Curvature of Surfaces

The Gaussian curvature of a surface $S \subset \mathbb{R}^3$ at a point $\mathbf{p}$ says a lot about the behavior of the surface at that point. Let's think again about how the Gauss map may contain information about S. We'll assume S is an orientable smooth surface, with Gauss map $N : S \to S^2$. Recall that $K(\mathbf{p}) = \det \mathrm{d}N(\mathbf{p})$ is the Gaussian curvature at $\mathbf{p}$.

First, let's just think about the sign of $K(\mathbf{p})$. You should already have realized that if the determinant of a 2×2 matrix is zero, then any pair of basis vectors are "squished on top of each other" somehow. What you might not have realized is that when the determinant is non-zero its sign tells you how the relative position of the basis vectors is affected. If the sign is positive, then the orientation of the principle angle between the vectors is preserved. If the sign is negative, then the orientation is switched.

What this means in terms of the Gauss map: if $K(\mathbf{p})$ is positive, then moving along the surface in the direction of a chosen pair of basis vectors changes the normal vector in either the same or the opposite direction for *both* basis vectors. If $K(\mathbf{p})$ is negative, then the change in the normal vector is the *same* for one basis vector and *opposite* for the other.

The following three examples require progressively more use of the definition of Gaussian curvature, and are very important examples of surfaces with constant curvature.

Example (Spheres of different radii). Let S_r^2 denote the sphere in $\mathbb{R}^3$ of radius r (i.e., the locus of the equation $x^2 + y^2 + z^2 = r^2$). As with the ordinary unit sphere, the normal vectors to S_r^2 extend radially outward from the sphere. The Gauss map $N : S_r^2 \to S^2$ simply scales by $1/r$, i.e.,

$$N : \mathbf{x} \to \frac{1}{r}\mathbf{x}.$$

This map is linear on $\mathbb{R}^3$, so its derivative on all of $\mathbb{R}^3$ is itself. In particular, its restriction to $T_{\mathbf{p}}S_r^2$, the tangent plane of S_r^2 at $\mathbf{p}$, is $1/r$ times the identity. Since the derivative of the Gauss map is a linear function from a two-dimensional space to a two-dimensional space, $[\mathrm{d}N(\mathbf{p})]$ in this case is $1/r$ times the 2×2 identity matrix, which has determinant $1/r^2$. By definition, this means the Gaussian curvature at every point is $K \equiv 1/r^2$.

Observe that as the radius of the sphere increases, the curvature decreases. This makes sense, since the surface of the Earth appears "flatter" than the surface of a baseball. But it also highlights the fact that *curvature depends on the choice of units.* The Earth appears much more curved if our units are AUs (an AU is an astronomical unit, the (average) distance from the Earth to the sun) than if they're meters.

Example (Cylinder). We need to compute the derivative of the Gauss map on an ordinary cylinder (say the one with equation $x^2 + y^2 = 1$ in $\mathbb{R}^3$). We do this by choosing two tangent directions at a point, looking at curves on the cylinder that have those directions as tangent vectors, and seeing what the curve looks like when composed with the Gauss map. (Recall that this is entirely analogous to the directional derivative in "flat" space, i.e., $\mathbb{R}^n$.) Two obvious directions are along a line parallel to the cylinder's axis (a *generatrix*), and along a circle around the cylinder (a *directrix*). Going around the directrix, the normal vector traces out a circle (actually, the equator) on S^2. However, going along the generatrix, the normal vector doesn't change at all. Therefore the corresponding basis vector in the tangent plane is scaled by zero, which means the derivative of the Gauss map must have determinant zero. We conclude that the cylinder has constant Gaussian curvature $K \equiv 0$.

Example (Pseudosphere). The pseudosphere is a surface of revolution, obtained by rotating a *tractrix* around its asymptotic line. The tractrix (asymptotic to the z-axis and lying in the (x, z)-plane in $\mathbb{R}^3$) can be parametrized by

$$\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ \tanh^{-1}\sqrt{1-e^{-2t}} - \sqrt{1-e^{-2t}} \end{pmatrix}.$$

This parametrization has the benefit that the tangent vector has unit length (as can be verified using the fact that $(\tanh^{-1})'(x) = 1/(1-x^2)$). We're mostly interested in the tangent vector, so we'll write its form:

$$\begin{pmatrix} x'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ \sqrt{1-e^{-2t}} \end{pmatrix}.$$

Because the pseudosphere is a surface of revolution, the normal vector at $(x(t), 0, z(t))^\top$ is simply the tangent vector rotated $90°$ (we choose clockwise):

$$N\begin{pmatrix} x(t) \\ 0 \\ z(t) \end{pmatrix} = \begin{pmatrix} z'(t) \\ 0 \\ -x'(t) \end{pmatrix} = \begin{pmatrix} \sqrt{1-e^{-2t}} \\ 0 \\ e^{-t} \end{pmatrix}.$$

It's now easy to compute how this varies with t:

$$\left[\mathrm{d}N\begin{pmatrix} x(t) \\ 0 \\ z(t) \end{pmatrix}\right]\begin{pmatrix} -e^{-t} \\ 0 \\ \sqrt{1-e^{-2t}} \end{pmatrix} = \begin{pmatrix} \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \\ 0 \\ -e^{-t} \end{pmatrix}.$$

This has multiplied the tangent vector to the tractrix by $\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}$.

To complete our computation of $\mathrm{d}N$, we need to choose another direction in the tangent plane, or, equivalently, another curve lying in the pseudosphere and passing through our point. A nice choice is the circle that the point traces as it's rotated around the z-axis. This choice is nice for two big reasons: it's easy to parametrize, and its tangent vector is $(0, e^{-t}, 0)^\top$, which is orthogonal to the first tangent vector. We parametrize by θ.

$$N\begin{pmatrix} x(t)\cos\theta \\ x(t)\sin\theta \\ z(t) \end{pmatrix} = \begin{pmatrix} \sqrt{1-e^{-2t}}\cos\theta \\ \sqrt{1-e^{-2t}}\sin\theta \\ e^{-t} \end{pmatrix}$$

Now we differentiate with respect to θ and evaluate at $\theta = 0$:

$$\left[\mathrm{d}N\begin{pmatrix} x(t) \\ 0 \\ z(t) \end{pmatrix}\right]\begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{1-e^{-2t}}\sin\theta \\ \sqrt{1-e^{-2t}}\cos\theta \\ 0 \end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix} 0 \\ \sqrt{1-e^{-2t}} \\ 0 \end{pmatrix}.$$

This has multiplied the tangent vector to the circle by $\frac{\sqrt{1-e^{-2t}}}{e^{-t}}$.

The matrix of $\mathrm{d}N(x(t), 0, z(t))^\top$ in our chosen basis $\{(-e^{-t}, 0, \sqrt{1-e^{-2t}})^\top, (0, e^{-t}, 0)^\top\}$ is therefore

$$\left[\mathrm{d}N\begin{pmatrix} x(t) \\ 0 \\ z(t) \end{pmatrix}\right] = \begin{bmatrix} \frac{-e^{-t}}{\sqrt{1-e^{-2t}}} & 0 \\ 0 & \frac{\sqrt{1-e^{-2t}}}{e^{-t}} \end{bmatrix},$$

which has determinant -1. We have thus proved that the pseudosphere has constant Gaussian curvature $K \equiv -1$.