 4s}{(s-2)^2}$ (ℓ) $\dfrac{s}{s^2 - 3}$

9. The answer to #8(ℓ) was $\cosh\sqrt{3}\,t$. Do you remember what $\cosh\sqrt{3}\,t$ is in terms of exponential functions.

10. Let

$$F(s) = \frac{1}{s^2 + 2s + 4}$$

(a) Find the inverse transform by completing the square
(b) Would it work to factor the denominator into *non-real* linear factors and use (17) in the tables.

SECTION 5.4 SOLVING DIFFERENTIAL EQUATIONS USING TRANSFORMS

transforms of derivatives

(1)	$f'(t) \leftrightarrow sF(s) - f(0)$
(2)	$f''(t) \leftrightarrow s^2 F(s) - sf(0) - f'(0)$

proof

$$\pounds\, f'(t) = \int_{t=0}^{\infty} f'(t)\, e^{-st}\, dt$$

Now use integration by parts with

$$u = e^{-st}, \quad dv = f'(t), \quad du = -se^{-st}\, dt, \quad v = f(t)$$

to get

$$\pounds\, f'(t) = \underbrace{e^{-st} f(t) \Big|_{t=0}^{\infty}}_{\substack{0 - f(0) \\ \text{see footnote}}} + \underbrace{s \int_{t=0}^{\infty} f(t)\, e^{-st}\, dt}_{F(s)} = sF(s) - f(0)$$

footnote
I'm assuming that $f(t)$ has a transform in the first place. So the improper integral $\int_{t=0}^{\infty} e^{-st} f(t)\, dt$ must exist for say $s > 0$. It can't exist unless $e^{-st} f(t) \to 0$ as $t \to \infty$ (in fact it must $\to 0$ quickly). So plugging $t=\infty$ into $e^{-st} f(t)$ gives 0.

This proves (1). To get (2), think of f'' as $(f')'$. Then

$$\pounds\, f''(t) = \pounds\,(f')' = s\, \pounds\, f'(t) - f'(0) \quad \text{by (1)}$$

$$= s\left[\, sF(s) - f(0) \,\right] - f'(0) \quad \text{by (1) again}$$

$$= s^2 F(s) - sf(0) - f'(0)$$

solving a DE with IC using transforms

Look at

$$y'' - 3y' + 2y = 2e^{-t} \quad \text{with IC} \quad y(0) = 2,\ y'(0) = -1.$$

The idea is to take transforms on both sides of the DE to get a new algebraic equation with unknown $Y(s)$. You can solve (easily) for $Y(s)$ and then (less easily) take the inverse transform to get $y(t)$, the solution to the DE. In the process the IC will be used automatically.

Transforming both sides of the DE gives

$$s^2 Y - sy(0) - y'(0) - 3[\, sY - y(0) \,] + 2Y = \frac{2}{s+1}$$

Use the IC $y(0) = 2$, $y'(0) = -1$ to get

$$s^2Y - 2s + 1 - 3\left[sY - 2\right] + 2Y = \frac{2}{s+1}$$

warning Don't leave out the brackets and write $-3sY - 2$ when it should be $-3[sY - 2]$

Then

$$(s^2 - 3s + 2)Y = \frac{2}{s+1} + 2s - 7$$

$$Y = \frac{2s^2 - 5s - 5}{(s+1)(s^2 - 3s + 2)} = \frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)}$$

You can split this into

$$\frac{2s^2}{(s+1)(s-1)(s-2)} - \frac{5s}{(s+1)(s-1)(s-2)} - \frac{5}{(s+1)(s-1)(s-2)}$$

and use (24),(23),(22) in the tables or you can decompose into partial fractions

$$\frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)} = \frac{1/3}{s+1} + \frac{4}{s-1} + \frac{-7/3}{s-2}$$

and then invert. Either way the answer is

$$y(t) = \left[\ \frac{1}{3}e^{-t} + 4e^{t} - \frac{7}{3}e^{2t}\right] u(t)$$

example 1

Use transforms to solve

$$y'' + 4y' + 3y = 0 \quad \text{with IC } y(0) = 3,\ y'(0) = 1$$

solution Take transforms on both sides of the DE to get

$$s^2Y - 3s - 1 + 4(sY - 3) + 3Y = 0 \quad \text{(the transform of 0 is 0)}$$

Then

$$Y = \frac{3s + 13}{s^2 + 4s + 3} = \frac{3s + 13}{(s+3)(s+1)} = \frac{3s}{(s+3)(s+1)} + \frac{13}{(s+3)(s+1)}$$

Use the tables to get

$$y(t) = (-2e^{-3t} + 5e^{-t})\ u(t)$$

example 2

Use transforms to solve

$$y'' + 3y' + 2y = f(t) \quad \text{with IC } y(0) = 0,\ y'(0) = 0$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

solution f(t) can be written as u(t) − u(t−4). Take transforms in the DE:

$$s^2Y + 3sY + 2Y = \frac{1}{s} - \frac{e^{-4s}}{s}$$

$$(3) \qquad Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-4s}}{s(s^2+3s+2)}$$

$$= \frac{1}{s(s+2)(s+1)} - \frac{e^{-4s}}{s(s+2)(s+1)}$$

Then (use tables and t−shifting)

$$(4) \qquad y(t) = \left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right] u(t) - \left[\frac{1}{2} - e^{-(t-4)} + \frac{1}{2}e^{-2(t-4)}\right] u(t-4)$$

writing without the u notation

If y = \$\$\$ u(t) + ### u(t−a) then

$$y = \begin{cases} 0 & \text{if } t \le 0 \\ \$\$\$ & \text{if } 0 \le t \le a \\ \$\$\$ + \#\#\# & \text{if } t \ge a \end{cases}$$

For example, if

$$y = u(t) + (t-7)u(t-7) + t^5 u(t-8)$$

then

$$(5) \qquad y = \begin{cases} 1 & \text{if } 0 \le t \le 7 \\ 1 + t-7 = t-6 & \text{if } 7 \le t \le 8 \\ t-6+t^5 & \text{if } t \ge 8 \end{cases}$$

warning

If $y = u(t) + (t-7)u(t-7) + t^5 u(t-8)$ it is *not* correct to write

$$\text{WRONG WRONG} \qquad y = \begin{cases} 1 & \text{if } 0 \le t \le 7 \\ t-7 & \text{if } 7 \le t \le 8 \\ t^5 & \text{if } t \ge 8 \end{cases} \qquad \text{WRONG WRONG}$$

The right version is in (5).

example 2 continued

The solution in (4) can be rewritten as

$$(6)\quad y(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} & \text{for } 0 \le t \le 4 \\ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - \left[\frac{1}{2} - e^{-(t-4)} + \frac{1}{2}e^{-2(t-4)}\right] & \text{for } t \ge 4 \end{cases}$$

warning

When you write (4), don't leave out the u(t) and especially not the u(t−4). If you do then your "answer" is

$$\text{WRONG} \qquad y = e^{-t} + \frac{1}{2}e^{-2t} + e^{-(t-4)} - \frac{1}{2}e^{-2(t-4)} \qquad \text{WRONG}$$

which is very different from the correct answer in (6).

review of Cramer's rule

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The determinant of coefficients is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If this determinant is non−zero then the solution is

$$x = \frac{\text{coeff determinant but with column 1 replaced by } \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}}{\text{coeff determinant}}$$

$$y = \frac{\text{coeff determinant but with column 2 replaced by } \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}}{\text{coeff determinant}}$$

$$z = \frac{\text{coeff determinant but with column 3 replaced by } \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}}{\text{coeff determinant}}$$

solving a system of DE with IC using transforms

Consider the system

$$x'' = -3x + 2y$$
$$y'' = 6x - 7y$$

with IC

$$x(0) = 0,\ y(0) = 1,\ x'(0) = 1,\ y'(0) = 2.$$

The unknowns are the functions x(t) and y(t).

Take transforms and collect terms:

$$s^2 X - 1 = -3X + 2Y$$
$$s^2 Y - s - 2 = 6X - 7Y$$

$$(s^2 + 3)X - 2Y = 1$$
$$-6X + (s^2 + 7)Y = s + 2$$

Use Cramer's rule to solve for X and Y:

$$X = \frac{\begin{vmatrix} 1 & -2 \\ s+2 & s^2+7 \end{vmatrix}}{\begin{vmatrix} s^2+3 & -2 \\ -6 & s^2+7 \end{vmatrix}} = \frac{s^2 + 2s + 11}{s^4 + 10s^2 + 9} = \frac{s^2 + 2s + 11}{(s^2 + 9)(s^2 + 1)}$$

$$Y = \frac{\begin{vmatrix} s^2+3 & 1 \\ -6 & s+2 \end{vmatrix}}{(s^2 + 9)(s^2 + 1)} = \frac{s^3 + 2s^2 + 3s + 12}{(s^2 + 9)(s^2 + 1)}$$

To find inverse transforms, decompose (or better still, get a larger set of tables or use Mathematica):

$$X = \frac{-\frac{1}{4}s - \frac{1}{4}}{s^2 + 9} + \frac{\frac{1}{4}s + \frac{5}{4}}{s^2 + 1}$$

$$Y = \frac{\frac{3}{4}s + \frac{3}{4}}{s^2 + 9} + \frac{\frac{1}{4}s + \frac{5}{4}}{s^2 + 1}$$

Then

$$x = \left[-\frac{1}{4}\cos 3t - \frac{1}{12}\sin 3t + \frac{1}{4}\cos t + \frac{5}{4}\sin t \right] u(t)$$

$$y = \left[\frac{3}{4}\cos 3t + \frac{1}{4}\sin 3t + \frac{1}{4}\cos t + \frac{5}{4}\sin t \right] u(t)$$

finding the impulse response using transforms

Look at the system where input f(t) and output y(t) are related by

$$2y'' - 4y' - 6y = f(t)$$

I want to find the impulse response h(t) of the system. This means solving

(7) $\quad 2y'' - 4y' - 6y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$

Take transforms in (7) to get

$$2s^2 Y - 4sY - 6Y = 1$$

$$Y = \frac{1}{2s^2 - 4s - 6}$$

This is the transform H(s) of the impulse response h(t). Factor and use tables (18) (or decompose):

$$H(s) = \frac{1}{2}\,\frac{1}{(s+1)(s-3)}$$

$$h(t) = \left(-\frac{1}{8}e^{-t} + \frac{1}{8}e^{3t}\right)u(t)$$

footnote
For comparison, here's the method from Chapter 2 for finding the impulse response. Switch from (7) to

$$2y'' - 4y' - 6y = 0 \text{ with IC } y(0) = 0,\ y'(0) = \frac{1}{2}$$

Then $2m^2 - 4m - 6 = 0$, $m = 3,-1$, $y_h = Ae^{3t} + Be^{-t}$.

The IC make $A = \frac{1}{8}$, $B = -\frac{1}{8}$ so

$$h(t) = \frac{1}{8}e^{3t} - \frac{1}{8}e^{-t} \quad \text{for } t \ge 0$$

In general, the transform H(s) of the impulse response h(t) is referred to as the system's *transfer function*. If inputs f(t) and outputs y(t) are related by

$$ay'' + by' + cy = f(t)$$

then

(8) $$H(s) = \frac{1}{as^2 + bs + c}$$

example 3
Solve

$$2y'' + 3y' + y = \cos t \text{ with IC } y(0) = 0,\ y'(0) = 0.$$

method 1 (as in examples 1 and 2) Take transforms on both sides of the DE:

$$s^2Y + 3sY + Y = \frac{s}{s^2+1}$$

$$Y = \frac{s}{(s^2+3s+1)(s^2+1)}$$

Now take the inverse transform to get solution y(t). I'm not going to bother doing it. (Mathematica did it in a split second. It would take me 15 minutes just to type the inverse transform.)

method 2 By (8)

$$H(s) = \frac{1}{s^2+3s+1}$$

Then by (9),

$$Y(s) = H(s)\,\mathcal{L}\cos t = \frac{1}{s^2+3s+1}\cdot\frac{s}{s^2+1}$$

Same now as method 1.

PROBLEMS FOR SECTION 5.4

1. Transform the DE, solve for Y and then stop (so that someone who had a large set of inverse transform tables and/or a computer to decompose could easily finish the problem)

$$2y'' + 3y' + 4y = e^{-8t} \sin 3t \quad \text{with IC } y(0) = -5,\ y'(0) = 6$$

2. Use transforms to solve

(a) $y'' + y = \sin 3t$ with IC $y(0) = 0$, $y'(0) = 0$

(b) $y'' + y = 2 \cos t$ with IC $y(0) = 2$, $y'(0) = 0$

(c) $i'(t) + 5i(t) = 25 \sin 5t$ with IC $i(0) = 0$
In particular, find the steady state solution

(d) $y'' + 3y' + 2y = e^{-t}$ with IC $y(0) = 0$, $y'(0) = 0$

3. Use transforms to solve

(a) $y'' + 2y = f(t)$ with IC $y(0) = 0$, $y'(0) = 0$ where $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $y'' + 4y = \begin{cases} \sin t & \text{if } 0 \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$ with IC $y(0) = 0$, $y'(0) = 0$

4. Solve the system of DE using transforms

(a) $x' = 7x + 6y$
$y' = 2x + 6y$ with IC $x(0) = 2$, $y(0) = 1$

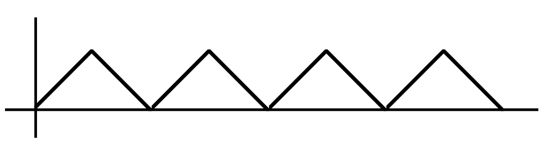
(b) $x' = 2x - 2y$, $y' = x$ with IC $x(0) = 2$, $y(0) = 2$

(c) $y_1' = 10y_2 - 20y_1 + 100$, $y_2' = 10y_1 - 20y_2$ with IC $y_1(0) = 0$, $y_2(0) = 0$

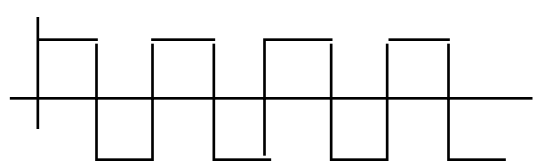
5. If the input $f(t)$ and response $y(t)$ are related by the given DE, use transforms to find (and then sketch) the impulse response.

(a) $2y'' + 3y = f(t)$ (b) $y'' + 5y' + 6y = f(t)$ (c) $y' + y = f(t)$

6. Let $f(t)$ be the triangular wave in the diagram. Its derivative is the square wave $g(t)$. Suppose you know $G(s)$. How would you find $F(s)$.



f(t)



g(t)

Problem 6

HONORS

7. A function f(t) has many antiderivatives. The partiular antiderivative whose value is 0 when t=0 is $\int_0^t f(t)\,dt$.

Analogous to the transform rules for the derivatives f'(t) and f"(t) it can be shown that there is a transform rule for the antiderivative $\int_0^t f(t)\,dt$, namely

$$(*) \qquad \int_0^t f(t)\,dt \quad \leftrightarrow \quad \frac{1}{s} F(s)$$

(a) Use it to solve the following integral equation

$$2y + \int_0^t y(t)\,dt = f(t)$$

where

$$f(t) = \begin{cases} 4 & \text{for } a \le t \le b \\ 0 & \text{otherwise} \end{cases}$$

(b) The rule in (*) read from from right to left says that to find the inverse of

$$\frac{1}{s} \times \text{something},$$

invert the something and then take $\int_0^t$. Use this to show how (11) in the inverse transform table can be derived from (10).

SECTION 5.5 CONVOLUTION

transform of a convolution

Remember that

$$f(t)*g(t) = \int_{u=-\infty}^{\infty} f(t-u)\,g(u)\;du$$

Now, in addition, let f(t) and g(t) be 0 for t < 0 (as has been the case throughout this chapter). Then

$$\boxed{f(t)*g(t) \quad\leftrightarrow\quad F(s)G(s)}$$

> In other words, convolving functions (that start at t = 0) in the t world corresponds to multiplication in the transform world. So to find the convolution f(t) * g(t)
> step 1 find the transforms F(s) and G(s)
> step 2 multiply the transforms
> step 3 take the inverse transform

proof (slippery)

$$\pounds\, f(t)*g(t) = \int_{t=0}^{\infty}\left[\int_{u=0}^{\infty} f(u)\,g(t-u)\;du\right] e^{-st}\;dt$$

It's OK to use u=0 as the lower limit in the convolution integral instead of $-\infty$ since f(u) = 0 for u ≤ 0. Now rewrite e^{-st} as $e^{-su}e^{-s(t-u)}$ and rearrange to get

$$\pounds\, f(t)*g(t) = \int_{u=0}^{\infty}\left[\int_{t=0}^{\infty} g(t-u)\,e^{-s(t-u)}\;dt\right] f(u)\;e^{-su}\;du$$

Substitute w = t−u, dw = dt in the inner integral to get

$$\pounds\, f(t)*g(t) = \int_{u=0}^{\infty}\left[\int_{w=-u}^{\infty} g(w)\;e^{-sw}\;dw\right] f(u)\;e^{-su}\;du$$

Since g(w) = 0 for w ≤ 0 we can change the lower limit on the inner integral from w = −u to w = 0. So

$$\pounds\, f(t)*g(t) = \int_{u=0}^{\infty}\underbrace{\left[\int_{w=0}^{\infty} g(w)\;e^{-sw}\;dw\right]}_{G(s)} f(u)\;e^{-su}\;du$$

$$= G(s)\int_{u=0}^{\infty} f(u)\;e^{-su}\;du$$

$$= G(s)F(s) \qquad \text{QED}$$

example 1

Let f(t) = t u(t) and g(t) = sin t u(t). Find f(t) $*$ g(t) using transforms

First take the transforms of f(t) and g(t):

$$\sin t\ u(t) \quad \leftrightarrow \quad \frac{1}{s^2+1}$$

$$tu(t) \quad \leftrightarrow \quad \frac{1}{s^2}$$

Multiply the transforms to get

$$F(s)G(s) = \frac{1}{s^2(s^2+1)}$$

From the transform tables,

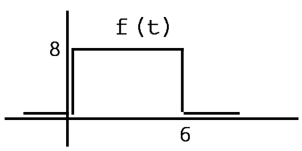
$$\frac{1}{s^2(s^2+1)} \quad \leftrightarrow \quad (t - \sin t)\ u(t)$$

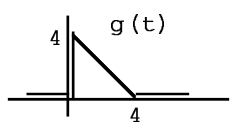
So

$$f(t) * g(t) = (t - \sin t)\ u(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t - \sin t & \text{if } t \ge 0 \end{cases}$$

example 2

Use transforms to find the convolution of the following functions and write the final answer without using u(t) notation.





First find the transforms and multiply them.

$$f(t) = 8u(t) - 8u(t-6) \quad \text{and} \quad g(t) = 4u(t) - r(t) + r(t-4)$$

$$F(s) = \frac{8}{s} - \frac{8}{s}e^{-6s} \quad \text{and} \quad G(s) = -\frac{1}{s^2} + \frac{4}{s} + \frac{1}{s^2}e^{-4s}$$

$$F(s)G(s) = -\frac{8}{s^3} + \frac{32}{s^2} + \frac{8}{s^3}e^{-4s} + \frac{8}{s^3}e^{-6s} - \frac{32}{s^2}e^{-6s} - \frac{8}{s^3}e^{-10s}$$

Now invert.

$$f*g = \left[-\frac{8t^2}{2!} + 32t\right]u(t) + \frac{8(t-4)^2}{2!}u(t-4) + \left[\frac{8(t-6)^2}{2!} - 32(t-6)\right]u(t-6)$$

$$- \frac{8(t-10)^2}{2!}\ u(t-10)$$

So

if $0 \le t \le 4$ then $f*g = -\frac{8t^2}{2!} + 32t = -4t^2 + 32t$

if $4 \le t \le 6$ then $f*g = -4t^2 + 32t + \frac{8(t-4)^2}{2!} = 64$

if $6 \le t \le 10$ then $f*g = 64 + \frac{8(t-6)^2}{2!} - 32(t-6) = 4t^2 - 80t + 400$

if $t \ge 10$ then $f*g = 4t^2 - 80t + 400 - \frac{8(t-10)^2}{2!} = 0$,

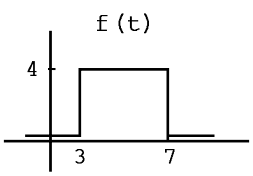
i.e.,

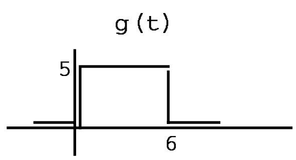
$$f*g = \begin{cases} 0 & \text{if } t \le 0 \\ -4t^2 + 32t & \text{if } 0 \le t \le 4 \\ 64 & \text{if } 4 \le t \le 6 \\ 4t^2 - 80t + 400 & \text{if } 6 \le t \le 10 \\ 0 & \text{if } t \ge 10 \end{cases}$$

PROBLEMS FOR SECTION 5.5

1. Let $g(t) = e^{-t}\,u(t)$ and $f(t) = 2-t$ for $0 \le t \le 2$ (f is 0 otherwise)
   Find $f(t)*g(t)$ using transforms. Give the answer with the u notation and then again without the u notation.

2. Use transforms to find $f(t)*g(t)$





3. Use transforms to convolve $f(t) = e^{-t}u(t)$ and $g(t) = tu(t)$

4. Let $f(t) = \lambda e^{-\lambda t}\,u(t)$ (where λ is just a fixed constant.)

   Find $f(t)*f(t)*f(t)*f(t)$

5. Take transforms to show that the solution to

$$y'' + a^2y = f(t) \text{ with } y(0) = K_1,\ \ y'(0) = K_2$$

is

$$y = \frac{1}{a}\sin at * f(t) + K_1 \cos at + \frac{K_2}{a}\sin at \quad \text{for } t \ge 0$$

HONORS

6. Let $f(t)$ be an arbitrary function (starting at $t=0$).

(a) Find the convolution $\delta(t)*f(t)$ directly (using the definition of convolution) and then again with transforms.

Interpret the result physically by thinking of $f(t)$ as an input into an initially-at-rest-system which has impulse response $\delta(t)$.

(b) Find $\delta(t-a)*f(t)$ directly and then again with transforms.

Interpret the result physically.

REVIEW PROBLEMS FOR CHAPTER 5

1\. Rewrite the following without the step notation and sketch the graph.

(a) $f(t) = e^{-5t}[u(t-1) - u(t-3)]$ (b) $f(t) = e^{-5t}u(t) + e^{-5(t-2)}\,u(t-2)$

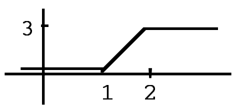
2\. Solve using transforms.

(a) $y'' + y = \cos t$ with IC $y(0) = 0$, $y'(0) = 0$

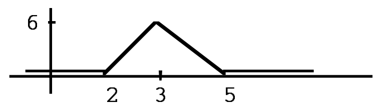
(b) $y'' + 4y' + 5y = 5$ with IC $y(0) = 1$, $y'(0) = 2$

3\. Find the transform.

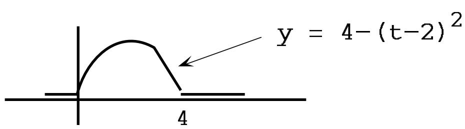
(a)



(b)



(c)

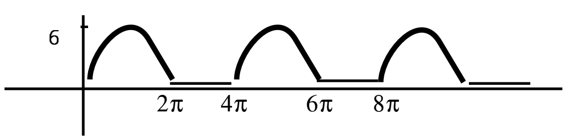


4\. (a) Let $a \ge 0$. Show that $\mathcal{L} f(at) = \frac{1}{a} F(\frac{s}{a})$ (called the scaling rule).

(b) Suppose $\mathcal{L} \sin t \sinh t = \dfrac{2s}{s^4 + 4}$. Use part (a) to find $\mathcal{L} \sin at \sinh at$.

5\. Use transforms to find the impulse response of the system whose input f(t) and response y(t) are related by $y'' + y' + 7y = f(t)$.

6\. Find the transform of the following function which is periodic for $t \ge 0$. The non-zero pieces are sines.



7\. Find the inverse tranform (a) $\dfrac{1}{s^4}$ (b) $\dfrac{1}{(s+2)^3}$ (c) $\dfrac{5}{(s-4)^2}$

(d) $\dfrac{1}{3s + 4}$ (e) $\dfrac{e^{-4s}}{s^3}$ (f) $\dfrac{s}{(s+1)(s^2 + 1)}$ (g) $\dfrac{s}{s^2 + 2}$

(h) $\dfrac{s}{s^2 - 1}$ (i) $\dfrac{1}{s^2(s-2)}$

8\. Find these integrals by inspection.

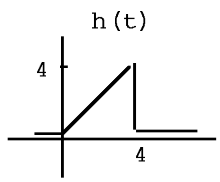
(a) $\displaystyle\int_0^\infty e^{-3t}\, t^4\, dt$ (b) $\displaystyle\int_0^\infty e^{-st}\, e^{-3t}\, t^4\, dt$

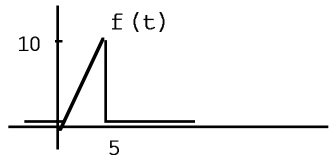
9. Solve for $x(t)$ and $y(t)$ if

$$x'' = -5x + 4y, \quad y'' = 4x - 5y$$

with IC $x(0) = 1$, $y(0) = -1$, $x'(0) = 0$, $y'(0) = 0$

10. Use transform to find $h(t) * f(t)$.





APPENDIX 1 FINDING TRANSFORMS AND INVERSE TRANSFORMS WITH MATHEMATICA

Load the transform package and the package containing the unit step function.

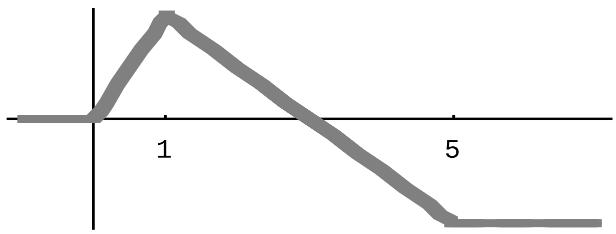
```
<<:Calculus:LaplaceTransform.m

<<:Calculus:DiracDelta.m
```

For convenience, introduce shorthand notation for the unit step and unit ramp function

```
u[t_]:= UnitStep[t]; r[t_]:= t u[t];
```

Here's the graph and the transform of a function built out of ramps.

```
Plot[ 2r[t] - 3r[t-1] + r[t-5],{t,-1,7},Ticks->{{1,5},None},
   PlotStyle->{{GrayLevel[.5], Thickness[.02]}}];
```




```
LaplaceTransform[2r[t] - 3r[t-1] + r[t-5],t,s]

2       1        3
-- + ------- - -----
 2    5 s  2    s  2
s    E    s    E  s
```

Here's the graph of an exponential pulse and its transform

```
Plot[E^(2t) (u[t] - u[t-2]),{t,-2,10}, Ticks->{{2},None},
   PlotStyle->{{GrayLevel[.5], Thickness[.02]}}];
```


```
LaplaceTransform[E^(2t) (u[t] - u[t-2]),t,s]

  1                1
------ - --------------------
-2 + s    2 (-2 + s)
         E            (-2 + s)
```

But Mathematica couldn't take the transform of a sine pulse. It gave an incomplete answer

```
LaplaceTransform[Sin[t] (u[t] - u[t-2]),t,s]

  1
------ - LaplaceTransform[Sin[t] UnitStep[-2 + t], t, s]
     2
1 + s
```

Here's the inverse transform of $\frac{e^{-5s}}{s^4}$. Mathematica gives the answer $\frac{(t-5)^3}{3!}$ u(t−5) but all cubed out.

```
InverseLaplaceTransform[E^(-5s)/s^4, s,t]
```

```
                        2     3
      125     25 t    5 t     t
(-(---) + ---- - ---- + --) UnitStep[-5 + t]
       6       2       2      6
```

Here's the inverse transform of $\frac{1}{(s-a)^{10}}$. Answer is $\frac{e^{at}\ t^9}{9!}$ but Mathematica multiples out the factorial.

```
InverseLaplaceTransform[1/(s-a)^10,s,t]
```

```
 a t  9
E    t
-------
362880
```

Here's one where Mathematica doesn't get the simplest possible answer until you do a little algebra yourself and then make it use some trig.

```
InverseLaplaceTransform[Pi E^(-s)/(s^2 + Pi^2),s,t]
```

```
Sin[Pi (-1 + t)] UnitStep[-1 + t]
```

```
Sin[-Pi + Pi t] u[-1 + t]//TrigReduce
```

```
-(Sin[Pi t] UnitStep[-1 + t])
```

APPENDIX 2 PARTIAL FRACTION DECOMPOSITION

decomposition with non-repeated linear factors

Let $F(s) = \dfrac{s^2 + 2s - 2}{s(s+2)(s-4)}$.

There is a decomposition of the form

$$\frac{s^2 + 2s - 2}{s(s+2)(s-4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4} \tag{3}$$

Since the factors are not repeated (i.e., there are no factors in the denominator like $(s+2)^2$ or s^3) it's easy to get A,B,C. You can use any method you may remember from calculus but here's how I do it.
To find A, delete the factor s from the left side of (3) and set s = 0
To find B, delete the factor s+2 from the left side of (3) and set s = −2
To find C, delete the factor s−4 from the left side of (3) and set s = 4.
So

$$A = \left.\frac{s^2 + 2s - 2}{(s+2)(s-4)}\right|_{s=0} = \frac{-2}{-8} = \frac{1}{4}$$

warning This method only works for *non-repeated* linear factors.

$$B = \left.\frac{s^2 + 2s - 2}{s(s-4)}\right|_{s=-2} = \frac{-2}{12} = -\frac{1}{6}$$

$$C = \left.\frac{s^2 + 2s - 2}{s(s+2)}\right|_{s=4} = \frac{11}{12}$$

So

$$F(s) = \frac{1/4}{s} - \frac{1/6}{s+2} + \frac{11/12}{s-4}$$

and

$$f(t) = \left[\frac{1}{4} - \frac{1}{6}e^{-2t} + \frac{11}{12}e^{4t}\right] u(t)$$

decomposition with non-repeated non-factorable quadratic factors

Let $F(s) = \dfrac{1}{s(s^2 - 2s + 5)}$

The factor $s^2 - 2s + 5$ doesn't factor (because $b^2 - 4ac < 0$). There is a decomposition of the form

$$\frac{1}{s(s^2 - 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5} \tag{4}$$

Then

$$A = \left.\frac{1}{(s^2 - 2s + 5)}\right|_{s=0} = \frac{1}{5}$$

To find B and C, multiply by $s(s^2 - 2s + 5)$ in (4) to get

$$1 = A(s^2 - 2s + 5) + (Bs + C)s$$

Equate coeffs of s^2: $0 = A + B, \quad B = -A = -\frac{1}{5}$

Equate coeffs of s: $0 = -2A + C, \quad C = 2A = \frac{2}{5}$

So

$$F(s) = \frac{1/5}{s} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5}$$

The first term inverts to $\frac{1}{5}\, u(t)$

For the second term, complete the square in the denom

$$(*) \qquad \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5} = \frac{-\frac{1}{5}s + \frac{2}{5}}{(s-1)^2 + 4}$$

and then rearrange the numerator as follows to "match" :

$$\frac{-\frac{1}{5}s + \frac{2}{5}}{(s-1)^2 + 4} = -\frac{1}{5}\,\frac{(s-1) - 1}{(s-1)^2 + 4} = -\frac{1}{5}\left[\frac{s-1}{(s-1)^2 + 4} - \frac{1}{(s-1)^2 + 4}\right]$$

Use s-shifting to get the inverse transform

$$-\frac{1}{5}\, e^t \cos 2t\, u(t) + \frac{1}{5}\,\frac{1}{2}\, e^t \sin 2t\, u(t)$$

Then the final inverse transform is

$$\left[\frac{1}{5} - \frac{1}{5}\, e^t \cos 2t + \frac{1}{10}\, e^t \sin 2t\right] u(t)$$

decomposition with repeated linear factors

$$\frac{s}{(s-5)^4} = \frac{A}{s-5} + \frac{B}{(s-5)^2} + \frac{C}{(s-5)^3} + \frac{D}{(s-5)^4}$$

$$s = A(s-5)^3 + B(s-5)^2 + C(s-5) + D$$

set $s = 5$: $\quad 5 = D$

equate s^3 coeffs: $\quad 0 = A$

equate s^2 coeffs: $\quad (\text{who cares})\cdot A + B = 0,\ B = 0$

set $s = 6$: $\quad 6 = A + B + C + D\ ,\ C = 1$

So

$$\frac{s}{(s-5)^4} = \frac{1}{(s-5)^3} + \frac{5}{(s-5)^4} \leftrightarrow \left[\frac{t^2}{2} + \frac{5t^3}{3\,!}\right] e^{5t}\, u(t)$$

decomposition with repeated non-factorable quadratic factors

Too ugly.

CHAPTER 6 PARTIAL DIFFERENTIAL EQUATIONS

SECTION 6.1 THE HEAT EQUATION AND FOURIER SINE SERIES

the heat equation and its physical significance

The 1-dimensional heat equation is

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad \text{(k is a fixed positive constant)}$$

Consider the rod in Fig 1 with net temperature $u(x,t)$ at position x and time t (net means degrees above room temperature). If the lateral surface of the rod is insulated (the ends may or may not be insulated) and k is a constant determined by the composition of the rod then it can be shown that $u(x,t)$ satisfies the heat equation. (So do lots of other things too.)

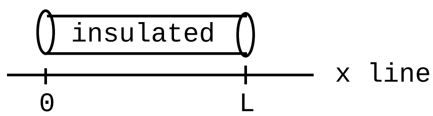


FIG 1

The heat equation comes with one initial condition (IC) of the form

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L$$

describing the initial temperature distribution along the rod.

There are two boundary conditions (BC) describing what happens at the ends of the rod. For example the BC

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all } t$$

correspond to maintaining the ends of the rod at (net) temperature 0.

To understand other BC you have to understand the significance of the partial derivative $\partial u/\partial x$. For any fixed value of t, the graph of $u(x,t)$ shows the temperature distribution in the rod at time t, and $\partial u/\partial x$ is the slope on the temperature hill (Fig 2).

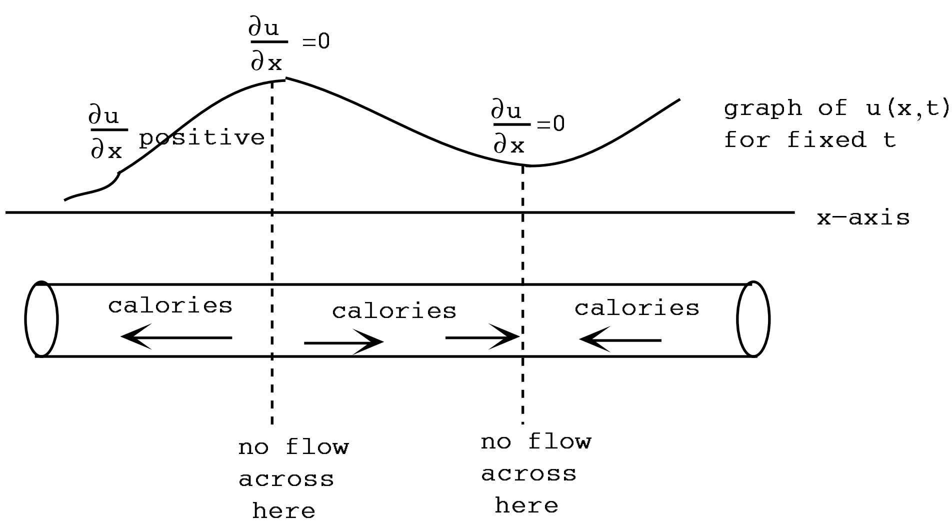


FIG 2

Physicists say that calories flow down temperature hills (from hot to cold) and so there's the following correspondence between $\frac{\partial u}{\partial x}$ and calory flow:

$\frac{\partial u}{\partial x}$ positive at x_0 A calory at this point flows to the left

$\frac{\partial u}{\partial x} = 0$ at x_0 A calory at this point doesn't move

$\frac{\partial u}{\partial x}$ negative at x_0 A calory at this point flows to the right

Physicists call $-\partial u/\partial x$ the heat flux density; it measures the calories per second per unit cross sectional area flowing in the rod from left to right.

The BC

$$\frac{\partial u}{\partial x}(L,t) = 0$$

for the rod in Fig 1 corresponds to an insulated right end; no calories flow across the right end.

The BC

$$-\frac{\partial u}{\partial x}(L,t) = C\,u(L,t)$$

where C is a fixed positive constant describes a flow of heat out the right end in proportion to the temperature at the right end (convection).

Two popular sets of BC are

(i) $u(0,t) = 0$ and $u(L,t) = 0$ for all t (both ends fixed at temperature 0)

(ii) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$ for all t (both ends insulated)

example 1

I'll solve the heat equation with

BC $u(0,t) = 0$, $u(6,t) = 0$ for all t

IC $u(x,0) = x$ for x in $[0,6]$

(A rod of length 6 with insulated lateral surface is initially heated so that at time 0 the temperature is 0 at the left end and increases steadily to 6 at the right end. Thereafter the ends are maintained at temperature 0. I'll find the temperature in the rod at position x and time t.)

Part I Separate the variables

To "separate variables" in the heat equation try a solution of the form

$$u(x,t) = X(x)T(t),$$

i.e., a solution containing x's and t's but with each variable appearing in a separate factor. The aim is to find a bunch of solutions containing arbitrary constants. When I try to satisfy the BC and IC I'll come back to this bunch of solutions to find one that fits.

Substitute into the heat equ to get

$$XT' = kX''T$$

Rearrange to get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$$

The left side has no t's, the right side has no x's so each side has no t's *and* no x's, i.e., each side is a constant. In other words (*key idea*), a function of x can't equal a function of t unless both functions are constant. So

$$\frac{X''}{X} = \frac{T'}{kT} = \text{constant}$$

This is really a pair of equations:

$$\frac{X''}{X} = \text{constant}, \qquad \frac{T'}{kT} = \text{same constant}.$$
$$X'' - \text{constant } X = 0, \qquad T' - k \text{ constant } T = 0.$$

For the X equation, $m = \pm\sqrt{\text{constant}}$. The X solution depends on whether the m's are real unequal, real equal or nonreal and this in turn depends on the sign of what's under the square root sign. So you need cases in order to continue.

case 1 The constant is negative. Call it $-\lambda^2$. Then

$X'' + \lambda^2 X = 0$, $m = \pm\lambda i$, $X = A\cos\lambda x + B\sin\lambda x$

$T' + k\lambda^2 T = 0$, $m = -k\lambda^2$, $T = Ce^{-k\lambda^2 t}$

case 2 The constant is positive. Call it λ^2 for convenience. Then

$X'' - \lambda^2 X = 0$, $m = \pm\lambda$, $X = De^{\lambda x} + Fe^{-\lambda x}$

$T' - k\lambda^2 T = 0$, $m = k\lambda^2$, $T = Ge^{k\lambda^2 t}$

case 3 The constant is 0. Then

$X'' = 0$, $X = Px + Q$,

$T' = 0$, $T = K$

Not only does the heat equation itself separate but the homogeneous BC separate as follows.

The boundary condition

$u(0,t) = 0$ for all t

becomes

$X(0)T(t) = 0$ for all t.

So

$X(0) = 0$ or $T(t) = 0$ for all t.

But if $T(t) = 0$ for all t then $u(x,t) = X(x)T(t) = 0$ for $0 \le x \le 6$, all t, a useless solution.

> Here's why it's useless. We eventually have to get a solution satisfying the IC. To do this, we want a "general" solution with arbitrary constants. We'll get one by using a superposition principle, i.e., we'll add all the solutions satisfying the (homogeneous) PDE and the (homogeneous) BC. It is of no consequence to include $u = 0$ in that sum.

So for all practical purposes, $X(0) = 0$ is the only useful possibility. Similarly, the BC $u(6,t) = 0$ for all t becomes $X(6) = 0$. So all in all, the BC separate to

$$X(0) = 0, \quad X(6) = 0$$

Part II Plug in the BC

case 1 Of the three cases, this is the one which has $u \to 0$ as $t \to \infty$; it is the most appropriate for a heat problem in which the ends of the rod are not insulated and all the calories eventually flow out the ends.
From $X(0) = 0$ you get $A = 0$
From $X(6) = 0$ you get $B \sin 6\lambda = 0$.

So either $B = 0$ or $\sin 6\lambda = 0$. But $B = 0$ together with $A = 0$ makes $X = 0$ and produces only the solution $u = 0$ which is not useful. So continue with

$$\sin 6\lambda = 0,$$

$$6\lambda = n\pi,$$

$$\lambda = n\pi/6 \text{ for any nonzero integer } n$$

(n can't be 0 since this is the case where $-\lambda^2$ is negative). So

$$X = B \sin \frac{n\pi x}{6} \quad \text{for } n = 1,2,3,\ldots\ ; \quad \text{any } B.$$

Nothing extra is gained by using $n = -1,-2,-3,\ldots$ so forget them.

footnote Here's why I'm ignoring $\lambda = -\pi x/6,\ -2\pi x/6, \ldots$ etc.

First of all, $\sin \frac{-n\pi x}{6} = -\sin \frac{n\pi x}{6}$ so $B \sin \frac{-n\pi x}{6} = -B \sin \frac{n\pi x}{6}$. Since B is an arbitrary constant, this is no different from $B \sin \frac{n\pi x}{6}$. So you don't get any more solutions for X by considering the negative λ's.

Second, $Ce^{-k(-n\pi/6)^2 t} = Ce^{-k(n\pi/6)^2 t}$ so you don't get any more solutions for T by considering the negative λ's.

case 2 $X = Ae^{\lambda x} + Be^{-\lambda x}$, $T = Ce^{k\lambda^2 t}$

$X(0) = 0$ makes $A + B = 0$, $B = -A$

$X(6) = 0$ makes $Ae^{6\lambda} + Be^{-6\lambda} = 0$
So

$$Ae^{6\lambda} - Ae^{-6\lambda} = 0$$

$$A(e^{6\lambda} - e^{-6\lambda}) = 0$$

$$A = 0 \text{ or } e^{6\lambda} = e^{-6\lambda}$$

So

$$A = 0 \quad \text{or} \quad \lambda = 0$$

This is the case where λ^2 is positive so λ can't be 0. So $A = 0$. But then $B = -A = 0$, $X = 0$, $u = 0$ (the trivial solution). Nothing useful comes out of this case.

case 3 $X = Ax + B$, $T = C$
$X(0) = 0$ makes $B = 0$
$X(6) = 0$ makes $6A = 0$, $A = 0$
So $X = 0$, $u = X(x)T(t) = 0$, a trivial solution.
This case is not useful.

Part III Get a general solution and plug in the IC
From the one productive case you have all the solutions

$$u(x,t) = X(x)T(t) = B e^{-k(\frac{n\pi}{6})^2 t} \sin \frac{n\pi x}{6} \quad , n = 1,2,3,...; \text{ any } B.$$

What happened to the C? It got absorbed by the B.
The rule for arbitrary constants is BC = D or in sloppy notation, BC = B

For instance some solutions are

$$u_1(x,t) = B_1 e^{-k(\frac{\pi}{6})^2 t} \sin \frac{\pi x}{6}$$

$$u_2(x,t) = B_2 e^{-k(\frac{2\pi}{6})^2 t} \sin \frac{2\pi x}{6} \qquad \text{etc.}$$

Before continuing, I need some superposition principles.

superposition rule for a linear homogeneous PDE

There was a superposition rule for a linear homogeneous *ordinary* DE , i.e., an equation of the form ay'' + by' + cy = 0.
There is a similar principle for a linear homogeneous *partial* DE, say with unknown u(x,t), an equation of the form

$$(*) \qquad a_1 \frac{\partial u^2}{\partial x^2} + a_2 \frac{\partial^2 u}{\partial t^2} + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial t} + cu(x,t) = 0$$

If u_1 and u_2 are solutions of (*) then $u_1 + u_2$ and Au_1 are also solutions .

superposition rule for linear homogeneous BC

If u_1 and u_2 both satisfy say BC u(6,t) = 0 then $u_1 + u_2$ also satisfies that BC.
In other words:

If u_1 and u_2 are both 0 at an end of the rod then $u_1 + u_2$ is also 0 at that end.

(This *doesn't* hold for *non*homog BC: If u_1 and u_2 are both say 4 at one end then $u_1 + u_2$ is 8, not 4 at that end.)

More generally, there's a superposition principle for the homog linear BC

$$a \frac{\partial u}{\partial x} (L,t) + bu(L,t) = 0$$

For example, if u_1 and u_2 both satisfy the BC $\frac{\partial u}{\partial x}(6,t) = 0$ then $u_1 + u_2$ also satisfies that BC (this kind of BC turns up in the next section).
For example, if u_1 and u_2 both satisfy the BC $\frac{\partial u}{\partial x}(0,t) = -5u(0,t)$ then $u_1 + u_2$ also satisfies that BC.

back to example 1

The heat equation is a linear homogeneous *partial* DE and our BC are homogeneous.

If you add all the solutions you have so far then by superposition the sum also satisfies the heat equation and the two homog BC. So we have "general" solution

(1) $$u(x,t) = \sum_{n=1}^{\infty} B_n\, e^{-k\left(\frac{n\pi}{6}\right)^2 t} \sin\frac{n\pi x}{6}$$

Now you have to determine the constants in (1) to satisfy the IC.
You need $u(x,0) = x$ for x in $[0,6]$ so set $t = 0$, $u = x$ in (1):

(2) $$x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{6} \quad \text{for } x \text{ in } [0,6]$$

Now you have to find constants B_n to satisfy (2).
I'll digress for a while to figure out how and then get back to the heat equation.

finding Fourier sine coefficients

More generally, given any $f(x)$, I want to be able to find constants B_n so that

(3) $$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L]$$

Assuming that it's possible to get (3), here's how to find the constants. I'll find B_3 first and then you'll see the pattern for all the other constants. Multiply both sides in (3) by $\sin\frac{3\pi x}{L}$ and integrate from 0 to L.

(4)
$$\begin{aligned}\int_0^L f(x)\sin\frac{3\pi x}{L}\,dx &= B_1 \underbrace{\int_0^L \sin\frac{\pi x}{L}\sin\frac{3\pi x}{L}\,dx}_{0} + B_2 \underbrace{\int_0^L \sin\frac{2\pi x}{L}\sin\frac{3\pi x}{L}\,dx}_{0} \\ &\quad + B_3 \underbrace{\int_0^L \sin\frac{3\pi x}{L}\sin\frac{3\pi x}{L}\,dx}_{L/2} + B_4 \underbrace{\int_0^L \sin\frac{4\pi x}{L}\sin\frac{3\pi x}{L}\,dx}_{0} + \dots\end{aligned}$$

All the integrals on the right side of (4), except the third one, can be done using (Q) in the tables and they all come out to be 0. The third integral is done using (A) in the tables and it comes out to be L/2. So (4) turns into

$$\int_0^L f(x)\sin\frac{3\pi x}{L}\,dx = B_3\,\frac{L}{2}$$

and you get this formula for B_3: $\quad B_3 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{3\pi x}{L}\,dx$

In general:

> The constants that will make
> $$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L],$$
> are
> (5) $$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

example 1 continued

To satisfy (2) you need

$$B_n = \frac{2}{6}\int_0^6 x \sin\frac{n\pi x}{6}\,dx = \begin{cases} \dfrac{-12}{n\pi} & \text{if } n \text{ is even} \\[2ex] \dfrac{12}{n\pi} & \text{if } n \text{ is odd} \quad \text{(see (2) in tables)} \end{cases}$$

Substitute these constants into (1) to get the final answer

$$u = \frac{12}{\pi} e^{-k\left(\frac{\pi}{6}\right)^2 t} \sin\frac{\pi x}{6} - \frac{12}{2\pi} e^{-k\left(\frac{2\pi}{6}\right)^2 t} \sin\frac{2\pi x}{6}$$

$$+ \frac{12}{3\pi} e^{-k\left(\frac{3\pi}{6}\right)^2 t} \sin\frac{3\pi x}{6} - \ldots \quad \text{for } 0 \le x \le 6,\ t \ge 0$$

Here are some graphs of 5 terms worth of $u(x,t)$ for various values of t and with $k=1$ (Fig 3). You can see that the temperature starts off sort of looking like the line $u = x$ (the graph doesn't look exactly like the line $u=x$ because only 5 terms worth of the u series were used). As time goes on, the temperature peak moves left and the temperature values die down. The steady state solution is 0 (i.e., the temperature $\to 0$ as $t \to \infty$).

```
solution5 = Sum[(-1)^(n+1) 12/(n Pi) E^(- (n Pi/6)^2t) Sin[n Pi x/6],
            {n,1,5}];
solution5Time0  = solution5/.{t->0};
solution5Time05 = solution05/.{t->1/2};
solution5Time2  = solution5/.{t->2};
solution5Time10 = solution5/.{t->10};
Plot[{solution5Time0,solution5Time05, solution5Time2,
          solution5Time10}, {x,0,6}, Ticks->{{3,6},{3,6}}]
```

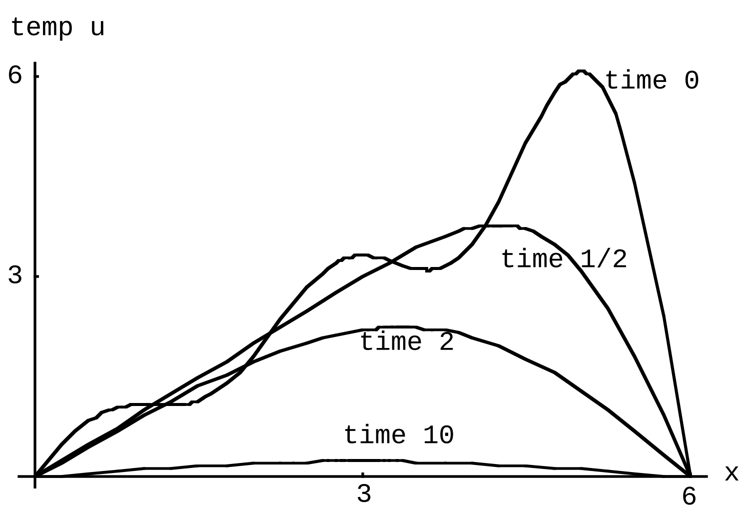


FIG 3

PROBLEMS FOR SECTION 6.1

1. In order to earn the privilege of using the reference tables
   (a) derive (1) on the reference page
   (b) derive (4a) on the reference page

2. Get familiar with the tables on the reference page by using them to find these integrals.

(a) $\frac{2}{4}\int_0^4 5x \sin\frac{n\pi x}{4}$

(b) $\frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi x}{6}\,dx$ where $f(x) = \begin{cases} 5 & \text{if } 0 \le x \le 3 \\ 0 & \text{if } 3 \le x \le 6 \end{cases}$

(c) $\frac{2}{6}\int_0^6 f(x)\sin\frac{n\pi x}{3}\,dx$ where $f(x) = \begin{cases} 5 & \text{if } 0 \le x \le 3 \\ 0 & \text{if } 3 \le x \le 6 \end{cases}$

(d) $\frac{2}{6}\int_0^6 f(x)\sin\frac{n\pi x}{6}\,dx$ where $f(x) = \begin{cases} 5 & \text{if } 0 \le x \le 2 \\ 0 & \text{if } 2 \le x \le 6 \end{cases}$

(e) $\frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi x}{8}\,dx$ where $f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 4 \\ -3x + 24 & \text{if } 4 \le x \le 8 \end{cases}$

3. Solve the heat equation with BC

$$u(0,t) = 0,\ u(4,t) = 0 \text{ for all } t$$

and each of the following IC. Write out enough terms of each solution to make the pattern clear.

(a) $u(x,0) = 8$ for x in $[0,4]$

(b) $u(x,0) = f(x)$ where $f(x) = \begin{cases} 6 & \text{for } x \text{ in } [0,2] \\ 0 & \text{for } x \text{ in } [2,4] \end{cases}$

(c) $u(x,0) = 5 \sin 2\pi x + 6 \sin 5\pi x$ for x in $[0,4]$

If you stop and *think* in part (c), you can get the constants you need by inspection.

SECTION 6.2 THE HEAT EQUATION AND FOURIER COSINES SERIES

example 1

Solve the heat equation with

$$\text{BC} \qquad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(8,t) = 0 \quad \text{for all } t$$

$$\text{IC} \qquad u(x,0) = f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 4 \\ 2 & \text{if } 4 \le x \le 8 \end{cases}$$

(The ends of the rod are insulated, as well as the lateral surface -- see §6.1. Initially, the left half of the rod is at 0° and the right half is 2°)

Part I Separate the variables

Let $u(x,t) = X(x)T(t)$. The separation was already done in Part I of example 1 in the last section so I won't repeat the whole thing. Here are the 3 cases worth of solutions.

case 1 (the constant is negative and named $-\lambda^2$)

$$X = A \cos \lambda x + B \sin \lambda x, \quad T = Ce^{-k\lambda^2 t}$$

case 2 (the constant is positive and named λ^2)

$$X = De^{\lambda x} + Fe^{-\lambda x}$$

$$T = Ge^{k\lambda^2 t}$$

case 3 (the constant is 0)

$$X = Px + Q,$$
$$T = K$$

Now separate the homog conditions. The boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all } t$$

becomes

$$X'(0)T(t) = 0 \quad \text{for all } t.$$

So

$$X'(0) = 0 \quad \text{or} \quad T(t) = 0 \text{ for all } t.$$

But if $T(t) = 0$ for all t then $u(x,t) = X(x)T(t) = 0$ for all t, a trivial solution .

> Here's why it's useless. We eventually have to get a solution satisfying the IC. To do this, we want a "general" solutions with arbitrary constants. We'll get one by using a superposition principle, i.e., we'll add all the solutions satisfying the (homogeneous) PDE and the (homogeneous) BC. It is of no consequence to include $u = 0$ in that sum.

So for all practical purposes you are left with $X'(0) = 0$ as the only useful possibility. Similarly, the BC

$$\frac{\partial u}{\partial x}(8,t) = 0 \quad \text{for all } t$$

becomes $X'(8) = 0$. So all in all, the BC separate to

$$X'(0) = 0, \quad X'(8) = 0$$

Part II Plug in the BC $X'(0) = 0$, $X'(8) = 0$

case 1

First you need X' :

$$X' = -\lambda A \sin \lambda x + B\lambda \cos \lambda x$$

To get $X'(0) = 0$ you need $B\lambda = 0$, $B = 0$ or $\lambda = 0$. But $\lambda \neq 0$ in this case since $-\lambda^2$ represents a negative number so $B = 0$.
To get $X'(8) = 0$ you need

$$-\lambda A \sin 8\lambda = 0.$$

So either $\lambda = 0$ (not possible in this case) or $A = 0$ (which together with $B = 0$ produces only the trivial solution $u = 0$) or

$$\sin 8\lambda = 0$$

$$8\lambda = n\pi$$

$$\lambda = \frac{n\pi}{8}, \ n = 1,2,3,\ldots$$

So

$$X = A \cos \frac{n\pi x}{8} \quad \text{for } n = 1,2,3,\ldots; \text{ any } A$$

$$T = Ce^{-k\left(\frac{n\pi}{8}\right)^2 t} \quad \text{for } n = 1,2,3,\ldots; \text{ any } C$$

Now you have all the solutions

$$u(x,t) = X(x)(T(t) = A_n e^{-k\left(\frac{n\pi}{8}\right)^2 t} \cos \frac{n\pi x}{8} \quad \text{for } n = 1,2,3,\ldots$$

case 2

physical argument In this case, $u \to \infty$ as $t \to \infty$ which is not physically realistic. So forget about this case.

mathematical argument

First you need X':

$$X' = \lambda De^{\lambda x} - \lambda Fe^{-\lambda x}$$

To get $X'(0) = 0$ you need $\lambda D - \lambda F = 0$.

To get $X'(8) = 0$ you need $\lambda De^{8\lambda} - \lambda Fe^{-8\lambda} = 0$.

From $\lambda D - \lambda F = 0$ we have $\lambda(D-F) = 0$. So either $\lambda = 0$ (impossible since this is the case where λ^2 is a positive number) or $D = F$.

If $D = F$ then the second equation is $\lambda D(e^{8\lambda} - e^{-8\lambda}) = 0$.

So either $\lambda=0$ (impossible in this case) or $D = 0$ (which makes $F = 0$ which produces the trivial solution $u = 0$) or $e^{8\lambda} + e^{-8\lambda}$. But $e^{8\lambda}$ can only equal $e^{-8\lambda}$ if $\lambda = 0$ which is impossible in this case. So all we get here is the useless trivial solution.

case 3

In this case,

$$X = Px + Q, \quad T = K, \quad X' = P.$$

The two BC $X'(0) = 0$ and $X'(8) = 0$ are satisfied by taking $P = 0$. So from this case you have the solution

$$u = X(x)T(t) = QK = A_0$$

Part III

By the superposition principle in the preceding section, *if u_1 and u_2 each satisfy the BC "deriv w.r.t. x is 0 at an end" then u_1+u_2 also satisfies that BC.*

So

(1) $$u = A_0 + \sum_{n=1}^{\infty} A_n \, e^{-k\left(\frac{n\pi}{8}\right)^2 t} \cos \frac{n\pi x}{8}$$

satisfies the heat equation and the two BC.

Determine the constants so that (1) satisfies the IC.

You need $u(x,0) = f(x)$ for x in $[0,8]$ so set $t = 0$, $u = f(x)$ in (1):

(2) $$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{8} \qquad \text{for } x \text{ in } [0,8]$$

Now you have to find constants A_0, A_n to satisfy (2).

I'll come back and finish the example after getting the formula for the constants.

finding Fourier cosine coefficients

More generally, if $f(x)$ is an arbitrary function, I want constants so that

(3) $$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \qquad \text{for } x \text{ in } [0,L]$$

The formulas for the constants can be derived in the same manner as the sine coeffs were derived in Section 6.1. Better still, in Section 6.7 there will be some general formulas for Fourier series and the cosine coeffs will be a special case of those formulas so wait until then for the explanation. Here are the formulas themselves.

The constants that will make

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \qquad \text{for } x \text{ in } [0,L]$$

are

(4)
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \text{average value of } f(x) \text{ in } [0,L] \quad \text{footnote}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad \text{for } n = 1,2,3,\ldots$$

footnote

Here's why the A_0 formula is the average value of $f(x)$ on $[0,L]$:

$$A_0 = \frac{\int_0^L f(x)\,dx}{L} = \frac{\text{area of region in Fig 1}}{\text{base of region}}$$

$= \text{average height of region}$

$= \text{average value of } f(x) \text{ for } 0 \le x \le L$

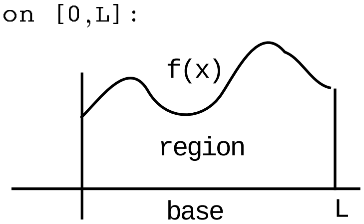


FIG 1

example 1 continued

To satisfy (2) you need

A_0 = average value of $f(x)$ in $[0,8]$ = 1

$$A_n = \frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi x}{8}\,dx = \begin{cases} -\dfrac{4}{n\pi} & \text{if } n = 1,5,9,\ldots \\[2ex] \dfrac{4}{n\pi} & \text{if } n = 3,7,11,\ldots \\[2ex] 0 & \text{if } n \text{ is even} \end{cases}$$

(Use (4) in the tables with a = 0, b = 2)

Plug these constants into (1) to get the final answer:

$$u = 1 - \frac{4}{\pi} e^{-k(\frac{\pi}{8})^2 t}\cos\frac{\pi x}{8} + \frac{4}{3\pi} e^{-k(\frac{3\pi}{8})^2 t}\cos\frac{3\pi x}{8} \tag{5}$$

$$- \frac{4}{5\pi} e^{-k(\frac{5\pi}{8})^2 t}\cos\frac{5\pi x}{8} + \ldots \text{ for } 0 \le x \le 8,\ t \ge 0$$

summary of the procedure for solving the heat equation and the upcoming wave equation with very simple BC

Part I Separate the PDE. (Not every PDE separates but the physically important ones do.)

And separate the homogeneous BC (*non*homogeneous conditions can't be separated). For instance

$\frac{\partial u}{\partial x}(5,t) = 0$ for all t becomes $X'(5) = 0$

$u(0,t) = 0$ for all t becomes $X(0) = 0$

Part II Solve the separate X problem with BC.
And solve the corresponding T equation.

So far, two separated X problems have turned up. And they will turn up again so you might as well notice now which cases were useless and avoid them in the future.

problem 1 $X'' = \text{constant}\cdot X$ with BC $X(0) = 0$, $X(L) = 0$

The case con = 0 has only the solution X=0. Ignore it.

The case con = λ^2 (i.e., positive constant) has only the solution X=0. Ignore it

The case con = $-\lambda^2$ (i.e., negative constant) has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = n\pi/L$ and the corresponding solution is $\sin\frac{n\pi x}{L}$ (and any multiple thereof).]

problem 2 $X'' = \text{constant}\cdot X$ (same equ as problem 1) with BC $X'(0) = 0$, $X'(L) = 0$

The case con = λ^2 has only the solution X=0. Ignore it.

The case con = 0 had a nonzero solution.

[It turns out that a solution is X = 1 and more generally X = A where A is an arbitrary constant.]

The case con = $-\lambda^2$ has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = n\pi/L$ and the corresponding solution is $\cos\frac{n\pi x}{L}$ (and any multiple thereof).]

Part III Collect the X(x)T(t) solutions and add them all up to get a solution (by superposition) with many constants.

Plug in the *non*homog IC to determine the constants in the solution.

summary of Fourier sine and cosine series

The functions

$$\sin\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \sin\frac{3\pi x}{L},\ \ldots$$

can be used to make a series that will do anything you want on the interval [0,L]. The way to make

$$B_1 \sin\frac{\pi x}{L} + B_2 \sin\frac{2\pi x}{L} + B_3 \sin\frac{3\pi x}{L} + \ldots \tag{6}$$

look like f(x) for $0 \le x \le L$ is to use

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L}\ dx$$

Similarly the functions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \ldots$$

can be used to make a series that will do anything you want on the interval [0,L]. The way to make

$$A_0\cdot 1 + A_1 \cos\frac{\pi x}{L} + A_2 \cos\frac{2\pi x}{L} + A_3 \cos\frac{3\pi x}{L} + \ldots \tag{7}$$

look like f(x) for $0 \le x \le L$ is to use

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx = \text{average value of } f(x) \text{ on the interval}$$

and for the other A's use

$$A_n = \frac{2}{L}\int_0^L f(x)\ \cos\frac{n\pi x}{L}\ dx$$

You need to know how to make various series converge to f(x) on an interval in order to determine constants so as to satisfy BC and/or IC conditions when you solve a PDE.

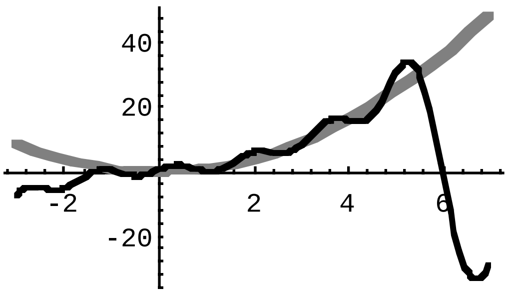
Here are some pictures of the sine and cosine series looking like the function x^2 for $0 \le x \le 6$ in case you don't believe they can do it.

Fig 1 is the graph of x^2 (the fat gray curve) and the sum of 7 terms of the sine series for x^2. Fig 2 is the graph of x^2 and the sum of 14 terms of the sine series for x^2. The more sines you add, the closer the sum gets to x^2. But no matter how many terms you add, the sum doesn't look like x^2 once $x \ge 6$ or $x \le 0$.

Fig 3 shows the sum of 6 terms (the constant term plus 5 cosines) of the cosine series for x^2. It's very much like x^2 not just for $0 \le x \le 6$ but actually for $-6 \le x \le 6$.

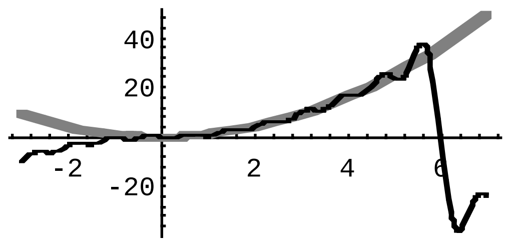
```
B[n_] := 2/6 Integrate[x^2 Sin[n Pi x/6], {x,0,6}]
sinSeries7 = Sum[B[n] Sin[n Pi x/6],{n,1,7}];
curve1 =  Plot[x^2,{x,-3,7},
      PlotStyle->{{Thickness[.02],GrayLevel[.5]}},
      DisplayFunction->Identity];

curve2 = Plot[sinSeries7,{x,-3,7},
        PlotStyle->{{Thickness[.01]}},DisplayFunction->Identity];
```

```
Show[{curve1, curve2},DisplayFunction->$DisplayFunction];
sinSeries14 = Sum[B[n] Sin[n Pi x/6],{n,1,14}];
curve3 =  Plot[sinSeries14,{x,-3,7},
     PlotStyle->{{Thickness[.01]}},DisplayFunction->Identity];

Show[{curve1, curve3},DisplayFunction->$DisplayFunction];
```



sum of 7 sines vs. x^2

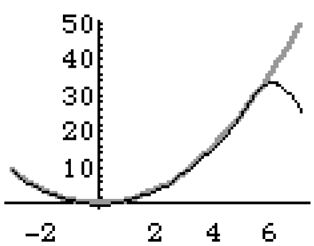
FIG 1



sum of 14 sines vs. x^2

FIG 2

```
K[n_]:= 2/6 Integrate[x^2 Cos[n Pi x/6],{x,0,6}]
K0 = 1/6 Integrate[x^2,{x,0,6}];
cosSeries6 = K0 + Sum[K[n] Cos[n Pi x/6],{n,1,5}];
curve4 =  Plot[ cosSeries6,{x,-3,7},
      PlotStyle->{{Thickness[.01]}},DisplayFunction->Identity];
 curve1 =  Plot[x^2,{x,-3,7},
       PlotStyle->{{Thickness[.02],GrayLevel[.5]}},
       DisplayFunction->Identity];
Show[{curve1, curve4},DisplayFunction->$DisplayFunction];
```



sum of 6 cosine terms vs. x^2

FIG 3

PROBLEMS FOR SECTION 6.2

1. Let y be a function of x and t. Assume $y(x,t) = X(x)T(t)$. Separate the following conditions if possible.

(a) $\frac{\partial y}{\partial x}(0,t) = 0$ (b) $\frac{\partial y}{\partial x}(5,t) = 0$ (c) $\frac{\partial y}{\partial x}(0,t) = 5$

(d) $y(4,t) = 3$ (e) $y(4,t) = 0$ (f) $y(x,0) = 2x$ (g) $y(x,0) = 0$

2. (a) Solve the heat equ with

BC $\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(6,t) = 0$ for all t

IC $u(x,0) = f(x) = \begin{cases} 5 & \text{if } x \text{ is in } [0,3] \\ 9 & \text{if } x \text{ is in } [3,6] \end{cases}$

(b) Look at the heat equation with the same BC as in part (a) and the IC

$$u(x,0) = 2 \text{ for } x \text{ in } [0,6]$$

(i) Solve by inspection by thinking about the physical significance of the BC and IC
(ii) For practice, solve by going through the procedure of this section

3 (a new heat equation). The heat equation satisfied by the net temperature in a rod whose lateral surface is *not* insulated is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

Find the solution satisfying

BC $u(0,t) = 0, \quad u(L,t) = 0$

IC $u(x,0) = 8$ for $0 \le x \le L$

4. (a) Find the steady state solution in example 1; i.e., find $u(x,\infty)$.

(b) The generalization of example 1 is the heat equation with

BC $\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$ for all t

IC $u(x,0) = f(x)$ for $0 \le x \le L$ where $f(x)$ is an arbitrary function.

Find the steady state solution using a physical argument and then get it mathematically.

5. Suppose $v(p,q) = P(p)Q(q)$. Separate this BC very slowly and give key reasons.

$$\frac{\partial v}{\partial p}(5,q) = 0 \text{ for all } q$$

INTEGRATING WITH THE DELTA FUNCTION

integrating $\delta(t-a)$

$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ because the area under the delta function is 1.

$$\int_{-\infty}^{\infty} 5\delta(t-a)\,dt = 5$$

the functions $f(t)\delta(t)$ and $f(t)\delta(t-a)$

Multiplying $\delta(t-3)$ by $f(t)$ leaves the zero heights on the $\delta(t-3)$ graph unchanged and multiplies the impulse at $t = 3$ by $f(3)$ so that the area becomes $f(3)$ instead of 1. In fact, $f(t)\delta(t-3)$ simplifies to $f(3)\delta(t-3)$.

