Special Mathematics Lecture
Differential geometry

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Handwritten notes taken by L. Zhang

Differential Geometry

Extrinsic / Intrinsic ways to study DG (they're not so different)
from outside of the manifold — on the manifold

Extrinsic : to look at curves or surfaces from outside in a bigger space
(in Calculus II) simple for visualization

Intrinsic : no more any ambiant space, like a 2D animal in a flatland ~~witho~~ without a 3rd dimension, useful in general relativity & universe
(mostly used in this course)

However, a manifold can always embedded in a higher dimensional space (not always one more)
(Nash embedding thm)

I) Differentiable manifolds

I.1 Topological manifolds (+ topology)

Def. a TOPOLOGICAL MANIFOLD of dimension n is a $\boxed{\text{topological space}}$ $\mathcal{M}$ s.t. :

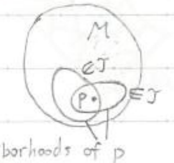
1) $\mathcal{M}$ is $\boxed{\text{Hausdorff}}$
2) Any $p \in \mathcal{M}$ has a $\boxed{\text{neighborhood}}$ V $\boxed{\text{homeorphic}}$ to an $\boxed{\text{open set}}$ $U \subset \mathbb{R}^n$.
3) $\mathcal{M}$ is $\boxed{\text{second countable}}$

$\boxed{\text{Def}}$ a TOPOLOGICAL SPACE $\mathcal{M} \equiv (\mathcal{M}, \mathcal{T})$ (curly T)
is a set $\mathcal{M}$ together with a collection $\mathcal{T}$ of subsets satisfying :

1) $\phi, \mathcal{M} \in \mathcal{T}$ (empty set 空集)
2) If $V_\alpha \in \mathcal{T}$, then $\bigcup_\alpha V_\alpha \in \mathcal{T}$ (union 並集) ($\mathcal{T}$ is STABLE FOR ARBITRARY UNION)
3) If $V_1, \cdots, V_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} V_i \in \mathcal{T}$ (intersection 交集) (↳ UNDER FINITE INTERSECTION)

The elements of $\mathcal{T}$ are called the OPEN SETS.
Their complement ($\mathcal{M} \setminus V$, $V \in \mathcal{T}$) is called a CLOSED SET.

$\boxed{\text{Def}}$ Let $(\mathcal{M}, \mathcal{T})$ be a topological space (t.s.), and let $p \in \mathcal{M}$.
a NEIBORHOOD of p is any open set containing p.
We write $\mathcal{V}_p$ (Greek nu) for the set of all neighborhoods of p.



neiborhoods of p

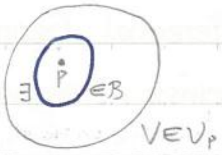
$\boxed{\text{Def}}$ $(\mathcal{M}, \mathcal{T})$ is HAUSDORFF if

$$\forall p_1, p_2 \in \mathcal{M},\ p_1 \neq p_2 : \exists V_1 \in \mathcal{V}_{p_1},\ V_2 \in \mathcal{V}_{p_2} : V_1 \cap V_2 = \phi$$

Hausdorff

It is often difficult to describe all open sets in $(\mathcal{M}, \mathcal{T})$
$\Rightarrow$ Introduce the notion of a basis. (related to Second Countable)

Def. A subset $\mathcal{B} := \{V_\alpha\} \subset \mathcal{T}$ is a BASIS of $(M, \mathcal{T})$ if



$$\forall p \in M \ \forall V \in \mathcal{V}_p : \exists U \in \mathcal{B} : p \in U \subset V$$

Example: $M = \mathbb{R}^n$ with $\mathcal{T} = \{$all open sets in $\mathbb{R}^n\}$ is a topological manifold.

An OPEN SET in $\mathbb{R}^n$ is a set V s.t. $\forall p \in V$:

there is a small ball centered at p and contained in V.

We set $B(p, r)$ = a ball centered at p and of radius r.

norm in $\mathbb{R}^n$

$$B(p, r) := \{x \in \mathbb{R}^n \mid \|x - p\| < r\}$$

Then: (all balls centered at any point)

$\mathcal{B} := \{B(x, r) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis for $\mathbb{R}^n$.

in a 1 to 1 (= bijective) relation with $\mathbb{N}$.

$\boxed{\text{Def.}}$ $(M, \mathcal{T})$ is SECOND COUNTABLE if it has a countable basis.

For $\mathbb{R}^n$, we can set

$\mathcal{B} := \{B(x, \frac{1}{n}) \mid x \in \mathbb{Q}^n, n \in \mathbb{N}\}$ and it is a countable basis for $\mathbb{R}^n$.

$\Rightarrow \mathbb{R}^n$ is second countable.

$\boxed{\text{Def}}$ Let $(M, \mathcal{T})$, $(\mathcal{N}, \mathcal{S})$ be 2 t.s., and let $f : M \mapsto \mathcal{N}$.

f is CONTINUOUS if $f^{-1}(U) \in \mathcal{T} \ \forall U \in \mathcal{S}$

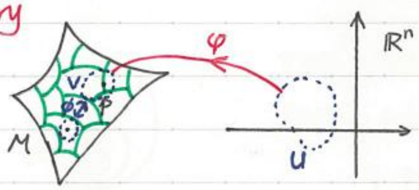
with the PRE-IMAGE $f^{-1}(U) := \{p \in M \mid f(p) \in U\}$

Exersise: When $M = \mathcal{N} = \mathbb{R}$ and $\mathcal{T} = \mathcal{S} = \{$open sets in $\mathbb{R}\}$, check if this def corresponds to the ε-δ def of coutinuousity.

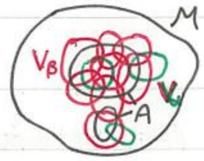
If f is bijective and f, f^{-1} are continuous,

we say that f is HOMEOMORPHIC.

Summary



Def. M is CONNECTED if it is not the disjoint union of 2 non-empty open sets.



Def. Let A be a subset of M.

1) An OPEN COVER for A is a subfamily $\{V_\alpha\} \subset \mathcal{T}$ s.t. $A \in \bigcup_\alpha V_\alpha$ finite or infinite
2) a SUBCOVER of an open cover for A (in which the green subsets are unnecessary) is a subfamily $\{V_\beta\} \subset \{V_\alpha\}$ which still covers A.
3) A is COMPACT (small in this setting) if any open cover of A admits a finite subcover (If $A = \mathbb{R}^n$, A is compact iff A is closed and bounded)

$(\mathbb{N}, \mathcal{T})$ topo. space

$\mathcal{T}_0 := \{[a,b] \cap \mathbb{N} \mid a \text{ is not odd and } b \text{ is not even}; a<b; a,b \in \mathbb{N} \cup \{\infty\}\}$

$\mathcal{T} := \{I \mid I = \bigcup_\alpha I_\alpha,\ \forall \alpha : I_\alpha \in \mathcal{T}_0\} \cup \{\emptyset\}$

$\mathcal{T} := \{(\bigcup_\alpha [A_\alpha, B_\alpha]) \cap \mathbb{N} \mid \forall \alpha : A_\alpha \text{ is not odd and } B_\alpha \text{ is not even}; A_\alpha < B_\alpha; A_\alpha, B_\alpha \in \mathbb{N} \cup \{\infty\}\} \cup \{\emptyset\}$

In the example on P_2, $\mathcal{B}=\{B(x,\frac{1}{m}) \mid x\in\mathbb{Q}^n, m\in\mathbb{N}\}$

Let us define a half-space :

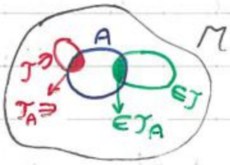
$$H^n := \{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_n \geq 0\}$$

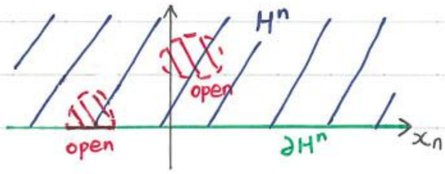
$$\partial H^n := \{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_n = 0\}$$ for the boundary.

Def. a TOPOLOGICAL MANIFOLD of dimension n with a boundary is a Hausdoff second-countable topological space M with each point $p\in M$ having a neighborhood V eigther homomorphic to an open subset of $H^n\setminus\partial H^n$ or to an open subset of H^n with the image of p inside ∂H^n.

$\boxed{\text{Remark}}$: If $(M,\mathcal{T})$ is a topo. space and if $A\subset M$.
Then the topology on A is given by $\mathcal{T}_A := \{V\cap A \mid V\in\mathcal{T}\}$
(called RELATIVE or SUBSPACE TOPOLOGY)

⚠ An open set for A (in $\mathcal{T}_A$) is not always an open set for M (in $\mathcal{T}$).





I.2 Smooth manifolds & Smooth maps

$\swarrow C^\infty$

Def. a SMOOTH (or C^∞) MANIFOLD M is a topo. manifold together with a family of homeomorphisms

$\varphi_\alpha : \mathbb{R}^n \supset \underset{\text{open}}{U_\alpha} \longmapsto M$ s.t.

1) $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$

2) If $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) =: V_{\alpha\beta} \neq \phi$ then

$$\begin{cases} \varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(V_{\alpha\beta}) \mapsto \varphi_\beta^{-1}(V_{\alpha\beta}) \\ \varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(V_{\alpha\beta}) \mapsto \varphi_\alpha^{-1}(V_{\alpha\beta}) \end{cases} \quad \text{(TRANSITION FUNCTIONS)}$$

are of C^∞ (from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$).

3) The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ is maximal.

$\mathcal{A}$ is called a C^∞ $\underbrace{\text{MAXIMAL}}_{3)}$ $\underbrace{\text{ATLAS}}_{1)\,\&\,2)}$.

MAXIMAL: If $\varphi : U \mapsto M$ ($U \subset \mathbb{R}^n$, open) satisfies $\varphi^{-1} \circ \varphi_\alpha$ and $\varphi_\alpha^{-1} \circ \varphi$ (whenever defined) is smooth then $(U, \varphi) \in \mathcal{A}$.

Remark: ◦ it is often easy to describe an atlas, but not the maximal one.

◦ A topological manifold can be endowed with different inequivalent maximal atlases. (see the P_1 on today's handout) (very deep)

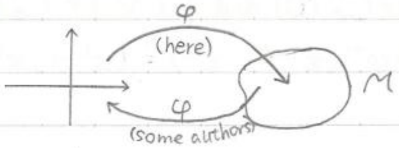
INEQUIVALENT: take 2 max atlases, if the union is not an atlas (some transision functions are not C^∞) then the 2 atlases are not equivalent.

Exercises

1) Provide an example of smooth manifolds with an atlas.
   (n-sphere, group of matrices, Lie groups, real projective space $P(\mathbb{R}^n)$, etc)
2) Show the uniqueness of the maximal atlas. (starting from a certain atlas)
3) Look at inequivalent atlases on n-sphere.
   $\parallel$
   differential structure

Remark:

1)



2) For $(U, \varphi) \in \mathcal{A}$ and $p \in \varphi(U) \subset M$, we set

$\varphi^{-1}(p) =: (x^1(p), x^2(p), \cdots, x^n(p))$

and call it a LOCAL COORDINATE of p. It means

$\varphi^{-1}(\cdot) =: (x^1(\cdot), x^2(\cdot), \cdots, x^n(\cdot))$ (a CHART or a LOCAL COORDINATE FUNCTION)

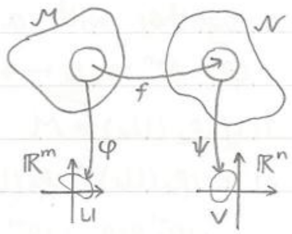
is an homeomorphism from an open subset of M to an open subset of $\mathbb{R}^n$.

Def: Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds of dim m and n respectively.

A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a SMOOTH MAP if

$\forall$ charts (U, φ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$:

$\psi \circ f \circ \varphi^{-1}$ is smooth wherever defined.



The function $\psi \circ f \circ \varphi^{-1}$ is called a LOCAL REPRESENTATION of f.

We set $C^\infty(\mathcal{M}, \mathcal{N}) :=$ the set of such smooth functions,

and $C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$.

Def. If $f \in C^\infty(\mathcal{M}, \mathcal{N})$ is bijective and if $f^{-1} \in C^\infty(\mathcal{N}, \mathcal{M})$, we call f a DIFFEOMORPHISM.

Remark: • a diffeomorphism is also a homeomorphism.

• A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a LOCAL DIFFEOMORPHISM at $p \in \mathcal{M}$ if

$\exists V \in \mathcal{V}_p$ and $W \in \mathcal{V}_{f(p)}$: $f|_V : V \mapsto W$ is a diffeomorphism.

Def. Let $f: \mathcal{M} \mapsto \mathcal{N}$ be a smooth function and let (U, φ) (V, ψ) be charts of $\mathcal{M}$ & $\mathcal{N}$ respectively.

For $p \in \mathcal{M}$, the RANK of f at p $(=: \mathrm{rank}(f)_p)$ corresponds to

the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix} (\varphi(p)) \quad \text{with } F := \psi \circ f \circ \varphi^{-1}$$

This rank is independent of the charts.

Thm. (not so easy) Framework as before. (Constant rank thm)

Suppose that $\mathrm{rank}(f)_p = k$ $\forall p \in \mathcal{M}$, with $k \in \mathbb{N}$. Then

$\forall p \in \mathcal{M}$ $\exists (U, \varphi), (V, \psi)$ charts of $\mathcal{M}, \mathcal{N}$ respectively s.t.

- $\varphi(p) = \mathbf{0} \in \mathbb{R}^m$ and $\psi(f(p)) = \mathbf{0} \in \mathbb{R}^n$;
- $\varphi(U) = C^m_\varepsilon(\mathbf{0})$ and $\psi(V) = C^n_\varepsilon(\mathbf{0})$ $\exists \varepsilon > 0$; Cube in $\mathbb{R}^n$ centered at $\mathbf{0}$ and with $x^i \in (-\varepsilon, \varepsilon)$
- $\psi \circ f \circ \varphi^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^k, \underbrace{0, \cdots, 0}_{n-k})$

I.3 Tangent Space

Recall that a PARAMETRIC SURFACE in $\mathbb{R}^3$ is a map $m: \mathbb{R}^2 \supset \Omega \mapsto \mathbb{R}^3$

Set $M := m(\Omega)$. For $p \in M$ and $c: (-\varepsilon, \varepsilon) \mapsto M \subset \mathbb{R}^3$ with $c(0) = p$ and if c is smooth,

$v := c'(0)$ is TANGENT to M at p.

The set of all such vectors generate the TANGENT PLANE.

Intrinsively, if M is a smooth manifold and if (U, φ) a chart at $p \in M$, then we could set

$v := \left[\frac{d}{dt}(\varphi \circ c)\right](0) \in \mathbb{R}^n$ and call it a tangent vector. (well-defined)

$\mathbb{R}^n \leftarrow M \leftarrow (-\varepsilon, \varepsilon)$

But it depends too much on the choice of a chart.

Def. For $p \in M$ (a s.m.) we denote by $C^\infty(p)$ the EQUIVALENCE CLASS of smooth real functions defined on a neighborhood of p.

Two functions are identified if they coincide (are identically same) on a neighborhood of p.

The elements of $C^\infty(p)$ are called GERMS of C^∞-function at p.

Observations: $C^\infty(p)$ is a vector space with the multiplication of functions

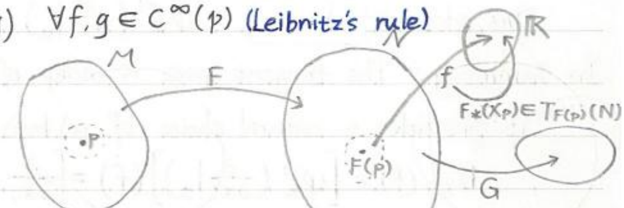
$\Leftrightarrow$ $C^\infty(p)$ is an algebra.

Def. The TANGENT SPACE $T_p(M)$ of M at p is the set of all maps

$X_p: C^\infty(p) \mapsto \mathbb{R}$ satisfying

1) $X_p(\alpha f + g) = \alpha X_p(f) + X_p(g) \quad \forall f, g \in C^\infty(p), \forall \alpha \in \mathbb{R}$

2) $X_p(fg) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g) \quad \forall f, g \in C^\infty(p)$ (Leibnitz's rule)



$T_p(M)$ is endowed with

1) $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$

2) $(\alpha X_p)(f) = \alpha X_p(f)$

which makes $T_p(M)$ a real vector space.

⚠ A tangent vector at p is any $X_p: C^\infty(p) \mapsto \mathbb{R}$.

Observe that this def is indep of any chart, and is intrusic.

Thm. 

(proof as exercise) (simple)

Let $F: M \mapsto N$ be a smooth map between smooth manifolds. For any $p \in M$:

$F^*: C^\infty(F(p)) \mapsto C^\infty(p), \quad F^*(f) := f \circ F$

$F_*: T_p(M) \mapsto T_{F(p)}(N), \quad [F_*(X_p)](f) := X_p(F^*(f)) = X_p(f \circ F) \quad \forall f \in C^\infty(F(p))$

Then F^* is a homomorphism of algebra $(F^*(f + \alpha g) = F^*(f) + \alpha F^*(g),\ F^*(fg) = F^*(f) F^*(g))$

and F_* is a homomorphism (means preserving structures) of vector space. $(F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p))$

If $H = G \circ F$, $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

F_* is called the DIFFRENTIAL of F and also denoted by $dF \equiv DF \equiv F'$

Now consider a local version of this result, with $\mathcal{N} = \mathbb{R}^n$.

Let $p \in M$ and (U, φ) a coordinate system ($\equiv$ a chart) at p

Then $\varphi_* : T_p(M) \mapsto T_{\varphi(p)}(\mathbb{R}^n)$ is a homomorphism $\forall p \in U$

If $a := \varphi(p) \in \varphi(U)$, then $\varphi^{-1}{}_* : T_a(\mathbb{R}^n) \mapsto T_p(M)$ is a homomorphism

It implies that φ_* and $\varphi^{-1}{}_*$ are isomorphisms.

$\rightsquigarrow$ We can borrow information from $T_a(\mathbb{R}^n)$

Lemma: $\forall X_a \in T_a(\mathbb{R}^n)$ $\exists! \, v \in \mathbb{R}^n$ s.t.

$: C^\infty(a) \mapsto \mathbb{R}$ There exists one unique

$$X_a(f) = \sum_{j=1}^{n} v_j \left(\frac{\partial F}{\partial x_j}\right)(a) = v \cdot [\nabla f](a) = [D_v f](a)$$ (directional derivative)

and any $v \in \mathbb{R}^n$ defines an element of $T_a(\mathbb{R}^n)$ by $X_a = D_v$.

In other words $T_a(\mathbb{R}^n) \ni X_a \xleftrightarrow{\text{bijective}} v \in \mathbb{R}^n$

$\xrightarrow[\text{simple}]{\text{less simple}}$ (to prove)

We conclude that $T_a(\mathbb{R}^n)$ is of dim n.

A basis of $T_a(\mathbb{R}^n)$ is given by $\left\{ \frac{\partial}{\partial x_1}\Big|_a, \frac{\partial}{\partial x_2}\Big|_a, \cdots, \frac{\partial}{\partial x_n}\Big|_a \right\}$

which can be written by $E_{i,a} = \frac{\partial}{\partial x_i}\Big|_a$ with $\{E_{i,a}\}_{i=1}^n$ a basis of $T_a(\mathbb{R}^n)$

$\Rightarrow$ For any coordinate system (U, φ) on M, the image

$\left\{ \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right) \right\}$ is a basis of $T_{\varphi^{-1}(a)}(M)$.

We also write $E_{i,p} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)$ and call these bases the COORDINATE FRAMES.

In summary: The tangent space is indep of any coordinate systems, but once one is given it provides a natural choice of a basis, namely if $f \in C^\infty(p)$, then

$$E_{i,p}(f) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)\right](f) = \left[\frac{\partial}{\partial x_i}(f \circ \varphi^{-1})\right](\varphi(p))$$

Exercise: if (V, ψ) is another coor. system, what are the relations between these bases?

Corollary: If $F: M \mapsto \mathcal{N}$ is smooth and if $p \in M$,

the rank of F at p is equal to the dim of $F_*(T_p(M))$ in $T_{F(p)}(\mathcal{N})$

(another def of rank indep of the coor. systems)

Back to curves :

are smooth manifolds

Consider $c : (-\varepsilon, \varepsilon) \mapsto M$ a smooth map.

On $(-\varepsilon,\varepsilon)$ all tangent vector at $t_0 \in (-\varepsilon,\varepsilon)$ are given by $v \underbrace{\frac{d}{dt}\Big|_{t_0}}_{\text{unit vector}}$ for $v \in \mathbb{R}$

Then $c_*\left(\frac{d}{dt}\Big|_{t_0}\right) f = \left[\frac{d}{dt}(f \circ c)\right](t_0) =: \circledast \quad (f \in C^\infty(c(t_0)))$

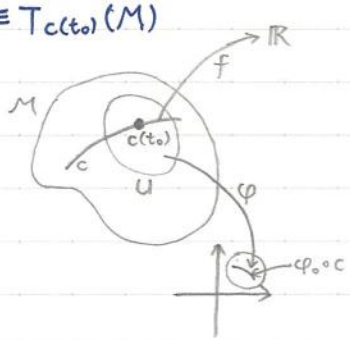
If (U, φ) is a coor. system at $c(t_0)$

and if we set $c^i := (\varphi \circ c)^i \quad \forall i = 1, \cdots, n$

$$\circledast = \left[\frac{d}{dt}\left(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n} \circ \underbrace{\varphi \circ c}_{\leftarrow (-\varepsilon,\varepsilon)}\right)\right](t_0) = \left[\frac{d}{dt}\left(f \circ \varphi^{-1}(c^1, c^2, \cdots, c^n)\right)\right](t_0) \leftarrow \text{Calculus II}$$

$$= \sum_{j=1}^n \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(\varphi \circ c(t_0))\, c^{j\prime}(t_0) = \sum_{j=1}^n c^{j\prime}(t_0)\, E_{j, c(t_0)}(f) \in T_{c(t_0)}(M)$$

$\Rightarrow$ A curve defines an element of $T_{c(t_0)}(M)$



The converse :

Lemma For any $p \in M$ and any $X_p \in T_p(M)$

$\exists c : (-\varepsilon, \varepsilon) \mapsto M$, smooth and with $c(0) = p$, s.t.

$c_*\left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$

I.4 Vector fields

We consider a map $X : M \mapsto \bigcup_{p \in M} T_p(M)$,
$p \mapsto X_p \in T_p(M)$

How can one impose some smoothness on X?

1^{st} solution: (best) (too abstract)

consider $T(M) = \bigcup_{p \in M} T_p(M)$ with a certain topology making a smooth manifold.

$T(M)$ is called TANGENT BUNDLE. → describe this: exercise for mathematiciens.

Then consider X as a smooth map between smooth manifolds.

2^{nd} solution:

for a coordinate system (U, φ) on M and for $p \in U$,

we consider the basis $\{E_{j,p}\}_{j=1}^{n}$

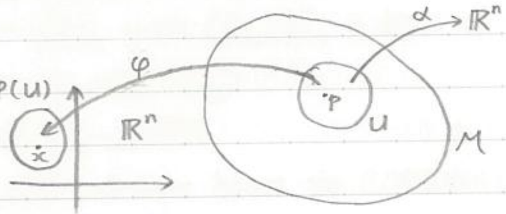
Then $X_p \in T_p(M)$ and $X_p = \sum_{j=1}^{n} \overbrace{\alpha_j(p)}^{\in \mathbb{R}} E_{j,p}$ (a decomposition of X_p on this basis)

By moving p in U, the coefficients $\alpha_j(p)$ is also varying.

So we can impose that

$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n$ is smooth.

This requirement $\Leftrightarrow$ first solution.



Def. a C^∞-VECTOR FIELD on M
is a map $X : M \mapsto T(M)$
whose components α_j in the
coordinate frame $\{E_{i,p}\}$ of any coordinate system satisfy

$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n$ is smooth.

The set of all C^∞-vector fields is denoted by $\mathfrak{X}(M)$.

Lemma: $X : M \mapsto T(M)$ is a C^∞-vector field iff

$\forall f \in C^\infty(M, \mathbb{R})$: $Xf : M \mapsto \mathbb{R}$, $[Xf](p) \equiv [Xf]_p := X_p f$ is smooth.

(another equivalent def) (could be an exercise)

Observe that in this lemma, X can be considered as a map

$C^\infty(M) \ni f \xmapsto{X} Xf \in C^\infty(M)$

Remark: $\mathfrak{X}(M)$ is a vector space and has ~~an~~ additional structures:

1) $\mathfrak{X}(M)$ is a $C^\infty(M)$-MODULE

$:\Leftrightarrow \forall f \in C^\infty(M) \forall X \in \mathfrak{X}(M) : \exists fX \in \mathfrak{X}(M)$ defined by $[fX]_p := f(p) X_p$

2) $\mathfrak{X}(M)$ is a Lie-algebra (very important)

$:\Leftrightarrow$ We can endow $\mathfrak{X}(M)$ with a Lie bracket:

$$\begin{array}{ccccc} \mathfrak{X}(M) & \times & \mathfrak{X}(M) & \longmapsto & \mathfrak{X}(M) \\ \Psi & & \Psi & & \Psi \\ (X & , & Y) & \longmapsto & [X,Y] := XY - YX \end{array}$$

given by ... satisfying

i) linearity in each element

ii) antisymmetry : $[X,Y] = -[Y,X]$

iii) Jacobi identity : $[X,[Y,Z]] = [Y,[Z,X]] = [Z,[X,Y]]$

Exercise : show that 1) and 2) hold

In particular check that $[X,Y]_p$ satisfies Leibniz's rule.

Recall that for any $X_p \in T_p(M)$ $\exists c : (-\varepsilon, \varepsilon) \mapsto M$ with $c(0) = p$ and

$$\dot{c}(0) := c_* \left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$$

Thm. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$.

there exists a unique

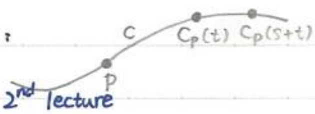
$$\forall p \in M \ \exists! c_p : (-\varepsilon, \varepsilon) \mapsto M \text{ with } c_p(0) = p \text{ and } \dot{c}_p(t) = X_{c_p(t)}$$

Remarks: The curve c_p is called the INTEGRAL CURVE of X at p,

and we call $c_p((-\varepsilon, \varepsilon))$ the ORBITAL of p.

⚠ The value ε depends on p.

Whenever it is well-defined, the following relation holds:



$$c_p(s+t) = c_{c_p(t)}(s)$$

in Appendix for 2^{nd} lecture

Thm. The orbit of p is either the single point p or an immersion of $(-\varepsilon, \varepsilon)$ in M.

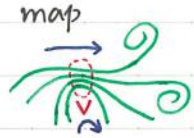
$\downarrow$ if $X_p = 0$ $\qquad$ $\downarrow$ if $X_p \neq 0$

Thm. For any $x \in \mathfrak{X}(M)$ and any $p \in M$ $\exists V \in \mathcal{V}_p$, $\varepsilon > 0$ and a smooth map

$\hookrightarrow := \{\text{open sets } \ni p\}$

$$F : \underbrace{(-\varepsilon, \varepsilon) \times V}_{\text{s.m.}} \longmapsto M \text{ satisfying}$$

$$F(0,q) = q \in V \text{ and } \dot{F}(t,q) = X_{F(t,q)} \ \forall \begin{smallmatrix} t \in (-\varepsilon, \varepsilon) \\ q \in V \end{smallmatrix}$$



⚠ The map F is called the LOCAL FLOW of X at p. Note that $F(t,p) = c_p(t)$

Def. Let $X \in \mathfrak{X}(M)$ and $p \in M$. If $X_p = 0$ then p is called a SINGULAR POINT of the vector field. Since $c_p(t) = p \ \forall t$ if p is sigular, these points are very special and we can study the integral curves around them.

The possible behaviors depend on the topology. Nice subject but we can't go further.

Def. A C^∞-vector field is COMPLETE if

at any $p \in M$, c_p is defined on all $\mathbb{R}$.

A complete vector field can contain some singular points.

Thm. Any C^∞-vector field on a compact manifold is complete.

Remark: Let $X \in \mathfrak{X}(M)$, $p \in M$ and c_p the corresponding integral curve.

Then for any $f \in C^\infty(p)$:

$$X_p f = \frac{d}{dt} f(c_p(t))\Big|_{t=0} = \lim_{t \to 0} \frac{f(c_p(t)) - f(p)}{t}$$

If $f \in C^\infty(M, \mathbb{R})$ recall that

$Xf \equiv L_X f$ is defined by $[Xf]_p = X_p f$

↳ called the LIE DERIVATIVE of f

interpreted as the derivative of f in the direction given by X.

If $Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y \in \mathfrak{X}(M)$ of Y is defined by

$$[L_X Y]_p := \lim_{t \to 0} \frac{1}{t}\Big(\underbrace{\overset{F(-t,\cdot):\, \to V_p}{F(-t, \underbrace{V_{c_p(t)}}_{\in V_{c_p(t)}})_*}}_{T_p(M) \leftarrow T_{c_p(t)}(M)} \underbrace{Y_{c_p(t)}}_{\in T_{c_p(t)}(M)} - \underbrace{Y_p}_{\in T_p(M)} \Big)$$

Lemma: $L_X Y = [X, Y]$

II: Tensors, tensor fields and differential forms

II.1 Tensors [Bo 199-214][GN 62-69]

Let V be a finite dimensional and real vector space ($\mathbb{R}^n$)
and let V^* be its DUAL.
(= the set of all linear maps $V \mapsto \mathbb{R}$, such a map is called a LINEAR FUNCTIONAL on V)

Prop. If $\dim V = n$, then $\dim V^* \overset{\text{exercise}}{=} n$

Def. a TENSOR ϕ on V is a multilinear map

$$\phi : \underbrace{V \times V \times V \times \cdots \times V}_{r \text{ terms}} \times \underbrace{V^* \times V^* \times \cdots \times V^*}_{s \text{ terms}} \longmapsto \mathbb{R}$$

e.g.

$$\phi(\overbrace{v_1, \alpha v_2 + \beta v_2'}^{\in V \times V}, \overbrace{\omega_1}^{\in V^*}) = \alpha \phi(v_1, v_2, \omega_1) + \beta \phi(v_1, v_2', \omega_1)$$

We say that ϕ is r-times COVARIANT and s-times CONTRAVARIANT.

