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PLUS SOLUTIONS TO ALL PROBLEMS

CHAPTER 1 VECTOR DERIVATIVES

SECTION 1.0 REVIEW

components of a vector

A vector is an arrow.

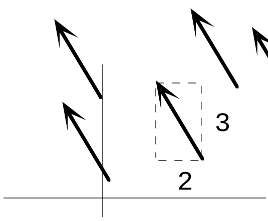
Every vector in 2−space has a pair of components giving "over" and "up" from tail to head. Similarly every vector in 3−space has 3 components.

Fig 1 shows several vectors all with components $(-2,3)$.

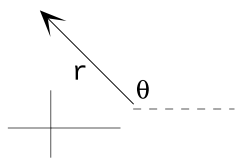
(1) In 2−space, the vector with length r and angle of inclination θ has components $(r\cos\theta,\ r\sin\theta)$ (Fig 2)

(2) In particular, the vector with length 1 and inclined at angle θ has components $(\cos\theta,\ \sin\theta)$

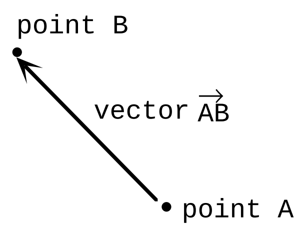
The vectors (0,0) and (0,0,0) are thought of as arrows with zero length and arbitrary direction and are called zero vectors. Both are denoted by $\vec{0}$.



vectors $(-2,3)$
FIG 1



vector $(r\cos\theta,\ r\sin\theta)$
FIG 2



vector $\overrightarrow{AB} = B - A$
FIG 3

Here's how to get the components of a vector when you know its head and tail:

vector components = head coordinates − tail coordinates

vector $\overrightarrow{AB}$ = point B − point A (Fig 3)

For example the vector with tail at (3,5,1) and head at (2,1,5) has components $(-1,-4,4)$

addition, subtraction and opposites of vectors

Vector addition, subtraction and oppositing is done componentwise: if

$$\vec{p} = (-3,5,6) \text{ and } \vec{q} = (6,7,2)$$

then

$$\vec{p} + \vec{q} = (3,12,8)$$

$$\vec{p} - \vec{q} = (-9,-2,\ 4)$$

$$-\vec{p} = (3,\ -5,\ -6)$$

Fig 4 shows some pictures of u,v and the corresponding u+v, u−v and −u.

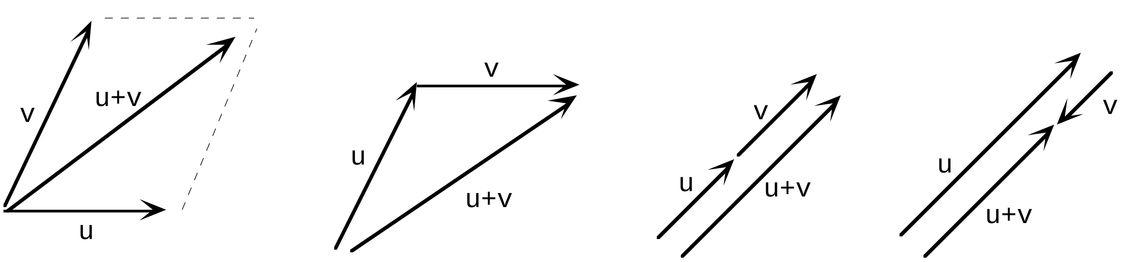


FIG 4

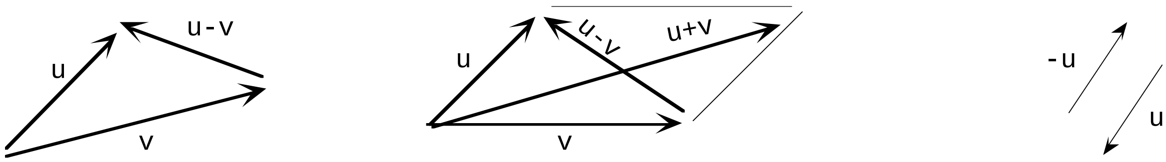


FIG 4 continued

scalar multiplication

To multiply a vector by a scalar (i.e., by a number), multiply all its components by the scalar.

For example, if $u = (-3,4,5)$ then $-2u = (6,-8,-10)$.

The vectors $\vec{cu}$ and $\vec{u}$ are parallel.
They point the same way if $c > 0$ and opposite ways if $c < 0$
The length of cu is $|c|$ times the length of u.

For example, $-2u$ is twice as long as u and points in the opposite direction; $3u$ is three times as long as u and points like u (Fig 5).

To decide if two vectors are parallel, see if one is a multiple of the other.

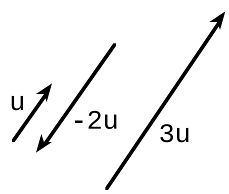


FIG 5

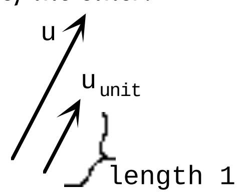


FIG 6

norms (magnitudes)

The norm of a vector is the length of the arrow.

If $u = (u_1,u_2,u_3)$ then $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

For example, if $u = (5,6)$ then $\|u\| = \sqrt{25 + 36} = \sqrt{61}$.

Here are some properties of norms.

(a) $\|cu\| = |c|\,\|u\|$ (the length of the vector cu is $|c|$ times the length of u)

(b) $\|u\| = 0$ if and only if $u = \vec{0}$. Otherwise, $\|u\| > 0$.

unit vectors

The vector $\dfrac{u}{\|u\|}$ is a unit vector (i.e., norm 1) which points like u.

I'll write it as u_{unit} (it's called the *normalized* u) (Fig 6). In other words,

$$u_{\text{unit}} = \frac{u}{\|u\|} = \left(\frac{u_1}{\|u\|}, \frac{u_2}{\|u\|}, \frac{u_3}{\|u\|} \right) = \text{vector with length 1 which points like } u$$

For example, if $u = (4,5)$ then $\|u\| = \sqrt{41}$ and $u_{\text{unit}} = \left(\dfrac{4}{\sqrt{41}}, \dfrac{5}{\sqrt{41}} \right)$ (Fig 6).

getting a vector with a given length and given direction

Suppose u has length 3 and the same direction as v. Then $u = 3v_{\text{unit}}$ since tripling the *unit* vector v_{unit} produces a vector with length 3, still pointing like v.

(3) $\boxed{\text{The vector which points like v and has length } \ell \text{ is } \ell \mathbf{v}_{\text{unit}}\text{, i.e., } \ell \dfrac{v}{\|v\|}}$

If u is supposed to have length 4 and the same direction as w = (1,3,2) then

$$u = 4w_{\text{unit}} = \left(\frac{4}{\sqrt{14}}, \frac{12}{\sqrt{14}}, \frac{8}{\sqrt{14}}\right)$$

the vectors i,j,k

The vector (1,0,0) is called i and is pictured as an arrow along the x-axis; the vector (0,1,0) is called j and (0,0,1) is called k (Fig 7)

In 2-space, i = (1,0) and j = (0,1).

It's easy to write vectors in terms of i,j,k. For example, the vector (2,3,4) can be written as 2i + 3j + 4k.

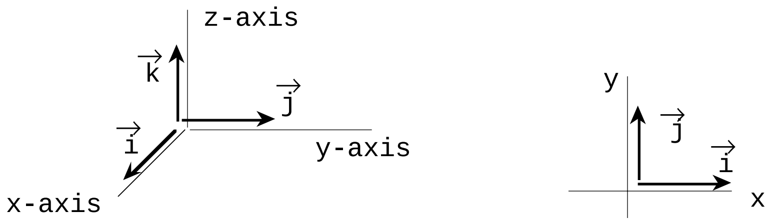


FIG 7

dot products

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

For example, if

$$u = 2i + 3j - 4k \text{ and } v = 5i - 3j + 2k$$

then

$$u \cdot v = 10 - 9 - 8 = -7$$

warning The dot product is the *number* −7, not the *vector* 10i − 9j − 8k

If θ is the angle between vectors u and v drawn with the same tail (Fig 8) then

$$\cos\theta = \frac{u \cdot v}{\|u\| \; \|v\|}$$

$u \cdot v$ is positive iff $0^\circ \le \theta < 90^\circ$

$u \cdot v$ is negative iff $90^\circ < \theta \le 180^\circ$

$\boxed{u \cdot v = 0 \text{ if and only if u and v are perpendicular}}$

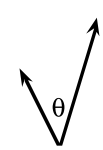


FIG 8

Here are some properties of the dot product:

(a) $u \cdot v = v \cdot u$

(b) $u \cdot (v + w) = u \cdot v + u \cdot w$

(c) $(u + v) \cdot (p + q) = u \cdot p + v \cdot p + u \cdot q + v \cdot q$

(d) $u \cdot \vec{0} = 0$

(e) $u \cdot u = \|u\|^2$

(f) $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$

(g) If u is a unit vector then $u \cdot u = 1$

(h) $u \cdot u = 0$ iff $u = \vec{0}$

warning

If u is a vector, there is no such thing as u^2. There's a dot product $u\cdot u$, and a norm $\|u\|$ and a norm squared, $\|u\|^2$, but nothing is denoted u^2.

left turns and right turns in two-space

If $\vec{u} = a\vec{i} + b\vec{j}$ then (Fig 9a) $u_{\text{left turn}} = -b\vec{i} + a\vec{j}$ and $u_{\text{right turn}} = b\vec{i} - a\vec{j}$.

Each of the vectors $u_{\text{left turn}}$ and $u_{\text{right turn}}$ is perp to u because $u\cdot u_{\text{left turn}} = 0$ and $u\cdot u_{\text{right turn}} = 0$.

For example, if $u = -2i + 4j$ then $u_{\text{left turn}} = -4i - 2j$ and $u_{\text{right turn}} = 4i + 2j$.

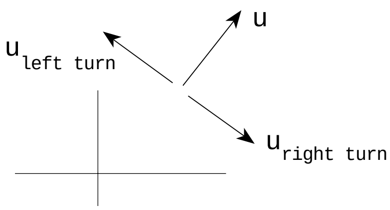
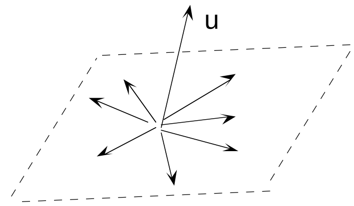


FIG 9a



can't pin down a left or right turn from u

FIG 9b

warning

There is no such thing as a left turn or right turn in *three*-space (Fig 9b).

the component of $\vec{u}$ in a direction (signed projections)

If u and v are drawn with a common tail then *the component of u in the direction of v* is the "signed projection" of u onto a line through v. It's positive if the angle between u and v is acute and it's negative if the angle is obtuse.

In Fig 10(a), the component of u in the direction of v is 6.

In Fig 10(b), the component of u in the direction of v is -7 (negative because u and v form an obtuse angle, i.e., because u is pulling against rather than with v).

Here's the formula for it.

(4)

> The component of u in the direction of v is $\dfrac{u\cdot v}{\|v\|}$.
>
> If v is a *unit* vector, the component of u in the direction v is just $u\cdot v$.

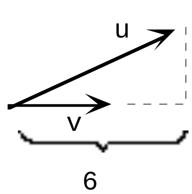


FIG 10(a)

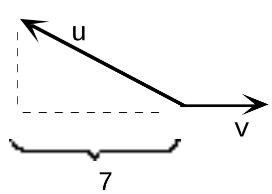


FIG 10(b)

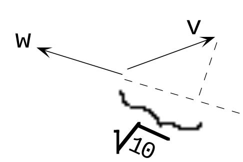


FIG 11

As a special case, the component of u in the direction of u itself is $\|u\|$.

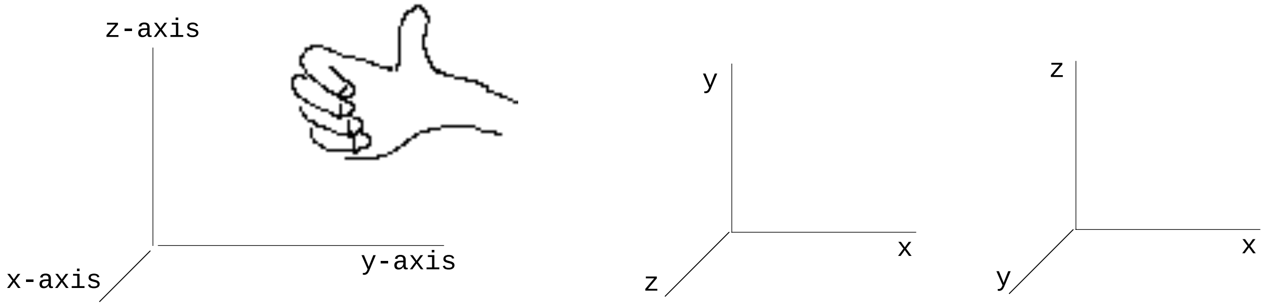
For example, if $v = (-5,5,1)$ and $w = (3,1,0)$ then

$$\text{component of v in direction of w} = \frac{v\cdot w}{\|w\|} = \frac{-10}{\sqrt{10}} = -\sqrt{10} \quad \text{(Fig 11)}$$

righthanded coordinate systems

Each coordinate system in 3-space is either righthanded or lefthanded (Fig 12) as follows: Hold your right hand so that your fingers curl from the positive x-axis toward the positive y-axis. If your thumb points in the direction of the positive z-axis then the system is righthanded. Otherwise the system is lefthanded.

We always use righthanded coordinate systems.



righthanded　　　righthanded　　　lefthanded

FIG 12

the cross product

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then

$$(5) \qquad \vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

This is how I remember the formula:

remember the minus

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} \cancel{u_1} & u_2 & u_3 \\ \cancel{v_1} & v_2 & v_3 \end{vmatrix}, \; -\begin{vmatrix} u_1 & \cancel{u_2} & u_3 \\ v_1 & \cancel{v_2} & v_3 \end{vmatrix}, \; \begin{vmatrix} u_1 & u_2 & \cancel{u_3} \\ v_1 & v_2 & \cancel{v_3} \end{vmatrix} \right)$$

You can also remember it like this:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

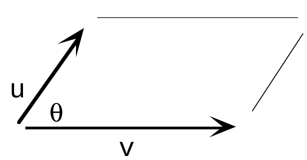
Cross products are defined for 3-dim vectors only but a 2-dim vector such as (2,3) can be considered to be (2,3,0) to make it crossable.

For example if $u = (3,1,2)$ and $v = (-4,5,6)$ then $u \times v = (-4, -26, 19)$.

Here are some properties of the cross product.

(a) $\|u \times v\| = \|u\|\,\|v\| \sin\theta$ where θ is the angle between u and v

$= $ area of parallelogram determined by $\vec{u}$ and $\vec{v}$ (Fig 13)



area of parallelogram is $\|u \times v\|$

FIG 13

(b) $u \times v = \vec{0}$ iff u and v are parallel
(But the best way to decide if two vectors are parallel is just to look at them and see if one is a multiple of the other -- page 2.)

(c) $u \times u = \vec{0}$ (the cross product of a vector with itself is the zero vector)
(d) $u \times v = -(v \times u)$

(e) $u \times \vec{0} = \vec{0} \times u = \vec{0}$
(f) $u \times (v + w) = u \times v + u \times w$

(g) $\vec{u} \times c\vec{v} = c\vec{u} \times \vec{v} = c(\vec{u} \times \vec{v})$

(h) $u \times v$ is perp to u and to v
But there are two directions perp to two given vectors u and v so this hasn't determined the direction of the cross product yet.
Which of the two directions the cross product takes depends on whether you are drawing in a righthanded or lefthanded coord system.
We always use righthanded coordinate systems in which case here is the rule:
$u \times v$ points like your thumb if the fingers of your right hand curl from u to v (Fig 14).

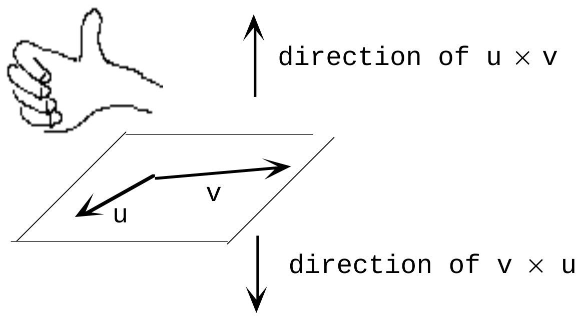


FIG 14

lines and planes in 3-space

The line through point (x_0,y_0,z_0) and parallel to the vector $a\vec{i} + b\vec{j} + c\vec{k}$ (Fig 15) has parametric equations

$$x = x_0 + at$$
$$y = y_0 + bt \qquad (6)$$
$$z = z_0 + ct$$

The plane through point (x_0,y_0,z_0) and perpendicular to the vector $a\vec{i} + b\vec{j} + c\vec{k}$ (Fig 16) has equation

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0. \qquad (7)$$

The graph of $ax + by + cz = d$ is a plane with normal vector $ai + bj + ck$.

