Squaring the Circle

A Case Study in the History
of Mathematics
Part III

Why?

By 1630 π had been calculated to 39 decimal places, yet even today this is far more accuracy than is needed in any application. So, what was driving people to continue this computation?

There are several "psycho-sociological" reasons for this, but there is also a mathematical question that could be answered by such a calculation.

It was not known whether or not π was a rational number (p/q with p and q integers, q not 0). If it was rational, then the decimal representation would eventually exhibit a repeating sequence. Furthermore, a rational π would mean that a quadrature of the circle is possible with straightedge and compass.

Calculation of π – Next Phase[1]

1671 – The Scottish mathematician James Gregory (1638-1675) obtained the infinite series:

$$\arctan x \ = \ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, (-1 \leqslant x \leqslant 1).$$

Gregory communicated this and other series to John Collins, secretary of the Royal Society, without any indication of how they were discovered. A study of Gregory's notes, found on the margins and other blank areas of letters he received, has led modern scholars to believe that Gregory used a version of the rule for Taylor's series first published over 40 years later.

These series, however, had been discovered in southern India perhaps 200 years earlier. They appear in Sanskrit verse in the Tantrasaṅgraha-vyākhyā (c. 1530). In another Indian work slightly later than this, credit for the arctan series is given to the mathematician Madhava (1340-1425).

Calculation of π – Next Phase

1671 – Not noted by Gregory was the fact that for x = 1, the series becomes:

$$\frac{\pi}{4} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

This very slowly converging series was known to Leibniz in 1674. It is not useful in calculating π.

1699   Abraham Sharp (1651 – 1742) however, used Gregory's arctangent series with

$$x \;=\; \sqrt{\frac{1}{3}}$$

giving π/6 to calculate π to 71 decimal places.

Calculation of π – Next Phase

1706 – John Machin (1680 – 1751) calculated π to 100 decimal places using the identity

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

and using Gregory's series to evaluate the arctangents.

1719 – The French mathematician T.F. De Lagny (1660 – 1734) repeated Sharp's computation with more diligence to obtain 112 correct places.

1767 – Johann Heinrich Lambert (1728 – 1777) proved that π is *irrational* by showing that if x is a non-zero rational number then tan x can not be rational. Thus, since tan $\pi/4 = 1$, $\pi/4$ and hence π can not be rational.

Irrationality of π

1794 – Adrien-Marie Legendre (1752 – 1833) proved that π^2 is irrational and gave a less satisfactory proof of the irrationality of π, than Lambert had*. Thus π is not the square root of a rational number either.

* In *Famous Problems of Elementary Geometry*, Felix Klein wrote

"After 1770 critical rigour gradually began to resume its rightful place. In this year appeared the work of Lambert: *Vorläufige Kenntnisse für die, so die Quadratur ... des Cirkulus suchen.* Among other matters the irrationality of π is discussed. In 1974 Legendre in his *Éléments de Géométrie* showed conclusively that π and π^2 are irrational numbers."[2]

The implication of this note is that Lambert did not discuss the irrationality of π conclusively and that Legendre did. However, both of these points of view are incorrect. Klein was simply reproducing the erroneous statements of the historian Rudio.

Irrationality of π^3

This error which appears in F. Rudio, *Archimedes, Huygens, Lambert, Legendre, vier Abhandlungen über die Kreismessung*. (Leipzig, 1892, p. 56f.), is also reproduced by B. Calò in Enriques's *Fragen der Elementargeometrie* (II. Teil, 1907, p. 315) and by D.E. Smith in Young's *Monographs on Topics of Modern Mathematics* (1911, p. 401).

    The matter was carefully studied by A. Pringsheim in "Über die ersten Beweise der Irrationalität von e und π" (Bayerische Akad. der Wissen,, *Sitzungsberichte*, mathem.-phys. Cl., v. 28, 1899, pp. 325-337). The work cited by Rudio contains some formulae without proof, and no analytical developments, and was rather intended to serve as a popular survey of the treatment of the topic. However, in Lambert's remarkable "Mémoire" of 1767, the proofs appear in minute detail, more precise than anything written in the next 50 years.

Irrationality of π

These results do not prove that the quadrature of the circle is impossible with straightedge and compass, but they do indicate that if such a construction is possible, it would be very complex.