We write $\phi \in \mathcal{T}^r_s(V) \equiv \mathcal{T}^{r,s}(V, V^*)$

Examples

1) $r = 1, s = 0$: $\phi : V \mapsto \mathbb{R}$ is an element of $V^* = \mathcal{T}^1_0(V)$
2) $r = 1, s = 1$: $\phi(v, \omega) \equiv \omega(v) \equiv \langle \omega, v \rangle$ related to scalar product

Lemma: $\mathcal{T}^r_s(V)$ is a vector space of dim n^{r+s}. (exercise)

Remark: If $\phi_j \in \mathcal{T}^{r_j}_0(V)$, $j = 1, 2$, we set

$\phi_1 \otimes \phi_2 \in \mathcal{T}^{r_1 + r_2}_0(V)$ with

$$\phi_1 \otimes \phi_2 (v_1, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_{r_1+r_2}) := \underbrace{\phi_1(v_1, \cdots, v_{r_1})}_{\in \mathbb{R}} \underbrace{\phi_2(v_{r_1+1}, \cdots, v_{r_1+r_2})}_{\in \mathbb{R}}$$

Similar def for $\phi_j \in \mathcal{T}^0_{s_j}(V)$, $j = 1, 2$

If $\phi_1 \in \mathcal{T}^r_0(V) =: \mathcal{T}^r(V)$, $\phi_2 \in \mathcal{T}^0_s(V) =: \mathcal{T}_s(V)$, then

$\phi_1 \otimes \phi_2 \in \mathcal{T}^r_s(V)$ with

$$\phi_1 \otimes \phi_2 (v_1, \cdots, v_r, \omega_1, \cdots, \omega_s) := \phi_1(v_1, \cdots, v_2)\, \phi_2(\omega_1, \cdots, \omega_2)$$

⚠ This product is not commutative!

Def. A tensor $\phi \in \mathcal{T}^r(V)$ is SYMMETRIC if invariant under the permutation of 2 argumen
(for example if $\phi(v_1, v_2) = \phi(v_2, v_1)$)
and ALTERNATING if it changes the sign under the permutation of 2 arguments.
(for example if $\phi(v_1, v_2) = -\phi(v_2, v_1)$)

Same def for $\phi \in \mathcal{T}_s(V)$.

We write $\Sigma^r(V)$ for the set of symmetric tensors in $\mathcal{T}^r(V)$
and $\Lambda^r(V)$ for " alternating " $\mathcal{T}^r(V)$.

Note that $\Sigma^r(V)$ and $\Lambda^r(V)$ are vector spaces.

Let S_k denote the group of all permutation of $\{1, \cdots, k\}$

$\sigma \in S_k$ if σ is a bijective map from $\{1, \cdots, k\}$ to itself
with $(1, \cdots, k) \longmapsto (\sigma(1), \cdots, \sigma(k))$

We set $\mathrm{sgn}(\sigma) = 1$ if σ corresponds to an even number of $\underbrace{\text{transposition}}_{\text{permutation of 2 element}}$,
and $\mathrm{sgn}(\sigma) = -1$ if " odd "

Def. On $\mathcal{T}^n(V)$ one set

$\mathcal{S}: \mathcal{T}^n(V) \mapsto \mathcal{T}^n(V)$ and $\mathcal{A}: \mathcal{T}^n(V) \mapsto \mathcal{T}^n(V)$ by

$$\left.\begin{aligned}
[\mathcal{S}\phi](v_1, \cdots, v_n) &:= \frac{1}{n!} \sum_{\sigma \in S_n} \phi(v_{\sigma(1)}, \cdots, v_{\sigma(n)}) \quad \text{(SYMMETRIZE)} \\
[\mathcal{A}\phi](v_1, \cdots, v_n) &:= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\phi(v_{\sigma(1)}, \cdots, v_{\sigma(n)}) \quad \text{(ANTI-SYMMETRIZE)}
\end{aligned}\right\} \text{linear}$$

Lemma: 1) $\mathcal{S}^2 = \mathcal{S}$, $\mathcal{A}^2 = \mathcal{A}$

2) $\mathcal{S}\mathcal{T}^n(V) = \Sigma^n(V)$, $\mathcal{A}\mathcal{T}^n(V) = \Lambda^n(V)$ if and only if

3) $\phi \in \Sigma^n(V)$ iff $\mathcal{S}\phi = \phi$, $\phi \in \Lambda^n(V)$ iff $\mathcal{A}\phi = \phi$.

Remark: If $F: V \mapsto W$ is a linear map between 2 vector spaces
then it induces a linear map

$F^*: \mathcal{T}^n(W) \mapsto \mathcal{T}^n(V)$ by

$$\underbrace{[F^*\phi]}_{\in \mathcal{T}^n(V)}\underbrace{(v_1, \cdots, v_n)}_{\in V^n} = \underbrace{\phi}_{\in \mathcal{T}^n(W)}\underbrace{(F(v_1), \cdots, F(v_n))}_{\in W^n} \ \forall \phi \in \mathcal{T}^n(W)$$

Now let us set

$$\begin{aligned}
\Lambda(V) &:= \overset{\mathbb{R}}{\underset{\parallel}{\Lambda^0}}(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^j(V) \oplus \cdots \\
&\subset \overset{\parallel}{\mathcal{T}^0}(V) \oplus \mathcal{T}^1(V) \oplus \mathcal{T}^2(V) \oplus \cdots \oplus \mathcal{T}^j(V) \oplus \cdots =: \mathcal{T}(V) \leftarrow \text{lemoi algebra over } V
\end{aligned}$$

The elements of $\Lambda(V)$ or $\mathcal{T}(V)$ cousint in finite "sums" ← only a notation

$$\phi^0 + \phi^1 + \phi^2 + \cdots + \phi^j + \cdots \equiv (\phi^0, \phi^1, \phi^2, \cdots, \phi^j, \cdots) =: \Phi$$

for $\phi^j \in \Lambda^j(V)$ or $\mathcal{T}^j(V)$; $\exists k \in \mathbb{N}\ \forall j \geq k: \phi^j = 0$ (k different for each Φ)

Lemma: $\mathcal{T}(V)$ is a vector space and an associative algebra with $\otimes$ ← extended by linearity
↳ $(\phi \otimes \psi) \otimes \varphi = \phi \otimes (\psi \otimes \varphi)$

E.g. $(\phi_0 + \phi_1) \otimes (\psi_0 + \psi_1 + \psi_2)$

$$\begin{aligned}
&= \underset{\underset{\mathbb{R}}{\cap}}{\phi_0 \otimes \psi_0} + \boxed{\phi_0 \otimes \psi_1} + \boxed{\phi_0 \otimes \psi_2} \\
&\quad + \underset{\underset{\mathcal{T}^1(V)}{\cap}}{\boxed{\phi_1 \otimes \psi_0}} + \underset{\underset{\mathcal{T}^2(V)}{\cap}}{\boxed{\phi_1 \otimes \psi_1}} + \underset{\underset{\mathcal{T}^3(V)}{\cap}}{\phi_1 \otimes \psi_2} \in \mathcal{T}(V)
\end{aligned}$$

What about $\Lambda(V)$? The product $\otimes$ does not generate altenating tensor

Def. For $\phi \in \mathcal{T}^r(V)$ and $\psi \in \mathcal{T}^s(V)$ we set

$\phi \wedge \psi \in \mathcal{T}^{r+s}(V)$ with

$\phi \wedge \psi := \frac{(r+s)!}{r!\,s!} \mathcal{A}(\phi \otimes \psi)$ called EXTERIOR PRODUCT or WEDGE PRODUCT

Lemma. the Wedge product is bilinear and associative.

Corollary: $\Lambda(V)$ with the wedge product is an associative algebra

↳ called EXTERIOR or GRASSMAN ALGEBRA over V

Lemma. If $\phi \in \Lambda^r(V)$ and $\psi \in \Lambda^s(V)$ then $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Thm. If $\dim V = n$

1) If $r > n$, then $\Lambda^r(V) = 0$

2) If $0 \leq r \leq n$, then $\dim \Lambda^r(V) = \binom{n}{r} := \frac{n!}{r!\,(n-r)!}$

In particular if $r=n$, $\dim \Lambda^n(V) = 1 \Rightarrow$ unicity of det

3) $\dim \Lambda(V) = 2^n$

(Next time: $M \mapsto \bigcup_{p \in M} \Lambda(T_pM)$)

II.2 About bases

Recall that if $\{E_1, \cdots, E_n\}$ is a basis of V, then $\exists!$ basis $\{\varphi_1, \cdots, \varphi_n\}$ of V^* s.t.

$$\varphi_j(E_k) = \delta_{jk} := \begin{cases} 1, & \text{if } j=k \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \forall v \in V: v = \sum_{j=1}^{n} \underbrace{\varphi_j(v)}_{\text{component of } v \text{ on } E_j} E_j$$

We call $\{\varphi_1, \cdots, \varphi_n\}$ the DUAL BASIS.

Consider M a smooth manifold, and (U, φ) a local chart.

For any $p \in U$ a basis of $T_p(M)$ is given by the coordinate frame $\{E_{1,p}, \cdots, E_{n,p}\}$ with

$$E_{j,p} := \varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)$$

Thus if we consider the dual space $T_p(M)^* \equiv T_p^*(M)$

there exists a dual basis for $\{E_{1,p}, \cdots, E_{n,p}\}$, usually denoted by $\{(dx^j)_p\}_{j=1}^n$

"Justification" for the notation (change of point of view)

Let $f \in C^\infty(p)$ and $X_p \in T_p(M)$. We set $(df)_p(X_p) := X_p f \in \mathbb{R}$ and in particular

$$(df)_p(E_{j,p}) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)\right](f) = \left[\frac{\partial}{\partial x_j}(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n})\right](\varphi(p))$$

If we choose $f = \varphi^i : V_p \ni V \mapsto \mathbb{R} \Rightarrow$ $= \frac{\partial}{\partial x_j}(x^i)(\varphi(p)) = \delta_{ij}$

Observe that $(df)_p : T_p(M) \mapsto \mathbb{R}$ is linear, and thus an element of $T_p^*(M)$

$\Rightarrow (d\varphi^i)_p$ is an element of the dual basis.

If $\mathcal{M} = \mathbb{R}^n$

then φ = identity, and if $f \in C^\infty(p)$ then $(df)_p = \sum_{j=1}^{n} \lambda_i (dx^i)_p$ with ← coef on a basis

$\lambda_i = E_{p,i}(f) = \frac{\partial f}{\partial x^i}(p)$

$\Rightarrow (df)_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(dx^i)_p$

Corresponds to $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n$, seen in Calculus II.

II.2 Tensor field

Recall that a vector field is a map

$$X : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M}) \equiv T(\mathcal{M}).$$

Def. a (r,s)-TENSOR FIELD on $\mathcal{M}$ is a map

$$\phi : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} \mathcal{T}^r_s(T_p(\mathcal{M}))$$

$$p \mapsto \phi(p) \in \mathcal{T}^r_s(T_p(\mathcal{M}))$$ of dimension n if $\dim \mathcal{M} = n$

Examples

1) A vector field $X : \mathcal{M} \mapsto T(\mathcal{M})$ is a $(0,1)$-tensor field. Indeed:

a $(0,1)$-tensor field ϕ is a map

$$\phi : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} \underbrace{\mathcal{T}^0_1(T_p(\mathcal{M}))}$$

linear map from $T^*_p(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T^{**}_p(\mathcal{M}) = T_p(\mathcal{M})$ ← an exercise

2) Reciprocally, a $(1,0)$-tensor field ϕ is a map

$$\phi : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} \underbrace{\mathcal{T}^1_0(T_p(\mathcal{M}))} = \bigcup_{p \in \mathcal{M}} T^*_p(\mathcal{M})$$

linear map from $T_p(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T^*_p(\mathcal{M})$

$\bigcup_{p \in \mathcal{M}} T^*_p(\mathcal{M})$ is called a COTANGENT BUNDLE. (exercise: it's a smooth manifold)

In this case ϕ is a COVECTOR FIELD.

3) A map $\phi : \mathcal{M} \mapsto \bigcup_{p \in \mathcal{M}} \mathcal{T}^2_0(T_p(\mathcal{M}))$ is called FIELD of BILINEAR FORMS.

$$\forall p \in \mathcal{M} : \phi_p : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \xrightarrow{\text{bilinear}} \mathbb{R}.$$

Observation: A bilinear map can be identified with a $n \times n$ matrix:

$$\alpha_{ij,p} := \phi_p(E_{i,p}, E_{j,p}) \quad (i,j \in \{1, \cdots, n\})$$

About smoothness

There are several equivalent defs for the smoothness for a tensor field.

For example, if $X_1, \cdots, X_r \in \mathfrak{X}(\mathcal{M}) = \{\text{smooth vector fields}\}$

and if $Y_1, \cdots, Y_s$ are smooth covector fields,

then one imposes that the map

$$\mathcal{M} \ni p \mapsto \phi_p(X_{1,p}, \cdots, X_{r,p}, Y_{1,p}, \cdots, Y_{s,p}) \in \mathbb{R} \text{ is smooth.}$$

Or, if (U, φ) is a chart, if $p \in U$ and if we consider $\{E_{j,p}\}_{j=1}^n$ and $\{(dx^i)_p\}_{i=1}^n$

the coordinate frames and coframes. Then we can write

$$\phi_p = \sum_{\substack{i_k=1 \\ j_\ell=1}}^{n} \underbrace{a_{i_1, \cdots, i_r}^{j_1, \cdots, j_s}(p)}_{\in \mathbb{R}} (dx^{i_1})_p \otimes \cdots \otimes (dx^{i_r})_p \otimes E_{j_1,p} \otimes \cdots \otimes E_{j_s,p}$$

(coefficient in a local basis)

and impose that the coefficients are C^∞ on U.

We call such smooth tensors C^∞-TENSOR FIELDS.

Def. The set of all smooth (r,s)-tensor fields on $\mathcal{M}$ is denoted by $\mathcal{T}_s^r(\mathcal{M})$.

Lemma: 1) $\mathcal{T}_s^r(\mathcal{M})$ is a vector field

2) $\mathcal{T}_0^r(\mathcal{M})$ is a $C^\infty(\mathcal{M})$-module $:\Leftrightarrow \phi(x_1, \cdots, f x_j, \cdots, x_n) = f\phi(x_1, \cdots, x_j, \cdots, x_n)$

3) If $\phi \in \mathcal{T}_0^r(\mathcal{M})$ and $\psi \in \mathcal{T}_0^s(\mathcal{M})$ then $\phi \otimes \psi \in \mathcal{T}_0^{r+s}(\mathcal{M})$

with $(\phi \otimes \psi)_p := \phi_p \otimes \psi_p$

Remarks

1) If $f \in C^\infty(\mathcal{M}) \equiv C^\infty(\mathcal{M}, \mathbb{R})$ then we define a covector field by the formula

$df : \mathcal{M} \mapsto \mathcal{T}^*(\mathcal{M}) = \cup_{p \in \mathcal{M}} T_p^*(\mathcal{M})$, $(\Leftrightarrow df \in \mathcal{T}_0^1(\mathcal{M}))$

$(df)_p(X_p) := X_p(f)$

called the DIFFERENTIAL of f

2) If $F : \mathcal{M} \mapsto \mathcal{N}$ a smooth map and if ϕ is a $(r,0)$-tensor field on $\mathcal{N}$

then we set $F^*\phi$ a $(r,0)$-tensor field on $\mathcal{M}$ by

$$(F^*\phi)_p(X_{1,p}, \cdots, X_{r,p}) := \phi_{F(p)}(\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(\mathcal{N})}, \cdots, F_*(X_{r,p}))$$

It means

$$F^* : \mathcal{T}_0^r(\mathcal{N}) \mapsto \mathcal{T}_0^r(\mathcal{M})$$

$\nearrow$ {sym. tensors}

Def. A tensor field $\phi \in \mathcal{T}^r_0(M)$ is SYMMETRIC if $\forall p \in M: \phi_p \in \Sigma^r(T_p(M))$

ALTERNATING if Λ

$\hookrightarrow$ {alt. tensors}

Remark: (Very important)

A symmetric tensor field $\underbrace{\phi \in \mathcal{T}^2_0(M)}_{\text{bilinear forms on } M}$ is POSITIVE DEFINITE if

$\forall p \in M \, \forall X_p \in T_p(M): \phi_p(X_p, X_p) \geq 0$; equality $\Leftrightarrow X_p = 0$

A manifold with a symmetric positive definite bilinear form is called a RIEMANN MANIFOLD; ϕ is called a RIEMANN METRIC. ($\Rightarrow$ Integration)

(Good for geometry)

II.3 Differential forms and exterior derivative

Def. A tensor field $\phi \in \mathcal{T}^r(M)$ which is alternating is called an EXTERIOR DIFFERENTIAL FORM of degree r ; or a r-FORM.

We write $\Lambda^r(M)$ for the set of all r-forms, and

$\Lambda(M) := \bigoplus_{r=0}^{n} \Lambda^r(M)$, with $\Lambda^0(M) := C^\infty(M)$.

Properties

1) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$ then $\underset{\underset{(-1)^{rs}\psi \wedge \phi}{\shortparallel}}{\phi \wedge \psi} \in \Lambda^{r+s}(M)$

2) $\Lambda(M)$ is an algebra with the Wedge product $\wedge$.

3) If (U, φ) is a local chart, and if $p \in U$, then the set

$\{(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p\}$ with $1 \leq i_1 < \cdots < i_r \leq n$

is a basis for $\Lambda^r(T_p(M))$, and accordingly

$\{(dx^{i_1}), \cdots, (dx^{i_r})\}$ is a basis for $\Lambda^r(U) \subset \Lambda^r(M)$.

$\Rightarrow$ $\Lambda(M)$ is the algebra of differential forms or exterior algebra.

Main result of this chapter (for def of grad, div, curl, etc)

Thm. Let M be a smooth manifold, and $\Lambda(M)$ the exterior algebra,
There is a unique linear map
$d : \Lambda(M) \mapsto \Lambda(M)$ satisfying $\underset{\uparrow}{d(f)}$ this map, differential of f
1) If $f \in \Lambda^0(M) = C^\infty(M)$, then $df = df \in \mathcal{T}^1_0(M)$, $(df)_p(X_p) = X_p(f)$
2) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, then
$$d(\phi \wedge \psi) = (d\phi)\wedge\psi + (-1)^r \phi \wedge (d\psi)$$
3) $d^2 = d \circ d = 0$

In local coordinates, we have an explicit formula for d:
Recall that if (U, φ) is a chart, $p \in U$, then
$\{E_{j,p}\}_{j=1}^n$ is a basis for $T_p(M)$ and $\{(dx^j)_p\}_{j=1}^n$ is a basis for $T_p^*(M)$.
Then $\phi \in \Lambda^r(M)$ can be represented by
$$\phi_p = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1, \dots, i_r}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p \quad \text{(a special case of } \mathcal{T}^r_s(M)\text{)}$$
$$=: \sum_I a_I(p)(dx^I)_p \quad \text{with } a_I : U \mapsto \mathbb{R} \text{ smooth.}$$
Then (def)
$$(d\phi)_p := \sum_I \overbrace{(da_I)_p}^{\in \mathcal{T}^1_0(M)} \wedge \overbrace{(dx^I)_p}^{\in \Lambda^r(M)} \in \Lambda^{r+1}(M)$$

Exercise: check that this def satisfies the 3 conditions [GN p74]

Remarks
1) d is a <u>local</u> operator: If $U \subset M$ and $\phi \in \Lambda(U) \subset \Lambda(M)$ then $d_U\phi = d_M\phi$
2) d maps $\Lambda^r(M)$ to $\Lambda^{r+1}(M)$
3) d is called the EXTERIOR DERIVATIVE

Exercise (Thm?)
If $\omega \in \Lambda^1(M)$ and $X, Y \in \mathfrak{X}(M) := \{C^\infty\text{-vector fields}\}$, then
$$d\omega(X,Y) = X\underbrace{\omega(Y)}_{\in C^\infty(M)} - Y\underbrace{\omega(X)}_{\in C^\infty(M)} - \underbrace{\omega([X,Y])}_{\in C^\infty(M)} \in C^\infty(M)$$

$$(\omega X)_p = \omega_p(X_p) \in \mathbb{R}$$

Proof: In a chart (U,φ), $\omega_p = \sum_{j=1}^n a_j(p)(dx^j)_p$
For shortness, we write $\omega_p = f\,dg$ for $f, g \in C^\infty(M)$
Then $d\omega(X,Y) = d(f\,dg)(X,Y) \overset{\text{by def}}{=} (df \wedge dg)(X,Y) \overset{\text{by def}}{=} df(X)dg(Y) - df(Y)dg(X)$
$= (Xf)(Yg) - (Yf)(Xg) \in C^\infty(M)$

$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(f\,dg(Y)) - Y(f\,dg(X)) - f\,dg([X,Y])$
$= X(fYg) - Y(fXg) - f(XY - YX)g \overset{\text{Leibniz}}{=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$
$= (Xf)(Yg) - (Yf)(Xg)$ $\square$

$$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(fdg(Y)) - Y(fdg(X)) - fdg([X,Y])$$
$$= X(fYg) - Y(fXg) - f(XY-YX)g \overset{\text{Leibniz}}{=\!=\!=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$$
$$= (Xf)(Yg) - (Yf)(Xg) \qquad \square$$

Generalization

Prop. ["GN" 3.8.2 p. 75 ~ 76] (independent of any coordinate systems)

Let $\phi \in \Lambda^r(M)$ and $X_1, \cdots, X_{n+1} \in \mathfrak{X}(M)$, then

$$[d\phi](X_1, \cdots, X_{r+1}) = \sum_{i=1}^{n+1} (-1)^{r+1} X_i\, \phi(X_1, \cdots, \overset{\text{omit}}{\hat{X_i}}, \cdots, X_{r+1}) + \sum_{i<j} (-1)^{i+j} \underbrace{\phi([X_i, X_j], X_1, \cdots, \hat{X_i}, \cdots, \hat{X_j}, \cdots, X_{r+1})}_{\in C^\infty(M)}$$

Exercise: This satisfies conditions 1~3 of Thm.

Recall that if $F: M \mapsto N$ a smooth map between manifolds, then

$F^*: \mathcal{T}^r(N) \mapsto \mathcal{T}^r(M)$ by

$$\underset{\uparrow \in M}{(F^*\phi)_p}(X_{1,p}, \cdots, \underbrace{X_{r,p}}_{\in T_p(M)}) := \phi_{F(p)}(\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(N)}, \cdots, F_*(X_{r,p}))$$

which is also $F^*: \Lambda^r(N) \mapsto \Lambda^r(M)$ (alternating property is preserved)

Lemma: In this framework

$F^* \circ d_N = d_M \circ F^*$ [Bo Thm 8.2 p. 223]

Exercise for mathematicians: about de Rham cohomology

[GN, p. 76 ex 5]

II.4 Orientation on a manifold (easier)

Let V be a real vector space of dim n, and $\{E_j\}_{j=1}^n$ and $\{F_j\}_{j=1}^n$ 2 bases

Set $A \in M_{n\times n}(\mathbb{R})$ by $F_i = \sum_{j=1}^n a_{ij} E_j$ coeff. of the change of basis

Def. The two bases has the SAME ORIENTATION if $\det(A) > 0$

and of OPPOSITE ORIENTATION if $\det(A) < 0$

$\Rightarrow$ There exist 2 classes of equivalence of bases.

We say either they are either POSITIVELY ORIENTED

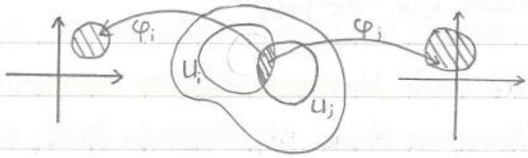
or NEGATIVELY ORIENTED.

Def. Let M be a smooth manifold of dim $n \geq 1$, ⚠ Convention changed

M is ORIENTABLE if there exists a covering ($\equiv$ atlas) $\{(U_j, \varphi_j)\}_j$ s.t. all transition maps
$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \mapsto \varphi_j(U_i \cap U_j)$ is ORIENTATION PRESERVING
$:\Leftrightarrow$ if $\det \mathrm{Jac}(\varphi_j \circ \varphi_i^{-1}) > 0$

Lemma: A connected orientable manifold of dim ≥ 1 has only 2 possible orientations.

Remark: If $M = \{p\}$ (of dim 0)
an orientation is a map from p to ± 1. We need this because
$$\int_a^b f'(x)dx = f(b) - f(a)$$
M of dim 0, $b \mapsto +1$, $a \mapsto -1$ are what we need from orientations.



Thm. [Bo. p.218] (very deep but intruisive) ϕ is called a VOLUME FORM
A manifold is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$ $(\phi_p \in \Lambda^n(T_p(M)))$

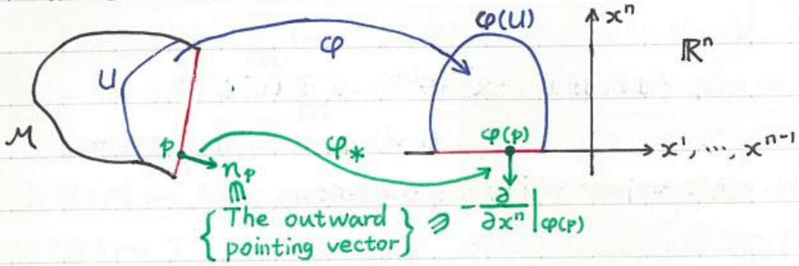
Recall that $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and
M is a smooth manifold with boundary if every chart
$(U_\alpha, \varphi_\alpha)$ with $\varphi_\alpha : U_\alpha \mapsto H^n$ is a homeomorphism. (+ atlas conditions)
The BOUNDARY of M is denoted by ∂M and is given by
$$\partial M := \bigcup_\alpha \varphi_\alpha^{-1}(\partial H_n \cap \varphi_\alpha(U_\alpha))$$
which is a smooth manifold with dim $(n-1)$

Next time: If M is oriented then it induces also an orientation on ∂M
(needed in Stroke's Thm)

Propositions

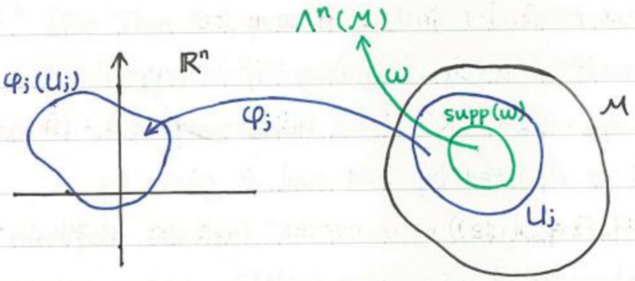
1) The boundary of a smooth manifold M of dim n is a smooth manifold ∂M of dim $(n-1)$.

2) If M is orientable then ∂M is also orientable.
More precisely, if an orientation is chosen on M, then there exists an INDUCED ORIENTATION on ∂M.



We set $\varphi_*^{-1}\left(-\frac{\partial}{\partial x^n}\Big|_{\varphi(p)}\right) =: n_p$

For a basis on ∂M, we choose a basis $\{e_1, \cdots, e_{n-1}\}$ of $T_p(\partial M)$ such that $\{n_p, e_1, \cdots, e_{n-1}\}$ generates a basis of $T_p(M)$ of the same orientation as on M.



III. Integration on manifolds

III.1 Integration of n-forms

Let $\mathcal{M}$ be an oriented $\Lambda^n(\mathcal{M})$ manifold and let $\{(U_j, \varphi_j)\}_j$ be an $\underbrace{\text{oriented}}^{\text{preserving}}$ atlas.

Let $\omega \in \Lambda^n(\mathcal{M})$ with $\mathrm{supp}(\omega) \subset U_j$ and with $\mathrm{supp}(\omega)$ compact.

$\Rightarrow \overset{(\text{or } \omega_p)}{\omega(p)} = a(p)(dx^1)_p \wedge \cdots \wedge (dx^n)_p$ with $a \in C^\infty(\mathcal{M})$

Recall that φ_j^{-1*} maps $\Lambda^n(\mathcal{M})$ to $\Lambda^n(\mathbb{R}^n)$

$\Rightarrow \varphi_j^{-1*}(\omega) = \underbrace{a \circ \varphi_j^{-1}}_{\text{function on } \varphi_j(U_j) \subset \mathbb{R}^n} dx^1 \wedge \cdots \wedge dx^n.$

Then we set

$$\int_{\mathcal{M}} \omega = \int_{U_j} \omega := \overset{\to \text{ usual Riemann integral in } \mathbb{R}^n}{\int_{\varphi_j(U_j)} a \circ \varphi_j^{-1} \, dx_1 \cdots dx_n} \equiv \int_{\varphi_j(U_j)} a(x) \, dV \qquad (*)$$

Lemma: If $\mathrm{supp}(\omega) \subset U_k$ for an other localization map (U_k, φ_k), then

$$\int_{\varphi_k(U_k)} a \circ \varphi_k^{-1} \, dx_1 \cdots dx_n = \int_{\varphi_j(U_j)} a \circ \varphi_j^{-1} \, dx_1 \cdots dx_n$$

(independence of the coordinate system) (proof as Exercise)

Def. Let $\mathcal{M}$ be an oriented smooth manifold, $\{(U_j, \varphi_j)\}_j$ a covering preserving the orientation, and $\omega \in \Lambda^n(\mathcal{M})$ with compact support.