> In general, $f(t)\delta(t-a)$ is the same as $f(a)\delta(t-a)$, an impulse of area $f(a)$ occurring at time $t = a$ (Fig 5).
> In particular, $f(t)\delta(t)$ is the same as $f(0)\delta(t)$, an impulse of area $f(0)$ occurring at $t=0$ (Fig 6)

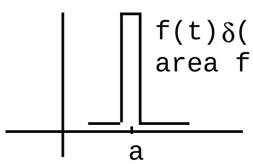


FIG 5

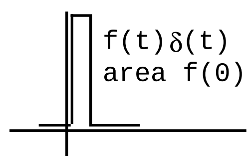


FIG 6

For example, $t^2\,\delta(t-4)$ is the same as $16\delta(t-4)$, an impulse of area 16 at $t=4$.
For example, $t^2\delta(t)$ is the same as $0\,\delta(t)$ which is just the zero function (carrying zero area). It isn't an impulse function anymore.

the sifting property of the delta function

From the box above, the area under the graph of $f(t)\delta(t-a)$ is $f(a)$ and it is all concentrated at $t = a$. So:

(1)
$$\int_{\text{interval containing } a} f(t)\delta(t-a)\,dt = f(a)$$
$$\int_{\text{interval not containing } a} f(t)\delta(t-a)\,dt = 0$$

In particular

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$
$$\int_{0}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$
$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$$

For example,

$$\int_0^{\pi} \delta(t - \tfrac{1}{2}\pi) \sin t \, dt = \sin \tfrac{1}{2}\pi = 1$$

$$\int_{\pi}^{2\pi} \delta(t - \tfrac{1}{2}\pi) \sin t \, dt = 0 \quad (\pi/2 \text{ is not in the interval of integration})$$

$$\int_{t=-\infty}^{\infty} \frac{t^3 + 3}{t^4 + 7} \delta(t) \, dt = \frac{0^3 + 3}{0^4 + 7} = \frac{3}{7}$$

$$\int_1^4 \delta(x - 2) \, dx = 1$$

$$\int_5^6 \delta(x - 2) \, dx = 0$$

SECTION 6.3 THE WAVE EQUATION

the wave equation and its physical significance

The 1-dim wave equation is

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad \text{(a is a fixed positive constant)}$$

Consider a vibrating string with small displacements $y(x,t)$ at position x and time t (Fig 1). If we ignore gravity and the retarding force of the medium, and a is a constant determined by the nature of the string, then it can be shown that the height $y(x,t)$ of the string satisfies the wave equation.

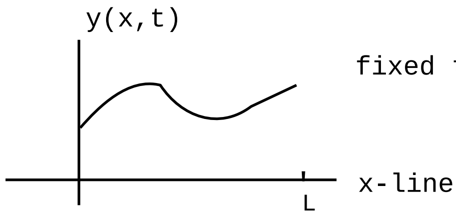


FIG 1

The wave equation comes with two IC:

$y(x,0) = f(x)$ (the initial shape of the string)

$\frac{\partial y}{\partial t}(x,0) = g(x)$ (the initial velocity of the string)

There are two popular types of BC, describing conditions at the ends of the string:

The BC $y(L,t) = 0$ means that the right end of the wire is fixed at height 0.

The BC $\frac{\partial y}{\partial x}(L,t) = 0$ corresponds to a string with zero slope at the right end. This can be accomplished by looping the right end around a pole (Fig 2) so that it is free to move up and down in response to any would-be vertical component of tension. In particular the right end responds by continually moving so as to maintain no vertical component of tension, i.e., the right end moves up and down so as to keep the slope zero at the right end (Fig 3).

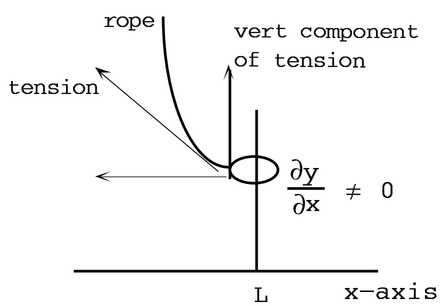


FIG 2 Right end moves up

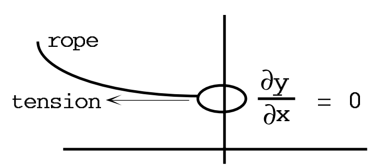


FIG 3 Equilibrium

example 1

I'll solve the wave equation with

BC $y(0,t) = 0$, $y(L,t) = 0$ for $t \ge 0$

IC $y(x,0) = f(x)$, $\frac{\partial y}{\partial t}(x,0) = g(x)$ for $0 \le x \le L$.

(The ends of the wire are nailed down at height 0, the wire has initial shape $f(x)$ and initial velocity $g(x)$.)

Part I Separate variables
Try a solution of the form

$$y(x,t) = X(x)T(t).$$

Substitute into the wave equation:

$$XT'' = a^2 X''T$$

$$\frac{X''}{X} = \frac{T''}{a^2T}$$

A function of x can't equal a function of t unless both functions are constant. So

$$\frac{X''}{X} = \frac{T''}{a^2T} = \text{constant}$$

This is the pair of equations

$$X'' = \text{constant } X, \qquad T'' = a^2 \text{ constant } T.$$

The BC separate to $X(0) = 0$, $X(L) = 0$.

So the X problem is

$$X'' = \text{constant } X \text{ with BC } X(0) = 0,\ X(L) = 0$$

This is problem 1 in the summary near the end of the preceding section.
We already know that the only way to get a nonzero solution for X is to use the case where the constant is negative, renamed $-\lambda^2$. Then

$$X'' + \lambda^2 X = 0, \qquad X = A\cos\lambda x + B\sin\lambda x$$

$$T'' + a^2\lambda^2 T = 0, \qquad T = C\cos\lambda at + D\sin\lambda at$$

Part II Plug in the BC
$X(0) = 0$ makes $A = 0$
$X(L) = 0$ makes $B \sin\lambda L = 0$.
Either $B = 0$ (which together with $A = 0$ produces only the trivial solution $y=0$) or

$$\sin\lambda L = 0$$

$$\lambda L = n\pi, \quad \lambda = \frac{n\pi}{L}$$

So

$$X = B_n \sin\frac{n\pi x}{L} \text{ for } n = 1,2,3,\ldots$$

So far we have a lot of solutions:

$$y(x,t) = X(x)T(t) = \left[C_n \cos\frac{n\pi at}{L} + D_n \sin\frac{n\pi at}{L}\right] \sin\frac{n\pi x}{L} \text{ for } n = 1,2,3,\ldots$$

Part III Use superposition and plug in the IC

Use superposition to get the solution

(1) $$\boxed{y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\frac{n\pi at}{L} + D_n \sin\frac{n\pi at}{L} \right] \sin\frac{n\pi x}{L} \quad \text{for } 0 \le x \le L,\ \ t \ge 0}$$

Now determine the constants to satisfy the two IC.
To get the first IC set $t = 0$, $y = f(x)$: you need

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L]$$

so

(2) $$\boxed{C_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

To satisfy the second IC first find

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left[-\frac{n\pi a}{L} C_n \sin\frac{n\pi at}{L} + \frac{n\pi a}{L} D_n \cos\frac{n\pi at}{L} \right] \sin\frac{n\pi x}{L}$$

Then plug in the second IC; set $t = 0$, $\partial y/\partial t = g(x)$. You need

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} D_n \sin\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L]$$

This is a sine series for $g(x)$ but with $\frac{n\pi a}{L} D_n$'s playing the role of the constants so

$$\frac{n\pi a}{L} D_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

(3) $$\boxed{D_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx}$$

The solution is in (1) with the constants in the solution given in (2) and (3).

example 1 continued

Suppose the wire has length 6, is not initially displaced but is initially moving up at 2 ft/sec. Then $L = 6$ and the IC are

$$y(x,0) = 0\ , \qquad \frac{\partial y}{\partial t}(x,0) = 2 \quad \text{for } x \text{ in } [0,L]$$

Use the solution in (1)–(3) with $f(x) = 0$ and $g(x) = 2$.
From (2), $C_n = 0$.
(Or you could separate the homogeneous IC $y(x,0) = 0$ to get $T(0) = 0$ and plug this into the solution $T = C\cos\lambda at + D\sin\lambda at$ to get $C = 0$.)
From (3),

$$D_n = \frac{6}{n\pi a}\ \frac{2}{6}\int_0^6 2\sin\frac{n\pi x}{6}\, dx = \frac{6}{n\pi a}\begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8}{n\pi} & \text{if } n \text{ is odd} \end{cases} \qquad \text{(tables (1))}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{48}{n^2\pi^2 a} & \text{if } n \text{ is odd} \end{cases}$$

The solution is

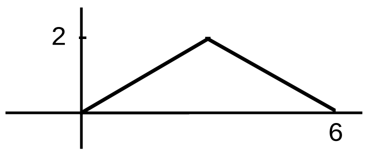
$$y(x,t) = \frac{48}{\pi^2 a} \sin\frac{\pi at}{6} \sin\frac{\pi x}{6} + \frac{48}{3^2 \pi^2 a} \sin\frac{3\pi at}{6} \sin\frac{3\pi x}{6}$$

$$+ \frac{48}{5^2 \pi^2 a} \sin\frac{5\pi at}{6} \sin\frac{5\pi x}{6} + \ldots \text{ for } 0 \le x \le 6,\ t \ge 0$$

PROBLEMS FOR SECTION 6.3

1. Use the work in example 1 to solve the wave equation with
   BC $y(0,t) = 0$, $y(6,t) = 0$ for all t
   IC $y(x,0)$ as given in the diagram, $\frac{\partial y}{\partial t}(x,0) = 0$ for x in [0,L]

(The string is initially plucked but not moving yet.)



string at time 0

2. Use the work in example 1 to solve the wave equation with

   BC $y(0,t) = 0$, $y(L,t) = 0$ for all t

   IC $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = \delta(x - \frac{1}{2} L)$ for x in [0,L]

(The string is initially not displaced but is given a large initial velocity in the middle.)

3. Solve the wave equation with

   BC $\frac{\partial y}{\partial x}(0,t) = 0$, $\frac{\partial y}{\partial x}(L,t) = 0$ for all t

   IC $y(x,0) = f(x)$, $\frac{\partial y}{\partial t}(x,0) = g(x)$ for x in [0,L]

4. Solve the wave equation with the following conditions and write out enough terms of the solution to make the pattern clear.

   BC $\frac{\partial y}{\partial x}(0,t) = 0$, $\frac{\partial y}{\partial x}(2,t) = 0$ for all t

   IC $y(x,0) = x$, $\frac{\partial y}{\partial t}(x,0) = 0$ for x in [0,2]

5. Consider the wave equation with

   BC $\frac{\partial y}{\partial x}(0,t) = 0$, $\frac{\partial y}{\partial x}(L,t) = 0$ for all t

   IC $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = 3$ for x in [0,L]

   (a) Think of the physical significance of the PDE, BC and IC and solve by inspection

   (b) For practice, go through the solving process of this section

SECTION 6.4 LAPLACE'S EQUATION

Laplace's equation and its physical significance

The 2–dimensional Laplace's equation is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Consider electric potential v(x,y) (i.e., voltage) at point (x,y) in the plane. If a region in the plane is charge-free then it can be shown that v(x,y) satisfies Laplace's equation for points (x,y) in that region (Fig 1). (Lots of other things besides voltage satisfy Laplace's equation, e.g., steady state temperature.)

Laplace's equation comes with boundary conditions describing what happens on the boundary of the region.

The BC $v = 0$ means zero voltage on the boundary.

To understand other BC you have to know what $\partial v/\partial x$ and $\partial v/\partial y$ mean. If you consider v as a function of x with y fixed then $\partial v/\partial x$ is the slope on a potential hill. Electric flux flows down potential hills so $-\partial v/\partial x$ is a measure of flux flowing horizontally from left to right. The BC $\partial v/\partial y = 0$ on a horizontal boundary (Fig 2) indicates no flow of flux across the boundary; i.e., the boundary is insulated.

Similarly $\partial v/\partial x = 0$ on a vertical boundary indicates no flow of flux across the boundary; i.e., the boundary is insulated (Fig 3).

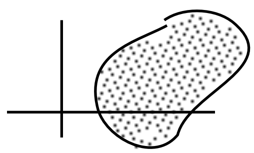


FIG 1

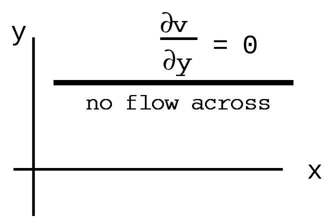


FIG 2

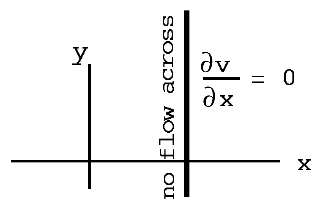


FIG 3

review of the functions cosh and sinh

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

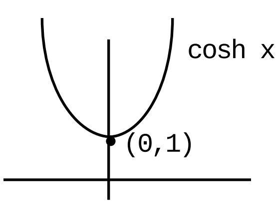


FIG 4

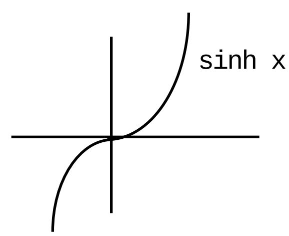


FIG 5

(1) $e^x = \cosh x + \sinh x$

(2) $e^{-x} = \cosh x - \sinh x$

(3) $D_x \cosh x = \sinh x$

(4) $D_x \sinh x = \cosh x$

(5) If A and B are arbitrary constants then

$$Ae^x + Be^{-x} = C \cosh x + D \sinh x$$

where C and D are new arbitrary constants

Here's a proof of (5):

$$\begin{aligned} Ae^x + Be^{-x} &= A(\cosh x + \sinh x) + B(\cosh x - \sinh x) \\ &= (A + B) \cosh x + (A - b) \sin h \\ &= C \cosh x + D \sinh x \end{aligned}$$

summary of the procedure for solving Laplace's equation on two types of regions with simple BC

Figs 6 and 7 show the two standard problems. Here are the three steps for solving them.

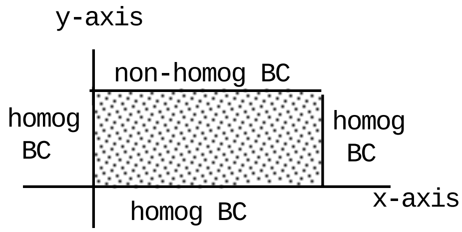


FIG 6 TYPE 1

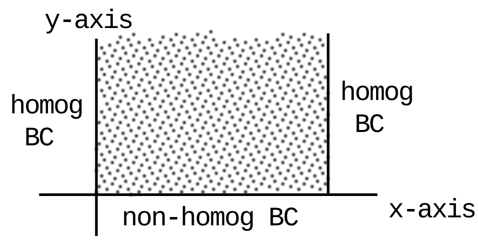


FIG 7 TYPE 2

Part I Separate the PDE and the homogeneous BC.

You have a choice of using $Y = Ce^{\lambda y} + De^{-\lambda y}$ or $Y = C \cosh \lambda y + D \sinh \lambda y$.

For convenience, *use the exponential version of Y in a type 2 problem and the cosh sinh version in type 1*

Part II Plug in the BC.

The nonhomog BC is along a *horizontal* boundary and will be of the form $v = f(x)$. This makes X the "important" function so use the case with a nice X solution, namely the case where $X = A \cos \lambda x + B \sin \lambda x$.

If the BC are $X(0) = 0$, $X(L) = 0$ then this one case is enough.

If the BC are $X'(0) = 0$, $X'(L) = 0$ then use the $\lambda = 0$ case also

The solution should stay finite to be physically realizable. In practice, this means that there's the additional BC $Y(\infty)$ finite.

For type 1, Y is in no danger of blowing up so don't worry about it. In this case it is more convenient algebraically to use the cosh sinh version of Y.

For type 2, Y *is* in danger of blowing up since $\cosh \lambda y$, $\sinh \lambda y$ and $e^{\lambda y}$ all blow up as $y \to \infty$. The best way to handle it is to use the exponential version of Y and toss out the $e^{\lambda y}$.

Part III Use superposition to add all the solutions and get a solution with many constants

Plug in the *non*homog BC to determine the remaining constants

example 1

Solve Laplace's equation for the region in Fig 8 if

$$f(x) = \begin{cases} x & \text{if } 0 < x < 3 \\ 6-x & \text{if } 3 \le x \le 6 \end{cases}$$

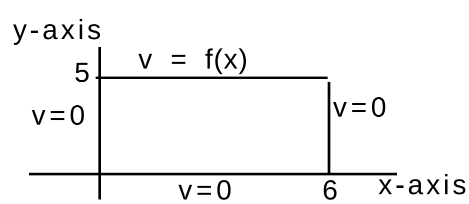


FIG 8

Part I Separate variables

Try $v(x,y) = X(x)Y(y)$

Then

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} \tag{6}$$

This is the pair of equations

(7) $X'' = \text{constant } X, \quad Y'' = -\text{constant } Y$

The left BC is $v(0,y) = 0$ which separates to $X(0) = 0$.
The right BC is $v(6,y) = 0$ which separates to $X(6) = 0$.
The lower BC is $v(x,0) = 0$ which separates to $Y(0) = 0$.

So the X problem is

$$X'' = \text{constant } X \text{ with BC } X(0) = 0,\ X(6) = 0$$

This is problem 1 in the summary on page 4 in Section 6.2.
We already know that the only way to get a nonzero solution for X is to use the case where the constant is negative, renamed $-\lambda^2$. Then

(8) $X'' + \lambda^2 X = 0, \qquad X = A\cos\lambda x + B\sin\lambda x$

(9) $Y'' - \lambda^2 Y = 0, \qquad Y = Ce^{\lambda y} + De^{-\lambda y} = E\cosh\lambda y + F\sinh\lambda y$

footnote Instead of (6), your separation could have been

$$-\frac{X''}{X} = \frac{Y''}{Y} = \text{constant}$$

in which case, your X problem is

$$X'' = -\text{constant } X \text{ with BC } X(0) = 0,\ X(6) = 0,$$

the only case with a nontrivial solution is the case where the constant is positive and renamed λ^2 and you still end up with (8) and (9).

Part II Plug the separated BC into (8) and (9).
$X(0) = 0$ makes $A = 0$

$X(6) = 0$ makes $B\sin 6\lambda = 0$, $6\lambda = n\pi$, $\lambda = \frac{n\pi}{6}$

Use the cosh sinh version of Y.
$Y(0) = 0$ makes $E = 0$

Part III Use superposition and plug in the IC.
By superposition,

(10) $v = \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi y}{6} \sin\frac{n\pi x}{6}$

> **warning** Don't leave out the summation sign. You can't go after the final BC until you *add* all the solutions to get a general solution

The top BC is $v(x,5) = f(x)$ for $0 \le x \le 6$ so set $y = 5$, $v = f(x)$ in (10). You need

$$f(x) = \sum_{n=1}^{\infty} D_n \sinh\frac{5n\pi}{6} \sin\frac{n\pi x}{6} \quad \text{for } 0 \le x \le 6$$

> **warning** The top BC is $v(x,\boxed{5}) = f(x)$. When you plug in the IC don't forget to set $y = 5$ as you set $v = f(x)$. *Don't leave it* y and don't just throw away the sinh factor.

To get this you need

$$D_n \sinh \frac{5n\pi}{6} = \frac{2}{6}\int_0^6 f(x) \quad \sin \frac{n\pi x}{6}\, dx$$

warning The left side is $D_n \sinh \frac{5n\pi}{6}$, not plain D_n

The graph of f(x) looks like the picture in (5) on the reference page, with L = 6, K = 3. So

$$D_n \sinh \frac{5n\pi}{6} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{24}{n^2\pi^2} & \text{if } n = 1,5,9,\ldots \\ -\dfrac{24}{n^2\pi^2} & \text{if } n = 3,7,11,\ldots \end{cases}$$

and

$$D_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{24}{n^2\pi^2 \sinh \frac{5n\pi}{6}} & \text{if } n = 1,5,9,\ldots \\ -\dfrac{24}{n^2\pi^2 \sinh \frac{5n\pi}{6}} & \text{if } n = 3,7,11,\ldots \end{cases}$$

Plug these into (10) to get final solution.

$$v = \frac{24}{\pi^2}\left[\frac{1}{\sinh \frac{5\pi}{6}} \sinh \frac{\pi y}{6} \sin \frac{\pi x}{6} - \frac{1}{9 \sinh \frac{15\pi}{6}} \sinh \frac{3\pi y}{6} \sin \frac{3\pi x}{6} + \frac{1}{25 \sinh \frac{25\pi}{6}} \sinh \frac{5\pi y}{6} \sin \frac{5\pi x}{6} + \ldots \right] \text{ for } 0 \le x \le 6,\ 0 \le y \le 5$$

warning The sinh's do not cancel out of the answer. The coeffs contain $\sinh \frac{\boxed{5}\, n\pi}{6}$ and the terms contain $\sinh \frac{n\pi \boxed{y}}{6}$

Fig 9 shows a 3D plot of v(x,y) (10 terms worth only). The y-axis goes back into the page. The x-axis goes from left to right. The v-axis is vertical
Fig 10 shows some contour curves of v(x,y)

```
solution10 = Sum[24/(n Pi)^2 Sin[n Pi/2] 1/Sinh[n Pi 5/6]
       Sinh[n Pi y/6] Sin[n Pi x/6],{n,1,10}];

Plot3D[solution10,{x,0,6},{y,0,5},Shading->False]
```

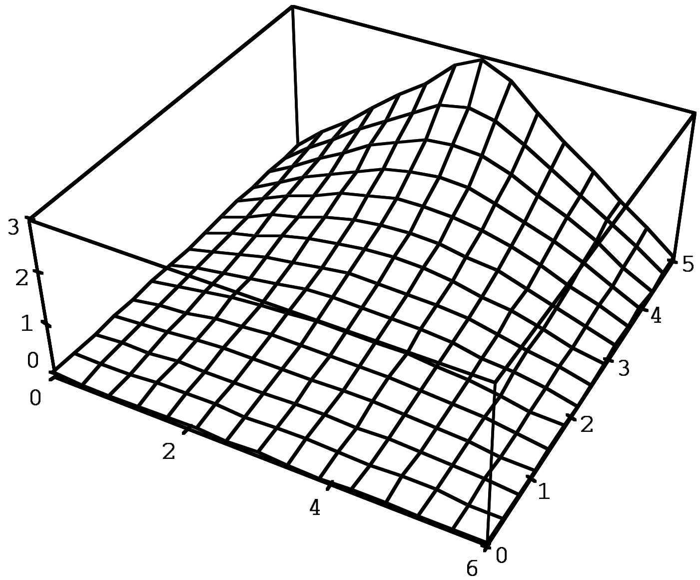


FIG 9

```
ContourPlot[solution10,{x,0,6},{y,0,5},ContourShading->False,
          FrameTicks->{{0,6},{0,5}},Contours->{.1,.5,1,1.5,2,2.5,2.8}];
```

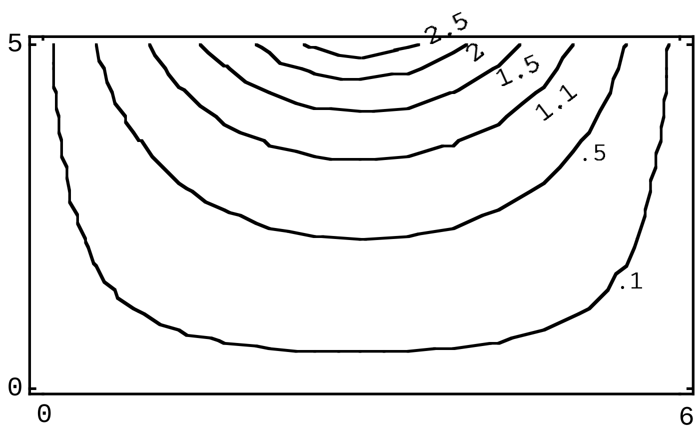


FIG 10

example 2

Solve Laplace's equation for the strip in Fig 11.

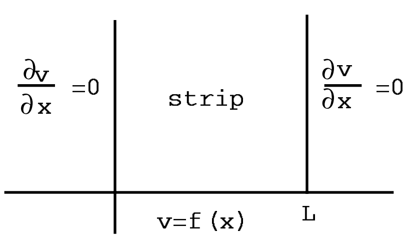


FIG 11

Part I Separate the PDE as in example 1 and get

$$X'' = \text{constant } X, \quad Y'' = -\text{constant } Y$$

The left BC is $\frac{\partial v}{\partial x}(0,y) = 0$. It separates to $X'(0) = 0$

The right BC is $\frac{\partial v}{\partial x}(L,y) = 0$. It separates to $X'(L) = 0$.

So the X problem is

$$X'' = \text{constant } X \text{ with BC } X'(0) = 0,\ X'(L) = 0$$

This is problem 2 in the summary near the end of the preceding section.
We already know that the only way to get a nonzero solution for X is to use these two cases:

case 1 The constant is negative and renamed $-\lambda^2$. Then

$$X = A\cos\lambda x + B\sin\lambda x, \quad Y = Ce^{\lambda y} + De^{-\lambda y}$$

case 2 The constant is 0. Then

$$X = Ex + F, \quad Y = Gy + H$$

Part II Plug in the homog BC.

case 1

$X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = 0$ makes $B = 0$

$X'(L) = 0$ makes

$$-A\sin\lambda L = 0$$

$$\lambda L = n\pi, \quad \lambda = \frac{n\pi}{L}$$

To keep Y finite when $y \to \infty$ choose $C = 0$.

case 2

$X'(0) = 0$ and $X'(L) = 0$ make $E = 0$.

$Y(\infty)$ finite makes $G = 0$

So from this case you get $v = X(x)Y(y) = FH = Q$.

Part III Satisfy the nonhomog BC.

By superposition, the solution is

$$\boxed{v = Q + \sum_{n=1}^{\infty} D_n\, e^{-\frac{n\pi y}{L}} \cos\frac{n\pi x}{L} \text{ for } 0 \le x \le 6,\ 0 \le y \le 5} \tag{11}$$

The nonhomog BC is $v(x,0) = f(x)$ for $0 \le x \le L$. Set $y = 0$, $v = f(x)$ in (10) to see that you need

$$f(x) = Q + \sum_{n=1}^{\infty} D_n \cos\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L] \tag{12}$$

which you can get with

$$\boxed{Q = \frac{1}{L}\int_0^L f(x)\,dx\ , \qquad D_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx} \tag{13}$$

The solution is (11) with the constants in the solution given in (13)

warning

Line (12) is not part of the solution. It's just an equation which determines the Q and D_n for the answer in (11).

summary of separation cases

1. Why are there cases?.

Cases turn up in *partial* differential equations because of the rules for solving *ordinary* differential equations. If you end up with

$$X'' = (5 + \text{con})X$$

then $m = \pm\sqrt{5 + \text{con}}$ and the solution depends on whether you get two *real* m's, two *nonreal* m's or *one* repeated m. This in turn depends on the sign of what's under the square root sign so the cases you need here are

case 1 $5 + \text{con} > 0$
case 2 $5 + \text{con} < 0$,
case 3 $5 + \text{con} = 0$

On the other hand, for the equation $X' = \text{con}\ X$, you have $m = \text{con}$ and $X = Ae^{\text{con}\ x}$ no matter what the sign of the con; no cases necessary.

2. Which cases will be useful (i.e., produce non-zero solutions).

First decide which letter to concentrate on. If the PDE comes with a nonhomog condition of the form $v(x,9) = f(x)$ then you want to end up with a nice X problem, one whose solutions can be used later to build a Fourier series for $f(x)$. So far this means ending up with problem 1 or problem 2 below..

If a PDE involves X and T, it will come with a nonhomog IC [i.e., a condition of the form $v(x,0) = f(x)$] and you need cases that have nice X solutions.

Second, say it is the letter X that is important and suppose you write the X equation so that the X" term has a positive coefficient (in other words you write $\frac{X''}{X} = -\frac{Y''}{Y} = \text{con}$ rather than $-\frac{X''}{X} = \frac{Y''}{Y} = \text{con}$). Then the con $= -\lambda^2$ will always be useful and the con $= \lambda^2$ case will *never* be useful. The zero case *might* produce a nontrivial solution; it depends on the particular BC.

Here is an list of what has turned up so far after the separation.

problem 1 $X'' = \text{constant}\cdot X$ with BC $X(0) = 0$, $X(L) = 0$

The case con = 0 has only the solution X=0. Ignore it.

The case con $= \lambda^2$ (i.e., positive constant) has only the solution X=0. Ignore it

The case con $= -\lambda^2$ (i.e., negative constant) has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = \frac{n\pi}{L}$ and the corresponding solution is $\sin\frac{n\pi x}{L}$ (and any multiple thereof).]

problem 2 $X'' = \text{constant}\cdot X$ with BC $X'(0) = 0$, $X'(L) = 0$ (same equ as problem 1 but different BC)

The case con $= \lambda^2$ has only the solution X=0. Ignore it.

The case con = 0 had a nonzero solution.

[It turns out that a solution is X = 1 and more generally X = A where A is an arbitrary constant.]

The case con $= -\lambda^2$ has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = \frac{n\pi}{L}$ and the corresponding solution is $\cos\frac{n\pi x}{L}$ (and any multiple thereof).]

Where did this come from? How do I *know* that the case con = λ^2 has only the trivial solution in problems 1 and 2 and that the case con = 0 is not useful in problem 1?

Because in Section 6.1 (example 1, part II), I tried all the cases in problem 1. And in Section 6.2 (example 1, part II), I tried all the cases in problem 2.

The general theory about which cases are useful and which aren't is stated in Section 6.7 (but is much too messy to prove there).

example 3

Solve Laplace's equation on a region with the BC in Fig 12. Leave integrals unevaluated at the end.

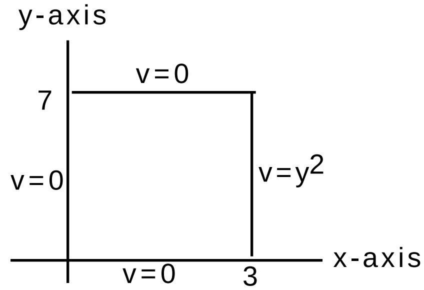


FIG 12

solution

Part I The nonhomog BC is $v(3,y) = y^2$ so you want to end up with nice Y solutions. Write (6) as

$$-\frac{X''}{X} = \frac{Y''}{Y} = \text{constant}$$

Then

$$X'' = -\text{ constant } X, \qquad Y'' = \text{constant } Y$$

The lower BC is $v(x,0) = 0$ which separates to $Y(0) = 0$.
The upper BC is $v(y,7) = 0$ which separates to $Y(7) = 0$.
The left BC is $v(0,y) = 0$ which separates to $X(0) = 0$.

So the Y problem is

$$Y'' = \text{constant } Y \text{ with BC } Y(0) = 0,\ Y(7) = 0$$

This is problem 1 in the summary.
The only way to get a nonzero solution for Y is to use the case where the constant is negative, renamed $-\lambda^2$. Then

(14) $\qquad Y'' + \lambda^2 Y = 0, \qquad\qquad Y = A\cos\lambda y + B\sin\lambda y$

(15) $\qquad X'' - \lambda^2 X = 0, \qquad\qquad X = Ce^{\lambda x} + De^{-\lambda x} = E\cosh\lambda x + F\sinh\lambda x$

Part II Plug the separated BC into (14) and (15).
$Y(0) = 0$ makes $A = 0$

$Y(7) = 0$ makes $B\sin 7\lambda = 0$, $7\lambda = n\pi$, $\lambda = \dfrac{n\pi}{7}$

Use the cosh sinh version of X.
$Y(0) = 0$ makes $E = 0$

Part III Use superposition and plug in the IC.
By superposition,

(16) $$\boxed{v = \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi x}{7} \sin \frac{n\pi y}{7} \text{ for } 0 \le \quad \le 3,\ 0 \le y \le 7}$$

The righthand BC is $v(3,y) = y^2$ for $0 \le y \le 7$ so set $x = 3$, $v = y^2$ in (16). You need

$$y^2 = \sum_{n=1}^{\infty} D_n \sinh \frac{3n\pi}{7} \sin \frac{n\pi y}{7} \quad \text{for } 0 \le y \le 7$$

To get this you need

$$D_n \sinh \frac{3n\pi}{7} = \frac{2}{7} \int_{y=0}^{7} y^2 \sin \frac{n\pi y}{7}\, dy$$

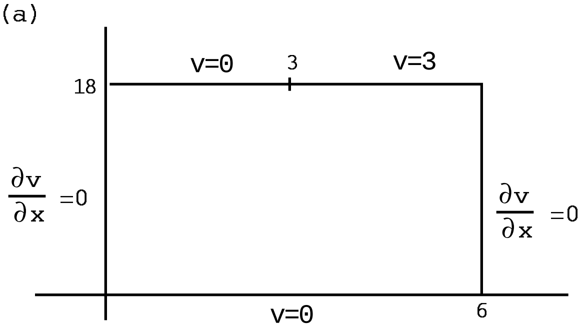
(17) $$\boxed{D_n = \frac{1}{\sinh 3n\pi/7} \frac{2}{7} \int_{y=0}^{7} y^2 \sin \frac{n\pi y}{7}\, dy}$$

The solution is (16) with the constants in the solution given in (17).

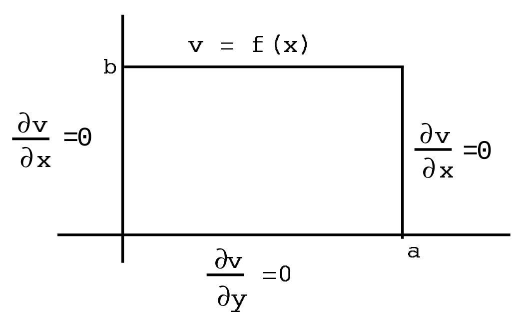
PROBLEMS FOR SECTION 6.4

1. Solve Laplace's equation

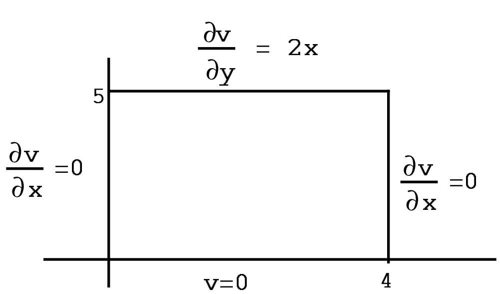
(a)



(b)



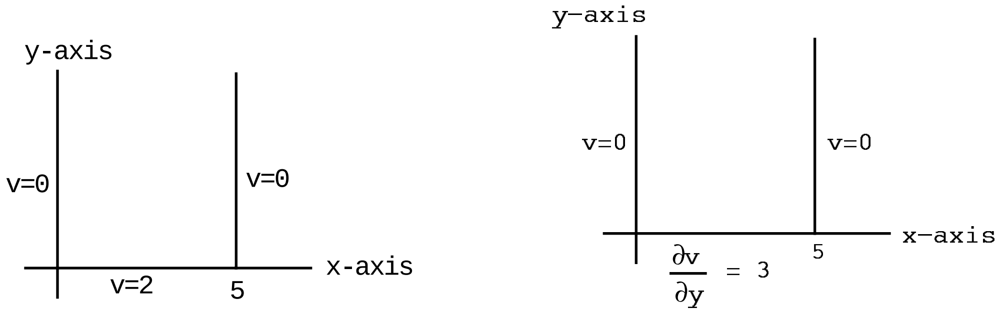
(c)



footnote
On a horizontal line, $-dv/dy$ is a measure of flux flowing up across the line. So the top BC here says that the flux across the top boundary is $2x$; e.g., the flux across the left end is 0, the flux across the midpoint is 4, the flux across the right end is 8

2. Solve Laplace's equation in the semi-infinite strip

(a) (b)



3. Show that plugging in $Y(0) = 0$ into $Y = C \cosh \lambda y + D \sinh \lambda y$ produces the same final result as plugging it into $Y = E e^{\lambda y} + Fe^{-\lambda y}$.

The rest of the problems in this section are about separating PDE and BC.

4. Separate and get X and Y solutions: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

Do only the useful cases assuming that later there will be a nonhomog condition of the form $u(x,y_o) = f(x)$

5. Look at the PDE $x\dfrac{\partial u}{\partial x} = y^2 \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y}$ with BC $u(0,y)=0$, $u(x,3) = 0$, $u(x,5) = x^2$.

Separate into an X problem and a Y problem (each with its own BC) and then stop. Don't try to solve.

6. Suppose the separation process in a PDE leads to

$$\frac{X'' + X}{X} = \frac{T'}{T}$$

Keep going with all possible cases.

7. Look at the PDE $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} - y$

(a) Here's the best way to separate: Let $y(x,t) = X(x)T(t)$. Then

$$XT'' = X'' T - XT$$

$$X(T'' + T) = X'' T$$

$$\frac{T'' + T}{T} = \frac{X''}{X} = \text{con},$$

Continue from here and just do the useful cases.

(b) Here's another way to separate:

$$XT'' = (X'' - X)T$$

$$\frac{T''}{T} = \frac{X'' - X}{X} = \text{con}$$

It's not as convenient but continue anyway and do the useful cases.

(c) And here's still another possibility for the separation:

$$XT'' = X''\,T - XT$$

$$X(T'' + T) = X''\,T$$

$$\frac{X}{X''} = \frac{T}{T'' + T} = \text{con}$$

Keep going in the useful cases.

8. Separate and get solutions (in all cases) $\dfrac{\partial u}{\partial x} = u - \dfrac{\partial u}{\partial y}$

9. Separate and get solutions (in all cases) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial x\,\partial y}$

10. Try but get stuck separating the PDE $\dfrac{\partial^2 u}{\partial x\,\partial y} + u = 4$.

11. Suppose $u = X(x)Y(y)$ for $0 \le x \le a$, $0 \le y \le b$. The condition

$$u(3,y) = 0 \text{ for } 0 \le y \le b$$

separates to $X(3) = 0$ but the condition

$$u(0,y) = 3 \text{ for } 0 \le y \le b$$

doesn't separate. Do you know why not?

12. Let $u(x,t) = X(x)T(t)$. Separate the BC

$$\frac{\partial u}{\partial x}(5,t) = -3u(5,t) \text{ for all } t$$

Honors

12. Here's Schrodinger's equation (from quantum mechanics) for the wave function $\psi(x,y,z,t)$ of a particle with mass m in a conservative force field with potential $V(x,y,z)$:

$$-\frac{h^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}\right)+V(x,y,z)\,\psi = ih\frac{\partial \psi}{\partial t}$$

(h and m are physical constants; i is the imaginary unit)

Separate t from x,y,z by assuming a solution of the form

$$\psi(x,y,z,t) = \phi(x,y,z)\;T(t)$$

to get two separate equations, one in the unknown $\phi(x,y,z)$ (this is called the time independent Schrodinger equation) and the other in the unknown $T(t)$.

The constant that turns up in the separation process is usually named E, not λ. Don't try to solve the separated equations (although the T one is easy to solve). Just find them.

SECTION 6.5 LAPLACE'S EQUATION IN POLAR COORDINATES AND FOURIER FULL SERIES

solution of $x^2y'' + axy' + by = 0$ (Euler's equation)

I'll need the solution to an Euler's equation before I can solve Laplace's equation in polar coordinates. Euler's equation is second-order, linear and homogeneous but with non-constant coeffs so there's a special method for it.

Look at an equation of the form

$$x^2y'' + axy' + by = 0$$

where y is a function of x. Substituting

$$x = e^t$$

turns it into the following new equation where y is now a function of t instead of x:

$$y''(t) + (a-1)y'(t) + by(t) = 0$$

(The x^2 and x disappear from the coeffs and a goes down by 1)

Solve the new DE for y(t) and then switch back to x's using

$$x = e^t, \qquad t = \ln x$$

to get the sol to the original DE.

proof

If $y = y(x)$ and $x = e^t$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy/dt}{e^t} = e^{-t}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(e^{-t}\frac{dy}{dt})}{dx} = \frac{d(e^{-t}\frac{dy}{dt})/dt}{dx/dt} = \frac{e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}}{e^t}$$

$$= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Take Euler's equation, replace y' and y'' by these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and replace x by e^t to get

$$e^{2t}e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + ae^t e^{-t}\frac{dy}{dt} + by = 0$$

which simplifies to

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0,$$

$$y'' + (a-1)y' + by = 0 \quad \text{where y is now y(t)} \quad \text{QED}$$

example 1

Solve $x^2y'' + 3xy' - 3y = 0$

This is an Euler's equation with $a = 3$, $b = -3$. Substituting $x = e^t$ turns it into

$$y''(t) + 2y'(t) - 3y(t) = 0.$$

Then

$$m^2 + 2m - 3 = 0, \qquad m = -3, 1,$$

(1) $$y = Ae^{-3t} + Be^{t}$$

Switch back to x's to get the final answer. One way to do it is to write y(t) as

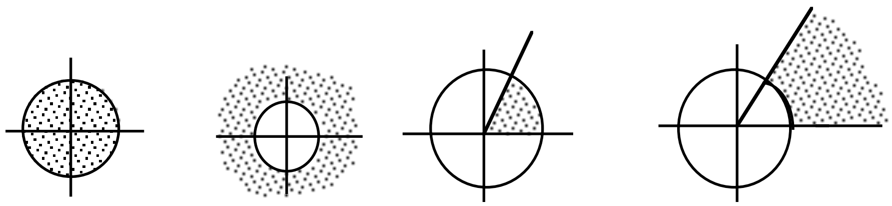
$$y(t) = A(e^t)^{-3} + Be^t.$$

Then substitute $x = e^t$ to get

(2) $$y = Ax^{-3} + Bx$$

summary of the procedure for solving Laplace's equation in polar coordinates on some standard regions with simple BC

Laplace's equation in polar coordinates is usually solved for the regions in Fig 1. Here are the three steps for solving them.



(a) (b) (c) (d)

FIG 1

Part I Separate the PDE and the homogeneous BC .

Part II Plug in the BC.

The nonhomog BC will be along a *circular* boundary and will be of the form $v = f(\boxed{\theta})$ This makes θ the "important" variable so use the case with the nice Θ solution, namely the case where $\Theta = A\cos\lambda\theta + B\sin\lambda\theta$.

If the BC are $\Theta(0) = 0$, $\Theta(L) = 0$ then this one case is enough.
If the BC are $\Theta'(0) = 0$, $\Theta'(L) = 0$ then the $\lambda = 0$ case also produces a solution.
If the region looks like (a) or (b), the $\lambda = 0$ case will produce a solution also.

The solution $v(r,\theta)$ should stay finite to be physically realizable. The dangerous spots are when $r = 0$ and $r = \infty$. In practice, when the region includes $r = \infty$ (namely (b) and (d)) you must make sure that $R(\infty)$ stays finite by throwing away the solution r^λ; and when the region includes $r = 0$ (namely (a) and (c)) you must make sure that $R(0)$ is finite by tossing out $r^{-\lambda}$.

For regions which include all θ between 0 and 2π (namely (a) and (b)) make the solution repeat every 2π with respect to θ, i.e., make the solution periodic.

Part III Use superposition to add all the solutions and get a solution with many constants .

Plug in the *non*homog BC to determine the remaining constants.

example 2

Laplace's equation in polar coords is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

I'll solve Laplace's equation on the region in Fig 2 radius 5 and angle $\pi/3$ with the indicated BC.

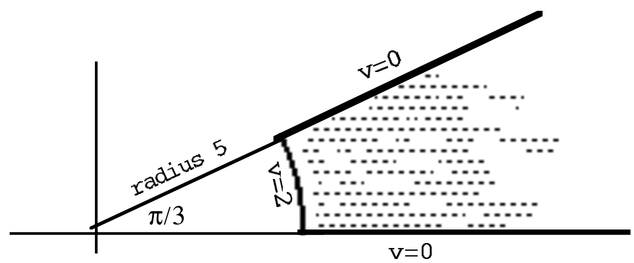


FIG 2

Part I Separate variables.
Try a solution of the form

$$v(r,\theta) = R(r)\Theta(\theta)$$

Then

$$R''\,\Theta + \frac{1}{r}R'\,\Theta + \frac{1}{r^2}R\,\Theta'' = 0$$

$$\Theta\left[R'' + \frac{1}{r}R'\right] = -\frac{1}{r^2}R\Theta''$$

$$\frac{-r^2\left[R'' + \frac{1}{r}R'\right]}{R} = \frac{\Theta''}{\Theta} = \text{constant}$$

$$\Theta'' - \text{constant}\;\Theta = 0, \qquad r^2R'' + rR' + \text{constant}\;R = 0$$

case 1 Constant = 0 The Θ equation is $\Theta'' = 0$ so $\Theta = A\theta + B$

The R equation is

$$r^2R'' + rR' = 0,$$

an Euler's equation with $a = 1$, $b = 0$. Substitute $r = e^t$ and get

$$R'' = 0, \quad R(t) = Ct + D.$$

Substitute back to get

$$R(r) = C\ \ell n\ r + D$$ (this is listed on the reference page).

case 2 Constant is positive. Ignore it. Not useful

case 3 Constant is negative. Call it $-\lambda^2$. Then for the Θ part we have

$$\Theta'' + \lambda^2\Theta = 0, \qquad \Theta = C\cos\lambda\theta + D\sin\lambda\theta$$

The R equation is

$$r^2R'' + rR' - \lambda^2R = 0,$$

an Euler's equation with $a = 1$, $b = -\lambda^2$. With the substitution $r = e^t$ it becomes

$$R'' - \lambda^2R = 0$$

$$R(t) = Ae^{\lambda t} + Be^{-\lambda t}$$

Substitute $t = \ell n\ r$ to get

$$R(r) = Ar^{\lambda} + Br^{-\lambda}$$ (this is on the reference page)

The BC on the first ray is $v(r,0) = 0$ which separates to $\Theta(0) = 0$
The BC on the second ray is $v(r, \pi/3) = 0$ which separates to $\Theta(\pi/3) = 0$

Part II Satisfy the homog BC.
Use the case where

$$\Theta = A \cos \lambda\theta + B \sin \lambda\theta, \qquad R = Cr^{\lambda} + Dr^{-\lambda}$$

$\Theta(0) = 0$ makes $A = 0$.
$\Theta(\pi/3) = 0$ makes

$$B \sin \frac{\lambda\pi}{3} = 0, \qquad \frac{\lambda\pi}{3} = n\pi, \quad \lambda = 3n.$$

To keep $R(\infty)$ finite, get rid of r^{3n} which blows up as $r \to \infty$. Choose $C = 0$.

Part III Satisfy the nonhomog BC.
By superposition,

(3) $$v = \sum_{n=1}^{\infty} B_n r^{-3n} \sin 3n\theta$$

The inner BC is $v(5,\theta) = 2$ for θ in $[0,\pi/3]$. To get it you need

(4) $$2 = \sum_{n=1}^{\infty} B_n 5^{-3n} \sin 3n\theta \quad \text{for } \theta \text{ in } [0,\pi/3]$$

warning When you plug in the nonhomog BC don't forget to set $r = 5$

Note that $\sin 3n\theta$ is of the form $\sin \frac{n\pi\theta}{L}$ where $L = \pi/3$. So (4) is a Fourier sine series and the coefficients formula is

$$B_n 5^{-3n} = \frac{2}{\pi/3} \int_{\theta=0}^{\pi/3} 2 \sin 3n\theta \, d\theta = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$B_{\text{even } n} = 0, \qquad B_{\text{odd } n} = \frac{8 \cdot 5^{3n}}{n\pi}$$

The final solution is

$$v = \frac{8}{\pi} \left[\left(\frac{5}{r}\right)^3 \sin 3\theta + \frac{1}{3} \left(\frac{5}{r}\right)^9 \sin 9\theta + \frac{1}{5} \left(\frac{5}{r}\right)^{15} \sin 15\theta + \ldots \right]$$

warning The coeffs contain $\boxed{5}^{3n}$ and the terms contain $\boxed{r}^{3n}$; they do *not* cancel .

example 3

I'll solve Laplace's equation for a disk with radius a, centered at the origin , and with BC $v = f(\theta)$ on the circular boundary (Fig 3)

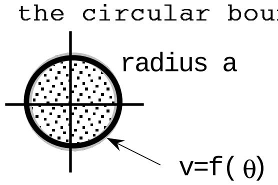


FIG 3

Part I Separate variables.
I won't repeat the whole separation. Here are the potentially useful cases.

case 1 $\Theta = C\cos\lambda\theta + D\sin\lambda\theta$, $R = Ar^{\lambda} + Br^{-\lambda}$

case 2 $\Theta = G\theta + H$, $R(r) = E\ln r + F$

Part II Plug in the homog BC
There are no homog BC but there are other, more subtle, conditions.
By the nature of polar coords, Θ must repeat every 2π. And to be realistic, R must stay finite.
In case 1, the period of the Θ solution is $2\pi/\lambda$. To guarantee that Θ repeats (at least) every 2π, λ must be an integer. For instance, $\sin 6\theta$ has period $2\pi/6$ and so it repeats every $\pi/3$; so there are 3 cycles every 2π so $\sin 6\theta$ also repeats every 2π. So $\lambda = n$ for $n = 1,2,3,\ldots$
To keep $R(0)$ finite get rid of r^{-n} since $1/r^n$ blows up as $r \to 0+$ Choose $B = 0$.
In case 2, to keep Θ periodic you need $G = 0$ and to keep R finite as $r \to 0+$ you need $E = 0$. So from this case you get $v = FH = K$.

Part III Satisfy the nonhomog condition.
By superposition

(5) $$\boxed{v = K + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)}$$

To get the BC $v(a,\theta) = f(\theta)$ you need

(6) $$f(\theta) = K + \sum_{n=1}^{\infty} a^n (C_n \cos n\theta + D_n \sin n\theta) \text{ for } 0 \le \theta \le 2\pi$$

Now you need constants K, C_n to satisfy (6). I'll come back and finish when I get the coefficient formulas.

finding Fourier full series coefficients

To get

(7) $$f(x) = C_0 + \sum_{n=1}^{\infty} \left[C_n \cos\frac{n\pi x}{L/2} + D_n \sin\frac{n\pi x}{L/2} \right] \text{ for } x \text{ in } [0,L]$$

use

(8)
$$C_0 = \frac{1}{L}\int_0^L f(x)\,dx = \text{average value of } f(x) \text{ in } [0,L]$$
$$C_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L/2}\,dx \quad \text{for } n = 1,2,3,\ldots$$
$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L/2}\,dx \quad \text{for } n = 1,2,3,\ldots$$

In Section 6.7 there will be some general formulas for Fourier series and the coeffs for the full series will be a special case of those formulas so you'll have to wait until then for the explanation.

warning

Note that in the full series, the sines and cosines are of $\frac{n\pi x}{\boxed{L/2}}$ not $\frac{n\pi x}{\boxed{L}}$.

The integral tables (1)-(5) on the reference page can be used to get coeffs for some sine and cosine series but not for a full series because the ingredients of a full series are $\sin \frac{n\pi x}{L/2}$ and $\cos \frac{n\pi x}{L/2}$ and all these formulas involve $\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$. But you can use the antiderivative formulas (A)-(K).

example 2 continued

To get the constants to satisfy (6), note that $\cos n\theta$ and $\sin n\theta$ are of the form $\cos \frac{n\pi\theta}{L/2}$ and $\sin \frac{n\pi\theta}{L/2}$ where $L = 2\pi$. So use the formulas in (8) with $L = 2\pi$. Then

$$\text{(9)} \qquad \boxed{K = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta}$$

$$a^n C_n = \frac{2}{2\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \qquad a^n D_n = \frac{2}{2\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta,$$

$$\text{(10)} \qquad \boxed{C_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta} \qquad \boxed{D_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta}$$

The solution is (5) with the constants in (9) and (10)

> **warning** Line (6) is *not* part of the answer. The answer is (5) along with (9) and (10)

summary of what functions blow up and how to keep your solution finite

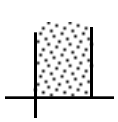
Let λ be a fixed positive number.

$e^{\lambda y}$ blows up as $y \to \infty$.
r^{λ} blows up as $r \to \infty$.
$r^{-\lambda}$ blows up as $r \to 0+$.

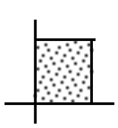
$\cosh \lambda y$ and $\sinh \lambda y$ both blow up as $y \to \infty$.

If $Y = A \cosh \lambda y + B \sinh \lambda y$ and the region includes $y=\infty$ (an unbounded vertical strip) it doesn't help to keep Y finite by making A=0 and B=0 since that leaves only the useless solution Y=0. Instead you can set $A = -B$ because $\cosh \lambda y$ and $\sinh \lambda y$ not only approach ∞ as $y \to \infty$ but they approach each other. Much better to use the alternate version $Y = Ce^{-\lambda y} + De^{\lambda y}$ and make $D = 0$.

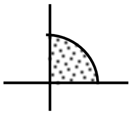
Here's how to avoid blowups when you solve Laplace's equation.



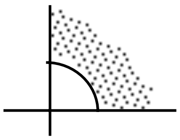
$$Y = Ce^{-\lambda y} + \cancel{De^{\lambda y}}$$



$Y = A \cosh \lambda y + B \sinh \lambda y$ (no blowup trouble, nothing to avoid)



$$R = \cancel{Ar^{-\lambda}} + Br^{\lambda}$$



$$R = Ar^{-\lambda} + \cancel{Br^{\lambda}}$$

warning

When λ is positive, as it always is, $e^{-\lambda y}$ does *not* blow up as $y \to 0$ or as $y \to \infty$. Don't see trouble where there is none.

summary of separation cases (continued from Section 6.4)

problem 1 $X'' = \text{constant}\cdot X$ with BC $X(0) = 0$, $X(L) = 0$

The case con = 0 has only the solution X=0. Ignore it.

The case con = λ^2 (i.e., positive constant) has only the solution X=0. Ignore it

The case con = $-\lambda^2$ (i.e., negative constant) has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = \frac{n\pi}{L}$ and the corresponding solution is $\sin\frac{n\pi x}{L}$ (and any multiple thereof).]

problem 2 $X'' = \text{constant}\cdot X$ with BC $X'(0) = 0$, $X'(L) = 0$ (same equ as problem 1 but different BC)

The case con = λ^2 has only the solution X=0. Ignore it.

The case con = 0 had a nonzero solution.

[It turns out that a solution is X = 1 and more generally X = A where A is an arbitrary constant.]

The case con = $-\lambda^2$ has nonzero X solutions for certain values of λ.

[It turns out that there is a nonzero sol iff $\lambda = \frac{n\pi}{L}$ and the corresponding solution is $\cos\frac{n\pi x}{L}$ (and any multiple thereof).]

problem 2 $\Theta'' = \text{con}\ \Theta$ with the condition that Θ have period 2π.

The case con = λ^2 has only the solution X=0. Ignore it.

The case con = 0 has a nonzero solution.

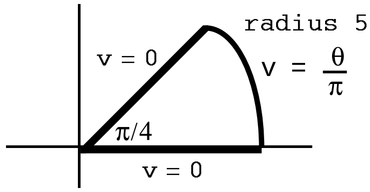
[It turns out that the solution is $\Theta = 1$ and more generally $\Theta = A$ where A is an arbitrary constant.]

The case con = $-\lambda^2$ has nonzero solutions for certain values of λ.

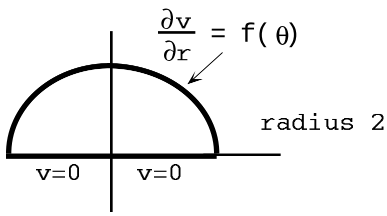
[It turns out that there is a nonzero sol iff $\lambda = n$ and the corresponding solutions are $\cos n\theta$ and $\sin n\theta$ (and any multiples thereof).]

PROBLEMS FOR SECTION 6.5

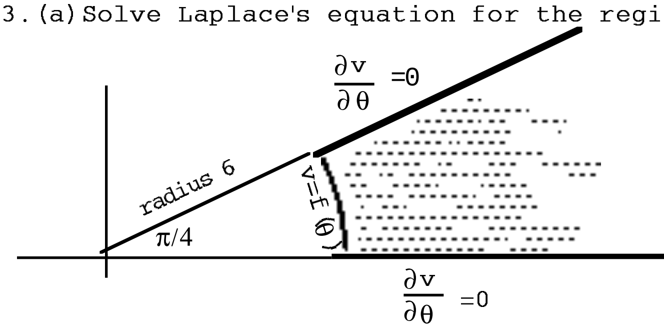
1.Solve Laplace's equation for the sector in the diagram



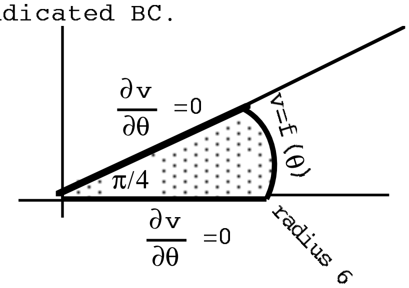
2.(a) Solve Laplace's equation for the semi-disk in the diagram
(b) Continue from part (a) but with $f(\theta) = 1$ in particular. Write out enough terms in the solution to make the pattern clear



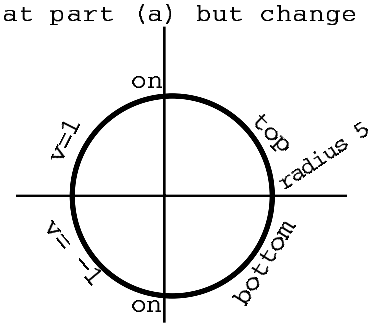
3.(a) Solve Laplace's equation for the region in the diagram with the indicated BC.



(b) Now solve Laplace's equation for the sector itself (easily, using part (a) with the indicated BC.



4. (a) Find $v(r,\theta)$ satisfying Laplace's equation for the region *inside* the disk in the diagram with the indicated BC and then again for the region *outside* the disk.
(b) Repeat part (a) but change the BC to $v = 4 \sin 3\theta$



5. Find a general solution.
   (a) $x^2y'' + 2xy' - 12y = 0$
   (b) $x^2y'' - 3xy' + 4y = 0$
   (c) $x^2y'' + 5xy' + 5y = 0$.
   (d) $x^2y'' - 3xy' + 4y = \ln x$.
   (e) $x^2y'' + 3xy' - 3y = 10x^2$.

SECTION 6.6 FOURIER TRIG SERIES FOR A PERIODIC FUNCTION

The Fourier trig series that you used to solve PDE can be used for an entirely different purpose, to represent periodic functions.

odd and even functions

A function $f(x)$ is called *even* if $f(x) = f(-x)$ for all x.
A function $f(x)$ is called *odd* if $f(-x) = -f(x)$ for all x.

The graph of an even function is symmetric with respect to the y–axis.
The graph of an odd function is symmetric with respect to the origin (Fig 1).

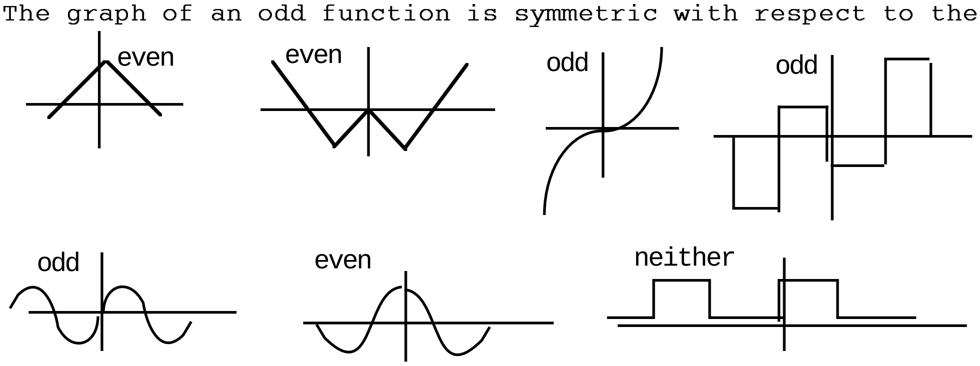


FIG 1

behavior of a Fourier trig series on $(-\infty,\infty)$

Look at the function x^2 for x in $[0,6]$ (Fig 2).
The function has three Fourier trig series representations, a sine series, a cosine series and a full series (I found all the coeffs):

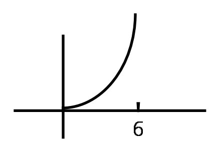


FIG 2

$$x^2 = \frac{72(\pi^2-4)}{\pi^3}\sin\frac{\pi x}{6} - \frac{72}{2\pi}\sin\frac{2\pi x}{6} + \frac{72(9\pi^2-4))}{27\pi^3}\sin\frac{3\pi x}{6} - \frac{72}{4\pi}\sin\frac{4\pi x}{6} + \frac{72(25\pi^2-4)}{125\pi^3}\sin\frac{5\pi x}{6} - \ \dots \quad \text{for } x \text{ in } [0,6]$$

$$x^2 = 12 - \frac{144}{\pi^2}\cos\frac{\pi x}{6} + \frac{144}{4\pi^2}\cos\frac{2\pi x}{6} - \frac{144}{9\pi^2}\cos\frac{3\pi x}{6} + \dots \text{ for } x \text{ in } [0,6]$$

$$x^2 = 12 + \frac{36}{\pi^2}\cos\frac{\pi x}{3} + \frac{36}{4\pi^2}\cos\frac{2\pi x}{3} + \frac{36}{9\pi^2}\cos\frac{3\pi x}{3} + \dots - \frac{36}{\pi}\sin\frac{\pi x}{3} - \frac{36}{2\pi}\sin\frac{2\pi x}{3} - \frac{36}{3\pi}\sin\frac{3\pi x}{3} - \dots \quad \text{for } x \text{ in } [0,6]$$

For x in $(-\infty,\infty)$, the sine series converges to the *odd periodic extension* of Fig 2 found by extending Fig 2 oddly to $[-6,6]$ (Fig 3a) and then extending the $[-6,6]$ piece periodically (Fig 3b). In other words, if you plot the sine series on a computer you'll get Fig 3b.

For x in $(-\infty,\infty)$, the cosine series converges to the *even periodic extension* of Fig 2, found by extending Fig 1 evenly to $[-6,6]$ (Fig 4a) and then extending the $[-6,6]$ piece periodically (Fig 4b).

For x in $(-\infty,\infty)$, the full series converges to the *periodic extension* of Fig 2, found by extending Fig 1 periodically (Fig 5).

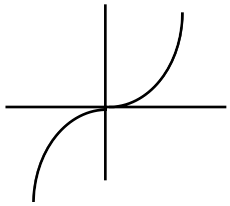


FIG 3a

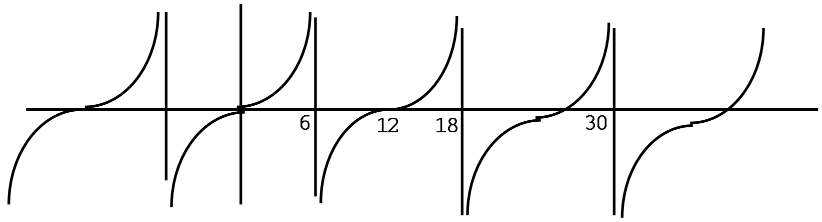


FIG 3b Odd periodic extension of Fig 2

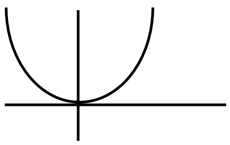


Fig 4a

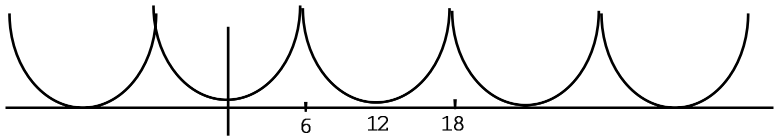


Fig 4b Even periodic extension of Fig 2

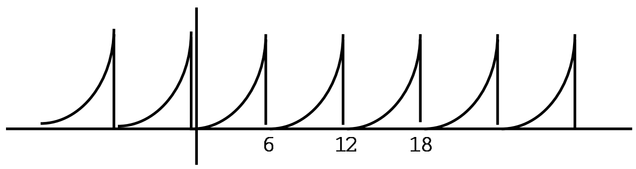


Fig 5 Periodic extension of Fig 2

proof

Here's *why* the sine series converges to Fig 3b (the explanations for the other series are similar).

Every term in the sine series is odd so the sum is also odd. Furthermore each term repeats every 12 units. (Fig 6 shows $\sin\frac{5\pi x}{6}$, with period $\frac{12}{5}$, repeating 5 times in 12 units and therefore repeating every 12 units.) So the sum repeats every 12.

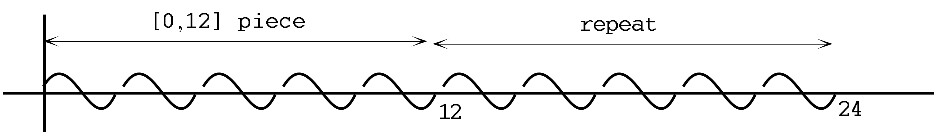


FIG 6

The series converges to x^2 for x in [0,6] (because of our choice of coeffs). It is also odd and repeats every 12. So it has no choice but to converge on $(-\infty,\infty)$ to the odd periodic extension in Fig 3b.

footnote

Fig 3b is ambiguous. It is not clear what the actual value of the series is at

$x = 6,18,30,\ldots$. The sine series for x^2 on [0,L] actually converges to the function in Fig 3c. Similarly, Fig 5 is ambiguous. The full series for x^2 on [0,6] actually converges to the function in Fig 5a.

In general, when the odd or plain periodic extension jumps, the correct value is the point in the "middle" of the jump.

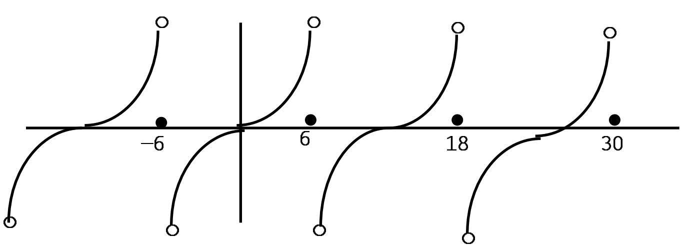


FIG 3c

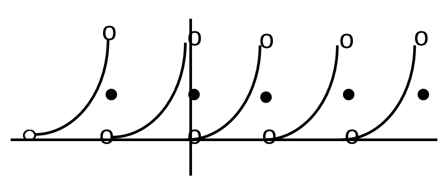


FIG 5a

the Fourier trig series for a periodic function

Given a periodic function f(x) on $(-\infty,\infty)$ you can find a Fourier trig series for it. The details depend on whether f is odd, even or neither.

case 1 How to do it for an *even* periodic function.
Look at the even function f(x) with period 8 in Fig 7.

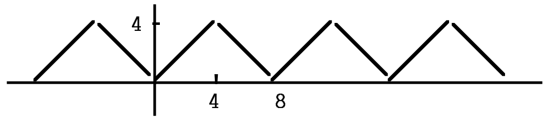


FIG 7

The cosine series for the [0,4] piece will converge on $(-\infty,\infty)$ to the even periodic extension of the [0,4] piece which is precisely Fig 7. So the series you want is the cosine series for the [0,4] piece:

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{4}$$

where

A_0 = average value of f(x) on [0,4] = 2

$$A_n = \frac{2}{4}\int_0^4 f(x) \cos \frac{n\pi x}{4}\, dx = \frac{2}{4}\int_0^4 x \cos \frac{n\pi x}{4}\, dx = -\frac{16}{n^2\pi^2} \quad \text{for odd } n \quad \text{(Tables (3))}$$

So

$$(1) \qquad f(x) = 2 - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{4} + \frac{1}{9}\cos\frac{3\pi x}{4} + \frac{1}{25}\cos\frac{5\pi x}{4} + \ldots \right].$$

The first cosine term in the series, called the fundamental harmonic, is $-\frac{16}{\pi^2}\cos\frac{\pi x}{4}$.

The fundamental harmonic frequency is 1/4 (cycles per sec) with amplitude $16/\pi^2$.

Similarly, the second harmonic (first overtone) is $-\frac{16}{9\pi^2}\cos\frac{3\pi x}{4}$. The first overtone frequency is 3/4 with amplitude $16/25\pi^2$.

review The term A sin bx (similarly A cos bx) has frequency $\frac{b}{2\pi}$ cycles per second, angular frequency b radians per second, amplitude |A| (amplitudes are always positive).

case 2 How to do it for an *odd* periodic function.
Look at the odd periodic function g(x) with period 8 in Fig 8.

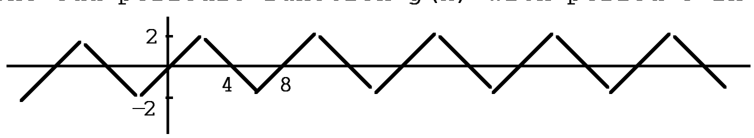
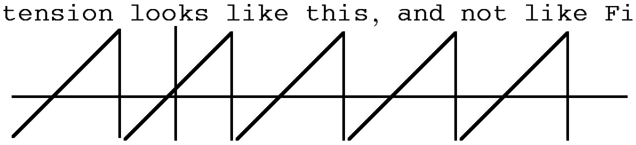


FIG 8 g(x)

The sine series for the [0,4] piece will converge on $(-\infty,\infty)$ to the odd periodic extension of the [0,4] piece which is precisely Fig 8.

warning Don't use the [0,2] piece because its odd periodic extension looks like this, and not like Fig 8



So

$$g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{4}$$

where

$$B_n = \frac{2}{4}\int_0^4 g(x) \sin \frac{n\pi x}{4}\, dx$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{16}{n^2\pi^2} & \text{if } n = 1,5,9,\ldots \\ -\dfrac{16}{n^2\pi^2} & \text{if } n = 3,7,11,.. \end{cases} \qquad \text{(Tables (5a) with K=2, L=4)}$$

So

$$(2) \qquad g(x) = \frac{16}{\pi^2}\left[\sin\frac{\pi x}{4} - \frac{1}{9}\sin\frac{3\pi x}{4} + \frac{1}{25}\sin\frac{5\pi x}{4} - \ldots \right]$$

The fundamental harmonic is $\frac{16}{\pi^2} \sin \frac{\pi x}{4}$ so the fundamental frequency is $\frac{1}{4}$ with amplitude $\frac{16}{\pi^2}$; the first overtone frequency is $\frac{3}{4}$ with amplitude $\frac{16}{9\pi^2}$ (same as for f(x) because f and g are the "same" wave, just in different locations).

case 3 How to do it for a *non-even non-odd* periodic function.
Look at the periodic non-even non-odd function h(x) with period 8 in Fig 9.