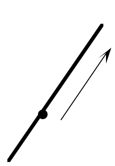


FIG 15

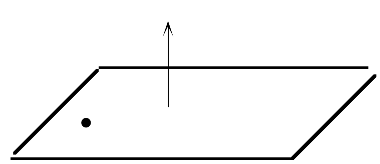


FIG 16

cylinders in 3-space (graphing equations in 3-space that are missing the letter z)

The graph of the equation $xy = 10$ in 2–space is a hyperbola (two branches) (Fig 17).

To get the graph of $xy = 10$ in *three*–space, first add a z-axis (Fig 18). A point such as (5, 2) on the hyperbola is now (5,2,0) and still satisfies the equation $xy = 10$. If you move that point up or down (i.e., in the z direction) its x and y coordinates don't change so the point still satisfies the equation $xy = 10$. The graph of $xy = 10$ is the surface swept out by moving the whole hyperbola up and down (Fig 18). The surface is called a cylindrical surface, a hyperbolic cylinder in particular.

The trick for drawing the cylinder $xy = 10$ in Fig 18 is to first draw the hyperbola $xy = 10$ in the x,y plane. Then copy it above (and below if you have room) and connect the copies and the original with suggestive lines parallel to the z–axis.

Similarly for equations missing the letter x. Or missing y. The graph is a cylindrical surface.

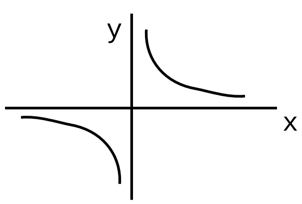


FIG 17

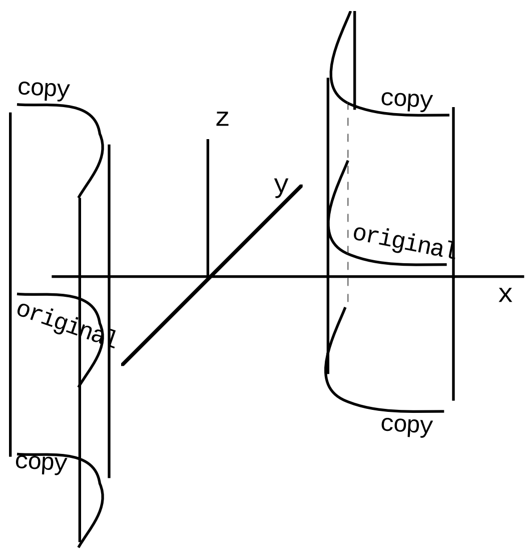


FIG 18

For example the plane in Fig 19 is a plane *cylinder* (the dotted lines are for perspective only). It can be traced out by moving the line AB forward and back.

The line considered in 2-space (the y,z plane) has equation $z = -4y + 8$

So the plane in 3-space has equation $z = -4y + 8$.

(The line considered in *three*-space has equations $z = -4y + 8$, $x = 0$ or equivalently has parametric equations $x = 0$, $y = y$, $z = -4y + 8$.)

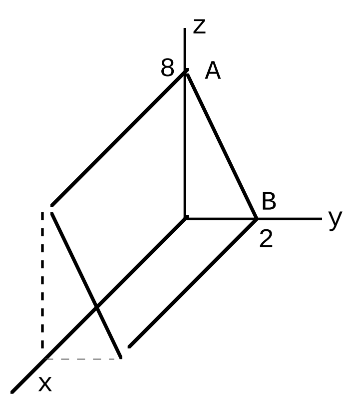


FIG 19

PROBLEMS FOR SECTION 1.0 (solutions are in the back of the notes)

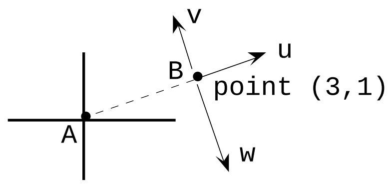
1. Find $u \times v$.
(a) $u = (6,-1,2)$, $v = (3,4,3)$
(b) $u = -2i - 3\vec{j} + 5\vec{k}$, $v = \vec{i} + \vec{j} + 4\vec{k}$
(c) $u = (6,1)$, $v = (3,4)$

2. Find $u \cdot v \times w$ if $u = (1,1,-2)$, $v = (0,3,2)$, $w = (-4,3,1)$.

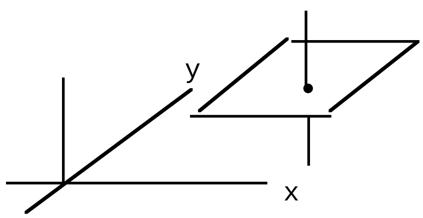
3. Show that the cross product of any two perpendicular unit vectors is another unit vector.

4. (a) Let $\vec{u} = (2,-7)$. Find some vectors perpendicular to u. How many are there.
(b) Let $\vec{u} = (2,-7,3)$. Find some vectors perpendicular to u. How many are there.
(c) Let $\vec{u} = (2,7,3)$, $\vec{v} = (1,2,-1)$. Find some vectors perp to u and v. How many are there.

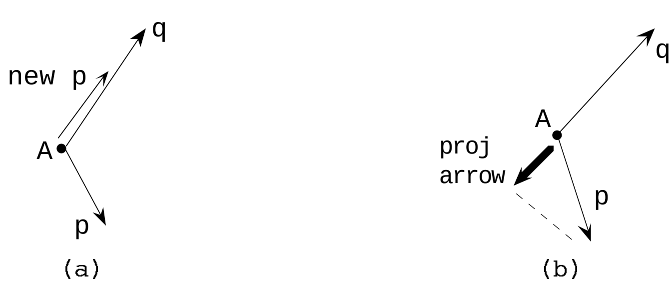
5. The vectors $\vec{u}$ and $\vec{v}$ in the diagram both have length 2 and are perpendicular. The vector w is perp to u and has length 7. Find the components of u, v and w.



6. Find equations (by inspection) for
(a) the line through point (2,3,4) and parallel to the z-axis.
(b) the plane through point (2,3,4) and perpendicular to the z-axis.



7. Let $\vec{p} = \vec{i} - \vec{j}$ and $\vec{q} = 2\vec{i} + 6\vec{j}$. The diagrams show p and q drawn with common initial point A.
(a) Suppose p is rotated around so that it still has its tail at A but now it lies on top of q. Find (easily) the components of the new p.
(b) Suppose arrow p is projected onto arrow q. Find the components of the projection arrow.



8. Let $v = (3,-1,2)$. Find the component of v in the direction of v itself
(a) by inspection (i.e., with a little common sense)
(b) using the formula in (4) (overkill)

SECTION 1.1 SCALAR FIELDS AND VECTOR FIELDS

scalar fields

A function which assigns a *scalar* (i.e., a number) to each point in the plane (or in space) is called a *scalar field*.

For example, if $f(x,y,z) = x^3 + 3yz$ then f is a scalar field. The scalar at the point (2,3,4) for instance is $f(2,3,4) = 44$.

The usual way to picture a scalar field f(x,y) is to draw the level sets:

> The *level sets of a function f(x,y)* are all the curves in 2-space of the form $f(x,y) = C$ where C is a constant.
>
> The *level sets of a function f(x,y,z)* are the 3-dim surfaces $f(x,y,z) = C$.

For example, let $f(x,y) = x^2 + y - 2$.
Fig 1 shows a few of the level sets.
The 2 level set is the curve $x^2 + y - 2 = 2$ (the parabola $y = -x^2 + 4$)
The 0 level set is the curve $x^2 + y - 2 = 0$ etc.
If f(x,y) represents the temperature at point (x,y) then the level sets are called isotherms. The 2 level set is the set of all points where the temperature is 2.
If f(x,y) is the air pressure at (x,y) then the level sets are isobars.
If you think of the x,y plane as the earth at sea level and think of f(x,y) as the altitude of the rolling plains above point (x,y) then the level sets of f are the contour curves on a topographic map.

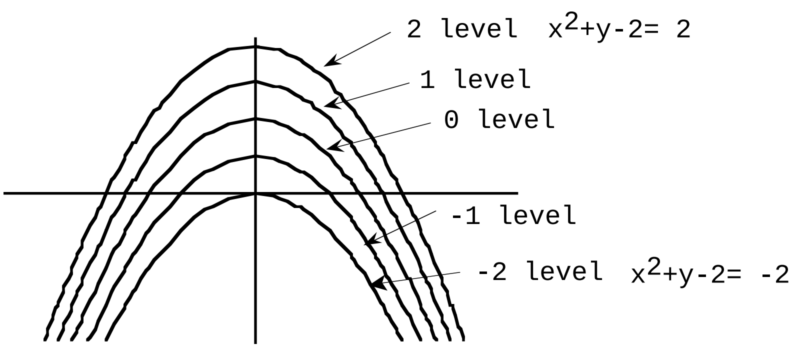


FIG 1

example 1

Fig 2 shows some of the level sets of a function f(x,y).
If the level sets are thought of as contour curves then the people standing at points A,B,C,D are all at altitude 60 (the 60 level consists of two curves).
If the level sets are thought of as isotherms then the points A,B,C,D all have temperature 60.

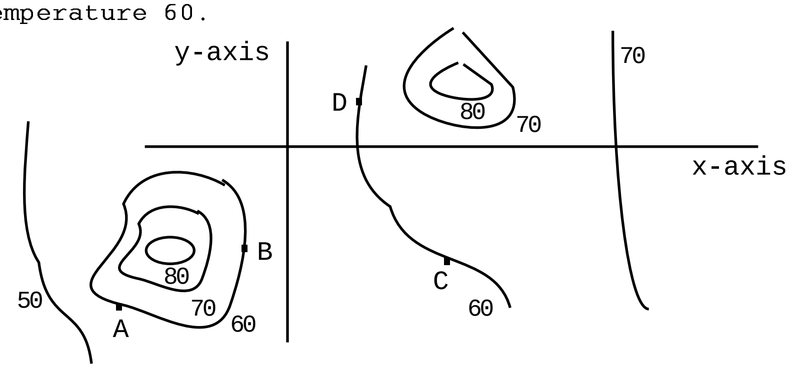


FIG 2

example 2

Let $f(x,y,z) = 2x + 3y + 6z - 10$.
The 14 level set is $2x + 3y + 6z - 10 = 14$.
The 15 level set is $2x + 3y + 6z - 10 = 15$ etc.
The level sets are parallel planes in 3-space (Fig 3).

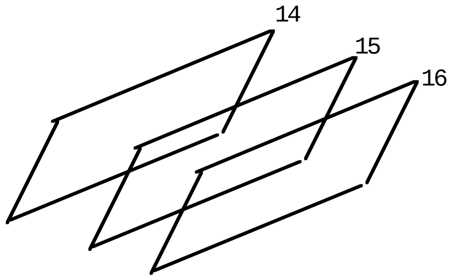


FIG 3

vector fields

A function which assigns a *vector* to each point in the plane (or space) is called a *vector field*. The field is sketched by drawing each vector with its tail at the corresponding point.

For example, let

$$\vec{F}(x,y) = (y + 3)\vec{i} + y\vec{j}.$$

At point (2,1) the $\vec{F}$ vector is $\vec{F}(2,1) = 4\vec{i} + \vec{j}$. In fact all the points on line $y=1$ get the vector $4\vec{i} + \vec{j}$.

The vector at $(-1,0)$ (and every other point where $y = 0$) is $3\vec{i}$ etc.

Fig 4 shows a portion of the vector field.

If $\vec{F}$ represents a *force field* then someone standing at point (2,1) feels force $4\vec{i}+\vec{j}$, a $\sqrt{17}$ pound push in the direction of arrow $4\vec{i}+\vec{j}$.

If $\vec{F}$ represents the *velocity field* of a fluid then the drop at point (2,1) is moving with velocity $4\vec{i}+\vec{j}$, i.e., moving in the direction of arrow $4\vec{i}+\vec{j}$ with speed $\sqrt{17}$.

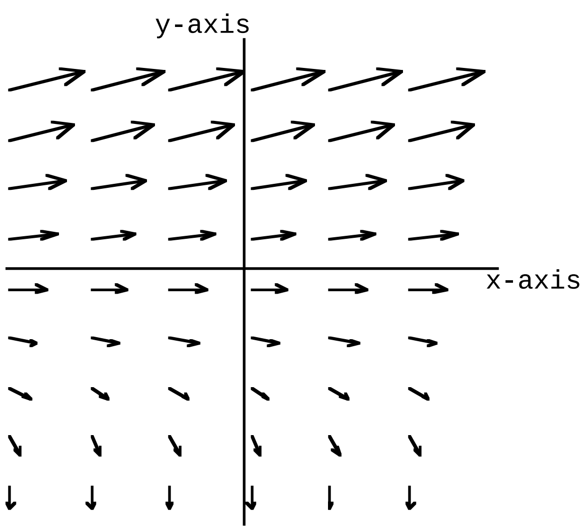


FIG 4

an away-from-the-origin field

Let

$$F(x,y) = x\vec{i} + y\vec{j}$$

If the vector $x\vec{i} + y\vec{j}$ has its tail at the origin then its head is at point (x,y).

So when the vector $x\vec{i} + y\vec{j}$ is drawn with its tail at (x,y) it points away from the origin (Fig 5).

For this particular field, the length of the vector at a point equals the distance from the point to the origin. The further from the origin, the longer the vector.

Similarly, in 3-space, an away-from-the-origin field is $F(x,y,z) = xi + yj + zk$.

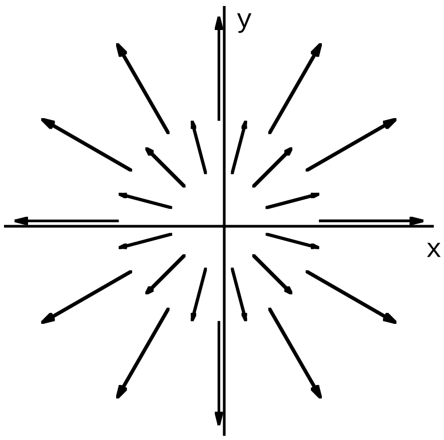


FIG 5

the 2-dim scalar field $f(x,y) = y-x^2$ versus the 3-dim scalar field $f(x,y,z) = y-x^2$

If you write $f(x,y) = y-x^2$ then you are thinking of $y-x^2$ as the temperature at point (x,y) in the plane. In that case the level sets are the parabolas in Fig 6. For example the 1-level set is $y - x^2 = 1$, i.e., $y = x^2 + 1$.

If you write $f(x,y,z) = y-x^2$ then you are thinking of $y-x^2$ as the temperature at point (x,y,z) in space (add a vertical z-axis to Fig 6). All the points on an old 2-dim level set can be raised up and down and will still have the same temp. The new level sets are the parabolic cylinders in Fig 7 (see the end of Section 1.0). I drew them again in Fig 8 with the z-axis coming out of the page at you.

The trick for drawing the parabolic cylinder $y = x^2 + 1$ (the 1-level set) in Fig 8 is to first draw the parabola $y = x^2 + 1$ in the x,y plane. Then copy it directly above (and below if you have room) and connect the copies and the original with suggestive lines parallel to the z-axis.

Learn how to do this.

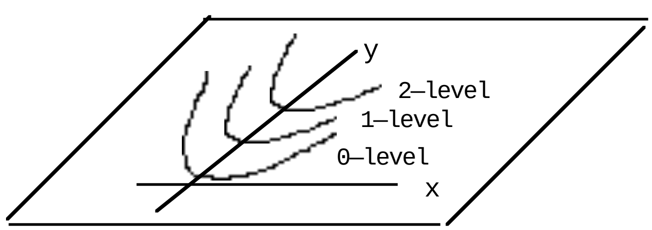


FIG 6

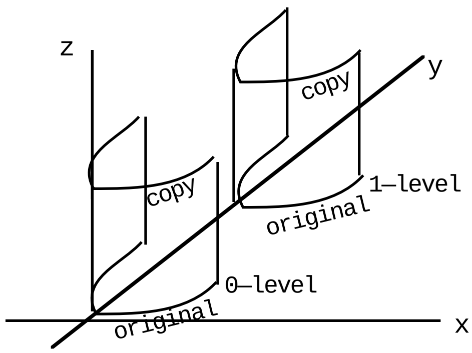


FIG 7

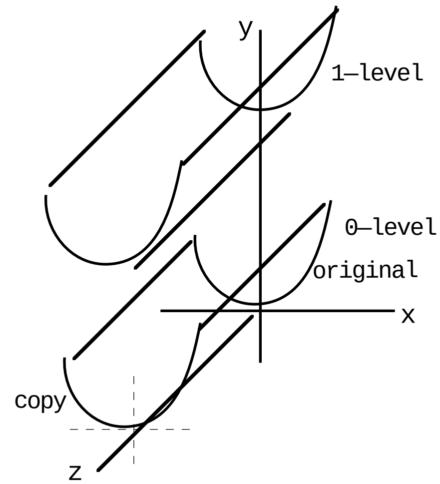


FIG 8

the 2-dim vector field $\vec{F}(x, y) = x\vec{i} + y\vec{j}$ versus the 3-dim vector field $\vec{F}(x,y,z) = x\vec{i} + y\vec{j}$

If $F(x,y) = x\vec{i} + y\vec{j}$ then you are thinking of $x\vec{i} + y\vec{j}$ as a vector at point (x,y) in the plane. In that case, the vector field is shown back in Fig 5.