To see why the quadrature is not ruled out, we need to examine the question of just what can be constructed with these instruments.
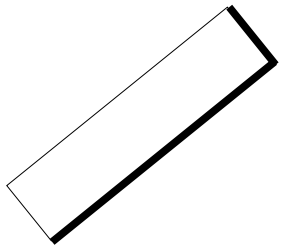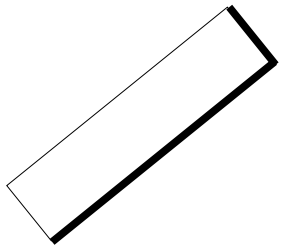
Straightedge and Compass
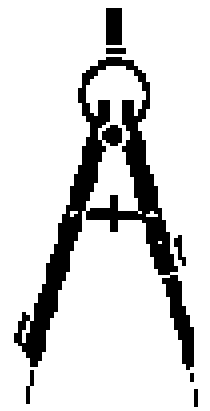
All constructions start with a given line segment (arbitrary but fixed) which is defined to be *one unit* in length. A real number α is ***constructible*** if one can construct a line segment of length $|\alpha|$ in a finite number of steps from the given unit segment by using a straightedge and compass.

Easy basic constructions show that if α and β are constructible then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$ and α/β when β is not 0. This implies that *the set of constructible numbers form a subfield of the field of real numbers* which contains the field of rational numbers, called the ***field of constructible numbers***.
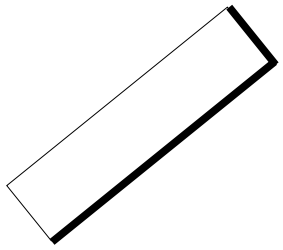
Straightedge and Compass
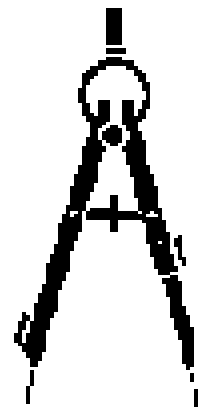
   The field of constructible numbers contains more than just the rationals. Using these tools, new points can be constructed only as 1) the intersection of two lines, 2) the intersection of a line and a circle or 3) the intersection of two circles. Algebraically, the second and third cases lead to quadratic equations whose solutions involve the square roots of constructible numbers.

   There is a standard compass and straightedge construction of the square root of a constructible number, and applying this to a non-square will produce an irrational constructible number.

The field of constructible numbers consists precisely of all real numbers which can be obtained from the rationals by taking square roots of positive numbers a finite number of times and applying a finite number of field operations.

Straightedge and Compass
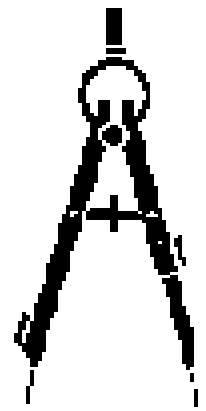
In modern field theory terms this is expressed as:

If α is a constructible number then
$$[\mathbb{Q}(\alpha):\mathbb{Q}] = 2^r \text{ for some integer } r \geq 0.$$

This implies that any constructible number must satisfy a polynomial equation with rational coefficients of degree 2^r.

More Calculations[4]

1841 – William Rutherford of England calculated π to 208 places, of which 152 were later found to be correct, using the identity:

$$\frac{\pi}{4} \; = \; 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{70}\right) + \arctan\left(\frac{1}{99}\right).$$

1844 – Zacharias Dase, the lightening calculator, found π correct to 200 places using the identity:

$$\frac{\pi}{4} \; = \; \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right).$$

Dase, born in Hamburg in 1824 was perhaps the most extraordinary mental calculator who ever lived. Among his performances were the mental calculation of the product of two 8 digit numbers in 54 seconds, two 20 digit numbers in 6 minutes, two 40 digit numbers in 40 minutes, and two 100 digit numbers in 8 hours and 45 min.

More Calculations

1853 – Rutherford returns to the problem and obtains 400 correct decimal places.

1873 – William Shanks of England, using Machin's formula, computed π to 707 places. In 1946, D.F. Ferguson of England discovered errors in Shanks' value starting in the 528th place. In January 1947 he gave a corrected value to 710 places.

1948 – In Jan. 1947, J.W. Wrench, Jr. of America, published an 808 place value, but Ferguson soon found an error in the 723rd place. In Jan. 1948, Ferguson and Wrench jointly published the corrected and checked value of π to 808 places.

The Transcendence of π

A real number which satisfies (is a root of, is a zero of) a polynomial with rational coefficients is called an ***algebraic number***.