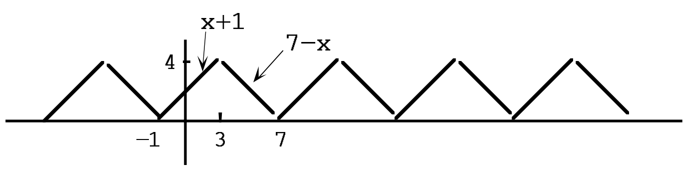


FIG 9 h(x)

The full series for the [0,8] piece will converge on $(-\infty,\infty)$ to the periodic extension of the [0,8] piece which is precisely Fig 9. So

$$h(x) = C_0 + \sum_{n=1}^{\infty}\left[C_n \cos\frac{n\pi x}{4} + D_n \sin\frac{n\pi x}{4}\right].$$

The standard coeff formulas use $\int_0^8$ but *you can integrate on any period's worth.* In this case it is more convenient to use $\int_{-1}^{7}$ (all the other L's in the coeff formulas remain 8) so

$$C_0 = \text{average value of } h(x) \text{ on } [-1,7] = 2$$

$$C_n = \frac{2}{8}\int_{-1}^{7} h(x) \cos\frac{n\pi x}{4}\, dx$$

$$= \frac{2}{8}\left[\int_{-1}^{3} (x+1) \cos\frac{n\pi x}{4}\, dx + \int_3^7 (7-x) \cos\frac{n\pi x}{4} dx\right]$$

$$D_n = \frac{2}{8}\left[\int_{-1}^{3} (x+1) \sin\frac{n\pi x}{4}\, dx + \int_3^7 (7-x) \sin\frac{n\pi x}{4} dx\right]$$

After a lot of integration (use (D) and (E) in the tables) you get

$$h(x) = 2 + \frac{8\sqrt{2}}{\pi^2}\left[\cos\frac{\pi x}{4} + \frac{1}{3}\cos\frac{3\pi x}{4} + \frac{1}{25}\cos\frac{5\pi x}{4} - \frac{1}{49}\cos\frac{7\pi x}{4} - \ldots + \sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{25}\sin\frac{5\pi x}{4} - \frac{1}{49}\sin\frac{7\pi x}{4} - \ldots\right]$$

The fundamental harmonic is

$$\frac{8\sqrt{2}}{\pi^2}\cos\frac{\pi x}{4} + \frac{8\sqrt{2}}{\pi^2}\sin\frac{\pi x}{4}$$

so the fundamental frequency is 1/4 with amplitude

$$\sqrt{\left[\frac{8\sqrt{2}}{\pi^2}\right]^2 + \left[\frac{8\sqrt{2}}{\pi^2}\right]^2} = \frac{16}{\pi^2}$$

review (see (2) in §1.5A) The sum $A\cos bx + B\sin bx$ describes harmonic oscillation with frequency $b/2\pi$ cycles per second, angular frequency b radians per second, amplitude $\sqrt{A^2 + B^2}$.

The first overtone frequency is $3\pi/4$ with amplitude

$$\sqrt{\left[\frac{8\sqrt{2}}{9\pi^2}\right]^2 + \left[\frac{8\sqrt{2}}{9\pi^2}\right]^2} = \frac{16}{9\pi^2}$$

(Figs 7, 8, 9 all have the same harmonics with the same respective amplitudes.)

In general, here's how to find a Fourier trig series for a periodic function f(x).

Look at the graph of f.

If f(x) is odd, identify the smallest piece whose odd periodic extension is the whole graph and find the cosine series for that piece.

If f(x) is even, identify the smallest piece whose even periodic extension is the whole graph and find the sine series for that piece.

If f(x) is neither even nor odd, identify the smallest piece whose (plain) extension is the whole graph and find the full series for that piece.

That's all I need.

It amounts to the following rule (if you like rules instead of art)

If f has period T find

the sine series with $L = \frac{1}{2}T$ if f is odd

the cos series with $L = \frac{1}{2}T$ if f is even

the full series with $L = T$ if f is neither

Sections 6.1–6.5 versus this section

There are two kinds of problems involving Fourier trig series.

1. (Sections 1–7) Given f(x) defined on an interval [0,L], find constants such that

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L]$$

or such that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \text{for } x \text{ in } [0,L]$$

or such that

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta) \quad \text{for } \theta \text{ in } [0,2\pi]$$

This type of problem turns up when you solve certain PDE's and end up needing one of these sets of coeffs. You don't get to choose which set of coeffs to find; the PDE tells you what to find. And you're not interested in the Fourier series itself; you want the constants so you can substitute them into the general solution of the PDE.

2.(this section) Given a *periodic* function, find its Fourier series.

This type of problem turns up when you want to express a signal as a superposition of harmonics. *You* must decide what kind of series will work (sines for an odd function, cosines for an even function, full for others) and what interval to build on (a half period for a sine or cosine series and a whole period for a full series).

inefficient methods

Look the even periodic function f(x) in Fig 7 again. I found its series in (1), a cosine series with L = 4. The series is

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{4}$$

where

$$A_0 = \frac{2}{4}\int_0^4 f(x)\,dx, \quad A_n = \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx$$

This works because Fig 7 is the even periodic extension of the [0,4] piece.

It's also correct to find a cosine series using L = 8. The series is

$$E_0 + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{8}$$

where

$$E_0 = \frac{2}{8}\int_0^8 f(x)\,dx, \quad E_n = \frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi x}{8}$$

This works because Fig 7 is also the even periodic extension of the [0,8] piece. The odd E's turn out to be 0 and this "second" series cancels down to the first series. But it's less efficient to compute because, for this f(x), $\int_0^8 f(x)\,dx$ is messier than $\int_0^4 f(x)\,dx$.

It's also correct to find a "full" series using L = 8. The series is

$$G_0 + \sum_{n=1}^{\infty} \left[G_n \cos\frac{n\pi x}{4} + H_n \sin\frac{n\pi x}{4} \right]$$

where

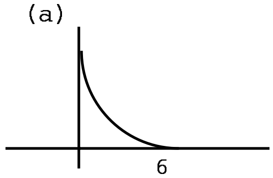
$$G_0 = \frac{2}{8}\int_0^8 f(x)\,dx, \quad G_n = \frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi x}{4}, \quad D_n = \frac{2}{8}\int_0^8 f(x)\sin\frac{n\pi x}{4}$$

This works because Fig 7 is also the (plain) periodic extension of the [0,8] piece. All the sine coeffs turn out to be 0 and this series turns into the original cosine series. But it's less efficient because there are more coeffs to compute. People will laugh at you for doing it this way.

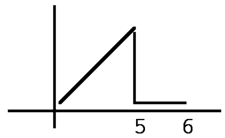
PROBLEMS FOR SECTION 6.6

1.Draw the even, odd and plain periodic extensions of these [0,6] pieces (with the x-axis calibrated).

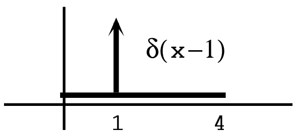
(a)



(b)



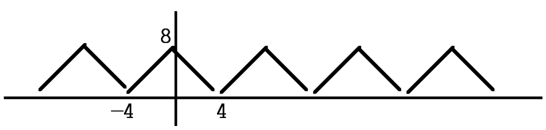
2. The function $\delta(x-1)$, for $0 \le x \le 4$, can be represented by a Fourier sine series, by a Fourier cosine series and by a Fourier full series.



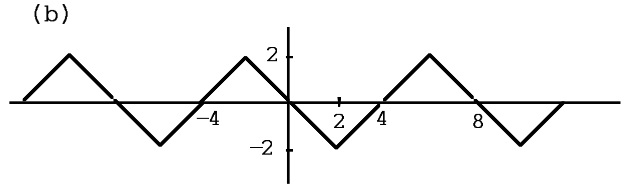
Sketch the graph of each series for all x.
Calibrate your axes and sketch enough of each picture to make the pattern clear.
(Don't *find* the three series; just draw a pretty picture of what each converges to.)

3. Find a Fourier series (as efficiently as possible) for each of these periodic function but stop before actually computing the coefficients. Just set it up

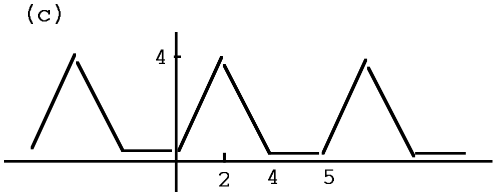
(a)



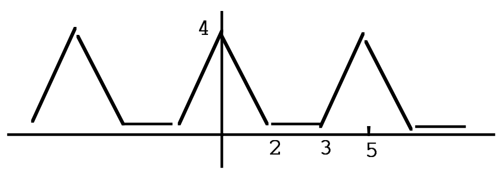
(b)



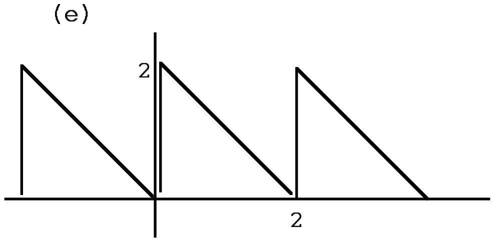
(c)



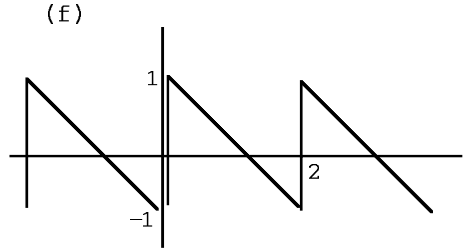
(d)



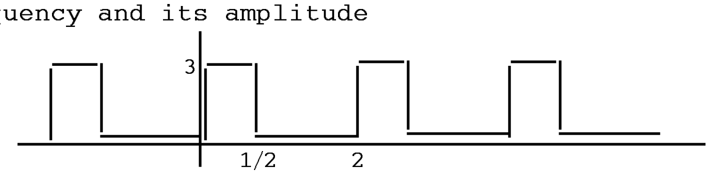
(e)



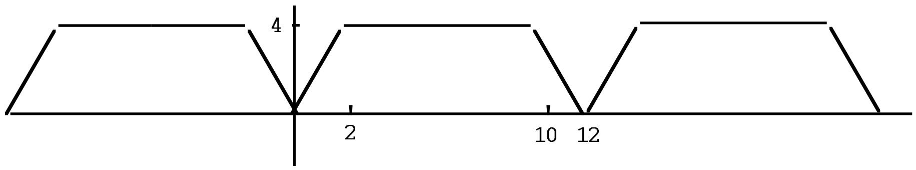
(f)



4. Find a Fourier series (efficiently) for the periodic function in the diagram. Then find the fundamental frequency and its amplitude, and the first overtone frequency and its amplitude

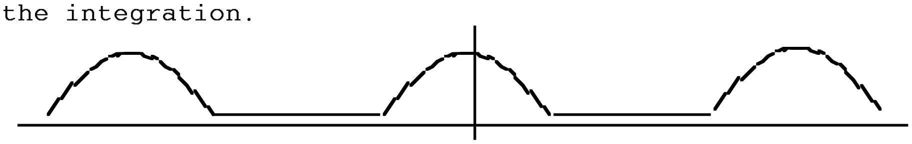


5. Find the fundamental frequency and its amplitude (but leave integrals unevaluated)

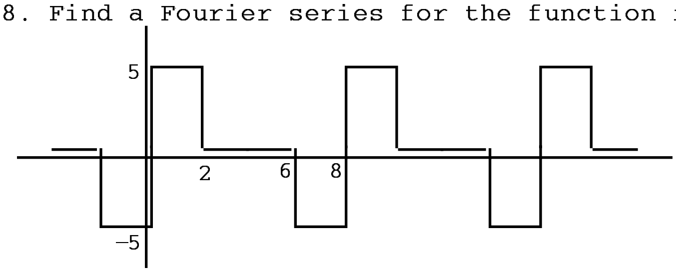


6. Find a Fourier series (efficiently) for $|\sin x|$ and write out enough terms to make the pattern clear

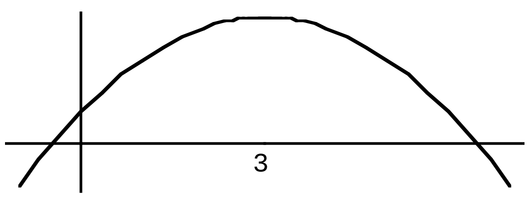
7. The function $\cos x$ passes through a half-wave rectifier which cuts off the lower pieces (see the diagram). Find a Fourier series for the result (efficiently) but skip the integration.



8. Find a Fourier series for the function in the diagram



9. Here's the graph of $3 + 6x - x^2$.



(a) Look at the sine series for the [0,3] piece, i.e., the series $\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{3}$

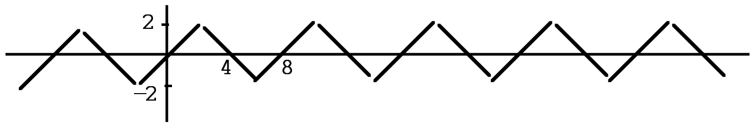
with $A_n = \frac{2}{3}\int_0^3 (3 + 6x - x^2) \sin \frac{n\pi x}{3}\, dx$.

(i) What does the series converge to on $(-\infty,\infty)$ (draw a picture).
(ii) Now go back and wherever you had a vertical line, be more precise and indicate without ambiguity exactly what the series does.
(iii) What's the value of the series when $x = 25$; i.e., what does the series converge to when $x = 25$.

(b) Repeat part (a) but with the cosine series for the [0,3] piece.

(c) Repeat part (a) but with the full series for the [0,3] piece.

10. The problem is to find a particular solution to the DE y" + 4y = g(x) where g is the periodic function in the diagram.



(a) Start by finding a particular solution to y" + 4y = sin kx where k is a constant, $k \neq 2$.

(b) The Fourier series for g(x) happens to be

$$g(x) = \frac{16}{\pi^2}\left[\sin\frac{\pi x}{4} - \frac{1}{9}\sin\frac{3\pi x}{4} + \frac{1}{25}\sin\frac{5\pi x}{4} - \ldots\right]$$

(I found this in (2).)

Use the Fourier series for g(x) plus a lot of superposition to find a particular solution to y" + 4y = g(x).

SECTION 6.7 COMPLETE SETS OF ORTHOGONAL FUNCTIONS

Every PDE problem ends like this: You have a general solution to a PDE and want to find the constants to make the solution satisfy a condition like

$$u(x,0) = f(x) \text{ for } a \le x \le b$$

After plugging in the condition you typically need constants so that

(1) $$f(x) = A_1\,\phi_1(x) + A_2\,\phi_2(x) + A_3\,\phi_3(x) + \dots \quad \text{for } a \le x \le b$$

for some functions $\phi_1(x)$, $\phi_2(x)$, ...

In this course the interval has been [0,L] and three sets of ϕ's turned up so far:

(2) (§6.1) $\sin\frac{\pi x}{L}$, $\sin\frac{2\pi x}{L}$, $\sin\frac{3\pi x}{L}$,...

(3) (§6.2) 1, $\cos\frac{\pi x}{L}$, $\cos\frac{2\pi x}{L}$, $\cos\frac{3\pi x}{L}$,...

(4) (§6.5) 1, $\cos\frac{\pi x}{L/2}$, $\cos\frac{2\pi x}{L/2}$, $\cos\frac{3\pi x}{L/2}$,..., $\sin\frac{\pi x}{L/2}$, $\sin\frac{2\pi x}{L/2}$, $\sin\frac{3\pi x}{L/2}$, ...

Here's the idea in general.

orthogonal functions

Two functions h(x) and k(x) are called orthogonal on the interval [a,b] if

$$\int_a^b h(x)\,k(x)\,dx = 0$$

The functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$,... are an orthogonal family on [a,b] if

$$\int_a^b \phi_i(x)\,\phi_j(x)\,dx = 0 \quad \text{for } i \ne j;$$

i.e., if any two different functions in the family are orthogonal on [a,b].

For example, the functions in (2) are an orthogonal family on [0,L]: for $n \ne m$,

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \left[\frac{\sin\frac{(n-m)\pi x}{L}}{2(n-m)\pi/L} - \frac{\sin\frac{(n+m)\pi x}{L}}{2(n+m)\pi/L}\right]_0^L = 0$$

And similarly the functions in (3) are orthogonal on [0,L]:

$$\int_0^L 1\cdot\cos\frac{n\pi x}{L}\,dx = 0$$

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \quad \text{for } n \ne m.$$

And similarly for the functions in (4).

example 1

Let $f(x) = 2x$ and $g(x) = x^2$. Are the functions orthogonal on [0,2]? on [−2,2]?

solution

$$\int_0^2 2x\cdot x^2\,dx = \int_0^2 2x^3\,dx = \left.\tfrac{1}{2}x^4\right|_0^2 \neq 0$$

$$\int_{-2}^2 2x\cdot x^2\,dx = \int_{-2}^2 2x^3\,dx = \left.\tfrac{1}{2}x^4\right|_{-2}^2 = 0$$

So the functions are *not* orthogonal on [0,2] and *are* orthogonal on [−2,2].

complete sets of orthogonal functions

A set of functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$,... that are orthogonal on interval [a,b] is called *complete* on the interval if given any f(x), you can find (unique) constants A_1, A_2, A_3,... so that

$$f(x) = A_1\,\phi_1(x) + A_2\,\phi_2(x) + A_3\,\phi_3(x) + \ldots \quad \text{for } a \le x \le b,$$

i.e., if you can make series out of the ϕ's that will converge to anything you want for $a \le x \le b$.

Not any old collection of orthogonal functions is complete. But it turns out (very hard to prove) that the sets of functions in (2)−(4) *are* complete on the interval [0,L]. And for all practical purposes whenever you need (1) in the context of a PDE, you can get it because the ϕ's will be a complete orthogonal family.

making series out of a complete set of orthogonal functions

If the ϕ's in (1) are complete and orthog on [a,b], here's how to get the coefficients in (1).

To get A_4 for instance, multiply both sides of (1) by $\phi_4(x)$ and integrate:

$$\int_a^b f(x)\,\phi_4(x)\,dx$$

$$= A_1 \underbrace{\int_a^b \phi_1(x)\phi_4(x)\,dx}_{0 \text{ by orthogonality}} + A_2 \underbrace{\int_a^b \phi_2(x)\phi_4(x)\,dx}_{0} + A_3 \underbrace{\int_a^b \phi_3(x)\phi_4(x)\,dx}_{0}$$

$$+ A_4 \underbrace{\int_a^b \phi_4(x)\phi_4(x)\,dx}_{\substack{\text{NOT } 0 \\ \phi_4 \text{ is not orthog to itself}}} + A_5 \underbrace{\int_a^b \phi_5(x)\phi_4(x)\,dx}_{0} + \ldots$$

Because of orthogonality, all the terms on the right except one drop out and you get

$$\int_a^b f(x)\,\phi_4(x)\,dx = A_4 \int_a^b \phi_4^2(x)\,dx$$

Solve for A_4 and you've got this formula:

$$A_4 = \frac{\int_a^b f(x)\,\phi_4(x)\,dx}{\int_a^b \phi_4^2(x)\,dx}$$

In general:

The constants that make (1) hold when the ϕ's are complete and orthogonal on [a,b] are

$$(5) \qquad A_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}$$

how to tell (in the middle of solving a PDE) when a set of functions is complete and orthogonal on an interval

When you reach stage (1) as you solve a PDE, look back to see where the ϕ's came from. They probably came from a problem that looked like (or could be rearranged to look like) this:

(6)
$$p(x)\,y''(x) + p'(x)\,y'(x) + q(x)y(x) = \text{constant}\cdot\, y(x)$$

plus certain BC on [a,b] where $p(x) \ge 0$ and $q(x) \le 0$ for $a \le x \le b$.

footnote
Those "certain" BC include
$y(L) = 0$
$y'(L) = 0$
$y'(L) = -3y(L)$
$y(0) = y(L)$
$y'(0) = y'(L)$
$y(0)$ is finite etc

Exactly the kind of BC that turn up in practice.

The DE in (6) is called *Sturm-Liouville form.*

It can be shown that the solutions are a complete orthogonal family on [a,b].

Solving (6) for y requires cases to get *all* the solution.

case 1 The constant is negative, renamed $-\lambda^2$

There are infinitely many solutions here; in particular there are infinitely many values of λ for which there are nonzero solutions to

$$p(x)\,y''(x) + p'(x)\,y'(x) + q(x)\,y(x) = -\lambda^2\,y(x) \quad \text{plus certain BC on [a,b]}$$

case 2 con = 0

Sometimes there will be a solution, i.e. you may get a nonzero solution from

$$p(x)\,y''(x) + p'(x)\,y'(x) + q(x)y(x) = 0 \text{ plus BC on [a,b]}$$

case 3 constant is positive

There will be no nonzero solutions for y. Forget this case

All this is hard to prove.

the special case of Fourier sine coeffs

The sines in (2) turned up when you so

$$X'' = \text{con } X \qquad \text{with BC } X(0) = 0,\ X(L) = 0$$

This is Sturm Liouville form with $p(x) = 1$, $q(x) = 0$. So the sines are a complete orthogonal family on $[0, L]$.

To get

$$f(x) = A_1 \sin\frac{\pi x}{L} + A_2 \sin\frac{2\pi x}{L} + A_3 \sin\frac{3\pi x}{L} + \ldots \text{ for } 0 \le x \le L,$$

the constants should be

$$A_n = \frac{\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}{\int_0^L \sin^2\frac{n\pi x}{L}\, dx} \qquad \text{(by (5))}$$

The denominator is $\left(\frac{x}{2} - \frac{L}{4n\pi}\sin\frac{2n\pi x}{L}\right)\Big|_0^L$ (antidriv tables) $= \ldots = L/2$

So the formula becomes

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \text{ (which agrees with the formula back in §6.1)}$$

the special case of Fourier cosine coeffs

The functions in (3) came from solving

$$X'' = \text{con } X \qquad \text{with BC } X'(0) = 0,\ X'(L) = 0$$

This is Sturm Liouville form with $p(x) = 1$, $q(x) = 0$.
So the functions in (3) are a complete orthogonal family on $[0, L]$.
By (5), to get

$$f(x) = A_0\cdot 1 + A_1 \cos\frac{\pi x}{L} + A_2 \cos\frac{2\pi x}{L} + A_3 \cos\frac{3\pi x}{L} + \ldots \quad \text{for } 0 \le x \le L,$$

the constants should be

$$A_0 = \frac{\int_0^L f(x)\ 1\ dx}{\int_0^L 1^2\, dx} = \frac{\int_0^L f(x)\ 1\ dx}{L} = \frac{1}{L}\int_0^L f(x)\, dx$$

$$A_n = \frac{\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx}{\int_0^L \cos^2\frac{n\pi x}{L}\, dx}$$

The denominator is $\left(\frac{x}{2} + \frac{L}{4n\pi}\cos\frac{2n\pi x}{L}\right)\Big|_0^L = L/2$

So the formula becomes

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx \quad \text{(which agrees with the formula in §6.2)}$$

the special case of Fourier full series coeffs

The functions in (4) are the solutions to the DE

$$y'' = \text{con } y \text{ with BC } y(0) = y(L),\ y'(0) = y'(L)$$

This is Sturm Liouville form with $p(\theta) = 1$, $q(\theta) = 0$ so the functions in (4) are a complete orthogonal family on [0,L]

By (5), to get

$$f(x) = A_0 \cdot 1 + A_1 \cos\frac{\pi x}{L/2} + A_2 \cos\frac{2\pi x}{L/2} + \ldots + B_1 \sin\frac{\pi x}{L/2} + B_2 \sin\frac{2\pi x}{L/2} + \ldots$$

for $0 \le x \le L$,

the constants should be

$$A_0 = \frac{\int_0^L f(x)\ 1\ dx}{\int_0^L 1^2\ dx} = \frac{\int_0^L f(x)\ 1\ dx}{L} = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_n = \frac{\int_0^L f(x) \cos\frac{n\pi x}{L/2}\ dx}{\int_0^L \cos^2\frac{n\pi x}{L/2}\ dx}$$

The denominator is $\left(\frac{x}{2} + \frac{L/2}{4n\pi}\cos\frac{2n\pi x}{L/2}\right)\Big|_0^L = L/2$

So the formula becomes

$$A_n = \frac{2}{L}\int_0^L f(x)\ \cos\frac{n\pi x}{L/2}\ dx$$

and similarly

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L/2}\ dx \quad \text{(the formulas from §6.5)}$$

footnote

The functions

$$1\ \cos\theta,\ \cos 2\theta,\ \cos 3\theta, \ldots,\ \sin\theta,\ \sin 2\theta,\ \sin 3\theta, \ldots$$

are a special case of (4). They turned up in Laplace's equation in polar coordinates (§6.5) where the letters were $\Theta(\theta)$ rather than $X(x)$, L was specifically 2π and we made Θ periodic which implies $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$

example 2

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

with BC

$$\frac{\partial u}{\partial x}(0,t) = 0 \text{ for all } t$$

$$-\frac{\partial u}{\partial x}(5,t) = 2u(5,t) \text{ for all } t$$

and IC $u(x,0) = f(x)$ for $0 \le x \le 5$

footnote The first BC says that the left end of the rod is insulated. $-\partial u/\partial x$ is the rate at which calories flow to the right in the rod so the second BC says that calories flow out the right end in proportion to the temperature at the right end (the hotter the right end, the more the calories flow out the right). I think this is called convection.

solution

Part 1 Separate

Try a solution of the form $u(x,t) = X(x)\,T(t)$.

Then

$XT' = kX''T$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{con}$$

case 1 $\text{con} = -\lambda^2$

$$X = A\cos\lambda x + B\sin\lambda x, \quad T = Ce^{-k\lambda^2 t}$$

case 2 $\text{con} = 0$

$$X = Px + Q, \quad T = D$$

The left boundary condition separates to $X'(0) = 0$.
The right boundary condition separates to $-X'(5) = 2X(5)$.

Part 2 Plug in separated conditions

case 1

$X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$.

To get $X'(0) = 0$ you need $B=0$.
To get $-X'(5) = 2X(5)$ you need

$$\lambda A\sin 5\lambda = 2A\cos 5\lambda.$$

Either A = 0 (which is not helpful) or $\lambda \sin 5\lambda - 2\cos 5\lambda$,

$$\tan 5\lambda = 2/\lambda.$$

There are infinitely many λ's satisfying this equation.
Fig 1 shows some of them, namely the x-coordinates of the points of intersection of the graph of $y = \tan 5\lambda$ (the dashed lines are its asymptotes) and the graph of $y = 2/\lambda$ (a hyperbola, with two branches).

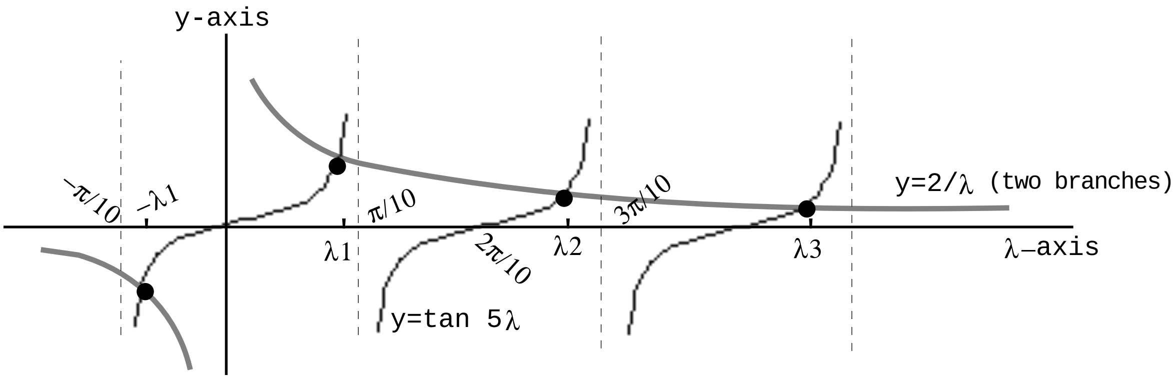


FIG 1

There are infinitely many points of intersection. Call the λ-coordinates of the intersection points on the right side of the diagram $\lambda_1, \lambda_2, \lambda_3, \ldots$. The λ-coords of the intersection points on the left side are $-\lambda_1, -\lambda_2, -\lambda_3, \ldots$.

Solutions in this case are $X = A_n \cos \lambda_n x$ for $n = 1,2,3,\ldots$, $T = C_n e^{-k\lambda_n^2 t}$

You don't get anything new by using the negative λ's because
$\cos(-\lambda x) = \cos \lambda x$ and $e^{-k[-\lambda]^2 t} = e^{-k\lambda^2 t}$.

So from this case we have solutions $u = C_n e^{-k\lambda_n^2 t} \cos \lambda_n x$.

case 2 con = 0
$X = Px + Q$
$T = D$
$X' = P$.
To get $X'(0) = 0$ you need $P = 0$.

To get $-X'(5) = 2X(5)$, you need $-0 = 2Q$, $Q = 0$.
There are no nonzero solutions from this case.

Why do I need to look at this case? Because Sturm Liouville theory says that there may be a nonzero solution here. The BC $X'(0) = 0$, $-X'(5) = 2X(5)$ haven't turned up in the course so far and until I try, I don't know whether or not there is a nonzero sol here.

Sturm Liouville theory says don't bother with the case where the constant is positive. All in all,

(7) $$u = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} \cos \lambda_n x \text{ for } 0 \le x \le 5, \text{ all } t, \lambda_n\text{'s in Fig 1}$$

To get the IC, you need

(8) $$f(x) = \sum_{n=1}^{\infty} C_n \cos \lambda_n x \text{ for } 0 \le x \le 5.$$

The functions $\cos \lambda_n x$ are a complete orthogonal family on [0,5] because they are the solutions to the Sturm Liouville problem

$$X'' = \text{con } X \qquad (\ p(x) = 1,\ q(x) = 0\ )$$

plus BC $\frac{\partial u}{\partial x}(0,t) = 0$ for all t

$$-\frac{\partial u}{\partial x}(5,t) = 2u(5,t) \text{ for all } t$$

So it is possible to get constants C_n that satisfy (8), namely

(9) $$C_n = \frac{\int_0^5 f(x) \cos \lambda_n x \, dx}{\int_0^5 \cos^2 \lambda_n x \, dx}$$

The solution consists of (7), (9) and Fig 1.

Note: The formula for C_n in (9) is *not* $\frac{2}{5}\int_0^5 f(x) \cos \lambda_n x \, dx$ because the denominator in (9) does not come out to be 5/2. That happened for the complete orthogonal family 1, $\cos \frac{n\pi x}{L}$ but this is a different family.

mathematical catechism

question 1 What does it mean to say that the functions $y_1, y_2, \ldots$ are orthogonal on the interval [a,b].

answer 1 It means that $\int_{x=a}^{b} y_i(x)\, y_j(x)\, dx = 0$ for $i \ne j$

question2 What does it mean to say that the functions $y_1, y_2, \ldots$ are a complete orthogonal family on the interval [a,b].

answer 2 It means that $\int_{x=a}^{b} y_i(x)\, y_j(x)\, dx = 0$ for $i \ne j$ (that's the orthog part) and that given any function f(x), you can find constants $A_1, A_2, \ldots$ such that

$$f(x) = A_1 y_1(x) + A_2 y_2(x) + \ldots \text{ for } a \le x \le b$$

(in fact the formula for the A's is in (5)).

PROBLEMS FOR SECTION 6.7

1. Suppose $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x) = x^4$, $\phi_4(x)$, $\phi_5(x)$,... is a complete orthogonal family on the interval [0,1]. Suppose we have the following Fourier series for $2x^5$:

$$2x^5 = A_1\phi_1(x) + A_2\phi_2(x) + A_3\phi_3(x) + A_4\phi_4(x) + \ldots \quad \text{for } 0 \le x \le 1$$

Find A_3.

2. Show that the functions 1 and $\cos\frac{\pi x}{L}$ are orthogonal on [0,L] as touted.

3.(a) Solve the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

with BC

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0$$

(the left end of the rod is insulated and the right end is maintained at temp 0)

and

IC $\quad u(x,0) = f(x)$ for $0 \le x \le L$

Along the way, identify the new complete set of orthogonal functions that turn up and the Sturm Liouville problem that produced them.

(b) Continue from part (a) and use the specific IC $u(x,0) = 7$ for $0 \le x \le L$.

4. Suppose you were asked to find constants A_1, A_2,... so that

$$(*) \quad f(x) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \quad \text{for } 0 \le x \le L$$

What would you say?

5. The functions $\sin\frac{x}{5}$, $\sin\frac{2x}{5}$, $\sin\frac{3x}{5}$, $\sin\frac{4x}{5}$,... are a complete orthogonal family on what interval.

SECTION 6.8 FOURIER BESSEL SERIES AND THE 2-DIM HEAT AND WAVE EQUATIONS

orthogonality with respect to a weight function

Suppose $w(x) \ge 0$ in the interval [a,b].
The functions $\phi_1(x)$ and $\phi_2(x)$ are called orthogonal on [a,b] *with respect to w(x)*, (called a *weight function*) if

$$\int_a^b \phi_1(x)\ \phi_2(x)\ w(x)\ dx = 0.$$

Plain orthogonality is the special case where $w(x) = 1$.

complete sets of functions orthogonal w.r.t. a weight function

A set of functions $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ that are orthogonal on interval [a,b] w.r.t. some $w(x)$ is called *complete* on the interval if given any $f(x)$, you can find (unique) constants $A_1, A_2, A_3, \ldots$ so that

$$(1) \qquad f(x) = A_1\ \phi_1(x) + A_2\ \phi_2(x) + A_3\,\phi_3(x) + \ldots \quad \text{for } a \le x \le b,$$

i.e., if you can make series out of the ϕ's that will converge to anything you want for $a \le x \le b$.

making series out of a complete set of functions orthogonal with respect to a weight function

If the ϕ's in (1) are orthog on [a,b] w.r.t. a weight function $w(x)$ and are complete, here's how to get the coefficients in (1). To get A_4 for instance, multiply both sides of (1) by $\phi_4(x)\,w(x)$ and integrate:

$$\int_a^b f(x)\,\phi_4(x)\,w(x)\ dx$$

$$= A_1 \underbrace{\int_a^b \phi_1(x)\phi_4(x)\ w(x)\ dx}_{\text{0 by orthogonality}} + A_2 \underbrace{\int_a^b \phi_2(x)\phi_4(x)w(x)\ dx}_{0} + A_3 \underbrace{\int_a^b \phi_3(x)\phi_4(x)\ w(x)\ dx}_{0}$$

$$+ A_4 \underbrace{\int_a^b \phi_4(x)\phi_4(x)\ w(x)\ dx}_{\text{NOT } 0} + A_5 \underbrace{\int_a^b \phi_5(x)\phi_4(x)\ w(x)\ dx}_{0} + \ldots$$

Because of orthogonality, all the terms on the right except one drop out and you get

$$\int_a^b f(x)\,\phi_4(x)\,w(x)\ dx = A_4 \int_a^b \phi_4^2(x)\,w(x)\ dx$$

Solve for A_4:

$$A_4 = \frac{\int_a^b f(x)\,\phi_4(x)\,w(x)\ dx}{\int_a^b \phi_4^2(x)\,w(x)\ dx}$$

In general:

The constants that make (1) hold when the ϕ's are complete and orthogonal w.r.t. $w(x)$ on $[a,b]$ are

$$(2) \qquad A_n = \frac{\int_a^b f(x)\, \phi_n(x)\, w(x)\, dx}{\int_a^b \phi_n^2(x)\, w(x)\, dx}$$

how to tell (in the middle of solving a PDE) whether a set of functions is complete and orthogonal w.r.t. some w(x) on an interval

Here's the more general Sturm Liouville problem:

(3) $p(x)\, y''(x) + p'(x)\, y'(x) + q(x)y(x) = \text{constant}\cdot w(x)y(x)$

plus certain BC on $[a,b]$ where $p(x) \ge 0$ and $q(x) \le 0$ for $a \le x \le b$.

It can be shown that the solutions are complete and orthogonal w.r.t. $w(x)$ on $[a,b]$.

Solving (3) for y requires cases to get *all* the solution.

case 1 The constant is negative, renamed $-\lambda^2$

There are infinitely many solutions here; in particular, there are infinitely many values of λ for which there are nonzero solutioins to

$$p(x)\, y''(x) + p'(x)\, y'(x) + q(x)\, y(x) = -\lambda^2\, w(x)y(x) \quad \text{plus certain BC on } [a,b]$$

case 2 con = 0

Sometimes there will be a solution, i.e. you may get a nonzero solution from

$$p(x)\, y''(x) + p'(x)\, y'(x) + q(x)\, y(x) = 0 \text{ plus BC on } [a,b]$$

case 3 constant is positive

There will be no nonzero solutions for y. Forget this case

All this is hard to prove.

Bessel's equation of order 0

Bessel's equation of order zero is

$$xy'' + y' + xy = 0$$

A general solution to the equation is

$$y = AJ_0(x) + BY_0(x) \qquad \text{(this is on the reference page)}$$

$J_0(x)$ is called Bessel's function of order 0 of the first kind and $Y_0(x)$ is a Bessel function of order 0 of the second kind. Figs 1 and 2 give their graphs.

```
Plot[BesselJ[0,x], {x,-10,30}]
Plot[BesselY[0,x], {x,0,30}]
```

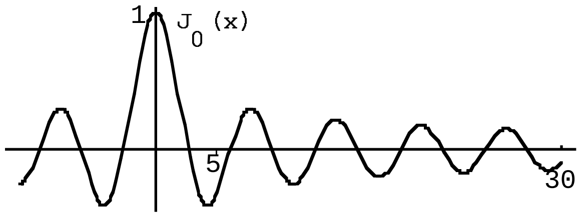


FIG 1 $J_0(x)$

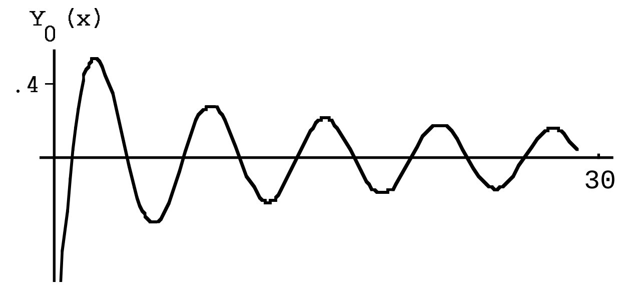


FIG 2 $Y_0(x)$

solution to $rR'' + R' + r\lambda^2 R = 0$ (this is on the reference page)

The general solution to

$$rR'' + R' + r\lambda^2 R = 0$$

is

$$R = AJ_0(\lambda r) + BY_0(\lambda r)$$

proof

I'll write the equation using letters x and y instead of r and R so that it's

(4) $$xy'' + y' + x\lambda^2 y = 0$$

To solve, let $t = \lambda x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(\frac{dy}{dx})}{dt} \quad \frac{dt}{dx} = \lambda \frac{d^2y}{dt^2} \cdot \lambda = \lambda^2 \frac{d^2y}{dt^2}$$

Substitute into (4):

$$\frac{t}{\lambda} \lambda^2 \frac{d^2}{dt^2} + \lambda \frac{dy}{dt} + \frac{t}{\lambda} \lambda^2 y = 0$$

(5) $$ty'' + y'' + ty = 0$$

where y is a function of t. The equation in (5) is Bessel's equation of order 0 and its solution is

$$y(t) = AJ_0(t) + BY_0(t)$$

Replace t by λx to get the solution to (4):

$$y(x) = AJ_0(\lambda x) + BY_0(\lambda x)$$

So, with a change of letters, the sol to $rR'' + R' + r\lambda^2 R = 0$ is $R = AJ_0(\lambda r) + BY_0(\lambda r)$

the 2-dim wave equation in polar coords

The 2-dim wave equation in polar coordinates is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right] \quad \text{(a is a fixed positive constant)}$$

Consider a vibrating circular membrane with small displacements $u(r,\theta,t)$ at point (r,θ) at time t. If you ignore gravity and the retarding force of the medium, and a is a constant determined by the nature of the membrane, then it can be shown that the height $u(r,\theta,t)$ satisfies the 2-dim wave equation (so do lots of other things).

In particular suppose the height is independent of θ and depends only on r and t (Fig 3). (For comparison, Fig 4 shows height *not* independent of θ). Then derivatives w.r.t. θ are 0 and the equation becomes

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right]$$

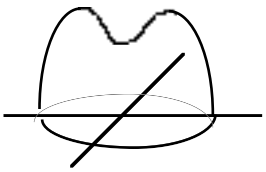


FIG 3
height is independent of θ

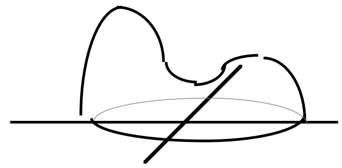


FIG 4
height is *not* independent of θ

the 2-dim heat equation in polar coords

The 2-dim heat equation in polar coordinates is

$$\frac{\partial u}{\partial t} = k \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right] \quad \text{(k is a fixed positive constant)}$$

Consider a disk with net temperature $u(r,\theta,t)$ at position (r,θ) at time t. If the face of the disk is insulated (the bounding edge may or may not be insulated) and k is a constant determined by the composition of the disk then it can be shown that $u(r,\theta,t)$ satisfies the heat equation (so do lots of other things).

In particular if the temperature is independent of θ and depends only on r and t then derivatives w.r.t. θ are 0 and the equation becomes

$$\frac{\partial u}{\partial t} = k \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right]$$

example 1

I'll solve the heat equation

$$\frac{\partial u}{\partial t} = k \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right]$$

for a disk with radius 8 centered at the origin (Fig 5) with

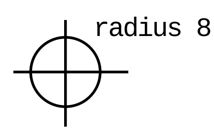


FIG 5

BC $u(8,t) = 0$ (the rim of the disk is kept at temperature 0)

IC $u(r,0) = f(r)$ for r in [0,8] (the initial temp in the disk is f(r))

solution Part I Separate variables.
Try $u(r,t) = R(t)T(t)$. Then

$$RT' = k\left[R''T + \frac{1}{r}R'\,T\right]$$

$$\frac{T'}{kT} = \frac{R'' + \frac{1}{r}R'}{R} = \text{constant}$$

$$T' - k\,\text{con}\,T = 0$$
(6) $$rR'' + R' - r\,\text{con}\,R = 0$$

The BC separatess to $R(8) = 0$

You need good R solutions to help get the nonhomog IC $u(r,0) = f(r)$.

case 1 The constant is negative and renamed $-\lambda^2$.
The equations become

$$T' + k\lambda^2 T = 0, \qquad rR'' + R' + r\lambda^2 R = 0$$

So

$$T = Ce^{-k\lambda^2 t}, \qquad R = A\,J_0(\lambda r) + BY_0(\lambda r)$$

case 2 The constant is 0.
Then $R = A\,\ell n\,r + B$ ((§ 6.5, p. 3) (this is on the ref page).

The BC $u(8,t) = 0$ separates to $R(8) = 0$

Part II Plug in the separable BC.
case 1 $R = A\,J_0(\lambda r) + BY_0(\lambda r)$

$Y_0(r)$ blows up at $r = 0$ so set $B = 0$ to keep R finite.
$R(8) = 0$ makes

$$AJ_0(8\lambda) = 0.$$

The function $J_0(x)$ is not periodic but it repeatedly crosses the x-axis at points I'll denote by a_1, a_2, a_3,... (Fig 6).

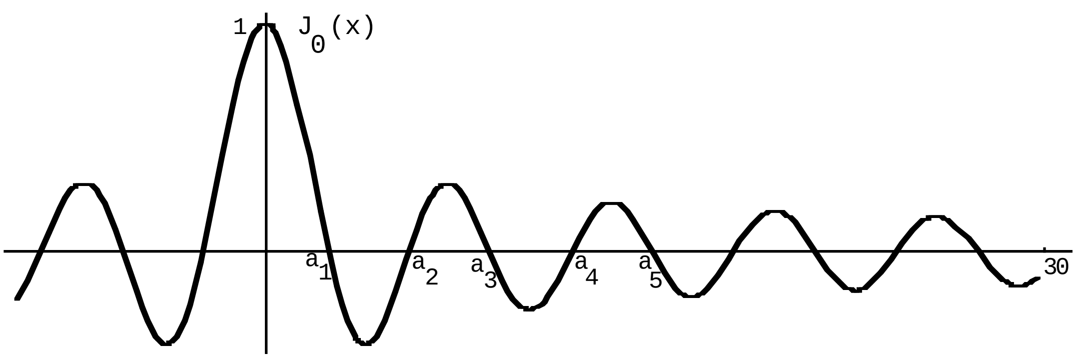


FIG 6 zeros of J_0

Then

$$8\lambda = a_n, \qquad \lambda = \frac{a_n}{8} \quad \text{for } n = 1,2,3,...$$

$$R = AJ_0\left(\frac{a_n r}{8}\right), \quad T = C)\ e^{-k\left(\frac{a_n}{8}\right)^2 t}$$

case 2 $R = A\,\ell n\,r + B$
To keep the solution finite when $r \to 0+$ you need $A = 0$. Then to get $R(8) = 0$ you need $B = 0$. So the only solution in this case is $R=0$. Not useful.

Part III Satisfy the nonhomog IC.
By superposition

(7) $$u = \sum_{n=1}^{\infty} D_n\, e^{-k\left(\frac{a_n}{8}\right)^2 t}\, J_0\left(\frac{a_n r}{8}\right) \quad \text{for } 0 \le r \le 8,\ t \ge 0$$

To satisfy the IC you need

(8) $$f(r) = \sum_{n=1}^{\infty} D_n\, J_0\left(\frac{a_n r}{8}\right) \quad \text{for } r \text{ in } [0,8]$$

The functions $J_0\left(\frac{a_n r}{8}\right)$ are the solutions to the DE

$$rR'' + R' = r \text{ con } R \quad \text{(see (6)) with BC } R(0) \text{ finite, } R(8) = 0$$

This is Sturm Liouville form with $p(r) = r$, $q(r) = 1$, $w(r) = r$.

So the functions $J_0\left(\frac{a_n r}{8}\right)$ are complete and orthogonal w.r.t. $w(r) = r$ on the interval [0,8]. And we can get (8) using the coeff formulas in (2):

(9) $$D_n = \frac{\int_0^8 f(r)\, J_0\left(\frac{a_n r}{8}\right) r\, dr}{\int_0^8 J_0^2\left(\frac{a_n r}{8}\right) r\, dr}$$

warning Don't leave out the weight function r in the integrals.

The final solution is (7), (9) and Fig 6.
(The solution does not include line (8). Line (8) is just part of the work.)

Fourier Bessel series

More generally, if the a_n's are the zeroes of J_0 (Fig 6) then any function f(r) can be written as

$$f(r) = \sum_{n=1}^{\infty} D_n\, J_0\left(\frac{a_n r}{L}\right) \quad \text{for } r \text{ in } [0,L] \quad \text{(Fourier Bessel series)}$$

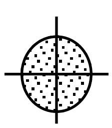
where

(10) $$D_n = \frac{\int_0^L f(r)\, J_0\left(\frac{a_n r}{L}\right) r\, dr}{\int_0^L J_0^2\left(\frac{a_n r}{L}\right) r\, dr}$$

summary continued from §6.5 of what functions blow up and how to keep your solution finite

$Y_0(\lambda r)$ blows up as $r \to 0+$.

Here's how to avoid blowups when you solve the heat equation and wave equation in polar coordinates.



$$R = A\, J_0(\lambda r) + \cancel{BY_0(\lambda r)}$$

example 2

I'll solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right]$$

for a disk with radius L centered at the origin (Fig 7) with

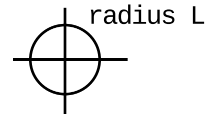


FIG 7

BC $u(L,t) = 0$
IC $u(r,0) = f(r)$ (initial position)
$\frac{\partial u}{\partial t}(r,0) = g(r)$ (initial velocity)

solution Part I Separate variables.
Try $u(r,t) = R(r)T(t)$. Then

$$RT'' = a^2 \left[R''T + \frac{1}{r} R'\, T \right]$$

$$\frac{T''}{a^2\, T} = \frac{R'' + \frac{1}{r} R'}{R} = \text{constant}$$

case 1 con $= -\lambda^2$
Then

$$rR'' + R' + r\lambda^2 R = 0, \quad T'' + a^2 \lambda^2 T = 0,$$

$$R = A\, J_0(\lambda r) + BY_0(\lambda r), \quad T = C \cos \lambda at + D \sin \lambda at$$

case 2 con $= 0$
$R = A \ln r + B$ (this is on the ref page).

The BC $u(L,t) = 0$ separates to $R(L) = 0$.

Part II Satisfy the separated BC.
case 1
To keep R finite as $r \to 0+$, set $B = 0$.
$R(L) = 0$ makes

$$AJ_0(\lambda L) = 0,$$

$$\lambda L = a_n$$

$$\lambda = \frac{a_n}{L}$$ where the a_n's are the zeros of the J_0.

case 2
As in example 1, no nonzero solutions.

Part III Satisfy the nonhomog IC.
By superposition

$$\boxed{u = \sum_{n=1}^{\infty} \left[C_n \cos \frac{a_n at}{L} + D_n \sin \frac{a_n at}{L} \right] J_0\!\left(\frac{a_n r}{L}\right)}$$

To get the first IC you need

$$f(r) = \sum_{n=1}^{\infty} C_n\, J_0\!\left(\frac{a_n r}{L}\right) \quad \text{for } r \text{ in } [0,L],$$

which you can get with

$$\boxed{C_n = \frac{\int_0^L f(r)\, J_0\left(\frac{a_n r}{L}\right) r\, dr}{\int_0^L J_0^2\left(\frac{a_n r}{L}\right) r\, dr}}$$

To get the second IC, first find

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[-C_n \frac{a_n a}{L} \sin \frac{a_n at}{L} + D_n \frac{a_n a}{L} \cos \frac{a_n at}{L} \right] J_0\left(\frac{a_n r}{L}\right)$$

To have $\frac{\partial u}{\partial t} = g(r)$ when $t = 0$ you need

$$g(r) = \sum_{n=1}^{\infty} D_n \frac{a_n a}{L} J_0\left(\frac{a_n r}{L}\right) \quad \text{for } r \text{ in } [0,L],$$

which you can get with

$$D_n \frac{a_n a}{L} = \frac{\int_0^L g(r)\, J_0\left(\frac{a_n r}{L}\right) r\, dr}{\int_0^L J_0^2\left(\frac{a_n r}{L}\right) r\, dr}$$

$$\boxed{D_n = \frac{L}{a_n a} \frac{\int_0^L g(r)\, J_0\left(\frac{a_n r}{L}\right) r\, dr}{\int_0^L J_0^2\left(\frac{a_n r}{L}\right) r\, dr}}$$

The solution is in the three boxes (and Fig 6).

PROBLEMS FOR SECTION 6.8

0. Go back and see if you can do examples 1 and 2 by yourself now without looking at the solution.

1. If the face of a disk is *not* insulated and the temperature u is independent of θ then u satisfies the PDE

$$\frac{\partial u}{\partial t} = k \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] - u$$

The problem here is just to get some separated solutions (there are no BC or IC yet).

Try $u = R(r)T(t)$. Then

$$RT' = k(R''T + \frac{1}{r} R'T) - RT$$

From here on it depends on how you factor.

(a) One possibility is

$$\frac{T' + T}{kT} = \frac{R'' + \frac{1}{r} R'}{R} = \text{constant}$$

Continue from here and get good solutions in the two potentially useful cases.

(b) Another possibility is

$$\frac{T'}{T} = \frac{kR'' + \frac{k}{r}R' - R}{R} = \text{con}$$

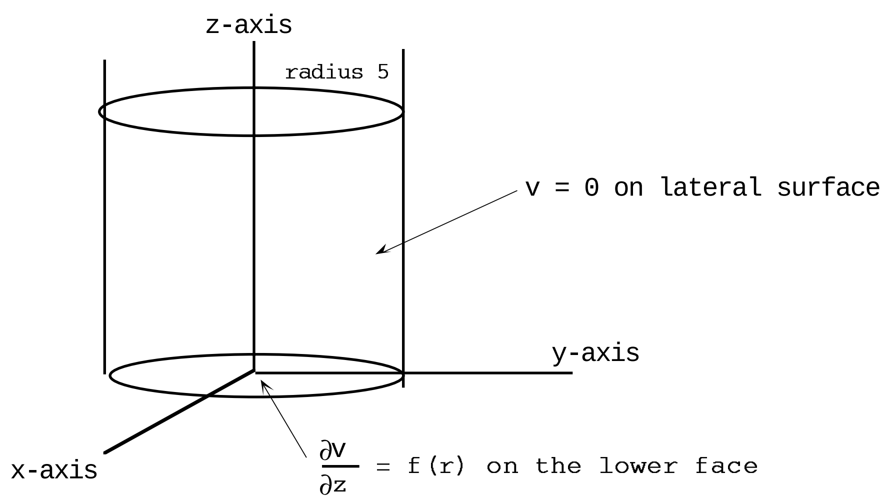
Continue from here and get good solutions in the two potentially useful cases.

2. Continue from the preceding problem and solve the PDE for a disk centered at the origin with radius L and with BC $u(L,t) = 0$ and IC $u(r,0) = f(r)$ for r in $[0,L]$

3. Here's Laplace's equation in cylindrical coordinates but independent of θ:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = 0$$

Solve the equation for the region inside the infinitely long cylinder (it has a bottom but no top) of radius 5 with the BC in the diagram.



4. Sketch a rough graph of $J_0\left(\frac{a_1 x}{100}\right)$ and compare it with $J_0(x)$.

REVIEW PROBLEMS FOR CHAPTER 6

1. Solve

$$\frac{\partial u}{\partial t} + u = k\,\frac{\partial^2 u}{\partial x^2}$$

(the heat equation for a rod with non-insulated lateral surface) with

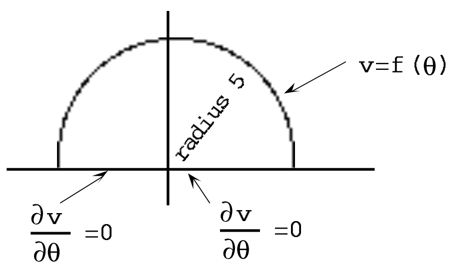
BC $\quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(4,t) = 0$ for all t

IC $\quad u(x,0) = f(x) = \begin{cases} 3 & \text{if } 0 \le x \le 2 \\ 7 & \text{if } 2 \le x \le 4 \end{cases}$

2.(a) Solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \quad \text{(Laplace's equation in polar coords)}$$

for the semi–disk with radius 5 in the diagram.



(b) Continue from part (a) and specifically use $f(\theta) = 3 + 6\cos 2\theta$.

3. Separate the following PDE into a t equation and an x equation and then stop (don't try to solve the equations):

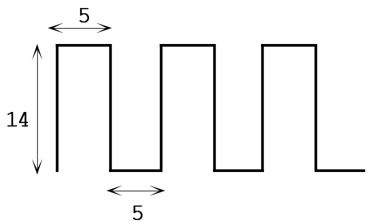
$$\frac{\partial^2 y}{\partial t^2} = gx\,\frac{\partial^2 y}{\partial x^2} + g\,\frac{\partial y}{\partial x} \quad \text{(g is a fixed positive constant)}$$

4. Separate to get solutions, useful cases only, assuming there will be a nonhomog IC to satisfy.

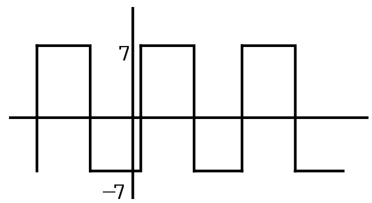
$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

Watch for the subcases within the main case.

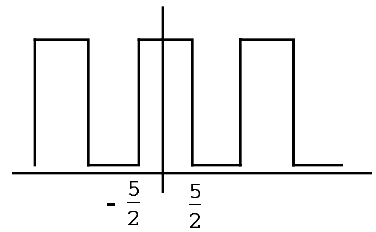
5. The first diagram shows a waveform. The problem is to find the fundamental frequency, the first overtone frequency and their amplitudes.
Try it three times for practice, with the axes inserted in the ways indicated in (a)–(c)



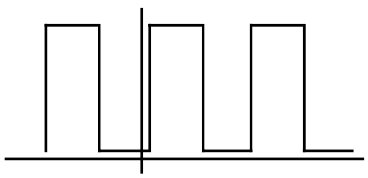
Problem 5



Problem 5(a)



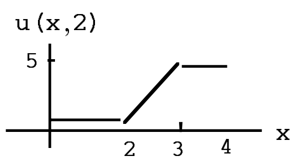
Problem 5(b)


Problem 5(c)

6. Pretend that the general solution to a PDE plus homog BC is

$$u(x,y) = A_0\, y^3 + \sum_{n=1}^{\infty} A_n\, e^{-ny} \cos \frac{n\pi x}{4} \quad \text{for } 0 \le x \le 4,\ y \ge 2$$

Find the solution (but leave the integrals unevaluated) so that u(x,2) is the function in the diagram.



(b) How would things be (terribly) different if in part (a) the general solution was

$$u(x,y) = A_0\, y^3 + \sum_{n=1}^{\infty} A_n\, e^{-ny} \cos \frac{n\pi x}{\boxed{3}} \quad \text{for } 0 \le x \le 4,\ y \ge 2$$

SOLUTIONS Section 1.1

1. (a) DE is $y'' + y' = -7x$ linear, nonhomog (because of the $-7x$)

(b) linear, homog
(c) non-linear because of the yy' term
(d) $x^2y''' + (\sin x)y = \cos x$ linear, nonhomog (because of the $\cos x$)
(e) $x^2y''' - (\sin x)y = 0$ Linear, homog
(f) nonlinear because of the $y \sin y$ term

2. (a) $3y'' + 2y' + xy = 2 \cos x$

(b) $3y'' + 2y' + xy = 3 \cos x$

(c) $3y'' + 2y' + xy = 0$

3. All are solutions to $3y'' + 2y' + 6y = 0$.

4. (a) sol is $3y_1(x)$ by superposition
(b) This is same as the given equation. Solution is y_1.
(c) equ is $ay'' + by' + cy = \frac{1}{3} x^2$. Sol is $\frac{1}{3} y_1$

5. If $y = x^3$ then $y' = 3x^2$ and yy' does equal $3x^5$.
If $y = e^{2x}$ then $y' = 2e^{2x}$ and yy' does equal $2e^{4x}$.

But if $y = x^3 + e^{2x}$ then $y' = 3x^2 + 2e^x$ and

$$yy' = (x^3 + e^{2x})(3x^2 + 2e^x) \text{ which is NOT } 3x^5 + 2e^{4x}$$

Superposition doesn't hold but it isn't a contradiction because the DE $yy' = f(x)$ isn't linear to begin with. A linear DE can't contain a yy' term.

SOLUTIONS Section 1.2

1.(a) $m^2 + 2m - 3 = 0$, $m = -3,1$, general $y = Ae^{-3x} + Be^{x}$

(b) $m^2 + 2m - 4 = 0$, $m = -1 \pm \sqrt{5}$, gen $y = Ae^{(-1+\sqrt{5})x} + Be^{(-1-\sqrt{5})x}$

(c) $4m^2 - 25 = 0$, $m = \pm\ 5/2$, gen $y = Ae^{5x/2} + Be^{-5x/2}$

(d) $m = 0,-2$, gen $y = A + Be^{-2x}$

2. $m = 3,-1$, gen $y = Ae^{3x} + Be^{-x}$.

Make $y(0) = 0$: $0 = A + B$

$y' = 3Ae^{3x} - Be^{-x}$

Make $y'(0) = 4$: $-4 = 3A - B$.

$A = -1$, $B = 1$. Answer is $y = -e^{3x} + e^{-x}$.

Honors

3.(a) Eccentric but OK.

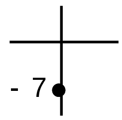
If A is an arbitrary constant then 17A is another arbitrary constant, say C. And If B is an arbitrary constant then $-\pi B$ is another arbitrary constant, say D. So her answer is really of the form $Ce^{3x} + De^{5x}$, which is the same as $Ae^{3x} + Be^{5x}$.

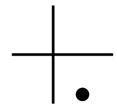
(b) She will get $A = -3/17$, $B = 16/\pi$ and our solutions will agree. We both get $y = -3e^{3x} - 16e^{5x}$.

SOLUTIONS Section 1.3

1.(a) $34 + 42i$ (b) $\frac{1}{8+3i} \cdot \frac{8-3i}{8-3i} = \frac{8}{73} - \frac{3}{73}\, i$ (c) $\frac{2+9i}{4-i} \cdot \frac{4+i}{4+i} = \frac{-1 + 38i}{17}$

2.(a) $r = 7$, $\theta = 3\pi/2$ (b) $r = 8$, $\tan\theta = -\sqrt{3}$, $\theta = -60^\circ$

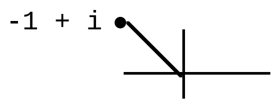




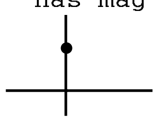
(c) $r = 5$, $\tan\theta = -\frac{3}{4}$, $\theta \approx 143^\circ$ (d) $r = 7$, $\theta = \pi$ (e) $r = 10\sqrt{2}$, $\theta = -\frac{\pi}{4}$

3. (a) $\pi/4$ because the angle between $-\pi/2$ and $\pi/2$ whose tangent is 1 is $\pi/4$
(b) $-3\pi/4$ because that's the angle of $-2-2i$ (using values between $-\pi$ and π)

4. (a) $-1 + i$ has mag $\sqrt{2}$ and (by inspection) angle $3\pi/4$



So $(-1+i)^6$ has mag $(\sqrt{2})^6 = 8$, angle $6 \cdot 3\pi/4 = 9\pi/2$. So it also has angle $\pi/2$.

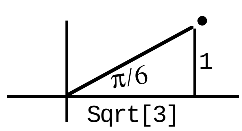


So, by inspection, $(-1 - i)^6 = 8i$
(b) $-1 + i$ has mag $\sqrt{2}$ and angle $3\pi/4$
So $(-1+i)^7$ has mag $(\sqrt{2})^7 = 8\sqrt{2}$, angle $7 \cdot 3\pi/4 = 21\pi/4$

so $(-1 + i)^7 = 8\sqrt{2}\ (\cos\frac{21\pi}{4} + i \sin\frac{121\pi}{4}) = 8\sqrt{2}\ (\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4})$

$= 8\sqrt{2}\ (-\frac{1}{2}\sqrt{2} + \frac{1}{2} i \sqrt{2}) = -8 + 8i$

(c) $\sqrt{3} + i$ has mag 2 and angle $\pi/6$



so $(\sqrt{3} + i)^3$ has mag 8 and angle $\pi/2$
So $(\sqrt{3} + i)^3 = 8i$

5. (a) $\sqrt{3} + i$ has mag 2 and angle $\pi/6$ so $(\sqrt{3} + i)^9$ has mag 2^9 and angle $9\pi/6 = 3\pi/2$.
So $(\sqrt{3} + i)^9 = -512i$.
Real part is 0, imag part is -512 (not $-512i$, just plain -512).
(b) $-1+i$ has mag $\sqrt{2}$ and angle $3\pi/4$.

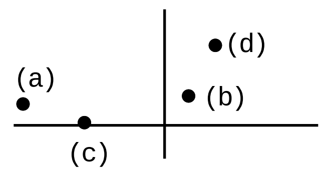
$(-1+i)^5$ has mag $(\sqrt{2})^5$ and angle $15\pi/4$. Might as well call the angle $-\pi/4$.
8 has mag 8 and angle 0.

So $\dfrac{8}{(-1+i)^5}$ has mag $\dfrac{8}{(\sqrt{2})^5} = \sqrt{2}$ and angle $0 - (-\pi/4) = \pi/4$.

Real part is $\sqrt{2}\cos\frac{\pi}{4} = 1$, imag part is $\sqrt{2}\sin\frac{\pi}{4} = 1$.

6.(a) $1 - i$ has maag $\sqrt{2}$ and angle $-\pi/4$ so $1-i = \sqrt{2}\, e^{-i\pi/4}$
(b) e^{3+5i} (c) $2e^{-\pi i/7}$

7. (a) mag is e^6, angle is 3 (radians)
(b) mag is 1, angle is $\pi/3$
(c) mag is e, angle is $-\pi$
(d) mag is 3, angle is 60°



8.(a) mag is e^2, angle is -3π
(b) mag is 2, angle is $\pi/4$
(c) 5i has mag 5 and angle $\pi/2$
e^{6ix} has mag 1 and angle 6x
So the product has mag $1\cdot 5 = 5$ and angle $6x + \frac{1}{2}\pi$

warning

This is different from $5e^{6ix}$ (no extra i in front) which has mag 5 and angle 6ix

9. (a) $e^{7ix} = \cos 7x + i\sin 7x$. Real part is $\cos 7x$, Imag part is $\sin 7x$

(b) $5e^{(2-3i)x} = 5e^{2x}(\cos 3x - i\sin 3x)$, Re $= 5e^{2x}\cos 3x$, Im $= -5e^{2x}\sin 3x$

(c) $(2+3i)\, e^{5ix} = (2+3i)(\cos 5x + i\sin 5x)$

Re $= 2\cos 5x - 3\sin 5x$, Im $= 3\cos 5x + 2\sin 5x$

(d) $(2+4i)e^{(1-2i)x} = (2+4i)\, e^x(\cos 2x - i\sin 2x)$
$= e^x(2\cos 2x + 4\sin 2x) + i\, e^x(4\cos 2x - 2\sin 2x)$

Re $= e^x(2\cos 2x + 4\sin 2x)$, Im $= e^x(4\cos 2x - 2\sin 2x)$

(e) $e^{3ix} + e^{-3ix} = \cos 3x + i\sin 3x + \cos 3x - i\sin 3x = 2\cos 3x$.
Re part is $2\cos 3x$, Im part is 0.

(f) $\dfrac{2}{7-4i} = \dfrac{2}{7-4i}\cdot\dfrac{7+4i}{7+4i} = \dfrac{14}{65} + \dfrac{8}{65}i$

$e^{4ix} = \cos 4x + i\sin 4x$

$\dfrac{2}{7-4i}e^{4ix}$ has real part $\frac{14}{65}\cos 4x - \frac{8}{65}\sin 4x$ and imag part $\frac{14}{65}\sin 4x + \frac{8}{65}\cos 4x$.

10. (a) $4i\cdot ie^{4ix} = -4e^{4ix}$

(b) $(6+12i)e^{(2+4i)x}$

(c) $12ie^{4ix} + 30ie^{6ix}$

(d) (product rule) $\pi ixe^{\pi ix} + e^{\pi ix}$

(e) $(2-i)(3+4i)e^{(3+4i)x} = (10+5i)e^{(3+4i)x}$

(f) (product rule) $3ix^3e^{3ix} + 3x^2e^{3ix}$

(g) $i(2-3i)e^{(2-3i)x} = (3+2i)e^{(2-3i)x}$

11. $\dfrac{d(e^{2ix})}{dx} = 2ie^{2ix}$, $\dfrac{d^2(e^{2ix})}{dx^2} = -4e^{2ix}$

12. Let $A = a + bi$. Then $B = a - bi$ and

$$Ae^{i\theta} + Be^{-i\theta} = (a+bi)(\cos\theta + i\sin\theta) + (a-bi)(\cos\theta - i\sin\theta)$$

$$= 2a\cos\theta - 2b\sin\theta$$

The i's cancelled out. The result is real.

Honors

13. If z_1 has real part x_1 and imag part y_1 and z_2 has real part x_2 and imag part y_2 then

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

so

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \text{ so}$$

z_1z_2 has real part $\boxed{x_1x_2 - y_1y_2}$ and imag part $\boxed{x_1y_2 + x_2y_1}$.

(Not pretty)

SOLUTIONS Section 1.4

1.(a) The forcing function is $4(\cos 3x + i \sin 3x)$
The real part of the forcing function is $4 \cos 3x$.
(The imag part which I'll use in part (b) is $4 \sin 3x$)

Rewrite the solution to be able to identify its real part.

$$\frac{4+2i}{i} = \frac{4+2i}{i}\,\frac{i}{i} = 2-4i$$

The given solution is $(2-4i)(\cos 3x + i \sin 3x)$.
The real part of the solution is $2 \cos 3x + 4 \sin 3x$

(The imag part, which I'll use in part (b) is $-4 \cos 3x + 2 \sin 3x$)

By the complex superposition principle in this section the real part of the solution goes with the real part of the forcing function.