Let $\{f_j\}$ be a partition of unity of $\mathcal{M}$ subordinated to U_j. ($:\Leftrightarrow \forall j: \mathrm{supp}(f_j) = U_j$) Then

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{M}} \sum_j f_j \, \omega = \sum_j \int_{\mathcal{M}} \underbrace{f_j \, \omega}_{\in \Lambda^n(\mathcal{M})} = \sum_j \int_{U_j} f_j \, \omega \text{ as defind in } (*).$$

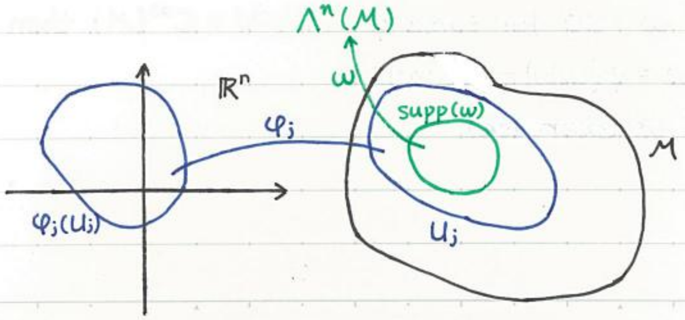
the sum is finite because supp(ω) is compact

Remarks

- $\int_{\mathcal{M}} \omega$ is independent of the choice of a partition of unity. (Exercise)
- The map $\Lambda^n(\mathcal{M}) \ni \omega \mapsto \int_{\mathcal{M}} \omega \in \mathbb{R}$ is a linear map.
- We can avoid the "comapactedly supported" but be careful about the convergence.
- If $F: \mathcal{M} \mapsto \mathcal{N}$ is a diffeomorphism and if $\omega \in \Lambda^n(\mathcal{N})$, compactedly supported,

$$\int_{\mathcal{M}} \underbrace{F^* \omega}_{\in \Lambda^n(\mathcal{M})} = \pm \int_{\mathcal{N}} \omega$$

(± depends on if F preserves the orientation or not)



Thm. (Stokes' Theorem) (The main thm of this chapter)

Let M be an oriented smooth manifold of dim n,

with boundary ∂M. (with induced orientation).

Let $i : \partial M \mapsto M$ be the inclusion map. (identity) $\Rightarrow i^* : \Lambda^{n-1}(M) \mapsto \Lambda^{n-1}(\partial M)$

Let $\omega \in \Lambda^{n-1}(M)$ with compact support. Then

$$\int_{\partial M} \underbrace{i^*\omega}_{\in \Lambda^{n-1}(\partial M)} = \int_M \underbrace{d\omega}_{\in \Lambda^n(M)}$$

Reference for the proof: [GN p. 82-84] [Bo p. 260-261]

Remark: 1) If $\partial M = \emptyset$ then $\int_M d\omega = 0$

2) The proof is similar to the one of Calculus II on $\mathbb{R}^2$ or $\mathbb{R}^3$,

and the main ingredient is $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Exercise: Show that the Green Thm, Stokes Thm in $\mathbb{R}^3$ or Divergence Thm are special cases of this theorem. See Bo p. 262-263.

Recall that M is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$.

Def. Let us fix one of them, and for any $f \in C^\infty(M)$ with compact support we set

$$\int_M f := \int_M f\phi$$ ⚠ This def depends on the choice of ϕ.

In particular if M is compact we set the volume of M as

$$\mathrm{Vol}(M) := \int_M 1\phi = \int_M \phi$$

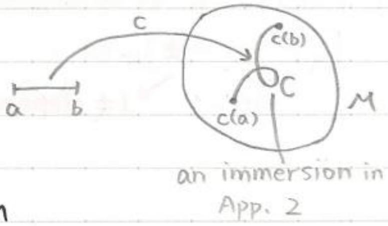
III.2 Line integrals

Let $c : [a,b] \mapsto M$ be a diffeomorphism and set $C = c([a,b])$ ↗Capital ; it is a manifold

If $\omega \in \Lambda^1(M)$ we set

$$\int_C \omega = \int_{[a,b]} \underbrace{c^*\omega}_{\in \Lambda^1([a,b])} = \int_a^b f(t)\,dt$$

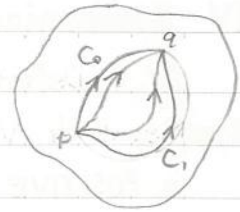
$$t \mapsto f(t)\,dt$$



Lemma: If $\omega = d\phi$ for some $\phi \in \Lambda^0(M) = C^\infty(M)$ then

$$\int_C \omega = \phi(c(b)) - \phi(c(a))$$

(Proof as exercise)



Consider a smooth map

$$H : [0,1] \times [a,b] \mapsto M$$

$\nwarrow$ parameter

with $H(s,a) = p \in M$ and $H(s,b) = q \in M \quad \forall s \in [0,1]$

We set $C_0 : [a,b] \mapsto M,\ C_0(t) = H(0,t)$
$C_1 : [a,b] \mapsto M,\ C_1(t) = H(1,t)$ } We say that C_0 and C_1 are HOMOTOPIC paths between p and q.

Thm. Let $\omega \in \Lambda^1(M)$ s.t. $d\omega = 0$ everywhere. Then

$$\int_{C_0} \omega = \int_{C_1} \omega$$

Remark: if $\omega = d\phi$ with $\phi \in C^\infty(M) = \Lambda^0(M)$, then $d\omega = d^2\phi = 0$ and the statement follows from the previous lemma.

- If M is of dim 2, the statement is "almost" a consequence of Stoke's Thm, but we don't have the smoothness of the boundary at p and q.
- More generously, see [Bo p. 271]

Remark: Smoothness can be relaxed in most of the statements.

IV Riemannian Manifolds

IV. 1 Definition and basic properties

Recall that if V is a real vector space of dimension n,

a POSITIVE DEFINATE BILINEAR FORM is a map $\phi : V \times V \mapsto \mathbb{R}$

which is linear in each argument

and s.t. $\phi(v,v) \geqslant 0 \ \forall v \in V$ and $\phi(v,v) = 0 \Leftrightarrow v = 0$.

ϕ is SYMMETRIC if $\phi(v_1, v_2) = \phi(v_2, v_1)$.

Def. A smooth manifold with a positive definite symmetric bilinear tensor field is called a RIEMANNIAN MANIFOLD.

$\Leftrightarrow \exists \phi \in \mathcal{J}^2(\mathcal{M})$:

$$\phi_p \in \Sigma^2(T_p(\mathcal{M})) \wedge [\forall X_p \in T_p(\mathcal{M}) : \phi_p(X_p, X_p) \geqslant 0 \text{ with } = 0 \Leftrightarrow X_p = 0]$$

We call ϕ a RIEMANNIAN METRIC.

Lemma: If $F : \mathcal{M} \mapsto \mathcal{N}$ is an IMMERTION $(:\Leftrightarrow \dim F(\mathcal{M}) = \dim \mathcal{M}$; see App. 2)

and if ϕ is a Riemannian metric on $\mathcal{N}$,

Then $F^*(\phi) \in \mathcal{J}^2(\mathcal{M})$ is a Riemannian metric on $\mathcal{M}$.

Proof as exercise; recall that

$$(F^*\phi)(X_p, Y_p) := \phi(F_*(X_p), \overbrace{F_*(Y_p)}^{\in T_p(\mathcal{N}),\ = 0 \text{ iff } Y_p = 0})$$

Thm. Any smooth manifold can be endowed with a Riemannian metric.

"2 proofs" : ① Use a covering + local coordinate system + Lemma above

② Use Whitney Imbedding Thm + Lemma above

Remark: For a Riemannian manifold, $T_p(\mathcal{M})$ has an inner product provided by ϕ

$\Rightarrow$ We can now define orthonormal bases on $T_p(\mathcal{M})$ at every $p \in \mathcal{M}$.

Thm. Let $(\mathcal{M}, \phi)$ be a Riemannian manifold which is oriented.

↳ fix ϕ from now on

Then $\exists!$ volume form Ω s.t. $\forall p \in \mathcal{M} : \Omega_p(F_{1,p}, \cdots, F_{n,p}) = 1$ (*)

whenever $\{F_{1,p}, \cdots, F_{n,p}\}$ is an oriented orthonormal basis of $T_p(\mathcal{M})$.

Proof: Since $\dim(\Lambda^n(T_p(\mathcal{M}))) = 1$, then Ω is uniquely defined by (*).

We have to show that it does not vanish.

Let $(\mathcal{U}, \varphi)$ be a local chart with $p \in \mathcal{U}$;

Let $\{E_{1,p}, \cdots, E_{n,p}\}$ be the corresponding basis for $T_p(\mathcal{M})$. (Coordinate frame at p)

Set $g_{ij}(p) := \phi_p(E_{i,p}, E_{j,p})$.

Since $E_{i,p} = \sum_{k=1}^{n} \alpha_i^k F_{k,p}$ and since $\phi_p(F_{i,p}, F_{j,p}) = \delta_{ij}$

$\Rightarrow$ $g_{i,j}(p) = \phi_p(E_{i,p}, E_{j,p}) = \phi_p\left(\sum_{k=1}^{n} \alpha_i^k F_{k,p}, \sum_{l=1}^{n} \alpha_j^l F_{k,p}\right)$

$= \sum_{k=1}^{n} \alpha_i^k \alpha_j^k = ({}^TAA)_{ij}$ with $A_{ij} = \alpha_j^i$

$\Rightarrow$ $\det(g_{ij}(p))_{ij} = \det({}^TAA) = (\det(A))^2 > 0$

$\Rightarrow$ $\sqrt{\det(g_{ij}(p))_{ij}} > 0$

$\Rightarrow$ $\Omega_p(E_{1,p}, \cdots, E_{n,p}) \overset{\text{exercise}}{=} \det(A) \underbrace{\Omega_p(F_{1,p}, \cdots, F_{n,p})}_{=1 \text{ by def}} = \det(A) = \sqrt{\det(g_{ij})} > 0$ — >0 by choice of orientation of $(F_{1,p}, \cdots, F_{n,p})$

Since p, (U, φ) are arbitrary, then Ω is a volume form.

Smoothness is automatic. □

Ω is called the NATURAL VOLUME ELEMENT on the oriented Riemannian manifold (M, ϕ).

We often see $\underbrace{\varphi^*\Omega}_{\in \Lambda^n(\mathbb{R}^n)} = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n$, $g := \det(g_{ij} \circ \varphi^{-1})$

Remark: We can use Ω to define

$\int_M f := \int_M f\Omega \quad \forall f \in C^\infty(M)$

Let $c: [a,b] \mapsto M$ be a smooth curve on a Riemannian manifold (M, ϕ).

The tangent vector is

$c_*\left(\frac{d}{dt}\Big|_t\right) =: \dot{c}(t) \in T_{c(t)}(M)$

Def. The LENGTH of the curve is defined by

$L := \int_a^b [\phi_{c(t)}(\dot{c}(t), \dot{c}(t))]^{\frac{1}{2}}\, dt$

Exercise: This is indep. of the parametrization.

The ARC LENGTH is defined by $s: [a,b] \mapsto [0, L]$,

$s(t) := \int_a^t [\phi_{c(\tau)}(\dot{c}(\tau), \dot{c}(\tau))]^{\frac{1}{2}}\, d\tau$

We often write $\left[\left(\frac{ds}{dt}\right)^2 = \phi(\dot{c}, \dot{c})\right]$

Thm. [Bo. p. 189~191] A connected manifold is a metric space with the metric defined by $d(p,q) = \inf$ on the length of all paths ($:=$ curves of C^1 or C^∞) between p and q.

The metric topology and the manifold topology coincide.

Reminder: a METRIC SPACE is a pair (M, d) with $d: M \times M \mapsto \mathbb{R}_+$ s.t.

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \Leftrightarrow x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z)$ ($\triangle$ inequality)

Def. Two R_o manifolds (M_1, ϕ_1) and (M_2, ϕ_2) are ISOMETRIC if
(Riemannian)

$\exists F : M_1 \mapsto M_2$ a diffeomorphism such that $F^* \phi_2 = \phi_1$

$\Rightarrow d_1(p,q) = d_2(F(p), F(q))$ (distance)

Remark: (Nash embedding thm) asserts that any R_o manifold can be isometrically embedded in $\mathbb{R}^d$, for $d \geq \frac{n(3n+11)}{2}$.

IV.2 Differentiation

Differentiation is important for the description of an evolution or a transport.

Example: In $\mathbb{R}^3$ for a fixed relavence system, $\dot{x}(t) = v$

One can also consider a moving reference system. (moving frame)

Example: We attach a reference system to a point moving in $\mathbb{R}^3$.

Let $s \mapsto c(s)$ be a curve in $\mathbb{R}^3$, with the arc length parameter.

Set $T(s) := c'(s)$, with the property $\|T(s)\| = 1$.

Then $\dot{T}(s) \equiv T'(s) \perp T(s)$ and set $T(s) = K(s) N(s)$ with $K(s) \geq 0$ and $\|N(s)\| =$

Consider $\{T(s), N(s), B(s)\}$ — the curvature — suppose $K(s) \neq 0$

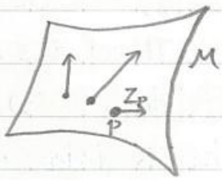
as a basis at $c(s)$ orthonormal

The equation of motion of this frame is given by the Sevret-Frenet formula

There are 2 parameters:

$K(s)$ = the curvature

$J(s)$ = the torsion



Example: Let M be a manifold of dim n in $\mathbb{R}^d$.

Let $Z \in \mathfrak{X}(\mathbb{R}^d)$ and let $p \in M$. $\Rightarrow Z_p \in T_p(\mathbb{R}^d)$ but not always $Z_p \in T_p(M)$.

If $Z_p \in T_p(M)$ (tangent to M at p) for any $p \in M$,

we say that Z is a tangent vector field.

Since $\mathbb{R}^d$ has a scalar product, it endows M with a scalar project

$\Rightarrow T_p(\mathbb{R}^d)$ has a scalar product, as well as $T_p(M)$.

$\Rightarrow T_p(\mathbb{R}^d) = T_p(M) \oplus T_p(M)^\perp$

$\Rightarrow \exists \Pi_p$ and $\Pi_p^\perp$ two orthogonal projections on $T_p(M)$ and $T_p(M)^\perp$.

Def. Let $Y \in \mathfrak{X}(M) \subset \mathfrak{X}(\mathbb{R}^d)$ and consider $t \mapsto c(t) \in M \subset \mathbb{R}^d$ a curve on M.

Set $Y(t) := Y_{c(t)} \in T_{c(t)}(M)$ and consider

$$\frac{DY}{dt}(t) := \Pi_{c(t)}\Big(\underbrace{\frac{d}{dt}Y(t)}_{\text{computation in } \mathbb{R}^d}\Big) \in T_{c(t)}(M)$$

called the COVARIANT DERIVATIVE of Y along c.

Thus, Y and $\frac{DY}{dt}$ belong to $\mathfrak{X}(M)$ but the defination of $\frac{DY}{dt}$ uses $\mathbb{R}^d$.

Prop. 1) $\frac{D}{dt}(Y_1 + Y_2) = \frac{DY_1}{dt} + \frac{DY_2}{dt}$

2) $\frac{D}{dt}(fY) = f'Y + f\frac{DY}{dt}$ with any $f \in C^\infty(M)$

$\frac{d}{dt}\langle Y_1, Y_2\rangle = \langle \frac{DY_1}{dt}, Y_2\rangle + \langle Y_1, \frac{DY_2}{dt}\rangle$ with $\boxed{Y_1 = Y_1 \circ c,\ Y_2 = Y_2 \circ c}$

⚠ $\frac{DY}{dt} = 0 \not\Leftrightarrow \frac{dY}{dt} = 0$

— End of example 3

Remark

If we consider $X_p \in T_p(M)$ and

if we choose a curve $t \longmapsto c(t) \in M$ with $c(t_0) = p$ and $\dot{c}(t_0) = X_p$

then $\frac{DY}{dt}(t_0)$ does not depend on $c(t)$ but only on X_p.

(proof as exercise)

It means we can define a map

$$\begin{array}{ccccc} T_p(M) & \times & \mathfrak{X}(M) & \longmapsto & T_p(M) \\ \Downarrow & & \Downarrow & & \Downarrow \\ X_p & & Y & & \frac{DY}{dt}(t_0) =: \nabla_{X_p} Y \end{array}$$

new notation

or more generally

$$\begin{array}{ccccc} \mathfrak{X}(M) & \times & \mathfrak{X}(M) & \longmapsto & \mathfrak{X}(M) \\ \Downarrow & & \Downarrow & & \Downarrow \\ X & & Y & & \nabla_X Y \end{array}$$

with $(\nabla_X Y)_p = \nabla_{X_p} Y$.

Def. An AFFINE CONNECTION on a smooth manifold M is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$$
$$X \quad Y \mapsto \nabla(X,Y) \equiv \nabla_X Y \text{ satisfying}$$

1) $\nabla_{fX} Y = f \nabla_X Y$ $C^\infty(M)$-linearity in the first variable
2) $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$

$\} \forall f \in C^\infty(M)$

Def. For any $X, Y \in \mathfrak{X}(M)$ we set

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M)$$

called the TORSION of the connection; and set

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$$

$\hookrightarrow : \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$; endomorphism 自同態

$$[\tilde{R}(X,Y,Z) := R(X,Y)Z]$$

called the CURVATURE of the connection.

Lemma:

$T(X,Y)$ is $C^\infty(M)$-linear in both arguments;
$\tilde{R}(X,Y,Z)$ is " " in the 3 arguments.

[Exercise; see Tu (geometry) p.44]

Def. On a R_0 manifold, a torsion free ($:\Leftrightarrow T(X,Y)=0 \ \forall X,Y$) connection satisfying

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \in C^\infty(M) \quad \forall X,Y,Z \in \mathfrak{X}(M)$$

is called a RIEMANNIAN CONNECTION or LEVI CIVITA CONNECTION.

(compability condition between the Riemannian metric ϕ and the connection ∇)

$\langle X,Y\rangle : M \mapsto \mathbb{R}$; $\langle X,Y\rangle_p := \phi_p(X_p, Y_p) \in \mathbb{R}$

Thm. On a Riemannian manifold $\exists!$ Riemannian connection.

This connection satisfies

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$$
$$- \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle$$

(Koszul formula)

Lemma: Let $\mathcal{M}$ be a smooth manifold of dim n (Riemannian not assumed) and let ∇ be an affine connection on $\mathcal{M}$.

Let (U, φ) be a chart and consider a coordinate frame on the tangent spaces.

Then ∇ is defined by n^3 functions

$\Gamma^k_{i,j} : U \longmapsto \mathbb{R}$ for $i, j, k \in \{1, \cdots, n\}$ called the CHRISTOFFEL SYMBOLS.

Proof: Let $X, Y \in \mathfrak{X}(\mathcal{M})$, and $\forall p \in U$:

$$X_p = \sum_{i=1}^n \underbrace{b^i(p)}_{\in \mathbb{R}} E_{i,p} \; ; \; Y_p = \sum_{i=1}^n \underbrace{a^i(p)}_{\in \mathbb{R}} E_{i,p} .$$

$\hookrightarrow$ bases of $T_p(\mathcal{M})$

Set

$$\overbrace{\nabla_{E_{i,p}} E_{j,p}}^{\in T_p(\mathcal{M})} =: \sum_{k=1}^n \Gamma^k_{i,j}(p) E_{k,p}$$

Then

$$\nabla_X Y = \nabla_{\sum_i b^i E_i} \sum_j a^j E_j \overset{\text{Linearity and 1)}}{=} \sum_{i,j} b^i \nabla_{E_i}(a^j E_j) \overset{2)}{=} \sum_{i,j} b^i \left\{ (E_i a^j) E_j + a^j \sum_k \Gamma^k_{i,j} E_k \right\}$$

$$= \sum_k \left(X a^k + \sum_{i,j} a^j b^i \Gamma^k_{i,j} \right) E_k \qquad (*)$$

$\Rightarrow \nabla$ can be expressed by $\Gamma^k_{i,j}$.

Conversely, if we start with $(*)$, it defines an affine connection. (5-min exercise) $\square$

Remark: 1) with these notations

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \sum_{ijk} (\Gamma^k_{i,j} - \Gamma^k_{j,i}) a^i b^j E_k .$$

Thus $\forall X, Y \in \mathfrak{X}(\mathcal{M}) : T(X,Y) = 0 \Leftrightarrow \forall i,j,k : \Gamma^k_{i,j} = \Gamma^k_{j,i}$

2) If $(\mathcal{M}, \phi)$ is Riemannian, recall that

$g_{ij}(p) = \phi_p(E_{i,p}, E_{j,p}) \; \forall i,j \in \{1, \cdots, n\}$ and then

$$\Gamma^k_{i,j} = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$\uparrow$ inverse matrix of (g_{ij})

(proof as exercise)

A new look at the covariant derivative:

Let $c: I \ni t \mapsto c(t) \in M$ be a smooth curve on M, and let $Y \in \mathfrak{X}(M)$.

Let (U, φ) be a local chart, and for $p \in U$

$$Y_p = \sum_{k=1}^{n} b^k(p) E_{k,p}.$$

Then we set

$$\frac{DY}{dt}(t) := \left[\nabla_{\dot{c}(t)} Y\right]_{c(t)} = \sum_{k=1}^{n} \Big(\dot{c}(t)\, b^k(c(t)) + \sum_{i,j} \Gamma^k_{i,j}(c(t))\, b^i(c(t))\, \dot{c}^j(t)\Big) E_{k,c(t)}$$

$\dot{c}(t) = \sum_k \dot{c}^k(t) E_{k,c(t)}$

Observe that

$$\dot{c}(t)\, b^k = c_*\left(\frac{d}{dt}\right)\Big|_t b^k = \frac{d}{dt}(b^k \circ c)\Big|_t = \frac{d}{dt} b^k(c(t)).$$

$$= \sum_{k=1}^{n} \left(\frac{db^k(c(t))}{dt} + \sum_{i,j} \Gamma^k_{i,j}(c(t))\, b^i(c(t))\, \dot{c}^j(t)\right) E_{k,c(t)} \qquad \triangle$$

Remark: only the values of Y on the curve are taken into account.

Def. Let $c: I \mapsto M$ be a curve on M, and ∇ an affine connection on M.

A vector field $Y: I \ni t \mapsto Y(t) \in T_{c(t)}(M)$ is PARALLEL along c if

$$\frac{DY}{dt}(t) = 0 \quad \forall t \in I.$$

Since $\triangle$ is a group of first-order differential equations we have:

Prop. 1) Given a smooth curve $c: (-\varepsilon, \varepsilon) \ni t \mapsto c(t) \in M$ and given $Y_{c(0)} \in T_{c(0)}(M)$ then

$$\exists!\, Y: (-\varepsilon, \varepsilon) \ni t \mapsto Y(t) \in T_{c(t)}(M) \text{ parallel to } c.$$

2) If (M, ϕ) is a Riemannian manifold and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_{c(0)}(M)$ then $\exists!$ orthonormal frame at $c(t)$ which is parallel to c.

More generally on Riemannian manifolds,

parallel transport preserves the length and the inner product.

IV.3 Geodesics

Let $c : I \mapsto M$ be a curve on M and ∇ be an affine connection.

($\xleftrightarrow{\text{locally}}$ a set of Christoffel's symbols)

Def. c is GEODESIC (with respect to ∇) if $\dot{c}$ is parallel along c, which means

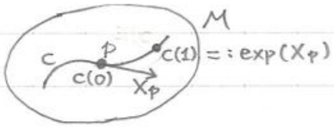
$$\frac{D\dot{c}}{dt}(t) = 0 \quad \forall t \in I$$

$$\Leftrightarrow \ddot{c}^k + \sum_{i,j} \Gamma_{i,j}^k \dot{c}^i \dot{c}^j = 0 \quad \forall k = 1, \cdots, n \quad \text{(geodesic equation)}$$

Remark: since the geodesic equation is a second-order differential equation,

given $p \in M$ and $X_p \in T_p(M)$,

$\exists!\, c : (-\varepsilon, \varepsilon) \mapsto M$ geodesic s.t. $c(0) = p$ and $\dot{c}(0) = X_p$.



Note that $\forall a > 0$, if we set $c_a : (-\frac{\varepsilon}{a}, \frac{\varepsilon}{a}) \mapsto M$ then

$c_a(0) = p$, $\dot{c}_a(0) = aX_p$ and c_a is again geodesic. Then

Def. $\exp(X_p) := c(1)$ whenever defined.

$:\Leftrightarrow \forall u \in U \ \forall a \in [0,1] : au \in U$

Prop. $\forall p \in M \ \exists$ open set $U \subset T_p(M)$ star-shaped with $0 \in U$ s.t.

$\exp : U \mapsto M$ is a diffeomorphism onto $V \subset M$ with $p \in V$.

The proof involves some uniformity.

$\operatorname{Exp}(U)$ is called a NORMAL NEIGHBORHOOD of p on M,

and exp is called the EXPONENTIAL MAP.

Remark: If (M, ϕ) is a Riemannian manifold,

and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_p(M)$, then

$$X_p = \sum_{j=1}^n \underbrace{x^j}_{\in \mathbb{R}} F_j \quad \text{(unique decomposition)}$$

Then

$$\varphi : \exp(U) \ni \exp(\underset{\overset{\cup}{\mathrm{U}}}{X_p}) = \exp\left(\sum_{j=1}^n x^j F_j\right) \longmapsto (x^1, \cdots, x^n) \in \mathbb{R}^n$$

and $(\exp(U), \varphi)$ is a coordinate system around p, called

the NORMAL COORDINATE SYSTEM around p.

(with special properties)

In summary, for a given $p \in M$ $\exists v \in \mathcal{V}_p$ (neighborhood) s.t.

any $q \in v$ can be joined to p by a unique geodesic.

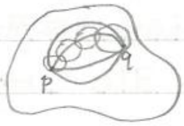
With more work one gets

Thm. If c is a piecewise differential path between p and q with

$\underset{\text{length of } c}{} \leftarrow L(c) = d(p,q) \rightarrow$ distance between p and q on the Riemannian manifold M (for defs of L and d, see p.27 in IV.1)

Then c is a geodesic when parametrized by its arc length.

Idea of proof: do it locally.



⚠ The distance is not always realized by a path.

Example: $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, $p = (0,1)$, $q = (0,-1)$

Thm. (Hopf and Rinow)

Let (M, ϕ) with Levi-Civita connection ∇.

Are equivalent:

1) exp is defined every on $T_p(M)$ $\forall p \in M$;

2) (M, d) is a COMPLETE metric space ($:\Leftrightarrow$ with "no holes")
↳ every Cauchy sequence $\subset M$ has a limit $\in M$

$\Updownarrow$

3) Every geodesic $c: I \mapsto M$ can be extended on $\mathbb{R}$.

Def. (M, ϕ) is GEODESICALLY COMPLETE

if one ($\Rightarrow$ all) of these conditions is satisfied.

Lemma. If (M, ϕ) is COMPACT then it is geodesically complete.

Proof: Based on the fact that any compact metric space is complete. □

$\underline{\text{V}}$ Curvature

$\underline{\text{V.1}}$ Several curvatures

Framework: $\mathcal{M}$ a smooth manifold with ∇ a connection.

If $(\mathcal{M}, \phi)$ is Riemannian, then ∇ is the Levi Civita connection.

Recall that the curvature R is defined on $X, Y \in \mathfrak{X}(\mathcal{M})$ by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(\mathcal{M}))$$

$$R(X,Y) : \mathfrak{X}(\mathcal{M}) \ni Z \longmapsto R(X,Y)Z \in \mathfrak{X}(\mathcal{M})$$

Lemma: If ∇ is torsion free then

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

[Bianchi identity ; GN p. 125]

True also for Levi Civita connection.

In local coordinates [= with a chart $(\mathcal{U}, \phi)$ and the coordinate frame $\{E_{j,p}\}_j$]

$$R(E_i, E_j)E_k = \sum_l R_{ijk}{}^l E_l$$

with $R_{ijk}{}^l = \frac{\partial}{\partial x^i}\Gamma^l_{jk} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} + \sum_m \Gamma^m_{jk}\Gamma^l_{im} - \sum_m \Gamma^m_{ik}\Gamma^l_{jm}$

$\uparrow$ components of R in a basis

⚠ It can be slightly different depending on the authors

For $(\mathcal{M}, \phi)$, let us also set

$$\underbrace{\phi(\underbrace{\underbrace{R(X,Y)}_{\in \mathrm{End}}Z}_{\in \mathfrak{X}(\mathcal{M})}, W)}_{\in C^\infty(\mathcal{M})} =: R(X,Y,Z,W) \in C^\infty(\mathcal{M})$$

$\langle \in \mathfrak{X}(\mathcal{M}), \mathfrak{X}(\mathcal{M}) \rangle$

$\uparrow \in \mathcal{T}^4(\mathcal{M})$; called the RIEMANNIAN CURVATURE TENSOR

and in local coordinates

$$R_{ijkl} := \phi(R(E_i, E_j)E_k, E_l) = \sum_m R_{ijk}{}^m g_{ml}$$

Lemma: For $(\mathcal{M}, \phi)$

1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

3) $R(X,Y,Z,W) = R(Z,W,X,Y)$

[Exercise ; see Boo p. 383 and GN p. 126]

For any $p \in M$, let us denote by Π a PLANE SECTION in $T_p(M)$,
it means Π is a 2D subspace of $T_p(M)$.

Let X_p, Y_p be 2 elements in $T_p(M)$ generating a basis of Π s.t. (X_p, Y_p) is an orthonormal basis of Π.

Def. The SECTIONAL CURVATURE $K(\Pi)_p$ of the section Π with basis (X_p, Y_p) is

$$K(\Pi)_p := -R(X_p, Y_p, X_p, Y_p) = -\phi_p(R(X_p, Y_p)X_p, Y_p)$$

Exercise: $K(\Pi)_p$ depends only on the plane Π and not on the choice of a basis.

Thm. For (M, ϕ) with $\dim(M) \geq 3$:
the Riemannian curvature tensor at p is uniquely determined by the values of all sectional curvatures at p.
[Exercise; see Boo p.385 and GN p.127]

Def. 1) (M, ϕ) is ISOTROPIC at p if

$$K(\Pi)_p = K_p = \text{constant } \forall \Pi;$$

2) (M, ϕ) is ISOTROPIC if it is isotropic at any $p \in M$;

3) If K_p is constant on any $p \in M$, we say that M has CONSTANT CURVATURE.