  The set of all algebraic real numbers has cardinality $\aleph_0$, which means that almost all real numbers are not algebraic. Numbers which are not algebraic are called ***transcendental numbers***.

  As we have seen, constructible numbers are algebraic numbers (but not all algebraic numbers are constructible), so the death knell of the quadrature problem will come if it can be shown that π is transcendental (actually, we want $\sqrt{\pi}$ to be transcendental, but if a number is algebraic then so is its square, so showing that π is transcendental will suffice.)

The Transcendence of π

In 1882, F. Lindemann (1852 – 1939) proved that π is transcendental. His proof closely followed the proof of C. Hermite (1822 – 1901) of the transcendence of e in 1873.

Many mathematical historians consider Hermite's work to be the more difficult and are saddened by the fact that Hermite didn't see the trick that permitted Lindemann to get the more famous result.[5]

Hermite's result, without the details, is proved as follows:[6]
  Assume that e is algebraic. Then there exists a polynomial equation that e satisfies, i.e.,

$$C_0 + C_1 e + C_2 e^2 + ... + C_n e^n = 0$$ where the C_i's are rational.

By multiplying through by the common denominator of the coefficients, we can assume wlog that the coefficients are integers.

The Transcendence of π

Multiply the equation through by a constant integer M and then break each term of the sum into an integral and a fractional part. If the sum of the integral parts is the integer M' and the sum of the fractional parts is F, the equation becomes M' + F = 0. What Hermite proves is that you can find an M so that 1) M' is not zero (shown by finding a prime which does not divide it evenly) and 2) F is as small as you like. With this M, the equation M' + F = 0 can clearly not have a solution, a contradiction.

What Lindemann did was to extend this technique and show that e could not satisfy an equation of the form

$$C_0 + C_1 x^{a_1} + C_2 x^{a_2} + \cdots + C_n x^{a_n} = 0$$

Where the C_i are algebraic numbers and the a_i are algebraic numbers which could be complex as well.

The Transcendence of π

Then, using the famous relation proved by Euler,
$$1 + e^{i\pi} = 0,$$
he concluded that $i\pi$ could not be algebraic, since the coefficients are algebraic. Since squares and square roots of algebraic numbers are algebraic, this implies that π is not algebraic.

Circle Squarers

"We have not placed in the above chronology of π any items from the vast literature supplied by sufferers of **morbus cyclometricus**, the circle-squaring disease. These contributions, often amusing and at times almost unbelievable, would require a publication all to themselves."

- Howard Eves, *An Introduction to the History of Mathematics*[7]

Circle squarers, angle trisectors, and cube duplicators are members of a curious social phenomenon that has plagued mathematicians since the earliest days of the science. They are generally older gentlemen who are mathematical amateurs (although some have had mathematical training) that upon hearing that something is impossible are driven by some inner compulsion to prove the authorities wrong.

Circle Squarers

In 1872, Augustus De Morgan's (1806-1871) widow edited and had published some notes that De Morgan had been preparing for a book, called *A Budget of Paradoxes*. A logician and teacher, De Morgan had been the first chair in mathematics of London University (from 1828). Besides his mathematical work, he wrote many reviews and expository articles and much on teaching mathematics. In the *Budget*, he examines his personal library and satirically barbs all the examples of weird and crackpot theories that he finds there. As he points out, these are just books that randomly came into his possession – he did not seek out any of this type of material. In the approximately 150 works he examined, there can be found 24 circle squarers and an additional 19 bogus values of π.[8]

Circle Squarers[9]

There are several characteristics of circle squarers (shared by others of their ilk) that may help you identify them.

1. *They are men*. Almost universally. Women seem to have more sense.

2. *They are old*. Often retired, having led a successful life in their chosen endeavors. Too much free time.

3. *They fail to understand what "impossible" means in mathematics*. The meaning is unfortunately not the same as the meaning in English. It is one of great failures of mathematics education that this essential difference is not made plain to students.

4. *They do not know much mathematics*. Often, high school is the last place they seen any formal mathematics.

Circle Squarers

5. *They think the problem is important*. Since Archimedes work, there has not been any need for such a construction, yet they persist in thinking that mathematics has been stymied by this lack.

6. *They believe that they will be richly rewarded for their work*. No one has ever put up a prize for a solution.