If $F(x,y,z) = x\vec{i} + y\vec{j}$ then you are thinking of $x\vec{i} + y\vec{j}$ as the vector $x\vec{i} + y\vec{j} + 0\vec{k}$ at point (x,y,z) in space. In that case the vector field is shown in Fig 9. The arrows all point away from the z-axis. The field in Fig 5 is one slice of the field in Fig 9.

The trick for drawing the picture in Fig 9 is to draw it once in the x,y plane and then copy and paste it above (and below) a few times.

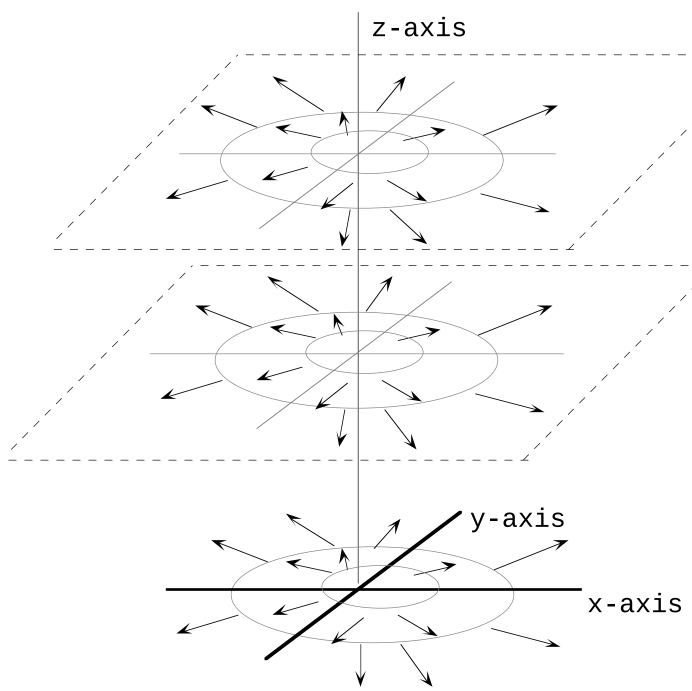


FIG 9

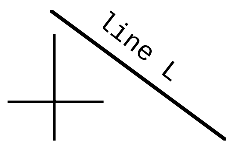
PROBLEMS FOR SECTION 1.1 (solutions are in the back of these notes)

1. Sketch (and label) some level sets of these 2-dim scalar fields.
   (a) $x^2 - y^2$
   (b) $1/(x+y)$
   (c) e^{x+y}
   (d) $e^{x^2+y^2}$
   (e) $3xy$
   (f) y^2

2. Sketch some level sets of these 3-dim scalar fields.
   (a) $f(x,y,z) = x^2 + y^2 + z^2$
   (b) $f(x,y,z) = y^2 - x$
   (c) $f(x,y,z) = y$

3. Let $f(x,y) = xy$. Is the point $(2,6)$ on a level set of f. If so, which one.

4. Let $f(x,y)$ be the distance from (x,y) to the line L in the diagram. Sketch some level sets of f.



5. Identify the equipotential surfaces (the level sets) if the potential energy at the point (x,y,z) is

$$\frac{2}{\sqrt{(x+2)^2 + (y-1)^2 + (z-3)^2}} .$$

6. Can two level sets of the same scalar field intersect.

7. Suppose $f(x,y) = 6$ for all (x,y). Describe the level sets.

8. Sketch the level sets of (a) $f(x,y) = ye^x$ (b) $f(x,y,z) = ye^x$

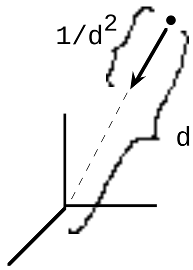
9. Sketch the vector field (draw enough arrows to make the pattern clear).
   (a) $F(x,y) = y\vec{j}$
   (b) $F(x,y) = 3\vec{i} + 6x\vec{j}$
   (c) $F(x,y) = e^{x+3y}\,\vec{i} + e^{x+3y}\,\vec{j}$

10. Fig 4 showed the vector field $\vec{F}(x,y) = (y + 3)\vec{i} + y\vec{j}$. Suppose all the arrows in Fig 4 are changed to have length 2. Find the new vector field.

11. Sketch the vector field (a) $F(x,y,z) = (x,y,z)$ (b) $F(x,y,z) = (x,y,0)$

12. Sketch the vector field (a) $F(x,y) = x\vec{i} + 3x\vec{j}$ (b) $F(x,y,z) = x\vec{i} + 3x\vec{j}$

13. (a) If the earth is at the origin then the arrows of a gravitational force field F(x,y,z) point toward the origin. At a point distance d from the origin, the length of the arrow is $1/d^2$ (see the diagram). Find the formula for F(x,y,z).