Report: manifolds with constant curvature are classified.

Remark: If $\dim(M) = 2$ then M is isotropic, and $K_p \equiv K(p)$ is called the GAUSS CURVATURE.

Report: on Gauss curvature or on Gauss-Bonnet Thm.

Lemma: If M is isotropic then locally

$$R_{ijkl}(p) = -K_p(g_{ik}g_{jl} - g_{il}g_{jk})(p)$$

The Riemannian curvature tensor, the sectional curvature, the Ricci curvature and the scalar curvature give some information on the local structure of the manifold.

Def. The RICCI CURVATURE tensor field

$\mathrm{Ric} \equiv R \equiv S \in \mathcal{T}^2(M)$ is defined on $X, Y \in \mathfrak{X}(M)$ by

$$S_p(X_p, Y_p) := \sum_j R(F_{j,p}, X, Y, F_{j,p}) \text{ with } \{F_{j,p}\}_j \text{ an orthonormal basis of } T_p(M)$$

Remark: 1) It is independent of the choice of a basis of $T_p(M)$.

Locally, $S_{ij} = S(E_i, E_j) = \sum_k R_{kij}{}^k$

2) The above operation is called a CONTRACTION of a tensor.

If we contract the Ricci curvature we get the SCALAR CURVATURE given

$$S(p) = \sum_j S(F_{j,p}, F_{j,p}) = \sum_{i,j} S_{i,j}\, g^{ij}(p)$$

V.2 Equation of structure

Recall that a connection ∇ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$$
$$X \quad Y \longmapsto \nabla_X Y$$

which is bilinear and satisfies

1) $\nabla_{fX} Y = f \nabla_X Y$
2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

∇ is torsion free if $\nabla_X Y - \nabla_Y X - [X,Y] \; (=: T(X,Y)) = 0$ and
∇ is compatible with the metric $\overset{\text{for}}{(M,\phi)}$ if
$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ making the parallel transport of 2 orthogonal vectors still orthogonal
$\Big\}$ ⊛

Let U be an open subset of M and
let $\{F_j\}_{j=1}^n$ be a C^∞-field of frames on U $\{F_{j,p}\}$ is a basis of $T_p(M)$ $\forall p \in U$; not necessarily orthonormal nor generated by a chart
e.g. the coordinate frames given by a chart (U, φ)

Let $\{\theta^j\}_{j=1}^n$ be a dual coframe, it means $\{\theta^j\}$ is a C^∞-field of frames on $T(M)^*$
and $\{\theta^j_p\}$ is a basis of $T_p(M)^*$ with $\theta^j_p(F_{k,p}) = \delta_{jk}$ Cronecker delta

Recall that ∇ is uniquely determined by $\{\Gamma_{ij}{}^k\}$ defined by

$$\nabla_{F_i} F_j = \sum_k \Gamma_{ij}{}^k F_k$$

Def. $\theta_j{}^k := \sum_l \Gamma_{lj}{}^k \theta^l \in \mathcal{J}^1(M)$ one form

$\{\theta_j{}^k\}$ are called CONNECTION FORMS. Clearly

$\Rightarrow \quad \theta_j{}^k(F_i) = \Gamma_{ij}{}^k$, and

if $T(M) \ni X = \sum_l b^l F_l$ then

$$\nabla_X F_j = \nabla_{\sum_l b^l F_l} F_j \xlongequal[\text{and 1)}]{\text{linear}} \sum_l \overset{\curvearrowright \in C^\infty(U)}{b^l} \nabla_{F_l} F_j \overset{\Gamma}{=} \sum_l b^l \sum_k \Gamma_{lj}{}^k F_k$$
$$\xlongequal[\text{of } \theta]{\text{Def}} \sum_k \sum_l b^l \theta_j{}^k (F_l) F_k \xlongequal[\text{of 1 forms}]{\text{linearity}} \sum_k \theta_j{}^k(X) F_k$$

Thus, $\theta_j{}^k(X)$ are the components of $\nabla_X F_j$ with respect to $\{F_k\}$.

For a R_0 manifold (M, ϕ) and for the Levi Civita connection ∇,
the n^2 connection form are not indep because of the relations ⊛.

Thm. (Structure Thm of Cartan) [GN p. 133]

Let (R, ϕ) be a R_0 manifold, ∇ the Levi Civita connection, $U, \{E_i\}, \{\theta^j\}$ above.

Then the connection forms $\{\theta_j{}^k\}$ are the unique solution of the equations:

1) $d\theta^i = \sum_j \theta^j \wedge \theta^i$ wedge product — equality between 2-forms

2) $\underbrace{dg_{ij}}_{\in C^\infty(M) \to \text{one-forms}} = \sum_k (g_{kj}\theta_i{}^k + g_{ki}\theta_j{}^k)$ equality between 1-forms

Remark: If $\{F_j\}$ is an orthonormal basis,

$g_{ij} := \phi(F_i, F_j) = \delta_{ij}$ and 2) becomes

2) $0 = \theta_i{}^j + \theta_j{}^i$

Similarly, one can introduce the CURVATURE FORM for $X, Y \in \mathfrak{X}(M)$:

$$\Omega_k^l(X,Y) := \underset{\in \mathfrak{X}(M)^*}{\theta^l}(\underbrace{R(X,Y)F_k}_{\in \mathfrak{X}(M)}) \in C^\infty(M) \Rightarrow \Omega_k^l \in \mathcal{T}^2(U) \subset \mathcal{T}^2(M)$$

which gives

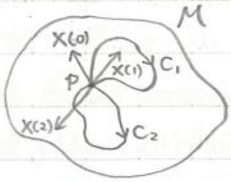
$$R(X,Y)F_k = \sum_j \Omega_k^j(X,Y)F_j$$

Thus $\Omega_k^l(X,Y)$ are components of $R(X,Y)F_k$ on the basis $\{F_j\}$

Remark: $\Omega_k^l \in \mathcal{T}^2(U) \subset \mathcal{T}^2(M)$ and one has

Thm. (Structure Thm of Cartan) [GN p. 135; Bo p. 391]

$$\Omega_i{}^j = d\theta_i{}^j - \sum_k \theta_i{}^k \wedge \theta_j{}^k$$ equality between 2-forms



V.3 Holonomy for a connected Riemannian manifold

⚠ Exists in a more general context of vector bundles or principal bundles.

Let $c: [0,1] \ni t \longmapsto c(t) \in M$ a smooth curve on (M, ϕ)

with $c(0) = c(1) = p$.

Let $X_p \in T_p(M)$ and let $X(t)$ be the parallel transport of X_p along c

with $X(0) = X_p$. Let

$P_c: T_p(M) \ni X_p = X_0 \longmapsto X_1 \in T_p(M)$, and clearly

$\underset{\text{composition of paths}}{P_{c_2 \circ c_1}} = P_{c_2} P_{c_1}$; $\underset{\text{backward}}{P_{c^{-1}}} = P_c^{-1}$ leading to the fact that it composes a group.

In addition $P_c \overset{\text{invertible matrices}}{\in} GL(T_p(M))$ because the parallel transport is a solution of a homogenous equation. $\Rightarrow$ linear in the initial condition

In fact $P_c \in O(T_p(M))$ orthogonal matrices on $T_p(M)$

because the parallel transport preserves norms and scalar products.

Remark: Instead of smooth curve, we can consider C^1-piecewise curves.

We have obtained that

$\{P_c\}_* \subset O(T_p(M))$ is a group e is the zero path

called the HOLONOMY GROUP at p and denoted $Hol(p)$.

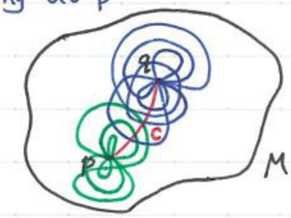
$* :=$ " c any C'-piecewise curve starting and ending at p"

If p and q are 2 points on M then

$Hol(p)$ is isomorphic to $Hol(q)$ since

$Hol(p) = P_c^{-1} \, Hol(q) \, P_c$

for some fixed path c between q and p.



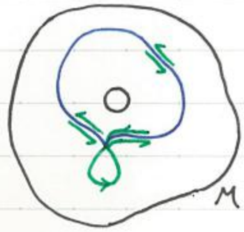
Def. We set $Hol(M) = Hol(p) \subset O(\ \ )$ for a fixed $p \in M$, and call it the HOLONOMY GROUP of M.

We also set $Hol^\circ(M)$ constructed only with C'-piecewise path which can be deformed to the zero path.

Remarks:

1) These groups are representations of the group of paths on M.

2) $Hol^\circ(M)$ is a normal subgroup of $Hol(M)$.



Lemma

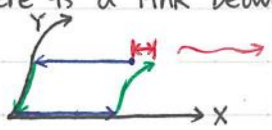
M is orientable iff $Hol(M) \subset SO(M)$ determinant 1 and never -1

Thm (deep notation)

$Hol^\circ(M)$ is compact (it is a closed set in $O(n)$)

Remark [see App. 12]

There is a link between holonomy and the curvature tensor $R(X,Y)$



They are not so many holonomy groups!

Thm. Let (M, ϕ) and suppose that $Hol^\circ(M) \subset O(n)$ is irreducible no invariant subspaces of $\mathbb{R}^n$.

(For a manifold made by product of two manifolds, this is not satisfied)

Suppose that M is not LOCALLY SYMMETRIC.

Then $Hol^\circ(M)$ is one of the following groups:

1) $SO(n)$ generic case
2) if $n=2m$, $Hol(M) = U(m)$
3) if $n=2m$, $Hol(M) = SU(m)$
4) $n=4m$, $Hol(M) = Sp(1)Sp(m)$
5) if $n=4m$, $Hol(M) = Sp(m)$
6) $n=16$, " $= Spin(9)$
7) $n=8$, " $= Spin(7)$
8) $n=7$, " $= G_2$

Later it is found that (6) does not actually appear in any manifolds.

4) ~ 8) are in qualernions extension of $\mathbb{C}$ 2D → 4D

Def. M is LOCALLY SYMMETRIC if for any $p \in M$:

the geodesic symmetry S_p is an isometry preserves the distance

namely, we have $S_p(c(t)) = c(-t)$ for any geodesic c with $c(0) = p$

Example: $\mathbb{R}^n$ is locally symmetric (easy to show). And

$Hol(\mathbb{R}^n) = Hol^\circ(\mathbb{R}^n) = \{e\}$ which is not one of the 8 kinds of groups above.

VI General relativity

Def. a PSEUDO-RIEMANNIAN MANIFOLD is a pair (M, ϕ) with

M a smooth manifold and $\phi \in \mathcal{T}^2(M)$, symmetric and non-degenerate.

$\phi(X,Y) = \phi(Y,X)$ | $\phi(X,Y) = 0 \ \forall Y \in \mathfrak{X}(M) \Rightarrow X = 0$

⚠ No 「positive definite」 required!

a LORENTZIAN MANIFOLD is a pseudo-Riemannian manifold

with $(g_{ij}) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ in suitable coordinates (locally). → signature (trace) $= n-2$

Facts for pseudo-Riemannian manifolds

1) unicity of Levi Civita connection when the 2 conditions are imposed.
2) Koszul formula still holds.
3) Hopf-Rinow thm + geodesically complete are no more valid.
   ⇒ We don't have a metric space anymore.
4) Cartan structure thm are still valid.

Recall that the length of a vector is not changed under parallel transport along a curve.

Geodesics c satisfy that $\dot{c}$ is parallel transported along c.

$\Rightarrow \phi(\dot{c}, \dot{c}) = \text{cst}$

Def. A geodesic c on a pseudo-Riemannian manifold (M, ϕ) is

TIMELIKE, NULL, or SPACELIKE if

$\phi(\dot{c}, \dot{c}) < 0$, $\phi(\dot{c}, \dot{c}) = 0$ or $\phi(\dot{c}, \dot{c}) > 0$

<0 and $=0$ are allowed by PSEUDO METRIC ϕ

Remark: these expressions come from special relativity with $M = \mathbb{R}^4$ and

$$\underbrace{(g_{\nu\mu})}_{\mu,\nu = 0,1,2,3} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

a special case of a Lorentzian manifold.

For a Lorentzian manifold (M, ϕ) of dim 4, with the Levi Civita connection, the Einstein field equation reads

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{for } \mu, \nu = 0,1,2,3 \qquad (*)$$

$R_{\mu\nu}$: Ricci curvature; R: scalar curvature; Λ: cosmological constant >0; $T_{\mu\nu}$: stress-energy or energy-momentum tensor

(about geometry) $G_{\mu\nu}$ Einstein tensor

contains the physics (energy + matter)

G = gravitation constant
c = speed of light

⚠ Not so much freedom for writing a meaningful equations.

This is a system of 10 equations because of symmetry between μ and ν.

In addition the thms of structure reduces the number of indep eq.

Remark: These equations define the pseudo metric tensor $g_{\mu\nu}$.

Indeed, $R_{\mu\nu\gamma}{}^{\delta}$ and $R_{\mu\nu}$ can be expressed in terms of $\Gamma_{\mu\nu}{}^{\delta}$ and its derivativ

And $\Gamma_{\mu\nu}{}^{\delta}$ can be expressed in terms of $g_{\mu\nu}$ and its derivatives.

$\Rightarrow$ ⊛ is a system of non linear partial differential equations for $g_{\mu\nu}$.

Schwarzschild solution

Assumptions:
- $T_{\mu\nu} = 0$
- $g_{\mu\nu}$ is time independent (static solution)
- spherically symmetric in space ($\equiv$ in the indices 1, 2, 3)
- $M = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^2$. $\mathbb{R}$ for t and $\mathbb{R}_+ \times \mathbb{S}^2$ is $\mathbb{R}^3$ in spherical coordinates

Suppose that

$$g = -A^2(r)\,dt \otimes dt + B^2(r)\,dr \otimes dr + r^2\,d\theta \otimes d\theta + r^2 \sin^2(\theta)\,d\varphi \otimes d\varphi \in \mathcal{T}^2(M)$$

with $A, B : \mathbb{R}_+ \mapsto \mathbb{R}$ unknown,

$(dt, dr, d\theta, d\varphi) \in \mathcal{T}^1(M)$ generate a basis of $T^*(M)$.

$\{(\frac{\partial}{\partial x^i})_p\}_{i=1}^n$ is a basis of $T_p(M)$, and $\{dx_p^i\}_{i=1}^n$ is a basis of $T_p^*(M)$.

$\Rightarrow \{dx^i \otimes dx^j\}_{i,j}$ is a basis of $\mathcal{T}^2(M)$.

Question: can we find A, B such that ⊛ is satisfied (with $T_{\mu\nu} = 0$)?

Two approaches:

1) Express $\Gamma_{\mu\nu}{}^{\delta} \rightsquigarrow R_{\mu\nu\delta}{}^{\gamma}$ and then $R_{\mu\nu}$ and R in terms of $g_{\mu\nu}$, and solve ⊛

2) Set

$$\left.\begin{array}{ll} \theta^0 := A(r)\,dt & \theta^2 := r\,d\theta \\ \theta^1 := B(r)\,dr & \theta^3 := r\sin(\theta)\,d\varphi \end{array}\right\} \in \mathcal{T}^1(M)$$

and observe that

$g = \sum_{\mu,\nu} \eta_{\mu,\nu}\, \theta^\mu \otimes \theta^\nu$ and $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ and that

$\{\theta^0, \theta^1, \theta^2, \theta^3\}$ is an orthonormal coframe a basis of $T^*(M)$

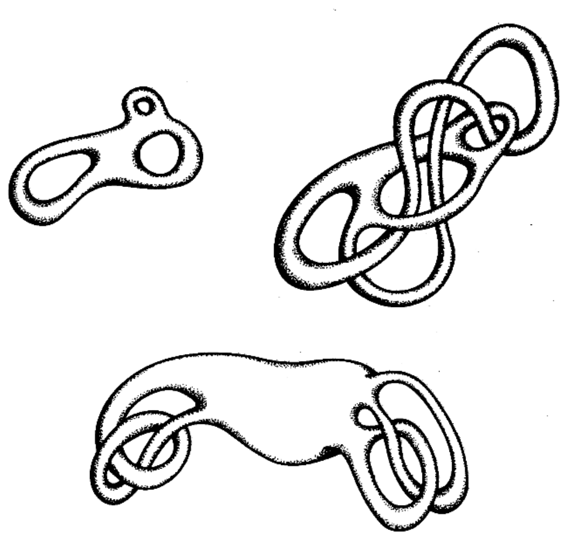
- Define $\theta_\mu{}^\nu$ and $\Omega_\mu{}^\nu$ (connection and curvature tensors) and write the structure relations of Cartan.
- One obtains some differential equations for A and B, which can be solved.
- $A(r) = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}$ and $B(r) = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}$ with $m \in \mathbb{R}$ an integration const

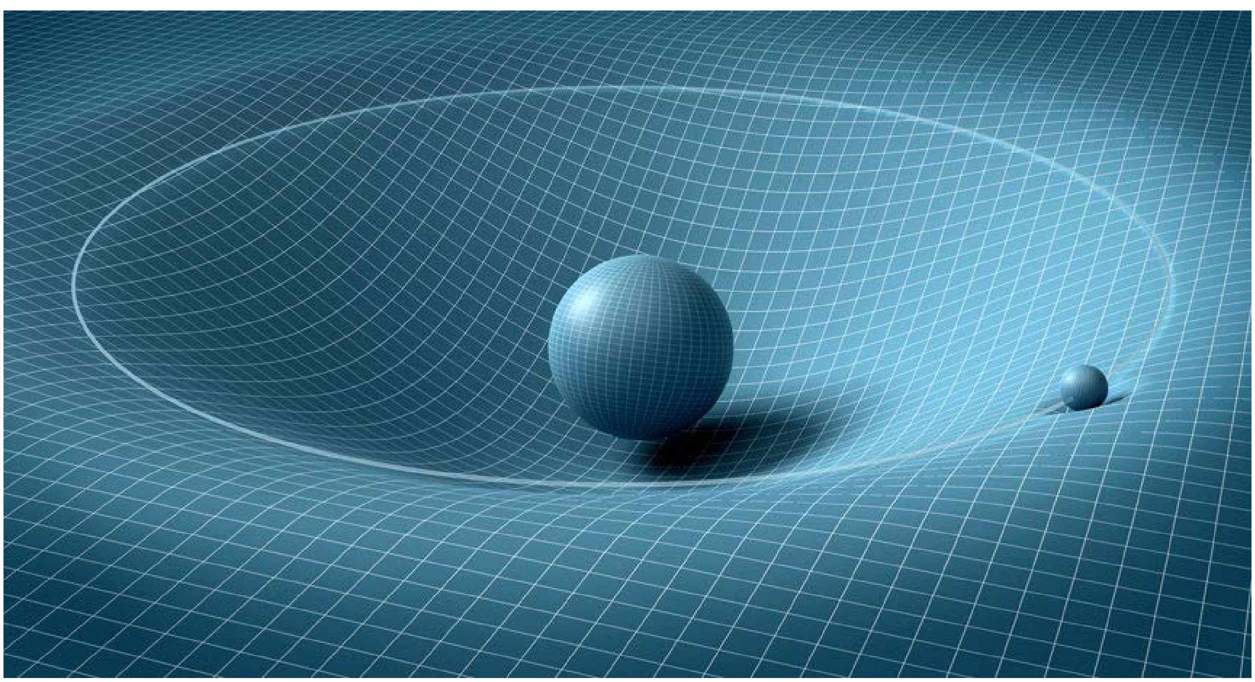
Conclusion

Textbooks on general relativity are now accessible (but still the theory is complicated).

Special Mathematics Lecture: Differential geometry

Extrinsic / **intrinsic**

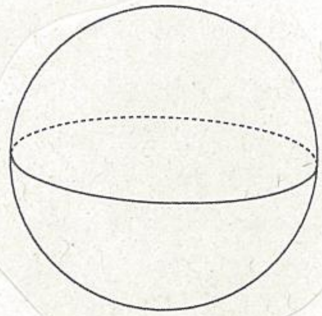




Example of topological manifolds

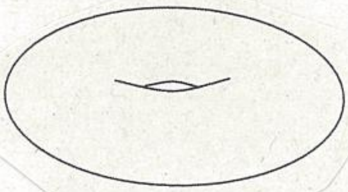
Circle

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$



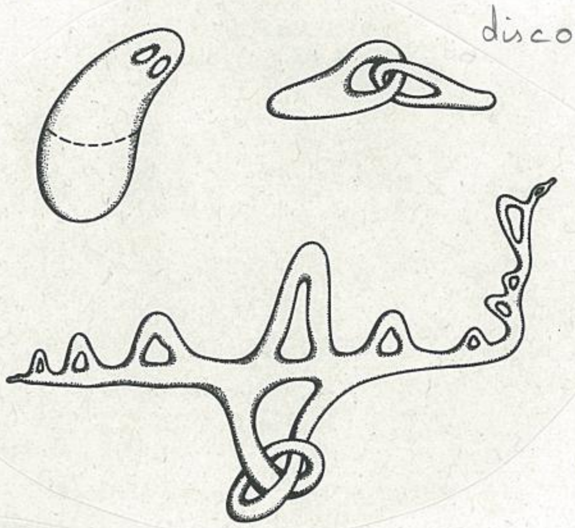
2 - sphere

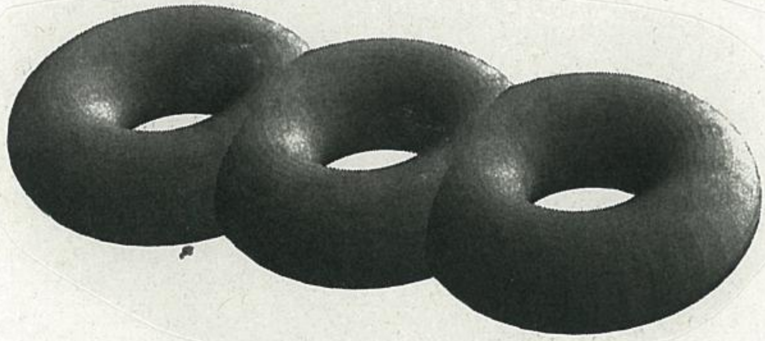
$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$



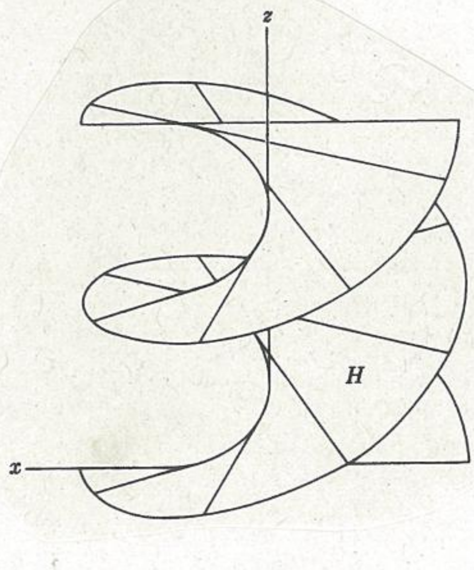
2 - torus

disconnected





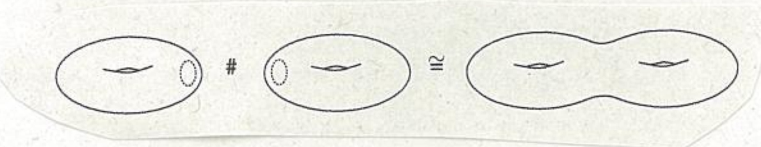
Hyperboloid (in 1 sheet)



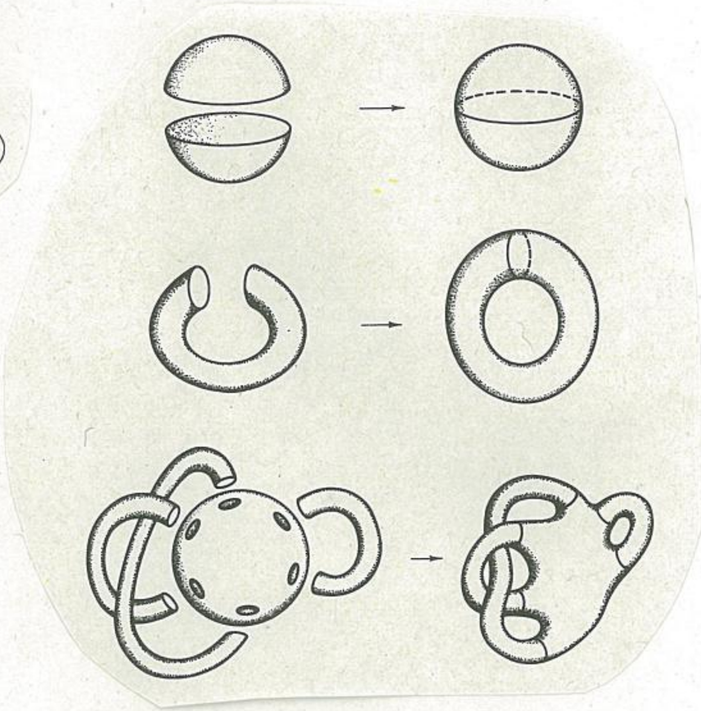
$$\begin{cases} x = s\cos(t) \\ y = s\sin(t) \\ z = bt \end{cases}$$

for fixed $b \neq 0$ and $(s,t) \in \mathbb{R}^2$

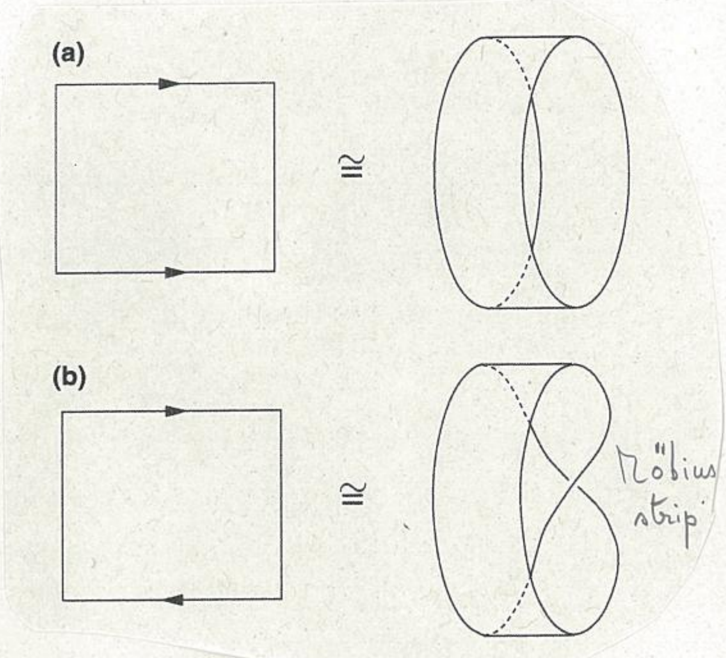
Cutting, pasting, quotienting

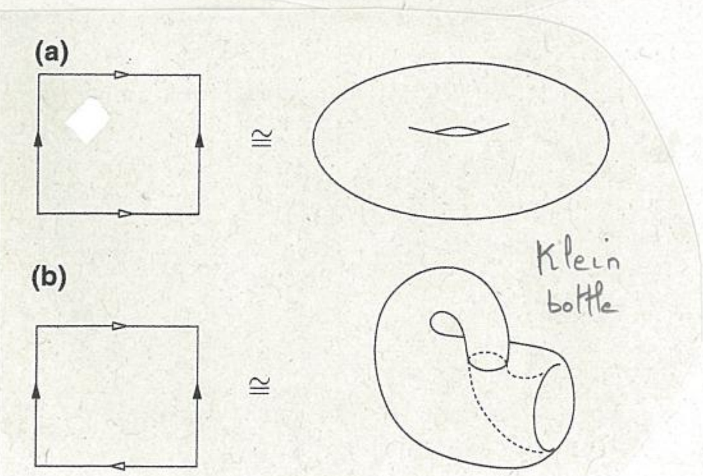


Cut and paste.



Quotient (= identification)





⚠ This operation (quotient) does not always produce a topological manifold.

Real projective space $P^n(\mathbb{R})$:

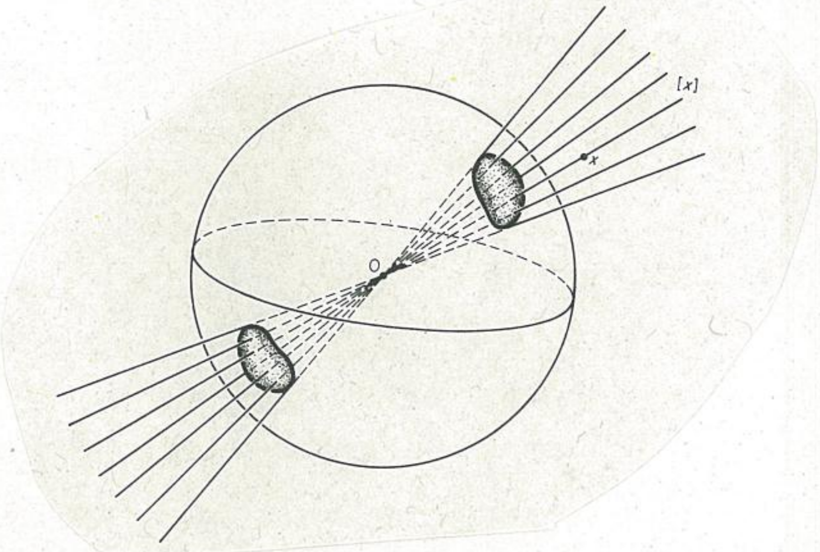
Consider $X := \mathbb{R}^{n+1} \setminus \{0\}$ and set

$(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$ if $\exists\, t \in \mathbb{R}^*$ s.t.

$(y_1, \dots, y_{n+1}) = (t x_1, \dots, t x_{n+1})$.