So $y = 2 \cos 3x + 4 \sin 3x$ is a solution to $ay'' + by' + cy = 4 \cos 3x$

By the ordinary superposition principle in Section 1.1,

$7/4 * (2 \cos 3x + 4 \sin 3x)$

is a solution to

$ay'' + by' + cy = (7/4)\ 4 \cos 3x$

i.e., to $ay'' + by' + cy = 7 \cos 3x$

So the answer is $y = 7/4 * (2 \cos 3x + 4 \sin 3x)$

(b) By the complex superposition principle in this section, the imag part of the solution goes with the imag part of the forcing function

So
$$y = -4 \cos 3x + 2 \sin 3x$$

is a solution to

$$ay'' + by' + cy = 4 \sin 3x$$

And by the ordinary superposition principle from Section 1.1,

$$y = 2*(-4 \cos 3x + 2 \sin 3x)$$

is a solution to

$$ay'' + by' + cy = 8 \sin 3x$$

And by more superposition

$$y = 7/4 * (2 \cos 3x + 4 \sin 3x) + 2*(-4 \cos 3x + 2 \sin 3x)$$

is a solution to

$$ay'' + by' + cy = 7 \cos 3x + 8 \sin 3x$$

Answer simplifies to $y = -\frac{9}{2} \cos 3x + 11 \sin 3x$

SOLUTIONS Section 1.5

1. (a) $e^{-3x}(A \cos 5x + B \sin 5x)$

   (b) $A \cos 2x + B \sin 2x$

   (c) $e^{3x} (A \cos 4x + B \sin 4x)$

2. (a) $m = \pm \pi i$, $y = A \cos \pi x + B \sin \pi x$
   (b) $m = \pm\pi$, $y = Ae^{\pi x} + Be^{-\pi x}$
   (c) $m = -1 \pm i\sqrt{3}$, $y = e^{-x}(A \cos\sqrt{3}\, x + B \sin\sqrt{3}\, x)$
3. (a) $m = -2 \pm i$, $y = e^{-2x}(A \cos x + B \sin x)$

   (b) $m = \pm 2i$, $y = A \cos 2x + B \sin 2x$

4. $m = \pm ki$, $y = A \cos kx + B \sin kx$

5. gen $y = Ae^{-x} + B \cos 2x + C \sin 2x$ Then
   $y' = -Ae^{-x} - 2B \sin 2x + 2C \cos 2x$, $y'' = Ae^{-x} - 4B \cos 2x - 4C \sin 2x$

To get the IC you need $0 = A + B$, $-1 = -A + 2C$, $5 = A - 4B$
Then $A = 1$, $B = -1$, $C = 0$ Answer is $y = e^{-x} - \cos 2x$

6. (a) $C_1 e^{-3x} + C_2 xe^{-3x} + C_3 x^2 e^{-3x} + C_4 e^{5x} + C_5 e^{-5x} + C_6 \cos 4x + C_7 \sin 4x$

$$+ e^{-2x}(C_8 \cos 3x + C_9 \sin 3x) + xe^{-2x}(C_{10} \cos 3x + C_{11} \sin 3x)$$

$$+ x^2e^{-2x}(C_{12} \cos 3x + C_{13} \sin 3x) + C_{14}$$

(b) $y_{gen} = A + Bx + Cx^2 + De^{3x}$

(c) $y_{gen} = Ae^{(2+\sqrt{5})x} + Be^{(2-\sqrt{5})x} + C \cos x + D \sin x + x(E \cos x + F \sin x)$

(d) $y_{gen} = e^{2x}(A \cos \sqrt{5}\, x + B \sin \sqrt{5}\, x) + C + De^{3x}$

(e) $y_{gen} = A \cos x + B \sin x + C \cos 2x + D \sin 2x + Ee^{x}$

7. $m = 0,0,-3$, $y = A + Bx + Ce^{-3x}$ Then $y' = B - 3Ce^{-3x}$
Plug in IC: $0 = A + C$, $2 = B - 3C$
Plug in $y(\infty) = 1$: $1 = B - 0$
So $B = 1$, $C = -1/3$, $A = 1/3$ answer is $y = \frac{1}{3} + x - \frac{1}{3} e^{-3x}$

8. (a) $m = 2,3$, $(m-2)(m-3) = 0$, $m^2 - 5m + 6 = 0$, DE is $y'' - 5y' + 6y = 0$

(b) $m = 0,0,1$, $m^2(m-1) = 0$, $y''' - y'' - 0$
(c) $m = 2,2$, $(m-2)^2 = 0$, $y'' - 4y' + 4y = 0$
(d) $m = 2 \pm 3i$, $(m - [2+3i])(m - [2-3i]) = 0$, $m^2 - 4m + 13 = 0$,

$y'' - 4y' + 13y = 0$

9. $m = -2$, $y = Ae^{-2x}$

10. (a) Think of a spring system. The system is initially disturbed (at time 0 its initial displacement is y_0 and its initial velocity is y_1) but there is no input as time goes on (because the forcing function is 0). So as time goes on, the effect of the IC should wear off and the spring (which is damped since $b > 0$) should move back toward its undisplaced (equilibrium) position. In other words, as $x \to \infty$, we should have $y(x) \to 0$.

(b) $$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The three possibilities are

$b^2 - 4ac < 0$	non-real m's
$b^2 - 4ac > 0$	real unequal m's
$b^2 - 4ac = 0$	real equal m's

The problem said to just consider the first case. Then

$$m = \frac{-b \pm i\sqrt{4ac-b^2}}{2a}.$$

To simplify notation, let $m = p \pm qi$ but note that p is *negative* because it equals $-b/2a$ where a and b are positive. Then the general solution is

$$y = e^{px}(A \cos qx + B \sin qx)$$

Let $x \to \infty$. Then

$e^{px} = e^{-\infty} = 0$ (actually 0+)
cos qx oscillates between −1 and 1.
$e^{px} \cos qx$ = (0+) times (oscillates between −1 and 1) = 0
(in particular, $e^{px} \cos qx$ oscillates above and below 0 with decreasing swing).
Similarly for $e^{px} \sin qx$.

So the whole y solution $\to 0$ as $x \to \infty$.

11.(a) The physical system is initially at rest (because the IC are 0). And no input ever comes in (because the forcing function is 0). So the system should never produces any response; i.e., the solution should be $y = 0$.
(b) Suppose the roots of the characteristic equ are real and unequal, say $m = m_1, m_2$.

Then $y_h = Ae^{m_1 x} + Be^{m_2 x}$

$y(0) = 0$ makes $A + B = 0$.
$y'(x) = m_1 Ae^{m_1 x} + m_2 Be^{m_2 x}$.

$y'(0) = 0$ makes $m_1 A + m_2 B = 0$.
Then $m_1 A + m_2(-A) = 0$, $(m_1 - m_2)A = 0$, $m_1 - m_2 = 0$ or $A = 0$. But $m_1 - m_2$ can't be 0 since this is the case where m_1 and m_2 are different. So $A = 0$. And since $B = -A$, B must be 0 also.
So the solution is $y = 0$.

Honors

12. Look at $ay'' + by' + cy = 0$ where $a,b,c > 0$.

The equation $am^2 + bm + c = 0$ has solution $m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The roots are repeated iff $b^2 - 4ac = 0$ in which case the repeated root is $m = -\frac{b}{2a}$.

Then $y = Ae^{-\frac{b}{2a}x} + Bxe^{-\frac{b}{2a}x}$.

(*) a and b are positive so $-b/2a$ is negative.

As $x \to \infty$, $e^{-\frac{b}{2a}x} = e^{-\infty} = 0$.

As $x \to \infty$, $xe^{-\frac{b}{2a}x} = \infty \times 0$ which is indeterminate.

Rewrite as $\dfrac{x}{e^{\frac{b}{2a}x}}$ which is $\frac{\infty}{\infty}$, still indeterminate.

Maybe you learned in calculus that if $m > 0$ then e^{mx} grows faster than x (has a higher order of magnitude) so that the limit here is 0.

Or you can use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{x}{e^{\frac{b}{2a}x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{\frac{b}{2a}\, e^{\frac{b}{2a}x}} \text{ (use L'Hopital)} = \frac{1}{\infty} = 0.$$

So no matter what A and B are (to be determined by IC), as $x \to \infty$,
$Ae^{-\frac{b}{2a}x} + Bxe^{-\frac{b}{2a}x} = A\cdot 0 + B\cdot 0 = 0$ so the sol is transient.

I used the hypothesis that a and b are positive in line (*). I didn't need the hypothesis that c is positive.

subtle point

The theorem really is this:

If $a,b > 0$ and the m's are repeated then the solution to $ay'' + by' + cy = 0$ is transient. Furthermore c will be positive also (because in the repeated roots case, once a and b are positive, c has to be positive also to make b^2-4ac equal 0).

So $c > 0$ is something you can prove from the other hypotheses, not something you need as part of the hypothesis.

14. (a) $y' = \dfrac{xv' - v}{x^2}$

$$y'' = \frac{x^2v'' - 2xv' + 2v}{x^3}.$$

(b) Replace y' by $\dfrac{xv' - v}{x^2}$, replace y'' by $\dfrac{x^2v'' - 2xv' + 2v}{x^3}$, and replace y by v/x to get the new DE

$$\frac{x^2v'' - 2xv' + 2v}{x^2} + 2\,\frac{xv' - v}{x^2} + 9v = 0$$

$v'' + 9v = 0$

Then $m = \pm 3i$

$v = A\cos 3x + B\sin 3x$

and

$$y = \frac{v(x)}{x} = \frac{A\cos 3x + B\sin 3x}{x}$$

15. *case 2* The roots are real and equal.
Call the root m. Then

$y_h = Ae^{mx} + Bxe^{mx}$

$y(0) = 0$ so $A = 0$
Then $y'(x) = B(mxe^{mx} + e^{mx})$

$y'(0) = 0$ so $B(0 + 1) = 0$, $B = 0$

So $y = 0$ QED

case 3 The roots are not real.
$m = a \pm bi$, $y_h = e^{ax}(A\cos bx + B\sin bx)$

$y(0) = 0$ makes $A = 0$.
Then $y'(x) = B(e^{ax}\, b\cos bx + ae^{ax}\sin bx)$

$y'(0) = 0$ makes $Bb = 0$. So $B = 0$ or $b = 0$. But b can't be 0 since $a \pm bi$ is a non-real number. So $B = 0$.
So $y = 0$.

SOLUTIONS Section 1.6

1.(a) $m = \frac{-3 \pm \sqrt{29}}{2}$, $y_h = Ae^{(-3+\sqrt{29})x/2} + Be^{(-3-\sqrt{29})x/2}$

Try $y_p = Ae^{2x}$. Substitute into the DE.

$4Ae^{2x} + 3\cdot 2Ae^{2x} - 5\cdot Ae^{2x} = 4Ae^{2x}$, $5Ae^{2x} = 4e^{2x}$, $5A = 4$, $A = \frac{4}{5}$

$y_{gen} = Ae^{(-3+\sqrt{29})x/2} + Be^{(-3-\sqrt{29})x/2} + \frac{4}{5}e^{2x}$

(b) $m = -2$, $y_h = Ae^{-2x}$. Try $y_p = Ae^{3x}$.

Need $3Ae^{3x} + 2Ae^{3x} = e^{3x}$, $5A = 1$, $A = \frac{1}{5}$

Answer is $y_{gen} = Ae^{-2x} + \frac{1}{5}e^{3x}$

2. (a) $m = \pm 3i$, $y_h = P\cos 3x + Q\sin 3x$. Try $y_p = Ax^2 + Bx + C$

Need $2A + 9\cdot(Ax^2 + Bx + C) = -162x^2$

Equate x^2 coeffs $9A = -162$, $A = -18$

Equate x coeffs $9B = 0$, $B = 0$

Equate constant terms $2A + 9C = 0$, $C = 4$

Answer is $y_{gen} = P\cos 3x + Q\sin 3x - 18x^2 + 4$

(b) $m = \pm 2$, $y_h = Ae^{2x} + Be^{-2x}$. Try $y_p = C$. Substitute into the DE.

Need $-4C = 2$, $C = -\frac{1}{2}$. Gen sol is $y = Ae^{2x} + Be^{-2x} - \frac{1}{2}$

3.(a) $y_h = Ae^{-x} + Be^{-2x}$. Try $y_p = Cx + D$. Need

$$3\cdot C + 2(Cx + D) = 2 - 4x$$

Match x coeffs $2C = -4$, $C = -2$

Match constant terms $3C + 2D = 2$, $D = 4$

A general sol is $y = Ae^{-x} + Be^{-2x} - 2x + 4$. Then $y' = -Ae^{-x} - 2Be^{-2x} - 2$

To get the IC we need $0 = A + B + 4$, $0 = -A - 2B - 2$. So $A = -6$, $B = 2$

Answer is $y = -6e^{-x} + 2e^{-2x} - 2x + 4$

(b) $m = \pm i$, $y_h = A\cos x + B\sin x$. Try $y_p = C$. Then $y_p'' = 0$; need $0 + C = 1$, $C = 1$.

Gen sol is $y = A\cos x + B\sin x + 1$.

$y' = -A\sin x + B\cos x$ so to get the IC we need $0 = 1 + A$, $2 = B$.

Answer is $y = 1 - \cos x + 2\sin x$

4. (a) $m = \frac{-1 \pm i\sqrt{3}}{2}$, $y_h = e^{-x/2}\,(A \cos \frac{1}{2}\sqrt{3}\,x + B \sin \frac{1}{2}\sqrt{3}\,x)$

To get y_p switch to $y'' + y' + y = 73e^{3ix}$. Try $y_p = De^{3ix}$. Need

$-9De^{3ix} + 3iDe^{3ix} + De^{3ix} = 73\,e^{3ix}$, $D = \frac{73}{-8+3i} = -8 - 3i$

Switched $y_p = (-8-3i)e^{3ix} = (-8-3i)(\cos 3x + i \sin 3x)$.

Take imag part to get original $y_p = -8 \sin 3x - 3 \cos 3x$.

Gen sol is $y = e^{-x/2}\,(A \cos \frac{1}{2}\sqrt{3}\,x + B \sin \frac{1}{2}\sqrt{3}\,x) - 8 \sin 3x - 3 \cos 3x$

(b) Like part (a) but take the real part of the switched y_p

answer is $y = e^{-x/2}\,(A \cos \frac{1}{2}\sqrt{3}\,x + B \sin \frac{1}{2}\sqrt{3}\,x) - 8 \cos 3x + 3 \sin 3x$

5. $m = -2 \pm i$, $y_h = e^{-2x}(A \cos x + B \sin x)$.
To get y_p, first find a particular sol to $y'' + 4y' + 5y = 8e^{ix}$ by trying $y_p = Ce^{ix}$.
Substitute into the new DE:

$-Ce^{ix} + 4Ce^{ix} + 5Ce^{ix} = 8e^{ix}$, $(4+4i)C = 8$, $C = \frac{8}{4+4i} = 1 - i$
switched $y_p = (1-i)e^{ix} = (1-i)(\cos x + i \sin x)$
Original y_p = imag part = $-\cos x + \sin x$

Gen sol is $y = e^{-2x}(A \cos x + B \sin x) - \cos x + \sin x$.

$y' = e^{-2x}(-A \sin x + B \cos x) - 2e^{-2x}(A \cos x + B \sin x) + \sin x + \cos x$

To get the IC we need $0 = A - 1$, $0 = B - 2A + 1$, so $A = 1$, $B = 1$

Answer is $y = e^{-2x}(\cos x + \sin x) - \cos x + \sin x$

Steady state solution is $-\cos x + \sin x$, which can also be written as
$\sqrt{2}\cos(x - \theta)$ where $\theta = \arctan[-1,1] = 3\pi/4$, *not* $\arctan \frac{1}{-1}$ which is $-\pi/4$.

6.(a) The answer $y = \cos 2x + 6 \sin 2x + x^2 - 5$ has $y(0) = 1 - 5 = -4$.

We have $y' = -2 \sin 2x + 12 \cos 2x + 2x$ so $y'(0) = 12$.

So the IC are $y(0) = -4$, $y'(0) = 12$
(b) Go backwards: $m = 2i$, $m^2 + 4 = 0$, DE is $y'' + 4y = f(x)$. The particular sol
$x^2 - 5x$ must satisfy the DE so plug it in to get
$2 + 4(x^2 - 5) = f(x)$, $f(x) = 4x^2 - 18$
So DE is $y'' + 4y = 4x^2 - 18$.

7. When you optimistically substitute into the DE you get $-3A + 2Ax = 2x$.
To make the x coeffs match you need $2A = 2$, $A = 1$.
To make the constant terms match you need $-3A = 0$, $A = 0$.
Impossible. So there is no particular solution of the form Ax.

8. $(5+3i)e^{2ix} = (5+3i)(\cos 2x + i \sin 2x)$

$e^{3ix} = \cos 3x + i \sin 3x$

So you are given that

(*) $\qquad (5+3i)(\cos 2x + i \sin 2x)$

is a solution to

$$ay'' + by' + cy = \cos 2x + i \sin 2x.$$

Now use a lot of superposition.

$5 \cos 2x - 3 \sin 2x$ (the real part of (*)) is a solution to $ay'' + by' + cy = \cos 2x$.

$3 \cos 2x + 5 \sin 2x$ (the imag part) is a solution to $ay'' + by' + cy = \sin 2x$.

$5(5 \cos 2x - 3 \sin 2x)$ is a solution to $ay'' + by' + cy = 5 \cos 2x$.

$7(3 \cos 2x + 5 \sin 2x)$ is a solution to $ay'' + by' + cy = 7 \sin 2x$.

$5(5 \cos 2x - 3 \sin 2x) + 7(3 \cos 2x + 5 \sin 2x)$ is a sol to $ay'' + by' + cy = 5 \cos 2x + 7 \sin 2x$.

So the answer is $46 \cos 2x + 20 \sin 2x$.

9.(a) $m = \pm i$, $y_h = A \cos x + B \sin x$

For $0 \le x \le \pi$ try $y_p = Ax + B$ and get $A = 1$, $B = 0$

For $x \ge \pi$ try $y_p = Ae^{\pi - x}$ and get $A = \pi/2$ So

$$y_{gen} = \begin{cases} A \cos x + B \sin x + x & \text{if } 0 \le x \le \pi \\ C \cos x + D \sin x + \frac{1}{2}\pi e^{\pi - x} & \text{if } x \ge \pi \end{cases}$$

The IC make $A = 0$, $B = 0$ so

$$y = \begin{cases} x & \text{if } 0 \le x \le \pi \\ C \cos x + D \sin x + \frac{1}{2}\pi e^{\pi - x} & \text{if } x \ge \pi \end{cases}$$

Make the y pieces agree when $x = \pi$: $\quad \pi = C \cos \pi + D \sin \pi + \frac{1}{2}\pi e^{\pi - x}$, $\quad C = -\frac{1}{2}\pi$.

Then

$$y' = \begin{cases} 1 & \text{if } 0 \le x \le \pi \\ \frac{1}{2}\pi \sin x + D \cos x - \frac{1}{2}\pi e^{\pi - x} & \text{if } x \ge \pi \end{cases}$$

Make the y' pieces match at $x = \pi$: $\quad 1 = -D - \frac{1}{2}\pi$, $D = -1 - \frac{1}{2}\pi$

Answer is

$$y = \begin{cases} x & \text{if } 0 \le x \le \pi \\ -\frac{1}{2}\pi \cos x + (-1 - \frac{1}{2}\pi) \sin x + \frac{1}{2}\pi e^{\pi - x} & \text{if } x \ge \pi \end{cases}$$

The term $\frac{1}{2}\pi\, e^{\pi - x} \to 0$ as $x \to \infty$ so the steady state response is

$-\frac{1}{2}\pi \cos x + (-1 - \frac{1}{2}\pi) \sin x$, harmonic oscillation with amplitude

$\sqrt{\frac{1}{4}\pi^2 + (-1 - \frac{1}{2}\pi)^2}$, period 2π, frequency $1/2\pi$ cycles per sec, angular frequency 1 cycle per 2π secs.

(b) $m = 3,-1$, $y_h = Ae^{-x} + Be^{3x}$

For $0 \le x \le 2$ try $y_p = K$ Need $-3K = -12$, $K = 4$ so $y = Ae^{-x} + Be^{3x} + 4$

To get the IC we need $A + B + 4 = 8$, $-A + 3B = 0$, so $A = 3$, $B = 1$,

$y = 3e^{-x} + e^{3x} + 4$, $y' = -3e^{-x} + 3e^{3x}$

For $x \ge 2$, $y = Ce^{-x} + De^{3x}$, $y' = -Ce^{-x} + 3De^{3x}$

Make the y pieces agree at $x = 2$:

(1) $\quad 3e^{-2} + e^{6} + 4 = Ce^{-2} + De^{6}$

Make the y' pieces agree at $x = 2$:

(2) $\quad -3e^{-2} + 3e^{6} = -Ce^{-2} + 3De^{6}$

The two equations in (1) and (2) are two ordinary equations in the two unknowns C and D. Solve them like you did in high school algebra

Rewrite your equations as

$$-e^{-2}\ C + 3e^{6}\ D = -3e^{-2} + 3e^{6}$$

$$e^{-2}\ C + e^{6}\ D = 3e^{-2} + e^{6} + 4$$

If you just add the equations, the C's drop out and you are left with

$$4e^{6}\ D = 4e^{6} + 4$$

Divide by $4e^{6}$ to get $D = 1 + e^{-6}$

Then substitute this value of D into say the first equation to get

$$-e^{-2}\ C + 3e^{6}(1 + e^{-6}) = -3e^{-2} + 3e^{6}$$

$$-e^{-2}\ C = -3e^{6}(1 + e^{-6}) - 3e^{-2} + 3e^{6}$$

$$-e^{-2}\ C = -3 - 3e^{-2}$$

Divide by $-e^{-2}$ to get

$$C = 3e^{2} + 3$$

Answer is

$$y = \begin{cases} 3e^{-x} + e^{3x} + 4 & \text{if } 0 \le x \le 2 \\ 3e^{-x} + 3e^{2-x} + e^{3x} + e^{3x-6} & \text{if } x \ge 2 \end{cases}$$

(c) $\quad m = -2$, $y_h = Ae^{-2x}$

For $0 \le x \le 1$ try $y_p = Px + Q$. Get $P = \frac{1}{2}$, $Q = -\frac{1}{4}$, $\quad y = Ae^{-2x} + \frac{1}{2}x - \frac{1}{4}$

The IC make $A = \frac{1}{4}$. So $y = \frac{1}{4}e^{-2x} + \frac{1}{2}x - \frac{1}{4}$

For $x \ge 1$, $\quad y = Ce^{-2x}$

Make the pieces agree when $x = 1$:

$$\frac{1}{4}e^{-2} + \frac{1}{2} - \frac{1}{4} = Ce^{-2}, \quad C = \frac{1}{4} + \frac{1}{4}e^{2}$$

Answer is $y = \begin{cases} \frac{1}{4} e^{-2x} + \frac{1}{2} x - \frac{1}{4} & \text{if } 0 \le x \le 1 \\ \frac{1}{4} e^{-2x} + \frac{1}{4} e^{2-2x} & \text{if } x \ge 1 \end{cases}$

$\frac{1}{4} e^{-2x} + \frac{1}{4} e^{2-2x}$ is transient so the steady state solution is $y = 0$.

(d) If $0 \le x \le 5$, $y_h = Ae^{-4x}$. Try $y_p = C$. Need $0 + 4C = 8$, $C = 2$.

$y_{gen} = Ae^{-4x} + 2$.

To get $y(0) = 1$ need $A + 2 = 1$, $A = -1$. so $y = -e^{-4x} + 2$.

If $x \ge 5$, $y_h = Ce^{-4x}$. Try $y_p = De^{-2x}$.

Need $-2Ce^{-2x} + 4Ce^{-2x} = 6e^{-2x}$, $2C = 6$, $C = 3$.

$y_{gen} = De^{-4x} + 3e^{-2x}$.

Make the pieces $-e^{-4x} + 2$ and $De^{-4x} + 3e^{-2x}$ agree at x=5. Need
$-e^{-20} + 2 = De^{-20} + 3e^{-10}$, $D = -1 + 2e^{20} - 3e^{10}$.

Final answer is $y = \begin{cases} -e^{-4x} + 2 & \text{if } 0 \le x \le 5 \\ (-1+2e^{20}-3e^{10})e^{-4x} + 3e^{-2x} & \text{if } x \ge 5 \end{cases}$

If $x \ge 5$, the whole solution is transient so the steady state sol is $y = 0$.
Here's a graph of the solution.

```
graph1 = Plot[-E^(-4x) + 2,{x,0,5}, DisplayFunction->Identity];
graph2 = Plot[(-1+2 E^20 - 3 E^10)E^(-4x) + 3 E^(-2x),{x,5,10},
                                   DisplayFunction->Identity];
Show[graph1,graph2, DisplayFunction->$DisplayFunction];
```

10. It solve $3y'' + 2y' + y = \cos x$ with IC $y(2) = 8$, $y'(2) = 11$

11. Use superposition and add to y_1 the solution to $y'' + 3y' - 4y =$ ZERO with IC $y(0) = -1$, $y'(0) = 3$

$m^2 + 3m - 4 = 0$, $m = -4,1$, $y = Ae^{-4x} + Be^x$.

To get $y(0) = -1$ you need $A + B = -1$

To get $y'(0) = 3$ you need $-4A + B = 3$

$A = -4/5$, $B = -1/5$, $y = -\frac{4}{5} Ae^{-4x} - \frac{1}{5} e^x$

Final answer is $y_1 - \frac{4}{5} Ae^{-4x} - \frac{1}{5} e^x$

honors

12.(a) Let $z = a+bi$. Then $\overline{z} = a-bi$ and

$$\frac{1}{2}(z + \overline{z}) = \frac{1}{2}\,(a+bi + a-bi) = a = \text{Re } z$$

$$-\frac{1}{2}i(z - \overline{z}) = -\frac{1}{2}\,i(a+bi - [a-bi]\,) = -\frac{1}{2}\,i\,2bi = b = \text{Im } z. \qquad \text{QED}$$

(b) $m^2 + 4 = 0$, $m = \pm 2i$, $y_h = A \cos 2x + B \sin 2x$.

method 1 for y_p I'll switch to the forcing function $-6e^{3ix}$ and try $y_p = De^{3ix}$. Substitute into the DE. You need

$$-9De^{3ix} + 4De^{3ix} = -6e^{3ix}$$

$$-5D = -6$$
$$E = 6/5.$$

So

(*) switched $y_p = \frac{6}{5}\,e^{3ix} = \frac{6}{5}\,(\cos 3x + i \sin 3x)$.

Take the real part (because the original problem was a cosine) to get the original $y_p = \frac{6}{5}\cos 3x$.

method 2 for y_p Try $y_p = A \cos 2x + B \sin 2x$ and you'll eventually get $A = 6/5$, $B = 0$.

Finally, $y_{gen} = A \cos 2x + B \sin 2x + \frac{6}{5}\cos 3x$.

(c) Mathematica got $y_h + y_p$. Their y_h is C[2] cos 2x − C[1] sin 2x; the arbitrary constants are named C[2] and −C[1] instead of A and B. It may be peculiar to have one of them named −C[1] but it's OK; −C[1] is just as arbitrary as C[1] or A or B.

It looks like Mathematica got the particular solution by using the complex exponential method. It first switched to the forcing function $-6e^{3ix}$ and got the switched y_p in (*). Then the program took the real part. But it didn't take the real part just by looking at (*) like a person would do; it took the real part using part (a):

$$\text{original } y_p = \frac{(*) + \overline{(*)}}{2}$$
$$= \frac{3(\text{Cos}[3\ x] + \text{I Sin}[3\ x])}{5} + \frac{3(\text{Cos}[3\ x] - \text{I Sin}[3\ x])}{5}$$

It simplified a little when it cancelled the 2 into the 6 to get 3 but it didn't combine terms after that to get the simplest form, $y_p = \frac{6}{5}\cos 3x$.

13. Just plug in any values you like for the arbitrary constants. Other particular solutions are

$$e^{-x}(3 \cos 2x + 5 \sin 2x) + x^2 - \frac{4}{5}x + \frac{8}{25}$$

$$8e^{-x} \cos 2x + x^2 - \frac{4}{5}x + \frac{8}{25}$$

$$e^{-x} \sin 2x + x^2 - \frac{4}{5}x + \frac{8}{25} \qquad \text{etc.}$$

14. The old general solution is

$$y = e^{-x}(A \cos 2x + B \sin 2x) + x^2 - \frac{4}{5}x + \frac{8}{25}$$

the new general solution (perfectly valid) is

$$y = e^{-x}(A \cos 2x + B \sin 2x) + 8e^{-x} \cos 2 + x^2 - \frac{4}{5}x + \frac{8}{25}$$

The two general solutions describe the *same* collection. In fact the second solution can be rewritten as

$$y = e^{-x}([A+8] \cos 2x + B \sin 2x) + 8e^{-x} \cos 2 + x^2 - \frac{4}{5}x + \frac{8}{25}$$

And if you replace the arbitrary constant A+8 by the arbitrary constant C, you get

$$y = e^{-x}(C \cos 2x + B \sin 2x) + x^2 - \frac{4}{5}x + \frac{8}{25}$$

which agrees with the old general solution.

SOLUTIONS Section 1.7

1. $m = 1,2$, $y_h = Ae^{-x} + Be^{2x}$. Ordinarily I would try $y_p = Ce^{-x}$. But Ce^{-x} is a homog sol so step up and try $y_p = Cxe^{-x}$. Then

$y_p' = -Cxe^{-x} + Ce^{-x}$ (product rule)

$y_p'' = Cxe^{-x} - Ce^{-x} - Ce^{-x} = Cxe^{-x} - 2Ce^{-x}$.

Substitute into the DE to determine C. Need

$$Cxe^{-x} - 2Ce^{-x} - (-Cxe^{-x} + Ce^{-x}) - 2Cxe^{-x} = 6e^{-x}$$

Equate xe^{-x} coeffs: Happened automatically. The coeff on each side is 0

Equate e^{-x} coeffs: $-3Ce^{-x} = 6e^{-x}$, $C = -2$

$y_{gen} = y_h + y_p = Ae^{-x} + Be^{2x} - 2xe^{-x}$

2. If you try $y_p = Ae^{2x}$ then $y_p' = 2Ae^{2x}$, $y_p'' = 4Ae^{2x}$ and you need

$4Ae^{2x} - 6Ae^{2x} + 2Ae^{2x} = 6e^{2x}$

$0Ae^{2x} = 6e^{2x}$.

You need $0A = 6$ but there is no such A (the solution is *not* $A = 0$) showing that there is no particular solution of the form Ae^{2x}.

(Naturally you can't make Ae^{2x} produce $6e^{2x}$ since Ae^{2x} is a homog sol and produces 0 when you substitute it into the left hand side.)

3.(a) Try $y_p = Ae^{3x}$ (no need to step up)

(b) Try $y_p = Axe^{3x}$ (step up because e^{3x} is a homog sol))

(c) Step up even more to $y_p = Ax^3e^{3x}$ because e^{3x}, xe^{3x} and x^2e^{3x} are all homog sols.

4. $m^3 - m = 0$, $m(m^2 - 1) = 0$, $m = 0, \pm 1$, $y_h = P + Qe^x + Se^{-x}$

Ordinarily you would try $y_p = Ax + B$. But B is a homog sol since one of the m's is 1 so try $y_p = x(Ax + B) = Ax^2 + Bx$. Then $y_p' = 2Ax + B$, $y_p''' = 0$

Need $-2Ax - B = x$

Equate x coeffs: $-2A = 1$, $A = -1/2$

Equate constant terms: $-B = 0$, $B - 0$

Gen sol is $y = P + Qe^x + Se^{-x} - \frac{1}{2}x^2$

5.(a) $m^2 = 0$, $m = 0,0$ so $y_h = Ae^{0x} + Bxe^{0x} = A + Bx$

For y_p you would ordinarily try $Cx^2 + Dx + E$ but now you have to step up twice to escape from the homogeneous sol. Try

$$y_p = x^2(Cx^2 + Dx + E) = Cx^4 + Dx^3 + Ex^2$$

Then $y_p' = 4Cx^3 + 3Dx^2 + 2Ex$

$y_p'' = 12Cx^2 + 6Dx + 2E$

Substitute into the DE and make it fit. Need $12Cx^2 + 6Dx + 2E = 3x^2$

Equate x^2 coeffs: $12C = 3$, $C = 1/4$

Equate x coeffs: $6D = 0$, $D = 0$

Equate constant terms: $2E= 0$, $E = 0$

$$12C = 3,\quad 6D = 0,\quad 2E = 0$$

$$C = \frac{1}{4},\ D = 0,\ E = 0$$

So $y_p = \frac{1}{4}x^4$, $y_{gen} = y_h + y_p = A + Bx + \frac{1}{4}x^4$

(b) If $y'' = 3x^2$, just antidifferentiate once to get $y' = x^3 + C$ and then antidiff again to get $y = \frac{1}{4}x^4 + Cx + K$, same answer as part (a).

6.(a) $m = \pm 3i$, $y_h = C\cos 3x + D\sin 3x$.

To find a particular sol, switch to $y'' + 9y = 4e^{3ix}$. Ordinarily you would try $y_p = Ae^{3ix}$ but since one of the m's is 3i, e^{3ix} is a homog sol so try $y_p = Axe^{3ix}$. Then $y_p' = 3iAxe^{3ix} + Ae^{3ix}$, $y_p'' = -9Axe^{3ix} + 6iAe^{3ix}$ (product rule)

Need $-9Axe^{3ix} + 6iAe^{3ix} + 9Axe^{3ix} = 4e^{3ix}$.

The xe^{3ix} terms cancel out on the left side which matches the right side.

Equate e^{3ix} coeffs: $6iA = 4$, $A = -\frac{2}{3}i$

Switched $y_p = -\frac{2}{3}ixe^{3ix} = -\frac{2}{3}ix(\cos 3x + i\sin 3x)$.

Original y_p = real part $= \frac{2}{3}x\sin 3x$

Gen sol is $C\cos 3x + D\sin 3x + \frac{2}{3}x\sin 3x$

(b) $m = \pm 2i$, $y_h = A\cos 2x + B\sin 2x$.

To get y_p you'll have to step up.

method 1 for y_p Switch to the new problem $y'' + 4y = 6e^{2ix}$.

Try $y_p = Cxe^{2ix}$.

Then $y_p' = 2iCxe^{2ix} + Ce^{2ix}$ (product rule)

$y_p'' = -4Cxe^{2ix} + 2iCe^{2ix} + 2iCe^{2ix} = -4Cxe^{2ix} + 4iCe^{2ix}$

Substitute into the switched DE. Need

$$-4Cxe^{2ix} + 4iCe^{2ix} + 4Cxe^{2ix} = 6e^{2ix}$$

The xe^{2ix} terms match since they cancel out on the left side.

Make the e^{2ix} terms match: $4iC = 6$, $C = \frac{6}{4i} = -\frac{3}{2}i$

switched $y_p = -\frac{3}{2}ixe^{2ix} = -\frac{3}{2}ix(\cos 2x + i\sin 2x)$

To get the original y_p, take the imag part (because the original problem had a sine forcing function). So $y_p = -\frac{3}{2}x\cos 2x$.

method 2 for y_p Try $y_p = x(C\cos 2x + D\sin 2x)$. You should end up with $C = -\frac{3}{2}$, $D=0$.

And finally, $y_{gen} = y_h + y_p = A\cos 2x + B\sin 2x - \frac{3}{2}x\cos 2x$.

7.(a) $y_h = Ae^{-x} + Be^{2x}$. Try y_p $Ax^4 + Bx^3 + Cx^2 + Dx + E$.

(b) Step up because $1, x, x^2, x^3, x^4$ are all homog sols.

Try $y_p = x^5(Ax^3 + Bx^2 + Cx + D) = Ax^8 + Bx^7 + Cx^6 + Dx^5$.

(c) Switch to forcing function e^{2ix}, try $y_p = Ae^{2ix}$ and take imag part.

(d) Switch to forcing function $2e^{4ix}$, try $y_p = Ax^2e^{4ix}$ (step up twice because e^{4ix} and xe^{4ix} are both homog sols) and take real part.

8. (a) If one of the m's is -4, step up to $y_p = Axe^{-4x}$.

If both m's are -4, step up to $y_p = Ax^2e^{-4x}$.

(b) If one of the m's is 0 (making $y = C$ a homog sol) step up to

$y_p = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$

If $m = 0,0$ (making C and x homog sols) step up to $y_p = x^2(Ax^2 + Bx + C)$.

(c) If $m = \pm 2i$ so that the general complex homog solution is $Ae^{2ix} + Be^{-2ix}$, step up to $y_p = Axe^{2ix}$.

9. (a) $y_p' = A$, $y_p'' = 0$ and you need $0 + 2A = x + 4$.

Can't get it. The left side can't be made to be $x+4$ since it has no x term at all.

warning It makes no sense to conclude that $A = \frac{1}{2}(x+4)$. A is a *constant* (you treated it as a constant when you differentiated the trial y_p) so it can't have x's in it.

(b) Try $y_p = x(Ax + B) = Ax^2 + Bx$.

Then you need $2A + 2(2Ax + B) = x + 4$ which you can get with

$$4A = 1$$
$$2A + 2B = 4$$

$$A = \frac{1}{4},\ B = \frac{7}{4}$$

So $y_{gen} = C + De^{-2x} + \frac{1}{4}x^2 + \frac{7}{4}x$.

(c) Since there's no y term in (1), you can antidifferentiate on both sides (if there is a y term then it doesn't help to try to antidiff on both sides since you don't know the antideriv of y):

$$y' + 2y = \frac{1}{2}x^2 + 4x + K \quad \text{(K is the arbitrary constant of integration)}$$

Now it's a *first* order equation with $y_h = De^{-2x}$. Try

$$y_p = Ax^2 + Bx + C \quad \text{(no need to step up)}$$

Substitute into the DE:

$$2Ax + B + 2(Ax^2 + Bx + C) = \frac{1}{2}x^2 + 4x + K$$

You need

$$2A = \frac{1}{2} \quad \text{(match the } x^2 \text{ coeffs)}$$
$$2A + 2B = 4 \quad \text{(match the x coeffs)}$$
$$B + 2C = K \quad \text{(match the constant terms)}$$
$$A = \frac{1}{4},\ B = \frac{7}{4},\ C = \frac{1}{2}K - \frac{7}{8}$$
$$y_{gen} = De^{-2x} + \frac{1}{4}x^2 + \frac{7}{4}x + \underbrace{\frac{1}{2}K - \frac{7}{8}}_{C}$$

Since K is an arbitrary constant, $\frac{1}{2}K - \frac{7}{8}$ is just as arbitrary and can be renamed C and you can see that this answer agrees with the answer in (b).

HONORS

10. (a) When you substitute your trial y_p into the DE, you get three kinds of terms: an e^{-5x} term, an xe^{-5x} term and an x^2e^{-5x} term. So you will get three equations in A,B,C. When you solve the equations you will get B = 17, C = 0, D = 0. Same end result as me. You got away with it.

(b) When you substitute your trial y_p into the left side of the DE, you will get three kinds of terms: an x^2 term, an x term and a constant term. So you will get 3 equations in the one unknown D. It will turn out that there is no solution (no single value of D will work in all 3 equations). This shows that there is no particular solution of the form Dx^2. Your trial y_p was no good.

SOLUTIONS Section 1.8

1.(a) $m = \pm 3i$, $y_h = A\cos 3x + B\sin 3x$. Try $y_p = Ce^x + Dx + E$.

Then $yp' = Ce^x + D$, $yp'' = Ce^x$.

Need $Ce^x + 9(Ce^x + Dx + E) = 5e^x + 3x$

Equate x coeffs: $9D = 3$, $D = 1/3$

Equate e^x coeffs: $10C = 5$, $C = 1/2$

Equate constant terms: $9E = 0$, $E = 0$

$y_{gen} = A\cos 3x + B\sin 3x + \frac{1}{2}e^x + \frac{1}{3}x$

(b) $m = \pm 2$, $y_h = Ae^{2x} + Be^{-2x}$. Try $y_p = Axe^{2x} + B$ (step up the first part).

Then $yp' = A(2xe^{2x} + e^{2x}$, $yp'' = A(4xe^{2x} + 2e^{2x} + 2e^{2x})$

Need $A(4xe^{2x} + 4e^{2x}) - 4(Axe^{2x} + B) = e^{2x} + 2$

The xe^{2x} terms drop out.

Equate e^{2x} coeffs: $4A = 1, A = 1/4$

Equate constants: $-4B = 2$, $B = -1/2$

$y_{gen} = Ae^{2x} + Be^{-2x} + \frac{1}{4}xe^{2x} - \frac{1}{2}$

2. $m = -1 \pm 3i$, $y_h = e^{-x}(A\cos 3x + B\sin 3x)$. No stepping up in either (a) or (b).

(a) *method 1* Switch to the forcing function $6e^{3ix}$ and get a switched y_p by trying $y_p = Ae^{3ix}$ and determining A.

Then to get y_p for the forcing function $6\cos 3x$, take the real part.

To get a y_p for the forcing function $7\sin 3x$, no need to start again. Just take the imag part (which goes with forcing function $6\sin 3x$) *and multiply it by 7/6*.

Then add those two y_p's.

method 2 (like method 1 but neater)

Switch to the forcing function e^{3ix} (not $6e^{3ix}$ or $7e^{3ix}$ but plain e^{3ix}) and get a switched y_p by trying $y_p = Ae^{3ix}$ and determining A.

Then to get a y_p for the forcing function $6\cos 3x$, take the real part of the switched y_p *and multiply it by 6*.

To get a y_p for the forcing function $7\sin 3x$, take the imag part of the switched y_p *and multiply it by 7*.

And finally, add those two y_p's. In other words,

original solution = $6 \times$ real part of switched y_p + $7 \times$ imag part of switched y_p

method 3 (without the complex exponential) Try $y_p = A\cos 3x + B\sin 3x$ and determine A and B.

(b) *step 1* Get y_{p1} to go with the forcing function $6\cos 3x$.

Switch to forcing function $6e^{3ix}$, Get a y_p for the switched forcing function by trying $y_p = Ae^{3ix}$ and take its *real* part.

step 2 Get y_{p2} to go with the forcing function $7\sin 4x$.

Switch to forcing function $7e^{4ix}$, get y_p for the switched forcing function by trying $y_p = Ae^{4ix}$, and take its *imag* part.

step 3 Add y_{p1} and y_{p2}.

3.(a) $m = -3, -1$, $y_h = Ae^{-3x} + Be^{-x}$.

Switch to $y'' + 4y' + 3y = 2e^{2x}e^{4ix} = 2e^{(2+4i)x}$ and try $y_p = Ce^{(2+4i)x}$.

Then $y_p{}' = (2+4i)Ce^{(2+4i)x}$, $y_p{}'' = (2+4i)^2\, Ce^{(2+4i)x}$.

Need $(2+4i)^2\, Ae^{(2+4i)x} + 4(2+4i)\, Ae^{(2+4i)x} + 3Ae^{(2+4i)x} = 2e^{(2+4i)x}$

$(-1 + 32i)Ae^{(2+4i)x} = 2e^{(2+4i)x}$

Equate coeffs of $e^{(2+3i)x}$: $A = \dfrac{2}{-1+32i} = \dfrac{-2\ -64i}{1025}$

Switched $y_p = \dfrac{-2\ -64i}{1025}\, e^{(2+4i)x} = \dfrac{-2\ -64i}{1025}\, e^{2x}(\cos 4x + i \sin 4x)$.

Take real part to get original y_p.

Answer is $y_{gen} = Ae^{-3x} + Be^{-x} + e^{2x}(\dfrac{-2}{1025}\cos 4x + \dfrac{64}{1025}\sin 4x)$

(b) From part (a), the imag part of the switched y_p is a particular sol for $y'' + 4y' + 3y = 2e^{2x}\sin 4x$. Take $\frac{5}{2}\times$ the imag part to get a particular sol for $y'' + 4y' + 3y = 5e^{2x}\sin 4x$. So

$$y_{gen} = Ae^{-3x} + Be^{-x} + \frac{5}{2}\, e^{2x}\, (-\frac{64}{1025}\cos 4x - \frac{2}{1025}\sin 4x)$$

(c) $y_h = Ce^{(3+\sqrt{5})x/2} + De^{(3-\sqrt{5})x/2}$

Get a particular sol to $y'' - 3y' + y = 3e^{(1+i)x}$ by trying $y_p = Ae^{(1+i)x}$. Need

$2iAe^{(1+i)x} - 3(1+i)Ae^{(1+i)x} + Ae^{(1+i)x} = 3e^{(1+i)x}$,

$(-2-i)A = 3$, $A = \dfrac{3}{-2-i} = \dfrac{-6+3i}{5}$

$y_p = \dfrac{-6+3i}{5}\, e^{(1+ix)} = \dfrac{-6+3i}{5}\, e^{x}(\cos x + i \sin x)$

Take imag part to get original y_p.

Answer is $y_{gen} = Ce^{(3+\sqrt{5})x/2} + De^{(3-\sqrt{5})x/2} + e^{x}(\frac{3}{5}\cos x - \frac{6}{5}\sin x)$

(d) $y_h = Ce^{x} + De^{-x}$.

Try $y_p = x(Ax + B)e^{x} = (Ax^2 + Bx)e^{x}$ (step up bcause e^{x} is a homog sol)

Then $y_p' = (Ax^2 + Bx)e^{x} + (2Ax + B)e^{x}$

$y_p'' = (Ax^2 + Bx)e^{x} + 2(Ax + B)e^{x} + 2Ae^{x}$

Need $(4Ax + 2B)e^{x} + 2Ae^{x} = xe^{x}$, $4A = 1$, $2B + 2A = 0$, $A = \frac{1}{4}$, $B = -\frac{1}{4}$

$y_{gen} = De^{x} + De^{-x} + (\frac{1}{4}x^2 - \frac{1}{4}x)e^{x}$

4.(a) $y_p = (Ax^2 + Bx + C)e^{2x}$

(b) $y_p = x^2(Ax^2 + Bx + C)e^{2x} = (Ax^4 + Bx^3 + Cx^2)e^{2x}$

(Step up twice because e^{2x} and xe^{2x} are homog sols)

(c) Switch to forcing function $e^{(3+4i)x}$. Try $y_p = Ae^{(3+4i)x}$ and eventually take real part. No stepping up.

(d) Switch to forcing function $e^{(3+4i)x}$, try $y_p = Axe^{(3+4i)x}$ and eventually take *imag* part. (Step up because $e^{(3+4ix)}$ is a homog sol)

(e) Switch to forcing function $e^{(3+4i)x}$, try $y_p = Ae^{(3+4i)x}$ and eventually take real part

(f) $y_p = xe^x(Ax^2 + Bx + C) = e^x(Ax^3 + Bx^2 + Cx)$

(Step up because Ce^x is a homog sol.)

(g) Switch to forcing function x^2e^{ix}. Try $y_p = (Ax^2 + Bx + C)e^{ix}$ and eventually take imag part.

(h) $y_p = (Ax^2 + Bc + C)\ e^x$

5. (a) The forcing function is the product $(-x^2 + 2)e^{2x}$. Try $y_p = (Bx^2 + Cx + D)e^{2x}$

(b) $y_h = Ae^{2x} + Bxe^x$. Since e^{2x} and xe^{2x} are homog sols, step up the part (a) answer and try

$$y_p = x^2(Bx^2 + Cx + D)e^{2x} = (Bx^4 + Cx^3 + Dx^2)e^{2x}$$

6.(a) Get a particular sol to $2y'' + 2y = 3xe^{ix}$. Ordinarily you would try $y_p = (Ax + B)e^{ix}$ but $m = \pm i$ so e^{ix} is a homog sol so step up to

$$y_p = x(Ax + B)e^{ix} = (Ax^2 + Bx)e^{ix}.$$

then

$$y_p' = i(Ax^2 + Bx)e^{ix} + (2Ax + B)e^{ix} = (iAx^2 + iBx + 2Ax + B)e^{ix}$$

$$y_p'' = i(iAx^2 + iBx + 2Ax + B)e^{ix} + (2iAx + iB + 2A)e^{ix}$$
$$= (2A + 2iB + 4iAx - Bx - Ax^2)e^{ix}$$

Substitute into the DE:

$$2(2A + 2iB + 4iAx - Bx - Ax^2)e^{ix} + 2(Ax^2 + Bx)e^{ix} = 3xe^{ix}$$

The $x^2\ e^{ix}$ terms on the left cancel out.

Equate the coeffs of xe^{ix}: $8iA - 2B + 2B = 3$, $A = \dfrac{3}{8i} = -\dfrac{3}{8}i$

Equate the coeffs of e^{ix}: $4A + 4iB = 0$, $B = -\dfrac{A}{i} = \dfrac{3}{8}$.

So the particular sol is $(-\frac{3}{8}i\ x^2 + \frac{3}{8}x)\ e^{ix} = (-\frac{3}{8}i\ x^2 + \frac{3}{8}x)(\cos x + i\ \sin x)$

Take the real part to get a sol to the original equation. Answer is

$$y_p = \frac{3}{8}x\ \cos x + \frac{3}{8}x^2\ \sin x$$

(b) Ordinarily you would try

$$y_p = (Ax + B)\sin x + (Cx + D)\cos x$$

but $m = \pm i$ so $\cos x$ and $\sin x$ are homog sols, so step up to

$$y_p = x(Ax + B)\sin x + x(Cx + D)\cos x = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x$$

Then

$$y_p'' = (2C + 2B + 4Ax - Dx - Cx^2)\cos x + (2A - Bx - Ax^2 - 2D - 4Cx)\sin x$$

Substitute into the DE:

$$2\Big[(2C + 2B + 4Ax - Dx - Cx^2)\cos x + (2A - Bx - Ax^2 - 2D - 4Cx)\sin x\Big]$$
$$+ 2\Big[(Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x\Big] = 3x\cos x$$

The $x^2\cos x$ terms and the $x^2\sin x$ terms on the left cancel out.

Equate coeffs of $x\cos x$: $8A - 2D + 2D = 3$, $A = \frac{3}{8}$

Equate coeffs of $x\sin x$: $-2B - 8C + 2B = 0$, $C = 0$

Equate coeffs of $\cos x$: $4C + 4B = 0$, $B - 0$

Equate coeffs of $\sin x$: $4A - 4D = 0$, $D = \frac{3}{8}$

$y_p = \frac{3}{8}x\cos x + \frac{3}{8}x^2\sin x$ as in part (a).

SOLUTIONS review problems for Chapter 1

1. $m = \pm 1$, $y_h = Ce^x + De^{-x}$. Ordinarily you would try $y_p = (Ax + B)e^x$. But e^x is a homog sol so try $y_p = x(Ax + B)e^x = (Ax^2 + Bx)e^x$ (step up)

Then $y_p' = (Ax^2 + Bx)e^x + (2Ax + B)e^x$

$y_p'' = (Ax^2 + Bx)e^x + (2Ax + B)e^x + (2Ax + B)e^x + 2Ae^x$

$= Ax^2 e^x + (B + 4A)xe^x + (2B + 2A)e^x$

Substitute to get

$Ax^2e^x + (B+4A)xe^x + (2B+2A)e^x - (Ax^2 + Bx)e^x = xe^x$

The x^2e^x terms drop out.
Need $2A + 2B = 0$, $4A = 1$. So $A = 1/4$, $B = -1/4$

$y_{gen} = Ce^x + De^{-x} + \frac{1}{4} e^x(x^2 - x)$

$y' = C^x - De^{-x} + \frac{1}{4} e^x (2x-1) + \frac{1}{4} e^x (x^2 - x)$

The IC make $C + D = 1$, $C - D - \frac{1}{4} = 0$. So $C = \frac{5}{8}$, $D = \frac{3}{8}$

Answer is $y = \frac{5}{8} e^x + \frac{3}{8} e^{-x} + \frac{1}{4} e^x(x^2 - x)$

2. (a) $m = -1,-1$, $y_h = Ae^{-x} + Bxe^{-x}$.

Switch to $y'' + 2y' + y = 3e^{2ix}$ and try $y_p = Ce^{2ix}$.

Need $-4Ce^{2ix} + 4iCe^{2ix} + Ce^{2ix} = 3e^{2ix}$

EQuate coeffs of e^{2ix}: $(-3+4i)C = 3$, $C = \dfrac{-9-12i}{25}$

(*) Switched $y_p = \dfrac{-9-12i}{25} e^{2ix} = \dfrac{-9-12i}{25} (\cos 2x + i \sin 2x)$

Take real part to get the original y_p.

General sol is $y = Ae^{-x} + Bxe^{-x} - \frac{9}{25} \cos 2x + \frac{12}{25} \sin 2x$

(b) The imag part of (*) is a particular sol to $y'' + 2y' + y = 3 \sin 2x$. By superposition $2 \times$ the imag part is a particular sol to $y'' + 2y' + y = 6 \sin 2x$

General sol is $y = Ae^{-x} + Bxe^{-x} + 2(-\frac{12}{25} \cos 2x - \frac{9}{25} \sin 2x)$

3. $y_h = e^{-3x}(C \cos x + D \sin x)$

Part 1 For $0 \le x \le \pi$, try $y_p = Ax + B$. Need

$6A + 10(Ax + B) = 2x$, $6A + 10B = 0$, $10A = 50$.

So $A = 5$, $B = -3$ and $y = e^{-3x}(C \cos x + D \sin x) + 5x - 3$.

The IC make $C = 4$, $D = 9$ so

$y = e^{-3x}(4 \cos x + 9 \sin x) + 5x - 3$

$y' = e^{-3x}(-4 \sin x + 9 \cos x) - 3e^{-3x}(4 \cos x + 9 \sin x) + 5$

Part 2 For $x \ge \pi$, try $y_p = E$ Substitute into the DE to get $10E = 10$, $E = 1$,

$y = e^{-3x}(M \cos x + N \sin x) + 1$,

$y' = e^{-3x}(-M \sin x + N \cos x) - 3e^{-3x}(M \cos x + N \sin x)$

Part 3 Make the two y pieces agree at $x = \pi$:

$$-4e^{-3\pi} + 5\pi - 3 = -Me^{-3\pi} + 1, \qquad \boxed{M = (4-5\pi)e^{3\pi} + 4}$$

Make the two y' pieces agree at $x = \pi$:

$$-9e^{-3\pi} + 12e^{-3\pi} + 5 = -Ne^{-3\pi} + 3Me^{-3\pi}, \qquad \boxed{N = 9 + (7 - 15\pi)e^{3\pi}}$$

Answer is $y = \begin{cases} e^{-3x}(4 \cos x + 9 \sin x) + 5x - 3 & \text{if } 0 \le x \le \pi \\ e^{-3x}(M \cos x + N \sin x) + 1 & \text{if } x \ge \pi \end{cases}$

where M and N are given in the boxes

4. The general sol must be $Ae^{-2x} + Be^{-x} + 3 \sin x$
Then $m = -2,-1$, $m^2 + 3m + 2 = 0$. So DE is of the form
$y'' + 3y' + 2y = f(x)$. Since $3 \sin x$ is a solution, substitute it into the DE to determine $f(x)$:

$$-3 \sin x + 9 \cos x + 6 \sin x = f(x)$$

$$f(x) = 9 \cos x - 3 \sin x.$$

Answer is $y'' + 3y' + 2y = 9 \cos x + 3 \sin x$

5. Add any homog solution to $3x^2$. The general homog sol is $Ae^{-4x} + Be^{x}$.

So other particular solutions are

$2x^{-4x} + 5e^{x} + 3x^2$
$e^{-4x} + 3x^2$
$8e^{x} + 3x^2$
etc.

6. When y_1 is substituted into the left side of the DE, it produces $f(x)$.
And y_1 satisfies IC $y(0) = 2$, $y'(0) = 3$.

Let $y_2 = 4x^2 + 1$, the thing that was tacked on.
When y_2 is substituted into the left side of the DE, it produces

$$y_2'' + 3y_2' - 4y_2 = 8 + 3(8x) - 4(4x^2 + 1) = 4 + 24x - 16x^2.$$

And $y_2(0) = 1$, $y_2'(0) = 0$ so y_2 satisfies IC $y(0) = 1$, $y'(0) = 0$

By superposition, $y_1 + 4x^2 + 1$ produces $f(x) + 4 + 24x - 16x^2$ and satisfies IC $y(0) = 2+1 = 3$, $y'(0) = 3+0 = 3$.
In other words, $y_1 + 4x^2 + 1$ is a solution to

$$y'' + 3y' - 4y = f(x) + 4 + 24x - 16x^2 \quad \text{with IC } y(0) = 3,\ y'(0) = 3$$

7. (a) Can be written as $y' - y = 0$. Linear and homog
(b) Linear but not homog (the forcing function is x)
(c) Can be written as $y'' - y = 0$. Linear and homog
(d) Not linear because of the yy'' term
(d) Can be written as $xy'' - y = 0$. Linear and homog (but with a variable coefficient).

8. (a) Sometimes. It's true if $f(x)$ is 0 so that the equation is homog. It's not true otherwise.

(b) Always.

By superposition, $y_1 - y_2$ is a solution of $ay'' + bu' + cy = f(x) - f(x) = 0$.

9. (a) $y' + y = 0$, $m = -1$, $y = Ae^{-x}$

(b) $y'' - y = x$, $m = \pm 1$, $y_h = Ae^{x} + Be^{-x}$

Try $y_p = Cx + D$. Need $0 - (Cx + D) = x$, $-C = 1$, $D = 0$,

$y_{gen} = Ae^{x} + Be^{-x} - x$

10. the differential equation is $mv' + cv = mg$. The unknown you're solving for is $v(t)$. I'm going to use the letter λ instead of m since m is already used here as the mass.

$$m\lambda + c\lambda = 0, \ \lambda = -\frac{c}{m}, \ \ y_h = Be^{-ct/m}$$

Try $y_p = A$. Get $A = \frac{mg}{c}$. So $y = \frac{mg}{c} + Be^{-ct/m}$. Plug in the IC.

Set $t = 0$, $v = 0$ to get $A = -mg$. Answer is $v = \frac{mg}{c}(1 - e^{-ct/m})$.

$v(\infty) = \frac{mg}{c}$.

11. If you try $y_p = Ax + B$ then $y_p' = A$, $y_p'' = 0$. Need $0 + 2A = x + 4$.

But you can't make this happen because there is no x term on the left side, i.e., $0 + 2A$ can never be $x+4$.

warning You can't do it by making $A = \frac{1}{2}(x + 4)$ because A is a *constant*; you treated it like a constant when you found the derivative of y_p and you can't change your mind now.

So the conclusion is that there is no particular solution of the form $Ax + B$.

SOLUTIONS Section 2.1

1.(a) Solve

(*) $2y'' + 2y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$.

To do this, switch to $2y'' + 2y = 0$ with IC $y(0) = 0$, $y'(0) = 1/2$.

$m = \pm i$, $y_h = A \cos x + B \sin x$. The IC make $A = 0$, $B = 1/2$

Answer is $h(t) = \frac{1}{2} \sin t$ for $t \ge 0$

(b) To solve

(**) $2y'' - y' - y = \delta(t)$ with $y(0) = 0$, $y'(0) = 0$

switch to $2y'' - y' - y = 0$ with $y(0) = 0$, $y'(0) = 1/2$.

$m = -1/2, 1$; $y = Ae^{-t/2} + Be^{t}$. The IC make $A = -1/3$, $B = 1/3$.

$h(t) = -\frac{1}{3} e^{-t/2} + \frac{1}{3} e^{t}$ for $t \ge 0$.

Question I get asked a lot In (*) and (**), do you always use IC $y(0) = 0$, $y'(0) = 0$. The problem didn't say anything about IC.

Answer If you want to find the impulse response then, yes, you must use IC $y(0) = 0$, $y'(0) = 0$ and use $\delta(t)$ as the forcing function. The impulse response is *defined* as the response of an *initially-at-rest system* to the delta function input.

You can use the delta function as the input into a system that is not initially at rest but then the response is not called the impulse response.

2.(a) Solve $y'' + 4y = 0$ with IC $y(0) = 0$, $y'(0) = 1$

$m = \pm 2i$, $y = A \cos 2x + B \sin 2x$. The IC make $A = 0$, $B = 1/2$

$h(t) = \frac{1}{2} \sin 2t$ for $t \ge 0$

(b) Take the $h(t)$ from part (a), multiply by 6 and delay.

$$\text{Answer is } y(t) = 6h(t-2) = \begin{cases} 0 & \text{if } t \le 2 \\ 3 \sin 2(t-2) & \text{if } t \ge 2 \end{cases}$$

3. Response $y(t)$ to $\delta(t)$ is $h(t)$.
   Response $y(t)$ to $6\delta(t)$ is $6h(t)$
   Response $y(t)$ to $\delta(t-2)$ is $h(t-2)$
   (a) $y(3) = h(3) = 1/9$
   (b) Response $y(3) = 6h(3) = 6/9$
   (c) Response $y(3) = h(3-2) = h(1) = 1$

4. To get the location of the max value of $h(t)$, find $h'(t)$ and set it equal to 0.

$$-\frac{1}{4} e^{-t} + \frac{1}{4} 3e^{-3t} = 0$$

$$e^{2t} = 3$$

$$2t = \ln 3$$

$$t = \frac{1}{2} \ln 3.$$

This is is either a relative max or a rel min or a point of inflection but since we already have the graph, this value of t must go with a rel max.

HONORS

5. (See the superposition rule for IC in Section 1.6.)
To get the solution to

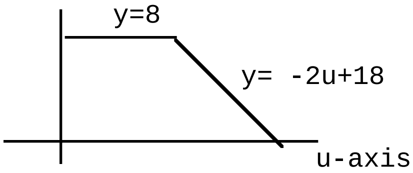
$$ay'' + by' + cy = \delta(t) \text{ with IC } y(0) = 4,\ y'(0) = 5$$

take h(t) and add the solution to

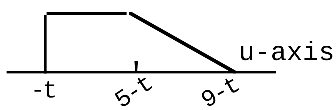
$$ay'' + by' + cy = \text{ZERO with IC } y(0) = 4,\ y'(0) = 5$$

SOLUTIONS Section 2.3

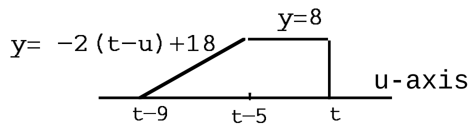
1.(a)



Graph of h(u)



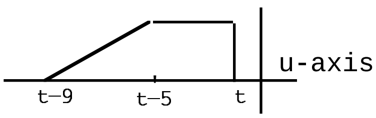
Intermediate step



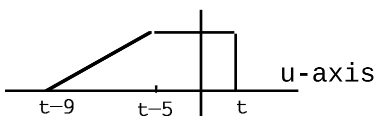
Graph of h(t−u)

Question There a y-axis (vertical axis) in the lefthand diagram.
Why isn't there a y-axis in the other two diagrams.
Answer Where the y-axis goes in the last two diagrams depends on the size of t.
Here is how it works for the righthand diagram.

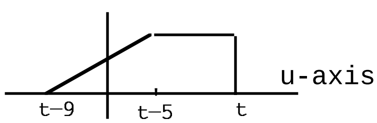
case 1 $t \le 0$



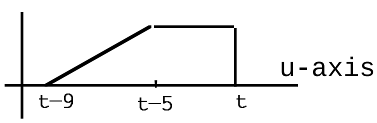
case 2 $0 \le t \le 5$ ({so that $t < 0$ but $t-5 \le 0$)



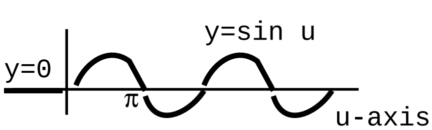
case 3 $5 \le t \le 9$ (so that $t-5 \ge 0$ but $t-9 \le 0$)



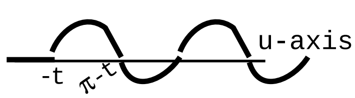
case 4 $t \ge 9$ (so that $t-9 \ge 0$)



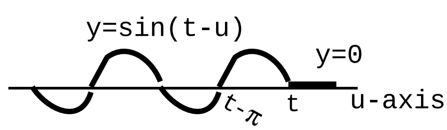
(b)



Graph of h(u)

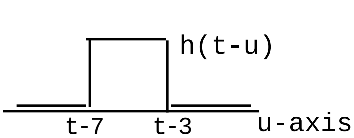


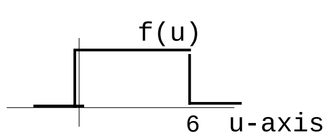
Intermediate step



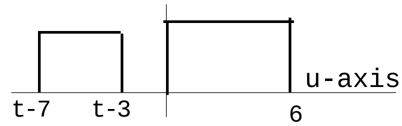
Graph of h(t−u)

2. The response is $h(t)*f(t)$. I'll use the version $\int_{u=-\infty}^{\infty} h(t-u)f(u)\,du$
Here are the graphs of $h(t-u)$ and $f(u)$





case 1 $t - 3 \le 0$, i.e., $\boxed{t \le 3}$



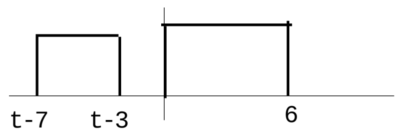
$$h(t)*f(t) = \int_{u=-\infty}^{\infty} 0 \, du = 0$$

(no overlap yet)

warning In this case, $h(t)*f(t)$ is *not* $\int_{-\infty}^{\infty} 4\cdot 5 \, du$. It's $\int_{u=-\infty}^{\infty} 0 \, du$

case 2 $t-7 \le 0$ and $t-3 \ge 0$,

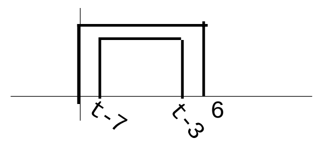
i.e., $\boxed{3 \le t \le 7}$



$$h(t)*f(t) = \int_{u=0}^{t-3} 5\cdot 4 \, du = 20t - 60$$

case 3 $t-7 \ge 0$ and $t-3 \le 6$,

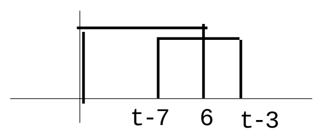
i.e., $\boxed{7 \le t \le 9}$



$$h(t)*f(t) = \int_{u=t-7}^{t-3} 20 \, du = 80$$

case 4 $t-7 \le 6$ and $t-3 \ge 6$,

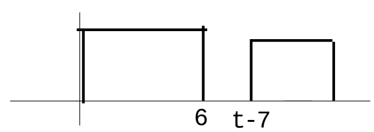
i.e., $\boxed{9 \le t \le 13}$



$$h(t)*f(t) = \int_{u=t-7}^{6} 20 \, du = -20t + 260$$

case 5 $t-7 \ge 6$, i.e., $\boxed{t \ge 13}$

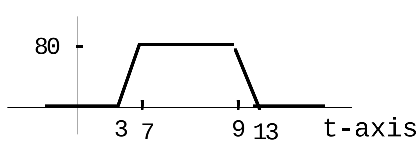
No more overlap.



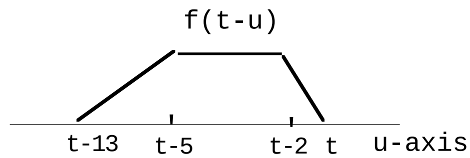
$$h(t)*f(t) = \int_{u=-\infty}^{\infty} 0 \, du = 0$$

All in all,

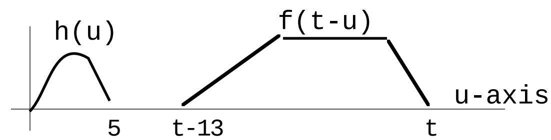
$$h(t)*f(t) = \begin{cases} 0 & \text{if } t \le 3 \\ 20t - 60 & \text{if } 3 \le t \le 7 \\ 80 & \text{if } 7 \le t \le 9 \\ -20t + 260 & \text{if } 9 \le t \le 13 \\ 0 & \text{if } t \ge 13 \end{cases}$$



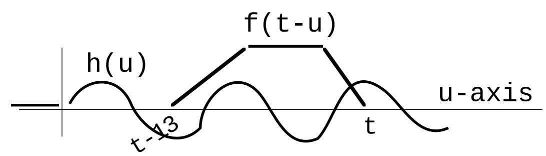
3. The response is f(t)*h(t).
I'll use the version which flips f.
You should get the same final answer no matter which you flip.