(b) Suppose that at a point (x,y) in 2–space at distance d from the origin there is an arrow G(x,y) pointing toward the origin with length $1/d^2$ (see the diagram) . Find the formula for G(x,y).

```
<< Graphics`PlotField`

PlotVectorField[{-x/(x^2 + y^2)^(3/2), -y/(x^2 + y^2)^(3/2)},
{x, -2, 2, 1}, {y, -2, 2, 1}, Axes -> True, Ticks -> None]
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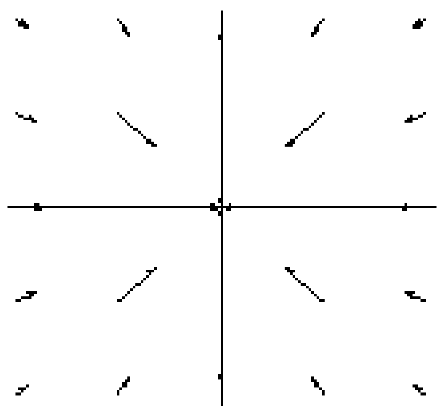
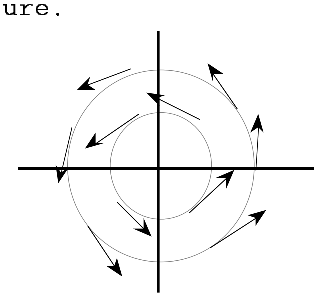
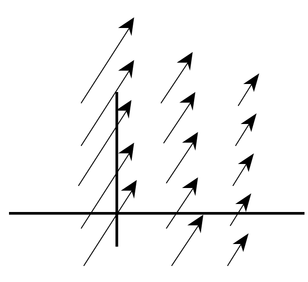


14. (a) The lefthand diagram shows a vector field F(x,y). All the arrows have length 3 and point "around counterclockwise". Find the F formula.
Suggestion: To get an "around ccl" vector, take a left turn (see Section 1.0) from an away-from-the-origin vector.
(b) The righthand diagram shows another vector field. The arrows all have slope 3 and get shorter out to the right. Make up an F(x,y) that roughly goes with the picture.



(a)



(b)

15. An arbitrary vector field F is of the form $p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k}$ where p, q, r are arbitrary scalar fields.
How would you represent an arbitrary vector field F where all the arrows have the same length.

SECTION 1.2 THE GRADIENT

In this section, to be more concrete, I'll usually think of the scalar field $f(x,y)$ as the temperature at point (x,y). And I'll measure distance in meters.

definition of the gradient of a scalar field

Let $f(x,y)$ be a *scalar* field. A *vector* field called the gradient of f, and denoted by ∇f, is defined like this:

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}}$$

Similarly, the gradient of $f(x,y,z)$ is

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}}$$

Remember, you take the gradient of a *scalar* field and the result is a *vector* field. There's a gradient *vector* at every point.

If

$$f(x,y) = \frac{x^2}{y}$$

then

$$\nabla f = \frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j}$$

At the point $A = (3,1)$, the gradient vector is $\nabla f(A) = 6\vec{i} - 9\vec{j}$.
Fig 1 shows this particular gradient as an arrow with its tail at point A.
Fig 2 shows a bunch of gradients (Mathematica shortened all the arrows so that they'd fit on the page).

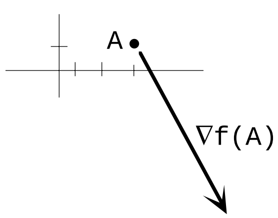


FIG 1

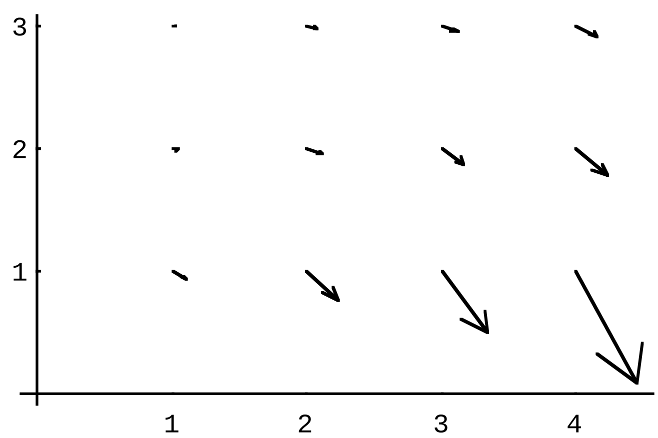


FIG 2

warning

∇f is *not* the *scalar* field $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$. It's the *vector* field $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

Big difference.

application of the gradient to rates of change of f

Let f(x,y) be the temperature at point (x,y) in 2-space.
Or let f(x,y,z) be the temperature at point (x,y,z) in 3-space.
Look at a fixed point and the ∇f vector at that point.

(A) If a particle moves through the point in the direction of a vector $\vec{u}$ then it feels the temperature changing by

$$\frac{\nabla f \cdot \vec{u}}{\|\vec{u}\|} \text{ degrees per meter}$$

(Remember that $\nabla f \cdot \vec{u}/\|\vec{u}\|$ is the component of ∇f in the u direction.)

Temps are going up in a direction u if ∇f makes an acute angle with u (i.e., $\nabla f \cdot u$ is positive)

Temps are going down in a direction u if ∇f makes an obtuse angle with u (i.e., $\nabla f \cdot u$ is negative)

For example, if $\nabla f \cdot \vec{u}/\|\vec{u}\| = -7$ at a point then as you walk through the point in the u direction, temp is *dropping* (instantaneously) by 7 degrees/meter; if $\nabla f \cdot \vec{u}/\|\vec{u}\| = 7$ then temp is going *up* by 7 degrees/meter.

(B) If a particle moves through the point in the direction of ∇f it feels temp increasing by $\|\nabla f\|$ degrees per meter and that's the biggest possible rate of change. The gradient is said to point in the direction of *steepest ascent* (Figs 3,4).

Similarly, in direction $-\nabla f$, temp is *dropping* by $\|\nabla f\|$ degrees per meter, the biggest possible drop rate.

If a particle moves through the point in any direction perpendicular to ∇f, it feels no change in temperature.

(C) The gradient at a point is perpendicular to the level set of f through the point (Figs 3, 4). And of the two perpendicular directions, ∇f points toward higher levels.

The rate of change of f in the direction of a vector u, given by the formula in (A), is called the *directional derivative* of f in direction u and is denoted by $D_u f$ or df/du or df/ds (s represents distance). Part (B) says that the directional derivative of f is max in the direction of ∇f, min (most negative) in the direction of $-\nabla f$ and zero in any direction perpendicular to ∇f.

The rate in (A) is "geographically instantaneous" meaning that the particle walking through the fixed point in direction u feels this rate just as it goes through the fixed point.

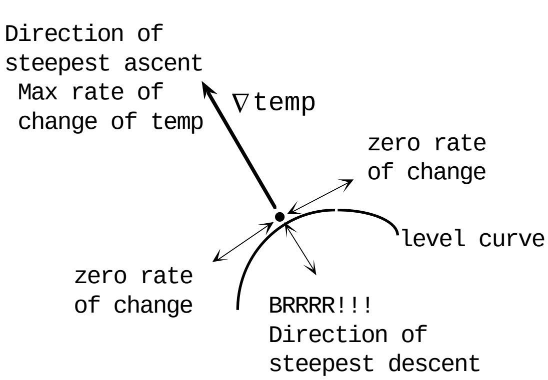


FIG 3 two-space

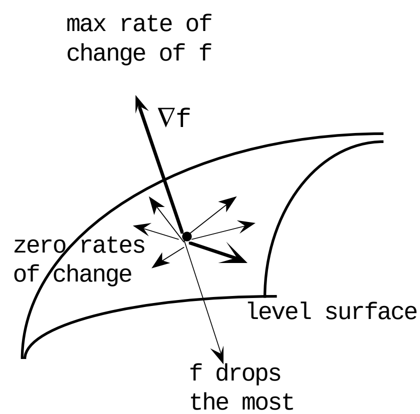


FIG 4 three-space

why (A) works

I'll justify the general rate-of-change formula in (A) by starting with some specific numbers.

Suppose that at point A in the plane,

$$\frac{\partial f}{\partial x} = 7 \text{ and } \frac{\partial f}{\partial y} = 8$$

$\partial f/\partial x$ is the rate of change of f (i.e., degrees) w.r.t. x. So on an east path through A, the temperature is rising by 7° per meter, and on a north path the temperature is rising by 8° per meter. Let's find the rate of change of temperature in the direction of

$$\vec{u} = 2\vec{i} + 3\vec{j}$$

Fig 5 shows a step of $\sqrt{13}$ meters in the $\vec{u}$ direction, visualized as the superposition of a 2 meter east step and a 3 meter north step.

If the temp were to rise by 7° per meter on the whole east leg (but it actually doesn't because $\partial f/\partial x$ is 7 just at point A) and by 8° per meter on the whole north leg, then on the original step of $\sqrt{13}$ meters, the temperature rises by

$$7 \times 2 \ + \ 8 \times 3 \text{ degrees,}$$

which amounts to a *rate* of

(1) $$\frac{7 \times 2 \ + \ 8 \times 3}{\sqrt{13}} \text{ degrees per meter}$$

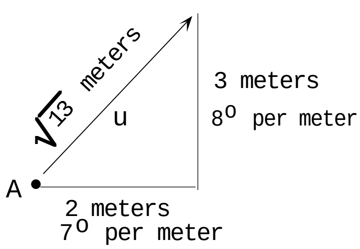


FIG 5

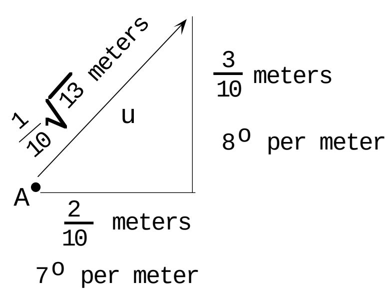


FIG 6

To figure out an *instantaneous* rate of change of f *at* A in the $\vec{u}$ direction, take a much smaller step in the $\vec{u}$ direction, say a step of size $\frac{1}{10}\sqrt{13}$, by superimposing an east step of $\frac{1}{10} \times 2$ meters and a north step of $\frac{1}{10} \times 3$ meters (Fig 6). Then the rate of change of temperature along the hypotenuse, in the direction of $\vec{u}$, is

$$\frac{(7 \times \frac{1}{10} \times 2) \ + \ (8 \times \frac{1}{10} \times 3)}{\frac{1}{10} \times \sqrt{13}} \text{ degrees per meter}$$

which cancels down to (1) again. As the computation is repeated for smaller and smaller steps in the u direction, you get (1) each time. So you can take (1) to be the *instantaneous* rate of change.

You can generalize (1) as follows: If $\vec{u} = u_1\vec{i} + u_2\vec{j}$ then the instantaneous rate of change of f in the $\vec{u}$ direction is

$$(2) \qquad \frac{\frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2}{\|\vec{u}\|} \text{ degrees per meter}$$

The numerator in (2) can be written as $\nabla f \cdot \vec{u}$, so (2) turns into the formula in (A). QED

why (B) works

By (1), the rate of change of f in a direction is the component (signed projection) of ∇f in that direction.

The biggest projection that any vector can have is its own length and it comes when you project the vector in its own direction. So the biggest rate of change of f is $\|\nabla f\|$ and it come from moving in the direction of ∇f itself.

A vector has projection 0 if you project it onto a direction perpendicular to itself. So the component of ∇f in any direction perp to ∇f is 0. So if you move perpendicular to ∇f you momentarily feel zero change in f.

why (C) works

Look at the fixed point and level set through it in Fig 3.
If you start to walk through the point *along* the level set, temp will not change.
So the no-temp-change directions are tangent to the level set.
By (B), ∇f is perp to the no-temp-change directions.
So ∇f is perp to the level set

example 1

Let

$$f(x,y) = x^2/y$$

and let A = (3,1). Then

$$\nabla f = \frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j}$$

$$\nabla f(A) = 6\vec{i} - 9\vec{j}.$$

warning about "at (x,y)" vs. "at a specific point"

Do *not* write

$$\nabla f = \frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j} = 6\vec{i} - 9\vec{j}.$$

There is a difference between ∇f *at (x,y)*, i.e., $\frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j}$ and ∇f *at point A in particular*, i.e., $6\vec{i} - 9\vec{j}$.

Use these two separate sentences:

$$\nabla f = \frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j}$$

$$\nabla f(A) = 6\vec{i} - 9\vec{j}.$$

Or squeeze it into one sentence like this:

$$\nabla f_{\text{at } A} = \left(\frac{2x}{y}\vec{i} - \frac{x^2}{y^2}\vec{j}\right)_{\text{at } A} = 6\vec{i} - 9\vec{j}$$

I'll see how f changes as you move through point A in several directions.

(a) Suppose you walk northeast through point A, i.e., in the direction of $\vec{u} = \vec{i} + \vec{j}$ (Fig 7).

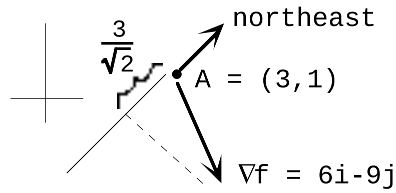


FIG 7

$$\frac{df}{d\,NE} = \frac{\nabla f \cdot u}{\|u\|} = -\frac{3}{\sqrt{2}}$$

If you walk northeast through A the temperature is dropping at the rate of $3/\sqrt{2}$ degrees per meter. It's as if you're going down a temperature hill which has slope $3/\sqrt{2}$.

warning about unintended double negatives

In this example you should say that as you walk NE,

temp is dropping by $3/\sqrt{2}$ degrees per meter.

It is also correct to say that as you walk NE,

temp is *in*creasing by $-3/\sqrt{2}$ degrees per meter

but that sounds peculiar.

But *don't* say

temp is dropping by $-3/\sqrt{2}$ degrees per meter

and *don't* say

temp is changing by $-3/\sqrt{2}$ degrees per meter and it is dropping

because *dropping* by *minus* $3/\sqrt{2}$ amounts to *increasing* by *plus* $3/\sqrt{2}$ degrees per meter which is not what you mean.

Fig 7 shows that ∇f make an obtuse angle with the northeast direction, corresponding to the fact that df/dNE is negative. The projection of ∇f onto the northeast direction has length $3/\sqrt{2}$ and the *signed* projection is $-3/\sqrt{2}$.

(b) Suppose you walk through A at an angle of 70° with the positive x-axis (Fig 8). A (unit) vector in the direction of the path is

$$v = \cos 70^\circ \,\vec{i} + \sin 70^\circ \,\vec{j}$$

and

$$\frac{df}{dv} = \frac{\nabla f \cdot v}{\|v\|} = 6\cos 70^\circ - 9\sin 70^\circ$$

$6\cos 70^\circ - 9\sin 70^\circ$ is negative (it's approx -6.4) so you feel temp dropping by $-(6\cos 70^\circ - 9\sin 70^\circ)$ (about 6.4) degrees/meter.

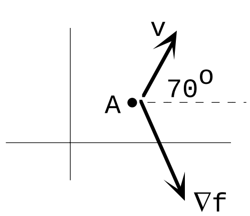


FIG 8

(c) Suppose you walk from A toward the point B = (1,−7) (Fig 9). This is the direction of $\overrightarrow{AB} = -2\vec{i} - 8\vec{j}$.

$$\frac{df}{d\overrightarrow{AB}} = \frac{\nabla f \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{60}{\sqrt{68}}$$

Just as you leave A you feel temperature increasing by $60/\sqrt{68}$ degrees/meter. You're going up a temperature hill whose slope is $60/\sqrt{68}$.

Note that the rate of change $60/\sqrt{68}$ degrees/meter is an "instantaneous" rate *at* A. Once you take a small step past A, it's a new point with a new gradient and there's a new rate of change. The temperature does not necessarily *continue* to rise by $60/\sqrt{68}$ degrees per meter as you move along to B.

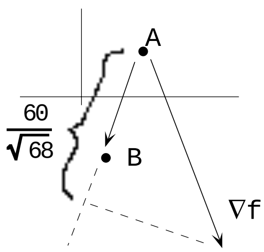


FIG 9

example 1 continued

If you want to leave A in the direction in which temp is increasing the most, walk in the direction of $\nabla f = 6i - 9j$. When you do, you'll momentarily feel temp increase by $\|\nabla f\| = \sqrt{117}$ degrees per meter (Fig 10).

In the direction of $-6i + 9j$ the temp falls by $\sqrt{117}$ degrees/meter, the biggest drop.

In the two directions perp to ∇f, $3i+2j$ and $-3i-2j$, the directional deriv is zero, i.e., temp is instantaneously not changing (Fig 10).

You know that point A is on the 9 level set because $f(3,1) = 9$. The 9 level set is $x^2/y = 9$; this is the parabola $y = \frac{1}{9}x^2$. The gradient at A is perp to the level set (Fig 10).

Fig 11 shows two level sets and a portion of the gradient vector field. The arrows are perp to the level sets (and point toward higher levels).

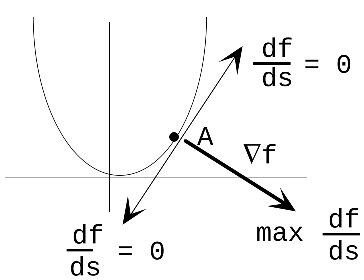


FIG 10

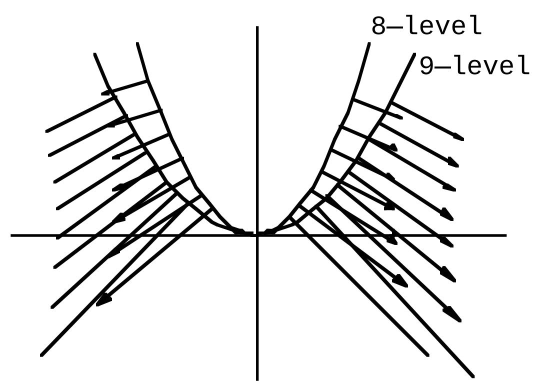


FIG 11

normals to a surface

As a by-product of (C), given a surface in space you can find a normal, i.e., perpendicular vector, at any point.

Look at point P = (6,1,1) on the paraboloid

$$x = y^2 + z^2 + 4 \quad \text{(Fig 12)}$$

Here's how to find a normal vector at P.

Rewrite the equation as say

$$x - y^2 - z^2 = 4$$

Let

$$h(x,y,z) = x - y^2 - z^2$$

Now you can think of the paraboloid as the 4 level set of the scalar field h. Then

$$\nabla h = \vec{i} - 2y\,\vec{j} - 2z\,\vec{k}$$

$$\nabla h \text{ at point } P = i - 2j - 2k$$

Since the gradient at a point is perp to the level set through the point, the vector $i - 2j - 2k$ is perp to the paraboloid at point P (Fig 12) (it happens to be an inner perp).

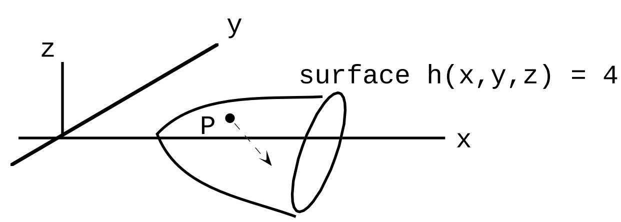


FIG 12

You can also rewrite the equation as $y^2 + z^2 - x = -4$ and take the gradient of $y^2 + z^2 - x$.

You can also rewrite the equation as $x - y^2 - z^2 - 4 = 0$ and take the gradient of $x - y^2 - z^2 - 4$

You can also rewrite the equation as $\frac{1}{4}x - \frac{1}{4}y^2 - \frac{1}{4}z^2 = 1$ and take the gradient of $\frac{1}{4}x - \frac{1}{4}y^2 - \frac{1}{4}z^2$.

You will always end up with a normal, either $i - 2j - 2k$ itself or a multiple of it.

To find a normal at point P on a surface in 3-space, write the equation of the surface with all variables are on one side so that the surface can be thought of as a level set of a scalar field. The normal is the gradient of that scalar field, evaluated at the given point.

example 2

A mountain surface has equation

$$z = 9 - x^2 - 2y^2.$$

Point P = (−2,−1,3) is on the mountain (it satisfies the equation).

Here's how to find a normal to the mountain at P.

Rewrite the equation of the surface to make it look like a level set, say as

$$-x^2 - 2y^2 - z = -9$$

Let $h(x,y,z) = -x^2 - 2y^2 - z$. Then

$$\nabla h = -2x\,\vec{i} - 4y\,\vec{j} - \vec{k}$$

warning (introducing new letters)