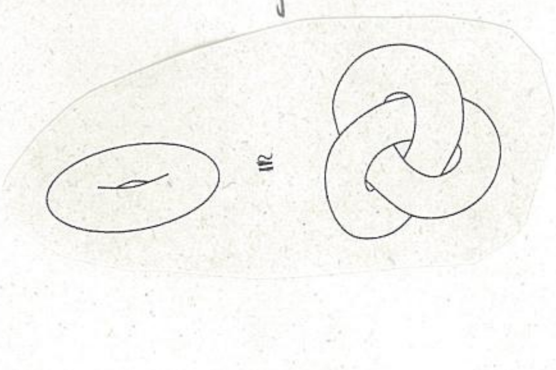
Then $P^n(\mathbb{R}) := X/\sim$

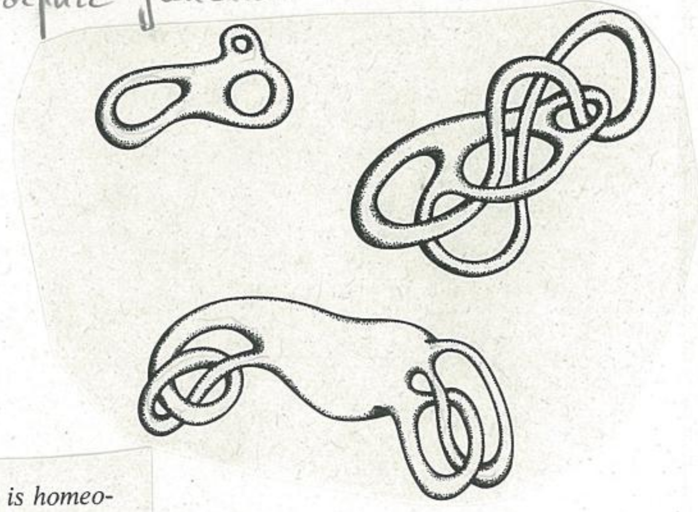
The equivalence class $[x]$ can be visualized as lines through the origin.



Homeomorphic topological manifolds

if there exists a homeomorphic function between them



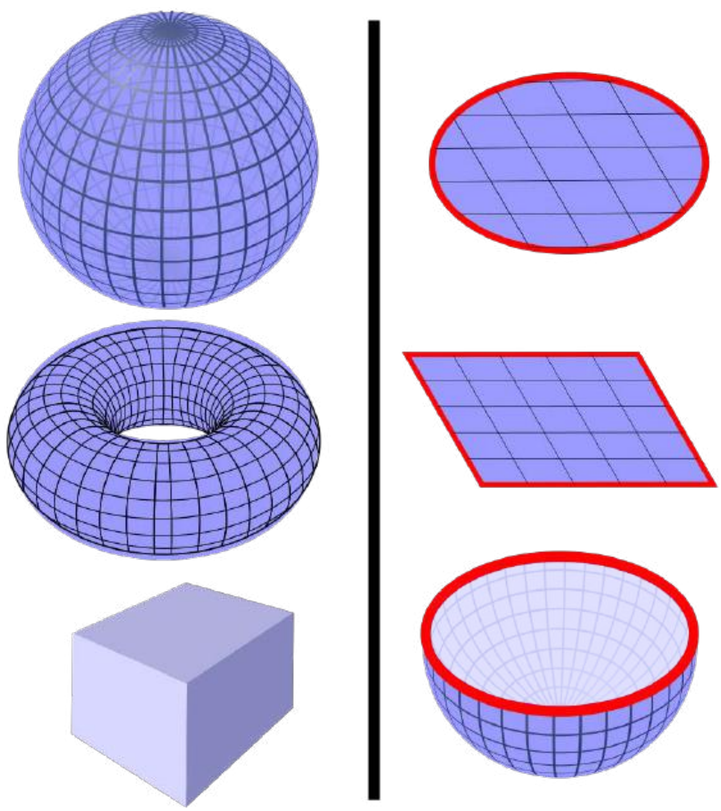


Theorem *Every compact, connected, orientable 2-manifold is homeomorphic to a sphere with handles added. Two such manifolds with the same number of handles are homeomorphic and conversely, so that the number of handles (called the genus) is the only topological invariant.*

[Boothby, p 14]

Homeomorphic manifolds

Topological manifolds without or with boundary



Differential structures on spheres of dimension 1 to 20 [edit]

The following table lists the number of smooth types of the topological m–sphere $\mathbf{S}^m$ for the values of the dimension m from 1 up to 20. Spheres with a smooth, i.e. C^∞–differential structure not smoothly diffeomorphic to the usual one are known as exotic spheres.

Dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

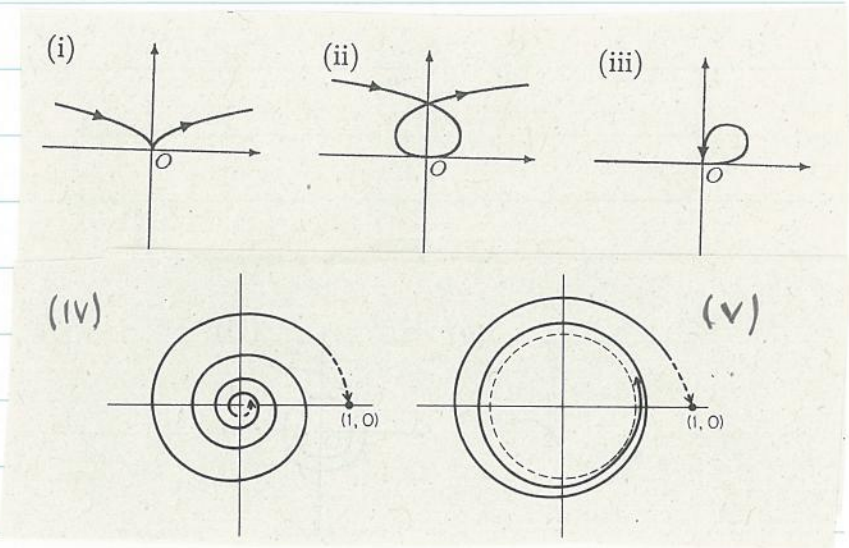
Immersion, submersion, submanifold

⚠ These definitions are not universal and can be slightly different depending on the authors.

<u>Def</u>: Let $f : \mathcal{M} \to N$ be a smooth map between smooth manifolds of dim m and n respectively.

- f is an <u>immersion</u> if $\mathrm{rank}(f)_p = m$ for any $p \in \mathcal{M}$.
- f is a <u>submersion</u> if $\mathrm{rank}(f)_p = n$ for any $p \in \mathcal{M}$.

<u>Examples</u>: $\mathcal{M} = \mathbb{R}$, $N = \mathbb{R}^2$



i) $f : \mathbb{R} \ni t \mapsto (t^3, t^2) \in \mathbb{R}^2$ is not an immersion since $\left[\frac{d}{dt} f\right](0) = (0,0)$.

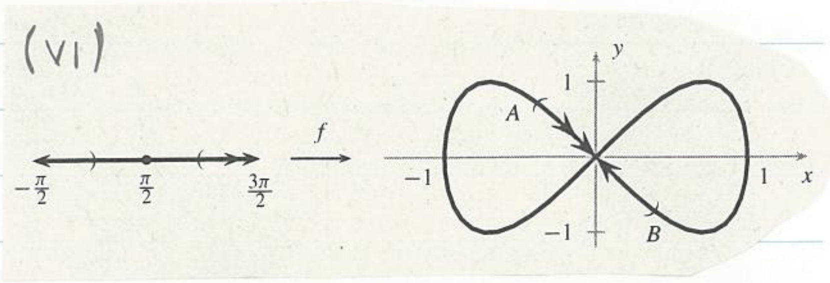
All the other examples are immersion since $\left[\frac{d}{dt} f\right](t) = f'(t) \neq (0,0)$ $\forall t \in \mathbb{R}$.

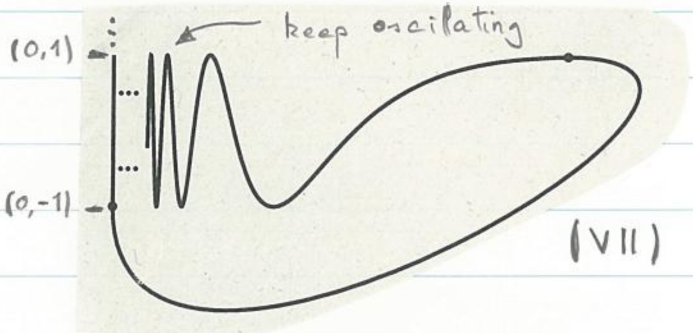
<u>Remarks</u>: 1) An immersion can be injective, or not (like in (ii)).

2) If f is an injective immersion, then $f(\mathcal{M})$ can be endowed with the topology and the differential structure of $\mathcal{M}$, but this is not really interesting since N does not play any role. In this situation, $f(\mathcal{M})$ is called a submanifold or immersed submanifold.

Injective immersions can be of different nature when the topology of N is taken into account, even through the subspace ($\equiv$ relative) topology.

(VI)

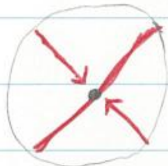




(VII)

For example, look at $(0,0)$ in (VI). In the relative topology any neighborhood of $(0,0)$ contain 3 parts parts:

neighborhood of $(0,0)$ in the relative topology



neighborhood of $(0,0)$ in the topology of $f(\mathbb{R})$.

The same phenomenon takes place on any point on $(0, x)$ with $x \in (-1, 1)$ in (VII), but the number of disconnected parts is even infinite!

<u>Definition</u>: A smooth map $f: M \to N$ is called an <u>imbedding</u> (or <u>embedding</u>) if it is an injective immersion and f defines a homeomorphism between M and $f(M)$ when $f(M)$ is endowed with the relative topology inherited from N. $f(M)$ is called an <u>imbedded manifold</u>.

Recall (from p 1) that a submanifold is the image of a manifold $\mathcal{R}$ through a smooth map $f : \mathcal{R} \to N$, when $f(\mathcal{R})$ is endowed with the structure from $\mathcal{R}$. In this case f is a diffeomorphism between $\mathcal{R}$ and $f(\mathcal{R})$.

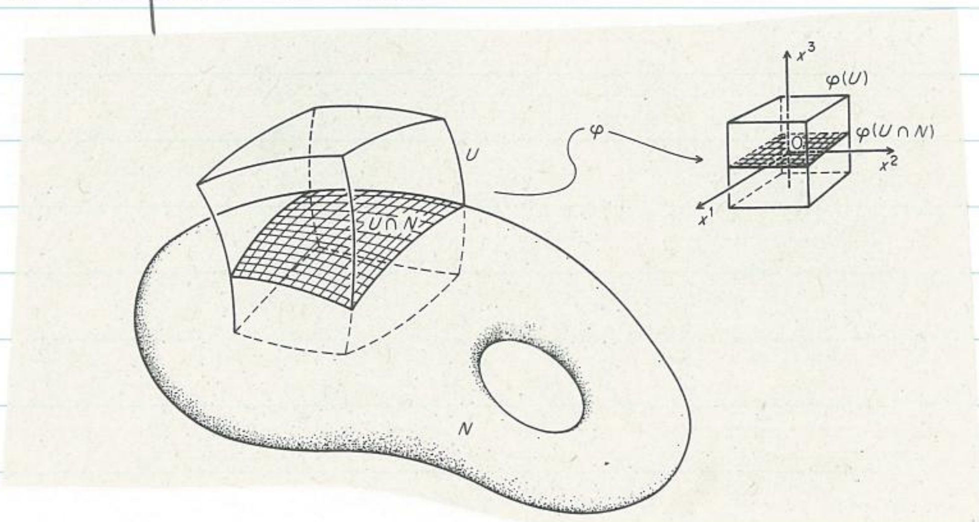
(immersed submanifold $\equiv$ submanifold)

<u>Def:</u> A subset N of a C^∞-manifold $\mathcal{R}$ has the <u>n-submanifold property</u> if $\forall p \in N$, there exists a chart (U, φ) on $\mathcal{R}$ with $p \in U$ s.t.

1) $\varphi(p) = 0 \in \mathbb{R}^m$
2) $\varphi(U) = C^m_\varepsilon(0)$ ← cube in $\mathbb{R}^m$, centered at 0 and of side 2ε.
3) $\varphi(U \cap N) = \{x \in C^m_\varepsilon(0) \mid x^{n+1} = x^{n+2} = \dots = x^m = 0\}$.

Such a chart is called <u>preferred coordinates</u> or <u>adapted chart</u> with respect to N.

Example with $\mathcal{R} = \mathbb{R}^3$.



Note that an immersed submanifold does not always have this property.

<u>Def</u> : <u>A regular submanifold</u> of a smooth manifold $\mathcal{M}$ is a subset N of $\mathcal{M}$ with the n-submanifold property and with the C^∞-structure provided by the preferred coordinates charts.

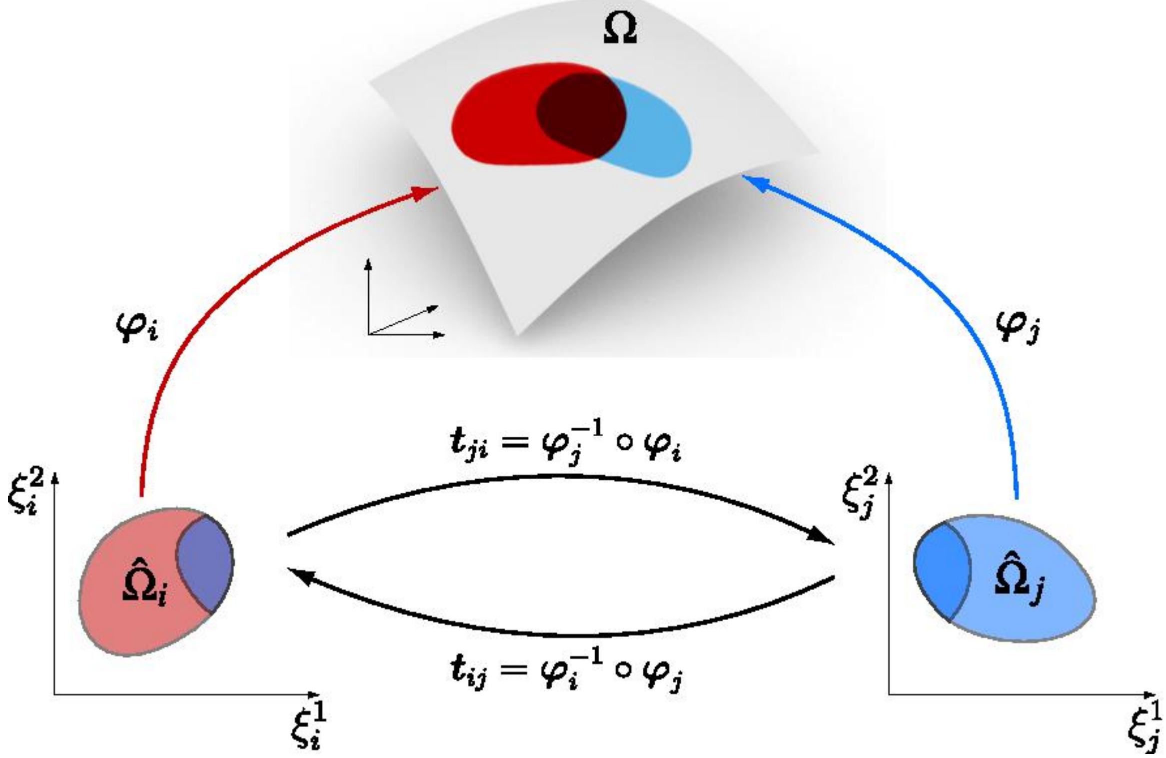
Examples of regular submanifolds are provided in [Tu, p 97].

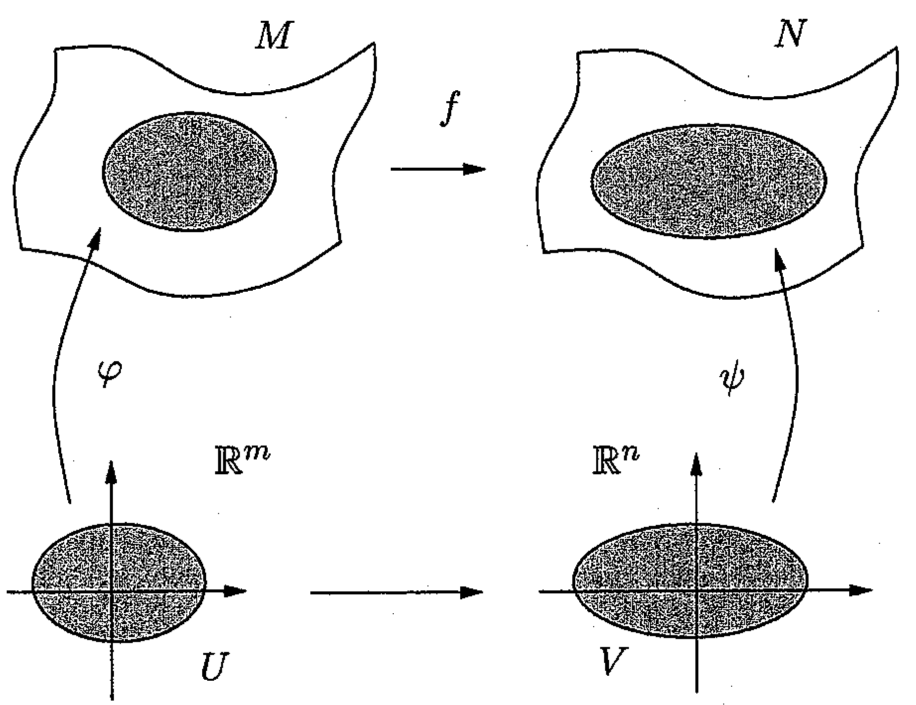
The first version of the following theorem has been proved by Whitney in 1936. It has then been simplified but it is still called the <u>Whitney imbedding theorem</u> :

<u>Thm</u> : Any smooth manifold $\mathcal{M}$ of dimension n can be imbedded in $\mathbb{R}^{2n}$. The image in $\mathbb{R}^{2n}$ is closed.

extrinsic and intrinsic approach are related, but this imbedding is not always so useful.

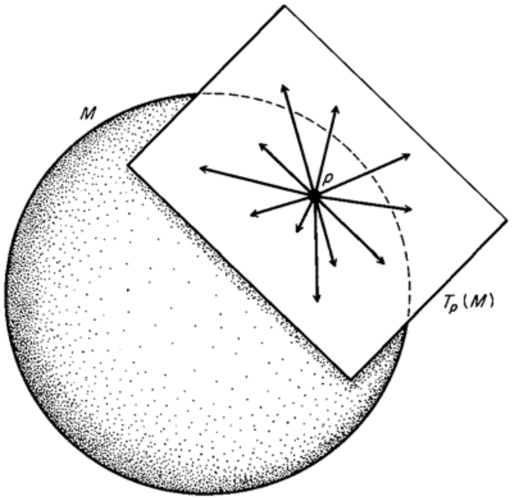
Smooth manifold and transition maps

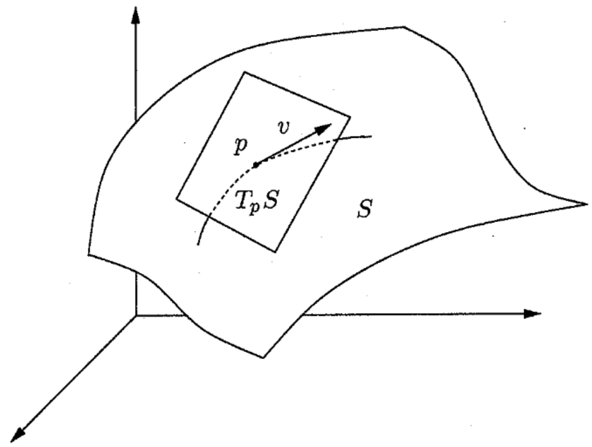




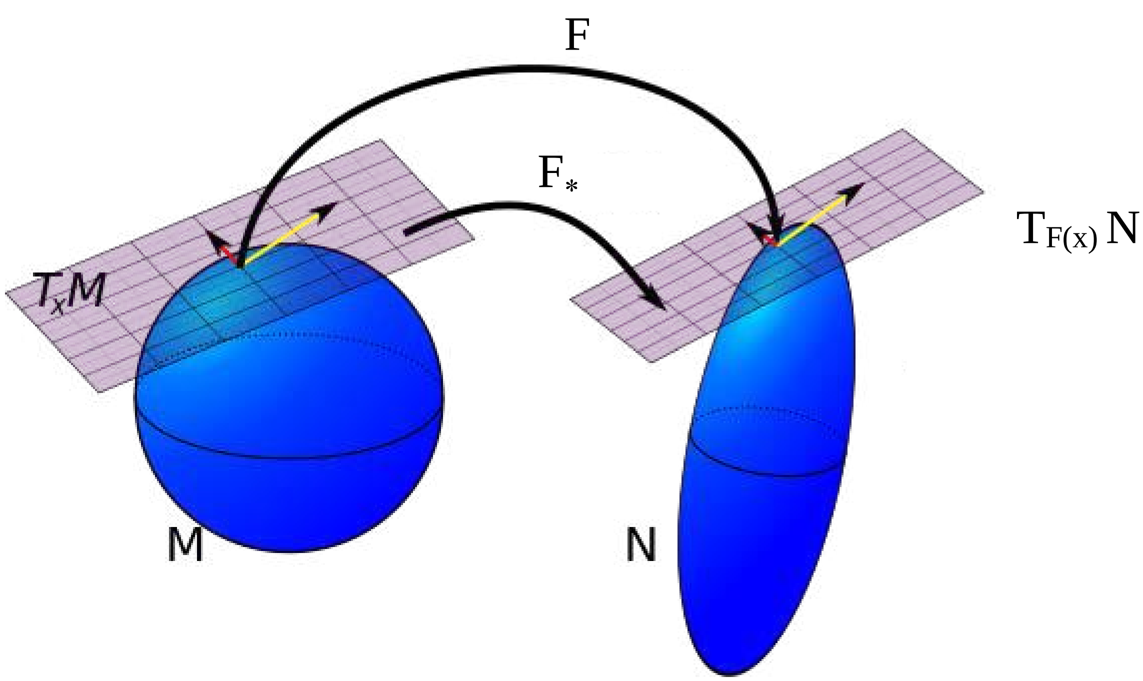
A smooth function f and its local representation given by the composition of three functions

Two tangent planes in the extrinsic representation

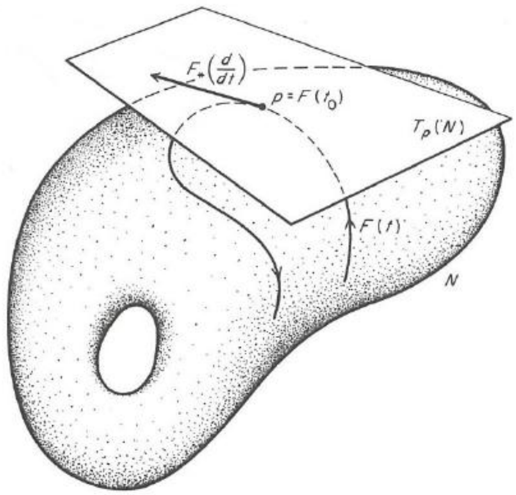


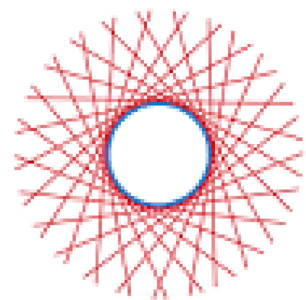


A function F and its differential $F_* = dF$

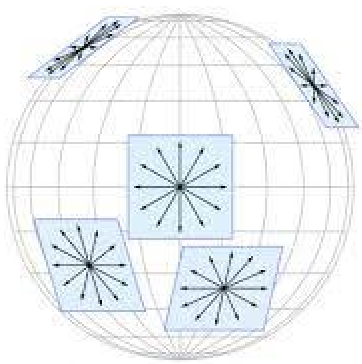


A curve (denoted by F) on a manifold and a representation of the corresponding tangent vector at the point p = F(t_o)

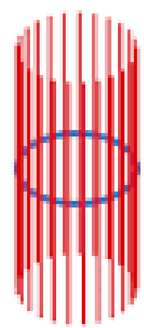




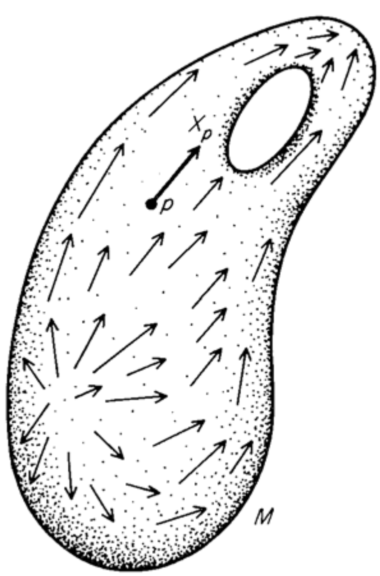
Some tangent bundles



Tangent bundle on a 2-sphere

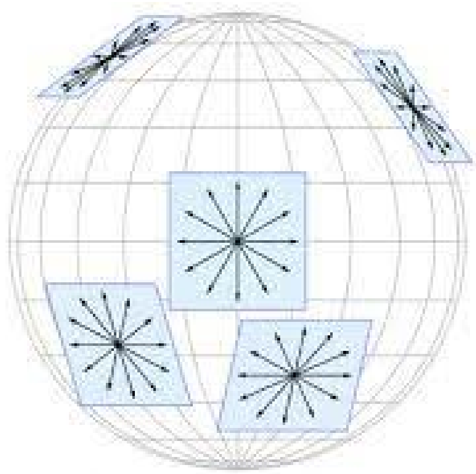


The tangent spaces have been reoriented

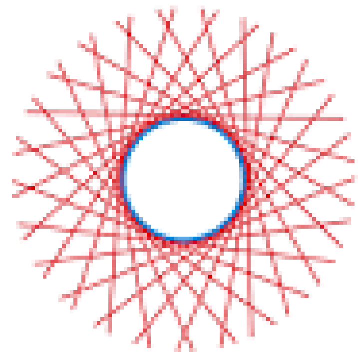


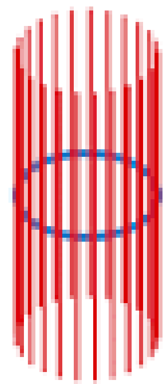
A vector field on a smooth manifold M

Some tangent bundles



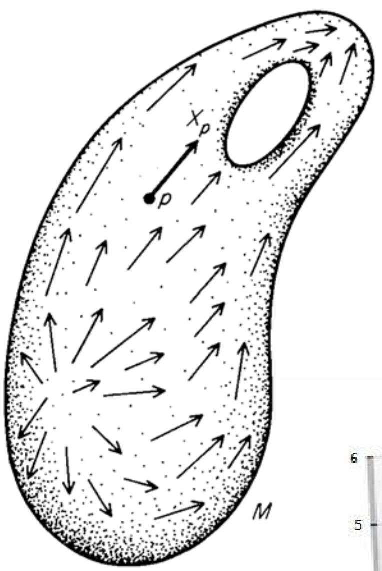
Tangent bundle on a 2-sphere

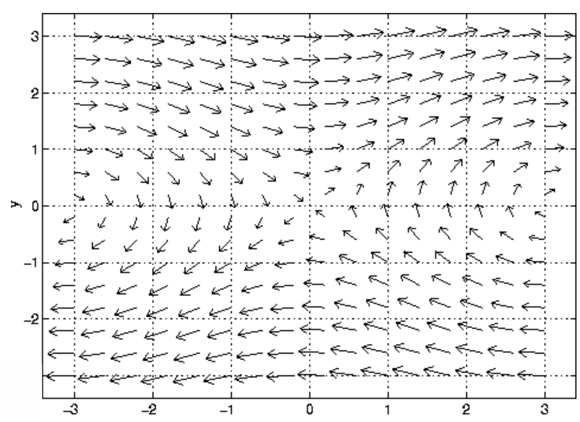


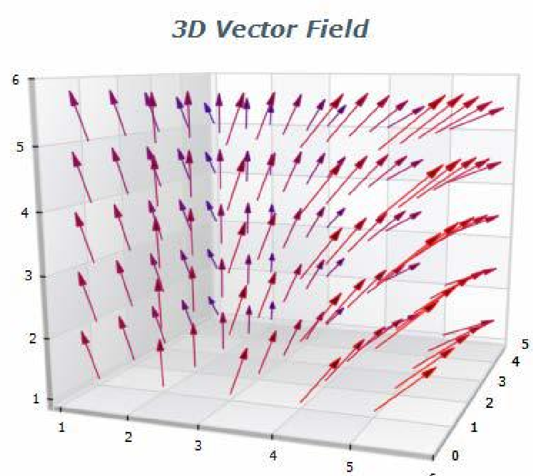


The tangent spaces have been reoriented

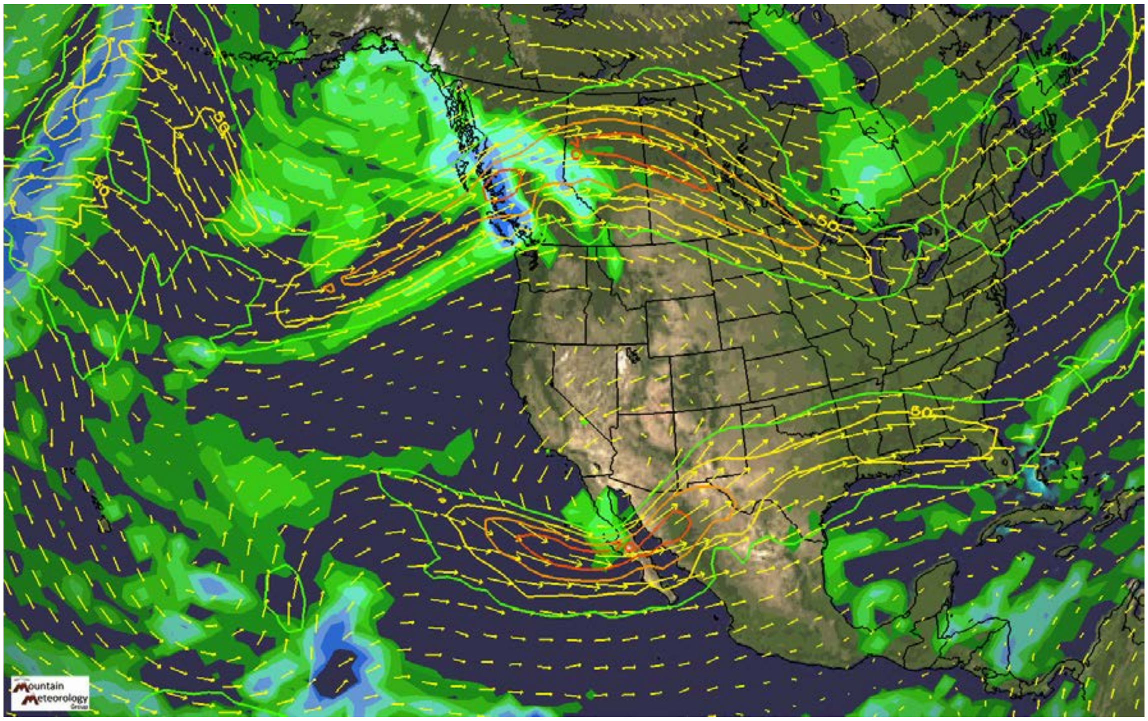
Some vector fields







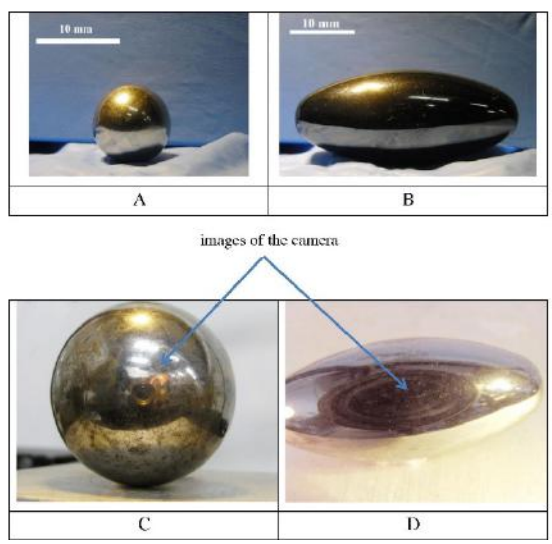
A vector field in daily life



The so-called Hairy ball theorem (for n = 2) :

Theorem 1. *Suppose* $\mathbf{v}$ *is a continuous vector field on* S^2. *Then there is* $\mathbf{p} \in S^2$ *such that* $\mathbf{v}(\mathbf{p}) = 0$.