"An agricultural laborer squared the circle, and brought the proceeds to London. He left his papers with me, one of which was the copy of a letter to the Lord Chancellor, desiring his Lordship to hand over forthwith 100,000 pounds, the amount of the alleged offer of reward. He did not go quite as far as M. de Vausenville, who, I think in 1778, brought an action against the Academy of Sciences to recover a reward to which he held himself entitled."

- De Morgan, *A Budget of Paradoxes*, Vol I, pg. 12

Circle Squarers

7. *They are not logical*. For instance,

 "Mr. Smith's method of proving that every circle is 3 1/8 diameters is to assume that it is so, - 'if you dislike the term datum, then by hypothesis, let 8 circumferences be exactly equal to 25 diameters,' - and then to show that every other supposition is thereby made absurd.

 'I think you will not dare to dispute my right to this hypothesis, when I can prove by means of it that every other value of π will lead to the grossest absurdities; unless, indeed, you are prepared to dispute the right of Euclid hypothetically for the purpose of a '*reductio ad absurdum*' demonstration, in pure geometry.' "
                    - *Budget*, Vol 2, pg. 117.

Circle Squarers

8. *They are loners*. They work by themselves, sometimes using books, but never discuss their work until it is completed. Even though they do not communicate with each other, they do tend to swarm.

For instance, in 1754, Jean Étienne Montucla, an early French historian of mathematics, wrote a legitimate history of the quadrature problem. A year later, the French Academy of Sciences was forced to publicly announce that it would no longer examine any solutions of the quadrature problem.[10]

Circle Squarers

9. *They are prolific writers*. Here is what De Morgan says about Milan whose method gave $\pi = 3.2$ in 1855:

[The circle-squarer] is active and able, with nothing wrong with him except his paradoxes. In the second tract named he has given the testimonials of crowned heads and ministers, etc. as follows. Louis Napoleon gives thanks. The minister at Turin refers it to the Academy of Sciences and hopes so much labor will be judged worthy of esteem. The Vice-Chancellor of Oxford – a blunt Englishman – begs to say that the University has never proposed the problem, as some affirm. The Prince Regent of Baden has received the work with lively interest. The Academy of Vienna is not in a position to enter into the question. The Academy of Turin offers the most *distinct* thanks. The Academy della Crusca attends only to literature, but gives thanks. The Queen of Spain has received the work with the highest appreciation. The University of Salamanca gives infinite thanks, and feels true satisfaction in having the book.

Circle Squarers

Lord Palmerston gives thanks. The Viceroy of Egypt, not yet being up in Italian, will spend his first moments of leisure in studying the book, when it shall have been translated into French: in the mean time he congratulates the author upon his victory over a problem so long held insoluble. All this is seriously published as a rate in aid of demonstration. If those royal compliments cannot make the circumference about 2 per cent larger than geometry will have it – which is all that is wanted – no wonder that thrones are shaky.
- *Budget*, Vol. 2, pp. 61-2.

Circle Squarers

Now, will you know a Circle Squarer when you see one coming? And will you know what to do?

Hint: What you do involves your legs.

No, you do **not** kick him![11]

Legislating π^{12}

And then there is the mathematical equivalent of an urban legend that House Bill No. 246 of the Indiana State Legislature in 1897 tried to legislate the value of π.

It is true that the Indiana House of Representatives did pass this bill with no dissenting votes, but the bill only gave the state the privilege of using the "proper" value of π for free!

After passing the House, the bill went to the Senate for consideration. Even though senators were and remained ignorant about what the value of π was, enough fuss was made in the press that they, being politicians, sensed that the bill should not pass. Action on the bill by the Senate was tabled indefinitely, and has remained so for over one hundred years.

House Bill No. 246

House Bill No. 246, Indiana, 1897

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897.

Section 1. Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is entirely wrong, as it represents the circle's area one and one-fifth times the area of a square whose perimeter is equal to the circumference of the circle. This is because one-fifth of the diameter fails to be represented four times in the circle's circumference. For example: if we multiply the perimeter of a square by one-fourth of any line one-fifth greater than one side, we can in like manner make the square's area to appear one fifth greater than the fact, as is done by taking the diameter for the linear unit instead of the quadrant of the circle's circumference.

House Bill No. 246

But why was this bill ever considered? A hint lies in the last section of the bill.

and because of these facts and the further fact that the rule in present use fails to work both ways, mathematically, it should be discarded as wholly wanting and misleading in its practical applications.

Section 3. In further proof of the value of the author's proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country. And be it remembered that these noted problems had been long since given up by scientific bodies as unsolvable mysteries and above man's ability to comprehend.