Don't write $\nabla h = -2x\,\vec{i} - 4y\,\vec{j} - k$ without first saying what h is. New letters, like the h here, have to be introduced (defined). And have to be introduced properly. The introduction usually begins with "Let" If you leave out the "let" and just write $h(x,y,z) = -x^2 - 2y^2 - z$, the implication is that there already was an h in the problem and you are *concluding* that it equals $-x^2 - 2y^2 - z$.

At point $(-2,-1,3)$ this is $4\vec{i} + 4\vec{j} - \vec{k}$. This is a normal to the surface (Fig 13, top part). It happens to be an *inner* normal because its z-component is negative and you can tell just from looking at the upside cup that a normal with a negative z-component will point in, not out.

Wait a minute!
∇z is $4i + 4j$.
Isn't ∇z supposed to be a perp??? So why isn't the answer here $4i + 4j$.

Yes, ∇z is a perp but ∇z is *two*-dim and is perp to a level set of z, i.e., a level set of $9-x^2-2y^2$, at Q (a contour curve of the mountain) (Fig 13, bottom part).

The significance of ∇z to the mountain climber is that if you start at point P on the mountain and walk in direction ∇z *while staying on the mountain* then you will ascend on the steepest mountain path through P. See the appendix at the end of the section if this still bothers you.)

It's the 3-dim vector $4i + 4j - k$ that is perp to the *surface*.

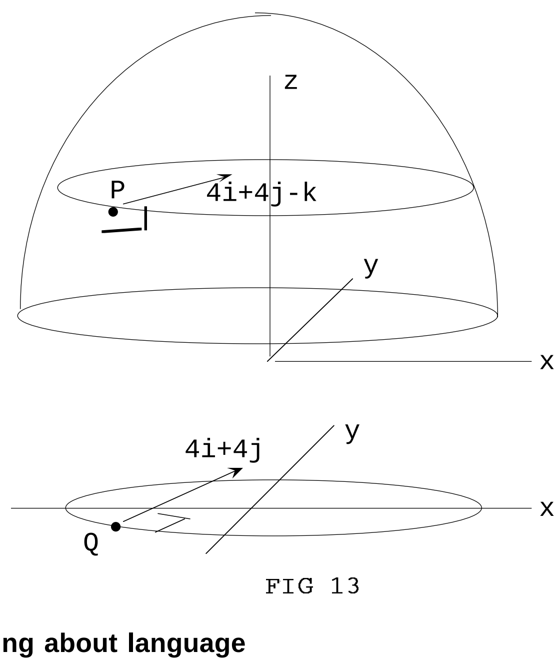


FIG 13

warning about language

In example 2,

I did *not* find the gradient of the *mountain.*

I did *not* find the gradient of the *equation* $z = 9 - x^2 - 2y^2$.

I did *not* find the gradient of the *surface*.

I found the gradient of the *scalar field* h.

The only thing you can take the gradient of is a *scalar field*.

Notation

f_x means $\partial f/\partial x$

f_y means $\partial f/\partial y$

f_{xy} means $\dfrac{\partial^2 f}{\partial y\ \partial x}$ (partial first w.r.t. x and then w.r.t. y)

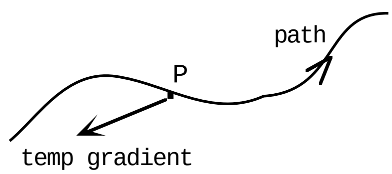
etc

PROBLEMS FOR SECTION 1.2

1. The temperature at the point (x,y) is $xy^2 + 6x + 3$.
Is the temperature increasing or decreasing as you move through point P = (1,2) in the following directions and at what rate (include units).
 (a) southwest
 (b) toward the point Q = (3,−4)
 (c) toward the x-axis
 (d) in the direction of ∇temp
 (e) WNW, i.e., at an angle that is halfway between W and NW
 (f) away from the origin
 (g) toward the origin

2. The temperature at point (x,y) is x^2y. A relay runner going northeast passes the baton at the point (2,3) to a teammate who continues northeast down the track. For each runner, is temp increasing or decreasing at the handoff and at what rate.

3. The diagram shows a temp gradient at point P. If a particle moves on the path in the diagram (from left to right) does it feel the temperature increasing or decreasing as it passes through P.



4. The temperature at point (x,y) is x^2y. Look at point (2,3).
(a) Find the direction of steepest ascent of temperature.
(b) How is temp changing in that direction of steepest ascent.
(c) Imagine the temperature hill at the point.
 (i) How high is the hill.
 (ii) What's the slope on the hill on the path of steepest ascent.
 (iii) What's the slope on the hill on a northwest path.

5. Let $f(x,y,z) = xy - y^2 + z$ and let A = (5,2,1).
(a) If a particle moves through A perpendicularly away from the z-axis, does it feel f increase or decrease and at what rate (include units).
(b) In what direction(s) from A is f instantaneously not changing.
(c) Suppose a particle at A and a second particle at B = (6,4,2) start moving toward one another.
 (i) How does each particle feel f changing initially.
 (ii) If they move until they meet midway, what rate of change of f will each feel as they pass one another.
(d) If f(x,y,z) is air pressure at (x,y,z) then a cloud at point A will move in the direction in which air pressure is decreasing most rapidly. Which way does it move initially and what rate of change of air pressure does it feel.

6. Suppose that from point P, the max rate of change of the scalar field f(x,y) is 2 and it occurs if you move in the direction of $3\vec{i} + 2\vec{j}$.

Find the rate of change of f at P

(a) in the northeast direction

(b) in the north direction

7. Let A = (1,2). If a particle moves through A toward the point (1,1), the temp is rising initially by 2° per meter. If it moves through A toward the point (7,10) the temp is initially dropping by 4° per meter.

In what direction from A is the temp rising most rapidly and what is that max rate.

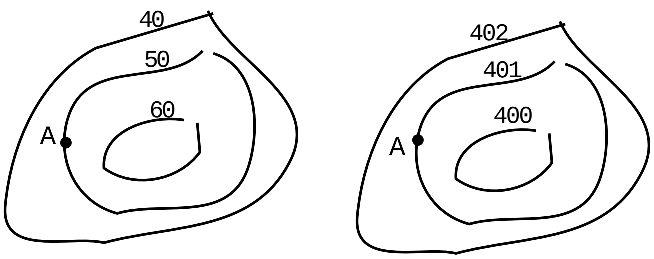
8. Find ∇f at point P and sketch the gradient vector and the level set of f through P.

(a) $f(x,y) = x^2y$, $P = (-1,2)$

(b) $f(x,y,z) = x^2 + 2y^2 - z^2 + 4$, $P = (1,2,1)$

9. The diagram show some level sets of f and g.

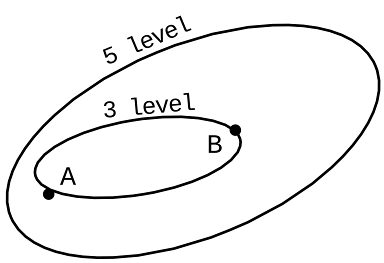
Find the direction of ∇f and of ∇g at A and decide which gradient is longer.



level sets of f

level sets of g

10. The diagram shows some level sets of f. Sketch ∇f at A and at B and decide which gradient is longer.



11. Let f(x,y,z) be the distance from (x,y,z) to a fixed line L. Without trying to find a formula for f, sketch enough level sets and gradient vectors of f to show the pattern.

12. There are (dull) sum and scalar product rules for the gradient operator:

$\nabla(f + g) = \nabla f + \nabla g$

$\nabla(\alpha f) = \alpha \nabla f$ where α is a scalar

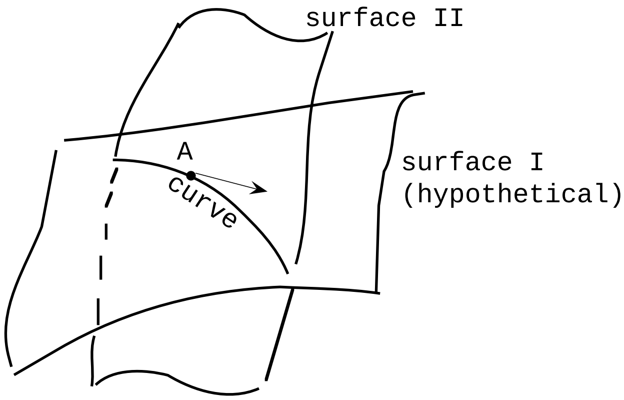
Prove the first one.

13. Given the surface $z = 3x^2 - 2y^2$. Let $P = (1,-1,1)$.

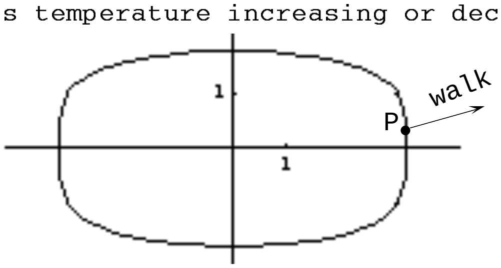
(a) Check that P is on the surface.

(b) Find a vector normal to the surface at P.

14. Given two surfaces $x^4 + y^4 + z^4 = 33$ and $z = xy$. The surfaces intersect in some curve. I know that point A = (1,2,2) is on that curve of intersection since it satisfies both equations. Find a vector tangent to the curve at point A.



15. The temperature in the plane is $x^2 + 3y$.
You are standing at point P = (3, 1) on the curve $x^2 + y^4 = 10$.
You walk perpendicularly away from the curve as shown in the diagram.
Is temperature increasing or decreasing as you start your walk and at what rate.



APPENDIX TO SECTION 1.2

(This is not really relevant in this course but I'm including it for completeness.)

slope on a path on a surface Let $f(x,y) = 9 - x^2 - 2y^2$.

Suppose $f(x,y)$ is the temperature at point (x,y).

Let Q be point $(-2,-1)$ in the x,y plane.

Then $\nabla f = -2x\,\vec{i} - 4y\,\vec{j}$ and $\nabla f(Q) = 4\vec{i} + 4\vec{j}$.

Let $\vec{u} = -5i - j$

If you walk through Q in direction u (mostly west and slightly south) (lower part of Fig A) then you feel temp changing (instantaneously) by

$$\frac{\nabla f \cdot u}{\|u\|} = -\frac{16}{\sqrt{26}} \text{ degrees/foot.}$$

Now look at the "temperature mountain", i.e., the graph of $f(x,y)$ in space. The mountain has equation $z = 9 - x^2 - 2y^2$. Fig A shows the mountain in 3-space, and its contour curves (level sets of f) in the x,y plane.

If $x = -2$, $y = -1$ then $z = 3$ so the point on the mountain directly above Q is $P = (-2,-1,3)$. If you walk from P in direction u *while staying on the mountain*, you walk (instantaneously) on a path with slope $-16/\sqrt{26}$, *descending* at a rate of $16/\sqrt{26}$ vertical feet for every foot in the u direction. In particular, the path in Fig A is the intersection of the mountain surface with the plane perpendicular to the x,y plane and containing the line through Q [and P] parallel to u.

question I often get asked

If you are at point P *in space,* how can it be meaningful to walk in the direction of the *2-dim* vector u?

Answer The "upness" which you may think is missing is determined by the fact that you have to *stay* on the mountain. The 2-dim u direction *plus* the requirement that you stay on the mountain determine the path in Fig A.

On the other hand, if you were at point P in space and could fly off the mountain, or burrow into the mountain or walk on the mountain, then it's a different game and you need a 3-dim vector to describe your direction (that's when you would take the gradient of $z + x^2 + 2y^2$).

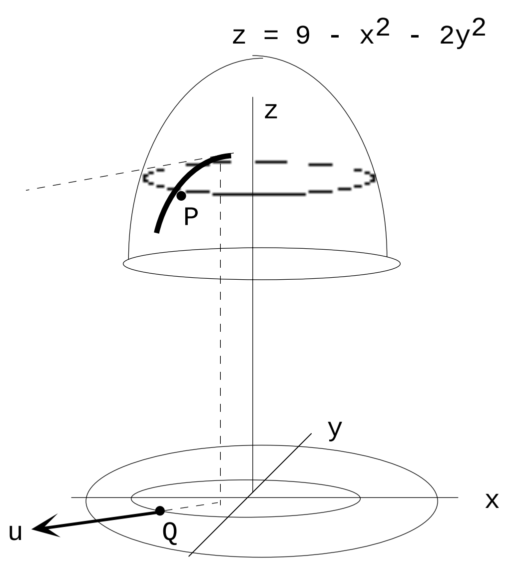


FIG A

slope on path at P is $\dfrac{\nabla z \cdot u}{\|u\|}$

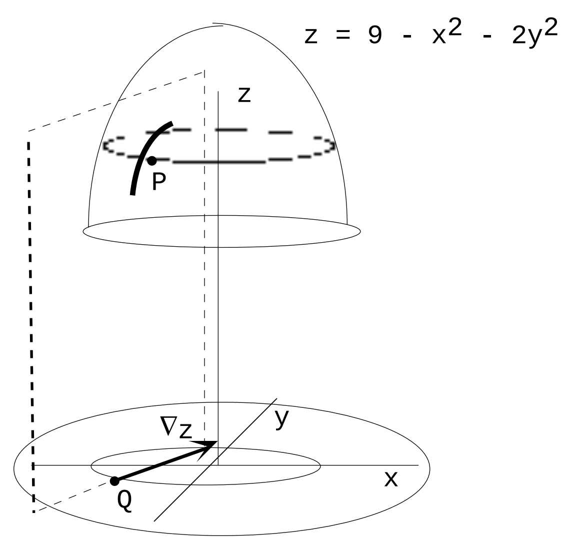


FIG B

path of steepest ascent

slope on a path on a surface continued (path of steepest ascent)

Continue with temperature f(x,y) and point Q as above.

The direction in which temp is increasing most rapidly from Q is $\nabla f(Q) = 4\vec{i} + 4\vec{j}$. In that direction (northeast), temp is changing instantaneously at the rate of $\sqrt{32} = 4\sqrt{2}$ degrees/foot.

Suppose you are at point $P = (-2,-1,3)$ on the mountain surface $z = 9 - x^2 - 2y^2$ (Fig B). If you walk northeast while staying on the mountain, you climb the mountain on the steepest mountain path through P. The path has slope $4\sqrt{2}$ at P.

SECTION 1.3 FLUX DENSITIES

mass flux density

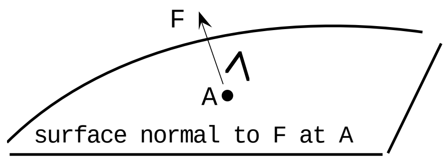
I'll measure mass in kilograms, time in seconds and length in meters.
Suppose a fluid is flowing in space.
The mass/sec flowing through a surface (i.e., a window) is called *mass flux*.

Here's what it means for a vector field F to be called the *mass flux density*. The units on F are kg/sec per square meter.

Here's how a mass flux density F works.
Look at a fixed point A.

(1) Mass flows through point A in the direction of the arrow F(A).

(2) $\|F(A)\|$ is the kilograms/sec (i.e., the mass flux) per square meter at point A flowing in the F direction through a surface normal to F (Fig 1).



At A, flow through window in the F direction is $\|F(A)\|$ kg/sec per m^2

FIG 1

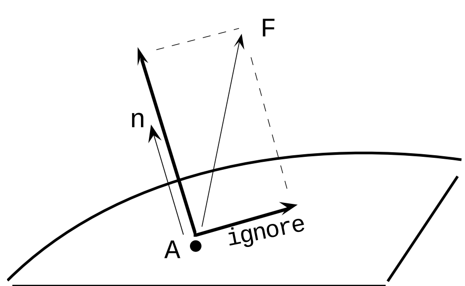
Suppose a surface through point A is not necessarily normal to F (Fig 2). Then F(A) is considered to be a sum of two flows, one in the normal direction (this is the only one that counts) and the other gliding along the window. So in effect, *stuff always flows through a surface in a normal direction* and here's how to find that rate of flow:

Suppose that at point A, a window has normal n (Fig 2).

(3) $\dfrac{F(A)\cdot n}{\|n\|}$ is the kg/sec per sq meter flowing through the window at A in the n direction

(Remember that $\dfrac{F\cdot n}{\|n\|}$ is the component of F in the n direction.)

If N is a *unit* normal then you can use $F\cdot N$ instead of $\dfrac{F\cdot n}{\|n\|}$



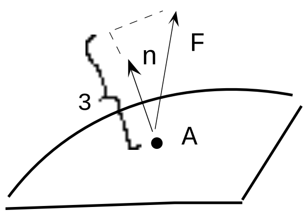
At A, flow through window in the n direction is $\dfrac{F\cdot n}{\|n\|}$ kg/sec per m^2

FIG 2

For example, look at the mass flux density F in Figs 3 and 4.

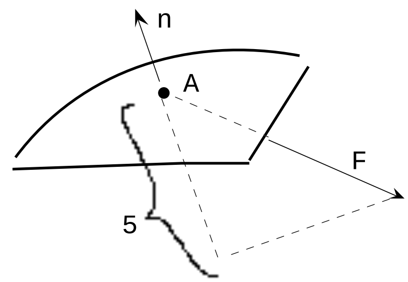
In Fig 3, at point A, the flow through the window is in the n direction at the rate of 3 kg/sec per square meter.

In Fig 4 the flow through the surface in the n direction is −5 kg/sec per square meter at point A so the flow is really in the −n direction at the rate of 5 kg/sec per square meter.



At A, flow through window is
3 kg/sec per m^2 in n direction

FIG 3



At A, flow through window is
5 kg/sec per m^2 in −n direction

FIG 4

flux through a SMALL almost-flat window

Let F(x,y,z) be a mass flux density.

The rule in (3) is about the flux *density*, i.e., the flow across a window *per square meter* at a given point.

Suppose a *small almost-flat* surface (a little window) has surface area dS and normal n (Fig 5).

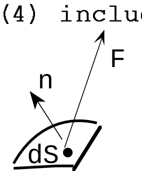
I want the surface to be small so that F does not change much on it from one point to the next.
And I want the surface to be almost−flat so that n does not change much on it from one point to the next.

Evaluate F at a point on the surface.

By (3), $F\cdot N$ kg/sec *per cubic meter* flow through the window at the point. And this little window has surface area dS cubic meters. So:

(4) Flux through a small surface in the direction of normal n (Fig 5) is $\dfrac{F\cdot n}{\|n\|}\, dS$.
The units are kg/sec.

Again, if N is a *unit* normal then you can use $F\cdot N$ instead of $\dfrac{F\cdot n}{\|n\|}$.
Note that the version in (4) includes a factor dS and has different units from (3).



Flux through window in n direction is $\dfrac{F\cdot n}{\|n\|}$ dS kg/sec

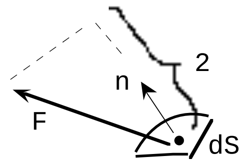
FIG 5

As a special case:

(5) Flux through a small surface normal to F itself = $\|F\|$ dS.
The units are kg/sec

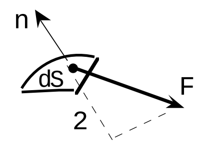
For example, look at the mass flux density F in Figs 6 and 7.
In Fig 6, the flux through the window in the n direction is 2 dS. The units are kg/sec.

In Fig 7, $\frac{F \cdot n}{\|n\|} = -2$. The flux through the window in the n direction is -2 dS.
So the flux is actually 2 dS in the $-n$ direction. The units are kg/sec.



Flux through window in n direction is 2 dS kg/sec

FIG 6



Flux through window in $-n$ direction is 2 dS kg/sec

FIG 7

warning about unintended double negatives

If you are asked specifically for the flux *in the n direction* through the window in Fig 7 then your answer should be -2 dS kg/sec.

If you are just asked for the flux through the window then say

flux = 2 dS kg/sec in the $-n$ direction

Don't say

flux = -2 dS kg/sec and it's in the $-n$ direction

because this amounts to saying +2 dS in the n direction which is not what you mean.

example 1

Let $F(x,y,z) = xi + xyj + xyzk$ be the mass flux density of a fluid.
Let $Q = (1, -1, 2)$

(a) Which way is the drop of mass at point Q moving.

(b) $\|F\|$ is $\sqrt{6}$ at point Q. So what? (What's the physical significance.)

(c) What units does the $\sqrt{6}$ from part (b) have.

(d) A little window at Q has normal $u = i + j + 3k$.
What's the flux through the window. Include units and direction.

solution

(a) $F(Q) = i - j - 2k$. That's the direction of the drop at point Q.

(b) There are two ways you can use the $\sqrt{6}$:
If a little window at point Q with area dS is perp to F then the flux through the window in the F direction is $\sqrt{6}$ dS kg/sec.
If a window through Q is perp to F then the flow through the window at Q in the F direction is $\sqrt{6}$ kg/sec *per square meter*.

(c) The units on $\|F\|$ are kg/sec per square meter.

(d) $\frac{F(Q)\cdot u}{\|u\|} = \frac{1 - 1 - 6}{\sqrt{11}} = \frac{-6}{\sqrt{11}}$

The flux through the window is $\frac{6}{\sqrt{11}}$ dS in the direction of $-u$ (where dS is the surface area of the small window).
The units are kg/sec .

warning (1 + 1 is 2, not 2 dS)

In example 1(d), do *not* write like this:

$$\frac{F(Q)\cdot u}{\|u\|} = \frac{1-1-6}{\sqrt{11}} = \frac{-6}{\sqrt{11}}\ dS\ \text{kg/sec}$$