Two applications of this theorem:



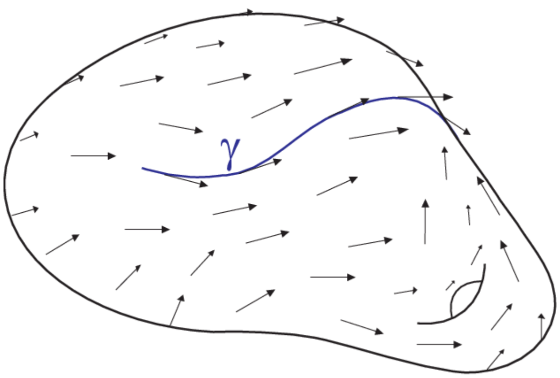
1) *No way to avoid reflection of light*

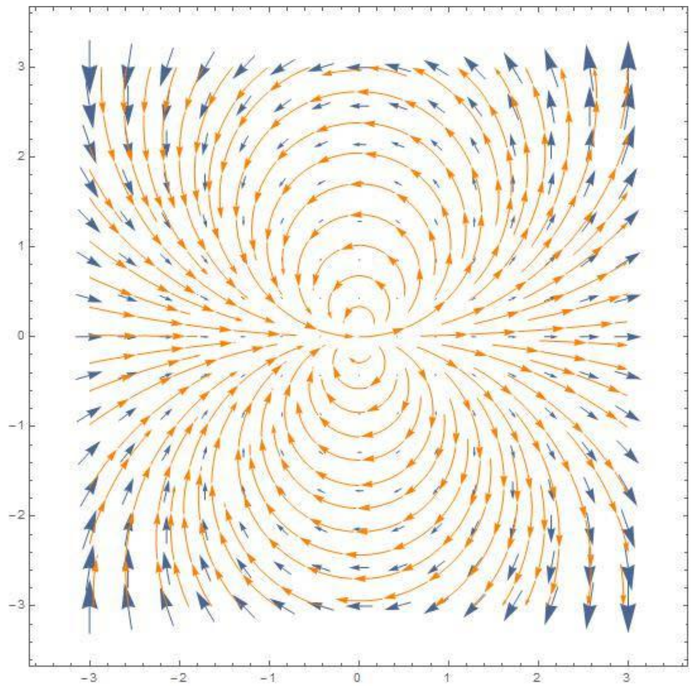
E. Bormashenko, A. Kazachkov
Results in Physics
Volume 6, 2016, Pages 76-77

2) *At any time, there is at least one place on earth with no wind*

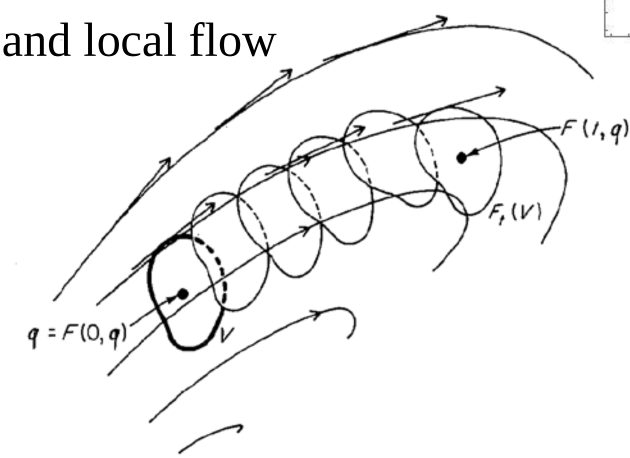
Appendix IV

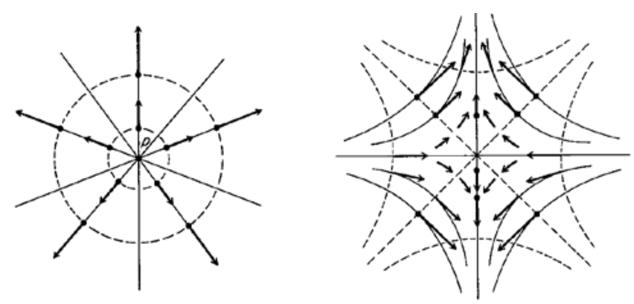
Integral curves





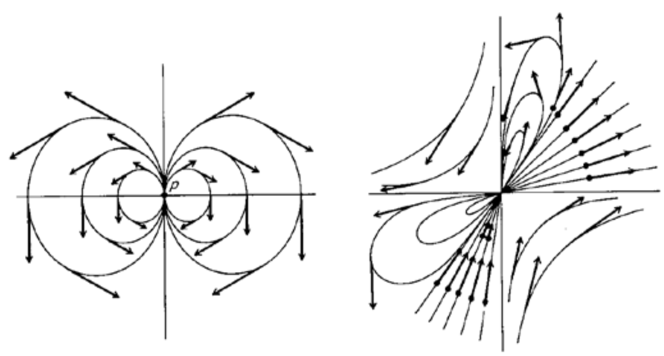
and local flow





(a) (b)

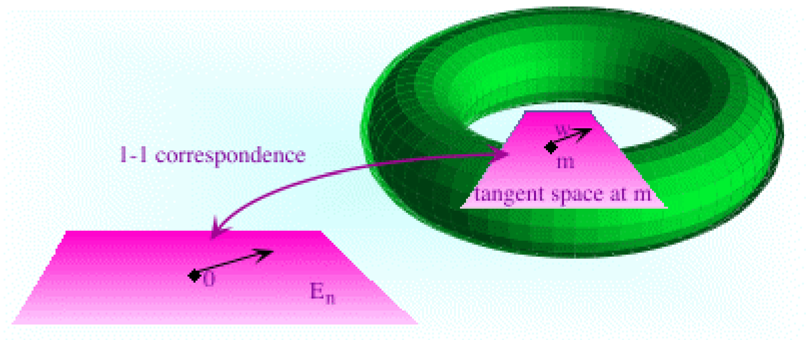
Critical points and integral curves around them

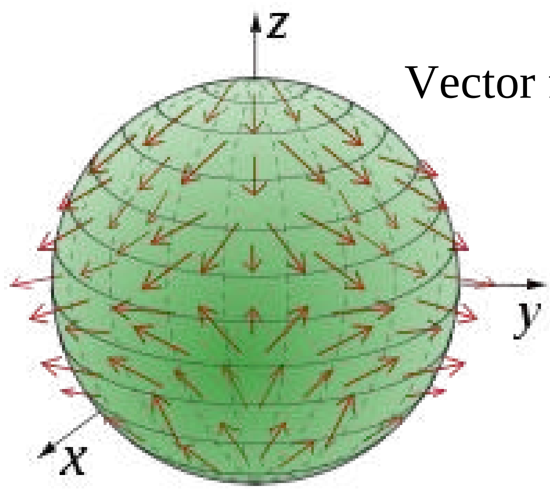


(c) (d)

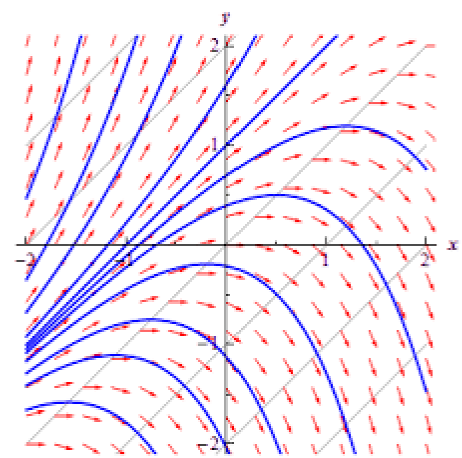
Additional exercises: [GN] Ex. & 12 p. 33 9 (see also [Bo] Thm 7.9 p. 155 for 9)
[Bo] Thm 3.6 p. 125, Thm 7.12 p. 156, Ex. 3 p. 157 & 11 p. 158

Tangent space

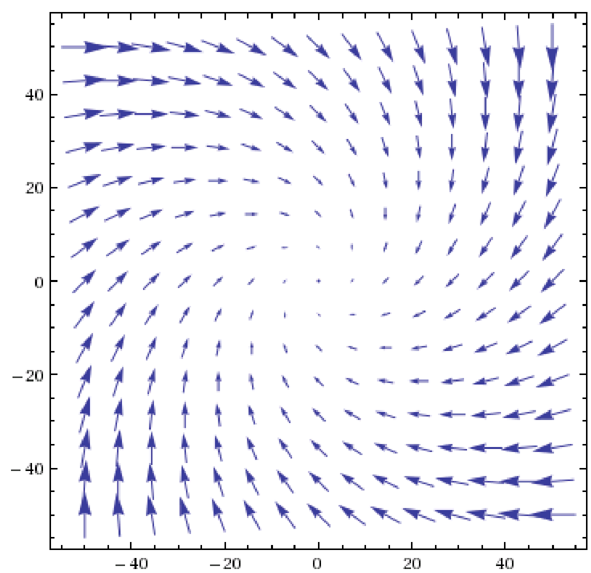




Vector field



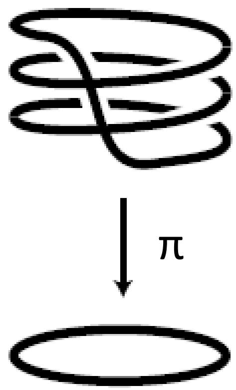
Integral curves



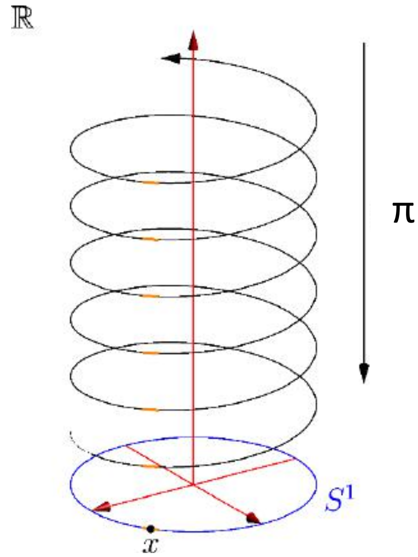
A singular point

About covering

Definition A smooth **covering** of a differentiable manifold B is a pair (M, π), where M is a connected differentiable manifold, $\pi : M \to B$ is a surjective local diffeomorphism, and, for each $p \in B$, there exists a connected neighborhood U of p in B such that $\pi^{-1}(U)$ is the union of disjoint open sets $U_\alpha \subset M$ (called **slices**), and the restrictions π_α of π to U_α are diffeomorphisms onto U. The map π is called a **covering map** and M is called a **covering manifold**.



Cover with 3 slices

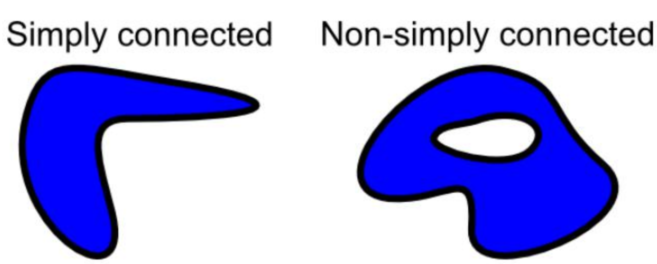


A cover with an infinite number of slices. ℝ is also the universal cover of S^1.

A diffeomorphism $h : M \to M$, where M is a covering manifold, is called a **deck transformation** (or **covering transformation**) if $\pi \circ h = \pi$, or, equivalently, if each set $\pi^{-1}(p)$ is carried to itself by h. It can be shown that the group G of all covering transformations is a discrete Lie g er.

Recall that

A simply connected domain is a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining in the domain.



Definition: A covering manifold which is simply connected is called a *universal covering*.

Summary on tangent space + vector fields

Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. A <u>tangent vector</u> X_p at p is a map

$$X_p : C^\infty(p) \to \mathbb{R}$$

satisfying 1) $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$

2) $X_p(fg) = X_p(f)\, g(p) + f(p)\, X_p(g)$.

1) = linearity , 2) = Leibniz's rule

The set of all tangent vectors at p is denoted by $\underline{T_p(\mathcal{M})}$ and is a vector space.

<u>Remark 1</u>: If $F : \mathcal{M} \to N$ is a smooth map between smooth manifolds, it defines a homomorphism $F_* : T_p(\mathcal{M}) \to T_{F(p)}(N)$ by

$$[F_*(X_p)](f) := X_p(f \circ F) \quad \forall f \in C^\infty(F(p)).$$

<u>Remark 2</u>: If $\mathcal{M} = \mathbb{R}^n$, $T_p(\mathbb{R}^n)$ is naturally identified with $\mathbb{R}^n$ by the relation:

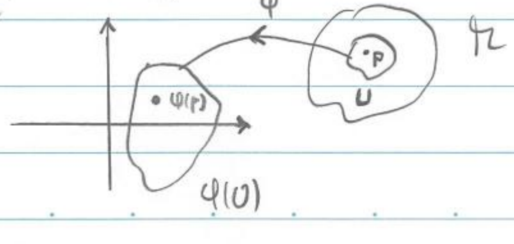
$$X_p(f) = [D_v f](p) = [v \cdot \nabla f](p)$$

for some $v \in \mathbb{R}^n$. ↖ directional derivative in direction v.

A basis for $T_p(\mathbb{R}^n)$ is given by $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}_{j=1}^n$

since $X_p(f) = \sum_{j=1}^n v_j \left[\frac{\partial}{\partial x^j} f\right](p) \equiv \sum_{j=1}^n v_j \frac{\partial}{\partial x_j}\Big|_p f$.

By Remarks 1 and 2, one infers that if (U, φ) is a local coordinate system for $\mathcal{M}$, and $p \in U$, a basis for $T_p(\mathcal{M})$ is given by



$$\Big\{ \underbrace{\varphi^{-1}_* \left(\frac{\partial}{\partial x^j}\Big|_{\varphi(p)} \right)}_{:= E_{j,p}} \Big\}_{j=1}^n$$

Observe that for $\mathcal{M} = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ the tangent space (1 dimensional) at any $t_0 \in (-\varepsilon, \varepsilon)$ is generated by $\frac{d}{dt}\big|_{t_0}$ since it satisfies $\frac{d}{dt}(\alpha f + \beta g)](t_0) =$
$= \alpha[\frac{d}{dt} f](t_0) + \beta[\frac{d}{dt} g](t_0)$ (linearity) and $[\frac{d}{dt}(fg)](t_0)$
$= [\frac{d}{dt} f](t_0)\, g(t_0) + f(t_0)[\frac{d}{dt} g](t_0)$ (Leibniz's rule).
Thus, if $c : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is a map between smooth manifolds, and if (U, φ) is a coordinate system around $c(t_0)$, then

$$[c_* (\frac{d}{dt}\big|_{t_0})](f) = \sum_{j=1}^{n} \dot{c}^j(t_0)\, \underbrace{E_{j, c(t_0)}}_{\text{element of the basis of } T_{c(t_0)}(\mathcal{M})}(f)$$

with $c^j = (\underbrace{\varphi \circ c}_{\text{curve in } \mathbb{R}^n})^j$ j component of this curve.

and $\dot{c}^j = \frac{d}{dt}(\varphi \circ c)^j$.
As a conclusion, derivatives of curves in $\mathcal{M}$ passing to a point also generate the tangent space, once a coordinate system has been chosen.

A <u>vector field</u> X is a map $X : \mathcal{M} \to \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$ which is smooth in a suitable sense. Note that $T(\mathcal{M}) := \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$ can be endowed with a topology which makes it a smooth manifold, and is called the <u>tangent bundle</u>.
The set of all smooth vector field is denoted by $\mathfrak{X}(\mathcal{M})$. One observes that any $X \in \mathfrak{X}(\mathcal{M})$ can be seen as a map from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})$ with $[Xf](p) = X_p(f)$. With this observation,

$\mathfrak{X}(\mathcal{R})$ becomes a Lie algebra, with Lie bracket $[X, Y] = XY - YX \quad \forall\, X, Y \in \mathfrak{X}(\mathcal{R})$.

To any $X \in \mathfrak{X}(\mathcal{R})$ one can associate a <u>local flow</u> $F : (-\varepsilon, \varepsilon) \times V \to \mathcal{R}$ satisfying

$\uparrow$ open set in $\mathcal{R}$

$$F(0, q) = q \in V \quad \text{and} \quad \dot{F}(t, q) = X_{c_q(t)}$$

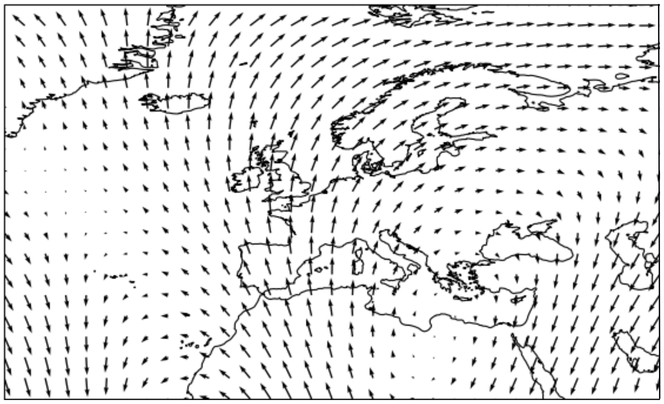
$$\dot{F}(t,q) \equiv \dot{c}_q(t) = c_q^*\left(\tfrac{d}{dt}\Big|_t\right)$$

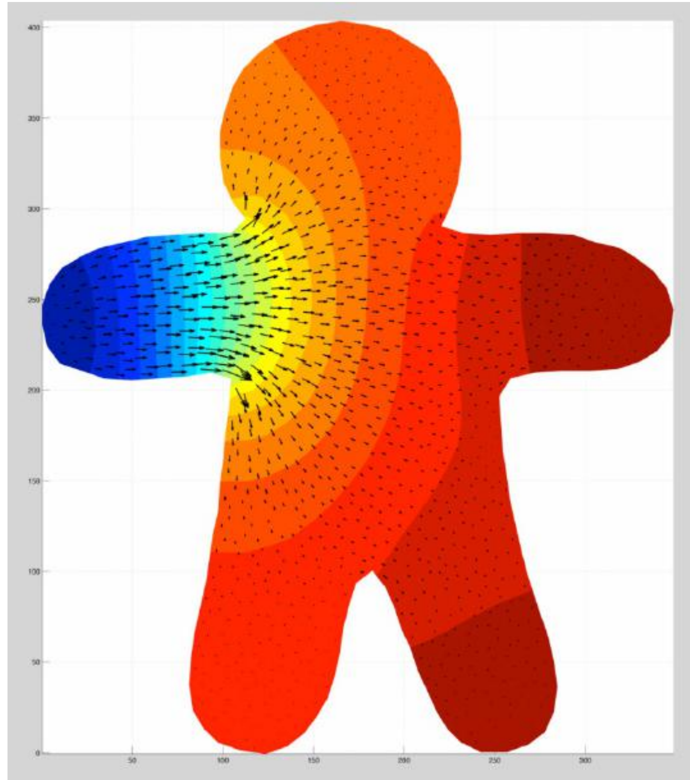
c_q is the curve passing by q at $t = 0$.

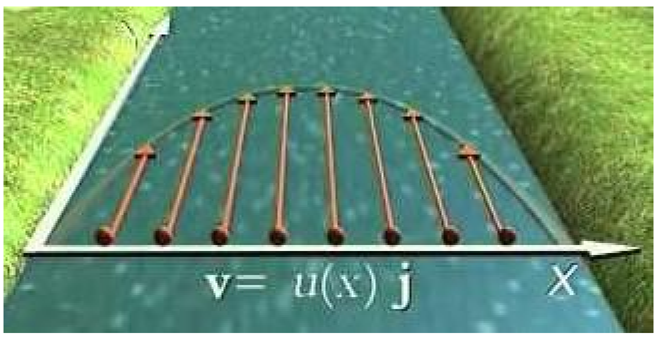
In other words, the vector field is tangent to the flow at any point of $\mathcal{R}$. The corresponding curves are called <u>integral curves</u>.

Appendix VI

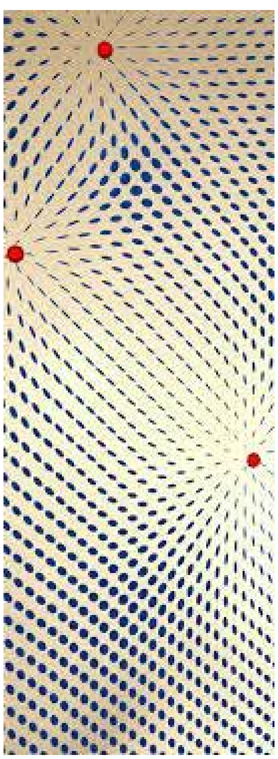
Vector fields

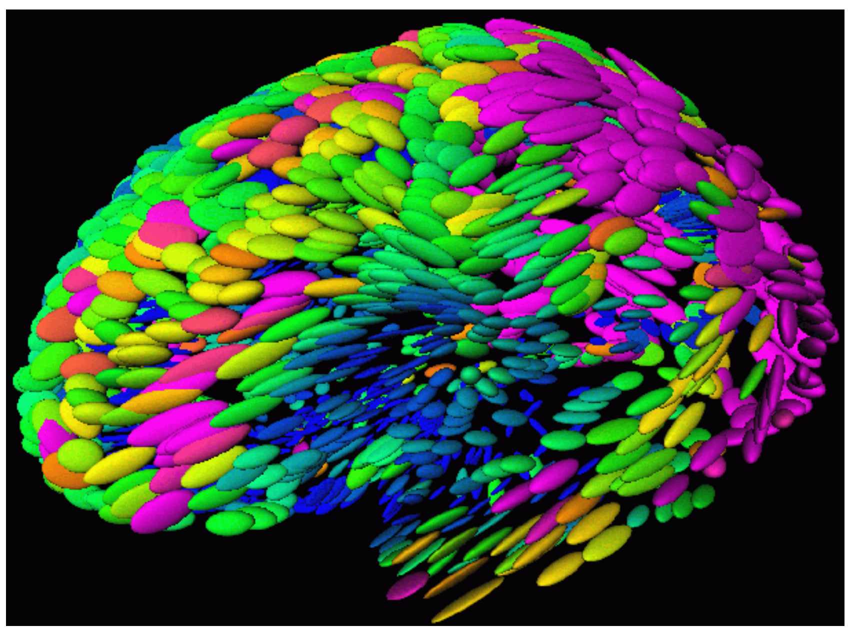






Tensor fields





Striking differences are found, even among normal human subjects, in the gyral patterns of the cerebral cortex. **Tensor maps** can be used to visualize these complex patterns of anatomical variation. In these maps (*below*), color distinguishes regions of high variability (*pink colors*) from areas of low variability (*blue*). Rectangular glyphs indicate the principal directions of variation - they are most elongated along directions where there is greatest anatomic variation across subjects. Each glyph represents the covariance tensor of the vector fields that map individual subjects onto their group average anatomic representation. The maps are based on a group of 40 normal subjects. The resulting information can be leveraged to distinguish normal from abnormal anatomical variants using random tensor field algorithms.

http://users.loni.usc.edu/~thompson/TMP/tensor.html

Reminder from Calculus II

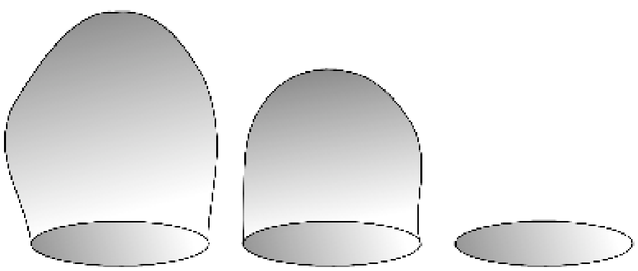
Stokes' Theorem:

$$\boxed{\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint (\underbrace{\boldsymbol{\nabla} \times \vec{\mathbf{F}}}_{\text{curl}\vec{\mathbf{F}}}) \cdot \hat{\mathbf{n}}\, dS}$$

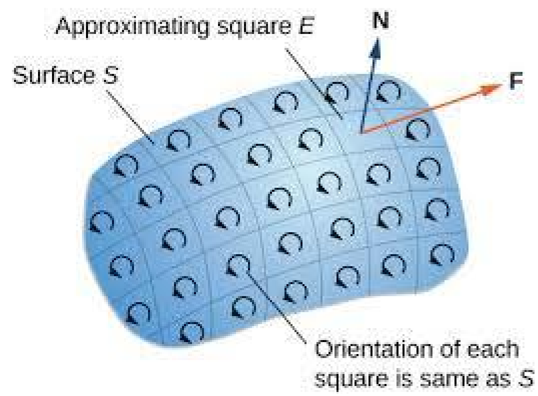
if C closed curve

S = any surface bounded by C

and $\vec{\mathbf{F}}$ defined everywhere in S



Different surfaces, same boundary



Integration	Differentiation	Both
• Line* $\int_\gamma \vec{F} \cdot d\vec{r}$	• grad $\vec{F} = -\vec{\nabla} \cdot \phi$	• Gradient Theorem: $\int_{\gamma \text{ from } \vec{a} \to \vec{b}} \left(-\vec{\nabla}\phi\right) \cdot d\vec{r} = \phi(\vec{a}) - \phi(\vec{b})$ independent of path γ
• Surface* $\iint_S \vec{F} \cdot d\vec{S}$	• curl $\vec{B} = \vec{\nabla} \times \vec{A}$	• Stoke's Theorem: $\iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{r}$
• Volume $\iiint_V \rho\, dV$	• div $\rho = \vec{\nabla} \cdot \vec{E}$	• Divergence Theorem: $\iiint_V \left(\vec{\nabla} \cdot \vec{E}\right) dV = \oiint_{\partial V} \vec{E} \cdot d\vec{S}$

* Also for scalar fields: $\int_\gamma \lambda\, |d\vec{r}|$ and $\iint_S \sigma\, \left|d\vec{S}\right|$

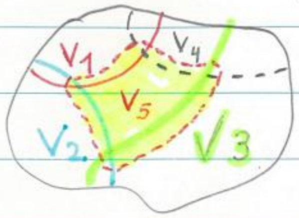
Important: It does not matter how I call my fields (F, E, A, ...). The theorems apply independent of the letter I use.

All summarized in: $$\int_{\partial\Omega} \omega = \int_\Omega d\omega\,.$$

Partition of unity

Recall that a <u>cover of $\mathcal{M}$</u> is a collection $\{V_\alpha\}_\alpha$ of subsets of $\mathcal{M}$ such that $\bigcup_\alpha V_\alpha = \mathcal{M}$.

union of all V_α



The cover is <u>locally finite</u> if $\forall p \in \mathcal{M}$, there exits a neighborhood U of p such that $U \cap V_\alpha = \emptyset$ except for a finite number of α.

The intersection of U with most of the V_α is empty.

Recall also that if $f : \mathcal{M} \to \mathbb{R}$, its support is defined by $\operatorname{supp}(f) := \overline{\{p \in \mathcal{M} \mid f(p) \neq 0\}}$

we take the closure of this set $\{\ldots\}$, it means we consider the smallest closed set containing $\{\ldots\}$.