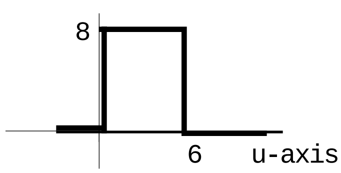
(a) No overlap iff $t-13 \geq 5$, i.e.,
if $t \geq 18$ then the product $h(u)f(t-u) = 0$
So response dies at time $t = 18$



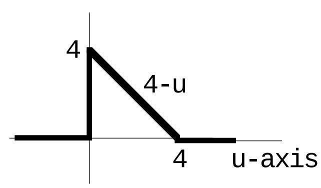
(b) Overlap never stops.
Response never dies out



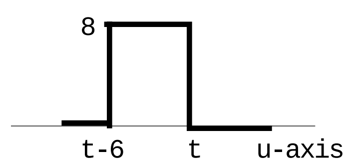
4.



graph of f(u)



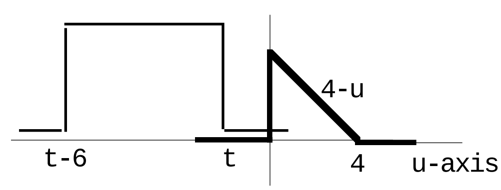
graph of h(u)



graph of f(t−u)

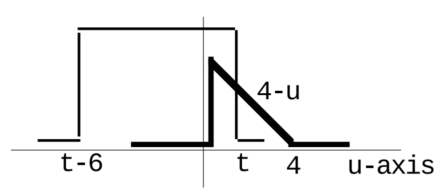
case 1 $\boxed{t \leq 0}$

$y(t) = 0$



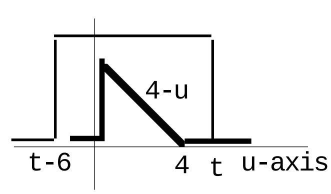
case 2 $\boxed{0 \leq t \leq 4}$

$$y(t) = \int_{u=0}^{t} 8(4-u)\,du = -4t^2 + 32t$$

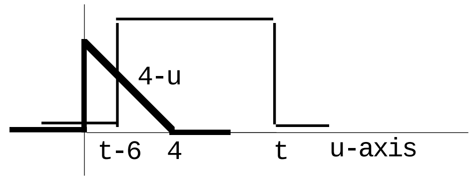


case 3 $t-6 \leq 0$, $t \geq 4$
i.e., $\boxed{4 \leq t \leq 6}$

$$y(t) = \int_{u=0}^{4} 8(4-u)\,du = 64$$

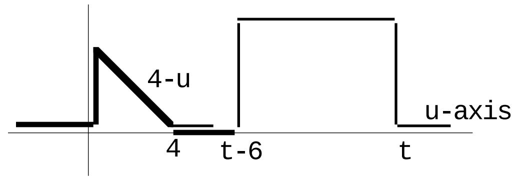


case 4 $0 \le t-6 \le 4$, i.e., $\boxed{6 \le t \le 10}$



$$y(t) = \int_{u=t-6}^{4} 8(4-u)\,du = 4t^2 - 80t + 400$$

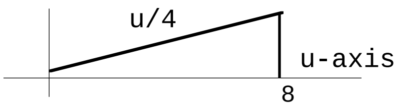
case 5 $t-6 \ge 4$, i.e., $\boxed{t \ge 10}$



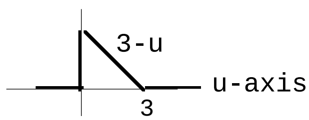
$y(t) = 0$ (no more overlap)

5. If the system has impulse response f(t) then input g(t) produces output c(t).
If the system has impulse response g(t) then input f(t) produces output c(t).

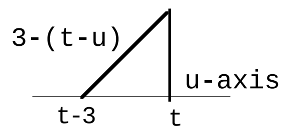
6. The response is $y(t) = ht)*f(t)$. I'll use the version $\int_{-\infty}^{\infty} h(t-u)f(u)\,du$



Graph of f(u)



Graph of h(u)



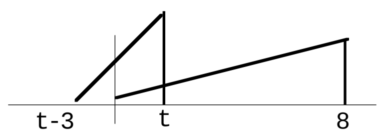
Graph of h(t−u)

case 1 $\boxed{t \le 0}$

$y(t) = 0$ (no overlap yet)

case 2 $t \ge 0$ and $t-3 \le 0$, i.e.,

$\boxed{0 \le t \le 3}$

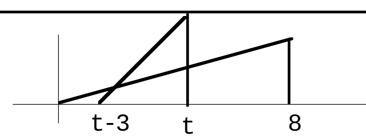


$$y(t) = \int_{u=0}^{t} \frac{1}{4} u\,(3+u-t)\,du = \frac{1}{4}\left(\frac{3}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{2}tu^2\right)\Bigg|_{u=0}^{u=t} = \frac{3}{8}t^2 - \frac{1}{24}t^3$$

warning The integrand contains $f(\boxed{u}) = \frac{1}{4}\boxed{u}$, *not* $\frac{1}{4}t$

case 3 $t \le 8$ and $t-3 \ge 0$, i.e.,

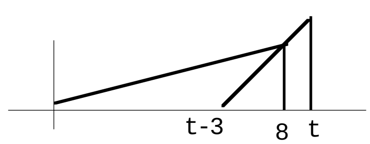
$\boxed{3 \le t \le 8}$



$$y(t) = \int_{u=t-3}^{t} \frac{1}{4} u\,(3+u-t)\,du = \frac{1}{4}\left(\frac{3}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{2}tu^2\right)\Bigg|_{u=t-3}^{u=t} = \frac{9}{8}t - \frac{9}{8}$$

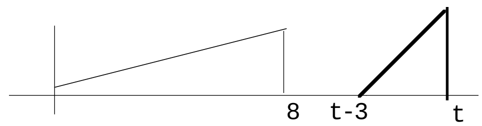
case 4 $t \ge 8$ and $t-3 \le 8$, i.e.,

$\boxed{8 \le t \le 11}$



$$y(t) = \int_{u=t-3}^{8} \frac{1}{4} u\,(3+u-t)\,du = \frac{1}{4}\left(\frac{3}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{2}tu^2\right)\Bigg|_{u=t-3}^{8}$$

case 5 $t-3 \geq 8$, i.e., $\boxed{t \geq 11}$



$y(t) = 0$

7. (a) Response is $f(t)*h(t)$ or equivalently $h(t)*f(t)$.

(b) By superposition, the solution to
$2y'' + 8y' + 6y = f(t)$ with IC $y(0) = 7$, $y'(0) = 9$

is the sum of the solutions to the following two problems:

(1) $2y'' + 8y' + 6y = f(t)$ with IC $y(0) = \text{ZERO}$, $y'(0) = \text{ZERO}$

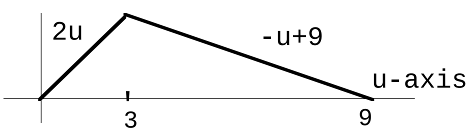
(2) $2y'' + 8y' + 6y = \text{ZERO}$ with IC $y(0) = 7$, $y'(0) = 9$

Solution to (1) is h(t){*f(t).

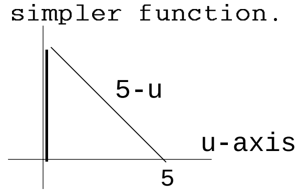
To get solution to (2):
$2m^2 + 8m + 7 = 0$, $m = -1, -3$, $y_h = Ae^{-t} + Be^{-3t}$. To get the IC, need
$A + B = 7$, $-A - 3B = 9$, $A = 15$, $B = -8$, $y = 15e^{-t} - 8e^{-3t}$

Final answer is h(t){*f(t) + $15e^{-t} - 8e^{-3t}$

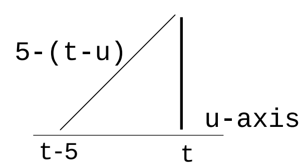
8. (a) I'll flip p because it's the simpler function.



Graph of q(u)



Graph of p(u)

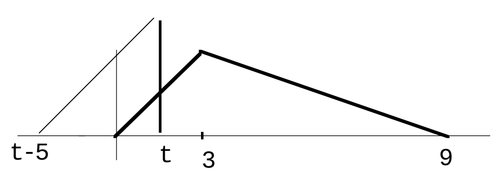


Graph of p(t−u)

case 1 $\boxed{t \leq 0}$
$p*q(t) = 0$ (no overlap yet)

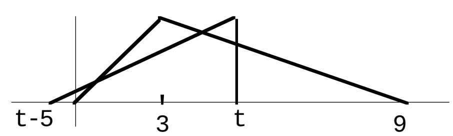
case 2 $\boxed{0 \leq t \leq 3}$

$$p*q(t) = \int_{u=0}^{t} 2u(u - t + 5)\, du$$
$$= \int_{u=0}^{t} 2u^2\, du + (5-t)\int_{u=0}^{t} 2u\, du$$
$$= 5t^2 - \frac{1}{3}t^3$$



case 3 $t-5 \leq 0$ and $t \geq 3$, i.e.,

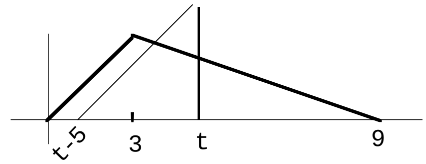
$\boxed{3 \leq t \leq 5}$



$$p*q(t) = \int_{u=0}^{3} 2u(u-t+5)\, du + \int_{3}^{t} (-u+9)(u-t+5)\, du$$

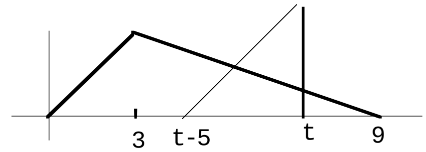
case 4 $0 \le t-5 \le 3$, i.e., $\boxed{5 \le t \le 8}$

$$p*q(t) = \int_{u=t-5}^{3} 2u(u-t+5)\,du + \int_{3}^{t} (-u+9)(u-t+5)\,du$$



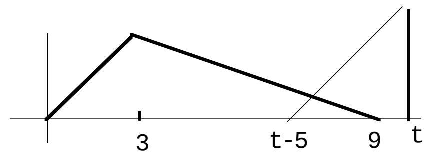
case 5 $t-5 \ge 3$ and $t \le 9$, i.e., $\boxed{8 \le t \le 9}$

$$p*q(t) = \int_{u=t-5}^{t} (-u+9)(u-t+5)\,du$$



case 6 $t-5 \le 9$, $t \ge 9$, i.e., $\boxed{9 \le t \le 14}$

$$p*q(t) = \int_{u=t-5}^{9} (-u+9)(u-t+5)\,du$$



case 7 $t-5 \ge 9$, i.e., $\boxed{t \ge 14}$ No more overlap

$p*q(t) = 0$

(b) $p(t)$ is the response of the system to the input $\delta(t)$ when the system is initially at rest.
(c) $p(t)*q(t)$ is the response of the system to input $q(t)$ if the system is initially at rest.

9. Let $v = t-u$, $dv = -du$. Then

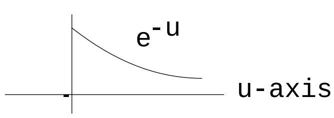
$$\int_{u=-\infty}^{\infty} h(t-u)\, f(u)\, du = \int_{v=\infty}^{-\infty} h(v)\, f(t-v) \cdot -dv$$

$$= \int_{v=-\infty}^{\infty} h(v)\, f(t-v)\, dv$$

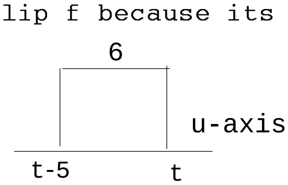
(reversing the limits of integration changes the sign of the integral)

$$= \int_{u=-\infty}^{\infty} h(u)\, f(t-u)\, du$$ (change from the dummy variable v to dummy variable u)

10. (a) The response is $f(t)*h(t)$. I'll flip f because its the simpler function.



Graph of h(u)



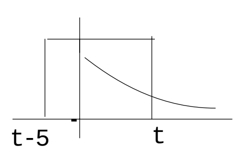
Graph of f(t−u)

case 1 $\boxed{t \le 0}$

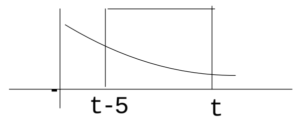
$f(t)*h(t) = 0$ (no overlap yet)

case 2 $t-5 \le 0$, $t > 0$, i.e., $\boxed{0 \le t \le 5}$

$$f(t)*h(t) = \int_{u=0}^{t} 6e^{-u}\,du = 6 - 6e^{-t}$$



case 3 $t - 5 \geq 0$, i.e., $\boxed{t \geq 5}$

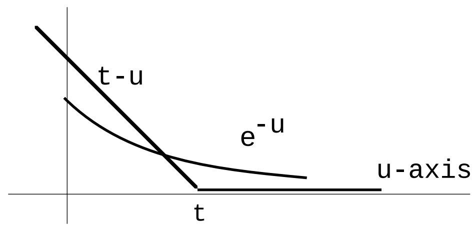


$$f(t)*h(t) = \int_{u=t-5}^{t} 6e^{-u}\,du$$
$$= -6e^{-t} + 6e^{5-t}$$

(b) The response to $\delta(t)$ is the given impulse response $h(t)$. That's the whole point of the impulse response.

11.(a) I flipped f.

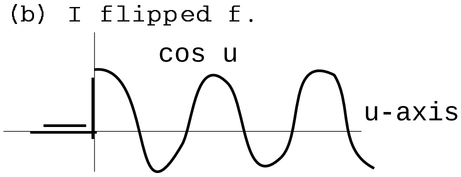
If $t \leq 0$ then $f(t)*h(t) = 0$



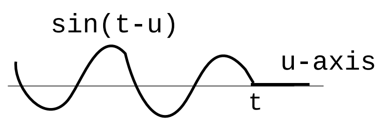
Suppose $t \geq 0$.
Then

$$f(t)*g(t) = \int_{u=0}^{t} (t-u)\,e^{-u}\,du$$
$$= \Big[-te^{-u} - (-ue^{-u} - e^{-u})\Big]_{u=0}^{t}$$
$$= e^{-t} - 1 + t$$

(b) I flipped f.



Graph of g(u)

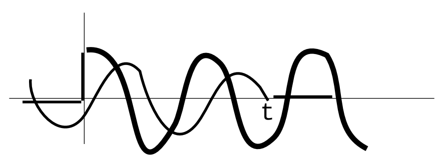


Graph of f(t−u)

If $t \leq 0$ then $f(t)*h(t) = 0$.

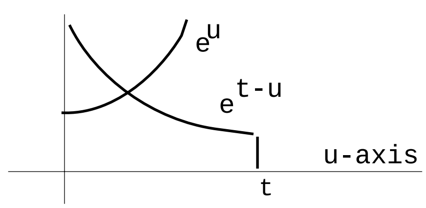
Suppose $t \geq 0$ Then

$$f(t)*g(t) = \int_{u=0}^{t} \sin(t-u)\cos u\,du$$
$$= \frac{1}{2}\int_{0}^{t} [\sin t + \sin(t-2u)]\,du$$
$$= \frac{1}{2}\left[\;u\sin t + \frac{1}{4}\cos(t-2u)\;\right]\Bigg|_{u=0}^{t}$$
$$= \frac{1}{2}\,t\sin t$$



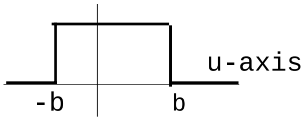
12.(a)
If $t \leq 0$ then $f*f(t) = 0$

Suppose $t \geq 0$. Then

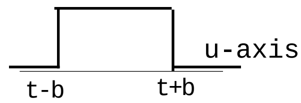


$$f(t)*f(t) = \int_{u=0}^{t} e^{t-u}\,e^{u}\,du$$
$$= e^{t}\int_{0}^{t} du = te^{t}$$

(b)



Graph of f(u)



Graph of f(t−u)

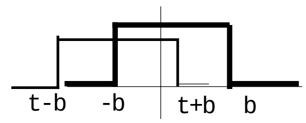
case 1 $t + b < -b$, i.e., $\boxed{t \le -2b}$

$f*f = 0$

No overlap yet

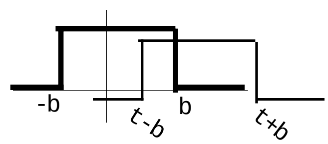
case 2 $-b \le t+b \le b$, i.e., $\boxed{-2b \le t \le 0}$

$$f(t)*f(t) = \; = \int_{u=-b}^{t+b} a^2 \, du = a^2(t + 2b)$$



case 3 $-b \le t-b \le b$, i.e., $\boxed{0 \le t \le 2b}$

$$f(t)*f(t) = \int_{u=t-b}^{b} a^2 \, du = a^2(2b-t)$$

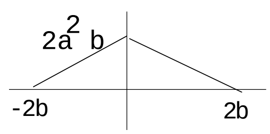


case 4 $t-b \ge b$, i.e., $\boxed{t \ge 2b}$

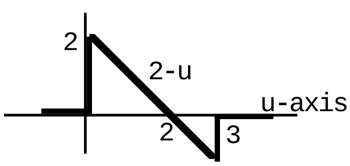
$f(t)*f(t) = 0$

No more overlap

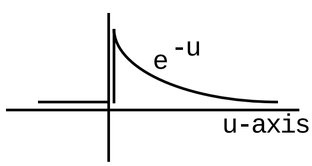
Here's the graph of $f(t)*f(t)$



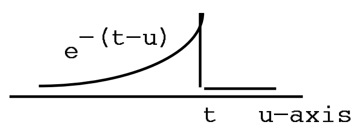
13. The function in the diagram is the impulse response so call it h(t). The response to f(t) is f(t)*h(t).



Graph of h(u)



Graph of f(u)



Graph of f(t−u)

case 1 $\boxed{t \le 0}$

$f(t)*h(t) = 0$

No overlap

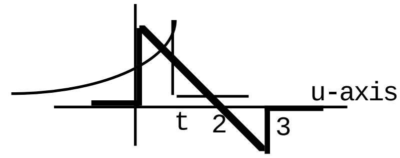
case 2 $\boxed{0 \le t \le 3}$

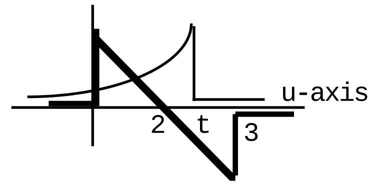
$$f(t)*h(t) = \int_0^t e^{u-t}\,(2-u)\,du$$

$$= e^{-t}\int_0^t (2e^u - ue^u)\,du$$

$$= e^{-t}(2e^u - ue^u + e^u)\Big|_{u=0}^{t}$$

$$= 3 - t - 3e^{-t}$$



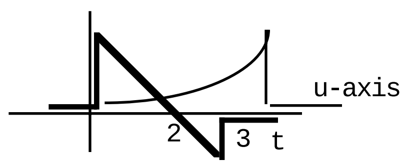


Note You don't need separate cases for $0 \le t \le 2$ and then $2 \le t \le 3$. No matter which of the two pictures you look at you still get the integral. The fact that the $h(u)$ graph goes below the axis at $u=2$ is irrelevant. What is relevant is when a function changes formula, which $h(u)$ does at $u=0$ and then again at $u=3$ but not at $u=2$.

If you do use the two cases $0 \le t \le 2$ and then $2 \le t \le 3$ you will get the *same* integral and the same final answer for each case which is the signal that you only needed one case in the first place.

case 3 $\boxed{t \ge 3}$

$$f(t)*h(t) = \int_{u=0}^{3} e^{u-t}\,(2-u)\,du$$



$$= e^{-t}(2e^u - ue^u + e^u)\Big|_{u=0}^{3}$$

$$= -3e^{-t}$$

Steady state solution is 0 since $-3e^{-t} \to 0$ as $t \to \infty$.

HONORS

14. There are two ways to think of the convolution $\delta(t)*g(t)$ (see problem 5). So there are two ways to do this problem.

method 1 (easier)

$\delta(t)*g(t)$ is the response of an initially-at-rest system to input $\delta(t)$ provided the system has inpulse response $g(t)$.

But the response of an initially-at-rest system to input $\delta(t)$ *is* the impulse response.

So $\delta(t)*g(t) = g(t)$.

(Makes your head spin a little.)

OMIT THIS SECOND EXPLANATION.

Here's the same explanation, phrased slightly differently.

Look at this question.

A system has impulse response $g(t)$.
Find the response of the system to $\delta(t)$.

The answer is $\delta(t)*g(t)$.

But the answer is also $g(t)$ because the response to $\delta(t)$ *is* the impulse response.

So the two "answers" $\delta(t)*g(t)$ and $g(t)$ must agree. So $\delta(t)*g(t) = g(t)$.

Footnote For the operation "convolution", the unit impulse function $\delta(t)$ is the identity element. Just as for the operation "ordinary multiplication", the number 1 is the identity element.

method 2

Think of $\delta(t)*g(t)$ as the response of an initially-at-rest system to input $g(t)$ provided the system has impulse response $\delta(t)$ (and, as usual, the system satisfies superposition and is time invariant).

Since the system has impulse response $\delta(t)$ this means that the input $\delta(t)$ produces the output $\delta(t)$ (it's a copy-cat system so far).

And the response to $\delta(t-4)$ is $\delta(t-4)$ (see "response to a delayed impulse" in Section 2.1).

And by superposition, if the input is $2\delta(t)$ then the output is $2\delta(t)$.

So any kind of input delta produces the same output delta (more copy-cat).

Now what about the output of this system when $g(t)$ is the input (because that's what the problem says to find).

You can think of $g(t)$ as a sum of various deltas (Fig A) (delayed and not necessarily unit deltas but it doesn't matter).

Each of the various delta inputs produces a copy of itself as an output.

By superposition, if you put in the sum of various deltas you get out a sum of the same deltas, which is $g(t)$ again. So when you input $g(t)$, you get out $g(t)$; the system copy-cats every input, not just deltas.

So $\delta(t)*g(t)$ [the response of the system to $g(t)$] $= g(t)$.

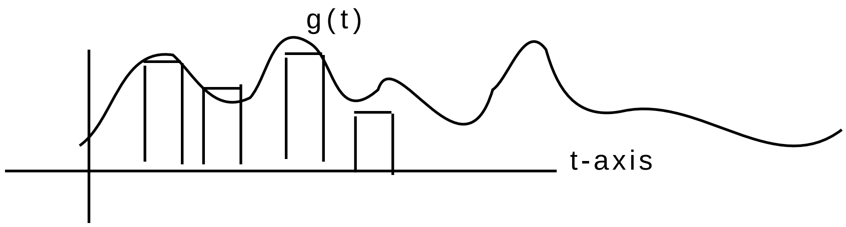


FIG A

15. *method 1* The input f(t) can be thought of as a sum of delta functions. The typical delta function in the sum occurs at time t=u, has width du, area f(u)du and is named f(u) du δ(t−u) (Fig A).

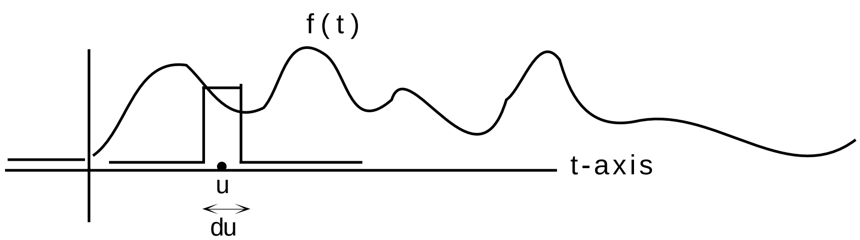


FIG A

So f(t) is a sum (integral) of f(u)du δ(t−u)'s.

If the response of the system to δ(t) is δ(t) then the response to δ(t−u) is δ(t−u) (see "response to a delayed impulse" in Section 2.1).

If the response to δ(t−u) is δ(t−u) then the response to f(u)du δ(t−u) is f(u)du δ(t−u) by superposition.

And the response to the *sum* of f(u)du δ(t−u)'s is that same sum of f(u)du δ(t−u)'s (more superposition). But the sum *is* f(t). So the response to f(t) is f(t). QED

method 2

The response to f(t) is h(t)*f(t) which in this case is δ(t)*f(t).
Use the previously proved fact (method 1) (or prove it again using transforms) that $\delta(t)*f(t) = f(t)$

16. (a) Solve $2y'' + 8y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$.
To do this, switch to

$$2y'' + 8y = 0 \text{ with IC } y(0) = 0,\ y'(0) = 1/2$$

$$2m^2 + 8 = 0,\ m^2 = -4,\ m = \pm 2i$$

$$y = A \cos 2t + B \sin 2t$$

To get $y(0) = 0$ you need $0 = A$
Then $y'(t) = 2B \cos 2t$

To get $y'(0) = 1/2$ you need $1/2 = 2B$, $B = 1/4$

Impulse response is $h(t) = \frac{1}{4} \sin 2t$

(b) The convolution $\frac{1}{4} \sin 2t * \dfrac{t^5 \tan t}{1 + t^2}$ is the solution to

$$2y'' + 8y = \frac{t^5 \tan t}{1 + t^2} \quad \text{with IC } y(0) = \text{ZERO},\ y'(0) = \text{ZERO}.$$

Use superposition to get the solution to

$$2y'' + 8y = \frac{t^5 \tan t}{1 + t^2} \quad \text{with IC } y(0) = \text{TWO},\ y'(0) = \text{THREE}$$

Add to the convolution the solution to

$2y'' + 8y = \text{ZERO}$ with IC $y(0) = \text{TWO}$, $y'(0) = \text{THREE}$

$y_{gen} = A \cos 2t + B \sin 2t$
To get $y(0) = 2$ you need $A = 2$.
Then $y' = -4 \sin 2t + 2B \cos 2t$.
To get $y'(0) = 3$ you need $2B = 3$, $B = 3/2$.
Solution here is $y = 2 \cos 2t + \frac{3}{2} \sin 2t$.

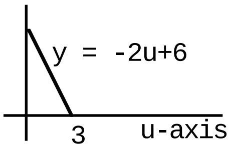
Final answer is $\frac{1}{4} \sin 2t * \frac{t^5 \tan t}{1 + t^2} + 2 \cos 2t + \frac{3}{2} \sin 2t$

SOLUTIONS review problems for Chapter 2

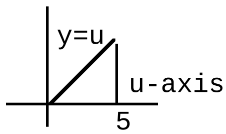
1.(a) Solve $4y'' + y = 0$ with IC $y(0) = 0$, $y'(0) = 1/4$.
$y = A \cos t/2 + B \sin t/2$.
The IC make $A = 0$, $B = \frac{1}{2}$ so $h(t) = \frac{1}{2} \sin t/2$ for $t \geq 0$

(b) $$h(t-11) = \begin{cases} 0 & \text{for } t \leq 11 \\ \frac{1}{2} \sin \frac{1}{2}(t-11) & \text{for } t \geq 11 \end{cases}$$

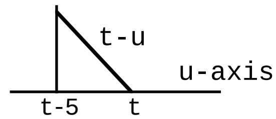
2. The response is $h(t)*f(t)$. I'll use the version $\int_{-\infty}^{\infty} h(u)\, f(t-u)\, du$.



Graph of h(u)



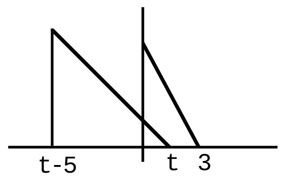
Graph of f(u)



Graph of f(t−u)

case 1 $\boxed{t \leq 0}$ $h(t)*f(t)) = 0$

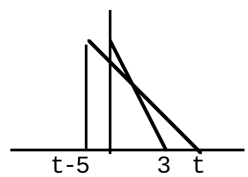
case 2 $\boxed{0 \leq t \leq 3}$



$$h(t)*f(t) = \int_{u=0}^{t} (-2u+6)(t-u)\, du = \left(-tu^2 + 6tu + \frac{2}{3}u^3 - 3u^2\right)\Bigg|_{u=0}^{u=t}$$

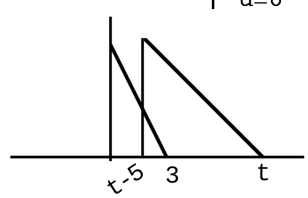
$$= 3t^2 - \frac{1}{3} t^3$$

case 3 $t \geq 3$ and $t-5 \leq 0$, i.e. $\boxed{3 \leq t \leq 5}$



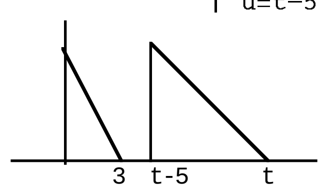
$$h(t)*f(t) = \int_{u=0}^{3} (-2u+6)(t-u)\, du = \left(-tu^2 + 6tu + \frac{2}{3}u^3 - 3u^2\right)\Bigg|_{u=0}^{u=3} = 9t - 9$$

case 4 $0 \leq t-5 \leq 3$, i.e., $\boxed{5 \leq t \leq 8}$



$$h(t)*f(t) = \int_{u=t-5}^{3} (-2u+6)(t-u)\, du = \left(-tu^2 + 6tu + \frac{2}{3}u^3 - 3u^2\right)\Bigg|_{u=t-5}^{u=3}$$

case 5 $t-5 \geq 3$, i.e., $\boxed{t \geq 8}$



No more overlap. $h(t)*f(t) = 0$

SOLUTIONS Section 3.1

1. $y_2 = y_1 + y_0 = 1 + 0 = 1$, $y_3 = y_2 + y_1 = 1 + 1 = 2$,

$y_4 = y_3 + y_2 = 2 + 1 = 3$, $y_5 = y_4 + y_3 = 3 + 2 = 5$

2. $S_{n+1} = S_n + n+1$ or equivalently $S_n = S_{n-1} + n$ with IC $S_1 = 1$

3. Need 3 terms, say y_1, y_2, y_3 to get started. Then $y_4 = \dfrac{2y_2 - 3y_1}{6}$,

$y_5 = \dfrac{2y_3 - 3y_2}{6}$ etc. The rr is 3-rd order and can be rewritten as

$6y_{n+3} - 2y_{n+1} + 3y_n = 0$

4. Set n = 1. Then $y_3 = 1^3 + 2y_1 = 1 + 2\cdot 2 = 5$

Set n = 2. Then $y_4 = 2^3 + 2y_2 = 8 + 2\cdot -3 = 2$

Set n = 3. Then $y_5 = 3^3 + 2y_3 = 27 + 2\cdot 5 = 37$

5. LHS = $(n + 2)\ 2^{n+2} - 4(n+1)\ 2^{n+1} + 4n\ 2^n = (n+2)\ 2^n\ 2^2 - 4(n+1)2^n\cdot 2 + 4n\ 2^n$

$= (4n+8)\ 2^n - (8n+8)\ 2^n + 4n\ 2^n = 0$ QED

6. (a) $3y_{n+4} + 5y_{n+1} - 2y_n = \sin \pi n + \sin \pi n = 2 \sin \pi n$

(b) $3y_{n+4} + 5y_{n+1} - 2y_n = 3 \sin \pi n$

(c) $3y_{n+4} + 5y_{n+1} - 2y_n = \sin \pi n - \sin \pi n = 0$

7. All are solutions to $ay_{n+2} + by_{n+1} + cy_n = 0$

8. The rr is always supposed to say

term = preceding term + (the number of the term)^2

(a) $S_n = S_{n-1} + n^2$

(b) $S_{n+6} = S_{n+5} + (n+6)^2$

SOLUTIONS Section 3.2

1. (a) $\lambda^2 - 3\lambda - 10 = 0$, $\lambda = -2, 5$, $y_n = A(-2)^n + B\,5^n$

(b) $\lambda^2 + 3\lambda - 4 = 0$, $\lambda = 1, -4$, $y_n = A + B(-4)^n$

(c) $2\lambda^2 + 2\lambda - 1 = 0$, $\lambda = \dfrac{-1 \pm \sqrt{3}}{2}$

$$y_n = A\left[\frac{-1+\sqrt{3}}{2}\right]^n + B\left[\frac{-1-\sqrt{3}}{2}\right]^n$$

(d) same as (b)

2. $\lambda^2 + 2\lambda - 15 = 0$, $\lambda = -5, 3$, gen $y_n = A(-5)^n + B\,3^n$

Need $A + B = 0$, $-5A + 3B = 1$. So $A = -\frac{1}{8}$, $B = \frac{1}{8}$. Answer is $-\frac{1}{8}(-5)^n + \frac{1}{8}\cdot 3^n$

3.(a) $y_{n+2} = y_{n+1} + 6y_n$ so $y_2 = y_1 + 6y_0 = 0 + 6\cdot 1 = 6$

$y_3 = y_2 + 6y_1 = 6 + 6\cdot 0 = 6$

(b) $\lambda^2 - \lambda - 6 = 0$, $\lambda = 3, -2$. Gen sol is $y_n = A\,3^n + B(-2)^n$

Need $1 = A + B$, $0 = 3A - 2B$ so $A = \frac{2}{5}$, $B = \frac{3}{5}$. Sol is $y_n = \frac{2}{5}\cdot 3^n + \frac{3}{5}(-2)^n$

(c) $y_3 = \frac{2}{5}\cdot 3^3 + \frac{3}{5}(-2)^3 = 6$

4. (a) $\lambda^2 + 2\lambda + 2 = 0$, $\lambda = -1 \pm i$. The number $-1 + i$ has mag $\sqrt{2}$ and angle $\frac{3\pi}{4}$ so

$y_n = (\sqrt{2})^n \left(A \cos \frac{3n\pi}{4} + B \sin \frac{3n\pi}{4}\right)$

(b) $\lambda^2 + \lambda + 1 = 0$, $\lambda = \dfrac{-1 \pm i\sqrt{3}}{2}$. Mag of $\dfrac{-1 + i\sqrt{3}}{2}$ is 1, angle is $\frac{2\pi}{3}$ so

$y_n = A \cos \frac{2n\pi}{3} + B \sin \frac{2n\pi}{3}$

5. $\lambda = -2 \pm 2i$. The number $-2 + 2i$ has mag $\sqrt{8}$ and angle $\frac{3\pi}{4}$ so

$y_n = (\sqrt{8})^n \left(A \cos \frac{3n\pi}{4} + B \sin \frac{3n\pi}{4}\right)$

To get $y_0 = 0$ need $A = 0$. To get $y_1 = 2$ need $2 = \sqrt{8}\,B\cdot\frac{1}{2}\sqrt{2}$, $B = 1$.

So $y_n = (\sqrt{8})^n \sin \frac{3n\pi}{4}$ and $y_{102} = (\sqrt{8})^{102} \sin \frac{153\pi}{2} = (\sqrt{8})^{102} = 8^{51}$

6.(a) $-\sqrt{3}+i$ has mag 2 and angle $\frac{5\pi}{6}$.

$$y_n = A(-3)^n + B\,4^n + Cn\,4^n + 2^n\left(D\cos\frac{5\pi n}{6} + E\sin\frac{5\pi n}{6}\right)$$

(b) 2i has mag 2 and angle $\pi/2$.

$$y_n = A + B\,2^n + C(-2)^n + D\,3^n + 2^n\left(E\cos\frac{n\pi}{2} + F\sin\frac{n\pi}{2}\right) + n2^n\left(G\cos\frac{n\pi}{2} + H\sin\frac{n\pi}{2}\right)$$

7. $\lambda = \frac{1\pm\sqrt{5}}{2}$, gen sol is $y_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$.

To get the IC we need $A+B=0$, $\frac{1}{2}A(1+\sqrt{5}) + \frac{1}{2}B(1-\sqrt{5}) = 1$, $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$.

Answer is $y_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

8.(a) $y_1 = 2$, $y_2 = 5$ (or you can begin with $y_0 = 2$, $y_1 = 5$. Makes no difference)
Then $y_3 = \frac{2+5}{2} = \frac{7}{2}$, $y_4 = \frac{1}{2}(5+\frac{7}{2}) = \frac{17}{4}$, $y_5 = \frac{1}{2}(\frac{7}{2}+\frac{17}{4}) = \frac{31}{8}$

(b) $y_{n+2} = \frac{y_{n+1}+y_n}{2}$, $2y_{n+2} - y_{n+1} - y_n = 0$,

$\lambda = -\frac{1}{2}, 1$; general sol is $y_n = A\left(-\frac{1}{2}\right)^n + B$

Plug in IC $y_1 = 2$, $y_2 = 5$ to get $2 = -\frac{1}{2}A + B$, $5 = \frac{1}{4}A + B$.

So $B = 4$, $A = 4$ and answer is $y_n = 4\left(-\frac{1}{2}\right)^n + 4$.

(c) $y_5 = 4(-\frac{1}{2})^5 + 4 = \frac{31}{8}$

9.(a) $y_n = A(-3)^n + B\,4^n + Cn\,4^n$

(b) $y_n = A\,5^n + Bn\,5^n + Cn^2\,5^n + Dn^3\,5^n + E\,2^n$

(c) $y_n = A + Bn + Cn^2 + D\,6^n + E(-7)^n$

10. $\lambda = -3, -3$, $y_n = A(-3)^n + Bn(-3)^n$

11. $\lambda = 1,1,2$, $(\lambda-1)^2(\lambda-2) = 0$, $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$.

rr is $y_{n+3} - 4y_{n+2} + 5y_{n+1} - 2y_n = 0$.

12. $(\lambda - 1)^3 = 0$, $\lambda = 1,1,1$. Gen sol is $y_n = A + Bn + Cn^2$.

Need $A + B + C = 0$, $A + 2B + 4C = 1$, $A + 3B + 9C = 0$. So $A = -3$, $B = 4$, $C = -1$.

Sol is $y_n = -3 + 4n - n^2$.

13. (a) *by inspection* If $y_1 = 4$ and $y_{n+1} = y_n$ then $y_2 = 4$, $y_3 = 4$ and in general, $y_n = 4$ for every n.

overkill $\lambda = 1$. General sol is $y_n = A$. Plug in the IC $y_1 = 4$ to get $A = 4$. Sol is $y_n = 4$.

(b) *by inspection* If $y_0 = 0$, $y_1 = 0$ and thereafter $y_{n+2} = -\frac{b}{a}\,y_{n+1} - \frac{c}{a}\,y_n$ then every term is 0, i.e., $y_n = 0$ for all n.

overkill Say the roots of the auxiliary equ are λ_1 and λ_2. Then the general sol is $A\,\lambda_1{}^n + B\,\lambda_2{}^n$ (assuming $\lambda_1 \neq \lambda_2$ which is another story). Plug in the IC and we get $0 = A + B$, $0 = A\,\lambda_1 + B\,\lambda_2$.
The only solution for A and B is $A = 0$, $B = 0$ which makes $y_n = 0$.

14. (a) Sequence is 5,7,9,11,13,... Pattern is $y_n = 2n + 3$.

(b) $y_{n+1} = \dfrac{y_{n+2} + y_n}{2}$, $y_{n+2} - 2y_{n+1} + y_n = 0$, $\lambda = 1,1$, $y_n = A + Bn$.

To get the IC $y_1 = 5$, $y_2 = 7$ need $A + B = 5$, $A + 2B = 7$.
So $A = 3$, $B = 2$. Answer is $y_n = 3 + 2n$.

SOLUTIONS Section 3.3

1.(a) $y_1 = 2$, $y_2 = 2y_1 + 6\cdot 2 = 4 + 12 = 16$, $y_3 = 2y_2 + 6\cdot 3 = 50$,

$y_4 = 2y_3 + 6\cdot 4 = 124$

(b) $\lambda = 2$, $h_n = A\ 2^n$. Try $p_n = Bn + C$. Need

$$Bn + C - 2\Big[\ B(n-1) + C\Big] = 6n$$

Equate n coeffs $\quad - B = 6$
Equate constant terms $\quad 2B - C = 0$

$B = -6$, $C = -12$, $y_n = A\ 2^n - 6n - 12$

The IC makes $2 = 2A - 6 - 12$, $A = 10$. Answer is $y_n = 10\cdot 2^n - 6n - 12$

(c) $y_4 = 10\cdot 16 - 6\cdot 4 = 124$

2.(a) $\lambda = 2,-1$, $h_n = A\ 2^n + B(-1)^n$. Try $p_n = C$.

Then $p_{n+1} = C$, $p_{n+2} = C$. Substitute into rr to get $C - C - 2C = 1$, $C = -\frac{1}{2}$

General sol is $y_n = A\ 2^n + B(-1)^n - \frac{1}{2}$

Plugging in the IC makes $1 = 2A - B - \frac{1}{2}$, $\quad 3 = 4A + B - \frac{1}{2}$

So $A = \frac{5}{6}$, $B = \frac{1}{6}$. Answer is $y_n = \frac{5}{6}\cdot 2^n + \frac{1}{6}(-1)^n - \frac{1}{2}$

(b) $\lambda = -5,3$, $h_n = A\ 3^n + B(-5)^n$. Try $p_n = Cn + D$

Need $C(n+2) + D + 2\ [\ C(n+1) + D] - 15(Cn + D) = 6n + 10$

Equate n coeffs $\quad -12C = 6$, $C = -\frac{1}{2}$

Equate constant terms $\quad 4C - 12D = 10$, $D = -1$

General sol is $y_n = A\ 3^n + B(-5)^n - \frac{1}{2}n - 1$

The IC make $A = \frac{11}{8}$, $B = \frac{5}{8}$, Answer is $y_n = \frac{11}{8}\cdot 3^n + \frac{5}{8}(-5)^n - \frac{1}{2}n - 1$

3. $\lambda = \frac{3 \pm\sqrt{5}}{2}$, $h_n = A\left(\frac{3+\sqrt{5}}{2}\right)^n + B\left(\frac{3-\sqrt{5}}{2}\right)^n$

Try $p_n = C\ 4^n$. Need

$$C\ 4^{n+2} - 3C\ 4^{n+1} + C\ 4^n = 10\cdot 4^n$$

$$16C\ 4^n - 12C\ 4^n + C\ 4^n = 10\cdot 4^n$$

$$5C = 10,\quad C = 2$$

Gen sol is $y_n = A\left(\frac{3+\sqrt{5}}{2}\right)^n + B\left(\frac{3-\sqrt{5}}{2}\right)^n + 2\cdot 4^n$

4. $\lambda = 3, -2$, $h_n = D\,3^n + E(-2)^n$. Try $p_n = An^2 + Bn + C$. Need

$$A(n+2)^2 + B(n+2) + C - \Big(A(n+1)^2 + B(n+1) + C\Big) - 6(An^2 + Bn + C) = 18n^2 + 2$$

Match n^2 coeffs $\quad -6A = 18,\ A = -3$
Match n coeffs $\quad 2A - 6B = 0,\quad B = -1$
Match constant terms $\quad 3A + B - 6C = 2,\ C = -2$

Gen sol is $y_n = D\,3^n + E(-2)^n - 3n^2 - n - 2$

The IC make $D = \frac{8}{5}$, $E = -\frac{3}{5}$ Answer is $y_n = \frac{8}{5}\cdot 3^n - \frac{3}{5}(-2)^n - 3n^2 - n - 2$

5.(a) *method 1* Switch to $y_{n+2} - 2y_n = 5e^{n\pi i}$

Need the homog sol to see if I need to step up. $\lambda = \pm\sqrt{2}$, $h_n = A(\sqrt{2})^n + B(-\sqrt{2})^n$.
No interference. No stepping up,

Try $p_n = De^{n\pi i}$. Need

$$De^{(n+2)\pi i} - 2De^{n\pi i} = 5e^{n\pi i}$$

$$De^{2\pi i}e^{n\pi i} - 2De^{n\pi i} = 5e^{n\pi i}$$

$$De^{n\pi i} - 2De^{n\pi i} = 5e^{n\pi i} \quad \text{because } e^{2\pi i} = 1$$

So $D - 2D = 5$, $D = -5$. Switched $p_n = -5e^{n\pi i} = -5(\cos n\pi + i \sin n\pi)$.

Original p_n = real part = $-5 \cos n\pi$

method 2 Try $p_n = A \sin n\pi + B \cos n\pi$. Need

$$A \sin \pi(n+2) + B \cos \pi(n+2) - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi$$

$$A \sin (n\pi + 2\pi) + B(\cos(n\pi + 2\pi) - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi$$

$$A \sin n\pi + B \cos n\pi - 2(A \sin n\pi + B \cos n\pi) = 5 \cos n\pi$$

(use identities $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$)

$$-A \sin n\pi - B \cos n\pi = 5 \cos n\pi$$

$-A = 0$, $-B = 5$, $B = -5$, $p_n = -5 \cos n\pi$

(b) Try $p_n = D(-1)^n$ (no need to step up since $h_n = A(\sqrt{2})^n + B(-\sqrt{2})^n$)

Need $D(-1)^{n+2} - 2D(-1)^n = 5(-1)^n$
$D(-1)^2(-1)^n - 2D(-1)^n = 5(-1)^n$

Equate coeffs of $(-1)^n$: $D - 2D = 5$, $D = -5$

$p_n = -5(-1)^n$ which agrees with the solution $-5 \cos n\pi$ from part (a).

(c) *method 1* Switch to

$$y_{n+1} - 2y_n = 10e^{n\pi i/2}$$

and try

$$p_n = D\,e^{n\pi i/2}$$

Need

$$De^{(n+1)\pi i/2} - 2D\,e^{n\pi i/2} = 10e^{n\pi i/2}$$

$$De^{\pi i/2}e^{n\pi i/2} - 2D\,e^{n\pi i/2} = 10e^{n\pi i/2}$$

$$Die^{n\pi i/2} - 2D\,e^{n\pi i/2} = 10e^{n\pi i/2} \quad \text{(because } e^{\pi i/2} = i\text{) (mag 1, angle } \pi/2)$$

$(-2 + i)D = 10, \quad D = -4-2i$

Switched $p_n = (-4-2i)\ e^{n\pi i/2} = (-4-2i)(\cos\frac{n\pi}{2} + i \sin\frac{n\pi}{2})$

Take imag part to get original $p_n = -4 \sin\frac{n\pi}{2} - 2 \cos\frac{n\pi}{2}$

method 2 Try $p_n = A \sin\frac{n\pi}{2} + B \cos\frac{n\pi}{2}$. Need

$$A \sin\frac{(n+1)\pi}{2} + B \cos\frac{(n+1)\pi}{2} - 2\left[A \sin\frac{n\pi}{2} + B \sin\frac{n\pi}{2}\right] = 10 \sin\frac{n\pi}{2}$$

$$A\left[\sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2}\right] + B\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{2} - \sin\frac{n\pi}{2}\sin\frac{\pi}{2}\right]$$
$$-2\left[A \sin\frac{n\pi}{2} + B \cos\frac{n\pi}{2}\right] = 10 \sin\frac{n\pi}{2}$$

$$(-2A - B)\sin\frac{n\pi}{2} + (A-2B)\cos\frac{n\pi}{2} = 10 \sin\frac{n\pi}{2}$$

Match coeffs: $-2A-B = 10$, $A - 2B = 0$.

So $A = -4$, $B = -2$, $p_n = -4 \sin\frac{n\pi}{2} - 2 \cos\frac{n\pi}{2}$

6.(a) Try $p_n = An^4 + Bn^3 + Cn^2 + Dn + E$

(b) Ordinarily you would try $p_n = An^4 + Bn^3 + Cn^2 + Dn + E$ but since E, n, n^2, n^3 are all homog sols, try $p_n = n^4(An^4 + Bn^3 + Cn^2 + Dn + E) = An^8 + Bn^7 + Cn^6 + Dn^5 + En^4$

(c) Try $p_n = An\ 2^n$ (step up bcause 2^n is a homog sol)

(d) Try $p_n = A\ 2^n$

(e) Try $p_n = An^2\ 3^n$ (step up bcause 3^n and $n3^n$ are both homog sols)

(f) One method is to try $p_n = n(A \cos\frac{n\pi}{2} + B \sin\frac{n\pi}{2})$

(Step up because i has mag 1 and angle $\pi/2$ so $\cos\frac{n\pi}{2}$ and $\sin\frac{n\pi}{2}$ are homog sols)

Another method is to switch to forcing function $5e^{n\pi i/2}$, try $p_n = An\ e^{n\pi i/2}$ (step up here too) and eventually take real part

(g) Either try $p_n = A \cos\frac{n\pi}{2} + B \sin\frac{n\pi}{2}$ or switch to the forcing function $5e^{n\pi i/2}$ and try $p_n = A\ e^{n\pi i/2}$ and eventually take real part.

Why not step up here the way you had to in part (f)?

Because 2i has mag 2 and angle $\pi/2$ so the homog sols are $2^n \cos\frac{n\pi}{2}$ and $2^n \sin\frac{n\pi}{2}$

When the forcing function is $5 \cos\frac{n\pi}{2}$ you should step up p_n only if plain $\cos\frac{n\pi}{2}$ is a homog solution as it was in part (f), not if $2^n \cos\frac{n\pi}{2}$ is a homog sol.

7. $\lambda = \frac{1}{2}$, $h_n = A(\frac{1}{2})^n$. Try $p_n = Bn\ (\frac{1}{2})^n$ (step up) Need

$$2B(n+1)\ (\tfrac{1}{2})^{n+1} - Bn\ (\tfrac{1}{2})^n = (\tfrac{1}{2})^n$$

$$2B(n+1)\tfrac{1}{2}\ (\tfrac{1}{2})^n - Bn\ (\tfrac{1}{2})^n = (\tfrac{1}{2})^n$$

$$B\ (\tfrac{1}{2})^n = (\tfrac{1}{2})^n$$

So B = 1. General sol is $y_n = A(\frac{1}{2})^n + n\ (\frac{1}{2})^n$

The IC make A = 3. Answer is $y_n = 3(\frac{1}{2})^n + n(\frac{1}{2})^n$

8. $\lambda = 1,1$, $h_n = A + Bn$. Ordinarily you would try $p_n = C$ but since C and n are both homog sols, step up twice and try $p_n = Cn^2$. Need

$$C(n+2)^2 - 2C(n+1)^2 + Cn^2 = 1$$

The n^2 terms and n terms cancel out.

Equate constant terms: $2C = 1$, $C = \frac{1}{2}$

Gen sol is $y_n = A + Bn + \frac{1}{2}n^2$

The IC make A =1, $\frac{1}{2} = A + B + \frac{1}{2}$. So B = −1. Answer is $y_n = 1 - n + \frac{1}{2}n^2$

9. $S_{n+1} = S_n + (n+1)^2$, $S_{n+1} - S_n = (n+1)^2$ with IC $S_1 = 1$.
$\lambda = 1$, $h_n = D$. Try $p_n = n(An^2 + Bn + C) = An^3 + Bn^2 + Cn$. (Step up because C is a homog sol.) Need

$$A(n+1)^3 + B(n+1)^2 + C(n+1) - (An^3 + Bn^2 + Cn) = n^2 + 2n + 1$$

The n^3 coeffs drop out.

Equate n^2 coeffs $3A = 1$, $A = \frac{1}{3}$

Equate n coeffs $3A + 2B = 2$, $B = \frac{1}{2}$

Equate constant terms $A + B + C = 1$, $C = \frac{1}{6}$

Gen sol is $S_n = D + \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

The IC $S_1 = 1$ makes D = 0. Answer is $S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$, usually written as $S_n = \frac{n(n+1)(2n+1)}{6}$

10.(a) Need $C2^{n+2} - 3C\ 2^{n+1} + 2C\ 2^n = 6\cdot 2^n$, $4C\ 2^n - 6C\ 2^n + 2C\ 2^n = 6\cdot 2^n$,

The C's cancel out leaving $0 = 6\cdot 2^n$ which can't be satisfied. There is no value of C which makes $C\ 2^n$ work.

(b) Need $2A(n+2) + 3A(n+1) + 4An = 18n$, $9An + 7A = 18n$.

Equate n coeffs $9A = 18$, $A = 2$
Equate contant terms $7A = 0$, $A = 0$

Impossible. So there is no solution of the form An.

11. $\lambda = 2$, $h_n = A(-2)^n$. Try $p_n = B + C\,4^n$. Need

$$B + C\,4^{n+1} + 2(B + C\,4^n) = 3 + 4^n$$

Match coeffs $3B = 3$, $6C = 1$

So $B = 1$, $C = \frac{1}{6}$. General sol is $y_n = A(-2)^n + 1 + \frac{1}{6}\cdot 4^n$

IC make $2 = A + 1 + \frac{1}{6}$. So $A = \frac{5}{6}$. Answer is $y_n = \frac{5}{6}(-2)^n + 1 + \frac{1}{6}\cdot 4^n$

12. $\lambda^4 - 16 = 0$ $\lambda^2 = \pm 4$, $\lambda = \pm 2, \pm 2i$. The number $2i$ has mag 2 and angle $\pi/2$ so

$$h_n = A2^n + B(-2)^n + 2^n\left(C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2}\right)$$

Try $p_n = An + B + C\,3^n$. Need

$$A(n+4) + B + C\,3^{n+4} - 16(An + B + C\,3^n) = n + 3^n$$

$$A(n+4) + B + 81\,C\,3^n - 16(An + B + C\,3^n) = n + 3^n$$

Equate n coeffs $-15A = 1$, $A = -\frac{1}{15}$

Equate constant terms $4A - 15B = 0$, $B = -\frac{4}{225}$

Equate 3^n terms $65C = 1$, $C = \frac{1}{65}$

Gen sol is

$$y_n = A2^n + B(-2)^n + 2^n\left(C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2}\right) - \frac{1}{15}n - \frac{4}{225} + \frac{1}{65}3^n$$

13. $\lambda = \frac{1 \pm i\sqrt{3}}{2}$. Mag of $\frac{1 + i\sqrt{3}}{2}$ is 1, angle is $\pi/3$. So

$$h_n = A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}$$

Try $p_n = C\,2^n$ Need

$$C\,2^{n+2} - C\,2^{n+1} + C\,2^n = 2^n$$

$$4C\,2^n - 2C\,2^n + C\,2^n = 2^n$$

$$4C - 2C + C = 1, \quad C = \frac{1}{3}$$

Gen sol is $y = A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3} + \frac{1}{3}\cdot 2^n$

The IC make $1 = A + \frac{1}{3}$, $3 = A\cos\frac{\pi}{3} + B\sin\frac{\pi}{3}$, $A = \frac{2}{3}$, $B = \frac{4}{\sqrt{3}}$

Answer is $y_n = \frac{2}{3}\cos\frac{n\pi}{3} + \frac{4}{\sqrt{3}}\sin\frac{n\pi}{3} + \frac{1}{3}\cdot 2^n$

14. $\lambda = 2, 1$, $h_n = A\,2^n + B$. Try $p_n = (Cn + D)3^n$. Then

$$p_{n+1} = \Big[\, C(n+1) + D \,\Big]\ 3^{n+1} + (Cn + C + D)3 \cdot 3^n$$

$$p_{n+2} = \Big[\, C(n+2) + D \,\Big]\ 3^{n+2} = (Cn + 2C + D)3^2 \cdot 3^n$$

We need

$$(9Cn + 18C + 9D)3^n - 3(3Cn + 3C + 3D)3^n + 2(Cn + D)3^n = 8n3^n$$

$$2Cn\ 3^n + (9C + 2)3^n = 8n\ 3^n$$

$$2C = 8, \quad 9C + 2D = 0$$

$$C = 4, \ D = -18$$

General sol is $y_n = A\,2^n + B + (4n - 18)3^n$

The IC make $-16 = A + B$, $-40 = 2A + B - 42$

So $A = 0$, $B = 2$. Anwer is $y_n = 2 + (4n - 18)\ 3^n$

SOLUTIONS review problems for Chapter 3

1. $\lambda^2 - 9 = 0$, $\lambda = \pm 3$, $h_n = A\,3^n + B(-3)^n$

Try $p_n = Cn^2 + Dn + E$

$C(n+2)^2 + D(n+2) + E - 9(Cn^2 + Dn + E) = 56n^2$

equate n^2 coeffs $\qquad -8C = 56$, $C = -7$
equate n coeffs $\qquad 4C - 8D = 0$, $D = -7/2$
equate constant terms $\qquad 4C + 2D - 8E = 0$, $E = -35/8$

$y_n = h_n + p_n = A3^n + B(-3)^n - 7n^2 - \frac{7}{2}n - \frac{35}{8}$

2. $\lambda = 1 \pm i\sqrt{3}$. The number $1 + i\sqrt{3}$ has mag 2, angle $\pi/3$ so

$y_n = 2^n(A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3})$

3. $\lambda = -2$, $h_n = A(-2)^n$. Try $p_n = B\,7^n$.

Need $2B\,7^{n+1} + 4B\,7^n = 6 \cdot 7^n$

$14B\,7^n + 4B\,7^n = 6 \cdot 7^n$

So $18B = 6$, $B = \frac{1}{3}$, $y_n = A(-2)^n + \frac{1}{3} \cdot 7^n$

To get the IC you need $5 = -2A + \frac{7}{3}$, $A = -\frac{4}{3}$.

Answer is $y_n = -\frac{4}{3}(-2)^n + \frac{1}{3} \cdot 7^n$

4. $\lambda = \frac{-5 \pm \sqrt{29}}{2}$, $h_n = A\left[\frac{-5 + \sqrt{29}}{2}\right]^n + B\left[\frac{-5 - \sqrt{29}}{2}\right]^n$

Try $p_n = D$. Need $D + 5D = 6$, $D = \frac{6}{5}$.
Answer is $y_n = A\left[\frac{-5 + \sqrt{29}}{2}\right]^n + B\left[\frac{-5 - \sqrt{29}}{2}\right]^n + \frac{6}{5}$

5. $S_{n+1} = S_n + n+1$, $S_{n+1} - S_n = n+1$ with IC $S_1 = 1$.

$\lambda = 1$, $h_n = A$. Try $p_n = n(Bn + C)$ (step up) $= Bn^2 + Cn$
Need $B(n+1)^2 + C(n+1) - (Bn^2 + Cn) = n+1$
The n^2 terms drop out on each side
Equate n coeffs $\qquad 2B + C - C = 1$, $B = \frac{1}{2}$

Equate constant terms $\qquad B + C = 1$, $C = \frac{1}{2}$

$S_n = \quad S_n = A + \frac{1}{2}n^2 + \frac{1}{2}n$

To get $S_1 = 1$ you need $1 = A + \frac{1}{2} + \frac{1}{2}$, $A = 0$.

Answer is $S_n = \frac{1}{2}n^2 + \frac{1}{2}n$ usually written as $S_n = \frac{n(n+1)}{2}$

6. $\lambda = \pm 3$, $h_n = A\ 3^n + B(-3)^n$. Try $p_n = Dn\ 3^n$ (step up).

Need $D(n+2)\ 3^{n+2} - 9Dn\ 3^n = 5 \cdot 3^n$

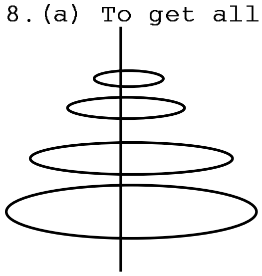
$9D(n+2)\ 3^n - 9Dn\ 3^n = 5 \cdot 3^n$

The $n3^n$ terms drop out.

Match the 3^n coeffs: $18D = 5$, $D = \frac{5}{18}$. Answer is $y_n = A\ 3^n + B(-3)^n + \frac{5}{18}\ n3^n$

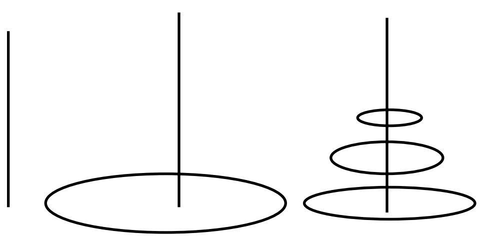
7. The rr can be written as $y_{n+1} - 2y_n = 0$ and it is only *first* order. It would come with only one IC and its general sol (namely $B\ 2^n$) should only have one constant. So nothing is wrong.

8.(a) To get all the rings moved you have to pass through the following stages



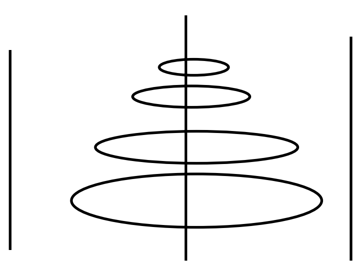
Start here

Move the top n−1 rings from peg 1 to peg 3 using peg 2 as storage.
Takes y_{n-1} moves to do it in the best way



Move the largest ring to peg 2

Takes one move



Move the n−1 rings from peg 3 to peg 2 using peg 1 as storage.

Takes y_{n-1} moves

So $y_n = 2y_{n-1} + 1$.

And since it only takes one move in a 1−ring game the IC is $y_1 = 1$.

(b) $\lambda = 2$, $h_n = A\ 2^n$. Try $p_n = B$.

Need $B - 2B = 1$, $B = -1$

Gen sol is $y_n = A\ 2^n - 1$. To get the IC you need $1 = 2A - 1$, $A = 1$.

Sol is $y_n = 2^n - 1$.

For example to move a 10-ring tower it takes a minimum of $2^{10} - 1$ moves.

SOLUTIONS Section 4.1

1. (a) $y' + \frac{4}{x+2} y = \frac{-6}{(x+2)^2}$, $P(x) = \frac{4}{x+2}$, $\int P(x)\,dx = 4 \ln(x+2) = \ln(x+2)^4$

$I = e^{\ln(x+2)^4} = (x+2)^4$, $(x+2)^4 y = \int -6(x+2)^2\,dx = \frac{-6(x+2)^3}{3} + k$,

answer is $y = \frac{-2}{x+2} + \frac{k}{(x+2)^4}$

(b) $y' + 4xy = x$, $P(x) = 4x$, $\int P(x)\,dx = 2x^2$, $I = e^{2x^2}$,

$ye^{2x^2} = \int xe^{2x^2}\,dx = \frac{1}{4} e^{2x^2} + K$. Answer is $y = \frac{1}{4} + Ke^{-2x^2}$

(c) *method 1* Rewrite the equation as $y' + y = \frac{1}{2} e^{2x}$. Then

$P(x) = 1$, $I = e^x$, $ye^x = \int e^x \frac{1}{2} e^{2x}\,dx = \frac{1}{2} \int e^{3x}\,dx = \frac{1}{6} e^{3x} + K$.

Answer is $y = \frac{1}{6} e^{2x} + Ke^{-x}$.

method 2 (since the coeffs are constant) (no need to rewrite the equation)

$m = -1$, $y_h = Ae^{-x}$.

Try $y_p = Be^{2x}$. Then $4Be^{2x} + 2Be^{2x} = e^{2x}$, $6B = 1$, $B = \frac{1}{6}$

$y_p = \frac{1}{6} e^{2x}$, answer is $y = Ae^{-x} + \frac{1}{6} e^{2x}$

(d) $y' - y \cot x = \csc x$, $P(x) = -\cot x$, $\int P(x)\,dx = -\ln \sin x = \ln \csc x$,

$I = e^{\ln \csc x} = \csc x$, $y \csc x = \int \csc^2 x\,dx = -\cot x + K$

Answer is $y = -\cos x + K \sin x$

2. Rewrite as $x(y')' + y' = 4x$, $(y')' + \frac{1}{x} y' = 4$. Then $\int P(x)\,dx = \ln x$,

$I = e^{\ln x} = x$, $xy' = \int 4x\,dx = 2x^2 + K$ so $y' = 2x + \frac{K}{x}$. Antidiff to get

$y = x^2 + K \ln x + C$ (two arbitrary constants)

3. (a) $y' + 4xy = x$

$P(x) = 4x$, $Q(x) = x$

$\int P(x)\,dx = x^2 + K$

$I = e^{x^2+K} = e^{x^2} e^K$ which you could call Ae^{x^2} but I'll just leave it $e^{x^2} e^K$ (doesn't make any difference)

$Iy = \int IQ$

$e^{x^2} e^K y = \int e^{x^2} e^K x\,dx = e^K \int e^{x^2} x\,dx$

The e^K on both sides cancels out now and you're left with $e^{x^2} y = \int e^{x^2} x\,dx$ etc.

Any K you put in at the $\int P(x)\,dx$ stage will cancel out later so why bother. The only constant you'll have left at the end of the problem is the one you put in when you find $\int IQ$.

(b) You get solution $y = 1/4$. This is *a* solution but the *general* solution. If you have an IC to satisfy, unless you are very very lucky, your one particular solution won't satisfy that IC and you have no constants available to *make* it satisfy the IC.

4. *method 1* $y' - ky = 0$, $m = k$, $y = Ae^{kx}$

method 2 $y' - ky = 0$, $P(x) = -k$, $\int P(x)\,dx = -kx$, $I = e^{-kx}$,

$ye^{-kx} = \int 0\,dx = C$, $y = Ce^{kx}$

5. *method 1* $P(x) = 1$, $\int P(x) = x$, $I = e^{x}$, $ye^{x} = \int 1\,dx = x + K$,

$y = xe^{-x} + Ke^{-x}$

To get $y(-1) = 3$ need $3 = -e + Ke$, $K = \dfrac{3 + e}{e}$

Answer is $y = xe^{-x} + \dfrac{3 + e}{e} e^{-x} = xe^{-x} + 3e^{-x-1} + e^{-x}$

method 2 (since coeffs are constant) $m = -1$, $y_h = Ae^{-x}$.

Try $y_p = Bxe^{-x}$ (step up). Need

$-Bxe^{-x} + Be^{-x} + Bxe^{-x} = e^{-x}$

xe^{-x} terms drop out.

Equate coeffs of the e^{-x} terms $B = 1$

Gen sol is $y = Ae^{-x} + xe^{-x}$.
The IC determine A as in method 1.

6. (a) e^{K} is not quite arbitrary. It can never be zero and it can never be negative. So it really shouldn't be turned into a C which is totally arbitrary. But most people pay no attention to these niceties. And it usually works out OK in the long run. You would probably find that the new not-so-arbitrary constant C (that used to be e^{K}) comes out positive anyway when you plug in a realistic condition.

(b) Yes because $\ell n\, K$ can take on any value, from very negative to very positive.

On the other hand, you have a slight problem because you shouldn't be taking log of an *arbitrary* K because you can't take ℓn of a negative number or zero. Most people don't worry about this either.

7. $y' + \dfrac{2}{x} y = \dfrac{x^2 + 1}{x}$. Then $P(x) = \dfrac{2}{x}$, $\int P(x) = 2\,\ell n\, x = \ell n\, x^2$, $I = x^2$,

$x^2 y = \int x(x^2 + 1)\,dx = \int (x^3 + x)\,dx = \dfrac{x^4}{4} + \dfrac{x^2}{2} + K$, $y = \dfrac{x^2}{4} + \dfrac{1}{2} + \dfrac{K}{x^2}$

8. First solve $y' - \dfrac{1}{x} y = x$. $P(x) = -\dfrac{1}{x}$, $\int P(x) = -\ell n\, x = \ell n\, x^{-1}$, $I = \dfrac{1}{x}$,

$\dfrac{1}{x} y = \int dx = x + C$, $y = x^2 + Cx$.

Then solve $y' - \dfrac{1}{x} y = 0$ getting $\dfrac{1}{x} y = \int 0\,dx = K$, $y = Kx$. So

$$y = \begin{cases} x^2 + Cx & \text{if } x \le 3 \\ Kx & \text{if } x > 3 \end{cases}$$

The condition $y(1) = 2$ makes $C = 1$.
To get the sol continuous, we want $x^2 + x = Kx$ when $x = 3$, $12 = 3K$, $K = 4$

Answer is $y = \begin{cases} x^2 + x & \text{if } x < 3 \\ 4x & \text{if } x > 3 \end{cases}$

HONORS

9.(a) The equation is $y' - ry = -h$ where r and h are (positive) constants. It's a linear first order DE with constant coeffs.

method 1 for solving the DE

$m = r$, $y_h = Ae^{rt}$

Try $y_p = B$. Need $0 - rB = -h$, $B = h/r$

$$y_{gen} = Ae^{rt} + \frac{h}{r}$$

method 2 for solving the DE

$P = -r$, $Q = -h$, $I = e^{-rt}$

$$e^{-rt}\, y = \int -h\, e^{-rt}\, dt = \frac{h}{r}\, e^{-rt} + A$$

$$y = \frac{h}{r} + Ae^{rt}$$

Whichever method you used, plug in the IC $y(0) = N$ to get $A = N - \frac{h}{r}$

Solution is $y(t) = \frac{h}{r} + (N - \frac{h}{r})e^{rt}$

(b) The solution now is $y(t) = \frac{1}{2}h + (40 - \frac{1}{2}h)e^{2t}$

Fished out means that as t gets larger, y hits 0 eventually. So the problem is to find which h's let y(t) reach 0.

Here's the graph point of view. The graph of $y(t) = \frac{1}{2}h + (40 - \frac{1}{2}h)e^{2t}$ starts at height 40 when t = 0. It's decreasing if

$$40 - \frac{1}{2}h < 0$$

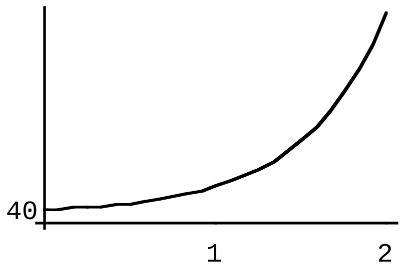
$$h > 80$$

and it's increasing if $h < 80$.

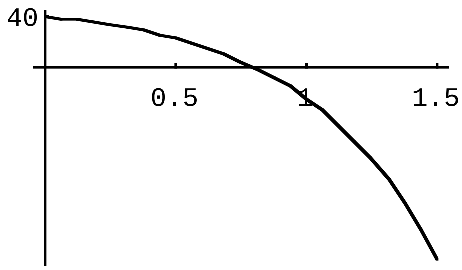
So the lake gets fished out if $h > 80$.