The number $\frac{1-1-6}{\sqrt{11}}$ equals $\frac{-6}{\sqrt{11}}$. It does not equal $\frac{-6}{\sqrt{11}}$ *times* dS.
dS is a small number. dS is not a unit that can just be tacked on.

clarification

In example 1, the drop of fluid at Q is moving in direction of F(Q) (part (a)).
But the flow through the *window* in part (d) is not considered to be going in direction F(Q). *Flow through any surface is always considered to be crossing in a normal direction* because when you find $\frac{F\cdot n}{\|n\|}$ dS you are regarding F as the sum of the normal flow and the tangential flow in Fig 2 and reporting the normal one.

other flux densities

There are other flux densities besides mass flux densities.
If F is a *heat* flux density then the units on F are *calories*/sec per square meter.
If F is a *charge* flux density then the units on F are *coulombs*/sec per square meter.
And so on

electric flux density (optional reading)

An electric force field $\vec{E}$ is thought of by physicists as an *electric flux density*. Electric flux is imagined to flow from positive charges to negative charges.

(Well, actually the electric flux density is $\varepsilon_0\vec{E}$ where ε_0 is a particular physical constant but I usually refer to $\vec{E}$ itself as the electric flux density.)

volume flux density

> Suppose F is the velocity field of a fluid in space meaning that at a point A in space the drop of fluid is moving in the direction of arrow F(A) with speed $\|F(A)\|$ meters/sec.
> Then F is the *volume flux density* of the fluid as follows.
> Suppose a small almost-flat window has surface area dS and unit normal N.
> Evaluate F at a point on the window.
> The flux through the window in the N direction is $F\cdot N$ dS.
> The units are cubic meters/sec.

Here's why.
Suppose that $\|F\| = 4$ so that the drops of fluid have speed 4 meters/sec. Look at a 1 meter × 1 meter window normal to the flow (Fig 8). Every second, a box of water 1 × 1 × 4 (i.e., *4 cubic meters*) flows through the *one-square-meter* window. So the fluid flows through a window normal to F at the rate of *4 cubic meters of fluid*/sec *per square meter of window* which makes F a *volume* flux density.

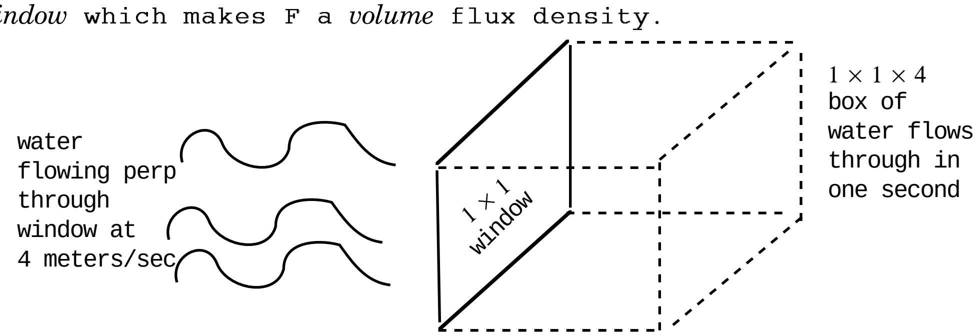


FIG 8

example 2

Suppose $F(x,y,z) = z\,i + xy\,k$ is the velocity field of a fluid.
Let $Q = (3,4,5)$
Find the flux through a small window at point Q tilted to have normal $n = 2j + 4k$

solution

$F(Q) = 5i + 12k$

$$\frac{F\cdot n}{\|n\|} = \frac{48}{\sqrt{20}}$$

The flux through the window is $\frac{48}{\sqrt{20}}\,dS$. It's in the n direction.

The units are cubic meters/sec.

warning Don't call the units here *kilograms*/sec.
Since F is a *volume* flux density, the units are *cubic meters*/sec.

heat flux density

Fourier's law of heat conduction says that

$$\boxed{\text{Heat Flux Density [abbreviated HFD]} = -\nabla\text{temperature}}$$

In other words:

Calories flow through a point in the direction of $-\nabla$temp at that point, i.e., in the direction of steepest *descent* of temperature (from hot to cold).

And at a point, $\|-\nabla\text{temp}\|$ (same as $\|\nabla\text{temp}\|$) is the calories/sec per square meter flowing through a surface perp to $-\nabla$temp.

footnote

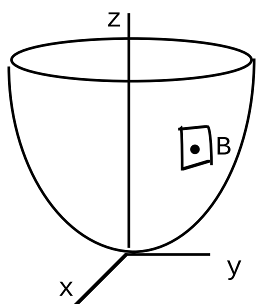
From Section 1.2, the units on $-\nabla$temp are degrees/meter. To get the heat flux density, whose units are $\frac{\text{calories}}{\text{sec meter}^2}$, you actually have to multiply $-\nabla$temp by a constant C, called the thermal conductivity, which depends on the material being heated, with units $\frac{\text{cal}}{\text{meter sec deg}}$. I'm going to ignore C.

> Suppose a small almost-flat surface has area dS and normal n.
> Evaluate the Heat Flux Density at a point on the surface.
> The flux through the surface in the n direction is $\frac{HFD\cdot n}{\|n\|}\,dS$.
> The units are cal/sec

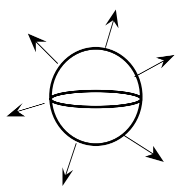
PROBLEMS FOR SECTION 1.3

1. Let $F = xi + y^2 j + yzk$ be the mass flux density of a fluid.
   Let $Q = (-1,3,2)$.
   A little window at point Q has normal $n = i - j + k$
   (a) In what direction is the drop of fluid at point Q flowing.
   (b) Find the flux through the window. Include direction and units.
   (c) What are the units on $\frac{F(Q)\cdot n}{\|n\|}$.

2. Let $\vec{F}(x,y,z) = zi + xj + yk$ be the mass flux density of a fluid.
Point B = (3,1,10) lies on paraboloid $z = x^2 + y^2$. The diagram shows a very small window at B lying on the paraboloid.
Is the fluid flowing into or out of this window on the paraboloid and at what rate.



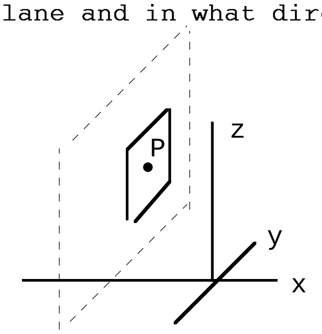
3. A sphere has radius 6. The mass flux density vectors at points on the sphere point perpendicularly out of the sphere and have length 2 (see the diagram). Find the flux out of the sphere.



4. Suppose F is a mass flux density and at point P, $\|F\| = 4$.
What does $\|F\|$ signify physically and what units does it have.

5. Let $F = (x + y + z)i + (x + y)j + xk$ be the velocity field of a fluid
Let Q = (3,2,1)
(a) There's a little window at point Q with normal $n = 2i - j - k$.
In what direction is the fluid flowing through the window and at what rate (include units).
(b) How should you tilt a window at point Q so that the flux through it is maximum. And what would that max be.

6. The temperature at point (x,y,z) is $2x + 3x^2 + xz$
Let P = (−1,1,2)
(a) In what direction do calories flow through point P.
(b) Find the heat flux through a small window at P normal to the flow. Include direction and units.
(c) How many calories per second flow through a small window at P in a plane parallel to the y,z plane and in what direction.



(d) Plane $x - y - 10z = -22$ goes through point P. Find the rate at which heat flows through the plane *at* P. Include direction and units.

SECTION 1.4 DIVERGENCE

definition of the divergence of a vector field

If

$$\vec{F}(x,y,z) = p(x,y,z)\ \vec{i} + q(x,y,z)\ \vec{j} + r(x,y,z)\ \vec{k}$$

then

(1)
$$\boxed{\operatorname{div} \vec{F} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}}$$

For example if

$$\vec{F}(x,y,z) = xy\ \vec{i} + (2x+3y+4z)\vec{j} + \sin xyz\ \vec{k}$$

then

$$\operatorname{div} \vec{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial(2x+3y+4z)}{\partial y} + \frac{\partial(\sin xyz)}{\partial z}$$
$$= y + 3 + xy \cos xyz$$

You take the divergence of a *vector* field and the result is a *scalar* field. There is a divergence *number* at every point.

More generally, a vector field might be a function not only of position (x,y,z) but also of time t. For example, the vector at point (x,y,z) at time t might be

$$F(x,y,z,t) = (2x+y+t)\vec{i} + tz\vec{j} + 3xze^{-t}\ \vec{k}$$

The divergence is still defined as in (1): $\operatorname{div} F = 2 + 3xe^{-t}$.

When you see an F formula without t's in it, then either F is not changing with time or F is the field *at a particular time* so that the variable t has been replaced by a specific value and you no longer see any t's.

using divergence to find the flux out of a small *closed* surface, i.e. out of a box

I'll illustrate the idea with a *mass* flux density F.

If F is a mass flux density then the flux through a *small almost-flat* surface with unit normal N is F·N dS.

But suppose a small surface is *closed* (a little box) instead of almost-flat. Then the N varies so much that you can't find the flux through the surface with just one F·N dS. You need a sum of several F·N dS's. And it turns out that when we add these F·N dS's we will end up with the following neat conclusion.

(2)

> Suppose a small closed surface (a box) encloses volume dV.
> Evaluate div F at a point in the box. Then
>
> div F dV is the flux *out* of the box.
> The units are kg/sec

For example suppose div F = 7 at a point in 3-space.

If a little box at the point has volume dV then 7 dV kg/sec flow out. (The flux is 7 dV, the units are kg/sec).

If div F = −3 at a point then flux flows *into* a little box at the point at the rate of 3 dV kg/sec.

(3)

> Suppose F is a mass flux density.
> By (2), the units on div F dV are kilograms/sec.
> The units on dV are cubic meters.
> So the units on div F itself are kg/sec per cubic meter
> For example, iff div F = −3 at point P then mass flows *into* a box around point P at the rate of 3 kg/sec per cubic meter

why (2) works

The rule in (2) works for a box of any shape. But I'm going to use a rectangular box here. Let

$$F(x,y,z) = p(x,y,z)\ i + q(x,y,z)\ j + r(x,y,z)\ k$$

be a mass flux density. Consider the rectangular box in Fig 1 cornered at point A = (x,y,z), with dimensions dx, dy, dz. To find the flux out of the box, I'll find the flux out of all six faces.

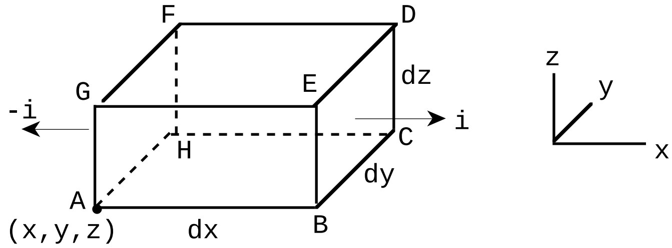


FIG 1

By (4) in the preceding section,
flux out of left face AHFG

$= (\text{component of } \vec{F} \text{ in the } -i \text{ direction at left face}) \times \text{area of face}$
$= (-p \text{ evaluated at a point on the left face, say point A}) \times \text{area of face}$
$= -\,p(x,y,z)\; dy\, dz$

flux out of the right face BCDE

$= (\text{component of } \vec{F} \text{ in the } \vec{i} \text{ direction at right face}) \times \text{area of face}$
$= (p \text{ evaluated at a point on the right face, say point B}) \times \text{area of face}$
$= p(x+dx,\ y,\ z)\; dy\, dz.$

So
net flux out of the left and right faces

$$= p(x+dx,\ y,\ z)\; dy\, dz - p(x,y,z)\; dy\, dz$$

$$= \frac{p(x+dx,\ y,\ z) - p(x,y,z)}{dx}\; dx\, dy\, dz \qquad \text{(multiply and divide by dx)}$$

$$= \frac{\partial p}{\partial x}\, dV$$

Similar results hold for the flux out the top and bottom, and the flux out the front and back. So

$$\text{net mass/sec out of the box} = \frac{\partial p}{\partial x}\, dV + \frac{\partial q}{\partial y}\, dV + \frac{\partial r}{\partial z}\, dV = \text{div}\ \vec{F}\, dV \qquad \text{QED}$$

example 1

Let

$$F(x,y,z) = xy\, i + (xy + 1)\, j + (x^2 + 3y + 4z^2)\, k$$

be the mass flux density of a fluid at a fixed time.

Let Q = (1,3,−1).

(a) A little window at point Q has normal vector u = 2i+3j−2k. Which way does mass flux flow through the window. At what rate (include units).

(b) Does mass flow in or out of a little box at Q and at what rate (include units).

solution

(a) $\dfrac{F(Q)\cdot u}{\|u\|} = \dfrac{6 + 12 - 28}{\|17\|} = \dfrac{-10}{\sqrt{17}}$.

Mass flows in the −u direction at the rate of $10/\sqrt{17}$ dS kg/sec (where dS is the surface area of the little window).

(The numerical answer is $10/\sqrt{17}$ dS. The units are kg/sec.)

(b) Div F = y + x + 8z.

(Div F)(Q) = 3 + 1 + 8(−1) = −4.

Mass flows into the box at the rate of 4 dV kg/sec (where dV is the volue of the little box.

(The numerical answer is 4 dV. The units are kg/sec.)

warning about "at (x,y)" versus "at a specific point"

In part (b) above, do *not* write

$$\text{div } F = y + x + 8z = -4.$$

There is a difference between div F at (x,y), i.e., y+x+8z, and div F at point Q in particular, i.e., −4. Use these two separate sentences instead:

Div F = y + x + 8z

Div F at Q = −4

Or squeeze it into one sentence like this: $(\text{div } F)_{\text{at } Q} = (y + x + 8z)_{\text{at } Q} = -4$.

warning about unintended double negatives

In part (b) above you can write (but it seems unnatural)

flux *out* = −4 dV kg/sec

The answer I want is

flux *in* = 4 dV kg/sec

But *don't* write

flow = −4 dV kg/sec and it's *in*

because this amounts to saying +4 dV kg/sec *out* which is not what you mean.

divergence of a velocity field in particular

Suppose F is the velocity field of a fluid in 3-space.
Then F is a vol flux density.

Suppose a small box encloses volume dV.
Evaluate div F at a point in the box. Then

div F dV = cubic meters/sec flowing *out* of the box

Div F by itself has units cubic meters of fluid/sec per cubic meter of space.

example 2

Suppose the velocity field of a fluid is

$$F(x,y,z) = xz\,\vec{i} + 3xz\,\vec{k}$$

Find the net flow out of a little box at point P = (1,2,3).

solution

Div F = z + 3x

Div F at P = 3 + 3 = 6

If the box has volume dV then 6 dV cubic meters/sec flow out.

clarification

The numerical answer here is 6 dV.
The units are cubic meters/sec
The direction of the flow is "out of the box".

warning that 1 + 1 is 2, not 2 dV or 2 dS or 2 ds

In example 2 do *not* write like this.

$$\underbrace{\text{Div F at P = 3 + 3}}_{\text{no dV in here}} = \underbrace{\text{6 dV}}_{\text{dV in here}} \text{ cubic meters/sec out}$$

It's wrong because 3 + 3 doesn't equal 6 dV. It equals plain 6.
6 dV is a much smaller number than 6 because in this context, dV is a very small number (dV is a *number*, not a unit).

Don't string together some equal things and then suddenly tack a dV (or a dA or a dS or a ds) onto just one of them. Use these two separate sentences instead:

Div F at P = 3 + 3 = 6
Flux out = 6 dV

Or tack the dV on everywhere (which I think is a nuisance) like this:

flux out = (Div F at P) dV = (3 + 3) dV = 6 dV

accounting for net mass flux in or out of a little box

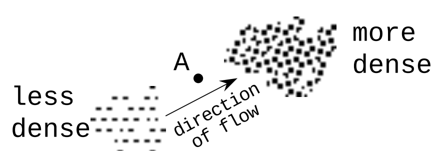
Suppose F is a mass flux density.

Suppose div F is positive at a point so that there is a net flux out of a box at the point.

Physicists say that there are only two ways to account for this.

One possibility is that there is a source of mass at the point, i.e., new mass is being created at the point (from some external source).

The other possibility is that the mass flowing through the point gets less dense as time goes on (Fig 2). Then at any fixed time, denser stuff is in the process of coming out and less-dense stuff is going in so there is a net mass flux out of a little box at the point.



Density is *decreasing* at point A as time goes on.
There is a net flow *out* of a little box at A.

FIG 2

Suppose the mass is *incompressible* so that the mass density can't change.

Then physicists say that the net flow in/out of a little box at a point is due to a sink/source at the point:

net flux	div F	what's happening at the point
out	pos	source (creating mass)
in	neg	sink (destroying mass)

It's not very interesting if div F is 0 at just *one* point but *if div F = 0 at all points then F represents an incompressible flow with no sources or sinks, so that for any box (large or small) the net flow out is 0; i.e., whatever flows in eventually flows out*.

optional reading