The following definition has many aplications:

<u>Def :</u> A <u>smooth partition of unity of M</u> is a collection $\{f_\beta\}_\beta$ of smooth functions on M satisfying :

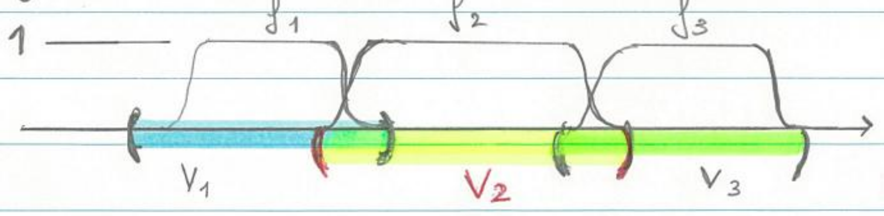
1) $f_\beta \geq 0$ the functions are positive

2) $\{\mathrm{supp}(f_\beta)\}_\beta$ is a locally finite cover of M,

3) $\sum_\beta f_\beta(p) = 1 \quad \forall p \in M$.

for every p this sum is finite because of 2)

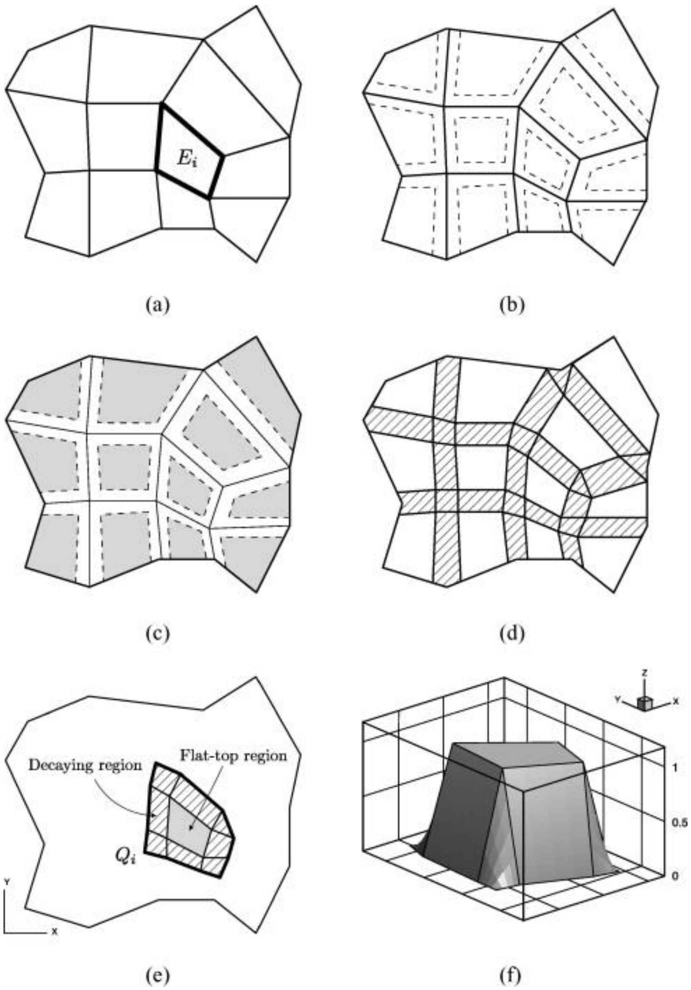
This partition is <u>subordinate</u> to an open cover $\{V_\alpha\}_\alpha$ of M if $\forall \beta \; \exists V_\alpha$ with $\mathrm{supp}(f_\beta) \subset V_\alpha$.

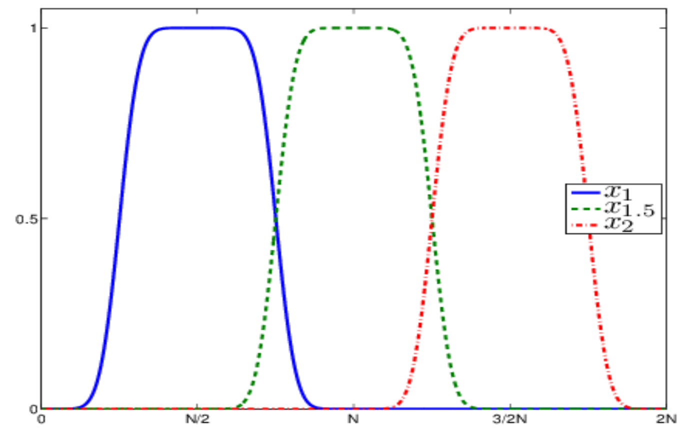


<u>Thm :</u> For any smooth manifold M and for any open cover of M there exists a smooth partition of unity of M subordinated to the open cover.

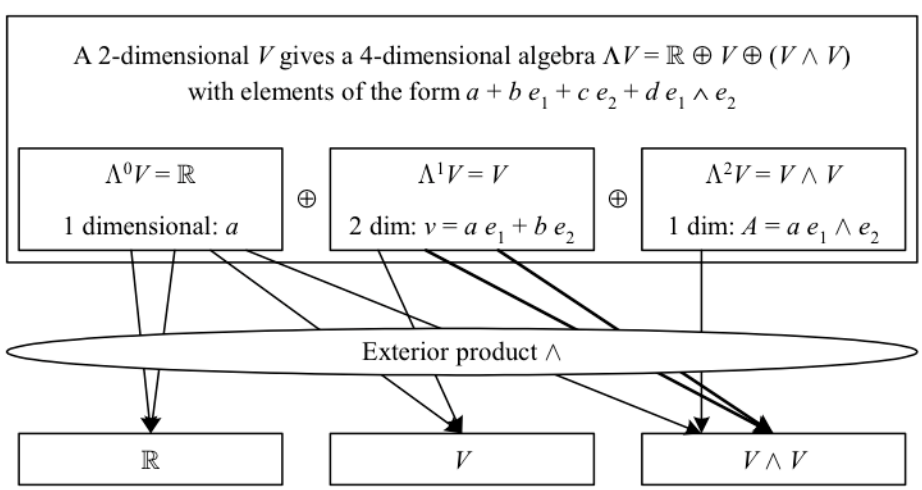
Partition of unity

A constructive approach





Example of an exterior algebra



$$v \wedge w = (a\, e_1 + b\, e_2) \wedge (c\, e_1 + d\, e_2) = (ad - bc)\, e_1 \wedge e_2$$

The exterior derivative

$$
\begin{aligned}
&d(\omega + \eta) = d\omega + d\eta \\
&d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \, \omega \wedge d\eta \qquad (p = \deg \omega) \\
&d(d\omega) = 0.
\end{aligned}
$$

Definition de Rham cohomology

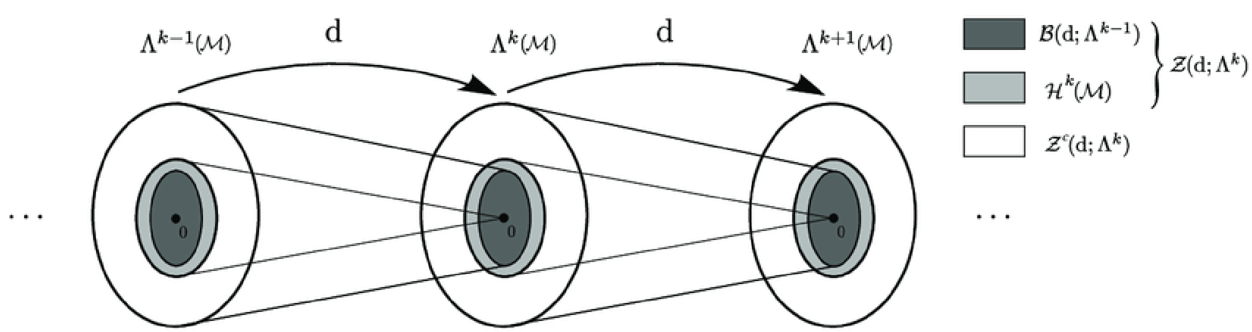
- Kernel and image of a map
- Space of all forms on a smooth manifold and exterior derivative

$$d : \Lambda^i(M) \to \Lambda^{i+1}(M) \qquad\qquad d^2 = 0$$

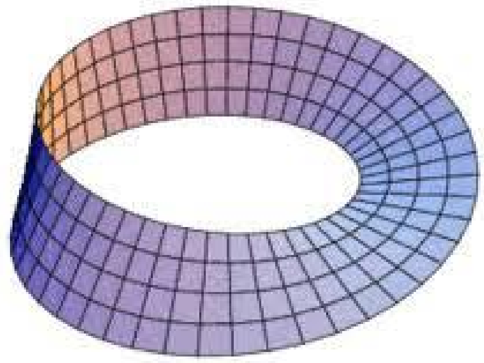
- Definition of de Rham cohomology:

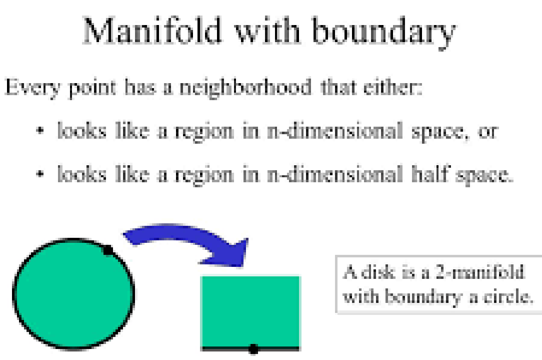
$$H^i(M) := \frac{ker(d : \Lambda^i \to \Lambda^{i+1})}{im(d : \Lambda^{i-1} \to \Lambda^i)}$$

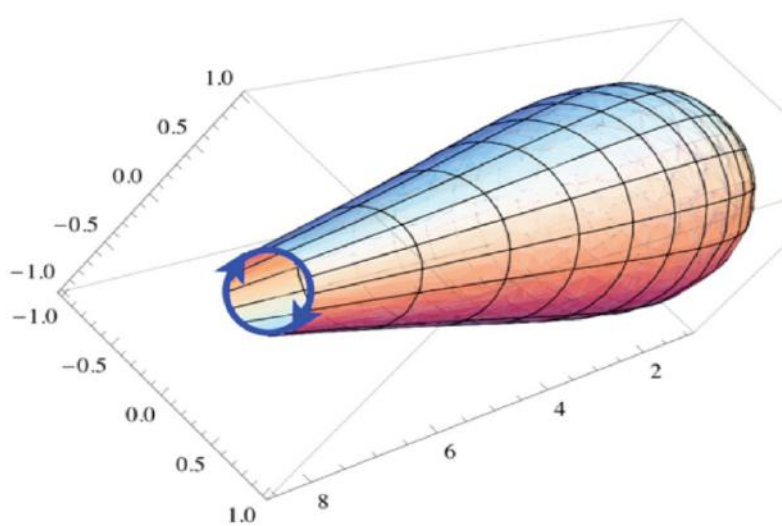
- Can think of de Rham cohomology intuitively as counting the numbers of i-dimensional holes



Non orientable manifolds



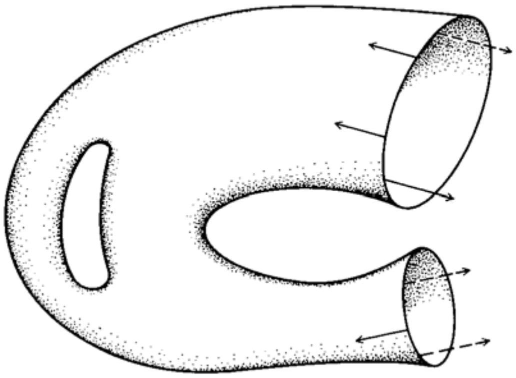




Summary + keywords

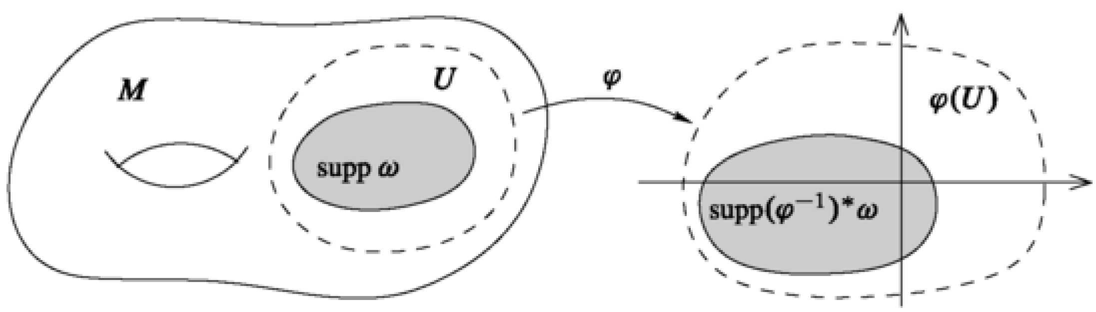
- Extrinsic / intrinsic (differential geometry)
- Topological manifold — open sets, neighborhood, homeomorphism, relative topology.
- Smooth manifold — transition functions, atlas chart or local coordinate system, diffeomorphism.
- Tangent space — $T_p(\mathcal{M})$, basis of the tangent space, coordinate frames, $\varphi_*^{-1}\left(\frac{\partial}{\partial x^j}\Big|_{\varphi(p)}\right)$.
- Vector fields — One tangent vector at each point of $\mathcal{M}$, smooth vector fields $\mathfrak{X}(\mathcal{M})$, a vector field maps smooth functions to smooth functions, local flow, integral curves, regular point.
- Tensor fields — Tensor, symmetric, alternating, $\Lambda(V)$, wedge product, dual space $\{(dx^j)_p\}_j$, a tensor at every point of $\mathcal{M}$, smooth tensor fields $\mathcal{T}^r_s(\mathcal{M})$, $\Lambda(\mathcal{M})$ the exterior algebra.

Appendix VIII



Manifold with boundary with inward and outward pointing vectors at the boundary

Differential form ω with compact support, and its local representation

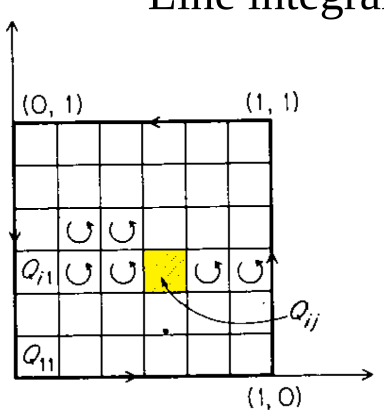


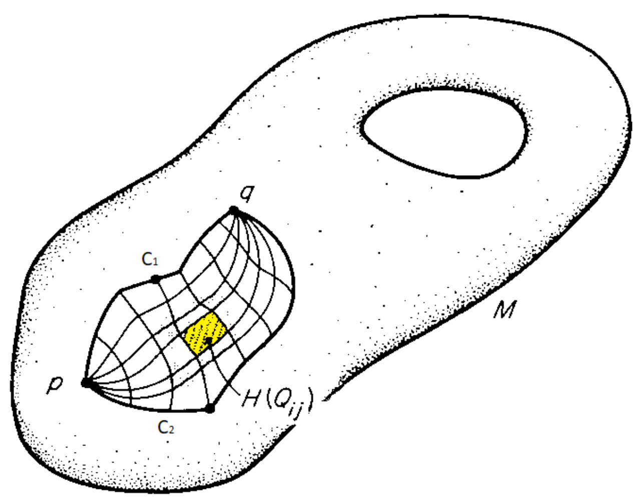
Stokes's theorem

Theorem. *(Stokes–Cartan) If ω is a smooth $(n-1)$-form with compact support on smooth n-dimensional manifold-with-boundary Ω, $\partial\Omega$ denotes the boundary of Ω given the induced orientation, and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then*

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^*\omega.$$

Line integrals





and independence with respect to the path if dω = 0

Remark on Stokes's theorem, from [Bo] p 263.

It is important to note that the version of Stokes's theorem proved above is deficient in the following sense: it holds only for smooth manifolds with smooth boundary. Thus, for example, our proof does not even include the case of a square in $\boldsymbol{R}^2$ or an open set of $\boldsymbol{R}^3$ bounded by a polyhedron. The difficulty in these cases is not so much with the analysis and integration theory, as with describing the regions of integration to be admitted and with giving precise definitions of orientability and induced orientation of the boundary. The search for reasonable domains of integration to validate Stokes's theorem usually leads to the concept of a simplicial or polyhedral complex, that is, a space made up by fastening together along their faces a number of simplices (line segments, triangles, tetrahedra, and their generalizations) (Fig. VI.4) or more general polyhedra (cubes, for example). Since it can be shown (see Munkres [1]) that any C^∞ manifold M may be "triangulated," which means that it is homeomorphic (even with considerable smoothness) to such a complex, the integral over M becomes the sum of the integrals over the pieces, which are images of simplices, cubes, or other polyhedra as the case may be (compare Remark 2.7). The strategy is then to reduce the theory (including Stokes's theorem) to the case of polyhedral domains of $\boldsymbol{R}^n$. This approach is particularly important for those interested in algebraic topology and de Rham's theorem. It is very clearly set forth, for example, by Singer and Thorpe [1] or Warner [1].

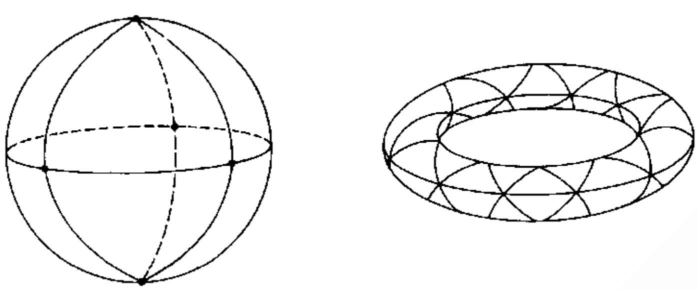
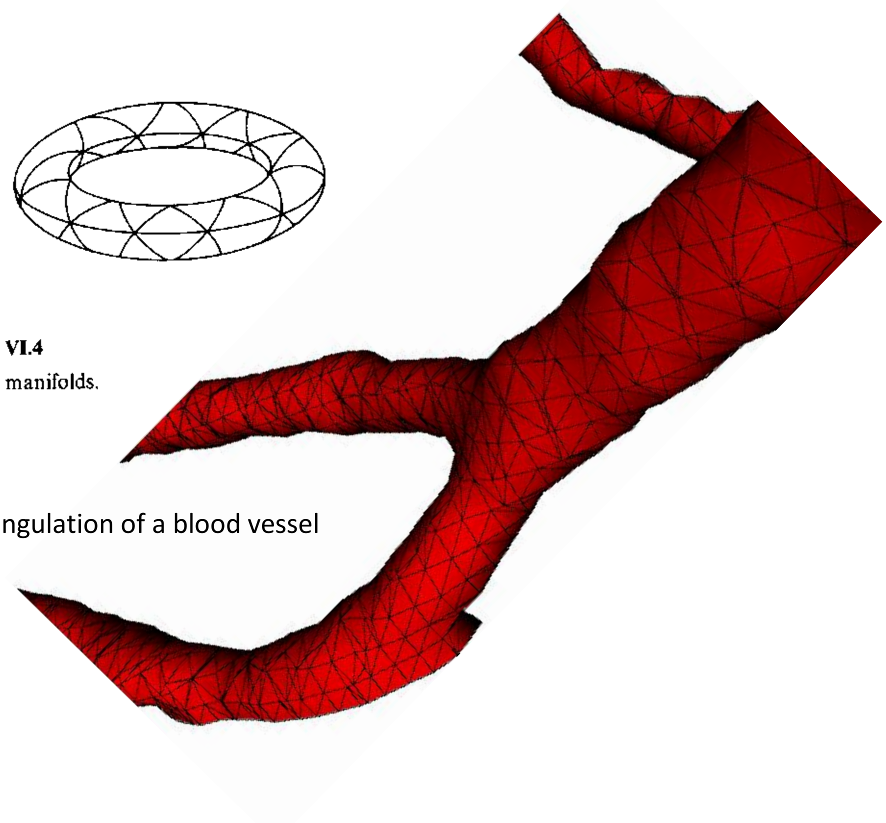
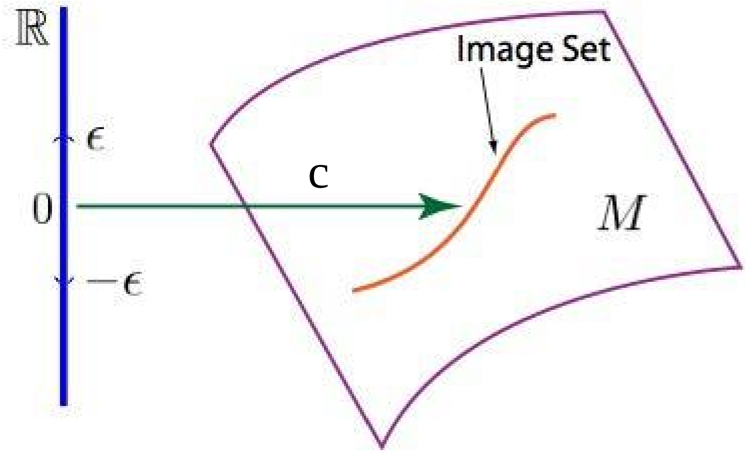


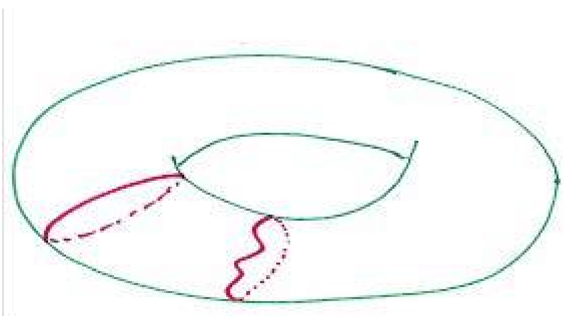
Figure VI.4
Triangulated manifolds.



Triangulation of a blood vessel

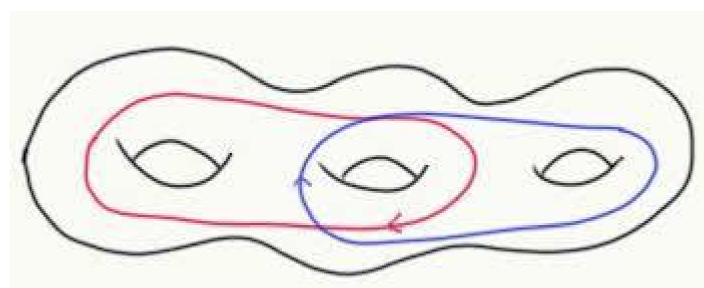
Curve on a manifold

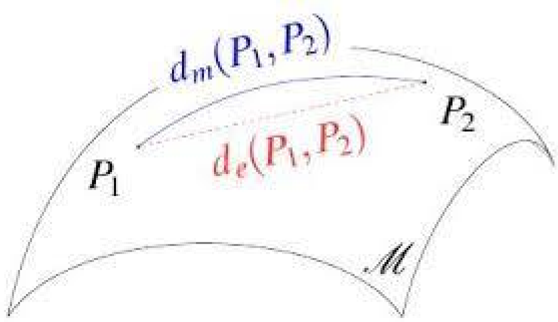




Homotopic curves

Non homotopic curves





Metric on a manifold, and

Euclidean metric

Curve in $\mathbb{R}^3$: The Frenet frame

Consider a parametric curve in $\mathbb{R}^3$, it means a map $c : [a,b] \to \mathbb{R}^3$, $t \mapsto c(t)$, and we assume it smooth.

This curve is regular if $\dot{c}(t) = \frac{d}{dt} c(t) \neq 0$.

The arc length is defined by $s = s(t) = \int_a^t \|\dot{c}(t)\| \, dt$ with $\|\dot{c}(t)\|$ the Euclidean norm of $\dot{c}(t)$ in $\mathbb{R}^3$.

Let $L := \int_a^b \|\dot{c}(t)\| \, dt$ the length of the curve.

Lemma: If c is regular, $\exists$ a diffeomorphism $\phi : [0,L] \to [a,b]$ such that $\|(c \circ \phi)'(s)\| = 1 \quad \forall s \in [0,L)$.

We say that the curve is parameterized by its arc length, and in this parametrization the tangent vector is of length 1.

see any course of calculus II for the proof, or [Klingenberg] p 9.

Whenever the letter s is used for the parametrization of a curve, it means that it is the arc length parametrization.

Let us set $T(s) := (c\circ\phi)'(s)$, and observe that

constant function scalar product in $\mathbb{R}^3$

$$0 = \frac{d}{ds} 1 = \frac{d}{ds}\|T(s)\|^2 = \frac{d}{ds}\langle T(s), T(s)\rangle =$$

$$= \langle \dot T(s), T(s)\rangle + \langle T(s), \dot T(s)\rangle = 2\langle T(s), \dot T(s)\rangle$$

symmetry of the scalar product in $\mathbb{R}^3$

with $\dot T(s) := \frac{d}{ds} T(s)$.

Thus $\underline{\dot T(s) \perp T(s)}$ ($\dot T(s)$ is perpendicular to $T(s)$ since their scalar product is 0).

Set $\kappa(s) := \|\dot T(s)\|$ and call it the <u>curvature</u>.

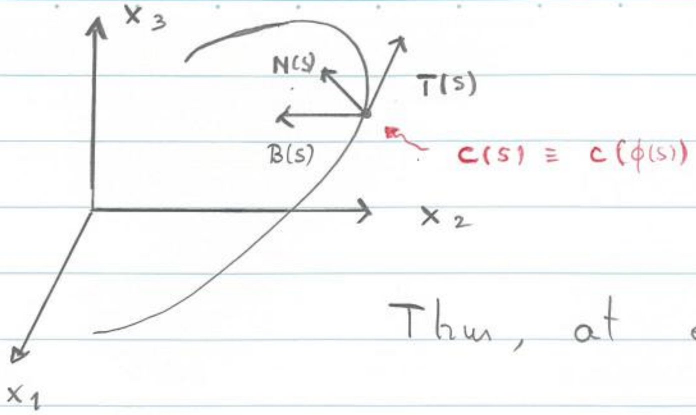
If $\kappa(s) \neq 0$, we set $N(s)$ for the vector of norm 1

positive scalar

satisfying $\dot T(s) = \kappa(s)\, N(s)$

vector in $\mathbb{R}^3$ vector in $\mathbb{R}^3$

If $\kappa(s) \neq 0$ we also set $B(s)$ $\in \mathbb{R}^3$ for the unique vector of norm 1 such that $\{T(s), N(s), B(s)\}$ is a basis of $\mathbb{R}^3$ (with a positive orientation).



Thus, at every point of the curve where the curvature $K(s) \neq 0$, one can define an orthonormal basis, it corresponds to a field of orthonormal frames.

<u>Observation</u>: One can show that $K(s) = 0 \ \forall s \in I \iff$ the curve is a straight line on this interval I.

Let us set $F_1(s) := T(s)$, $F_2(s) = N(s)$, $F_3(s) := B(s)$ (we assume $K(s) \neq 0$). Since these vectors generate an orthonormal basis, one has $\langle F_i(s), F_j(s) \rangle = \delta_{ij}$

and $\frac{d}{ds} \langle F_i(s), F_j(s) \rangle = \langle \dot{F}_i(s), F_j(s) \rangle + \langle F_i(s), \dot{F}_j(s) \rangle = 0$ ⊛

Since $\dot{F}_i(s)$ is a linear combination of the 3 vectors $F_1(s), F_2(s), F_3(s)$ one has $\dot{F}_j(s) = \sum_{k=1}^{3} \underset{\underset{\mathbb{R}}{\cap}}{a_j^k} F_k(s)$ for $j = 1, 2, 3$.

$a_j^k(s)$

By inserting this in ⊛ one gets

$$\langle \sum_k a_i^k F_k(s) , F_j(s) \rangle + \langle F_i(s) , \sum_k a_j^k F_k(s) \rangle = 0$$

$\Leftrightarrow \quad a_i^j(s) + a_j^i(s) = 0 \quad \Rightarrow \quad (a_i^j(s))_{i,j}$ is a skew-symmetric matrix (in particular $a_i^i(s) = 0$)

Also, since $\dot{F}_1(s) = \dot{T}(s) = \kappa(s)\, N(s) = \kappa(s)\, F_2(s) \Rightarrow a_1^2(s) = \kappa(s)$ and $a_1^3(s) = 0$. Let us finally set $a_2^3(s) =: \tau(s)$ and call it the <u>torsion</u>. One finally gets the system

$$\begin{cases} \dot{T}(s) = \kappa(s)\, N(s) \\ \dot{N}(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s) \\ \dot{B}(s) = -\tau(s)\, N(s) \end{cases}$$

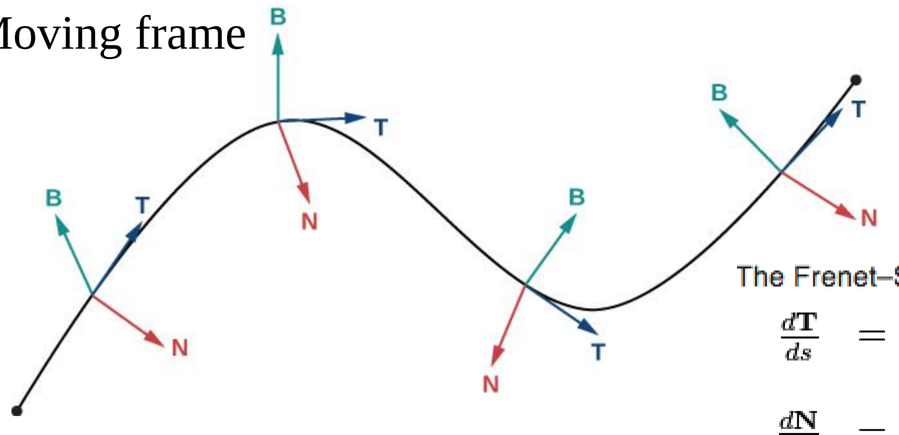
Frenet-Serret formulas

This system determines the evolution of the tangent vector $N(s)$, the normal vector $N(s)$ and the binormal vector $B(s)$ along the curve c.

<u>Lemma</u>: The curve lies in a plane iff $\tau(s) = 0 \ \forall s$.