To illustrate, here's the graph of y(t) for h = 60 (and you can see the fish population taking off) and again for h = 90 where you see the population hit 0 (at which point the mathematical model ceases to apply.)

```
Plot[60/2 + (40 - 60/2)E^(2t),{t,0,2}];
```




```
Plot[100/2 + (40 - 100/2)E^(2t),{t,0,1.5}, Ticks->{{.5,1,1.5},{20,40}}];
```



(c) $\frac{100}{2} + (40 - \frac{100}{2})e^{2t} = 0$, $e^{2t} = 5$, $t = \frac{1}{2}\ln 5$

In the second diagram, the graph crosses the horizontal axis at $t = \frac{1}{2}\ln 5$. That's when the lake is fished out.

SOLUTIONS Section 4.2

1. (a) $\cos y\, dy = -x\, dx$, $\quad \sin y = -\frac{1}{2}x^2 + C$ (implicit sol)

(b) $y\, dy = -\frac{dx}{x^3}$, $\quad \frac{1}{2}y^2 = \frac{1}{2x^2} + C$, $\quad y = \pm\sqrt{\frac{1}{x^2} + D}$

(c) $y^4\, dy = -x^2\, dx$, $\quad \frac{1}{5}y^5 = -\frac{1}{3}x^3 + C$, $\quad y = \sqrt[5]{K - \frac{5}{3}x^3}$

(d) $\frac{dy}{y} = \frac{dx}{2x+3}$, $\quad \ell n\, Ky = \frac{1}{2}\ell n(2x+3) = \ell n\sqrt{2x+3}$,

$Ky = \sqrt{2x+3}$, $\quad y = A\sqrt{2x+3}$

---

warning It's OK to write

$$\ell n\, y = \frac{1}{2}\ell n(2x+3) + C$$

but when you take exp on both sides it is *wrong* to get

$$y = \sqrt{2x+3} \text{ PLUS } e^C \qquad \text{WRONG}$$

which turns into

$$y = \sqrt{2x+3} + A$$

The *right* way to take exp is to get

$$y = e^{\ell n(2x+3)^{1/2} + C} \qquad \text{RIGHT}$$

which turns into

$$y = e^{\ell n(2x+3)^{1/2}} \text{ TIMES } e^C = A\sqrt{2x+3}$$

(e) $e^{-y}\, dy = \frac{dx}{x^2}$, $\quad -e^{-y} = -\frac{1}{x} + C$, $\quad e^{-y} = \frac{1}{x} + D$, $\quad -y = \ell n(\frac{1}{x} + D)$,

$y = -\ell n(\frac{1}{x} + D)$

(f) $y\, dy = (5x+3)\, dx$, $\quad \frac{1}{2}y^2 = \frac{5}{2}x^2 + 3x + C$, $\quad y = \pm\sqrt{5x^2 + 6x + D}$

(g) not separable

(h) $(y+1) = \frac{1}{x}\, dx$, $\quad \frac{1}{2}y^2 + y = \ell n\, Kx$ (implicit sol),

$y = \frac{-2 \pm \sqrt{4 + 8\,\ell n Kx}}{2} = -1 \pm \sqrt{1 = 2\,\ell\, Kx}$ (explicit sol)

2. For (e), $y = -\ln\left(\frac{1}{x} + D\right)$

$$x^2\,dy = x^2\,y'(x)\,dx = x^2 \cdot \frac{-1}{\frac{1}{x} + D} \cdot -\frac{1}{x^2}\,dx = \frac{dx}{\frac{1}{x} + D}$$

$$e^{-y}\,dx = e^{-\ln\left(\frac{1}{x} + D\right)} = e^{\ln\left(\frac{1}{x} + D\right)^{-1}}\,dx = \left(\frac{1}{x} + D\right)^{-1}\,dx$$

So $x^2\,dy$ does equal $e^{-y}\,dx$, QED.

For (f), $y = \pm\sqrt{5x^2 + 6x + D}$, $y' = \dfrac{10x + 6}{\pm 2\sqrt{5x^2 + 6x + D}} = \dfrac{5x + 3}{\pm\sqrt{5x^2 + 6x + D}}$

so y' does equal $\dfrac{5x + 3}{y}$, QED.

3.(a) $\dfrac{dy}{y} = x\,dx$, $\quad \ln Ky = \frac{1}{2}x^2$, $\quad Ky = e^{x^2/2}$, $\quad y = Ae^{x^2/2}$

Use the condition to get $3 = Ae^{1/2}$, $A = 3e^{-1/2}$,

Sol is $y = 3e^{-1/2}\,e^{x^2}$, $\quad y = 3e^{(x^2-1)/2}$

(b) $y\,dy = (3 - 5x)\,dx$, $\quad \frac{1}{2}y^2 = 3x - \frac{5}{2}x^2 + C$. Set $x = 2$, $y = 4$ to get $C = 12$.

Then $\frac{1}{2}y^2 = 3x - \frac{5}{2}x^2 + 12$, $\quad y = \sqrt{6x - 5x^2 + 24}$

(Choose the *positive* square root since y is *positive* when x = 2.)

(c) $e^y\,dy = 3x\,dx$, $\quad e^y = \frac{3}{2}x^2 + C$. Set $x = 0$, $y = 2$ to get $C = e^2$.

Then $e^y = \frac{3}{2}x^2 + e^2$, $\quad y = \ln\left(\frac{3}{2}x^2 + e^2\right)$

(d) $\dfrac{dy}{y^4} = \cos x\,dx$, $\quad -\dfrac{1}{3y^3} = \sin x + C$. Set $x = 0$, $y = 2$ to get $C = -\dfrac{1}{24}$.

Sol is $y \quad \dfrac{-1}{\sqrt[3]{3\sin x - \frac{1}{8}}}$

4. *method 12* (works because the coeffs are constant)

$w' + \frac{1}{5}w = 2$, $m = -\frac{1}{5}$, $w_h = Ae^{-t/5}$

Try $w_p = B$. Substitute into the DE to get $0 + \frac{1}{5}B = 2$, $B = 10$

$w_{gen} = Ae^{-t/5} + 10$ (which is the same as the answer $10 - De^{-t/5}$ from example 1 because the arbitrary constant A is the same as the arbitrary constant $-D$)

method 2 $P = \frac{1}{5}$, $Q = 2$, $I = e^{\int P\,dt} = e^{t/5}$

$Iw = \int IQ = \int 2e^{t/5}\,dt$, $\quad e^{t/5}\,w = 10e^{t/5} + K$, $w = 10 + Ke^{-t/5}$

5. The equation can be written as $xy^2 - y - 7 = 0$ and treated as a quadratic equation of the form $ay^2 + by + c = 0$ where $a = x$, $b = -1$, $c = -7$. So

$y = \dfrac{1 \pm \sqrt{1+28x}}{2x}$, not really one implicit solution, but two. (And they real only if $x > -1/28$ and they aren't defined for $x = 0$.)

6.(a) $\frac{dy}{y} = -10\,dx$, $\ln Ky = -10x$, $Ky = e^{-10x}$, $y = Ae^{-10x}$.

Plug in the IC $y(0) = G$ to get $y = Ge^{-10x}$.

Now let $y = G/2$ and find x: $G/2 = Ge^{-10x}$, $\frac{1}{2} = e^{-10x}$, $-10x = \ln 1/2$,

$x = \frac{1}{10}\ln 2$. So the half life is $\frac{1}{10}\ln 2$.

(b) Let the constant of proportionality be called C. As in part (a), the half life is $\frac{1}{C}\ln 2$. If you want this to be 3, choose $C = \frac{1}{3}\ln 2$.

7. (a) Differentiate w.r.t. x to get the differential equation of the family $xy = K$.

$$xy' + y = 0$$
$$y' = -y/x$$

The orthog family has differential equation $y' = x/y$. Solve it to find the orthogonal family.

$$y\,dy = x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + A$$
$$y^2 - x^2 = C$$

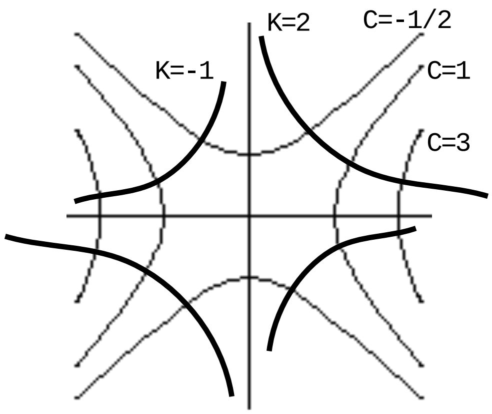
Both families are hyperbolas (each hyperbola has two branches). In the diagram, the original family is in darker type.

(b) $\frac{y}{x^2} = A$

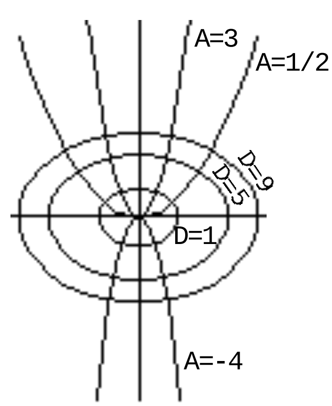
$$x^{-2}y' - 2x^{-3}y = 0$$
$$y' = \frac{2y}{x}$$

The orthog family has DE $y' = -\frac{x}{2y}$. Solve to get the orthog family.

$$2yy' = -x$$
$$2y\,dy = -x\,dx$$
$$y^2 = -\frac{1}{2}x^2 + C$$
$$x^2 + 2y^2 = D \quad \text{(a family of ellipses)}$$



Problem 7(a)



Problem 7(b)

SOLUTIONS Section 4.3

1. $d(\frac{y}{x})$ comes out to be (22) immediately by the quotient rule and similarly for (23)

$$d(x^2 + y^2)^{-1} = -(x^2 + x^2)^{-2} \; d(x^2 + y^2)) \qquad \text{(chain rule)}$$

$$= -(x^2 + x^2)^{-2} \; (2x\,dx + 2y\,dy) \text{ which is (24)}$$

$$d(\pm\sqrt{x^2 + x^2}\,) = \pm \frac{1}{2}\,(x^2 + y^2)^{-1/2}\, d(x^2 + y^2) \qquad \text{(chain rule)}$$

$$= \frac{2x\,dx + 2y\,dy}{\pm 2\,\sqrt{x^2 + x^2}} \quad \text{which cancels to (25)}$$

$$d(\ell n\,(x^2 + y^2) = \frac{1}{x^2 + y^2}\; d(x^2 + y^2) \quad \text{(chain rule)} = \frac{2x\,dx + 2y\,dy}{(x^2 + y^2)}$$

$$d(\arctan y/x) = \frac{1}{1 + (y/x)^2}\, d(\frac{y}{x}) = \frac{1}{1 + (y/x)^2}\; \frac{x\,dy - y\,dx}{x^2} = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

2. $x = r\cos\theta$ so by (1), $dx = \cos\theta\,dr - r\sin\theta\,d\theta$
$y = r\sin\theta$, $dy = \sin\theta\,dr + r\cos\theta\,d\theta$

3.(a) $p = 2xy$, $q = y$, $\frac{\partial q}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 2x$. Not equal so form is not exact

(b) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} = 3x^2$. Exact. Antidiff p w.r.t. x to get $\frac{1}{4}x^4 + x^3y$. Diff this temporary answer w.r.t. y to get x^3. Compare with q and see that you should tack on $\frac{1}{4}y^4$. Answer is

$$f(x,y) = \frac{1}{4}x^4 + x^3y + \frac{1}{4}y^4$$

(c) $f(x,y) = -\frac{y}{x} + 5y$

4. Need $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, $\frac{\partial q}{\partial x} = 3xy^2$, $q = \frac{3}{2}x^2y^2 + \text{any } f(y)$

For example q could be $\frac{3}{2}x^2y^2 + \sin y + 7$

5. (a) $d(2x^3 + xy^2 + y^3) = 0$, implicit sol is $2x^3 + xy^2 + y^3 = C$

(b) $d(x^3 + xy) = 0$ Implicit sol is $x^3 + xy = C$. Explicit sol is $y = \frac{C - x^3}{x}$

(c) $(x-y\cos x)\,dx - (y + \sin x)\,dy = 0$

$$d(\frac{1}{2}x^2 - y\sin x - \frac{1}{2}y^2) = 0$$

Implicit solution is $\frac{1}{2}x^2 - y\sin x - \frac{1}{2}y^2 = K$

(d) $e^{xy}\,dx - dy = 0$ Not exact since $\frac{\partial q}{\partial x} = 0$ but $\frac{\partial p}{\partial y} = xe^{xy}$

(e) $(2r\cos\theta - 1)dr - r^2\sin\theta\,d\theta = 0$
$d(r^2\cos\theta - r) = 0$
Implicit sol is $r^2\cos\theta - r = K$

(f) not exact

(g) $d(\sin x\cos y) = d(\frac{1}{4}x^4)$ Implicit sol is $\sin x\cos y = \frac{1}{4}x^4 + C$

(h) $(ye^{-x} - \sin x)\ dx - (e^{-x} + 2y)\ dy = 0$
$d(-ye^{-x} + \cos x - y^2) = 0$
Implicit sol is $-ye^{-x} + \cos x - y^2 = C$

6. Take differentials throughout $-ye^{-x} + \cos x - y^2 = C$ to get

$$-y\cdot e^{-x}\ dx + e^{-x}\cdot -dy - \sin x\ dx - 2y\ dy = 0$$

Collect terms: $(ye^{-x} - \sin x)\ dx = (e^{-x} + 2y)\ dy$, QED

7.(a) $d(x^2y + \frac{1}{2}y^2) = 0$
Implicit sol is $x^2y + \frac{1}{2}y^2 = C$
Set $x = 1$, $y = 4$ to get $C = 12$. Implicit particular sol is is $x^2y + \frac{1}{2}y^2 = 12$

(b) $d(-\cos(2x + 3y)) = 0$ Implicit sol is $-\cos(2x + 3y) = C$

Set $x = 0$, $y = \pi/2$ to get $C = 0$. Implicit particular sol is $\cos(2x + 3y) = 0$

(c) $d\ \ell n(x + y) = d(x)$. Implicit sol is $\ell n(x + y) = x + C$.

Set $x = 0$, $y = 1$ to get $C = 0$. Implicit particular sol is $\ell n(x + y) = x$.

Then $x + y = e^x$ so explicit solution is $y = e^x - x$

8.(a) $d(\frac{1}{3}x^3 + 2x + \frac{3}{2}y^2) = 0$ Implicit sol is $\frac{1}{3}x^3 + 2x + \frac{3}{2}y^2 = K$

(b) $(x^2 + 2)dx = -3y\ dy$, $\frac{1}{3}x^3 + 2x = -\frac{3}{2}y^2 + K$

9. (a) It doesn't do any good to collect the dx terms and get
$(x^2 + y^2 + y)\ dx - x\ dy = 0$ because this arrangement isn't exact.
Instead, look at (27). Use integrating factor $\dfrac{1}{x^2 + y^2}$. The equation becomes

$$dx = \frac{x\ dy - y\ dx}{x^2 + y^2} \quad \text{so } x = \tan^{-1}\frac{y}{x} + K$$

(b) See (22). Use integrating factor $1/y^2$.

$$\frac{y\ dx - x\ dy}{y^2} = dx, \qquad \frac{x}{y} = x + K, \qquad y = \frac{x}{x + K}$$

(c) See (25). Use integrating factor $\dfrac{1}{\sqrt{x^2 + y^2}}$.

$$dy = \frac{x\ dx + y\ dy}{\sqrt{x^2 + y^2}}, \quad y = \sqrt{x^2 + y^2} + K$$

(d) $x\ dx = (x^2 + y^2 - y)\ dy$
$x\ dx + y\ dy = (x^2 + y^2)dy$

See (26). Use integrating factor $\dfrac{2}{x^2 + y^2}$.

$$\frac{2x\,dx + 2y\,dy}{x^2 + y^2} = 2\,dy, \quad \ln(x^2 + y^2) = 2y + K$$

(e) See (23). Multiply by $1/x^2$ to get $\dfrac{x\,dy - y\,dx}{x^2} = 2x\,dx + 2y\,dy$,

$d\left(\dfrac{y}{x}\right) = d(x^2 + y^2)$, $\dfrac{y}{x} = x^2 + y^2 + K$. Implicit sol is $y = x^3 + xy^2 + Kx$.

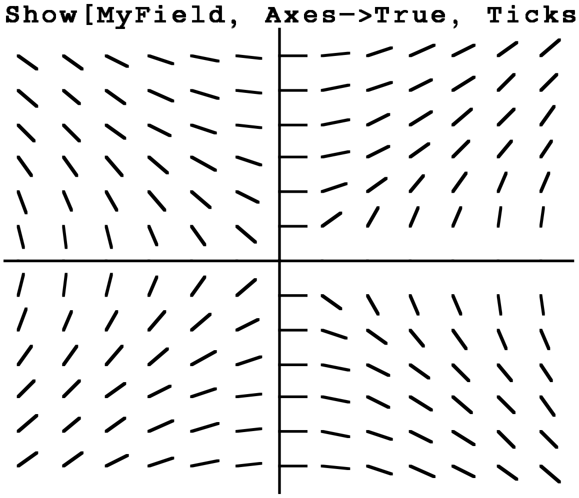
SOLUTIONS Section 4.4

1. Remember that at each point (x,y), the idea is to draw a little segment with slope x/y.
On the line y = x, all the segments have slope 1.
In quadrants I and III the segments all have positive slope.
In quadrants II and IV the segments have negative slope

As you move right on a horizontal line in quadrants I, y stays the same and x gets larger so the segments get steeper. As you move left on a horizontal line in quadrant II, y stays the same and x gets negatively larger so the segments get steeper (but with negative slope).

As you move up a vertical line in quadrant I, x stays the same and y gets larger so the segments get less steep. etc.

```
MyField = directionField[x/y, {x,-3,3},{y,-3,3},.5,.3];
Show[MyField, Axes->True, Ticks->None];
```



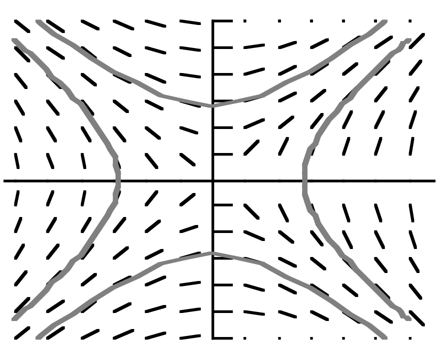
$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + K$$

$$y^2 - x^2 = A$$

Each solution is a hyperbola. Here's a picture of the direction field along with the two particular solutions $y^2 - x^2 = 2$, $y^2 - x^2 = -2$

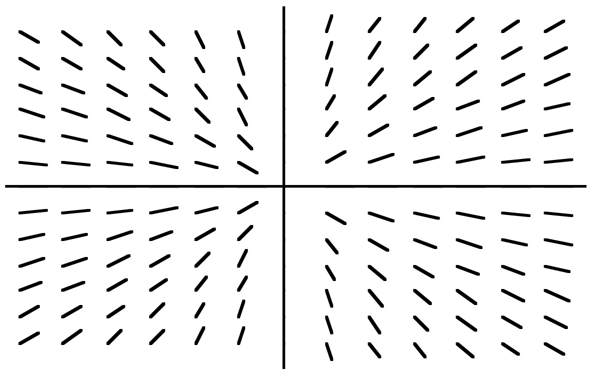
```
SomeSols =  ImplicitPlot[{y^2 - x^2== -2,y^2 - x^2 ==2},{x,-3,3},
      PlotStyle->{{GrayLevel[.5],Thickness[.01]}},
      PlotRange->{-3,3},DisplayFunction->Identity];

Show[{MyField,SomeSols}, Axes->True,
         Ticks->None,DisplayFunction->$DisplayFunction];
```



(b) When $y = x$, the little segments have slope 1.

As you move right on a horizontal line in quadrant I, y stays fixed and x increases so the segments get less steep.

As you move up a vertical line in quadrant I, x stays fixed and y increases so the segments get steeper.



$$\frac{dy}{y} = \frac{dx}{x}$$

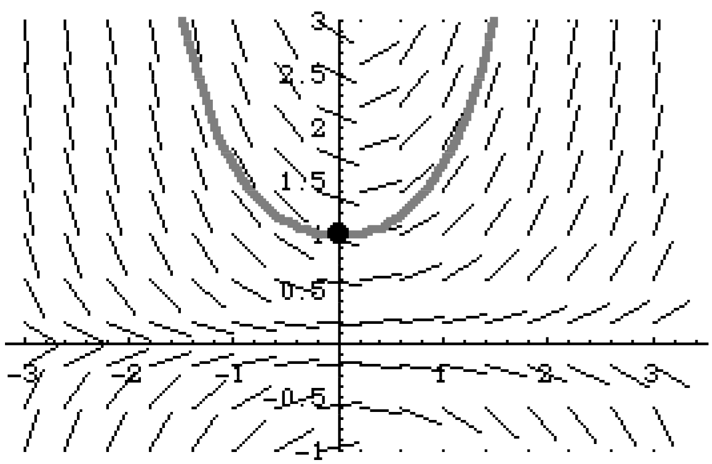
$$\ln Ky = \ln x$$

$$Ky = x$$

$$y = Ax$$

The solutions are all lines through the origin.

2. Draw a curve through the point (0,1) using the little segments.
(By the way, the direction field in the problem happens to be that of the DE $y' = xy$.)



3.(a). The segment at point (x_0, y_0) is supposed to have slope x_0y_0. If you solved the DE and found the particular solution satisfying the condition $y(x_0) = y_0$, in the vicinity of point (x_0, y_0) it's graph should look like the little segment.

(b) The equation is separable and linear first order and exact. I'll separate.

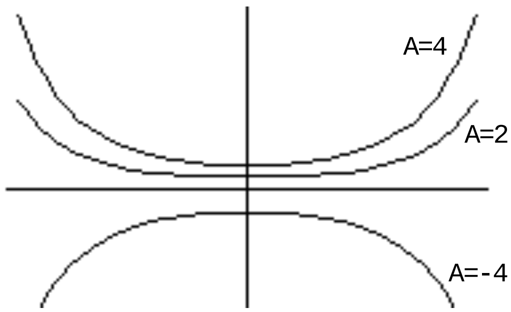
$$\frac{dy}{y} = x\,dx$$

$$\ln Ky = \frac{1}{2}x^2$$

$$Ky = e^{x^2/2}$$

$$y = Ae^{x^2/2}$$

(c) I plotted $2e^{x^2/2}$, $4e^{x^2/2}$ and $-4e^{x^2/2}$.



(d) Plug in the condition y(4) = 3; i.e., set x=4, y=3 to determine A.

$3 = Ae^8$, $A = 3e^{-8}$. The curve is $y = 3e^{-8}\ e^{x^2/2}$; equivalently $y = 3e^{-8+x^2/2}$.

SOLUTIONS review problems for Chapter 4

1.(a) *method 1* (exact) $d(\frac{1}{3}x^3 + 2x + \frac{3}{2}y^2) = 0$, $\frac{1}{3}x^3 + 2x + \frac{3}{2}y^2 = K$

$y = \pm\sqrt{\frac{2}{3}K - \frac{2}{9}x^3 - \frac{4}{3}x}$

method 2 (separable) $(x^2 + 2)\,dx = -3y\,dy$, $\frac{1}{3}x^3 + 2x = -\frac{3}{2}y^2 + K$,
same y as in method 1 now.

(b) *method 1* (separable) $\frac{dy}{y} = -\,dx$, $\ln Ky = -x$, $Ky = e^{-x}$, $y = Ae^{-x}$

method 2 (exact) $y\,dx + dy = 0$ is NOT exact but $dx + \frac{1}{y}\,dy = 0$ *is* exact.
Then $d(x + \ln y) = 0$, $x + \ln y = K$, $\ln y = K - x$, $y = e^{K-x}$,

$y = e^K e^{-x}$, $y = Ae^{-x}$

method 3 (linear, constant coeffs) $y' + y = 0$, $m = -1$, $y = Ae^{-x}$

method 4 (linear first order) $y' + y = 0$, $P = 1$, $Q = 0$, $I = e^{\int P} = e^x$,

$e^x y = \int 0\,dx = K$, $y = Ke^{-x}$

(c) *method 1* (can be arranged to be exact)
$(y-2x)\,dx + x\,dy = 0$, $d(xy - x^2) = 0$, $xy - x^2 = C$
If $x = 1$, $y = 2$ then $C = 1$. Sol is $xy - x^2 = 1$. Explicit sol is $y = x + \frac{1}{x}$

method 2 (first order linear) $y' + \frac{1}{x}y = 2$, $I = e^{\int(1/x)} = e^{\ln x} = x$,

$xy = \int 2x\,dx = x^2 + K$, $y = x + \frac{K}{x}$. If $y(1) = 2$ then $K = 1$. Sol is $y = x + \frac{1}{x}$

(d) (linear) $y'' - y = 0$, $m = \pm 1$, $y = Ae^x + Be^{-x}$

(e) *method 1* $y'' - 3y' = 12$, $m = 0,3$, $y_h = A + Be^{3x}$

Try $y_p = Cx$ (step up) Get $C = -4$. Answer is $y = A + Be^{3x} - 4x$

method 2 If you think of y' as the variable this is first order.
Let $y' = u$ Then DE is $u' = 3u + 12$ This is exact, also separable,. also linear with constant coeffs (P,Q stuff). Here's the separation method:

$\frac{du}{3u + 12} = dx$,

$\frac{1}{3}\ln K(3u + 12) = x$,

$\ln K(3u + 12) = 3x$,

$K(3u + 12) = e^{3x}$

$u = Be^{3x} - 4$
So $y' = Be^{3x} - 4$ and (antidiff)

$y = \frac{1}{3}Be^{3x} - 4x + C$, $y = De^{3x} - 4x + C$, same as in other method

(f) *method 1* (separable) $\frac{dy}{dx} = e^x e^y$, $e^{-y}\,dy = e^x\,dx$, $-e^{-y} = e^x + K$,

$e^{-y} = A - e^x$, $-y = \ln(A - e^x)$, $y = -\ln(A - e^x)$

method 2 (exact) $e^x\,dx - e^{-y}\,dy = 0$, $d(e^x + e^{-y}) = 0$, $e^x + e^{-y} = K$,
$y = -\ln(K - e^x)$

(g) (Second order, variable coeffs, can't use m's)

method 1 Consider y' the variable (call it w if you like). Then the equation is first order separable:

$$xw' - w = 1, \quad \frac{dw}{w+1} = \frac{dx}{x}, \quad \ln(w+1) = \ln Kx, \quad w + 1 = Kx, \; w = Kx - 1$$

So $y' = Kx - 1$, $y = Kx^2 - x + C$

The IC make $K = 2$, $C = 1$ Answer is $y = 2x^2 - x + 1$

method 2 Consider y' the variable (call it w if you like). Then the equation is first order linear:

$$w' - \frac{1}{x}\,w = \frac{1}{x}$$

$$I = e^{\int P(x)\,dx} = e^{-\ln x} = e^{\ln 1/x} = \frac{1}{x}$$

$$\frac{1}{x}\,w = \int \frac{1}{x}\frac{1}{x}\,dx = -\frac{1}{x} + K. \quad \text{So} \quad y' = -1 + Kx, \quad y = -x + \frac{1}{2}Kx^2 + C\ .$$

The IC make $K = 4$, $C = 1$,
Answer is $y = -x + 2x^2 + 1$

2. *method 1* (separable)

$$m\frac{dv}{dt} = mg - cv$$

$$\frac{dv}{vc - mg} = -\frac{dt}{m}$$

$$\frac{1}{c}\ln(cv - mg) = -\frac{t}{m}$$

$$K(cv - mg) = e^{-ct/m}$$

$$cv - mg = A\,e^{-ct/m}$$

$$v = \frac{mg}{c} + \frac{A}{c}\,e^{-ct/m}$$

method 2 Use the P,Q method. $v' + \frac{c}{m}\,v = g$, $I = e^{\int c/m\ ex} = e^{ct/m}$,

$$e^{ct/m}\,y = \int g e^{ct/m}\,dt = \frac{gm}{c}\,e^{ct/m} + K, \qquad y = \frac{gm}{c} + Ke^{-ct/m} \quad \text{etc.}$$

method 3 (harder to notice) $m\,dv + (cv - mg)\,dt = 0$ is not exact but

$$\underbrace{\frac{m}{cv - mg}}_{p}\,dv + \underbrace{1}_{q}\,dt = 0$$

is exact ($\partial q/\partial v = \partial p/\partial t = 0$). The equation can be written as

$$d\left(\frac{m}{c}\,\ell n(cv - mg) + t\right) = 0$$

Implicit solution is

$$\frac{m}{c}\,\ell n(cv - mg) + t = K$$

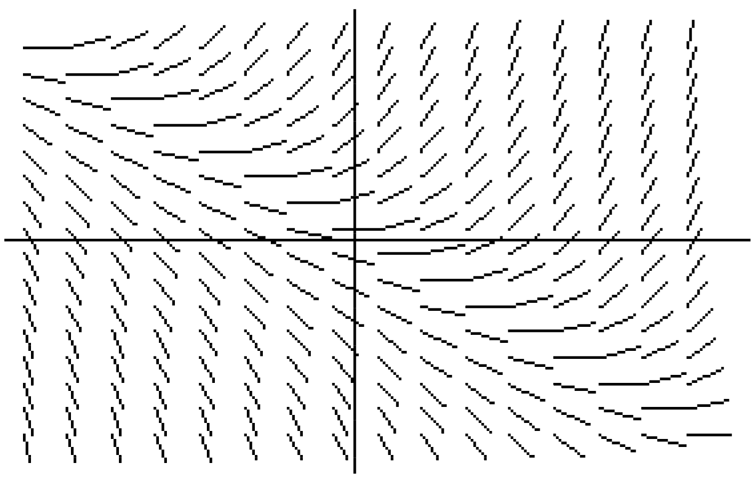
Solve for v to get explicit sol:

$$\ell n(cv - mg) = \frac{Kc}{m} - \frac{ct}{m}$$

$$cv - mg = e^{Kc/m - ct/m} = e^{Kc/m}\, e^{-ct/m} = A\, e^{-ct/m}$$

$$v = \frac{mg}{c} + \frac{A}{c}\, e^{-ct/m}$$

3. At point (x,y) draw a little segment with slope x+y.



method 1 for solving

The equation is $y' - y = x$. This is linear first order with *constant* coeffs so the methods of Chapter 1 work.

$m = 1$, $y_h = Ae^x$.

Try $y_p = Bx + C$. Substitute into the DE. You need $B - (Bx + C) = x$.

Equate the x coeffs: $-B = c$, $B = -1$.

Equate the constant terms: $B - C = 0$, $C = -1$.

So $y_p = -x-1$, $y_{gen} = y_h + y_p = Ae^x - x - x$.

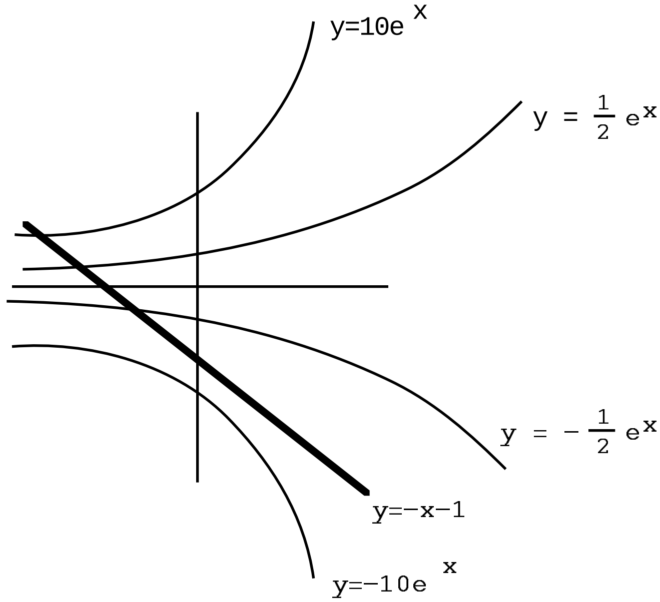
method 2 for solving

The equation is linear. It can be written as $y' - y = x$.

$P(x) = -1$, $Q(x) = x$, $e^{\int P(x)\,dx} = e^{-x}$, $ye^{-x} = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

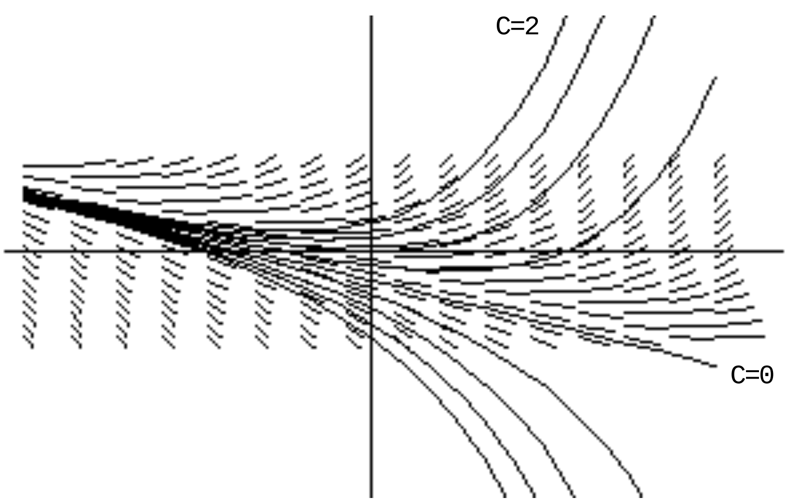
Solution is $y = -x - 1 + Ce^x$.

Yes you should be able to sketch the graph of this family. I would first sketch $y=-x-1$ (a line) and $y = Ce^x$ separately.



Then add the line heights to each of the other curves. Way out to the left, Ce^{x} is near 0 so the sum is like the line. Way out to the right, the exponential heights are much larger in absolute value than the line heights so the sum is like the exponential. When C = 0, the sum *is* the line.

Here are some of the curves in the family $y = -x - 1 + e^{x}$, along with the direction field.



4. Rewrite the DE as $\frac{y}{x^3} = A$ and then differentiate w.r.t. x on both sides.

$$x^{-3}\, y' - 3x^{-4}\, y = 0$$

$$y' = \frac{3y}{x}$$

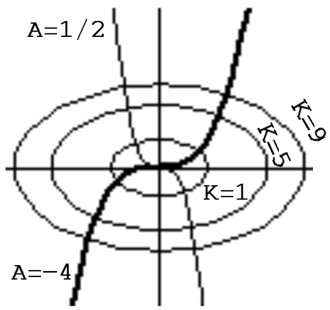
The orthogonal family has differential equation $y' = -\frac{x}{3y}$. Now solve it.

$$3y\, dy = -\, x\, dx$$

$$\frac{3y^2}{2} = -\frac{x^2}{2} + C$$

$$x^2 + 3y^2 = 2C$$

$$x^2 + 3y^2 = K \quad \text{(an ellipse family)}$$



5. A separable DE can be written as

(*) x-stuff dx = y-stuff dy and rearranged to look like

(**) $$\underbrace{\text{x-stuff}}_{p}\, dx + \underbrace{-\text{y-stuff}}_{q}\, dy = 0$$

Then $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ (both are 0) so (**) is exact.

To disprove the converse, i.e., to show that not every exact DE is also separable, all you have to do is produce one counterexample. Problem 1(c) is exact but not separable. QED

SOLUTIONS Section 5.1

1.(a) $\frac{5!}{s^6}$ (b) $\frac{3!}{s^4}$ (c) $\frac{1}{s-3}$ (d) $\frac{1}{s+4}$ (e) $\frac{4}{s^2+16}$ (f) $\frac{s}{s^2+25}$

2. $\int_0^\infty e^{-st}\, e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt = -\frac{1}{s+a} e^{-(s+a)t} \Big|_{t=0}^{\infty} = \frac{1}{s+a}$

assuming $s > -a$ so that $e^{-(s+a)\infty} = 0$

3.(a) $\cosh t = \frac{e^t + e^{-t}}{2}$ so transform is $\frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right]$

(b) $\sin^2 4t = \frac{1 - \cos 8t}{2}$ so transform is $\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+64}\right]$

(c) $\cos(at + b) = \cos at \cos b - \sin at \sin b$ (note that $\cos b$ and $\sin b$ are just constants). Transform is

$$\cos b \cdot \frac{s}{s^2+a^2} - \sin b \cdot \frac{a}{s^2+a^2} = \frac{s \cos b - a \sin b}{s^2+a^2}$$

(d) $\frac{16}{s^3} + \frac{2}{s^2} - \frac{3}{s}$

(e) $\frac{2}{s-3} - \frac{\pi}{s^2+\pi^2}$

(f) $5\, u(t) \leftrightarrow \frac{5}{s}$

(g) $-2u(t) \leftrightarrow -\frac{2}{s}$

(h) $\frac{1}{2}\, r(t) \leftrightarrow \frac{1}{2s^2}$

(i) $7r(t) \leftrightarrow \frac{7}{s^2}$

(j) $e^{3t+4} = e^4\, e^{3t}$ so transform is $e^4 \cdot \frac{1}{s-3}$

(k) $\mathcal{L}\ \delta(t) = 1$ so $\mathcal{L}\ 6\delta(t) = 6$

(ℓ) $\frac{24}{s^4} - \frac{6}{s^3} + \frac{5}{s^2} + \frac{2}{s}$

4.(a) $\cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$

(b) 0 (the impulse occurs at 7, outside the interval of integration)

(c) $6^3 = 216$ (d) $\cos 0 = 1$ (e) 6 (f) 6

(g) $0^2 = 0$ (h) $e^0 = 1$ (i) $2^2 = 4$ (j) e^2

5.(a) This is the transform of t^4 so it is $\frac{4!}{s^5}$

(b) This is the same integral as part (a) but with dummy variable of integration u instead of t. Answer is still $\frac{4!}{s^5}$.

(c) This is like part (a) but with w playing the role of s. Answer is $\frac{4!}{w^5}$.

(d) $e^{3s^2 t}$ can be written as $e^{-(-3s^2)t}$ so this is like part (a) but with $-3s^2$ playing the role of s. Answer is $\frac{4!}{(-3s^2)^5}$ (provided $-3s^2 > 0$, i.e., $3s^2 \le 0$, so that the integral converges to begin with).

Honors

6. (a) Let $u = t^n$, $du = nt^{n-1}\,dt$, $dv = e^{-st}\,dt$, $v = -\frac{1}{s}e^{-st}$. Then

$$\pounds\, t^n\, u(t) = \int_0^\infty t^n e^{-st}\,dt$$

$$= \underbrace{-\frac{1}{s} t^n e^{-st}\Big|_{t=0}^{\infty}}_{0 \text{ (see below)}} + \frac{n}{s}\underbrace{\int_0^\infty t^{n-1} e^{-st}\,dt}_{\pounds\, t^{n-1}\, u(t)}$$

Here's why the first term is 0. If you plug in t=0, you get 0 because of the t^n factor.

If you plug $t=\infty$ into $\frac{t^n}{e^{st}}$ you get $\frac{\infty}{\infty}$ but in this case it's 0 because the exponential in the denominator grows much faster than the power function in the numerator.

So

$$\pounds\, t^n\, u(t) = \frac{n}{s}\,\pounds\, t^{n-1}\, u(t)$$

(b) $\pounds\, t^2\, u(t) = \frac{2}{s}\,\pounds\, t\, u(t) = \frac{2}{s}\,\frac{1}{s^2} = \frac{2}{s^3}$

$$\pounds\, t^3\, u(t) = \frac{3}{s}\,\pounds\, t^2\, u(t) = \frac{3}{s}\,\frac{2}{s^3} = \frac{3\cdot 2}{s^4}$$

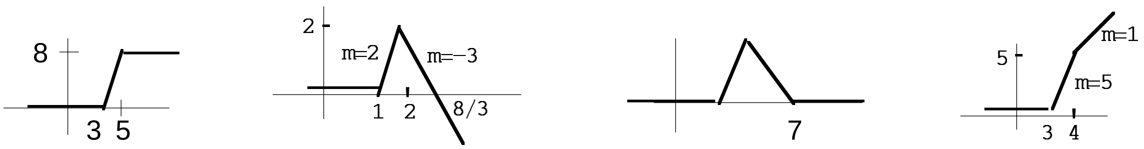
$$\pounds\, t^4\, u(t) = \frac{4}{s}\,\pounds\, t^3\, u(t) = \frac{4}{s}\,\frac{3\cdot 2}{s^4} = \frac{4!}{s^5}$$

$$\pounds\, t^n\, u(t) = \frac{n!}{s^{n+1}}$$

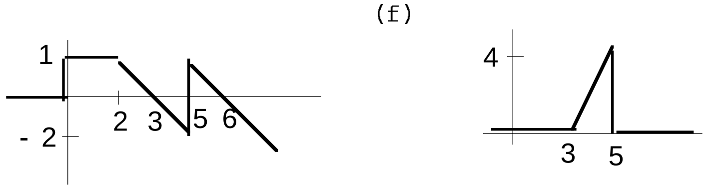
SOLUTIONS Section 5.2

1. (a) (b) (c) (d)

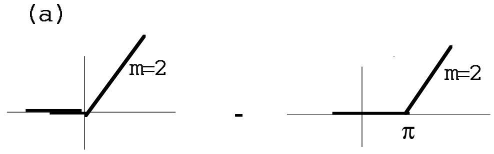
2. (a) (b) (c) (d)



(e) (f)

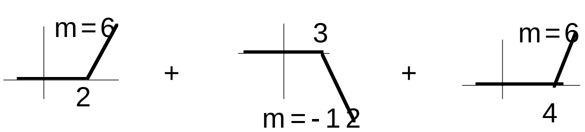


3. (a)



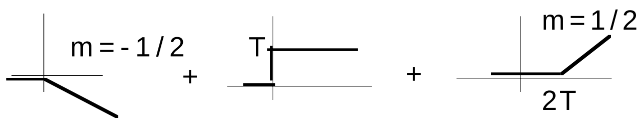
$2r(t) - 2r(t-\pi) \leftrightarrow \frac{2}{s^2} - \frac{2e^{-\pi s}}{s^2}$

(b)



$6r(t-2) - 12r(t-3) + 6r(t-4) \leftrightarrow \frac{6e^{-2s} - 12e^{-3s} + 6e^{-4s}}{s^2}$

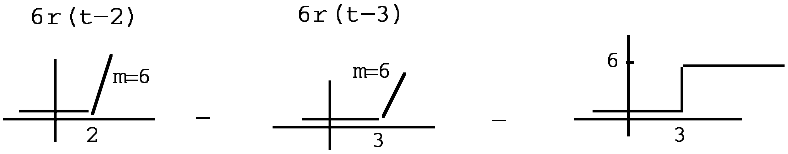
(c)



$-\frac{1}{2}r(t) + Tu(t) + \frac{1}{2}r(t-2T) \leftrightarrow -\frac{1/2}{s^2} + \frac{T}{s} + \frac{1/2}{s^2}e^{-2Ts}$

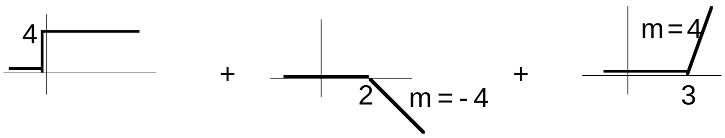
(d)

6r(t−2) 6r(t−3)



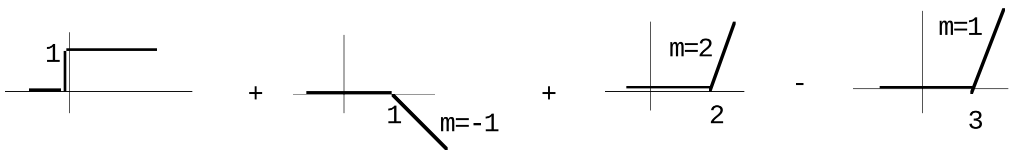
$6r(t-2) - 6r(t-3) - 6u(t-3) \leftrightarrow \frac{6e^{-2s}}{s^2} - \frac{6e^{-3s}}{s^2} - \frac{6e^{-3s}}{s}$

(e)



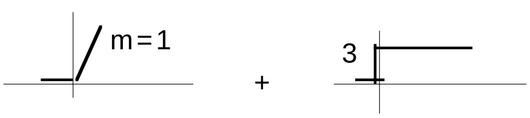
$$4u(t) - 4r(t-2) + 4r(t-3) \leftrightarrow \frac{4}{s} - \frac{4e^{-2s}}{s^2} + \frac{4e^{-3s}}{s^2}$$

(f)

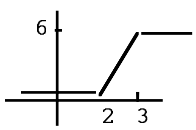


$$u(t) - r(t-1) + 2r(t-2) - r(t-3) \leftrightarrow \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{2e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2}$$

(g) $f(t) = (t+3)u(t) = tu(t) + 3u(t) \leftrightarrow \frac{1}{s^2} + \frac{3}{s}$

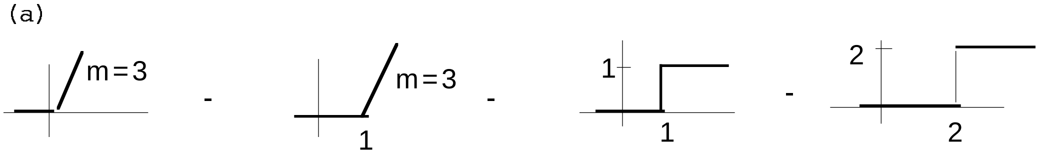


4. The decomposition $6r(t-2) - 6r(t-3)$ goes with this function, not $f(t)$:



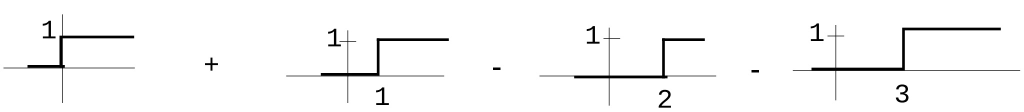
You have to subtract $6u(t-3)$ to make the function jump down.
The correct transform is $\frac{6e^{-2s}}{s^2} - \frac{6e^{-3s}}{s^2} - \frac{6e^{-3s}}{s}$.

5.(a)



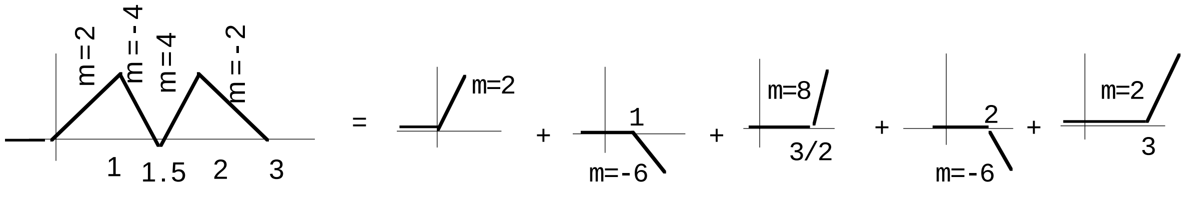
$$3r(t) - 3r(t-1) - u(t-1) - 2u(t-2) \leftrightarrow \frac{3}{s^2} - \frac{3e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

(b)



$$u(t) - u(t-1) - u(t-2) - u(t-3) \leftrightarrow \frac{1 + e^{-s} - e^{-2s} - e^{-3s}}{s}$$

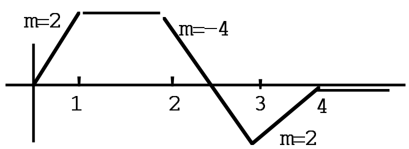
(c)

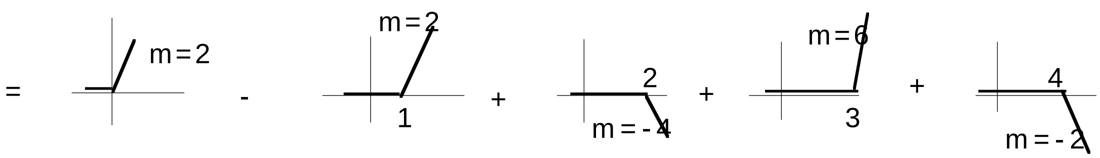


$$2r(t) - 6r(t-1) + 8r(t-\tfrac{3}{2}) - 6r(t-2) + 2r(t-3)$$

$$\leftrightarrow \quad \frac{2 - 6e^{-s} + 8e^{-3s/2} - 6e^{-2s} + 2e^{-3s}}{s^2}$$

(d)

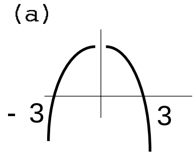




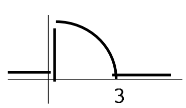
$$2r(t) - 2r(t-1) - 4r(t-2) + 6r(t-3) - 2r(t-4)$$

$$\leftrightarrow \quad \frac{2 - 2e^{-s} - 4e^{-2s} + 6e^{-3s} - 2e^{-4s}}{s^2}$$

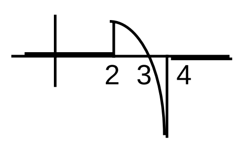
6. (a)



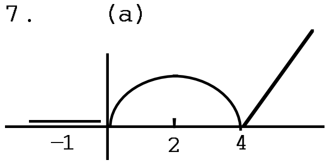
(b)



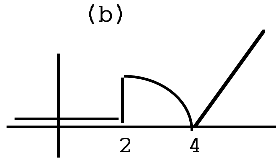
(c)



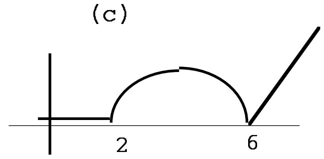
7. (a)



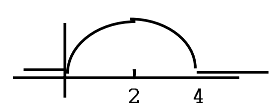
(b)



(c)



(d)



8. (a) $e^{-t}\left(u(t) - u(t-2) \right) = e^{-t}u(t) - e^{-t}u(t-2)$

$$= e^{-t}u(t) - e^{-(t-2)-2}\,u(t-2)$$

$$= e^{-t}u(t) - e^{-2}\,e^{-(t-2)}\,u(t-2)$$

$$\leftrightarrow \quad \frac{1}{s+1} - \frac{e^{-2}\,e^{-2s}}{s+1} = \frac{1 - e^{-2-2s}}{s+1}$$

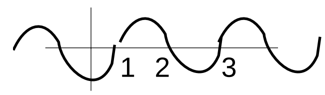
(b) $f(t) = t^3 \left(u(t) - u(t-2) \right) = t^3\, u(t) - t^3\, u(t-2)$

$$= t^3\, u(t) - \left[(t-2) + 2 \right]^3 u(t-2)$$

$$= t^3\, u(t) - \left((t-2)^3 + 6(t-2)^2 + 12(t-2) + 8 \right) u(t-2)$$

$$F(s) = \frac{3!}{s^4} - e^{-2s} \left[\frac{3!}{s^4} + \frac{6 \cdot 2!}{s^3} + \frac{12}{s^2} + \frac{8}{s} \right]$$

(c) The function f(t) in the problem is a piece of the sine curve $-\sin \pi t$. So



graph of $-\sin \pi t$

$$f(t) = -\sin \pi t \left[u(t-1) - u(t-3) \right]$$

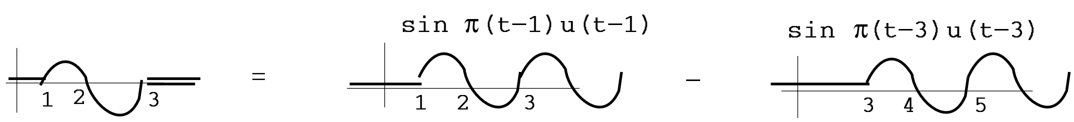
$$= u(t-1) - u(t-3)) = -\sin \pi t\, u(t-1) - \sin \pi t\, u(t-3)$$

$$= -\sin\left(\pi(t-1) + \pi \right) u(t-1) - \sin\left(\pi(t-3) + 3\pi \right) u(t-3)$$

$$= \sin \pi(t-1)\, u(t-1) + \sin \pi(t-3)\, u(t-3)$$

since $\sin(x+\pi) = \sin(x+3\pi) = -\sin x$

Can also get this by decomposing as follows:



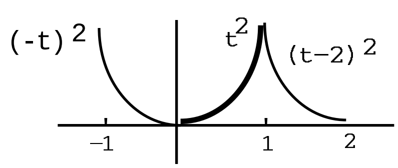
Then $F(s) = \dfrac{\pi e^{-s}}{s^2 + \pi^2} - \dfrac{\pi e^{-3s}}{s^2 + \pi^2}$

(d) Here's one way to find the mirror image.

The original piece is t^2 for $0 \le t \le 1$.

First reflect in the y-axis. The reflection is $(-t)^2$ for $-1 \le t \le 0$ which simplifies to t^2 for $-1 \le t \le 0$.

Then translate the reflection to the right 2 so that it starts at t=1 instead of t=−1. The result is $(t-2)^2$ for $1 \le t \le 2$. That's the mirror image.



So

$$f(t) = t^2 \left[u(t) - u(t-1) \right] + (t-2)^2 \left[u(t-1) - u(t-2) \right]$$

$$= t^2\, u(t) - t^2\, u(t-1) + (t-2)^2\, u(t-1) - (t-2)^2\, u(t-2)$$

$$= t^2\, u(t) - 4t\, u(t-1) + 4u(t-1) - (t-2)^2\, u(t-2)$$

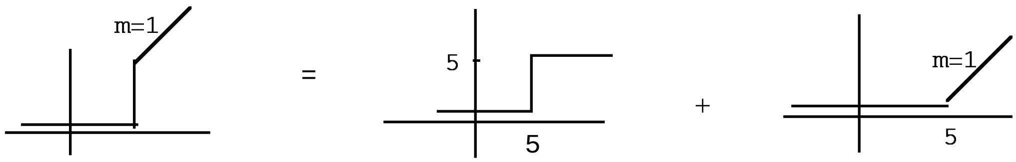
$$= t^2\, u(t) - 4\Big[\, (t-1) + 1 \Big]\, u(t-1) \;+ 4u(t-1) - (t-2)^2\, u(t-2)$$

$$= t^2\, u(t) - 4(t-1)\, u(t-1) \;- (t-2)^2\, u(t-2)$$

$$F(s) = \frac{2}{s^3} - \frac{4e^{-s}}{s^2} - \frac{2e^{-2s}}{s^3}$$

9. *method 1* The diagram shows that $tu(t-5) = 5u(t-5) + r(t-5)$

So $tu(t-5) \leftrightarrow \dfrac{5}{s}\, e^{-5s} + \dfrac{1}{s^2\, e^{-5s}}$



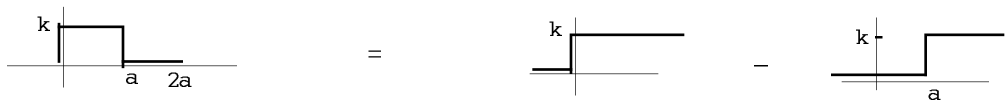
method 2 Use algebra to write

$$tu(t-5) = [(t-5)+5)]\; u(t-5) = \underbrace{(t-5)u(t-5)}_{r(t-5)} + 5u(t-5)$$

and now continue as in method 1.

10. (a) Period is 2a.

one period's worth



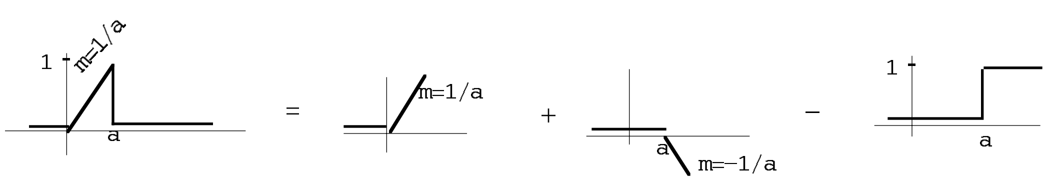
$$\leftrightarrow \quad \frac{k}{s} - \frac{ke^{-as}}{s} = \frac{k}{s}\,(1 - e^{-as})$$

Answer is $\dfrac{k}{s}\,\dfrac{1 - e^{-as}}{1 - e^{-2as}}$ which happens to simplify neatly to

$$\frac{k}{s}\,\frac{1 - e^{-as}}{(1-e^{-as})\,(1+e^{-as})} = \frac{k}{s}\,\frac{1}{1 + e^{-as}}$$

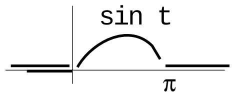
(No, you don't have to simplify on an exam)

(b) Period is a



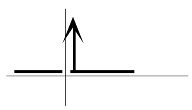
answer is $\dfrac{1}{1-e^{-as}}\left[\; \dfrac{1}{as^2} - \dfrac{e^{-as}}{as^2} - \dfrac{e^{-as}}{s} \;\right]$

(c) Period is π. Example 4 found that the transform of



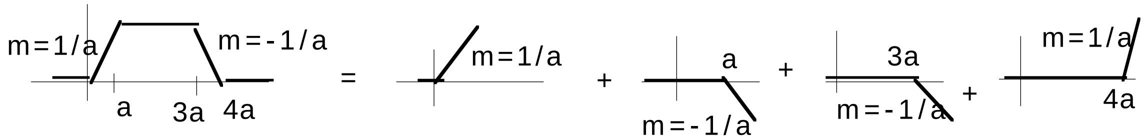
is $\dfrac{1 + e^{-\pi s}}{s^2 + 1}$. So answer is $\dfrac{1}{1 - e^{-\pi s}} \; \dfrac{1 + e^{-\pi s}}{s^2 + 1}$

(d) Period is a. Transform of



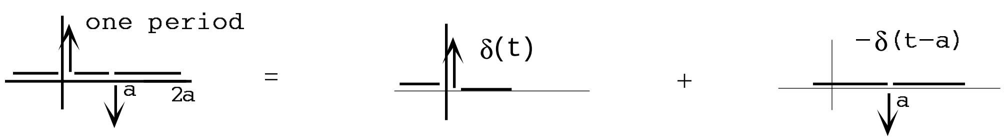
is 1. Answer is $\dfrac{1}{1 - e^{-as}}$

(e) Period is 4a



Answer is $\dfrac{1}{1-e^{-4as}} \; \dfrac{1}{as^2} \; \left[1 - e^{-as} - e^{-3as} + e^{-4as} \right]$

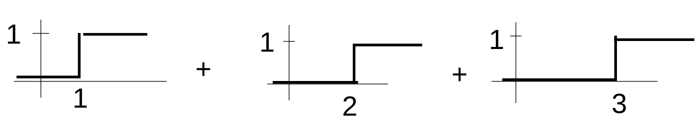
(f) Period is 2a



$\leftrightarrow 1 - 1 \cdot e^{-as}$

Answer is $\dfrac{1 - e^{-as}}{1-e^{-2as}} = \dfrac{1 - e^{-as}}{(1-e^{-as})(1+e^{-as})} = \dfrac{1}{1 + e^{-as}}$

11. (a)

staircase =  + ...

$= u(t-1) + u(t-2) + u(t-3) + \dots$

$\leftrightarrow \dfrac{1}{s}\left(e^{-s} + e^{-2s} + e^{-3s} + \dots\right)$

$= \dfrac{1}{s} \left[e^{-s} + (e^{-s})^2 + (e^{-s})^3 + \dots \right]$

The series in the brackets is a geometric series with $a = e^{-s}$, $r = e^{-s}$ and sum $\dfrac{e^{-s}}{1-e^{-s}}$. So the answer is $\dfrac{e^{-s}}{s(1-e^{-s})}$. It looks a little simpler if you multiply numerator and denom by e^{s} to get $\dfrac{1}{s(e^{s}-1)}$.

(b) $f(t) = e^{-2t}\left[u(t) - u(t-1)\right] + e^{-2t}\left[u(t-2) - u(t-3)\right]$

$$+ e^{-2t}\left[u(t-4) - u(t-5)\right] + \dots$$

$$= e^{-2t}u(t) - e^{-2t}u(t-1) + e^{-2t}u(t-2) - e^{-2t}u(t-3) + \dots$$

$$= e^{-2t}u(t) - e^{-2(t-1)-2}u(t-1) + e^{-2(t-2)-4}u(t-2) - e^{-2(t-3)-6}u(t-3) + \dots$$

$$= e^{-2t}u(t) - e^{-2}e^{-2(t-1)}u(t-1) + e^{4}e^{-2(t-2)}u(t-2) - e^{-6}e^{-2(t-3)}u(t-3) + \dots$$

$$F(s) = \frac{1}{s+2}\left(1 - e^{-2}e^{-s} + e^{-4}e^{-2s} - e^{-6}e^{-3s} + \dots\right)$$

$$= \frac{1}{s+2}\left[1 - e^{-(s+2)} + \left[e^{-(s+2)}\right]^2 - \left[e^{-(s+2)}\right]^3 + \dots\right]$$

The series in the brackets is geometric with $a = 1$, $r = -e^{-(s+2)}$.

If $s > -2$ then $-(s+2)$ is negative, r is between 0 and 1 and in that case the series converges and

$$F(s) = \frac{1}{s+2}\,\frac{1}{1+e^{-(s+2)}}$$

12. Use s-shifting (a) $\dfrac{s-3}{(s-3)^2+16}$ (b) $\dfrac{2}{(s+4)^3}$

13. The integral happens to be the transform of $(t-5)^7\,u(t-5)$. Here's why.
The transform of $(t-5)^7\,u(t-5)$ is

$$\int_{t=0}^{\infty} e^{-st}(t-5)^7\,u(t-5)\,dt$$

But $u(t-5)$ is 0 for $t \le 5$ and 1 if $t \ge 5$ so the integral becomes

$$\int_{t=5}^{\infty} e^{-st}(t-5)^7\,dt$$

So the answer is $\dfrac{7!}{s^8}e^{-5s}$.

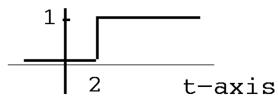
SOLUTIONS Section 5.3

1. (a) $5t\ u(t)$ (b) $\frac{5t^3}{3\,!}\ u(t)$

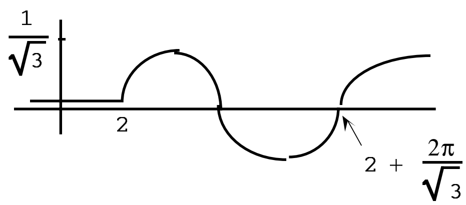
(c) $5e^{3t}\ u(t)$ (d) $\frac{4}{\sqrt{5}}\ \sin\sqrt{5}\ t\ u(t)$

(e) $\frac{-1}{s-3} \leftrightarrow -e^{3t}\ u(t)$ (f) $4\cos\sqrt{5}\ t\ u(t)$

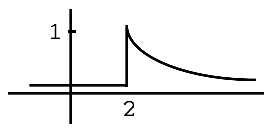
2. (a) $u(t-2)$



(b) $\frac{1}{\sqrt{3}}\ \sin\sqrt{3}(t-2)\ u(t-2)$

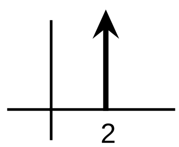


(c) $e^{-3(t-2)}\ u(t-2)$



(d) $1 \leftrightarrow \delta(t)$ so

$e^{-2s}\cdot 1 \leftrightarrow \delta(t-2)$



3.(a) $\frac{1}{2}\ t^2\ e^{3t}\ u(t)$ (b) $\frac{1}{6}\ t^3\ e^{-2t}\ u(t)$ (c) $te^{5t}\ u(t)$ (d) $e^{-6t}\ u(t)$

(e) You know that $\frac{1}{(s+6)^8} \leftrightarrow \frac{t^7}{7\,!}\ e^{-6t}\ u(t)$ by the s-shifting rule.

Now t-shift everything because of the e^{-3s} to get answer $\frac{(t-3)^7}{7\,!}\ e^{-6(t-3)}\ u(t-3)$

4. (a) $£^{-1}\ \frac{2}{3}\ \frac{1}{s+\frac{4}{3}} = \frac{2}{3}\ e^{-4t/3}\ u(t)$

(b) $\frac{1}{2}\ \frac{1}{s+\frac{1}{2}} \leftrightarrow \frac{1}{2}\ e^{-t/2}\ u(t)$

(c) $\frac{3s}{2s^2+5} = \frac{3}{2}\ \frac{s}{s^2+\frac{5}{2}} \leftrightarrow \frac{3}{2}\cos\sqrt{\frac{5}{2}}\,t$

(d) $\frac{1}{s^2+5} \leftrightarrow \frac{1}{\sqrt{5}}\ \sin\sqrt{5}\,t\ u(t)$

(e) $\frac{1}{\sqrt{5}}\ e^{-4t}\ \sin\sqrt{5}\,t\ u(t)$ (s-shifting rule)

(f) $\frac{1}{\sqrt{5}} \sin\sqrt{5}(t-2)\; u(t-2)$ (t-shifting rule)

(g) $e^{-4(t-2)} \frac{1}{\sqrt{5}} \sin\sqrt{5}(t-2)\; u(t-2)$ (s-shifting and t-shifting)

(h) $\frac{1}{s^2 + 2s} = \frac{1}{s(s+2)} \leftrightarrow \frac{1}{2}(1 - e^{-2t})\; u(t)$ (tables (17) with a=0, b=-2)

(i) $\frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2} \leftrightarrow te^{-t}\; u(t)$ (use s-shifting rule)

Can't use (17) in the tables with $a = -1$, $b = -1$ because then you end up dividing by 0 on the other side. Formula (17) is meant to be used only when $a \neq b$.

(j) $\frac{s+1}{s^2 - 3s + 3} = \frac{s+1}{(s-\frac{3}{2})^2 + \frac{3}{4}} = \frac{(s-\frac{3}{2})+\frac{5}{2}}{(s-\frac{3}{2})^2 + \frac{3}{4}}$

$$= \frac{s-\frac{3}{2}}{(s-\frac{3}{2})^2 + \frac{3}{4}} + \frac{\frac{5}{2}}{(s-\frac{3}{2})^2 + \frac{3}{4}}$$

$$\leftrightarrow e^{3t/2} \cos \tfrac{1}{2}\sqrt{3}\, t + \tfrac{5}{2} e^{3t/2} \frac{2}{\sqrt{3}} \sin \tfrac{1}{2}\sqrt{3}\, t$$

5. (a) (i) $\frac{1}{s^2 - a^2} = \frac{1}{(s-a)(s+a)} = \frac{1/2a}{s-a} + \frac{-1/2a}{s+a}$

Inverse trans is

$$\frac{1}{2a} e^{at}u(t) - \frac{1}{2a} e^{-at}\, u(t) = \frac{1}{a} \frac{e^{at} - e^{-at}}{2}\, u(t) = \frac{1}{a} \sinh at\; u(t)$$

(ii) $\frac{s+1}{(s-a)^2} = \frac{1}{s-a} + \frac{a}{(s-a)^2}$. Inverse transform is

$$e^{at}\, u(t) + ate^{at}\, u(t) \quad \text{(s-shifting)}$$

(b) You know $\frac{1}{s^2} \leftrightarrow tu(t)$ so $\frac{1}{(s-a)^2} \leftrightarrow te^{at}\, u(t)$ by s-shifting

6. $\frac{s}{s+1} = 1 - \frac{1}{s+1}$. Inverse trans is $\delta(t) - e^{-t}\, u(t)$

7. (a) Decomp is of the form $\frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2}$

Inverse trans is of the form $(A\frac{t^2}{2!} + Bt + C + De^{2t})\; u(t)$

(b) Decomp is of the form $\frac{A}{s-1} + \frac{B}{(s+2)^4} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^2} + \frac{E}{s+2}$

Inverse trans is of the form $\left(Ae^{t} + B\frac{t^3}{3!}e^{-2t} + C\frac{t^2}{2}e^{-2t} + Dte^{-2t} + Ee^{-2t}\right)u(t)$

8.(a) $$\frac{s^2+6s+5}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5} = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+1}$$

$$= \frac{A}{s} + \frac{B(s+2) - 2B + C}{(s+2)^2+1}$$

$$= \frac{A}{s} + \frac{B(s+2)}{(s+2)^2+1} + \frac{-2B+C}{(s+2)^2+1}$$

Inverse trans is $\left[A + Be^{-2t}\cos t + (C-2B)\,e^{-2t}\sin t\right]u(t)$ STOP HERE

It turns out that $A = 1,\ B = 0,\ C = 2$ so the actual answer is
$(1 + 2e^{-2t}\sin t)\,u(t)$

(b) Rewrite as $\frac{s}{s^2(s^2+1)} + \frac{1}{s^2(s^2+1)} = \frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)}$

Use tables (10),(11) to get inverse trans $(1 - \cos t + t - \sin t)u(t)$

(c) $$\frac{1}{(s+2)^2+3} \leftrightarrow \left[\frac{1}{\sqrt{3}}e^{-2t}\sin\sqrt{3}\,t\right]u(t)$$

(d) $$\frac{s}{s^2+3s+3} = \frac{(s+\frac{3}{2}) - \frac{3}{2}}{(s+\frac{3}{2})^2+\frac{3}{4}} = \frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{3}{4}} - \frac{\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{3}{4}}$$

$$\leftrightarrow e^{-\frac{3}{2}t}u(t)\left[\cos\sqrt{\frac{3}{4}}\,t - \frac{3}{2}\sqrt{\frac{4}{3}}\sin\sqrt{\frac{3}{4}}\,t\right]$$

(e) $$\frac{s}{s^2+2} + \frac{3}{s^2+2} \leftrightarrow \left[\cos\sqrt{2}\,t + \frac{3}{\sqrt{2}}\sin\sqrt{2}\,t\right]u(t)$$

(f) $$\frac{1}{2}\,\frac{s+4}{s^2+2s+\frac{5}{2}} = \frac{1}{2}\,\frac{(s+1)+3}{(s+1)^2+\frac{3}{2}}$$

$$\leftrightarrow \frac{1}{2}e^{-t}u(t)\left[\cos\sqrt{\frac{3}{2}}\,t + 3\sqrt{\frac{2}{3}}\sin\sqrt{\frac{3}{2}}\,t\right]$$

(g) $te^{-4t}u(t)$ (s-shifting)

(h) Use tables (21). Inverse transform is $e^{-4t}(1-4t)\,u(t)$

(i) $\frac{1}{2}\sin 2t\ u(t)$

(j) $\cos 2t\ u(t)$

(k) $$\frac{10}{(s-2)^2} - \frac{4s}{(s-2)^2} \leftrightarrow 10te^{2t}u(t) - 4e^{2t}(2t+1)\,u(t) \quad \text{(tables (21))}$$

(ℓ) Use tables (20). $\cosh\sqrt{3}\,t$

9. (see end of §5.1 for cosh and sinh formulas) $\cosh\sqrt{3}\,t = \dfrac{e^{\sqrt{3}\,t} + e^{-\sqrt{3}\,t}}{2}$

10. (a) $$\frac{1}{s^2 + 2s + 4} = \frac{1}{(s+1)^2 + 3} \leftrightarrow \frac{1}{\sqrt{3}} e^{-t} \sin\sqrt{3}\,t$$

(b) The factoring rule is

$$as^2 + bs + c = a\left[s - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[s - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]$$

so

$$\frac{1}{s^2 + 2s + 4} = \frac{1}{\left[s-(-1+i\sqrt{3})\right]\left[s-(-1-i\sqrt{3})\right]}$$

$$\leftrightarrow \frac{1}{2i\sqrt{3}}\left[e^{(-1+i\sqrt{3})t} - e^{(-1-i\sqrt{3})t}\right]$$

$$= \frac{1}{2i\sqrt{3}}\left[e^{-t}(\cos\sqrt{3}\,t + i\sin\sqrt{3}\,t) - e^{-t}(\cos\sqrt{3}\,t - i\sin\sqrt{3}\,t)\right]$$

$$= \frac{1}{2i\sqrt{3}}\, 2e^{-t}\, i\sin\sqrt{3}\,t$$

$$= \frac{1}{\sqrt{3}} e^{-t}\sin\sqrt{3}\,t \text{ (same answer as part (a))}$$

Yes, it works. When you convert the complex exponentials back to sines and cosines, the i's will cancel out.