Ahah! The author was a circle squarer.

The claim of acceptance that he makes is a bit of a stretch. An article of his did appear in the first volume of the American Mathematical Monthly, but ...

The Monthly Article

QUADRATURE OF THE CIRCLE
By Edward J. Goodwin, Solitude, Indiana
Published by the request of the author.

A circular area is equal to the square on a line equal to the quadrant of the circumference; and the area of a square is equal to the area of the circle whose circumference is equal to the perimeter of the square.

To quadrate the circle is to find the side of a square whose perimeter equals that of the given circle; rectification of the circle requires to find a right line equal to the circumference of the given circle. The square of a line equal to the arc of 90° fulfills both of the said requirements.

This was published in the "Queries and Information" section where all sorts of miscellania could be found. The editors of the fledgling *Monthly* probably needed material to fill space and thought this piece might provide entertainment for its readers.

Goodwin's π

It is a little difficult to determine what value Goodwin thought π had since his writing was so confused and confusing. From the opening lines of the Monthly article we get $\pi = 4$, and from a later section we can obtain $\pi = 3.232488...$.

David Singmaster succeeded in finding *nine* different values of π in Goodwin's bill.

Calculating π – The Computer Age[13]

The 808 places of π calculated by Ferguson and Wrench in 1948 was the last hand calculation ever attempted since in ...

1949 – The electronic computer ENIAC, at the Army Ballistic Research Laboratories in Aberdeen, Maryland, calculated π to 2037 decimal places.

1959 – François Genuys, in Paris, computed π to 16,167 places using an IBM 704.

1961 – Wrench and Daniel Shanks, of Washington, D.C., computed π to 100,265 places using an IBM 7090.

Calculating π – The Computer Age

1966 – M. Jean Guilloud and his co-workers at the Commisariat à l'Énergie Atomique in Paris obtained π to 250,000 places on a STRETCH (IBM) computer.

1967 – Exactly one year later, the above workers found 500,000 places on a CDC 6600.

1973 – Guilloud and his co-workers found 1,000,000 places using a CDC 7600.

1981 – The Japanese mathematicians Kazunori Miyoshi and Kazuhika Nakayama of the University of Tsukuba calculated 2,000,038 places in 137.30 hours on a FACOM M-200 computer.

Calculating π – The Computer Age

And ...

1986 – In January, D.H. Bailey of the NASA Ames Research Center in California ran a Cray-2 supercomputer for 28 hours to get 29,360,000 digits. His code was based on an algorithm by the J.M. and P.D. Borwein of Dalhousie University. Bailey checked his code against a slower algorithm, also developed by the Borweins, and verified the accuracy of his result.

A little later, Yasumasa Kanada of the University of Tokyo, using a NEC SX-2 supercomputer and the Borweins' algorithm computed 134,217,700 digits.

Where is π now?

And in 1989 the Chudnovsky brothers David and Gregory (Columbia University) entered the game with new faster algorithms based on ideas of Ramanujan and the race began.

Where is π now?[14]

Authors	Year	Exact digits	Computer
Chudnovskys	05-1989	480 000 000	CRAY-2
Chudnovskys	06-1989	525 229 270	IBM 3090
Kanada & Tamura	07-1989	536 870 898	Hitachi S-820/80
Chudnovskys	08-1989	1 011 196 691	IBM 3090 & CRAY-2
Kanada & Tamura	11-1989	1 073 741 799	Hitachi S-820/80
Chudnovskys	08-1991	2 260 000 000	m-zero
Chudnovskys	05-1994	4 044 000 000	m-zero
Kanada & Takahashi	06-1995	3 221 220 000	Hitachi S-3800/480
Kanada & Takahashi	08-1995	4 294 967 286	Hitachi S-3800/480
Kanada & Takahashi	10-1995	6 442 450 000	Hitachi S-3800/480
Chudnovskys	03-1996	8 000 000 000	m-zero ?
Kanada & Takahashi	04-1997	17 179 869 142	Hitachi SR2201
Kanada & Takahashi	06-1997	51 539 600 000	Hitachi SR2201
Kanada & Takahashi	04-1999	68 719 470 000	Hitachi SR8000
Kanada & Takahashi	09-1999	206 158 430 000	Hitachi SR8000
Kanada et al.	12-2002	**1 241 100 000 000**	Hitachi SR8000/MP