[Bo, Thm 1.9 p 303]

Moving frame



The Frenet–Serret formulas are

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

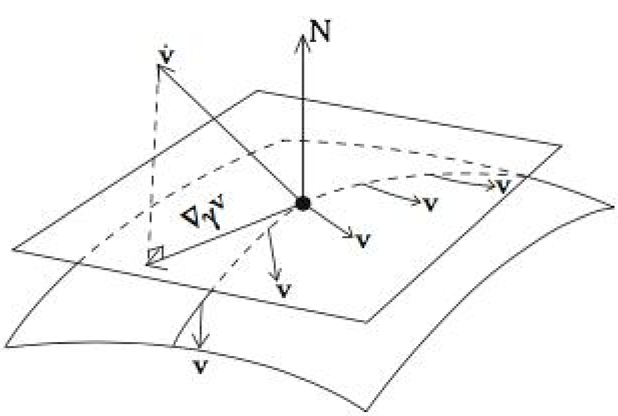
$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

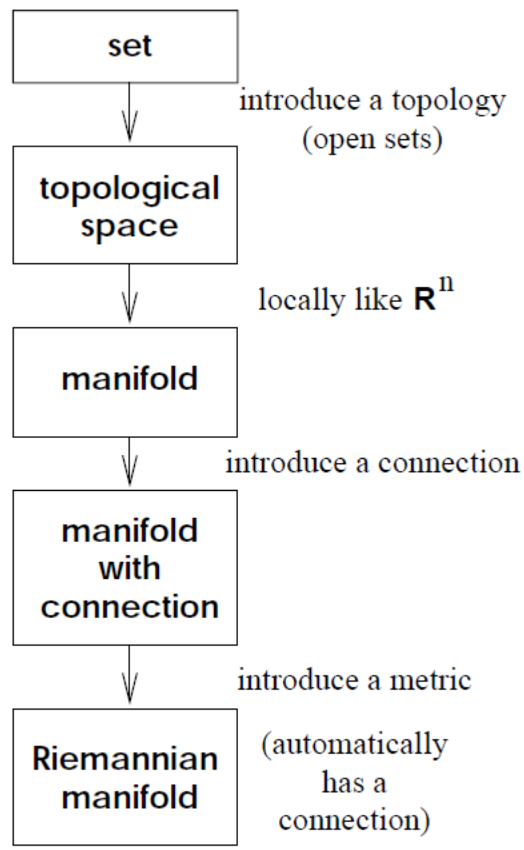
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

κ = curvature

τ = torsion

Covariant derivative along a curve γ





Appendix XI

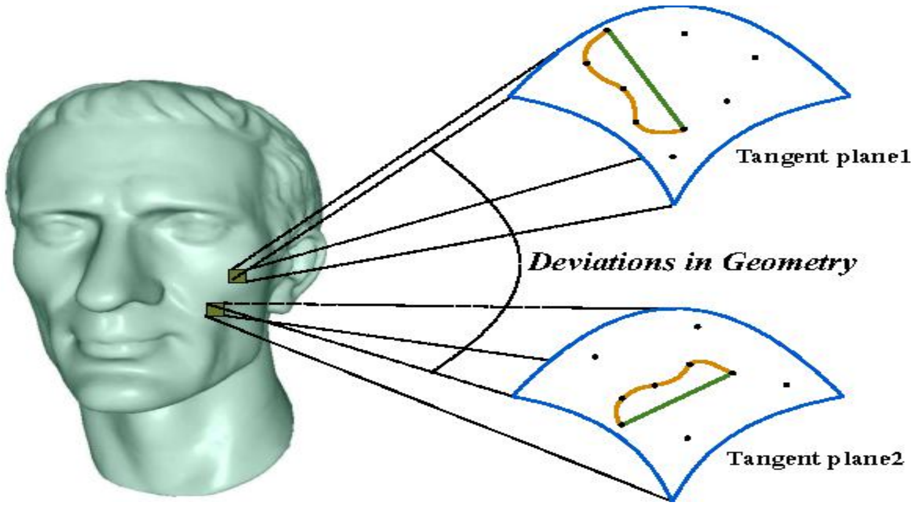


Fig. 4. The computation of Christoffel symbols for a 3D object is carried out on a pair of neighboring local tangent planes by computing the deviations in the metric tensor over the tangent planes.

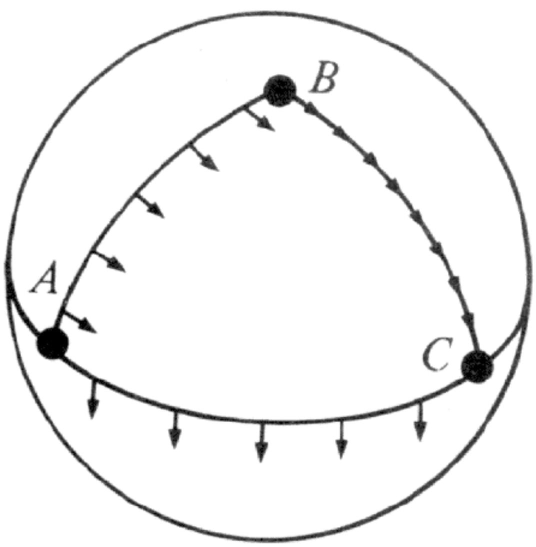
In:

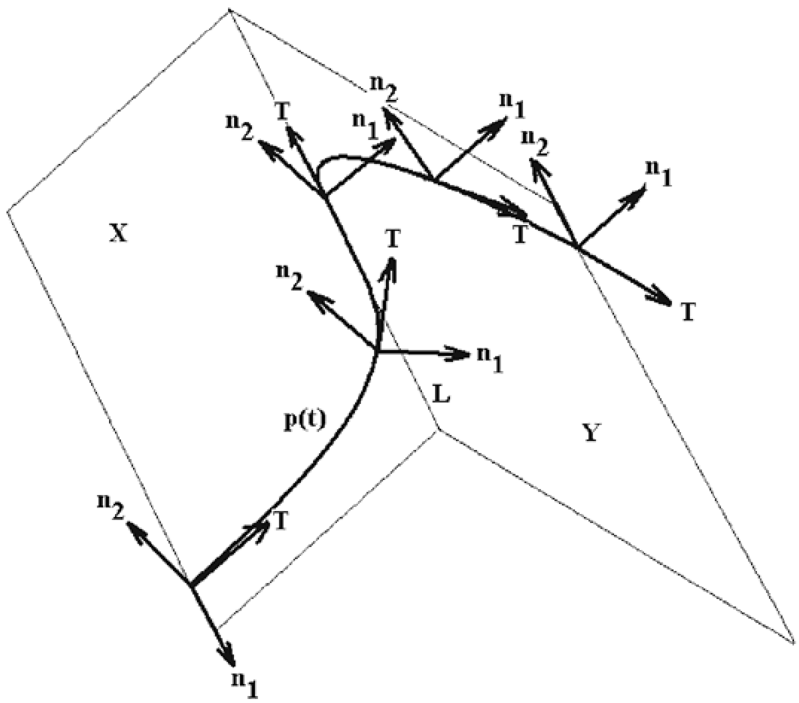
Metric tensor and Christoffel symbols based 3D object categorization

Syed Altaf Ganihar, Shreyas Joshi, Shankar Setty and Uma Mudenagudi

SIGGRAPH Posters, 2014

Parallel transport:

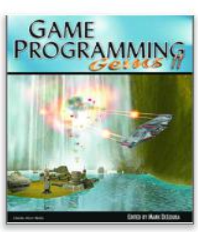




The Parallel Transport Frame

Authors: Carl Dougan

In book:



Game Programming Gems 2
Edited by Mark DeLoura

Charles River Media, 2001
ISBN 1-58450-054-9

See on Amazon

Pages: 215–219

Citation: Carl Dougan. “The Parallel Transport Frame”. In *Game Programming Gems 2*, Charles River Media, 2001, pp. 215–219.

Moving Path Following for Autonomous Robotic Vehicles

Tiago Oliveira[1] Pedro Encarnação[2] A. Pedro Aguiar[3]

***Abstract*— This paper introduces the moving path following (MPF) problem for autonomous robotic vehicles, in which the vehicle is required to converge to and follow a desired geometric moving path, without a specific temporal specification. This case generalizes the classical path following problem, where the given path is stationary. Possible tasks that can be formulated as a MPF problem include terrain/air vehicles target tracking and gas clouds monitoring, where the velocity of the target/cloud specifies the motion of the path. Using the concept of parallel-transport frame associated to the geometric path, we derive the MPF kinematic-error dynamics for 3D paths with arbitrary motion specified by its linear and angular velocity. An application is made to the problem of tracking a target on the ground using an Unmanned Aerial Vehicle. The control law is derived using Lyapunov methods. Formal convergence results are provided and hardware in the loop simulations demonstrate the effectiveness of the proposed method.**

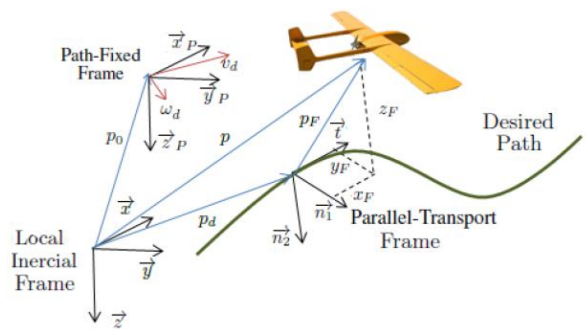
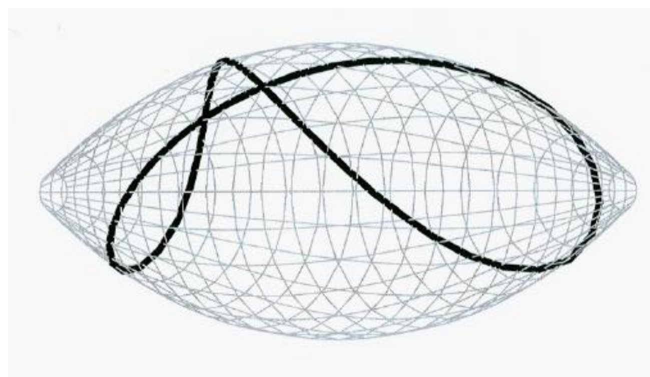
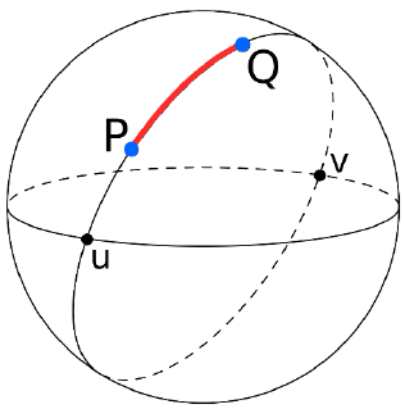


Fig. 1. Error space frames, illustrating for the case of an UAV.



Geodesics:

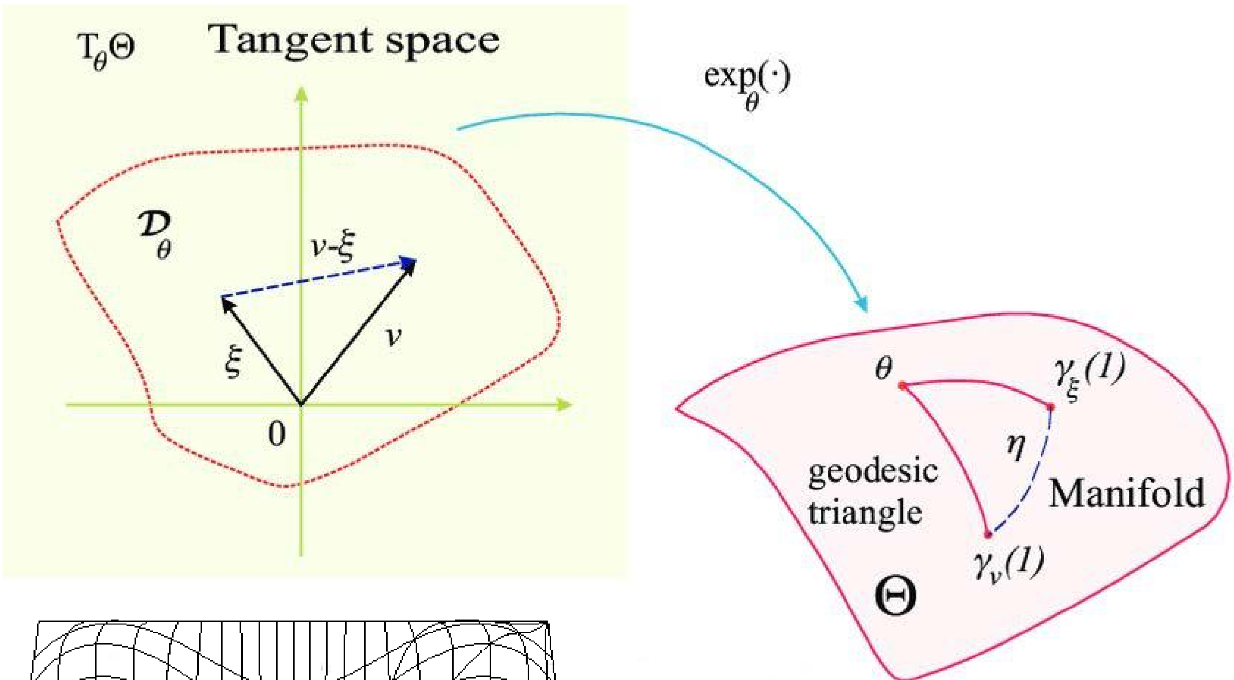


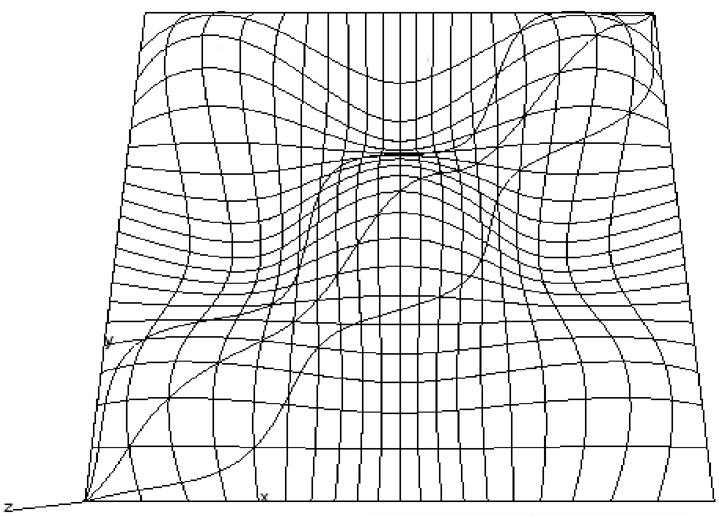
Geodesic equation:

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

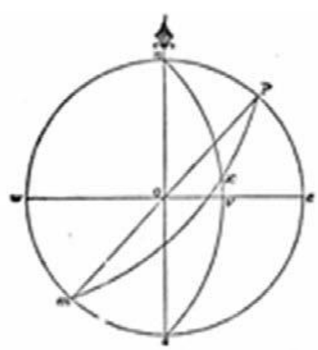
Appendix XI

The exponential map:

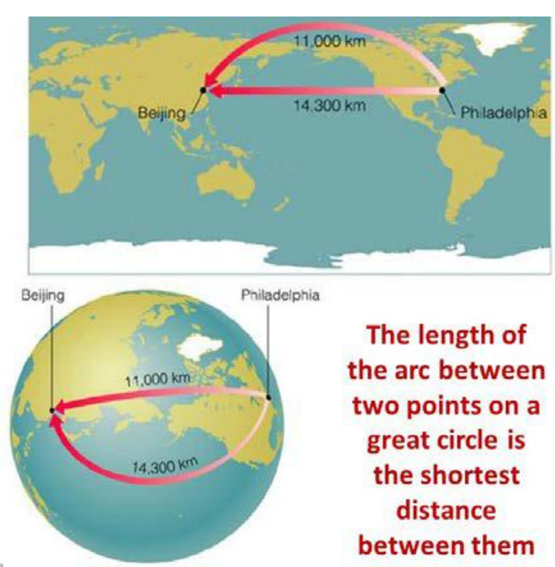




Geodesics: shortest paths



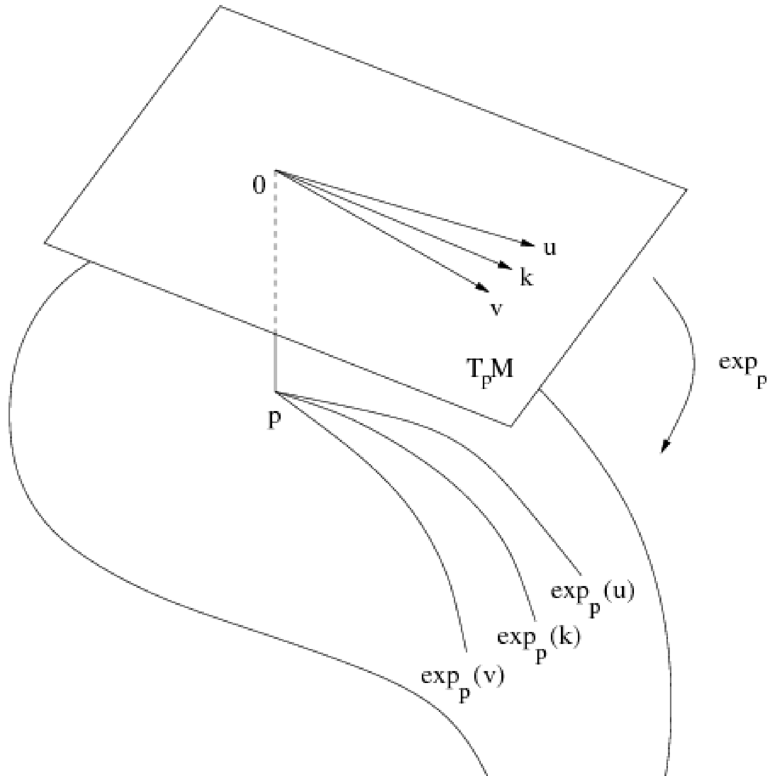
Great Circle is the circle on the surface of a sphere formed by intersecting with a plane passing through its Centre



The length of the arc between two points on a great circle is the shortest distance between them

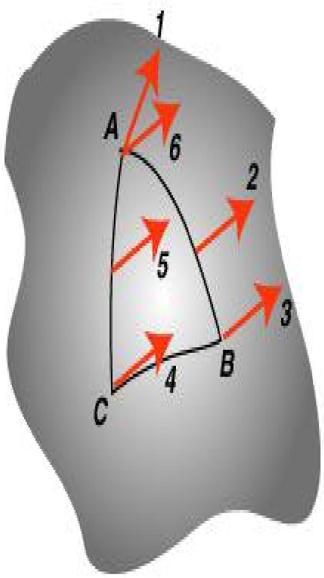
http://www.astro.umass.edu/~myun/teaching/a100/longlecture17.html

The exponential map



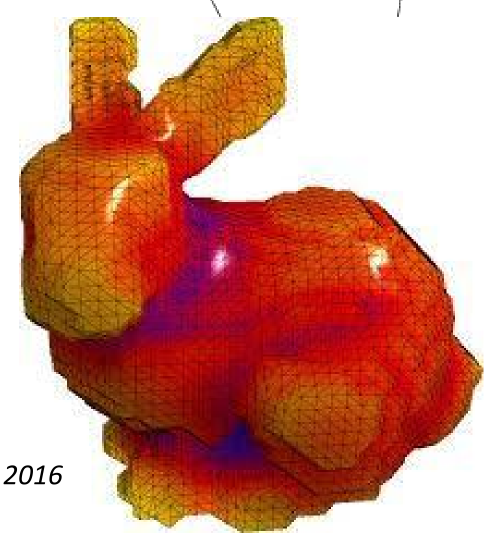
Curvature

HOW TO MEASURE THE CURVATURE BY THE INTRINSIC METHOD



Given a curved surface, make a parallel transport of an unit vector (red arrow) from A to B, C, and back to A again, along the triangle ABC. "Parallel" roughly means "Keep the same angle to the geodesic in question".

Thus, starting from 1, the vector comes back as 6. Notice the change of its direction! This change in comparison to the area of the triangle shows the curvature in this location (can be expressed in terms of Riemann curvature tensor). The curvature of a sphere, for example, can be recovered by this method.



Submitted: 2016

Interactive Curvature Tensor Visualization on Digital Surfaces*

Hélène Perrier[1], Jérémy Levallois[1,2], David Coeurjolly[1], Jean-Philippe Farrugia[1], Jean-Claude Iehl[1], and Jacques-Olivier Lachaud[2]

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LIRIS, UMR5205, F-69621, France
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LAMA, UMR5127, F-73776, France

Abstract. Interactive visualization is a very convenient tool to explore complex scientific data or to try different parameter settings for a given processing algorithm. In this article, we present a tool to efficiently analyze the curvature tensor on the boundary of potentially large and dynamic digital objects (mean and Gaussian curvatures, principal curvatures, principal directions and normal vector field). More precisely, we combine a fully parallel pipeline on GPU to extract an adaptive triangulated isosurface of the digital object, with a curvature tensor estimation at each surface point based on integral invariants. Integral invariants being parametrized by a given ball radius, our proposal allows to explore interactively different radii and thus select the appropriate scale at which the computation is performed and visualized.

Appendix XII

From Wikipedia: *Riemann curvature tensor*

Formally [edit]

When a vector in a Euclidean space is parallel transported around a loop, it will again point in the initial direction after returning to its original position. However, this property does not hold in the general case. The Riemann curvature tensor directly measures the failure of this in a general Riemannian manifold. This failure is known as the non-holonomy of the manifold.

Let x_t be a curve in a Riemannian manifold M. Denote by $\tau_{x_t} : T_{x_0}M \to T_{x_t}M$ the parallel transport map along x_t. The parallel transport maps are related to the covariant derivative by

$$\nabla_{\dot{x}_0} Y = \lim_{h\to 0} \frac{1}{h}\left(Y_{x_0} - \tau_{x_h}^{-1}(Y_{x_h})\right) = \frac{d}{dt}(\tau_{x_t} Y)\Big|_{t=0}$$

for each vector field Y defined along the curve.

Suppose that X and Y are a pair of commuting vector fields. Each of these fields generates a one-parameter group of diffeomorphisms in a neighborhood of x_0. Denote by τ_{tX} and τ_{tY}, respectively, the parallel transports along the flows of X and Y for time t. Parallel transport of a vector $Z \in T_{x_0}M$ around the quadrilateral with sides tY, sX, $-tY$, $-sX$ is given by

$$\tau_{sX}^{-1}\tau_{tY}^{-1}\tau_{sX}\tau_{tY}Z.$$

This measures the failure of parallel transport to return Z to its original position in the tangent space $T_{x_0}M$. Shrinking the loop by sending $s, t \to 0$ gives the infinitesimal description of this deviation:

$$\frac{d}{ds}\frac{d}{dt}\tau_{sX}^{-1}\tau_{tY}^{-1}\tau_{sX}\tau_{tY}Z\Big|_{s=t=0} = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z = R(X,Y)Z$$

where R is the Riemann curvature tensor.

From Wikipedia: *Constant curvature*

The Riemannian manifolds of constant curvature can be classified into the following three cases:

- elliptic geometry – constant positive sectional curvature
- Euclidean geometry – constant vanishing sectional curvature
- hyperbolic geometry – constant negative sectional curvature.

From Wikipedia: *Ricci curvature*

Direct geometric meaning [edit]

Near any point p in a Riemannian manifold (M,g), one can define preferred local coordinates, called geodesic normal coordinates. These are adapted to the metric so that geodesics through p correspond to straight lines through the origin, in such a manner that the geodesic distance from p corresponds to the Euclidean distance from the origin. In these coordinates, the metric tensor is well-approximated by the Euclidean metric, in the precise sense that

$$g_{ij} = \delta_{ij} + O\left(|x|^2\right).$$

In fact, by taking the Taylor expansion of the metric applied to a Jacobi field along a radial geodesic in the normal coordinate system, one has

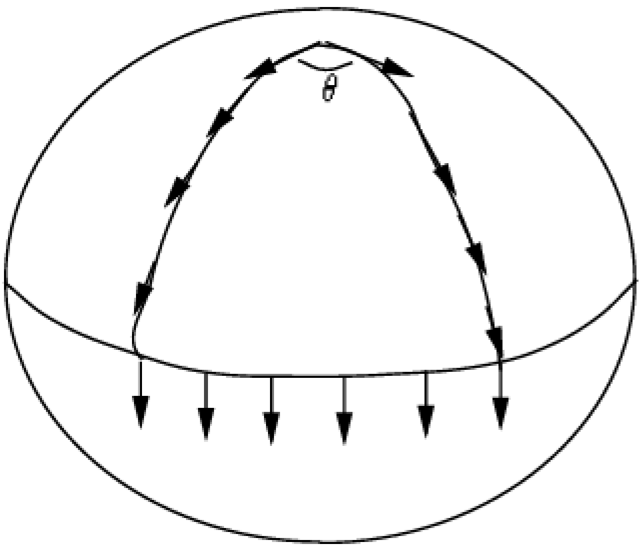
$$g_{ij} = \delta_{ij} - \tfrac{1}{3}R_{ikjl}x^k x^l + O\left(|x|^3\right).$$

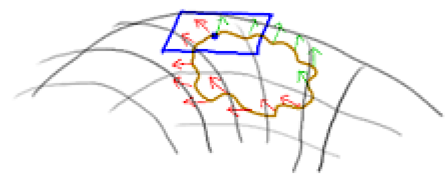
In these coordinates, the metric volume element then has the following expansion at p:

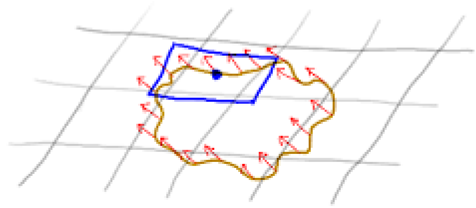
$$d\mu_g = \left[1 - \tfrac{1}{6}R_{jk}x^j x^k + O\left(|x|^3\right)\right] d\mu_{\text{Euclidean}},$$

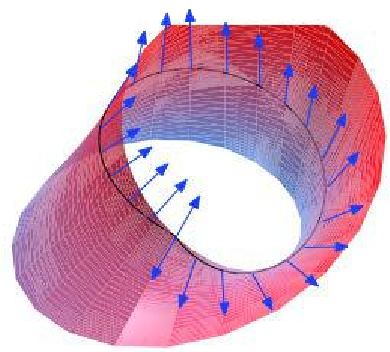
which follows by expanding the square root of the determinant of the metric.

Parallel transport and holonomy map:

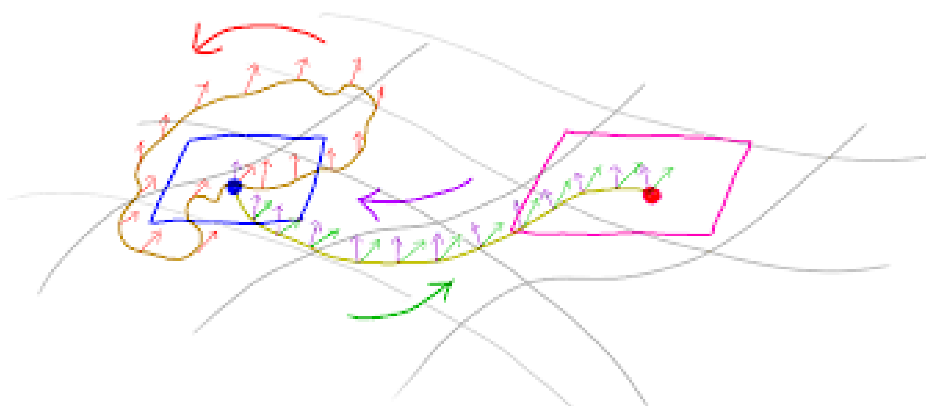








Independence with respect to initial point:



Only a few holonomy groups are possible:

Berger's classification

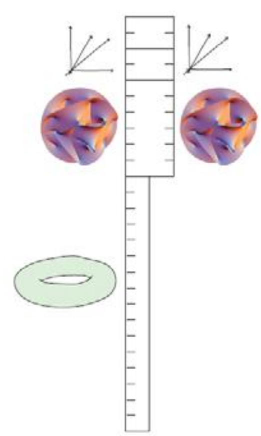
- Holonomy classification theorem by Berger (1955):

		Kähler	Calabi-Yau	Hyper-kähler	quat. Kähler	G_2-hol.	Spin(7)-hol.
	$\mathbf{SO(n)}$	$\mathbf{U(\frac{n}{2})}$	$\mathbf{SU(\frac{n}{2})}$	$\mathbf{Sp(\frac{n}{4})}$	$\mathbf{Sp(1)Sp(\frac{n}{4})}$	$\mathbf{G_2}$	$\mathbf{Spin(7)}$
$n = 2$:	✓	✓					
$n = 3$:	✓						
$n = 4$:	✓	✓	✓(2)				
$n = 5$:	✓						
$n = 6$:	✓	✓	✓(2)				
$n = 7$:	✓					✓(1)	
$n = 8$:	✓	✓	✓(2)	✓(3)	✓		✓(1)
$n = 4m + 1$:	✓						
$n = 4m + 2$:	✓	✓	✓(2)				for
$n = 4m + 3$:	✓						$m \geq 2$
$n = 4m + 4$:	✓	✓	✓(2)	✓$(m + 2)$	✓		

- Ricci-flatness (red cases) is related to the existence of globally defined, covariantly constant spinors. $\Rightarrow$ numbers in parenthesis

Special holonomy in compactification

- Classical usage: The equations for (partially) unbroken 4d SUSY after a Kaluza-Klein compactification reduce to the internal space admitting a parallel spinor, i.e. the internal space being Ricci-flat. (Candelas et al. 1985)
  $\rightsquigarrow$ Calabi-Yau compactification
- More recently (following the same basic idea):
  - G_2-compactification of 10d string theory
    $\Rightarrow$ 3d $N = 1$ toy models
  - G_2-compactification of 11d M-theory
    $\Rightarrow$ 4d $N = 1$ models
  - Spin(7)-compactification of 11d M-theory
    $\Rightarrow$ 3d $N = 1$ toy models



- However, spaces with special holonomy can also be used in higher dimensional gauge theory.