SOLUTIONS Section 5.4

1. $2[s^2Y - s\cdot -5 - 6] + 3[sY - 5] + 4Y = \dfrac{3}{(s+8)^2 + 9}$ (s−shifting)

$$Y = \frac{3}{\left[(s+8)^2 + 9\right]\left[2s^2 + 3s + 4\right]} + \frac{27 - 10s}{2s^2 + 3s + 4}$$

2. (a) $s^2Y + Y = \dfrac{3}{s^2 + 9}$, $Y = \dfrac{3}{(s^2 + 1)(s^2 + 9)}$

$y(t) = (\frac{3}{8}\sin t - \frac{1}{8}\sin 3t)\,u(t)$ (tables (14))

(b) $s^2Y - 2s + Y = \dfrac{2s}{s^2 + 1}$, $Y = \dfrac{2s}{s^2 + 1} + \dfrac{2s}{(s^2 + 1)^2}$

$y(t) = (2\cos t + t\sin t)u(t)$ (tables (6) and (8))

(c) $sI + 5I = \dfrac{125}{s^2 + 25}$,

$$I(s) = \frac{125}{(s+5)(s^2 + 25)}$$

$i(t) = 125\,\frac{1}{50}\,(e^{-5t} - \cos 5t + \sin 5t)\,u(t)$

Steady state solution is $-\frac{5}{2}\cos 5t + \frac{5}{2}\sin 5t$, harmonic oscillation with amplitude $\sqrt{(-\frac{5}{2})^2 + (\frac{5}{2})^2} = \frac{5}{2}\sqrt{2}$ and frequency 5 cycles per 2π seconds.

(d) $s^2 Y + 3sY + 2Y = \dfrac{1}{s+1}$

$$Y = \frac{1}{(s+1)(s^2 + 3s + 2)} = \frac{1}{(s+1)^2(s+2)}$$

$y(t) = (e^{-2t} - e^{-t} + te^{-t})\,u(t)$

3.(a) $f(t) = u(t) - u(t-1)$, $s^2Y + 2Y = \dfrac{1}{s} - \dfrac{1}{s}e^{-s}$

$$Y = \frac{1}{s(s^2 + 2)} - \frac{e^{-s}}{s(s^2 + 2)}$$

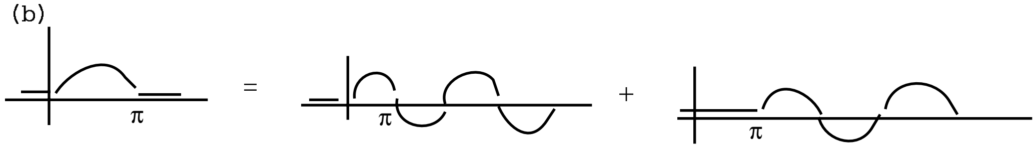
$y(t) = \frac{1}{2}(1 - \cos\sqrt{2}\,t)u(t) - \frac{1}{2}\left[1 - \cos\sqrt{2}\,(t-1)\right]u(t-1)$

(the second inverse is the same as the first but t−shifted)

So

$$y(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1}{2}(1 - \cos\sqrt{2}\,t) & \text{if } 0 \le t \le 1 \\ -\frac{1}{2}\cos\sqrt{2}\,t + \frac{1}{2}\cos\sqrt{2}\,(t-1) & \text{if } t \ge 1 \end{cases}$$

(b)

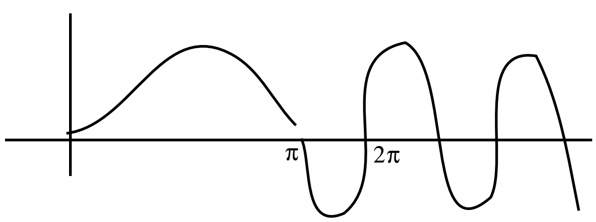


$f(t) = \sin t \; u(t) + \sin(t-\pi)\; u(t-\pi)$

$$s^2Y + 4Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}, \qquad Y = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+9)}$$

$$y(t) = \left[-\frac{1}{6}\sin 2t + \frac{1}{3}\sin t\right] u(t)$$
$$+ \left[-\frac{1}{6}\underbrace{\sin 2(t-\pi)}_{\sin 2t} + \frac{1}{3}\underbrace{\sin(t-\pi)}_{-\sin t}\right] u(t-\pi)$$

$$y(t) = \begin{cases} 0 & \text{if } t \le 0 \\ -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t & \text{if } 0 \le t \le \pi \\ -\frac{1}{3}\sin 2t & \text{if } t \ge \pi \end{cases}$$



Steady state sol is $-\frac{1}{3}\sin 2t$, the response for large t (in fact the response for $t \ge \pi$)

4.(a) $sX - 2 = 7X + 6Y, \quad sY - 1 = 2X + 6Y,$

$$\begin{aligned} (s-7)X - 6Y &= 2 \\ -2X + (s-6)Y &= 1 \end{aligned}$$

$$X = \frac{\begin{vmatrix} 2 & -6 \\ 1 & s-6 \end{vmatrix}}{\begin{vmatrix} s-7 & -6 \\ -2 & s-6 \end{vmatrix}} = \frac{2s-6}{s^2 - 13s + 30} = \frac{2}{s-10}, \quad x = 2e^{10t}\, u(t)$$

$$Y = \frac{\begin{vmatrix} s-7 & 2 \\ -2 & 1 \end{vmatrix}}{s^2 - 13s + 30} = \frac{1}{s-10}, \qquad y = e^{10t}\, u(t)$$

(b) $sX - 2 = 2X - 2Y, \quad sY - 2 = X$

$$\begin{aligned} (s-2)X + 2Y &= 2 \\ -X + sY &= 2 \end{aligned}$$

$$X = \frac{\begin{vmatrix} 2 & 2 \\ 2 & s \end{vmatrix}}{\begin{vmatrix} s-2 & 2 \\ -1 & s \end{vmatrix}} = \frac{2s-4}{s^2 - 2s + 2} = \frac{2(s-1) - 2}{(s-1)^2 + 1}, \quad x = (2e^t \cos t - 2e^t \sin t)u(t)$$

$$Y = \frac{\begin{vmatrix} s-2 & 2 \\ -1 & 2 \end{vmatrix}}{s^2 - 2s + 2} = \frac{2s-2}{s^2 - 2s + 2} = \frac{2(s-1)}{(s-1)^2 + 1}, \quad y = 2e^t \cos t \, u(t)$$

(c) $sY_1 = 10Y_2 - 20Y_1 + \frac{100}{s}$, $sY_2 = 10Y_1 - 20Y_2$

$$(s+20)Y_1 - 10Y_2 = \frac{100}{s}$$
$$-10Y_1 + (s+20)Y_2 = 0$$

$$Y_1 = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{100(s+2)}{s(s+30)(s+10)}$$

Now either decompose into $\frac{20/3}{s} + \frac{-1/5}{s+30} + \frac{-5}{s+10}$ or use tables (22) and (23)

$$y_1 = \left(\frac{20}{3} - \frac{5}{3} e^{-30t} - 5e^{-10t}\right) u(t)$$

$$Y_2 = \frac{\begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{1000}{s(s+30)(s+10)}$$

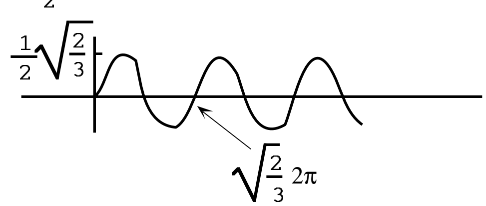
Now either decompose into $\frac{10/3}{s} + \frac{5/3}{s+30} - \frac{5}{s+10}$ or use tables

$$y_2 = \left(\frac{10}{3} + \frac{5}{3} e^{-30t} - 5e^{-10t}\right) u(t)$$

5. (a) Solve $2y'' + 3y = \delta(t)$ with IC $y(0) = 0$, $y'(0) = 0$. Take transforms to get

$2s^2Y + 3Y = 1$, $\quad Y = \frac{1}{2s^2 + 3} = \frac{1}{2} \frac{1}{s^2 + \frac{3}{2}}$ Invert to get impulse response

$$h(t) = \frac{1}{2}\sqrt{\frac{2}{3}} \sin \sqrt{\frac{3}{2}}\, t \quad u(t)$$



Question I often get asked

Do you always make the IC zero? The problem didn't say anything about IC.

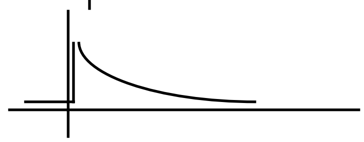
Answer If you want to find the impulse response then yes you make the IC zero and use $\delta(t)$ as the forcing function. The impulse response is *defined* as the response of an *initially-at-rest* system to the delta function input. So if you ask me on an exam if you should make the IC zero I won't answer because it's included in the definition of the impulse response.

You can use the delta function as the input into a system that is not initially at rest but then the response is not called the impulse response.

(b) $H(s) = \dfrac{1}{s^2 + 5s + 6} = \dfrac{-1}{s+3} + \dfrac{1}{s+2}$, $h(t) = (e^{-2t} - e^{-3t})\, u(t)$, i.e.,

$h(t) = e^{-2t} - e^{-3t}$ for $t \geq 0$

(c) $H(s) = \dfrac{1}{s+1}$, $h(t) = e^{-t}u(t)$



6. Given $g(t) \leftrightarrow G(s)$, i.e., $f'(t) \leftrightarrow G(s)$. But $f'(t) \leftrightarrow sF(s) - f(0)$

So $sF(s) - f(0) = G(s)$. And $f(0) = 0$ (see this from the graph) so $F(s) = \dfrac{G(s)}{s}$.

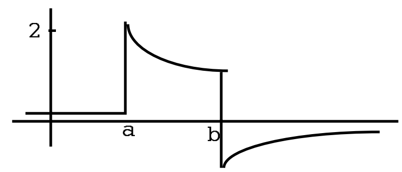
honors

7. $f(t) = 4\left[u(t-a) - u(t-b)\right]$

$$2Y + \frac{1}{s}Y = \frac{4}{s}(e^{-as} - e^{-bs}), \quad Y = \frac{4}{2s+1}(e^{-as} - e^{-bs}) = \frac{2}{s+\frac{1}{2}}e^{-as} - \frac{2}{s+\frac{1}{2}}e^{-bs}$$

$$y = 2e^{-\frac{1}{2}(t-a)}u(t-a) - 2e^{-\frac{1}{2}(t-b)}u(t-b)$$

$$= \begin{cases} 0 & \text{if } t \le a \\ 2e^{-\frac{1}{2}(t-a)} & \text{if } a \le t \le b \\ 2e^{-\frac{1}{2}(t-a)} - 2e^{-\frac{1}{2}(t-b)} & \text{if } t \ge b \end{cases}$$



(b) Delete $\frac{1}{s}$ to get $\frac{1}{s(s^2+a^2)}$. From (10) in the tables, the inverse is

$\frac{1}{a^2}(1 - \cos at)$. Then take $\int_0^t$. Final answer is

$$\frac{1}{a^2}\int_0^t (1 - \cos at)\, dt = \frac{1}{a^2}t - \frac{1}{a^3}\sin at \Big|_0^t = \frac{1}{a^3}(at - \sin at)$$

SOLUTIONS Section 5.5

1. $f(t) = 2u(t) - r(t) + r(t-2)$

$$F(s) = -\frac{1}{s^2} + \frac{2}{s} + \frac{1}{s^2}e^{-2s}$$

$$G(s) = \frac{1}{s+1}$$

$$F(s)G(s) = \frac{-1}{s^2(s+1)} + \frac{2}{s(s+1)} + e^{-2s}\frac{1}{s^2(s+1)}$$

$$= \frac{-1}{s^2(s+1)} + \frac{2}{s} - \frac{2}{s+1} + e^{-2s}\frac{1}{s^2(s+1)}$$

With the u notation

$f(t)*g(t) = -(e^{-t} + t - 1)u(t) + (2 - 2e^{-t})u(t) + (e^{-(t-2)} + (t-2) - 1)\ u(t-2).$

Without the u notation

If $0 \le t \le 2$ then $f*g = -e^{-t} - t + 1 + 2 - 2e^{-t} = 3 - t - 3e^{-t}$

If $t \ge 2$ then $f*g = 3 - t - 3e^{-t} + e^{-(t-2)} + (t-2) - 1 = -3e^{-t} + e^{-(t-2)}$

All in all

$$f*g = \begin{cases} 0 & \text{if } t \le 0 \\ 3 - t - 3e^{-t} & \text{if } 0 < t < 2 \\ -3e^{-t} + e^{-(t-2)} & \text{if } t > 2 \end{cases}$$

2. $f(t) = 4u(t-3) - 4u(t-7)$, $g(t) = 5u(t) - 5u(t-6)$

$$F(s) = \frac{4}{s}e^{-3s} - \frac{4}{s}e^{-7s}, \qquad G(s) = \frac{5}{s} - \frac{5}{s}e^{-6s}$$

$$F(s)G(s) = \frac{20}{s^2}e^{-3s} - \frac{20}{s^2}e^{-7s} - \frac{20}{s^2}e^{-9s} + \frac{20}{s^2}e^{-13s}$$

$$f(t)*g(t) = 20(t-3)u(t-3) - 20(t-7)u(t-7) - 20(t-9)u(t-9) + 20(t-13)u(t-13)$$

$$= \begin{cases} 0 & \text{if } t \le 3 \\ 20t - 60 & \text{if } 3 \le t \le 7 \\ 80 & \text{if } 7 \le t \le 9 \\ 260 - 20t & \text{if } 9 \le t \le 13 \\ 0 & \text{if } t \ge 13 \end{cases}$$

3. $F(s) = \dfrac{1}{s+1}$, $G(s) = \dfrac{1}{s^2}$, $F(s)G(s) = \dfrac{1}{s^2(s+1)}$.

$$f(t)*g(t) = (-1 + t + e^{-t})u(t)$$

4. $F(s) = \dfrac{\lambda}{s+\lambda}$, $[F(s)]^4 = \dfrac{\lambda^4}{(s+1)^4}$. Answer is inverse trans $\dfrac{\lambda^4 t^3}{3\,!}\ e^{-\lambda t}\ u(t)$

5. $s^2Y - sK_1 - K_2 + a^2Y = F(s)$

$$Y = \frac{F(s) + sK_1 + K_2}{s^2 + a^2} = \frac{1}{s^2 + a^2}\ F(s) + \frac{sK_1}{s^2 + a^2} + \frac{K_2}{s^2 + a^2}$$

$$\frac{1}{s^2 + a^2} \leftrightarrow \frac{1}{a} \sin at\ u(t)$$

$$F(s) \leftrightarrow f(t)\ u(t)$$

so $\dfrac{1}{s^2 + a^2}\ F(s) \leftrightarrow \dfrac{1}{a} \sin at\ u(t) * f(t)u(t)$

Then

$$y(t) = \frac{1}{a} \sin at\ u(t) * f(t)u(t) + K_1 \cos at\ u(t) + \frac{K_2}{a} \sin at\ u(t)$$

If you assume that $f(t) = 0$ for $t \le 0$ then the convolution is 0 until $t = 0$ and you can write this as

$$y(t) = \frac{1}{a} \sin at * f(t) + K_1 \cos at + \frac{K_2}{a} \sin at \quad \text{for } t \ge 0$$

Honors

6. (a) *directly*

$$\delta(t)*f(t) = \int_{u=-\infty}^{\infty} \delta(u)\,f(t-u)\,du = f(t-0) \quad \text{(sifting property)} \quad = f(t)$$

with transforms The transforms of $\delta(t)$ and $f(t)$ are 1 and $F(s)$. Multiply to get $F(s)$ and invert to get answer $f(t)$

To interpret physically, first remember that the impulse response $h(t)$ is the response of an at-rest system to the unit impulse $\delta(t)$, and $h(t)*f(t)$ is the at-rest system's response to $f(t)$.

Here's what the result says: Suppose you have a system where the impulse response is $\delta(t)$; i.e., when the input is $\delta(t)$, the output is the *same* as the input (it's a copy-cat system so far). Then for *any* input $f(t)$ into the at-rest system, the output is the same as the input (the system copy-cats every input, not just deltas). In other words, *the system with impulse response $\delta(t)$ just sends any input through untouched.*

(b) *directly* $\delta(t-a)*f(t) = \int_{u=-\infty}^{\infty} \delta(u-a)\,f(t-u)\,du = f(t-a)$ by sifting

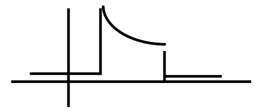
with transforms $\delta(t-a) \leftrightarrow e^{-as}$, $f(t) \leftrightarrow F(s)$ so $\delta(t-a)*f(t) \leftrightarrow e^{-as}F(s)$

Now take the inverse transform of $e^{-as}F(s)$ using s-shifting to get
$\delta(t-a)*f(t) = f(t-a)$

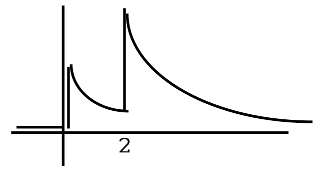
Here's the physical interpretation: Suppose a system has impulse response $\delta(t-a)$; i.e., the input $\delta(t)$ produces an identical-but-delayed response $\delta(t-a)$. Then for *any* input $f(t)$ into the at-rest system, the output is simply a delayed version of the input. *The system with impulse response $\delta(t-a)$ is just a delay.*

SOLUTIONS review problems for Chapter 5

1.(a) $f(t) = \begin{cases} e^{-5t} & \text{if } 1 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$



(b) $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-5t} & \text{if } 0 < t < 2 \\ e^{-5t} + e^{-5(t-2)} & \text{if } t \ge 2 \end{cases}$



2.(a) $s^2Y + Y = \dfrac{s}{s^2 + 2}$, $Y = \dfrac{s}{(s^2+1)^2}$, $y = \dfrac{1}{2}t \sin t \, u(t)$ (tables)

(b) $s^2 Y - s - 2 + 4(sY - 1) + 5Y = \dfrac{5}{s}$,

$$Y(s) = \frac{s^2 + 6s + 5}{s(s^2 + 4s + 5)} \quad \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$A = \left.\frac{s^2 + 6s + 5}{s^2 + 4s + 5}\right|_{s=0} = 1$$

Now get B and C.

$$s^2 + 6s + 5 = A(s^2 + 4s + 5) + (Bs + C)s$$

Equate s^2 coeffs: $1 = A + B$, $B = 1-A = 0$

Equate s coeffs: $6 = 4A + C$, $C = 2$

So $\dfrac{s^2 + 6s + 5}{s(s^2 + 4s + 5)} = \dfrac{1}{s} + \dfrac{2}{s^2 + 4s + 5} = \dfrac{1}{s} + \dfrac{2}{(s+2)^2 + 1}$

$y(t) = (1 + 2e^{-2t} \sin t)\, u(t)$

3.(a) The function is $3r(t-1) - 3r(t-2)$

trans is $\dfrac{3}{s^2}(e^{-s} - e^{-2s})$

(b) The function is $6r(t-2) - 9r(t-3) + 3r(t-5)$

Transform is $\dfrac{1}{s^2}(6e^{-2s} - 9e^{-3s} + 3e^{-5s})$

(c) function $= \left[4 - (t-2)^2\right]\left[(u(t) - u(t-4))\right]$

$$= \left[4 - (t-2)^2\right]u(t) - \Big[\underbrace{4 - [(t-4) + 2]^2}_{\text{TRICK}}\Big]\, u(t-4)$$

$$= (4t - t^2)u(t) - \left[-4(t-4) - (t-4)^2\right] u(t-4)$$

$$\leftrightarrow \quad \frac{4}{s^2} - \frac{2}{s^3} + e^{-4s}\left[\frac{4}{s^2} + \frac{2}{s^3}\right]$$

4.(a) $\mathcal{L}\, f(at) = \int_{t=0}^{\infty} f(at)\, e^{-st}\, dt$ by definition of the transform

Now let u = at, du = a dt.If t = 0 then u = 0; if t = ∞ then u = ∞. So

$$\mathcal{L}\, f(at) = \int_{u=0}^{\infty} f(u)\, e^{-s\frac{u}{a}} \frac{1}{a}\, du = \frac{1}{a}\int_{u=0}^{\infty} f(u)\, e^{\frac{-s}{a}u}\, du$$

The integral is the same as the integral for the transform of f(t) except that there is an $\frac{s}{a}$ instead of s (and the dummy variable of integration is u instead of t) So

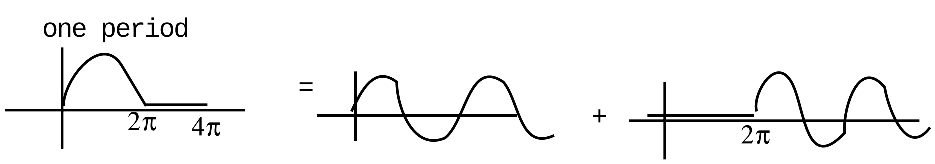
$$\mathcal{L}\, f(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

(b) $\mathcal{L}\, \sin at \sinh at = \dfrac{1}{a}\, \dfrac{2\frac{s}{a}}{\left(\frac{s}{a}\right)^4 + 4} = \dfrac{2a^2 s}{s^4 + 4a^4}$

5. $H(s) = \dfrac{1}{s^2 + s + 7} = \dfrac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{27}{4}}$,

$h(t) = e^{-t/2}\, \dfrac{2}{\sqrt{27}} \sin \frac{1}{2}\sqrt{27}\, t\; u(t)$ (s-shifting)

6. First find the transform of one period's worth (period is 4π, not 2π)



one period = $6 \sin \frac{1}{2} t\; u(t) + 6 \sin \frac{1}{2}(t-2\pi)\, u(t-2\pi)$

transform of one period is $\dfrac{3}{s^2 + \frac{1}{4}} + \dfrac{3e^{-2\pi s}}{s^2 + \frac{1}{4}}$

transform of the periodic function is $\dfrac{1}{1 - e^{-4\pi s}}\; \dfrac{3 + 3e^{-2\pi s}}{s^2 + \frac{1}{4}}$

7. (a) $\dfrac{t^3}{3!}\, u(t)$ (b) $\dfrac{t^2}{2}\, e^{-2t}\, u(t)$ (c) $5te^{4t}\, u(t)$

(d) $\dfrac{1}{3s + 4} = \dfrac{1}{3}\, \dfrac{1}{s + \frac{4}{3}} \leftrightarrow \dfrac{1}{3}\, e^{-4t/3}\, u(t)$

(e) $\frac{1}{2}(t-4)^2\ u(t-4)$

(f) $$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A = \left.\frac{s}{s^2+1}\right|_{x=-1} = -\frac{1}{2}$$

Now get B and C.

$s = A(s^2+1) + (Bs+C)(s+1)$

equate s^2 coeffs: $0 = A + B$, $B = -A = \frac{1}{2}$

equate constant terms: $0 = A + C$, $C = -A = \frac{1}{2}$

$$\frac{s}{(s+1)(s^2+1)} = \frac{-1/2}{s+1} + \frac{\frac{1}{2}s}{s^2+1} + \frac{1/2}{s^2+1}$$

inverse trans is $\left(-\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t\right)u(t)$

(g) $\cos\sqrt{2}\ t\ u(t)$

(h) $\frac{s}{s^2-1} = \frac{1/2}{s-1} + \frac{1/2}{s+1} \leftrightarrow \frac{1}{2}u(t)\ (e^t + e^{-t})$ (which is $\cosh t\ u(t)$)

(i) $\left(\frac{1}{4}e^{2t} - \frac{1}{2}t - \frac{1}{4}\right)u(t)$ (tables)

8\. (a) You know that $\int_0^\infty e^{-\boxed{s}t}\, t^4\, dt = \pounds\, t^4 = \frac{4!}{\boxed{s}^5}$ (by the definition of the transform). So $\int_0^\infty e^{-3t}\, t^4\, dt = \frac{4!}{3^5}$

(b) method 1 $$\int_0^\infty e^{-st}\, e^{-3t}\, t^4\, dt = \pounds\ e^{-3t}\, t^4 = \frac{4!}{(s+3)^5}$$

method 2 $$\int_0^\infty e^{-st}\, e^{-3t}\, t^4\, dt = \int_0^\infty e^{-(s+3)t}\, t^4\, dt = \frac{4!}{(s+3)^5}$$ as in part (a)

9\. $s^2X - s = -5X + 4Y$, $s^2Y + s = 4X - 5Y$

$$\begin{cases} (s^2+5)X - 4Y = s \\ -4X + (s^2+5)Y = -s \end{cases}$$

$$X = \frac{\begin{vmatrix} s & -4 \\ -s & s^2+5 \end{vmatrix}}{\begin{vmatrix} s^2+5 & -4 \\ -4 & s^2+5 \end{vmatrix}} = \frac{s(s^2+5)-4s}{(s^2+5)^2-16} = \frac{s(s^2+1)}{s^4+10s+9} = \frac{s}{s^2+9}$$

$$Y = \frac{\begin{vmatrix} s^2+5 & s \\ -4 & -s \end{vmatrix}}{s^4+10s+9} = \frac{-s}{s^2+9},$$

$x(t) = \cos 3t\ u(t)$, $y(t) = -3 \cos 3t\ u(t)$

10. $h(t) = r(t) - r(t-4) - 4u(t-4)$
$f(t) = 2r(t) - 2r(t-5) - 10\,u(t-5)$

$$H(s) = \frac{1}{s^2} - e^{-4s}\left[\frac{1}{s^2} - \frac{4}{s}\right],$$

$$F(s) = \frac{2}{s^2} - e^{-5s}\left[\frac{2}{s^2} - \frac{10}{s}\right]$$

$$F(s)H(s) = \frac{2}{s^4} - e^{-4s}\left[\frac{2}{s^4} + \frac{8}{s^3}\right] - e^{-5s}\left[\frac{2}{s^4} + \frac{10}{s^3}\right] + e^{-9s}\left[\frac{2}{s^4} + \frac{18}{s^3} + \frac{40}{s^2}\right]$$

$$f(t)*h(t) = \frac{2t^3}{3!}u(t) - \left[\frac{2(t-4)^3}{3!} + \frac{8(t-4)^2}{2!}\right]u(t-4)$$
$$- \left[\frac{2(t-5)^3}{3!} + \frac{10(t-5)^2}{2!}\right]u(t-5)$$
$$+ \left[\frac{2(t-9)^3}{3!} + \frac{18(t-9)^2}{2!} + 40(t-9)\right]u(t-9)$$

Now simplify.

If $0 \le t \le 4$ then $f*h = \frac{2t^3}{3!} = \frac{1}{3}t^3$

If $4 \le t \le 5$ then $f*h = \frac{1}{3}t^3 - \left[\frac{2(t-4)^3}{3!} + \frac{8(t-4)^2}{2!}\right] = 48t - \frac{128}{3}$

If $5 \le t \le 9$ then $f*h = 48t - \frac{128}{3} - \left[\frac{2(t-5)^3}{3!} + \frac{10(t-5)^2}{2!}\right] = -\frac{1}{3}t^3 + 73t - 126$

If $t \ge 9$ then $f*h = -\frac{1}{3}t^3 + 73t - 126 + \left[\frac{2(t-9)^3}{3!} + \frac{18(t-9)^2}{2!} + 40(t-9)\right] = 0$

In other words,

$$f(t)*h(t) = \begin{cases} \frac{1}{3}t^3 & \text{if } 0 \le t \le 4 \\ 48t - \frac{128}{3} & \text{if } 4 \le t \le 5 \\ -\frac{1}{3}t^3 + 73t - 126 & \text{if } 5 \le t \le 9 \\ 0 & \text{if } t \ge 9 \end{cases}$$

SOLUTIONS Section 6.1

1. (a) $\frac{2}{L}\int_0^L K \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}K\cdot -\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{2K}{n\pi}(\cos n\pi - 1)$

But $\cos n\pi = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

so $-\frac{2K}{n\pi}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4K}{n\pi} & \text{if } n \text{ is odd} \end{cases}$

(b) $\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[\int_0^{L/2} a \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^L b\sin\frac{n\pi x}{L}\,dx\right]$

$$= \frac{2}{L}\left[-a\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/2} - b\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}\right]$$

$$= \frac{2a}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) + \frac{2b}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

If n is odd then $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$ and this comes out to be $\frac{2(a+b)}{n\pi}$

If $n = 2,6,10,\ldots$ then $\cos\frac{n\pi}{2} = -1$, $\cos n\pi = 1$ and this comes out to be $\frac{4(b-a)}{n\pi}$

If $n = 4,8,12,\ldots$ then $\cos\frac{n\pi}{2} = 1$ and $\cos n\pi = 1$ and this comes out to be 0 QED

Now be grateful for the rest of the tables.

2. (a) Use (2) with L = 4 and multiply by 5 because of the $\boxed{5}x$.

integral $= \begin{cases} -40/n\pi & \text{if } n \text{ is even} \\ 40/n\pi & \text{if } n \text{ is odd} \end{cases}$

(b) Use (4b) with a = 5, b = 0, L = 6.

integral $= \begin{cases} 10/n\pi & \text{if } n = 1,5,9,\ldots \\ -10/n\pi & \text{if } n = 3,7,11,\ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$

(c) Can't use (2c) because the upper limit is 6 but it's $\sin\frac{n\pi x}{\boxed{3}}$.

integral $= \frac{2}{6}\left[\int_0^3 5\sin\frac{n\pi x}{3}\,dx + \int_3^6 0\sin\frac{n\pi x}{3}\,dx\right] = \frac{5}{3}\int_0^3 \sin\frac{n\pi x}{3}\,dx$

You can do this directly if you like or you can use (1) with L = 3 but multiply by L/2 because you're missing the 2/L in front.

$$\text{integral} = \begin{cases} \dfrac{5}{3}\,\dfrac{3}{2}\,\dfrac{4}{n\pi} & \text{if } n \text{ is odd} \\[2ex] 0 & \text{if } n \text{ is even} \end{cases}$$

(d) Can't use (4a) because f(x) breaks at 2, not in the middle of [0,6].

$$\text{integral} = \frac{2}{6}\left[\int_0^2 5 \sin\frac{n\pi x}{6}\,dx + \int_2^6 0 \sin\frac{n\pi x}{6}\,dx\right]$$

$$= \frac{5}{3}\int_0^2 \sin\frac{n\pi x}{6}\,dx = -\frac{5}{3}\,\frac{6}{n\pi}\cos\frac{n\pi x}{6}\Bigg|_0^2 = -\frac{10}{n\pi}\cos\frac{n\pi}{3} + \frac{10}{n\pi}$$

If $n = 1$ the integral is $\dfrac{5}{\pi}$

If $n = 2$ the integral is $\dfrac{15}{2\pi}$

If $n = 3$ the integral is $\dfrac{20}{3\pi}$

If $n = 4$ the integral is $\dfrac{5}{4\pi}$

If $n = 5$ the integral is $\dfrac{15}{5\pi}$

If $n = 6$ the integral is 0 etc.

(e) The graph of f(x) looks like the picture in (5) with L = 8, K =12

$$\text{integral} = \begin{cases} 0 & \text{if } n \text{ is odd or if } n = 4,8,12,\ldots \\[2ex] \dfrac{-192}{n^2\pi^2} & \text{if } n = 2,6,10,\ldots \end{cases}$$

3. (a) *Part I* The equation was separated in Part I of example 1 so I won't repeat it. The BC separate to X(0) = 0, X(4) = 0

Part II Look at the case where $X = A\cos\lambda x + B\sin\lambda x$, $T = Ce^{-k\lambda^2 t}$

X(0) = 0 makes A = 0

X(4) = 0 makes $B\sin 4\lambda = 0$, $4\lambda = n\pi$, $\lambda = \dfrac{n\pi}{4}$

So $X = B\sin\dfrac{n\pi x}{4}$ and $T = Ce^{-k\left(\frac{n\pi}{4}\right)^2 t}$.

Part III By superposition,

$$(*) \qquad u = \sum_{n=1}^{\infty} C_n\, e^{-k\left(\frac{n\pi}{4}\right)^2 t} \sin\frac{n\pi x}{4}$$

To get the IC you need

$$8 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{4} \quad \text{for } 0 \le x \le 4$$

which you can get with

$$C_n = \frac{2}{4}\int_0^4 8 \sin\frac{n\pi x}{4}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{32}{n\pi} & \text{if } n \text{ is odd} \end{cases} \qquad \text{(use tables (A) with K=8)}$$

Plug this into (*) to get final answer

$$u = \frac{32}{\pi} e^{-k\left(\frac{\pi}{4}\right)^2 t} \sin\frac{\pi x}{4} + \frac{32}{3\pi} e^{-k\left(\frac{3\pi}{4}\right)^2 t} \sin\frac{3\pi x}{4}$$

$$+ \frac{32}{5\pi} e^{-k\left(\frac{5\pi}{4}\right)^2 t} \sin\frac{5\pi x}{4} + \ldots \quad \text{for } 0 \le x \le 4,\ t \ge 0$$

(b) This is like part (a) but to satisfy the IC you need

$f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{4}$ for $0 \le x \le 4$ which you can get with

$$C_n = \frac{2}{4}\int_0^4 f(x) \sin\frac{n\pi x}{4}\,dx = \begin{cases} 0 & \text{if } n = 4,8,12,\ldots \\ \frac{24}{n\pi} & \text{if } n = 2,6,10,\ldots \\ \frac{12}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

(Use (4a) in the tables with $a = 6$, $b=0$.)
Final solution is

$$u = \frac{12}{\pi} e^{-k\left(\frac{\pi}{4}\right)^2 t} \sin\frac{\pi x}{4} + \frac{24}{2\pi} e^{-k\left(\frac{2\pi}{4}\right)^2 t} \sin\frac{2\pi x}{4}$$

$$+ \frac{12}{3\pi} e^{-k\left(\frac{3\pi}{4}\right)^2 t} \sin\frac{3\pi x}{4} + \frac{12}{5\pi} e^{-k\left(\frac{5\pi}{4}\right)^2 t} \sin\frac{5\pi x}{4}$$

$$+ \frac{24}{6\pi} e^{-k\left(\frac{6\pi}{4}\right)^2 t} \sin\frac{6\pi x}{4} + \ldots \quad \text{for } 0 \le x \le 4,\ t \ge 0$$

(c) Continuing as in part (a), for $0 \le x \le 4$ you need

$$5 \sin 2\pi x + 6 \sin 5\pi x = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{4}$$

$$= C_1 \sin\frac{\pi x}{4} + C_2 \sin\frac{2\pi x}{4} + C_3 \sin\frac{3\pi x}{4} + \ldots$$

You don't need the fancy formulas for Fourier sine coeffs to accomplish this. By inspection what you need is $C_8 = 5$, $C_{20} = 6$, other C's = 0.

Final answer is

$$u = 5\, e^{-k\left(\frac{8\pi}{4}\right)^2 t} \sin\frac{8\pi x}{4} + 6\, e^{-k\left(\frac{20\pi}{4}\right)^2 t} \sin\frac{20\pi x}{4} \quad \text{for } 0 \le x \le 4,\ t \ge 0$$

SOLUTIONS Section 6.2

1. (a) $X'(0) = 0$ (b) $X'(5) = 0$ (c) doesn't sep (d) doesn't sep (e) $X(4) = 0$ (f) doesn't sep (g) $T(0) = 0$

2.(a) I separated the equation in example 1 so I won't repeat it here.
The BC separate to $X'(0) = 0$, $X'(6) = 0$
Consider the case where

$$X = A\cos\lambda x + B\sin\lambda x, \quad T = Ce^{-k\lambda^2 t}, \quad X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x.$$

$X'(0) = 0$ makes $B = 0$

$X'(6) = 0$ makes $-\lambda A\sin 6\lambda = 0$, $6\lambda = n\pi$, $\lambda = \frac{n\pi}{6}$

Then $X = A\cos\frac{n\pi x}{6}$, $T = Ce^{-k\left(\frac{n\pi}{6}\right)^2 t}$.
Since the BC are of the form $X'(0) = 0$, $X'(L) = 0$ you should try the zero separation case; it produces the solution $X = Ax + B$, $T = C$, $X' = A$.

The BC make $A = 0$ so $u = X(x)T(t) = BC = Q$
By superposition,

$$(*) \qquad u = Q + \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{6}\right)^2 t}\cos\frac{n\pi x}{6}$$

To get the IC you need $f(x) = Q + \sum_{n=1}^{\infty} C_n\cos\frac{n\pi x}{6}$ for x in $[0,6]$
So Q = av value of $f = 7$

$$C_n = \frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi x}{6}\,dx = \begin{cases} \frac{-8}{n\pi} & \text{if } n = 1,5,9,\ldots \\ \frac{8}{n\pi} & \text{if } n = 3,7,11,\ldots \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad \text{(tables (4))}$$

Plug these into (*) to get the final answer

$$u = 7 - \frac{8}{\pi}e^{-k\left(\frac{\pi}{6}\right)^2 t}\cos\frac{\pi x}{6} + \frac{8}{3\pi}e^{-k\left(\frac{3\pi}{6}\right)^2 t}\cos\frac{3\pi x}{6} - \frac{8}{5\pi}e^{-k\left(\frac{5\pi}{6}\right)^2 t}\cos\frac{5\pi x}{6} + \ldots \text{ for } 0 \le x \le 6,\ t \ge 0$$

(b) (i) The rod is initially at 2°. The lateral surface of the rod is insulated and the ends are insulated so no calories flow out. In fact no calories flow anywhere in the rod since it is all at the same temp. So the rod stays at 2° for all time; i.e., sol is $u(x,t) = 2$ for $t \ge 0$, $0 \le x \le 6$
(ii) Continue as in (a). To get the IC you need

$$2 = Q + \sum_{n=1}^{\infty} C_n\cos\frac{n\pi x}{6} \text{ for } x \text{ in } [0,6]$$

By inspection you can get this with $Q = 2$, $C_n = 0$ (This is what you'd get if you do it the long way and use the formulas for the Fourier cosine coeffs.)
Plug these into (*) get final answer $u = 2$.

3. *Part I* Try $u(x,t) = X(x)T(t)$. Then $XT' = X''T - XT$. There are two ways to continue the separation.

method 1 $XT' = (X'' - X)T$, $\quad \dfrac{T'}{T} = \dfrac{X'' - X}{X} = \lambda$,

$$T' - \lambda T = 0, \qquad X'' - (1+\lambda)X = 0$$

The three cases to consider here are $1 + \lambda$ positive, negative, zero

method 2 $X''T = X(T' + T)$, $\quad \dfrac{X''}{X} = \dfrac{T' + T}{T} = \lambda$,

$$T' + (1-\lambda)T = 0, \qquad X'' - \lambda X = 0$$

The three cases here are λ positive, negative, zero.

The two methods will eventually produce the same collection of solutions but the second method is simpler since it makes the X part simpler (albeit at the expense of the T part). So continue with the separation in method 2.

In the case where λ is negative and renamed $-\lambda^2$ you have

$$X'' + \lambda^2 X = 0, \quad X = A\cos\lambda x + B\sin\lambda x,$$

$$T' + (1 + \lambda^2)T = 0, \quad T = Ce^{-(1+\lambda^2)t}$$

The BC separate to $X(0) = 0$ and $X(L) = 0$

Part II (plug in BC)

$X(0) = 0$ makes $A = 0$

$X(L) = 0$ makes $B\sin\lambda L = 0$, $\lambda = \dfrac{n\pi}{L}$

Part III (get a gen sol and go for the IC)
The solution is

$$(*) \qquad u = \sum_{n=1}^{\infty} C_n \, e^{-\left(1+\left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L}$$

To get the IC you need

$$8 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \qquad \text{for } x \text{ in } [0,L]$$

Can get this with $C_n = \dfrac{2}{L}\displaystyle\int_0^L 8\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{32}{n\pi} & \text{if } n \text{ is odd} \end{cases}$ (tables (1))

Plug these constants into (*). The answer is

$$u(x,t) = \frac{32}{\pi} e^{-\left(1+\left(\frac{\pi}{L}\right)^2\right)t} \sin\frac{\pi x}{L} + \frac{32}{3\pi} e^{-\left(1+\left(\frac{3\pi}{L}\right)^2\right)t} \sin\frac{3\pi x}{L}$$

$$+ \frac{32}{5\pi} e^{-\left(1+\left(\frac{5\pi}{L}\right)^2\right)t} \sin\frac{5\pi x}{L} + \ldots \text{ for } 0 \le x \le L,\ t \ge 0$$

4. (a) Plug $t = \infty$ into the solution in (5). The steady state sol is $u = 1$ because when $t \to \infty$, the exponentials all $\to 0$.

(Initially, the left half of the rod is at temp 0, the right half is at temp 2, the lateral surface and the ends are insulated, so calories flow within the rod until the temperature evens out at 1.)

(b) *physical argument*

The lateral surface of the rod is insulated, the ends are insulated, and the initial temp distribution is f(x). Calories flow within the rod (they can't escape) until the temperature "evens out". The steady state temperature distribution is a constant, namely is the average value of f(x).

mathematical version

The solution satisfying the heat equation and the BC look like (1) but with L instead of 8:

$$u = A_0 + \sum_{n=1}^{\infty} A_n \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L} \quad \text{for } 0 \le x \le L,\ t \ge 0$$

The steady state solution is the constant A_0 because when $t \to \infty$, the exponentials all $\to 0$.
To satisfy the IC you need

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \quad \text{for } 0 \le x \le L$$

so $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ = average value of f(x) on the interval [0,L]
So the steady state solution is the average value of f(x) on the interval [0,L].

5. $P'(5)Q(q) = 0$ for all q
$P'(5) = 0$ or $Q(q) = 0$ for all q

But if $Q(q) = 0$ for all q then $v(p,q) = 0$ for all q. So this is one possibility. But when it comes to using superposition to add solutions to get a good solution with lots of arbitrary constants so that some IC can be satisfied, adding in the solution $v(p,q) = 0$ will not be helpful.
So the only useful possibility is $P'(5) = 0$.
So $\frac{\partial v}{\partial p}(5,q) = 0$ for all q separates to $P'(5) = 0$.

SOLUTIONS Section 6.3

1. Use (1)−(3) with L = 6, f(x) as in the picture, g(x) = 0. Then $D_n = 0$. To find C_n use (5a) on the reference page with L = 6, K = 2:

$$C_n = \quad = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{16}{n^2\pi^2} & \text{if } n=1,5,9,\ldots \\ \dfrac{-16}{n^2\pi^2} & \text{if } n=3,7,11,\ldots \end{cases}$$

Solution is

$$y = \frac{16}{\pi^2}\left[\cos\frac{\pi at}{6}\sin\frac{\pi x}{6} - \frac{1}{9}\cos\frac{3\pi at}{6}\sin\frac{3\pi x}{6} + \frac{1}{25}\cos\frac{5\pi at}{6}\sin\frac{5\pi x}{6} - \ldots\right]$$

2. Use the solution in (1)−(3) with f(x)= 0, g(x) = $\delta(x-\frac{1}{2}L)$. Then $C_n = 0$.

$$D_n = \frac{L}{n\pi a}\ \frac{2}{L}\int_0^L \delta(x-\tfrac{1}{2}L)\sin\frac{n\pi x}{6}\,dx$$

$$= \frac{L}{n\pi a}\ \frac{2}{L}\ \sin\frac{n\pi L/2}{L} \qquad \text{(sifting property §6.2A)}$$

$$= \frac{2}{n\pi a}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2/n\pi a & \text{if } n = 1,5,9,\ldots \\ -2/n\pi a & \text{if } n = 3,7,11,\ldots \end{cases}$$

Solution is

$$y = \frac{2}{\pi a}\left[\sin\frac{\pi at}{L}\sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi at}{L}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi at}{L}\sin\frac{5\pi x}{L} - \ldots\right]$$

3. *Part I* Separate the variables

The equation was separated in Part I of example 1 so I won't repeat it all here. The two potentially useful cases are

case 3 $\quad X = A\cos\lambda x + B\sin\lambda x$
$\qquad\qquad T = C\cos\lambda at + D\sin\lambda at$

case 1 $\quad X = Ax + B$
$\qquad\qquad T = Ct + D$

The BC separate to $X'(0) = 0$, $X'(L) = 0$

Part II Plug in the homog BC

Begin with case 3 where the X solutions are sines and cosines. First find

$$X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$$

$X'(0) = 0$ makes $B = 0$

$X'(L) = 0$ makes $-\lambda A\sin\lambda L = 0$

Either A = 0 (which together with B = 0 produces only the trivial solution y = 0) or $\lambda = 0$ (impossible since in this case $-\lambda^2$ represents a negative number) or

$$\sin \lambda L = 0$$

$$\lambda L = n\pi, \quad \lambda = \frac{n\pi}{L}$$

$$X = A \cos \frac{n\pi x}{L}$$

Since the BC are $X'(0) = 0$, $X'(L) = 0$, anticipate that the zero separation case will produce a solution too. If $\lambda = 0$ then

$$X = Ax + B, \quad T = Ct + D,$$

The BC's $X'(0) = 0$, $X'(L) = 0$ force A = 0.

So X = B and from this case you have solutions

$$y(x,t) = X(x)T(t) = B(Ct + D) = Pt + Q \quad \text{for any } P, Q.$$

Part III Get a general sol and plug in the IC

By superposition, a general solution is

$$\boxed{(1) \quad y(x,t) = Pt + Q + \sum_{n=1}^{\infty} \left[C_n \cos \frac{n\pi a t}{L} + D_n \sin \frac{n\pi a t}{L} \right] \cos \frac{n\pi x}{L}}$$

Now you have to determine the constants to satisfy the IC.

To get $y(x,0) = f(x)$ for x in [0,L] you need

$$f(x) = Q + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} \quad \text{for x in } [0,L]$$

which you can get with

$$\boxed{(2) \quad Q = \frac{1}{L} \int_0^L f(x)\, dx, \qquad C_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx}$$

Now you have to satisfy the second IC. First, get

$$\frac{\partial y}{\partial t} = P + \sum_{n=1}^{\infty} \left[-\frac{n\pi a}{L} C_n \sin \frac{n\pi a t}{L} + \frac{n\pi a}{L} D_n \cos \frac{n\pi a t}{L} \right] \cos \frac{n\pi x}{L}$$

Then to get $\frac{\partial y}{\partial t}(x,0) = g(x)$ for x in [0,L] you need

$$g(x) = P + \sum_{n=1}^{\infty} \frac{n\pi a}{L} D_n \cos \frac{n\pi x}{L} \quad \text{for x in } [0,L].$$

To do this you need

$$\boxed{(3) \qquad P = \frac{1}{L}\int_0^L g(x)\, dx}$$

and
$$\frac{n\pi a}{L} D_n = \frac{2}{L}\int_0^L g(x) \cos\frac{n\pi x}{L}$$

$$\boxed{(4) \qquad D_n = \frac{2}{n\pi a}\int_0^L g(x) \cos\frac{n\pi x}{L}\, dx}$$

The solution is (1), with the constants in the solution given in (2)−(4).

footnote The solution in (1) is unrealistic in the sense that $y \to \infty$ as $t \to \infty$ (unless the coefficient P is 0). That's because the wave equation doesn't include a term representing gravity and when the ends are on rollers, it's possible for the idealized rope to move unboundedly high.

4. This is like problem 3 but with L = 2. I won't repeat the whole separation. Begin with the case where the X solutions are sines and cosines:

$$X = A\cos\lambda x + B\sin\lambda x, \qquad T = C\cos\lambda at + D\sin\lambda at$$

The BC $X'(0) = 0$, $X'(2) = 0$ make $B = 0$, $\lambda = \frac{n\pi}{2}$ so $X = A\cos\frac{n\pi x}{2}$

The second IC separates to $T'(0) = 0$ which makes $D = 0$ so $T = C\cos\frac{n\pi at}{2}$

From this case you have $y = E\cos\frac{n\pi at}{2}\cos\frac{n\pi x}{2}$

In the λ=0 case,

$$X = Ax + B, \quad T = Ct + D$$

The BC $X'(0) = 0$, $X'(2) = 0$ make $A = 0$

The IC $T'(0) = 0$ makes $C = 0$.

From this case you get $y = BD = Q$

By superposition, $y = Q + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi at}{2}\cos\frac{n\pi x}{2}$

To get the final (nohomog) IC you need

$$x = Q + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{2} \qquad \text{for } x \text{ in } [0,2]$$

Q = average value of x in [0,2] = 1

$$E_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases} \qquad \text{(ref page (3))}$$

Answer is

$$y = 1 - \frac{8}{\pi^2}\cos\frac{\pi at}{2}\cos\frac{\pi x}{2} - \frac{8}{9\pi^2}\cos\frac{3\pi at}{2}\cos\frac{3\pi x}{2} - \frac{8}{25\pi^2}\cos\frac{5\pi at}{2}\cos\frac{5\pi x}{2} - \ldots$$

5. (a) Initially, the wire lies flat on the x-axis but has velocity 3, i.e., is in the process of moving up at 3 meters per second. The ends are looped around poles, free to move up or down. There is no gravity or air resistance. So as time goes on the wire simply continues moving up at the rate of 3 meters per second. So $y(x,t) = 3t$

(b) I won't repeat the separation. One of the good cases is where
$X = A\cos\lambda x + B\sin\lambda x, \quad T = C\cos\lambda at + D\sin\lambda at$

$X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$

$X'(0) = 0$ makes $B = 0$

$X'(L) = 0$ makes $-\lambda A\sin L\lambda = 0, \quad L\lambda = n\pi, \ \lambda = \dfrac{n\pi}{L}$

The first IC is homog and it separates to $T(0) = 0$ which makes $C = 0$

In the $\lambda=0$ case,

$$X = Ax + B, \quad T = Ct + D$$

$X'(0) = 0$ ands $X'(L) = 0$ make $A = 0$
$T(0) = 0$ makes $D = 0$

Put it all together to get

$$(*) \quad y = Kt + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi at}{L}\cos\frac{n\pi x}{L}$$

Still need $\dfrac{\partial y}{\partial t}(x,0) = 3$ for x in $[0,L]$. We have

$$\frac{\partial y}{\partial t} = K + \sum_{n=1}^{\infty}\frac{n\pi a}{L}A_n\cos\frac{n\pi at}{L}\cos\frac{n\pi x}{L}$$

so you need $3 = K + \sum_{n=1}^{\infty}\dfrac{n\pi a}{L}A_n\cos\dfrac{n\pi x}{L}$ for x in $[0,L]$

You don't need fancy Fourier coeff formulas for this. By inspection, pick $K = 3$, $A_n = 0$. Plug this into (*) to get final answer $y = 3t$.

SOLUTIONS Section 6.4

1.(a) The equation was separated in Part I of example 1 so I won't repeat it. Use the case where $X = A\cos\lambda x + B\sin\lambda x$, $Y = C\cosh\lambda y + D\sinh\lambda y$

Then $X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = 0$ so $B = 0$

$X'(6) = 0$ so $-A\lambda\sin 6\lambda = 0$, $6\lambda = n\pi$, $\lambda = \frac{n\pi}{6}$

$Y(0) = 0$ so $C = 0$

Use the $\lambda=0$ separation case where $X = Ex + F$, $Y = Gy + H$

$X'(0) = 0$ and $X'(6) = 0$ make $E = 0$
$Y(0) = 0$ makes $H = 0$

This case produces solution $v = FGy = Ky$

By superposition,

$$v = Ky + \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi y}{6}\cos\frac{n\pi x}{6}$$

The top BC is $v(x,18) = f(x)$ for x in $[0,6]$ where $f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 3 \\ 3 & \text{if } 3 \le x \le 6 \end{cases}$

To get it you need

$$f(x) = 18K + \sum_{n=1}^{\infty} D_n \sinh 3n\pi \cos\frac{n\pi x}{6} \quad \text{for } x \text{ in } [0,6]$$

which you can get with

$$18K = \text{av value of } f = \frac{3}{2}$$

$$K = \frac{1}{12}$$

$$D_n \sinh 3n\pi = \frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi x}{6}\,dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ -6/n\pi & \text{if } n = 1,5,9,\ldots \\ 6/n\pi & \text{if } n = 3,7,11,\ldots \end{cases} \quad \text{(by (4b))}$$

so

$$D_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{1}{\sinh 3n\pi}\cdot -\dfrac{6}{n\pi} & \text{if } n = 1,5,9,\ldots \\ \dfrac{1}{\sinh 3n\pi}\cdot\dfrac{6}{n\pi} & \text{if } n = 3,7,11,\ldots \end{cases}$$

Solution is

$$\boxed{v = \frac{1}{12}y + \frac{6}{\pi}\left[-\frac{1}{\sinh 3\pi}\sinh\frac{\pi y}{6}\cos\frac{\pi x}{6} + \frac{1}{3\sinh 9\pi}\sinh\frac{3\pi y}{6}\cos\frac{3\pi x}{6} - \frac{1}{5\sinh 15\pi}\sinh\frac{5\pi y}{6}\cos\frac{5\pi x}{6} + \ldots\right]}$$

(b) I won't repeat the separation.
Use the case where $X = A\cos\lambda x + B\sin\lambda x$, $Y = C\cosh\lambda y + D\sinh\lambda y$.

Then $X' = -\lambda A \sin \lambda x + B \lambda \cos \lambda x$, $Y' = C\lambda \sinh \lambda y + D\lambda \cosh \lambda y$

$X'(0) = 0$ so $B = 0$

$X'(a) = 0$ so $-\lambda A \sin \lambda a = 0$, $\lambda = \frac{n\pi}{a}$

$Y'(0) = 0$ so $D = 0$

In the case where $\lambda = 0$ you have $X = Ex + F$, $Y = Gy + H$

$X'(0) = 0$ and $X'(a) = 0$ make $E = 0$
$Y'(0) = 0$ makes $G = 0$
From this case you get $v = Fh = K$

By superposition

$$\boxed{v = K + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}}$$

The last BC is $v(x,b) = f(x)$ so you need

$$f(x) = K + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi b}{a} \cos \frac{n\pi x}{a} \quad \text{for } x \text{ in } [0,a]$$

which you can get with

$$\boxed{K = \frac{1}{a} \int_0^a f(x)\, dx}, \qquad A_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx$$

$$\boxed{A_n = \frac{2}{a \cosh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx}$$

(c) I won't repeat the separation.
Use the case where $X = A \cos \lambda x + B \sin \lambda x$, $Y = C \cosh \lambda y + D \sinh \lambda y$

Then $X' = -\lambda A \sin \lambda x + B \lambda \cos \lambda x$

$X'(0) = 0$ so $B = 0$

$X'(4) = 0$ so $-\lambda A \sin 4\lambda = 0$, $\lambda = \frac{n\pi}{4}$

$Y(0) = 0$ so $C = 0$

Use the case where $X = Ex + F$, $Y = Gy + H$
$X'(0) = 0$ and $X'(4) = 0$ make $E = 0$

$Y(0) = 0$ makes $H = 0$

From this case you have $v = FGy = Ky$
By superposition

$$v = Ky + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{4} \cos \frac{n\pi x}{4}$$

Now plug in the final BC $\frac{\partial v}{\partial y}(x,5) = 2x$ for x in [0,4] . We have

$$\frac{\partial v}{\partial y} = K + \sum_{n=1}^{\infty} A_n \frac{n\pi}{4} \cosh \frac{n\pi y}{4} \cos \frac{n\pi x}{4}$$

so you need

$$2x = K + \sum_{n=1}^{\infty} A_n \frac{n\pi}{4} \cosh \frac{n\pi 5}{4} \cos \frac{n\pi x}{4} \quad \text{for } x \text{ in } [0,4],$$

K = av value of 2x in [0,4] = 4

$$\frac{n\pi}{4} A_n \cosh \frac{5n\pi}{4} = \frac{2}{4}\int_0^4 2x \cos \frac{n\pi x}{4}\, dx = \begin{cases} -\dfrac{32}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad \text{(tables (3))}$$

So $A_{\text{even } n} = 0$, $A_{\text{odd } n} = \dfrac{-128}{n^3\,\pi^3 \cosh \frac{5n\pi}{4}}$

and the final answer is

$$\boxed{v = 4y - \frac{128}{\pi^3 \cosh \frac{5\pi}{4}} \sinh \frac{\pi y}{4} \cos \frac{\pi x}{4} - \frac{128}{27\pi^3 \cosh \frac{15\pi}{4}} \sinh \frac{3\pi y}{4} \cos \frac{3\pi y}{4} - \ldots}$$

2.(a) I won't repeat the separation. Use the case where

$X = A \cos \lambda x + B \sin \lambda x$, $Y = Ce^{\lambda y} + De^{-\lambda y}$.

$X(0) = 0$ so $A = 0$

$X(5) = 0$ so $B \sin 5\lambda = 0$, $5\lambda = n\pi$, $\lambda = \frac{n\pi}{5}$

Make $C = 0$ to keep Y finite.

By superposition, $v = \sum_{n=1}^{\infty} D_n\, e^{\frac{-n\pi y}{5}} \sin \frac{n\pi x}{5}$

To get the last BC, $v(x,0) = 2$, you need

$$2 = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{5} \quad \text{for } x \text{ in } [0,5],$$

$$D_n = \frac{2}{5}\int_0^5 2 \sin \frac{n\pi x}{5}\, dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Solution is

$$\boxed{v = \frac{8}{\pi} e^{\frac{-\pi y}{5}} \sin \frac{\pi x}{5} + \frac{8}{3\pi} e^{\frac{-3\pi y}{5}} \sin \frac{3\pi x}{5} + \frac{8}{5\pi} e^{\frac{-5\pi y}{5}} \sin \frac{5\pi x}{5} + \ldots}$$

(b) I won't repeat the separation. Use the case where

$$X = A\cos\lambda x + B\sin\lambda x, \quad Y = Ce^{\lambda y} + De^{-\lambda y}.$$

$X(0) = 0$ so $A = 0$

$X(5) = 0$ so $B\sin 5\lambda = 0$, $5\lambda = n\pi$, $\lambda = \frac{n\pi}{5}$

Make $C = 0$ to keep Y finite.
By superposition,

$$(*) \qquad v = \sum_{n=1}^{\infty} D_n\, e^{\frac{-n\pi y}{5}} \sin\frac{n\pi x}{5}$$

The last BC is $\frac{\partial v}{\partial y}(x,0) = 3$ for $0 \le x \le 5$. First find

$$\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} D_n\, \frac{-n\pi}{5}\, e^{\frac{-n\pi y}{5}} \sin\frac{n\pi x}{5}$$

Then set $y = 0$, $\partial v/\partial y = 3$: you need

$$3 = \sum_{n=1}^{\infty} D_n\, \frac{-n\pi}{5} \sin\frac{n\pi x}{5} \quad \text{for } 0 \le x \le 5$$

which you can get with

$$\frac{-n\pi}{5} D_n = \frac{2}{5}\int_0^5 3\sin\frac{n\pi x}{5}\,dx = \; = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{12}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

So $D_n = \frac{-60}{n^2\pi^2}$ for odd n . Plug this back into (*) to get the solution

$$v = -\frac{60}{\pi^2}\left(e^{\frac{-\pi y}{5}} \sin\frac{\pi x}{5} + \frac{1}{9} e^{\frac{-3\pi y}{5}} \sin\frac{3\pi x}{5} + \frac{1}{25} e^{\frac{-5\pi y}{5}} \sin\frac{5\pi x}{5} + \dots \right)$$

3. Plugging it into the cosh sinh version makes $C = 0$ and leaves

$$Y = D\sinh\lambda y \quad \text{for any } D$$

Plugging it into the exp version makes $E + F = 0$, $F = -E$,

$$Y = E\,e^{\lambda y} - Ee^{-\lambda y} = E(e^{\lambda y} - e^{-\lambda y})$$

But $E(e^{\lambda y} - e^{-\lambda y}) = E\cdot 2\sinh\lambda y$ and $2E$ is just another arbitrary constant, say Q, so

$$Y = Q\sinh\lambda y \quad \text{for any } Q,$$

same as before

4. *method 1* Try $u(x,y) = X(x)Y(y)$. Then

$$X''Y + XY'' = XY$$

$$X''Y = X(Y - Y'')$$

$$\frac{X''}{X} = \frac{Y - Y''}{Y} = \lambda$$

You want good X solutions in anticipation of the condition involving $f(x)$.

case 1 λ is negative. Call it $-\lambda^2$

$\boxed{X = A\cos\lambda x + B\sin\lambda x}$

$Y'' - (1+\lambda^2)Y = 0$, $\boxed{Y = Ce^{\sqrt{1+\lambda^2}} + De^{-\sqrt{1+\lambda^2}}}$

case 2 $\lambda = 0$

$\boxed{X = Ax + B,\quad Y = Ce^{-y} + De^{y}}$

method 2 (The other way is algebraically easier). You can factor differently and end up with

$$\frac{X - X''}{X} = \frac{Y''}{Y} = \text{con}$$

$$X'' + (\text{con} - 1)X = 0, \quad Y'' - \text{con}\,Y = 0.$$

For the X problem here are the useful cases.

case 1 con $-$ 1 > 0 so let con $-$ 1 $= \lambda^2$

Then

$X'' + \lambda^2 X = 0$, $\boxed{X = A\cos\lambda x + B\sin\lambda x}$

$Y'' - (1+\lambda^2)Y = 0$ and since $1+\lambda^2$ is always positive, you have

$m = \pm\sqrt{1+\lambda^2}$, $\boxed{Y = Ce^{y\sqrt{1+\lambda^2}} + De^{-y\sqrt{1+\lambda^2}}}$

case 2 con $-$ 1 = 0, con = 1

Then

$X'' = 0$, $\boxed{X = Ax + B}$

$Y'' - Y = 0$, $\boxed{Y = Ce^{-y} + De^{y}}$

5. Try $u(x,y) = X(x)Y(y)$ Then

$xX'Y = y^2XY'' + XY'$, $xX'Y = X(y^2Y'' + Y')$, $\dfrac{xX''}{X} = \dfrac{y^2Y'' + Y'}{Y} = \lambda$,

$\boxed{xX'' = \lambda X \text{ with BC } X(0) = 0}$, $\boxed{y^2Y'' + Y' = \lambda Y \text{ with BC } Y(3) = 0}$

(The BC $u(x,5) = x^2$ doesn't separate.)

6. $\dfrac{X'' + X}{X} = \text{con}$, $\dfrac{T'}{T} = \text{con}$,

$X'' + (1-\text{con})X = 0$, $T' - \text{con}\,T = 0$

The T solution is always $Ce^{\text{con}\,t}$ but for the X part you have $m = \pm\sqrt{\text{con}-1}$ and the solution now depends on the sign of 1$-$con (not on the sign of con) *case 1* con $-1 = 0$, i.e., con $= 1$

$X'' = 0$, $\boxed{X = Ax + B}$, $\boxed{T = Ce^{t}}$

case 2 con-1 is negative. Call it $-\lambda^2$ so that con $= 1-\lambda^2$

$X'' + \lambda^2 X = 0$, $\boxed{X = A\cos\lambda x + B\sin\lambda x}$, $\boxed{T = Ce^{(1-\lambda^2)t}}$

case 3 con $- 1$ is positive. Call it λ^2 so that con $= 1+\lambda^2$

$X'' - \lambda^2 X = 0$, $\boxed{X = Ae^{\lambda x} + Be^{-\lambda x}}$, $\boxed{T = Ce^{(1+\lambda^2)t}}$

7. You want good X solutions (T is never the important variable).

(a) $X'' = \text{con } X$, $T'' + (1 - \text{con})T = 0$

case 1 con < 0, say con $= -\lambda^2$

$X'' = -\lambda^2 X$, $m = \pm\lambda i$, $\boxed{X = A\cos\lambda x + B\sin\lambda x}$

$T'' + (1+\lambda^2)T = 0$, $m = \pm i\sqrt{1+\lambda^2}$, $\boxed{T = C\cos t\sqrt{1+\lambda^2} + D\sin t\sqrt{1+\lambda^2}}$

case 2 con $= 0$

$X'' = 0$, $\boxed{X = Ax + B}$, $T'' + T = 0$, $m = \pm i$, $\boxed{T = C\cos t + D\sin t}$

(b) $X'' - X = \text{con } X$, $X'' - (1 + \text{con})X = 0$

$T'' = \text{con } T$

case 1 $1 + \text{con} < 0$, say $1 + \text{con} = -\lambda^2$ so that con $= -1-\lambda^2$

$X + \lambda^2 X = 0$, $\boxed{X = A\cos\lambda x + B\sin\lambda x}$

$T'' = -(1+\lambda^2)T = 0$, $m = \pm i\sqrt{1+\lambda^2}$,

$\boxed{T = C\cos t\sqrt{1+\lambda^2} + D\sin t\sqrt{1+\lambda^2}}$

case 2 $1 + \text{con} = 0$ so that con $= -1$

$X'' = 0$, $\boxed{X = Ax + B}$

$T'' = -T$, $\boxed{T = C\cos t + D\sin t}$

(c) $X'' - \dfrac{1}{\text{con}} X = 0$, $T'' + \left(1 - \dfrac{1}{\text{con}}\right) T = 0$

To get good X solutions (namely sines and cosines) here are the cases you need.

case 1 $\frac{1}{\text{con}} < 0$, say $\frac{1}{\text{con}} = -\lambda^2$ so that $\text{con} = -\frac{1}{\lambda^2}$

$X'' + \lambda^2 X + 0$, $\boxed{X = A\cos\lambda x + B\sin\lambda x}$

$T'' + (1+\lambda^2)T = 0$, $m = \pm i\sqrt{1+\lambda^2}$, $\boxed{T = C\cos t\sqrt{1+\lambda^2} + D\sin t\sqrt{1+\lambda^2}}$

case 2 $\frac{1}{\text{con}} = 0$, $\text{con} = \infty$

$X'' = 0$, $\boxed{X = Ax + B}$

$T'' = -T$, $\boxed{T = C\cos t + D\sin t}$

Notice that you get the same solutions no matter which way you separate. But the titles of the cases depend on how you separate. That's why there can't be a rule like "always use the case where $\text{con} = -\lambda^2$*".*
The rule is "pick cases that give you good solutions for the important variable". In this problem it means choose cases so that you end up with $X'' + \lambda^2 X = 0$, $X = A\cos\lambda x + B\sin\lambda x$.

8. Try $u(x,y) = X(x)Y(y)$ Then $X'Y = XY - XY'$, $\frac{X'}{X} = \frac{Y - Y'}{Y} = \lambda$,

$Y' + (\lambda-1)Y = 0$, $X' - \lambda X = 0$.

(No need for cases because both DE's are *first* order.)