On the other hand, suppose there are no mass sources or sinks. Then physicists say that a net flow in or out of a little box must be due to a mass density $\rho(x,y,z,t)$ [units are kg/cubic meter] which is changing with time t:

net flow	div F	density at point	$\partial\rho/\partial t$	
out	pos	decreases with t	neg	(Fig 2)
in	neg	increases with t	pos	

In particular they say that

$$\operatorname{div} \vec{F} = -\frac{\partial \rho}{\partial t} \qquad \text{(called the equation of continuity)}$$

accounting for net volume flux in or out of a little box

Suppose F is the velocity field of a fluid in space which makes F a volume flux density.

There is no compressibility option here. In one cubic meter of space you can have say 5 kg of mass and then by compressing the mass you can squeeze in 6 kg of mass. But you can never have anything but one cubic meter of fluid in one cubic meter of space. Can't compress fluid volume. So a net flux in or out can only be accounted for by sources and sinks:

Div F > 0 at a point means there is a source of fluid at the point.
div F < 0 at a point means there is a sink.

example 3

Suppose the temperature at point (x,y,z) is $\dfrac{x^3y}{z}$

Look at point Q = (2,1,1).

(a) In what direction does a calorie at point Q flow.

(b) Find the heat flux through a small window at Q, tilted so that it has normal vector $2\vec{i} + \vec{k}$.

(c) Are calories flowing in or out of a small box at Q and at what rate.

(d) If you leave point Q and walk toward the origin how is temperature changing initially.

solution

(a) $-\nabla$temp is the heat flux density (preceding section)

$$\nabla \text{temp} = \frac{3x^2y}{z}\vec{i} + \frac{x^3}{z}\vec{j} - \frac{x^3y}{z^2}\vec{k}$$

At point Q, HFD = $-12i - 8j + 8k$

The calorie moves in the direction of $-12i - 8j + 8k$.

(b) $$\frac{\text{HFD}\cdot(2i+k)}{\|2i+k\|} = -\frac{16}{\sqrt{5}}.$$

If the window has area dS then the heat flux through the window is $\dfrac{16}{\sqrt{5}}$ dS.

The units are cal/sec.

The flow is through the window is in the direction of $-2i-k$.

(c) div(heat flux density) = div($-\nabla$temp)

$$= \operatorname{div}\left(-\frac{3x^2y}{z}\vec{i} - \frac{x^3}{z}\vec{j} + \frac{x^3y}{z^2}\vec{k}\right)$$

$$= -\frac{6xy}{z} - \frac{2x^3y}{z^3}.$$

At point Q, div(heat flux density) = -28.

If the box has volume dV then 28 dV cal/sec flow *into* the box.

(d) (This is a Section 1.2 problem)
A toward-the-origin vector at point Q is $u = -2i - j - k$

$$\frac{\nabla \text{temp} \cdot u}{\|u\|} = \frac{-24}{\sqrt{6}}.$$

Temp is dropping by $24/\sqrt{6}$ degrees per meter as you start to walk.

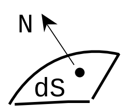
warning

If $f(x,y,z)$ is a *scalar* field then it doesn't have a divergence.
The expression $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ is tempting but it is *not* div f (or anything else useful for that matter).

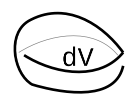
summary of through a small window (preceding section) vs. out of a small box (this section)

Let F be a mass flux density (similarly for other flux densities.)

(4) Flux through a small almost-flat surface with area dS in the direction of unit normal N (Fig 3)	$F \cdot N\ dS$ units are kg/sec
(5) Flux out of a small *closed* surface (box) containing volume dV (Fig 4)	div F dV units are kg/sec



kg/sec through in N direction = $F \cdot N\ dS$
FIG 3



kg/sec out = div F dV
FIG 4

> $F \cdot N$ by itself (without the factor dS) is kg/sec *per square meter* through a window in the N direction
> Div F by itself (without the factor dV) is kg/sec *per cubic meter* out of a box.

You'll see how to get flux through a *not*-small surface in Chapter 4, using surface integrals.

clarification

Question The flux is $F \cdot N\ dS$ in (4). How did it suddenly become div F dV in (5).
Answer The flux out of a small rectangular box is the sum of six $F \cdot N\ dS$'s, one for each face of the box. After some algebra the sum came out to be div F dV. [I did it for a convenient box in "why (2) works" in this section.]

estimating div F at a point from a picture of F

Given a vector field $F(x,y,z)$ in *three*-space, to estimate whether div F is positive or negative at a point, put a little box (any shape you like) around the point and see if net flux is going out (div F is positive) or coming in (div F is negative).

For example, suppose that at every point, F is a unit vector pointing away from the origin (Fig 5). In Fig 6, I drew a box around point P.
The faces ABJE and DCGH lie on a cones
The faces ADCB and EHGJ lies on spheres centered at the origin.
The faces ADHE and BCGJ lie on half-planes, swung around from the xz plane.

No flux flows through the cone faces or the plane faces since F is tangent to those faces.
Flux flows into face ABCD. Flux flows out of face EJGH.
More flows out than comes in because the arrows at those faces have the same length but face EJGH is larger than face ABCD.
So all in all, there is net flux out of the box. So div F at P should be positive.

Question Why draw the a box with the shape in Fig 6 rather than say a rectangular box like the one back in Fig 1.

Answer Because with the box in Fig 6, the field arrows are perp to some of the sides and tangent to other sides which makes it easier to see what's going in and out.

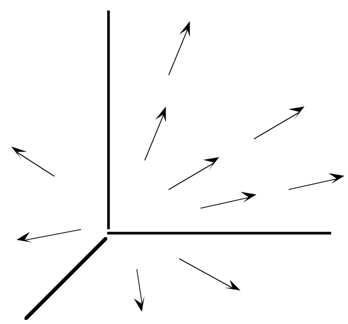


FIG 5

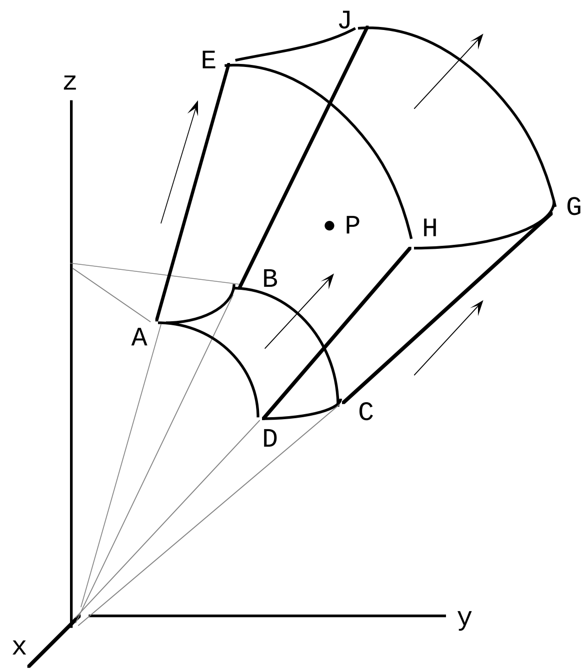


FIG 6

estimating div of a 2-dim F from a picture of F

Fig 7 shows a 2-dim vector field F.

To estimate div F at point P put a little 2-dim box at P

Fig 8 shows a convenient box (convenient usually means with sides perp and tangent to the field if possible).

Through sides BC and AD there is no flux.

There is more flux flowing in through side AB than out through side DC because AB and DC have the same length but the F arrows are longer on AB.

So there's a net flux *into* the box.

So div F is negative at point P.

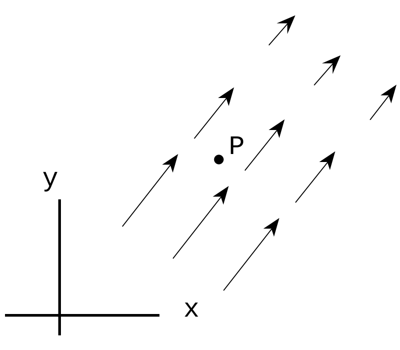


FIG 7

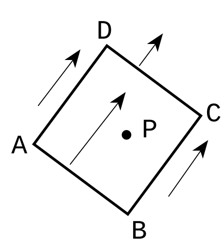


FIG 8

footnote

The 2−dim field in Fig 7 is really just a slice of the 3−dim field in Fig 9 below. In other words, I was looking at a vector field in 3-space after all, but a special one.

To find div F at a point P you really need a box at the point like the one in Fig 10 But the only flux in and out of the box is across faces ABEH and DCKG (the F arrows just glide along the other faces). And F doesn't change as you move up or down these faces. So you can make the decision about net flux out of the box by looking at the 2-dim flow out of the 2-dim box ABCD, which is what I did in Fig 8.

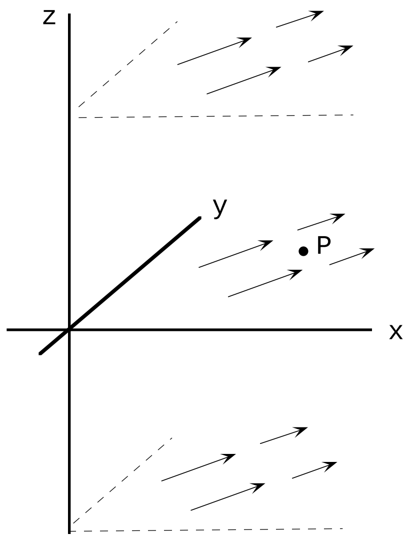


FIG 9

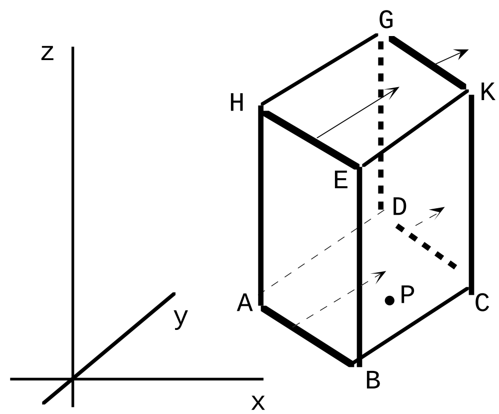


FIG 10

PROBLEMS FOR SECTION 1.4

1. Find the divergence of the vector field.

(a) $\vec{F}(x,y,z) = (xy,\ y^2,\ xyz)$ (b) $\vec{G}(x,y,z) = yz\vec{i} + yz\vec{j} + yz\vec{k}$

(c) $\nabla(x \sin y)$ (d) $\vec{F}(x,y) = (\sin xy,\ x^2 + y^3)$

2. Let $F(x,y,z) = xi + yj + zk$ be a mass flux density.
Let P be the point $(\pi, -\sqrt{17}, e)$.
Is mass going in or coming out of a little box at point P and at what rate.

3. Suppose $F(x,y,z) = xye^x\ \vec{i} + xye^x\ \vec{j} + 2yz\ \vec{k}$ is a mass flux density.
Is mass going in or out of a little box at point $(-1,2,3)$ and at what rate.

4. Suppose F is a mass flux density and at point P, $\|F\| = 4$ and div $F = -5$.
What does each number signify physically. What units does each have.

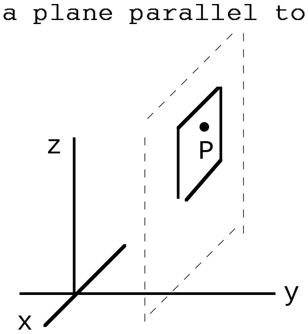
5. Suppose $F(x,y,z) = (y^2z^3,\ 2xyz^3,\ 3xy^2z^2)$ is a heat flux density. Let $P = (1,1,2)$.
Is there a net flow of calories in or out of a small box at P and at what rate.

6. Let $F(x,y,z) = x^2y\ \vec{i} + x^2y\ \vec{j} + yz\ \vec{k}$ be the mass flux density of a fluid.
Look at point $Q = (-1,2,2)$.
(a) In what direction is the drop of mass at point Q flowing.
(b) At what rate is mass flowing through a small window at Q normal to the flow.
(c) At what rate is mass flowing through a small window at Q in plane $2x-3y-5z = -18$. In what direction is it flowing.
(d) At what rate is mass flowing out of a little box at point Q.
(e) Suppose the fluid is incompressible. Is point Q a source or a sink..
(f) Suppose the fluid is compressible. Can you tell if Q is a source or a sink.

7. Suppose $F(x,y,z) = xyz\,\vec{i} + (x^2 + y^2 + 4)\vec{j} + xz^3\,\vec{k}$ is the velocity field of a fluid. Let P = (1,1,2).
(a) Find the volume flux through a small window at point P with area dS and normal u = i−j−k.
(b) Find the flux in/out of a box at point P (include units)
(c) Is P a source or a sink.

8. What's the difference between mass, mass flux and mass flux density.

9. The temperature at point (x,y,z) is $x^2y^3 + 2xz^3$.
Let P = (−1,1,2)
(a) In what direction does a calorie at point P flow.
(b) At what rate do calories flow through a small window at point P, and in what direction, if
(i) the window is normal to the flow
(ii) the window lies in a plane parallel to the x,z plane

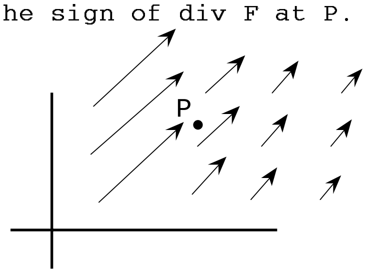


(iii) the window lies in plane x − y − z = −4

(c) Do calories flow into or out of a small box around P and at what rate.

10. The temp in space is xy^2z^3. Point Q is (−1,1,2).
(a) If you walk from Q toward point P = (1,1,1) how do you feel temperature changing initially.
(b) In what direction do calories flow through point Q. At what rate do they flow through a small normal−to−the−flow window.
(c) Do calories go into or out of a little box at point Q and at what rate.
(d) Find the heat flux per square meter at point Q through a window with normal
n = 3i + 2j + k

11. Look at the picture of a 2-dim vector field F.
Estimate the sign of div F at P.



12. Let F(x,y) = −yi + xj.
(a) Sketch the vector field.
(b) Estimate div F at the point (5,6) from your picture.
(c) Calculate div F and see if it agrees with your answer in (b).

SECTION 1.5 THE LAPLACIAN

definition of the Laplacian of a scalar field

If f(x,y,z) is a scalar field then

(1)
$$\boxed{\text{Lapl } f = \text{div } \nabla f}$$

So

$$\text{Lapl } f = \text{div } \left(\frac{\partial f}{\partial x}\,i + \frac{\partial f}{\partial y}\,j + \frac{\partial f}{\partial z}\,k\right)$$

(2)
$$\boxed{\text{Lapl } f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

Similarly if f(x,y) is a 2-dim scalar field then

$$\boxed{\text{Lapl } f = \text{div } \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}}$$

The Laplacian of a *scalar* field is another *scalar* field. There is a Lapl *number* at every point.

For example, if

$$f(x,y,z) = x^2y^3 + z^4 + 7$$

then

$$\text{Lapl } f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2y^3 + 6x^2y + 12z^2$$

harmonic functions

If Lapl f = 0 at every point then f is called *harmonic*.

example 1

Let $f(x,y) = x^3 - 3xy^2$. Show that f is harmonic.

solution

$$\text{Lapl } f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x + -6x = 0 \text{ so f is harmonic.}$$

Laplacian of temperature in 3-space

Lapl temp = div(∇temp).
$-\nabla$temp is the heat flux density.
Div(heat flux density) = div($-\nabla$temp) = $-$div(∇temp) = $-$Lapl temp.
So:

(3)

Lapl temp = *minus* div(heat flux density)

Lapl(temp) dV = calories/sec flowing *into* a small box

The units on Lapl(temp) itself are cal/sec per cubic meter

For example,
if Lapl temp = 3 at a point then 3 dV cal/sec flow *into* a little box at the point;
if Lapl temp = -7 then 7 dV cal/sec flow *out*..

warning

Div(heat flux density) = div($-\nabla$temp) and when it is *positive* at a point, calories flow *out* of a little box around the point. Standard rule for divergence of a flux density.

Lapl(temp) has an "opposite" rule because Lapl(temp) is the divergence of the "wrong" thing, ∇temp not $-\nabla$temp. When Lapl temp is *pos* at a point, calories flow *into* a little box around the point.

example 2

Suppose Lapl(temp) = 7 at a point in 3-space.
If a little box around the point has volume dV then the heat flux into the box is 7 dV. The units are cal/sec.

example 3

Suppose Lapl(temp) = 2x − 3z. Let Q = (−3, 2,1)
Find the heat flux in/out of a small box at point Q

solution

At point Q, Lapl(temp) = −9.
Heat flux goes *out* of the box at the rate of 9 dV cal/sec.

example 4

Suppose the temperature at point (x,y,z) is $x^2y^2 + x^2z^2$.
Look at point Q = (1,−1,1).

(a) In what direction do the calories flow through Q.
(b At what rate do calories flow through Q.
(c) What's the rate of flow of calories through a small window at Q, tilted so that it has normal vector $2\vec{i} + \vec{k}$.
(d) Are calories going in or out of a small box at Q and at what rate.

solution

(a) $\nabla\text{temp} = (2xy^2 + 2xz^2)\,i + 2x^2y\,j + 2x^2z\,k$
$\nabla\text{temp}(Q) = 4i + 2j + 2k$
Calories flow in the direction −4i −2j − 2k

(b) The heat flux density is $-\nabla$temp.
$\|\text{HFD}\|$ at Q = $\sqrt{24}$
The flow across a window at Q normal to −4i −2j − 2k is $\sqrt{24}$ cal/sec per square meter

(c) Let n = 2i + k. Then $\dfrac{\text{HFD}\cdot n}{\|n\|} = \dfrac{-10}{\sqrt{5}}$

If the window has area dS then the flow across is $\dfrac{10}{\sqrt{5}}$ dS cal/sec in the −n direction

Equivalently, the flow across the window is $10/\sqrt{5}$ cal/sec per square meter in the direction of −n.

(d) *method 1*

$$\text{Lapl temp} = \frac{\partial^2 \text{temp}}{\partial x^2} + \frac{\partial^2 \text{temp}}{\partial y^2} + \frac{\partial^2 \text{temp}}{\partial z^2}$$
$$= 2y^2 + 2z^2 + 2x^2 + 2x^2$$
$$= 4x^2 + 2y^2 + 2z^2$$

Lapl temp at Q = 8.
Calories flow *into* the box at the rate of 8 dV cal/sec

method 2

HFD = $-\nabla$temp = $-(2xy^2 + 2xz^2)$ i $- 2x^2y$ j $- 2x^2z$ k

div (HFD) = $-2y^2 - 2z^2 - 2x^2 - 2x^2 = -4x^2 - 2y^2 - 2z^2$

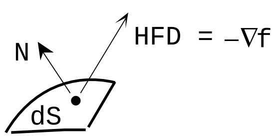
At Q, div(HFD) = -8.

Calories flow into the box at the rate of 8 dV cal/sec

summary of calories through a window vs. out of a box

Remember that the heat flux density (HFD) is $-\nabla$temp.

Heat flux through a small almost-flat surface (a window) with area dS in the direction of unit normal N (Fig 1)	HFD·N dS units are cal/sec
Heat flux in/out a small box with vol dV (Fig 2)	*version 1* div(HFD) dV out *version 2* Lapl(temp) dV *in* units are calories/sec



Heat flux through window in the N direction is HFD·N dS

FIG 1

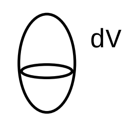


FIG 2

optional reading: accounting for a net calorie flow in or out of a little box

Suppose the Laplacian of temperature is positive at a point, so that calories flow into a box around the point.

One possibility is that the point is a sink which is destroying calories, taking them out of the system.