$Y = Be^{(1-\lambda)y}$, $X = Ae^{\lambda x}$, $u = Ae^{\lambda x}\,Be^{(1-\lambda)y} = \boxed{Ce^{\lambda x + (1-\lambda)y}}$

9. Try $u(x,y) = X(x)Y(y)$ Then

$X''Y = X'Y'$

$\frac{X''}{X'} = \frac{Y'}{Y} = \lambda$

$X'' = \lambda X'$, $Y' = \lambda Y$

case 1 $\lambda \neq 0$

For the X equ, $m^2 - \lambda m = 0$, $m = 0, \lambda$, $\boxed{X = A + Be^{\lambda x}}$

For the Y equ, $m = \lambda$, $\boxed{Y = Ce^{\lambda y}}$

case 2 $\lambda = 0$

$X'' = 0$, $\boxed{X = Ax + B}$

$Y' = 0$, $\boxed{Y = C}$

10. Try $u(x,y) = X(x)Y(y)$. Then $X'Y' + XY = 4$ and you just can't get any further. You can't get X's on one side and Y's on the other side.

11. Here's why the first one separates.
If $u(x,y) = X(x)Y(y)$ and $u(3,y) = 0$ for $0 \le y \le b$ then

$$X(3)Y(y) = 0 \text{ for } 0 \le y \le b.$$

So

$$X(3) = 0 \text{ or } Y(y) = 0 \text{ for } 0 \le y \le b.$$

If $Y(y) = 0$ for $0 \le y \le b$ then $u(x,y) = 0$ for $0 \le y \le b$ which is not a useful solution.

So use $X(3) = 0$.

Here's why the second one doesn't separate. If $u(0,y) = 3$ for $0 \le y \le b$ then

$$X(0)Y(y) = 3 \text{ for } 0 \le y \le b$$

But if the product of two factors is 3 then you can't conclude that one of the factors has to be 3. In fact you can't conclude anything about the individual factors. So this BC doesn't separate.

12. $X'(5)T(t) = -3X(5)T(t)$ for all t
$T(t)[X'(5) + 3X(5)] = 0$ for all t

Either $T(t) = 0$ for all t [which produces only the trivial solution for u so ignore it] or $X'(5) + 3X(5) = 0$. So the BC separates to $X'(5) = -3X(5)$.

Honors

12. Try a solution of the form $\Psi(x,y,z,t) = \phi(x,y,z)\ T(t)$ Then

$$-\frac{h^2}{2m}\left(\frac{\partial^2\phi}{\partial x^2}T + \frac{\partial^2\phi}{\partial y^2}T + \frac{\partial^2\phi}{\partial z^2}T\right) + V\phi T = ih\phi T'$$

$$\frac{-\frac{h^2}{2m}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + V\phi}{\phi} = \frac{ihT'}{T}$$

The left side has no t's in it and the right side has no x,y,z's in it so neither side has any variables in it so each side is a constant which I'll call E. So

$$\frac{-\frac{h^2}{2m}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + V\phi}{\phi} = \frac{ihT'}{T} = E$$

$$T' = \frac{E}{ih}T \qquad (\text{ sol is } T = Ae^{-i(E/h)t})$$

And the ϕ equation (the *time independent Schrödinger equation*) is

$$-\frac{h^2}{2m}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + V\phi = E\phi$$

SOLUTIONS Section 6.5

1. The equation was separated in Part I of example 2 so I won't repeat it.Try the case where $R = Ar^{\lambda} + Br^{-\lambda}$, $\Theta = C \cos \lambda\theta + D \sin \lambda\theta$

$\Theta(0) = 0$ so $C = 0$

$\Theta(\pi/4) = 0$ so $D \sin \frac{\pi\lambda}{4} = 0$, $\frac{\pi\lambda}{4} = n\pi$, $\lambda = 4n$ for $n = 1,2,3,\ldots$

To keep $R(0)$ finite set $B = 0$

By superposition, $v = \sum_{n=1}^{\infty} D_n r^{4n} \sin 4n\theta$

To get $v(5,\theta) = \frac{\theta}{\pi}$ for θ in $[0,\pi/4]$ you need

$$\frac{\theta}{\pi} = \sum_{n=1}^{\infty} D_n 5^{4n} \sin 4n\theta \quad \text{for } \theta \text{ in } [0,\pi/4]$$

Sin $4n\theta$ is of the form $\sin \frac{n\pi\theta}{L}$ where $L = \pi/4$ so you need

$$D_n 5^{4n} = \frac{2}{\pi/4} \int_0^{\pi/4} \frac{\theta}{\pi} \sin 4n\theta \, d\theta = \begin{cases} \dfrac{-1}{2n\pi} & \text{if } n \text{ is even} \\ \dfrac{1}{2n\pi} & \text{if } n \text{ is odd} \end{cases}$$

(use (2) in the tables with $L = \pi/4$ and an extra factor of $1/\pi$).

So $D_{\text{even } n} = -\dfrac{1}{5^{4n} \, 2n\pi}$, $D_{\text{odd } n} = \dfrac{1}{5^{4n} \, 2n\pi}$ and the solution is

$$\boxed{v = \frac{1}{2\pi}\left[\left(\frac{r}{5}\right)^4 \sin 4\theta - \frac{1}{2}\left(\frac{r}{5}\right)^8 \sin 8\theta + \frac{1}{3}\left(\frac{r}{5}\right)^{12} \sin 12\theta - \ldots \right]}$$

2.(a) I won't repeat the separation.
Use the case where $\Theta = A \cos \lambda\theta + B \sin \lambda\theta$, $R = Cr^{\lambda} + Dr^{-\lambda}$

$\Theta(0) = 0$ so $A = 0$

$\Theta(\pi) = 0$ so $B \sin \pi\lambda = 0$, $\pi\lambda = n\pi$, $\lambda = n$ for $n = 1,2,3,\ldots$

To keep $R(0)$ finite choose $D = 0$.

By superposition, $\boxed{v = \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$

Then $\frac{\partial v}{\partial r} = \sum_{n=1}^{\infty} B_n n r^{n-1} \sin n\theta$.

To get the last BC $\frac{\partial v}{\partial r}(2,\theta) = f(\theta)$ you need

$$f(\theta) = \sum_{n=1}^{\infty} n B_n 2^{n-1} \sin n\theta \quad \text{for } \theta \text{ in } [0,\pi]$$

So you need $nB_n 2^{n-1} = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta$, $\boxed{B_n = \dfrac{1}{n\, 2^{n-1}} \dfrac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta}$

(b) $$B_n = \frac{1}{n\,2^{n-1}} \frac{2}{\pi}\int_0^{\pi} \sin n\theta \, d\theta = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{n^2 \pi \, 2^{n-1}} & \text{if } n \text{ is odd} \end{cases} \qquad \text{(Tables (2))}$$

Solution is

$$v = \frac{8}{\pi}\left[\frac{r}{2}\sin\theta + \frac{1}{9}\left(\frac{r}{2}\right)^3 \sin 3\theta + \frac{1}{25}\left(\frac{r}{2}\right)^5 \sin 5\theta + \ldots \right]$$

3. (a) I won't repeat the separation.
Use the case where $\Theta = A\cos\lambda\theta + B\sin\lambda\theta$, $R = Cr^{\lambda} + Dr^{-\lambda}$

Then $\Theta' = -\lambda A \sin\lambda\theta + \lambda B\cos\lambda\theta$

$\Theta'(0) = 0$ so $B = 0$

$\Theta'(\pi/4) = 0$ so $-\lambda A \sin\frac{\lambda\pi}{4} = 0$, $\frac{\lambda\pi}{4} - n\pi$, $\lambda = 4n$ for $n = 1,2,3,\ldots$

To keep $R(\infty)$ finite make $C = 0$

Use the $\lambda = 0$ case where $\Theta = E\theta + F$, $R = G\ln r + H$

$\Theta'(0) = 0$ and $\Theta'(\pi/4) = 0$ make $E = 0$

To keep $R(\infty)$ finite make $G = 0$ From this case you get $v = FH = K$

By superposition, $\boxed{v = K + \sum_{n=1}^{\infty} C_n r^{-4n}\cos 4n\theta}$

To get $v(6,\theta) = f(\theta)$ for θ in $[0,\pi/4]$ you need

$$f(\theta) = K + \sum_{n=1}^{\infty} C_n 6^{-4n}\cos 4n\theta \quad \text{for } \theta \text{ in } [0,\pi/4],$$

$$\boxed{K = \frac{1}{\pi/4}\int_0^{\pi/4} f(\theta)\,d\theta}\ , \qquad C_n 6^{-4n} = \frac{2}{\pi/4}\int_0^{\pi/4} f(\theta)\cos 4n\theta\,d\theta,$$

$$\boxed{C_n = 6^{4n}\frac{2}{\pi/4}\int_0^{\pi/4} f(\theta)\cos 4n\theta\,d\theta}$$

(b) The only difference here is that you must keep $R(0)$ finite by making $D = 0$ in the first case and $G = 0$ again in the second case. The net effect is to have r^{4n} instead of r^{-4n} in the solution and 6^{4n} instead of 6^{-4n} in the C_n coeff formula. Solution is

$$\boxed{v = K + \sum_{n=1}^{\infty} C_n r^{4n}\cos 4n\theta}$$

where $\boxed{K = \frac{1}{\pi/4}\int_0^{\pi/4} f(\theta)\,d\theta}$ and $\boxed{C_n = 6^{-4n}\frac{2}{\pi/4}\int_0^{\pi/4} f(\theta)\cos 4n\theta\,d\theta}$

4. (a) Use the major case where $\Theta = C \cos \lambda\theta + D \sin \lambda\theta$, $R = Ar^{\lambda} + Br^{-\lambda}$ and the minor case where $R = E \ln r + F$, $\Theta = G\theta + H$.

For v inside, continue as in example 3. Need $\lambda = n$ and $G = 0$ to keep Θ periodic. Need $B = 0$ and $E = 0$ to keep R finite. By superposition

$$v = K + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$$

The BC is $v(5,\theta) = \begin{cases} 1 & \text{if } 0 \le \theta \le \pi \\ -1 & \text{if } \pi \le \theta \le 2\pi \end{cases}$

To get it you need

$$(*) \quad v(5,\theta) = K + \sum_{n=1}^{\infty} 5^n (C_n \cos n\theta + D_n \sin n\theta) \text{ for } 0 \le \theta \le 2\pi$$

$$K = \frac{1}{2\pi}\int_0^{2\pi} v(5,\theta)\, d\theta = \text{average value of } v(5,\theta) = 0$$

$$5^n C_n = \frac{2}{2\pi}\int_0^{2\pi} v(5,\theta) \cos n\theta\, d\theta = \frac{1}{\pi}\left[\int_0^{\pi} \cos n\theta\, d\theta + \int_{\pi}^{2\pi} -\cos n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\,\frac{1}{n} \sin n\theta\Big|_0^{\pi} - \frac{1}{\pi}\,\frac{1}{n} \sin n\theta\Big|_{\pi}^{2\pi} = 0$$

(Can't use (4b) in the tables because here $L = 2\pi$ but the cosine is not $\cos \frac{n\pi\theta}{2\pi}$.) Similarly

$$5^n D_n = \frac{2}{2\pi}\int_0^{2\pi} f(\theta) \sin n\theta\, d\theta = \frac{1}{\pi}\left[\int_0^{\pi} \sin n\theta\, d\theta + \int_{\pi}^{2\pi} -\sin n\theta\, d\theta\right]$$

and eventually you get

$$D_{\text{even } n} = 0, \quad D_{\text{odd } n} = \frac{4}{5^n\, n\pi}$$

so

$$v_{\text{inside}} = \frac{4}{\pi}\left[\frac{r}{5} \sin \theta + \frac{1}{3}\left(\frac{r}{5}\right)^3 \sin 3\theta + \frac{1}{5}\left(\frac{r}{5}\right)^5 \sin 5\theta + \ldots\right]$$

Finding v outside is like finding v inside but to keep R finite set $A = 0$ in the main case instead of B. The net effect is to have r^{-n} instead of r^n in the solution and 5^{-n} instead of 5^n in the $D_{\text{odd } n}$ coeff formula. All in all,

$$v(r,\theta) = \begin{cases} \frac{4}{\pi}\left(\frac{r}{5} \sin \theta + \frac{1}{3}\left(\frac{r}{5}\right)^3 \sin 3\theta + \frac{1}{5}\left(\frac{r}{5}\right)^5 \sin 5\theta + \ldots\right) & \text{for } r \le 5 \\ \frac{4}{\pi}\left(\frac{5}{r} \sin \theta + \frac{1}{3}\left(\frac{5}{r}\right)^3 \sin 3\theta + \frac{1}{5}\left(\frac{5}{r}\right)^5 \sin 5\theta + \ldots\right) & \text{for } r \ge 5 \end{cases}$$

(b) Continue as in part (a) until line (*) which becomes

$$4 \sin 3\theta = K + \sum_{n=1}^{\infty} 5^n (C_n \cos n\theta + D_n \sin n\theta) \text{ for } 0 \le \theta \le 2\pi$$

By inspection you can get this with $K = 0$, $5^n C_n = 0$, $C_n = 0$,

$$5^3 D_3 = 4, \quad D_3 = \frac{4}{5^3}, \quad \text{other } D_n\text{'s} = 0.$$

Solution is $v_{\text{inside}} = \frac{4}{5^3} r^3 \sin 3\theta = 4\left(\frac{r}{5}\right)^3 \sin 3\theta$

Similarly $v_{\text{outside}} = 4\left(\frac{5}{r}\right)^3 \sin 3\theta$.

5. (a) This is an Euler's equation with $a = 2$, $b = -12$. Let $x = e^t$.
Get $y'' + y' - 12y = 0$, $m = -4, 3$
$y(t) = Ae^{-4t} + Be^{3t}$, $y(x) = Ax^{-4} + Bx^3$

(b) Euler with $a = -3$, $b = 4$. Let $x = e^t$. Get $y'' - 4y' + 4y = 0$, $m = 2,2$,
$y(t) = Ae^{2t} + Bte^{2t}$, $y(x) = Ax^2 + Bx^2 \ln x$

(c) Euler with $a = 5$, $b = 5$. Let $x = e^t$. Get $y'' + 4y' + 5y = 0$, $m = -2 \pm i$,
$y(t) = e^{-2t}(A \cos t + B \sin t)$, $y(x) = x^{-2}(A \cos \ln x + B \sin \ln x)$

(d) Let $x = e^t$. Get $y'' - 4y' + 4y = t$, $y_h = Ae^{2t} + Bte^{2t}$
Try $y_p = At + B$, Need

$$-4A + 4At + 4B = t,$$

$$4A = 1, \; -4A + 4B = 0$$

$$A = \frac{1}{4}, \; B = \frac{1}{4}$$

$y(t) = Ae^{2t} + Bte^{2t} + \frac{1}{4}t + \frac{1}{4}$

$y(x) = Ax^2 + Bx^2 \ln x + \frac{1}{4} \ln x + \frac{1}{4}$

(e) Let $x = e^t$ to get

$$(*) \qquad y''(t) + 2y'(t) - 3y(t) = 10e^{2t}$$

We have

$$y_h = Ae^{-3t} + Be^t$$

as before. Try

$$y_p = Ce^{2t}$$

Substitute into (*) to get

$$4Ce^{2t} + 2 \cdot 2Ce^{2t} - 3Ce^{2t} = 10e^{2t}$$

$$5C = 10, \quad C = 2$$

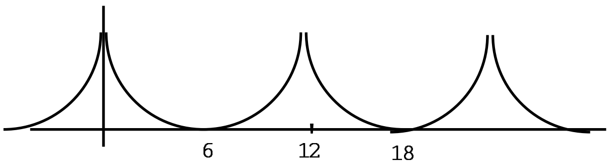
$$y_{gen}(t) = Ae^{-3t} + Be^t + 2e^{2t}$$

Now go back to x's. One way to do it is to think of e^{-3t} and e^{2t} as $(e^t)^{-3}$ and $(e^t)^2$. Substitute x for e^t to get the final answer:

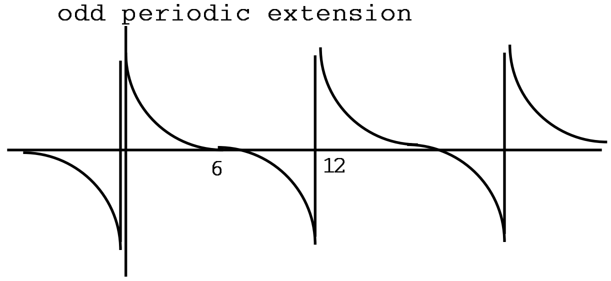
$$y_{gen}(x) = \frac{A}{x^3} + Bx + 2x^2$$

SOLUTIONS Section 6.6

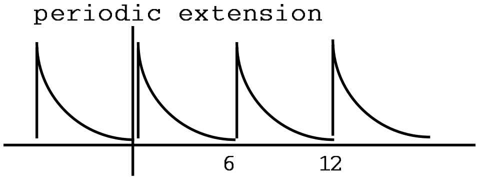
1.(a) even periodic extension



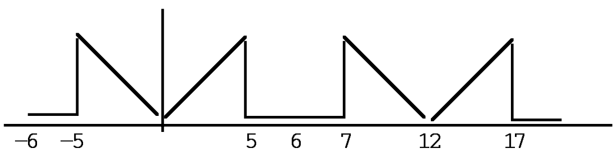
odd periodic extension



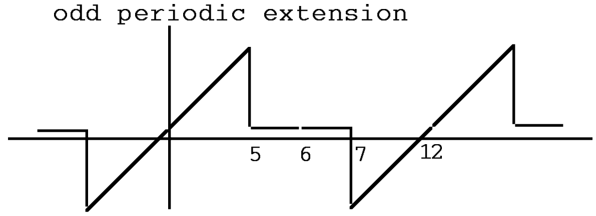
periodic extension



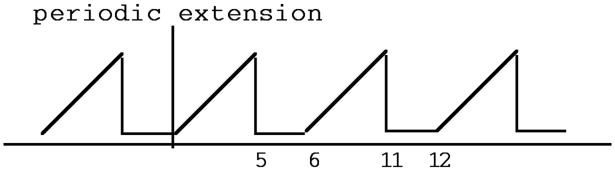
(b) even periodic extension



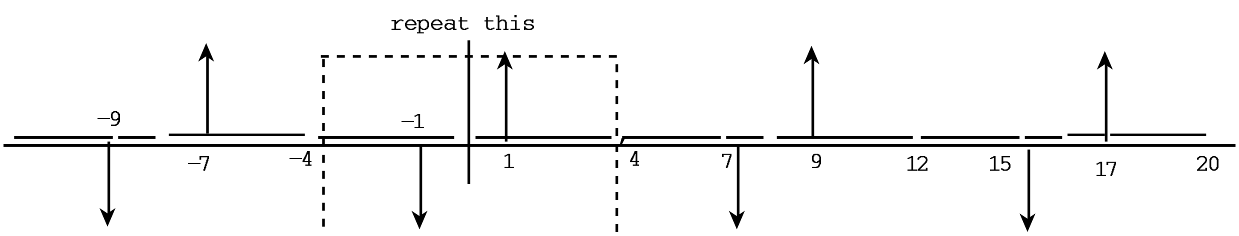
odd periodic extension



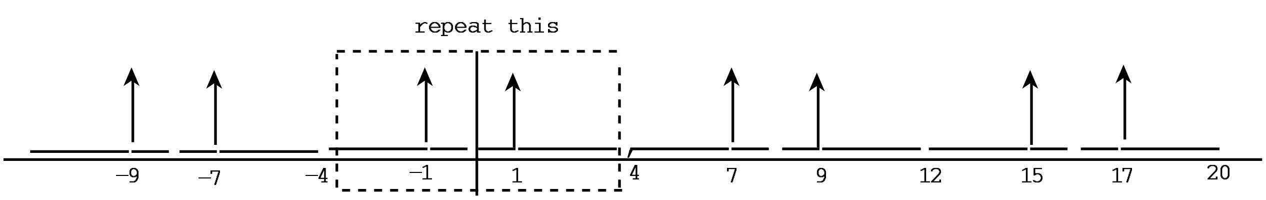
periodic extension



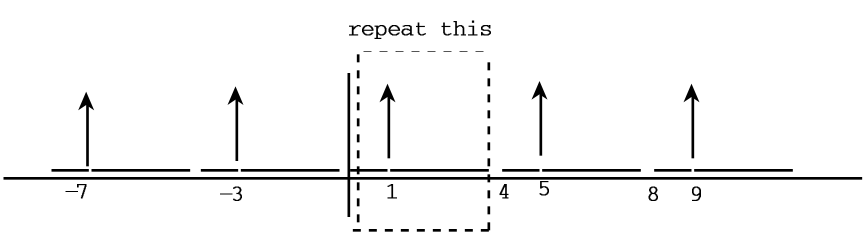
2. sine series converges to the odd periodic extension



cosine series converges to the even periodic extension



full series converges to the periodic extension



3. (a) The picture is the even periodic extension of the [0,4] piece (even function, T = 8, use cos series with L = 4). Series is $A_0 + \sum A_n \cos \frac{n\pi x}{4}$ where

$$A_0 = \frac{1}{4}\int_0^4 (8-2x)\,dx = \text{average value of } f(x) \text{ on } [0,4] \text{ which is 4 by inspection}$$

$$A_n = \frac{2}{4}\int_0^4 (8-2x)\cos\frac{n\pi x}{4}\,dx$$

(Can also *inefficiently* find cos series using [0,8] piece or full series using [0,8] piece. All three versions will turn out to be the same series.)

(b) The picture is the odd periodic extension of the [0,4] piece (odd function, T = 8, use sines with L = 4). Series is $\sum B_n \sin\frac{n\pi x}{4}$ where

$$B_n = \frac{2}{4}\int_0^4 f(x)\sin\frac{n\pi x}{4}\,dx = \frac{2}{4}\left[\int_0^2 -x\sin\frac{n\pi x}{4}\,dx + \int_2^4 (x-4)\sin\frac{n\pi x}{4}\,dx\right]$$

(can also, inefficiently, find full series for [0,8] piece)

(c) The picture is the periodic extension of the [0,5] piece (not-odd, not-even, T = 5, use full series with L = 5). Series is

$$A_0 + \sum\left[A_n\cos\frac{n\pi x}{5/2} + B_n\sin\frac{n\pi x}{5/2}\right]$$

where $A_0 = \frac{1}{5}\int_0^5 f(x)\,dx = \frac{1}{5}\left[\int_0^2 2x\,dx + \int_2^4 (8-2x)\,dx\right]$

(average value of f(x) on [0,5] is 8/5, by inspection)

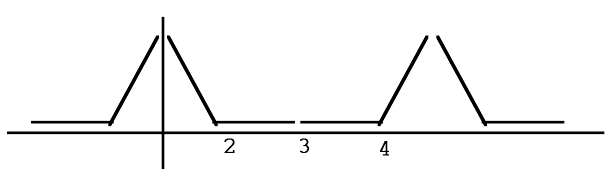
$$A_n = \frac{2}{5}\left[\int_0^2 2x\cos\frac{2n\pi x}{5}\,dx + \int_2^4 (8-2x)\cos\frac{2n\pi x}{5}\,dx\right]$$

B_n = ditto but with sines

(d) The picture is the even periodic extension of the $[0, 2\frac{1}{2}]$ piece; i.e., I'll find the cos series for

$$f(x) = \begin{cases} 4-2x & \text{if } 0 \le x \le 2 \\ 0 & \text{if } 2 \le x \le 2.5 \end{cases}$$

(even function, T = 5, use cos series with L = 5/2).

> **warning** It's wrong to use [0,3] because the even periodic extension of the [0,3] piece is
>
> 
>
>
> The flat pieces have length 2 not 1 so this is *not* the desired picture.
> *Check your choice of interval carefully*.

Series is $A_0 + \sum A_n \cos\frac{2n\pi x}{5}$ where

$$A_0 = \frac{1}{5/2}\int_0^{5/2} f(x)\,dx = \frac{2}{5}\int_0^2 (4-2x)\,dx$$

(average value of f(x) on [2, 5/2] is 8/5, by inspection)

$$A_n = \frac{2}{5/2}\int_0^{3/2} f(x)\,dx = \frac{4}{5}\int_0^2 (4-2x)\cos\frac{2n\pi x}{5}\,dx$$

(e) The picture is the periodic extension of the [0,2] piece (not-odd, not-even, T = 2, use full series with L = 2). Series is

$$A_0 + \sum (A_n \cos n\pi x + B_n \sin n\pi x)$$

where $A_0 = \frac{1}{2}\int_0^2 (2-x)\,dx$ (average f value on [0,2] is 1, by inspection)

$$A_n = \frac{2}{2}\int_0^2 (2-x)\cos n\pi x\,dx, \quad B_n = \frac{2}{2}\int_0^2 (2-x)\sin n\pi x\,dx$$

(f) Picture is the odd periodic extension of the [0,1] piece (odd function, T = 2, use sines with L = 1). Series is $\sum B_n \sin n\pi x$ where $B_n = 2\int_0^1 (1-x)\sin n\pi x\,dx$
(Can inefficiently find the sine series for the [0,2] piece or find the full series for the [0,2] piece)

4. Function is the periodic extension of the [0,2] piece (not-odd, not-even, T = 2, use full series with L = 2). Series is $A_0 + \sum (A_n \cos n\pi x + B_n \sin n\pi x)$ where

$$A_0 = \text{average value of } f(x) \text{ on } [0,2] = \frac{3}{4}$$

$$A_n = \frac{2}{2}\int_0^2 f(x)\cos n\pi x\,dx = \int_0^{1/2} 3\cos n\pi x\,dx = \frac{3}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{3}{n\pi} & \text{if } n = 1,5,9,\ldots \\ -\dfrac{3}{n\pi} & \text{if } n = 3,7,11,\ldots \end{cases}$$

$$B_n = \int_0^{1/2} 3\sin n\pi x\,dx = -\frac{3}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$= \begin{cases} \dfrac{3}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n = 4,8,12,\ldots \\ \dfrac{6}{n\pi} & \text{if } n = 2,6,10,\ldots \end{cases}$$

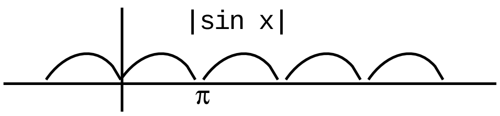
$$f(x) = \frac{3}{4} + \frac{3}{\pi}\left(\cos\pi x - \frac{1}{3}\cos 3\pi x + \frac{1}{5}\cos 5\pi x - \frac{1}{7}\cos 7\pi x + \ldots\right)$$
$$+ \frac{3}{\pi}\left(\sin\pi x + \frac{2}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \frac{2}{6}\sin 6\pi x + \frac{1}{7}\sin 7\pi x + \ldots\right)$$

The first harmonic is $\frac{3}{\pi}(\cos \pi x + \sin \pi x)$ so the fundamental frequency is 1 {cycle per second) with amp $\sqrt{\frac{9}{\pi^2} + \frac{9}{\pi^2}} = \frac{3}{\pi}\sqrt{2}$

The first overtone is $\frac{3}{\pi}\sin 2\pi x$ (there is no $\cos 2\pi x$ term) so first overtone frequency is 2 with amp $3/\pi$

5. Function is the even periodic extension of the [0,6] piece (even function, T = 12, use cosines with L = 6). Series is of the form $A_0 + \sum A_n \cos\frac{n\pi x}{6}$. You want the first nonzero cosine term. Assuming $A_1 \neq 0$, the fundamental frequency is 1/6 and its amplitude is $|A_1|$ where

$$A_1 = \frac{2}{6}\int_0^6 f(x)\cos\frac{\pi x}{6}\,dx = \frac{2}{6}\left[\int_0^2 2x\cos\frac{\pi x}{6}\,dx + \int_2^6 4\cos\frac{\pi x}{6}\,dx\right]$$



6. The function is the even periodic extension of the $\pi/2$ piece (even function, $T = \pi$, use cos series with $L = \pi/2$). Series is $A_0 + \sum A_n \cos 2nx$ where

$$A_0 = \frac{1}{\pi/2}\int_0^{\pi/2} \sin x\,dx = \frac{2}{\pi}$$

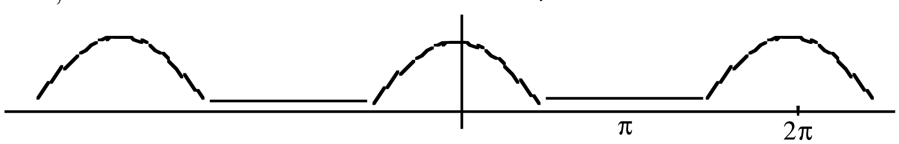
$$A_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x\cos 2nx\,dx = \frac{4}{\pi}\left[-\frac{\cos(1-2n)x}{2(1-2n)} - \frac{\cos(1+2n)x}{2(1+2n)}\right]_0^{\pi/2}$$

Note that $\cos(1-2n)\frac{\pi}{2}$ and $\cos(1+2n)\frac{\pi}{2}$ are 0. So

$$A_n = \frac{4}{\pi}\left[\frac{1}{2(1-2n)} + \frac{1}{2(1+2n)}\right] = \frac{-4}{\pi(2n-1)(2n+1)}$$

Series is $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1\cdot 3}\cos 2x + \frac{1}{3\cdot 5}\cos 4x + \frac{1}{5\cdot 7}\cos 6x + \ldots\right]$

7. The function is the even periodic extension of the $[0,\pi]$ piece (even function, $T = 2\pi$, use cosine series with $L = \pi$).



Series is $A_0 + \sum A_n \cos nx$ where

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} \cos x\,dx$$

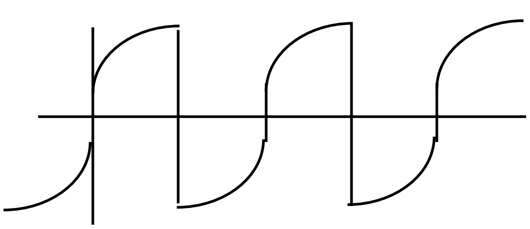
$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\cos nx\,dx$$

8. The function is the odd periodic extension of the $[0,4]$ piece (odd function, $T = 8$, use sine series with $L = 4$):

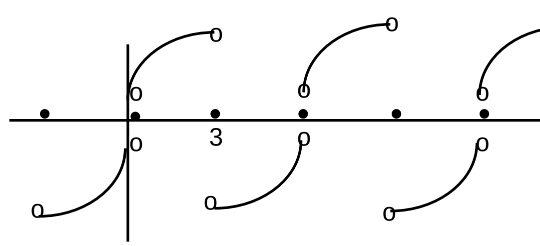
$$\sum B_n \sin\frac{n\pi x}{4} \quad \text{where} \quad B_n = \frac{2}{4}\int_0^2 5\sin\frac{n\pi x}{4}\,dx = \begin{cases} 10/n\pi & \text{if } n \text{ is odd} \\ 20/n\pi & \text{if } n = 2,6,10,\ldots \end{cases}$$

Series is $\frac{10}{\pi}\sin\frac{\pi x}{4} + \frac{20}{2\pi}\sin\frac{2\pi x}{4} + \frac{10}{3\pi}\sin\frac{3\pi x}{4} + \frac{10}{5\pi}\sin\frac{5\pi x}{4} + \ldots$

9. (a)(i) The series converges to the odd periodic extension of the $[0,3]$ piece
(ii) At a jump, the series converges to the middle value



Problem 9(a)(i)

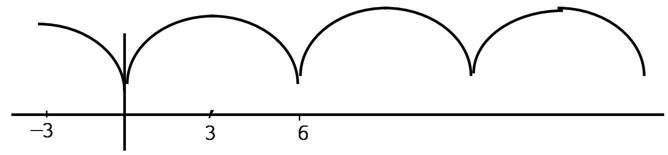


Problem 9(a)(ii)

(iii) The [24,30] piece is the same as the [0,6] piece. At $x = 29$ the series has the same value that it did at $x = 5$ and at $x = -1$. To get the value at $x = -1$, find the value at $x = 1$ and change signs. Answer is $-(3 + 6\cdot 1 - 1^2) = -8$.

(b) (i) The series converges to the even periodic extension of the [0,3] piece

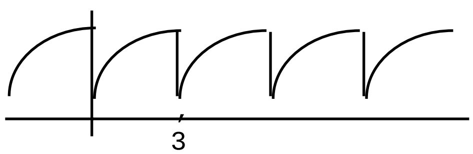
(ii) There are no jumps.



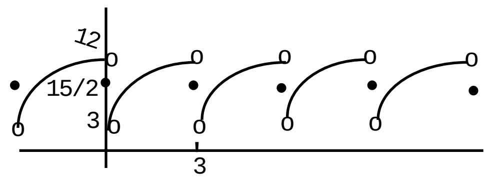
Problem 9(b)(i)

(iii) At $x = 29$ the series has the same value that it did at $x = -1$ which (since the picture is even) is the same as its value at $x = 1$. Answer is 8.

(c) (i) The series converges to the periodic extension of the [0,3] piece
(ii) See the diagram.



Problem 9(c)(i)



Problem 9(c)(ii)

(iii) The [27,29] piece is the same as the [0,3] piece.
At $x = 29$, the series has the same value as at $x = 2$. Answer is 11.

10. (a) I'll use the forcing function e^{ikx} and try $y_p = Ae^{ikx}$. Substituting the trial y_p into the DE gives

$$Ai^2 k^2 e^{ikx} + 4Ae^{ikx} = e^{ikx}, \quad A(4 - k^2) = 1, \ A = \frac{1}{4-k^2},$$

$$y_p = \frac{1}{4-k^2}(\cos kx + i \sin kx).$$

Take imag part to get the particular sol to the original equation:

$$y_{pk} = \frac{1}{4-k^2} \sin kx.$$

(b) By (2), the DE is

$$y'' + 4y = \frac{16}{\pi^2}\left[\sin\frac{\pi x}{4} - \frac{1}{9}\sin\frac{3\pi x}{4} + \frac{1}{25}\sin\frac{5\pi x}{4} - \dots \right]$$

By part (a) and superposition, the particular solution is

$$y_p = y_{p1} + y_{p3} + y_{p5} + \dots$$

$$= \frac{16}{\pi^2}\left[\frac{1}{4-(\frac{\pi}{4})^2}\sin\frac{\pi x}{4} - \frac{1}{9}\,\frac{1}{4-(\frac{3\pi}{4})^2}\sin\frac{3\pi x}{4} + \frac{1}{25}\,\frac{1}{4-(\frac{5\pi}{4})^2}\sin\frac{5\pi x}{4} - \dots \right]$$

SOLUTIONS Section 6.7

1. $$A_3 = \frac{\int_0^1 2x^5 \cdot x^4 \, dx}{\int_0^1 x^4 \cdot x^4 \, dx} = \frac{1/5}{1/9} = \frac{9}{5}$$

2. $$\int_0^L 1 \cdot \cos\frac{\pi x}{L}\, dx = \frac{L}{n\pi} \sin\frac{\pi x}{L}\Big|_0^L = \frac{L}{n\pi}(\sin \pi - \sin 0) = 0$$

3. (a) Part I Separate Try $u = X(x)\, T(t)$. Then $XT' = kX''T$, $\frac{X''}{X} = \frac{T'}{kT} = \text{constant}$.

The BC separate to $X'(0) = 0$, $X(L) = 0$

case 1 con $= -\lambda^2$

$$X'' = -\lambda^2 X,\quad T' = -\lambda^2 T,\quad X = A\cos\lambda x + B\sin\lambda x,\quad T = Ce^{-\lambda^2 kt}$$

case 2 con $= 0$

$$X'' = 0,\quad T' = 0,\quad X = Px + Q,\quad T = D$$

Part II Plug in the separated BC

case 1

$X' = -A\lambda \sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = 0$ makes $B = 0$

$X(L) = 0$ makes $A\cos\lambda L = 0$, $\lambda L = \frac{n\pi}{2}$ for *odd* n

$\lambda = \frac{n\pi}{2L}$ for $n = 1,3,5,\ldots$

$X = \cos\frac{n\pi x}{2L}$, $T = Ce^{-(n\pi/2L)t}$ for *odd* n

case 2

$X' = P$

$X'(0) = 0$ makes $P = 0$

$X(L) = 0$ makes $Q = 0$

Nothing useful here.

Part III

A general solution is

$$(*)\qquad u = \sum_{\text{odd } n} A_n\, e^{-k(n\pi/2L)^2 t} \cos\frac{n\pi x}{2L} \qquad \text{for } 0 \le x \le L,\ t \ge 0$$

To satisfy the IC you need

$$f(x) = \sum_{\text{odd } n} A_n \cos\frac{n\pi x}{2L} \quad \text{for } 0 \le x \le L$$

Note that the "ingredients" of the series (the ϕ's) are *not* the functions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \ldots$$

The ϕ's here are

$$(**)\qquad \cos\frac{\pi x}{2L},\ \cos\frac{3\pi x}{2L},\ \cos\frac{5\pi x}{2L},\ \ldots$$

This is a new complete orthogonal family on the interval [0,L]. They came from solving

$$X'' = \text{con}\cdot X \quad \text{with BC } X'(0) = 0,\ X(L) = 0$$

This is a Sturm Liouville problem with $p(x) = 1$, $q(x) = 0$, so the functions in (**) are orthogonal on the interval [0,L].

The solution is (*) with the constants given by the formula in (5):

$$A_{\text{odd } n} = \frac{\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx}{\int_0^L \cos^2\frac{n\pi x}{2L}\,dx}$$

(b) numerator of the $A_{\text{odd } n}$ formula

$$= \int_0^L 7\cos\frac{n\pi x}{2L}\,dx = \frac{14L}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 14L/n\pi & \text{if } n = 1,5,9,\ldots \\ -14L/n\pi & \text{if } n = 3,7,11,\ldots \end{cases}$$

$$\text{denom} = \frac{L}{2} + \frac{L\sin n\pi}{2n\pi} = \frac{L}{2}$$

Answer is

$$u = \frac{28}{\pi}\left[e^{-k(\pi/2L)^2 t}\cos\frac{\pi x}{2L} - \frac{1}{3}e^{-k(3\pi/2L)^2 t}\cos\frac{3\pi x}{2L} + \frac{1}{5}e^{-k(5\pi/2L)^2 t}\cos\frac{5\pi x}{2L} - \ldots\right]$$

for $0 \le x \le L,\ t \ge 0$

4. Can't be done in general. The series is an "incomplete" cosine series: it doesn't have a constant term. The building blocks of the series in (*) are

$$\cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \ldots$$

but the *complete* set of orthogonal functions that are used to make a series that will converge to any $f(x)$ for $0 \le x \le L$ is

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \ldots$$

Can't make the incomplete series do what you want it to do for $0 \le x \le L$. It is *not* correct to say the series will converge to $f(x)$ if $A_n = \frac{2}{L}\int_{x=0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$.

The only functions you can make the series in (*) converge to for $0 \le x \le L$ are functions whose average value on [0,L] is 0. The cosine series for that kind of function has $A_0 = 0$ anyway so their series are *supposed* to be missing a constant term.

5. The functions are of the form $\sin\frac{n\pi x}{L}$ where $L = 5\pi$. So the interval is $[0, 5\pi]$.

SOLUTIONS Section 6.8

1\. (a) $R(T' + T) = kT(R''T + \frac{1}{r}R')$

$$\frac{T' + T}{kT} = \frac{R'' + \frac{1}{r}R'}{R} = \text{con}$$

$T' + (1-\text{con } k)T = 0$, $T = Ce^{(\text{con } k - 1)t}$
$rR'' + R' = r \text{ con } R$

case 1 con $= -\lambda^2$

$rR'' + R' + r\lambda^2 R = 0$

$R = AJ_0(\lambda r) + BY_0(\lambda r)$, $T = Ce^{-(1+k\lambda^2)t}$

case 2 con $= 0$

$rR'' + R' = 0$, $R = A \ln r + B$ (ref page), $T = Ce^{-t}$

(b) $T' - \text{con } T = 0$, $T = Ce^{\text{con } t}$

$$rR'' + R' = \frac{1 + \text{con}}{k} rR$$

Sturm Liouville form with $p(r) = r$, $q(r) = 0$, $w(r) = r$ and $\frac{1+\text{con}}{k}$ playing the role of the constant.

case 1 $\frac{1 + \text{con}}{k} = -\lambda^2$, con $= -(1+k\lambda^2)$

$rR'' + R' + \lambda^2 rR = 0$, $R = AJ_0(\lambda r) + BY_0(\lambda r)$

$T = Ce^{-(1+k\lambda^2)t}$

case 2 $\frac{1 + \text{con}}{k} = 0$, con $= -1$

$rR'' + R' = 0$, $R = A \ln r + B$
$T = Ce^{-t}$

Ultimately these are the same solutions as part (a) but they were easier to get with the factoring in (a).

2\. The BC $u(L,t) = 0$ separates to $R(L) = 0$
Plug it in.
case 1 $R = AJ_0(\lambda r) + BY_0(\lambda r)$
Set $B = 0$ to keep R finite

$R(L) = 0$ makes $AJ_0(\lambda L) = 0$, $\lambda = \frac{a_n}{L}$ where the a_n's are the zeros of J_0.
case 2 $R = A \ln r + B$
Set $A = 0$ to keep R finite.
To get $R(8) = 0$ you need $B = 0$. So the only solution in this case is $R=0$.

Part III Use the IC.

By superposition,

$$(*) \qquad u = \sum_{n=1}^{\infty} A_n \; e^{-(1+k(a_n/L)^2)t} \; J_0\left(\frac{a_n r}{L}\right) \quad \text{for } 0 \le r \le L,\ t \ge 0$$

To get the IC, set t = 0, u = f(r). You need

$$f(r) = \sum_{n=1}^{\infty} A_n \; J_0\left(\frac{a_n r}{L}\right) \quad \text{for } r \text{ in } [0,L]$$

which you can get with

$$(**) \qquad A_n = \frac{\int_0^L f(r)\; J_0\left(\frac{a_n r}{L}\right) r\,dr}{\int_0^L J_0^2\left(\frac{a_n r}{L}\right) r\,dr}$$

The solution is (*) and (**) where the a_n's are the zeros of J_0 (Fig 6).

3. *Part I* Separate variables.

Try v(r,z) = R(r)Z(z). Then

$$R''Z + \frac{1}{r} R' Z + RZ'' = 0$$

$$\frac{R'' + \frac{1}{r} R'}{R} = -\frac{Z''}{Z} = \text{con}$$

$rR'' + R' = \text{con } r\, R$
$Z'' + \text{con } Z = 0$

The BC v(5,z) = 0 separates to R(5) = 0

case 1 con $= -\lambda^2$

$$R = A\, J_0(\lambda r) + BY_0(\lambda r), \quad Z = Ce^{\lambda z} + De^{-\lambda z}$$

case 2 con = 0

$R = A \ln R + B$, $Z = C \cos \lambda z + D \sin \lambda z$

Part II Satisfy the separable BC

case 1

Set B = 0 to keep R finite at r=0.

R(5) = 0 so $AJ_0(5\lambda) = 0$, $\lambda = \frac{a_n}{5}$ where the a_n's are the zeros of J_0.

Set C = 0 to keep Z finite as $z \to \infty$.

case 2

Nothing useful. Just get R = 0.

Part III Satisfy the nonhomog BC

By superposition

$$(*) \qquad v = \sum_{n=1}^{\infty} A_n \, J_0\left(\frac{a_n r}{5}\right) e^{-a_n z/5}$$

Then

$$\frac{\partial v}{\partial z} = \sum_{n=1}^{\infty} -\frac{a_n}{5} A_n \, J_0\left(\frac{a_n r}{5}\right) e^{-a_n z/5}$$

and to get the nonhomog BC you need

$$f(r) = \sum_{n=1}^{\infty} -\frac{a_n}{5} A_n \, J_0\left(\frac{a_n r}{5}\right) \quad \text{for } r \text{ in } [0,5],$$

which you can get with

$$-\frac{a_n}{5} A_n = \frac{\int_0^5 f(r)\, J_0\left(\frac{a_n r}{5}\right) r\,dr}{\int_0^5 J_0^2\left(\frac{a_n r}{5}\right) r\,dr}$$

$$(**) \qquad A_n = -\frac{5}{a_n} \; \frac{\int_0^5 f(r)\, J_0\left(\frac{a_n r}{5}\right) r\,dr}{\int_0^5 J_0^2\left(\frac{a_n r}{5}\right) r\,dr}$$

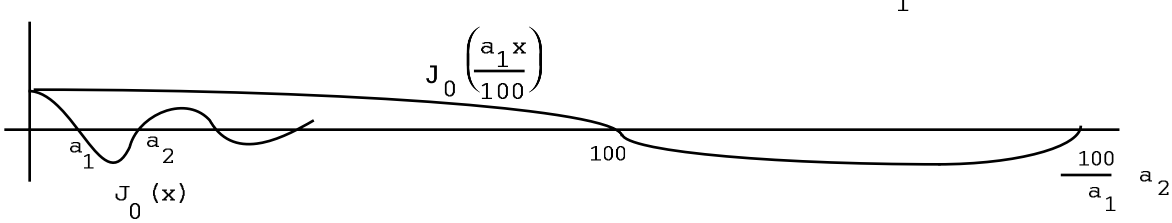
The solution is (*) and (**) where the a_n's are the zeros of J_0 (Fig 6).

4. The graph of $J_0(x)$ crosses the x-axis at $a_1, a_2, a_3, \ldots$

The graph of $J_0\left(\frac{a_1 x}{100}\right)$ crosses at $\frac{100}{a^1} a_1$ (= 100), $\frac{100}{a_1} a_2$, $\frac{100}{a_1} a_3$, ...

Think that *if* J_0 were periodic (which it isn't at the beginning but almost is eventually) then to get $J_0\left(\frac{a_1 x}{100}\right)$, multiply the old period by $\frac{100}{a_1}$ (just the way the period of $\sin \frac{1}{2} x$ is twice the period of sin x).

Footnote a_1 is approximately 2.4 so $\frac{100}{a_1}$ is approx 42.



SOLUTIONS review problems for Chapter 6

1. Let $u(x,t) = X(x)T(t)$. Then

$$XT' + XT = kX''T$$

$$\frac{X''}{X} = \frac{T' + T}{kT} = \text{con}$$

$$X'' = \text{con}\ X, \qquad T' + (1 - k\ \text{con})T = 0$$

Try the case where con is negative, renamed $-\lambda^2$. Then

$$X = A\cos\lambda x + B\sin\lambda x, \quad T = Ce^{-(1+k\lambda^2)t}$$

$$X' = A\lambda\sin\lambda x + B\lambda\cos\lambda x$$

$X'(0) = 0$ so $B = 0$

$X'(4) = 0$ so $-4\lambda\sin 4\lambda = 0$, $4\lambda = n\pi$, $\lambda = \frac{n\pi}{4}$ for $n = 1,2,3,\ldots$

Try the case where con = 0. Then $X'' = 0$, $T' + T = 0$, $X = Dx + E$, $T = Fe^{-t}$

$X'(0) = 0$ and $X'(4) = 0$ make $D = 0$.
Solution in this case is $u = EFe^{-t} = Ge^{-t}$

By superposition $u = Ge^{-t} + \sum_{n=1}^{\infty} A_n\, e^{-(1+k(n\pi/4)^2)t} \cos\frac{n\pi x}{4}$

To get the IC we need

$$f(x) = G + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{4} \quad \text{for } x \text{ in } [0,4]$$

$$C = \text{average value of } f(x) \text{ on } [0,4] = 5$$

$$A_n = \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx = \frac{2}{4}\left[\int_0^2 3\cos\frac{n\pi x}{4}\,dx + \int_2^4 7\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \begin{cases} -\dfrac{8}{n\pi} & \text{if } n = 1,5,9,\ldots \\ \dfrac{8}{n\pi} & \text{if } n = 3,7,11,\ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Solution is

$$u = 5e^{-t} - \frac{8}{\pi}\left[e^{-(1+k(\frac{\pi}{4})^2)t}\cos\frac{\pi x}{4} - \frac{1}{3}e^{-(1+k(\frac{3\pi}{4})^2)t}\cos\frac{3\pi x}{4} + \frac{1}{5}e^{-(1+k(\frac{5\pi}{4})^2)t}\cos\frac{5\pi x}{4} - \ldots\right]$$

2.(a) *Part I* Separate variables.
Try a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$
Then

$$R''\,\Theta + \frac{1}{r} R'\,\Theta + \frac{1}{r^2} R\,\Theta'' = 0$$

$$\Theta\left[R'' + \frac{1}{r} R'\right] = -\frac{1}{r^2} R\Theta''$$

$$\frac{-r^2\left[R'' + \frac{1}{r}R'\right]}{R} = \frac{\Theta''}{\Theta} = \text{constant}$$

$$\Theta'' - \text{constant}\,\Theta = 0, \qquad r^2 R'' + rR' + \text{constant}\,R = 0$$

case 1 con $= -\lambda^2$
$\Theta = C\cos\lambda\theta + D\sin\lambda\theta$, $R = Ar^{\lambda} + Br^{-\lambda}$.
case 2 Constant = 0
$\Theta'' = 0$, $\Theta = A\theta + B$
$r^2R'' + rR' = 0$, $R(r) = Ar^{\lambda} + Br^{-\lambda}$ (reference page)

The BC separate to $\Theta'(0) = 0$, $\Theta'(\pi) = 0$

Part II Plug in the separated BC.
case 1
$\Theta'(\theta) = -C\lambda\sin\lambda\theta + D\lambda\cos\lambda\theta$

$\Theta'(0) = 0$ so $D = 0$
$\Theta'(\pi) = 0$ so $-C\lambda\sin\lambda\pi = 0$, $\lambda = n$ for $n = 1,2,3,\ldots$
Need $B = 0$ to keep R finite

case 2
$\theta = A\theta + B$, $R = C\,\ell n\, r + D$
$\Theta'(0) = 0$ and $\Theta'(\pi) = 0$ make $A = 0$.
Need $C = 0$ to keep R finite From this case we have solution $v = BD = F_0$

By superposition, $\boxed{v = F_0 + \sum_{n=1}^{\infty} F_n r^n \cos n\theta}$

Part III Get the last (nonhomog) BC.
We need

$$f(\theta) = F_0 + \sum_{n=1}^{\infty} F_n 5^n \cos n\theta \quad \text{for } \theta \text{ in } [0,\pi],$$

$$\boxed{F_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta}$$

$$F_n\, 5^n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\,d\theta, \qquad \boxed{F_n = \frac{1}{5^n}\,\frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\,d\theta}$$

(b) Now we need $3 + 6\cos 2\theta = F_0 + \sum_{n=1}^{\infty} F_n 5^n \cos n\theta$ for θ in $[0,\pi]$,

By inspection we can get it with $F_0 = 3$, $F_2\, 5^2 = 6$, , $F_2 = \frac{6}{25}$, other F's = 0

Solution is $v = 3 + \frac{6}{25} r^2 \cos 2\theta$

3. Try $y = X(x)T(t)$ Then $XT'' = gxX''T + gX'\,T$, $\dfrac{T''}{gT} = \dfrac{xX'' + X'}{X}$ = constant,

$\boxed{T'' - \text{con}\; gT = 0}$, $\boxed{xX'' + X'' - \text{con}\; X = 0}$

4. Try $y = X(x)T(t)$ Then $XT'' + XT' = X''T$, $\dfrac{T'' + T'}{T} = \dfrac{X''}{X}$ = con

case 1 con is negative, call it $-\lambda^2$. Then

$X = A\cos\lambda x + B\sin\lambda x$

$T'' + T' = -\lambda^2 T$, $T'' + T' + \lambda^2 = 0$, $m = \dfrac{-1 \pm \sqrt{1 - 4\lambda^2}}{2}$ and we need subcases because the T solution depends on whether $1 - 4\lambda^2$ is pos, neg or zero.

case 1(a) $1 - 4\lambda^2$ positive, i.e., $0 \le \lambda^2 < \frac{1}{4}$

$$T = Ce^{\frac{-1 + \sqrt{1 - 4\lambda^2}}{2}t} + De^{\frac{-1 - \sqrt{1 - 4\lambda^2}}{2}t}$$

case 1(b) $1 - 4\lambda^2$ negative, i.e., $\lambda^2 > \frac{1}{4}$

$$T = e^{-t/2}\left[C\cos\frac{\sqrt{4\lambda^2 - 1}}{2}t + D\sin\frac{\sqrt{4\lambda^2 - 1}}{2}t\right]$$

case 1(c) $1 - 4\lambda^2 = 0$, i.e., $\lambda^2 = 1/4$

$$T = C\,e^{-t/2} + Dt\,e^{-t/2}$$

case 2 con = 0

$X = Ax + B$, $T'' + T' = 0$, $T = C + De^{-t}$

5. (a) With these axes, the picture is the odd periodic extension of the [0,5] piece (odd function, T = 10, use sines with L = 5)

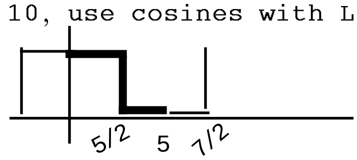
Series is $\sum B_n \sin\dfrac{n\pi x}{5}$ where $B_n = \dfrac{2}{5}\displaystyle\int_0^5 7\sin\frac{n\pi x}{5}\,dx = \frac{28}{n\pi}$ if n is odd

Series is $\dfrac{28}{\pi}\sin\dfrac{\pi x}{5} + \dfrac{28}{3\pi}\sin\dfrac{3\pi x}{5} + \dfrac{28}{5\pi}\sin\dfrac{5\pi x}{5} - \ldots$

Fund frequency is $\frac{1}{5}$ with amp $\frac{28}{\pi}$

First overtone freq is $\frac{3}{5}$ with amp $\frac{28}{3\pi}$

(b) With these axes the function is the even periodic extension of the [0,5] piece (even function, T = 10, use cosines with L = 5).



Series is $A_0 + \sum A_n \cos \frac{n\pi x}{5}$ where A_0 = average f(x) on [0,5] = 7,

$$A_n = \frac{2}{5}\int_0^5 f(x) \cos \frac{n\pi x}{5}\, dx = \frac{2}{5}\int_0^{5/2} 14 \cos \frac{n\pi x}{5}\, dx = \frac{28}{n\pi} \sin \frac{n\pi x}{5}\Big|_0^{5/2}$$

Series is $7 + \frac{28}{\pi}\cos\frac{\pi x}{5} - \frac{28}{3\pi}\cos\frac{3\pi x}{5} + \frac{28}{5\pi}\cos\frac{5\pi}{5} - \ldots$

Same frequencies and amplitudes as in method 1.

(c) Here's the long way to get the series.
The function is the periodic extension of the [0,10] piece (not-odd, not-even, T = 10, use full series with L = 10). Series is

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{5} + B_n \sin\frac{n\pi x}{5}\right) \text{ where}$$

A_0 = average value of the [0,10] piece = 7

I got this by inspection. Half the time, f(x) = 14 and half the time f(x) = 0.

You can also use $\frac{1}{10}\int_0^{10} f(x)\, dx = \frac{1}{10}\left[\int_0^5 14\, dx + \int_5^{10} 0\, dx\right]$

$$A_n = \frac{2}{10}\int_0^{10} f(x)\cos\frac{n\pi x}{5}\, dx = \frac{2}{10}\left[\int_0^5 14\cos\frac{n\pi x}{5}\, dx + \int_5^{10} 0 \cos\frac{n\pi x}{5}\, dx\right]$$

$$= \frac{2}{10}\cdot 14\cdot \frac{5}{n\pi}\sin\frac{n\pi x}{5}\Big|_0^5 = 0$$

$$B_n = \text{ditto but with sines} = \frac{2}{10}\cdot 14\cdot -\frac{5}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^5 = -\frac{14}{n\pi}(\cos n\pi - 1)$$

$$= \frac{28}{n\pi} \text{ if n is odd}$$

Series is $7 + \frac{28}{\pi}\sin\frac{\pi x}{5} + \frac{28}{3\pi}\sin\frac{3\pi x}{5} + \frac{28}{5\pi}\sin\frac{5\pi x}{5} - \ldots$

Like the series in part (a) but with the extra term 5 (which makes it a full series, not a sine series). Same frequencies and amps as in part (a)

The fast way to get the series is to notice that

function in Fig (c) = 7 + function in Fig (a)

(i.e., translate Fig (a) up 7 to get Fig (c))

So

series for (c) = 7 + series for (a)

6. (a) You need

$$u(x,2) = 8A_0 + \sum_{n=1}^{\infty} A_n e^{-2n} \cos\frac{n\pi x}{4} \quad \text{for } 0 \le x \le 4$$

which you can get with

$$8A_0 = \frac{1}{4}\int_0^4 u(x,2)\,dx$$

$$A_n e^{-2n} = \frac{2}{4}\int_0^4 u(x,2)\cos\frac{n\pi x}{4}\,dx$$

$$u(x,2) = \begin{cases} 0 & \text{if } 0 \le x \le 2 \\ 5x - 10 & \text{if } 2 \le x \le 3 \\ 5 & \text{if } 3 \le x \le 4 \end{cases}$$

so the solution is

$$u(x,y) = A_0\, y^3 + \sum_{n=1}^{\infty} A_n e^{-ny} \cos\frac{n\pi x}{4} \quad \text{for } 0 \le x \le 4,\ y \ge 2$$

where

$$A_0 = \frac{1}{8}\,\frac{1}{4}\left[\int_2^3 (5x-10)\,dx + \int_3^4 5\,dx\right]$$

$$A_n = e^{2n}\,\frac{2}{4}\left[\int_2^3 (5x-10)\cos\frac{n\pi x}{4}\,dx + \int_3^4 5\cos\frac{n\pi x}{4}\,dx\right]$$

(b) Now you would need

$$u(x,2) = 8A_0 + \sum_{n=1}^{\infty} A_n e^{-2n} \cos\frac{n\pi x}{\boxed{3}} \quad \text{for } 0 \le x \le \boxed{4}$$

But the functions 1, $\cos\frac{\pi x}{3}$, $\cos\frac{2\pi x}{3}$, $\cos\frac{3\pi x}{3}$, ... are not a complete set on the interval [0,4]. You can't make a series out of them that will do what you want it to do for $0 \le x \le 4$ (you can only control what happens for $0 \le x \le 3$). So you can't get the condition satisfied. This shouldn't happen in the course of solving a PDE which comes from a real life problem. You should realize how lucky you are.

REFERENCE PAGE FOR CHAPTERS 2,4,6

convolution integral

$$f(t)*g(t) = \int_{u=-\infty}^{\infty} f(t-u)\ g(u)\ du = \int_{u=-\infty}^{\infty} g(t-u)\ f(u)\ du$$

ANTIDERIVATIVE TABLES

(A) $\int \sin^2 ax\, dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax$

(B) $\int \cos^2 ax\, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax$

(C) $\int xe^{ax}\, dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$

(D) $\int x\cos ax\, dx = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax$

(E) $\int x\sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$

(F) $\int \sin mx \cos nx\, dx = \frac{\cos(n-m)x}{2(n-m)} - \frac{\cos(n+m)x}{2(n+m)}$ where $m \neq n$

(G) $\int \sin mx \cos mx\, dx = \frac{\sin^2 mx}{2m}$

(H) $\int \cos mx \cos nx\, dx = \frac{\sin(n-m)x}{2(n-m)} + \frac{\sin(n+m)x}{2(n+m)}$ where $m \neq n$

(I) $\int \sin mx \sin nx\, dx = \frac{\sin(n-m)x}{2(n-m)} - \frac{\sin(n+m)x}{2(n+m)}$ where $m \neq n$

(J) $\int e^{ax}\cos nx\, dx = \frac{e^{ax}(a\cos nx + n\sin nx)}{a^2+n^2}$

(K) $\int e^{ax}\sin nx\, dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2+n^2}$

a table of some exact differentials

(22) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(23) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(24) $\frac{-2x\,dx - 2y\,dy}{(x^2+y^2)^2} = d\left(\frac{1}{x^2+y^2}\right)$

(25) $\frac{x\,dx + y\,dy}{\pm\sqrt{x^2+y^2}} = d\left(\pm\sqrt{x^2+y^2}\right)$

(26) $\frac{2x\,dx + 2y\,dy}{x^2+y^2} = d\,\ell n(x^2+y^2)$

(27) $\frac{-y\,dx + x\,dy}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

COEFFS FOR FOURIER SINE SERIES

To get $f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ for x in $[0,L]$ choose $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

COEFFS FOR FOURIER COSINE SERIES

To get $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$ for x in $[0,L]$

choose $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ = average value of f in $[0,L]$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

COEFFS FOR FOURIER FULL SERIES

To get $f(x) = C_0 + \sum_{n=1}^{\infty}\left[C_n\cos\frac{n\pi x}{L/2} + D_n\sin\frac{n\pi x}{L/2}\right]$ for x in $[0,L]$

choose $C_0 = \frac{1}{L}\int_0^L f(x)\,dx$ = average value of f in $[0,L]$

$C_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L/2}\,dx$, $\quad D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L/2}\,dx$

OVER

COEFFS FOR FOURIER BESSSEL SERIES

To get $f(r) = \sum_{n=1}^{\infty} D_n \, J_0\left[\frac{a_n r}{L}\right]$ for r in [0,L], where the a_n's are the zeros of J_0,

$$\text{choose } D_n = \frac{\int_0^L f(r)\, J_0\left[\frac{a_n r}{L}\right] r\, dr}{\int_0^L J_0^2\left[\frac{a_n r}{L}\right] r\, dr}$$

SOME ORDINARY DE WITH VARIABLE COEFFICIENTS THAT TURN UP WHEN YOU SOLVE PDE

$rR'' + R' + r\lambda^2 R = 0$ has solution $R = AJ_0(\lambda r) + BY_0(\lambda r)$

$r^2 R'' + rR' - \lambda^2 R = 0$ has solution $R = Ar^{\lambda} + Br^{-\lambda}$

$r^2 R'' + rR' = 0$ has solution $R = C \ln r + D$

INTEGRAL TABLES

(1) $\frac{2}{L}\int_0^L K \sin\frac{n\pi x}{L}\,dx$ (where K is a constant) $= \begin{cases} 0 & \text{if n is even} \\ \frac{4K}{n\pi} & \text{if n is odd} \end{cases}$

(2) $\frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx = \begin{cases} \frac{-2L}{n\pi} & \text{if n is even} \\ \frac{2L}{n\pi} & \text{if n is odd} \end{cases}$

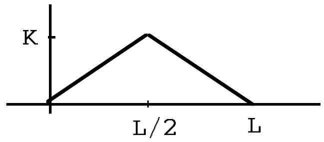
(3) $\frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if n is even} \\ \frac{-4L}{n^2\pi^2} & \text{if n is odd} \end{cases}$

(4) If $f(x) = \begin{cases} a & \text{for } 0 \le x \le L/2 \\ b & \text{for } L/2 \le x \le L \end{cases}$ where a and b are constants then

(a) $\frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n=4,8,12, \\ \frac{4(a-b)}{n\pi} & \text{if } n=2,6,10,\ldots \\ \frac{2(a+b)}{n\pi} & \text{if n is odd} \end{cases}$

(b) $\frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \begin{cases} \frac{2(a-b)}{n\pi} & \text{if } n=1,5,9, \\ \frac{-2(a-b)}{n\pi} & \text{if } n=3,7,11,\ldots \\ 0 & \text{if n is even} \end{cases}$

(5) If f(x) looks like this



then

(a) $\frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if n is even} \\ \frac{8K}{n^2\pi^2} & \text{if } n=1,5,9,\ldots \\ \frac{-8K}{n^2\pi^2} & \text{if } n=3,7,11,\ldots \end{cases}$

(b) $\frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if n is odd or if } n=4,8,12,\ldots \\ \frac{-16K}{n^2\pi^2} & \text{if } n=2,6,10,\ldots \end{cases}$

REFERENCE PAGE FOR CHAPTER 5

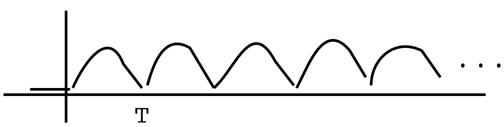
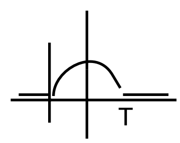
DEFINITION OF THE TRANSFORM $F(s) = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$

TRANSFORMS OF DERIVATIVES $f'(t) \leftrightarrow sF(s) - f(0)$,

$$f''(t) \leftrightarrow s^2 F(s) - sf(0) - f'(0)$$

TRANSFORM OF A CONVOLUTION $f(t)*g(t) \leftrightarrow F(s)G(s)$

TRANSFORM TABLE

$u(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
$t^n\, u(t)$	$\frac{n!}{s^{n+1}}$
$\sin at\; u(t)$	$\frac{a}{s^2 + a^2}$
$\cos at\; u(t)$	$\frac{s}{s^2 + a^2}$
$e^{at}\, u(t)$	$\frac{1}{s - a}$
$\delta(t)$	1
 T ...	$\frac{1}{1-e^{-Ts}} \times$ transform of  T

SHIFTING RULES

$f(t-a)u(t-a) \leftrightarrow e^{-as} F(s)$

$e^{at} f(t) \leftrightarrow F(s-a)$

INVERSE TRANSFORMS

It's understood that these inverse tranforms are good for any values of a,b,c (including non-real values) as long as you don't end up dividing by 0.

(1)	1	$\delta(t)$
(2)	$\frac{1}{s-a}$	$e^{at}\, u(t)$
(3)	$\frac{1}{s}$	$u(t)$
(4)	$\frac{1}{s^2}$	$t\; u(t)$
(5)	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}\, u(t)$
(6)	$\frac{s}{s^2 + a^2}$	$\cos at\; u(t)$

(7) $\dfrac{1}{s^2 + a^2}$ $\quad$ $\dfrac{1}{a} \sin at \; u(t)$

(8) $\dfrac{s}{(s^2 + a^2)^2}$ $\quad$ $\dfrac{1}{2a} \, t \sin at \; u(t)$

(9) $\dfrac{1}{(s^2 + a^2)^2}$ $\quad$ $\dfrac{1}{2a^3} (\sin at - at \cos at) \; u(t)$

(10) $\dfrac{1}{s(s^2 + a^2)}$ $\quad$ $\dfrac{1}{a^2} (1 - \cos at) \; u(t)$

(11) $\dfrac{1}{s^2(s^2 + a^2)}$ $\quad$ $\dfrac{1}{a^3} (at - \sin at) \; u(t)$

(12) $\dfrac{1}{s^3(s^2 + a^2)}$ $\quad$ $\left(\dfrac{1}{2a^2} t^2 + \dfrac{1}{a^4} \cos at - \dfrac{1}{a^4} \right) u(t)$

(13) $\dfrac{s^2}{(s^2 + a^2)^2}$ $\quad$ $\dfrac{1}{2a} (\sin at + at \cos at) \; u(t)$

(14) $\dfrac{1}{(s^2 + a^2)(s^2 + b^2)}$ $\quad$ $\dfrac{1}{b^2 - a^2} \left(\dfrac{1}{a} \sin at - \dfrac{1}{b} \sin bt \right) u(t)$

(15) $\dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$ $\quad$ $\dfrac{1}{b^2 - a^2} (\cos at - \cos bt) \; u(t)$

(16) $\dfrac{1}{s^2(s-a)}$ $\quad$ $\left(\dfrac{1}{a^2} e^{at} - \dfrac{t}{a} - \dfrac{1}{a^2} \right) u(t)$

(17) $\dfrac{1}{(s-a)(s-b)}$ $\quad$ $\dfrac{1}{a-b} (e^{at} - e^{bt}) \; u(t)$

(18) $\dfrac{s}{(s-a)(s-b)}$ $\quad$ $\dfrac{1}{a-b} (ae^{at} - be^{bt}) \; u(t)$

(19) $\dfrac{1}{s^2 - a^2}$ $\quad$ $\dfrac{1}{a} \sinh at \; u(t)$ (special case of (17))

(20) $\dfrac{s}{s^2 - a^2}$ $\quad$ $\cosh at \; u(t)$ (special case of (18))

(21) $\dfrac{s}{(s-a)^2}$ $\quad$ $(at + 1) \, e^{at} \, u(t)$

(22) $\dfrac{1}{(s-a)(s-b)(s-c)}$ $\quad$ $\left[\dfrac{e^{at}}{(a-b)(a-c)} + \dfrac{e^{bt}}{(b-a)(b-c)} + \dfrac{e^{ct}}{(c-a)(c-b)} \right] u(t)$

(23) $\dfrac{s}{(s-a)(s-b)(s-c)}$ $\quad$ $\left[\dfrac{ae^{at}}{(a-b)(a-c)} + \dfrac{be^{bt}}{(b-a)(b-c)} + \dfrac{ce^{ct}}{(c-a)(c-b)} \right] u(t)$

(24) $\dfrac{s^2}{(s-a)(s-b)(s-c)}$ $\quad$ $\left[\dfrac{a^2 \, e^{at}}{(a-b)(a-c)} + \dfrac{b^2 \, e^{bt}}{(a-b)(c-b)} + \dfrac{c^2 \, e^{ct}}{(a-c)(b-c)} \right] u(t)$

(25) $\dfrac{1}{(s-a)(s^2 + b^2)}$ $\quad$ $\dfrac{1}{a^2 + b^2} \left[e^{at} - \cos bt - \dfrac{a}{b} \sin bt \right] u(t)$

(26) $\dfrac{1}{(s-a)^2(s-b)}$ $\quad$ $\left[\dfrac{-e^{at}}{(a-b)^2} + \dfrac{e^{bt}}{(a-b)^2} + \dfrac{te^{at}}{a-b} \right] u(t)$