If there are no sinks then the calories stay in the system and raise the temperature at the point as time goes on so that d(temp)/dt is positive.

In general, if u(x,y,z,t) is the temperature in a solid at point (x,y,z) at time t, and if there are no heat sources or sinks, then Lapl u is proportional to $\partial u/\partial t$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = k\frac{\partial u}{\partial t} \tag{4}$$

The constant of proportionality k depends on the material that the solid is made of. The partial differential equation in (4) is called the *3-dim heat equation* (also called the *diffusion* equation).

If there are no heat sources or sinks and the temperature is steady state (i.e., not changing with time) then the temperature u satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{5}$$

This PDE is called *Laplace's equation*.
Note that "f satisfies Laplace's equation" is equivalent to "f is a harmonic function".

optional reading: taking the Laplacian of something besides temperature
It is useful to take the Laplacian of electric potential because it turns out that electric potential plays a role in the theory of electricity analogous to temperature in the theory of heat.

heat stuff	electric stuff (optional)
Heat flux density = $-\nabla$ temp	Electric flux density = $-\nabla$(electric potential)
Calories flow down temp hills in the direction of $-\nabla$temp.	Electric flux flows down potential hills in the direction of $-\nabla$(elec potential)
If Lapl of temp is positive in a little box then calories/sec flow in.	If Lapl of electric potential is pos in a little box then elec flux flows in.

More generally if a vector field F is $-\nabla f$ for some f then f is called the potential for F (more on this in Section 4.1), you can imagine some kind of stuff rolling down f hills and Lapl f dV is the stuff/sec flowing into a little box.
So it's not just temperature that physicists take the Laplacian of.

PROBLEMS FOR SECTION 1.5

1. Find the Laplacian.
   (a) $x^2 \sin y$
   (b) $x + 2xy^3 + 3x^2yz^3$
   (c) $e^{xy} + x^3 + y$
   (d) $xe^{xy} + x^2y^3$

2. Let $F(x,y) = \dfrac{x^3}{y}\,\vec{i} + (x^3 y - 6)\vec{j}$ Find Lapl div F

3. Show that $e^x \sin y$ is harmonic.

4. Let $f(x,y,z) = z^4 - 6y^2z^2 + y^4 + 3x$. Is f harmonic.

5. The temp in space is $x^2 + y^2 + z^2$.

Point Q is $(-1,-1,2)$.

(a) If you walk from Q toward point P = (1,1,1) how do you feel temperature changing initially.
(b) In what direction do calories flow through Q and at what rate through a window normal to the flow.
(c) Find the rate of flow of calories through a small window at point Q with normal $n = 3i + 2j + k$
(d) Are calories flowing into or out of a little box at point Q and at what rate.

6. The temp at point (x,y,z) is $xy \sin z$.
Let $P = (2, 1, \pi/2)$
Find the rate of calorie flow in/out of a small box at point P

7. Suppose the Laplacian of temp is $2x + 3y + 4z$ (this is not temp itself -- it's the *Laplacian* of temp)
Let $P = (1,2,3)$
Find whichever of the following you can
(a) flow across a small window at the point with normal $2i - j + k$
(b) flow out of a small box at the point

SECTION 1.6 CURL

definition of the curl of a vector field

If $\vec{F}(x,y,z) = p(x,y,z)\,\vec{i} + q(x,y,z)\,\vec{j} + r(x,y,z)\,\vec{k}$ then

(1) $$\boxed{\text{curl }\vec{F} = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z},\quad \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x},\quad \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right)}$$

You take the curl of a *vector* field and the result is another *vector* field. There is a curl vector at every point.

In the 2-dim case where $\vec{F}(x,y) = \big(p(x,y),\ q(x,y)\big)$ you can think of $F(x,y)$ as

$$\vec{F}(x,y,z) = \big(p(x,y),\ q(x,y),\ 0\big)$$

in which case

(2) $$\boxed{\text{curl }\vec{F} = \left(0,\ 0,\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)}$$

∇ notation (the best way to remember the curl formula)

Think of the symbol ∇ as standing for the pseudo vector (a vector *operator*)

$$\nabla = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$$

and furthermore think that $\frac{\partial}{\partial x}$ "times" p is the partial derivative $\frac{\partial p}{\partial x}$.

Then you can think that

$\text{curl }\vec{F} = \nabla \times \vec{F}$

$\text{div }\vec{F} = \nabla \cdot \vec{F}$

$\text{grad } f = \nabla f$

$\text{Lapl } f = \nabla \cdot \nabla f$ denoted $\nabla^2 f$

warning

$\nabla^2 f$ does not mean $\nabla\nabla f$ (there is no such thing since you can't take the gradient of the *vector* field ∇f)

The notation $\nabla^2 f$ means ∇ *dot* ∇f

For example, if

$$\vec{F}(x,y,z) = (x^2y,\ x + y + z,\ xyz)$$

then

$$\text{curl }\vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(x^2y,\ x+y+z,\ xyz\right)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -yz & 1-x^2 \end{vmatrix}$$

$$= \left(\begin{vmatrix} \cancel{\frac{\partial}{\partial x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cancel{x^2y} & x+y+z & xyz \end{vmatrix},\quad \text{MINUS} \begin{vmatrix} \frac{\partial}{\partial x} & \cancel{\frac{\partial}{\partial y}} & \frac{\partial}{\partial z} \\ x^2y & \cancel{x+y+z} & xyz \end{vmatrix},\quad \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \cancel{\frac{\partial}{\partial z}} \\ x^2y & x+y+z & \cancel{xyz} \end{vmatrix} \right)$$

$$= (xz-1,\ -yz,\ 1-x^2)$$

warning

Make sure you get the sign right for the middle component of curl F.

warning

Suppose $\vec{F}$ is the vector field pi + qj + rk and g is a scalar field.

There is a $\nabla\cdot\vec{F}$ (means div $\vec{F}$).

There is a $\nabla\times\vec{F}$ (means curl $\vec{F}$)
There is a ∇g (means grad g).

But there is no $\nabla\vec{F}$.

And the vector field $(\frac{\partial p}{\partial x}, \frac{\partial q}{\partial y}, \frac{\partial r}{\partial z})$ is a *nothing*.
It has no name and it has no use.

example 1

Let F = xyz i + (2x + 3y + 4z) j + xye^z k.

Here's my scratch work for computing curl F:

$$\begin{array}{ccc} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 2x+3y+4z & xye^z\ k \end{array}$$

Curl F = $(xe^z - 4)i + (xy - ye^zj + (xz - 2)k$.

definition of circulation of a velocity field F along a small almost-straight curve

(3)

Let F be the velocity field of a fluid in space.
Suppose a small almost-straight curve has length ds and unit tangent T (Fig 1).

I want the curve to be small so that F does not change much on it from one point to the next.
And I want the surface to be almost—straight so that T does not change much on it from one point to the next.

Evaluate F at a point on the curve. Then

circulation along the curve in the T direction = $F\cdot T\ ds$

(Remember that $F\cdot T$ is the component of F in the T direction.)

The units on a velocity field are meters/sec and the units on ds are meters so

circulation units are square meters/sec

You can think of circulation on a curve as "tendency for the fluid to be flowing along the curve".
In Fig 2, the component of F in the T direction is −2. If the length of curve AB is ds then the circ on the curve in the T direction is −2 ds. Bettere to say the circ on the curve is 2 ds in the −T direction.

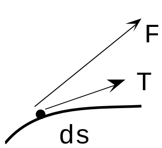


FIG 1

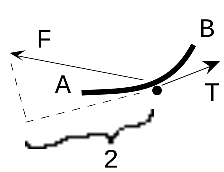


FIG 2

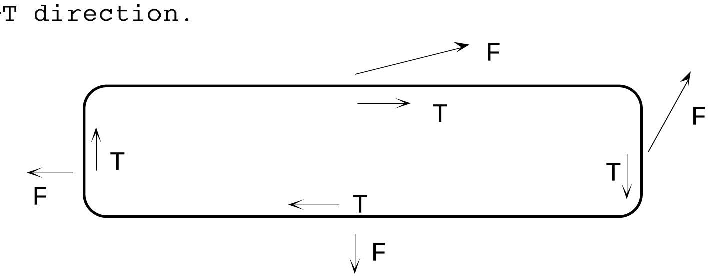


FIG 3

example 2

Look at the curve in Fig 3 (supposed to be very small) with a representative F arrow on each side and some clockwise T's.

total clockwise circulation

$$= \underbrace{F\cdot T}_{\text{about } 2} \text{ ds on top } + \underbrace{F\cdot T}_{\text{about } -2} \text{ ds on right side } + \underbrace{F\cdot T}_{0} \text{ ds on bottom } + \underbrace{F\cdot T}_{0} \text{ ds on left}$$

The top ds is larger than the ds on the right side so the sum is positive.
The total clockwise circ is positive.

using curl to find the circulation along a small *closed* curve (i.e. a loop)

Given the velocity field F, you can find the circulation along a small almost-straight curve with $F\cdot T$ ds.

But suppose a small curve is closed (i.e., is a loop) instead of almost-straight. Then the T varies so much that you can't find the circ along the curve with just one $F\cdot T$ ds. You need a sum of a few $F\cdot T$ ds's as in example 2. And it turns out (next page) that when we add these $F\cdot T$ ds's we end up with the following neat conclusion.

Let F be the velocity field of a fluid.

Look at a little loop at a point (Fig 4).
Imagine that the loop is on a surface and the surface has unit normal N. For short, I'll say the loop goes *around* unit vector N.
(4) And suppose the surface area bounded by the loop is dS.

Evaluate curl F at the point.
Use a right-handed coordinate system. Then

circ around the loop directed righthanded around N = curl $F\cdot N$ dS
The units are m^2/sec

(Remember that curl $F\cdot N$ is the component of curl F in the N direction.)

Fig 5 shows a small loop around the vector N and shows curl F.
The component of curl F in the N direction is 2.
The circ on the loop is 2 dS, directed righthanded around N.

In Fig 6, the component of curl F in the N direction is -3. The righthanded circulation around the N direction is -3 dS. So the circulation is really 3 dS m^2/sec directed lefthanded around N as shown in the diagram.

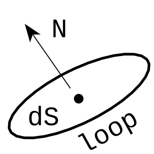
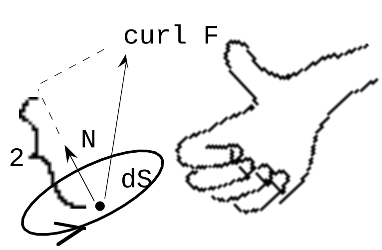
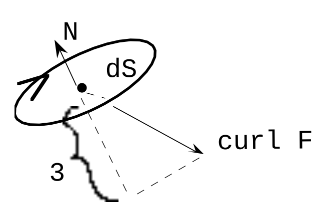


FIG 4



Circ is 2 dS
FIG 5



Circ is 3 dS
FIG 6

getting max circulation, no circulation

Look at all the ways a loop at a point can be tilted (Fig 7a).
If the loop is tilted to go around curl F (Fig 7b) (i.e., so that curl F is perp to the surface inside the curve) it gets the max circ.

In other words, at a fixed point, curl F points in the direction around which there is the most circulation. If the loop encloses area dS then the value of the max circ is $\|\text{curl } F\|$ dS. It is directed righthanded around curl F.

Suppose the loop is tilted so that it goes around a vector N that is perp to curl F (Fig 7c) (i.e., so that the loop and curl F lie in the same plane). Then curl F·N is 0 and there is *zero* circulation on the loop. There are many such tilts because there are infinitely many N's perp to curl F.

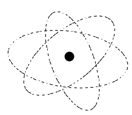
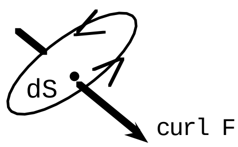
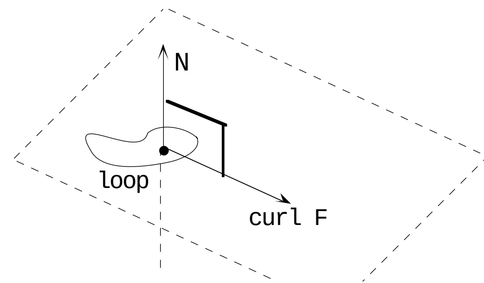


FIG 7a



Max circ
FIG 7b



Zero circ
FIG 7c

why (4) works

Let F(x,y,z) = (p(x,y,z), q(x,y,z), r(x,y,z))

Consider the point A = (x,y,z) and for my N I'll use $\vec{k}$. I'll use the rectangle ABCDA in Fig 8 as my loop around $\vec{k}$.

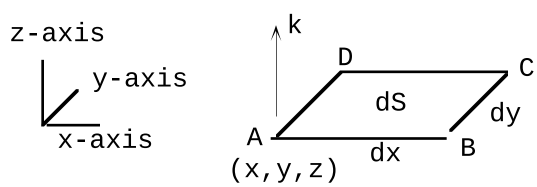


FIG 8

Circulation on ABCDA in the A to B to C to D to A direction

= A-to-B circ + B-to-C circ + C-to-D circ + D-to-A circ

= (comp of F in the $\overrightarrow{AB}$ direction)dx + (comp of F in the $\overrightarrow{BC}$ direction)dy
+ (comp of F in the $\overrightarrow{CD}$ direction)dx + (comp of F along $\overrightarrow{DA}$ direction)dy by (3)

= (p say at point A)dx + (q say at point B)dy
+ (−p say at point D)dx + (−q say at point A)dy

= p(x,y,z) dx + q(x+dx,y,z)dy − p(x,y+dy,z)dx − q(x,y,z)dy

$$= \left[\frac{q(x+dx,y,z) - q(x,y,z)}{dx} - \frac{p(x,y+dy,z) - p(x,y,z)}{dy} \right] dx\, dy \quad \text{(rearrange)}$$

$$= \underbrace{\left[\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right]}_{\text{3rd comp of curl F}} dS$$

This string of equalities began with the righthanded circ around $\vec{k}$ and ended with (k-component of curl F) dS. Which demonstrates (4).

example 3

Let $F(x,y,z) = (x^2y,\ x+y+z,\ xyz)$ be the velocity field of a fluid.

Fig 9 shows a small loop at point $B = (1,2,3)$ going around $\vec{u} = -\vec{i} + 3\vec{j} + \vec{k}$.

(In other words, u is perp to the *surface* the loop lies on.)

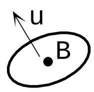


FIG 9

(a) Find the direction and speed of the drop of fluid at point B.
(b) Find the circulation along the loop (include units and direction)
(c) How should a small loop at point B be tilted to get the max circulation. What's the value of that max circulation .
(d) How should a small loop at B be tilted to get zero circulation.
(e) Find the rate of flow through the loop in Fig 9.

solution (a) $\vec{F}(B) = 2\vec{i} + 6\vec{j} + 6\vec{k}$, $\|\vec{F}\| = \sqrt{76}$

The drop is moving in the direction of arrow $2\vec{i} + 6\vec{j} + 6\vec{k}$ at speed $\sqrt{76}$ meters/sec.

(b) Curl $F = (xz - 1,\ -yz,\ 1-x^2)$.

Curl F at $B = 2\vec{i} - 6\vec{j}$.

$$\frac{\text{curl}\,\vec{F}\cdot\vec{u}}{\|\vec{u}\|} = -\frac{20}{\sqrt{11}}\ .$$

The circulation on the loop is $\frac{20}{\sqrt{11}}$ dS (where dS is the surface area enclosed).

The units are square meters/sec.
The direction of the circ is lefthanded around u (Fig 10) (clockwise as seen from the tip of u).

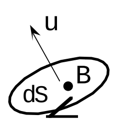


FIG 10

warning (AGAIN) about unintended double negatives

The number MINUS $\frac{20}{\sqrt{11}}$ dS is the *righthanded* circulation around u.

So the most sensible conclusion is that the circ on the loop is

PLUS $\frac{20}{\sqrt{11}}$ dS *lefthanded* around u

Above all, d*on't* write

circ on the loop is $-\frac{20}{\sqrt{11}}$ dS m^2/sec and it's lefthanded around u

because this amounts to saying PLUS $\frac{20}{\sqrt{11}}$ dS righthanded around u which is not what you mean.

(c) The loop should be tilted so that it goes around curl F (Fig 11) (i.e., tilt the loop so that curl F is perp to the surface inside the loop).

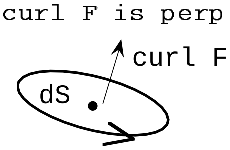


FIG 11

warning

Do *not* write that the curve should be tilted so that curl F is "perp to the *curve*". Perp to the *curve* is not the same thing as perp to the *surface* inside the curve. There are lots of perps to a curve in 3-space but essentially only one perp to a small surface.

$\|\text{Curl F}\| = \sqrt{40}$ so with that tilt, the circ on the loop is $\sqrt{40}$ dS (the max you can get), righthanded around curl F. The units are square meters/sec.

(d) Tilt the loop so that it goes around a vector N that is perp to curl F. Many possibilities. For instance you could tilt the loop to go around $6i + 2j + \pi k$.

(e) (This is a Section 1.3 question). F is a volume flux density. At point B,

$$F = 2\vec{i} + 6\vec{j} + 6\vec{k}$$

$$\frac{F \cdot u}{\|u\|} = \frac{22}{\sqrt{11}}$$

If the loop encloses area dS then the flux goes through the loop in the u direction at the rate of $\frac{22}{\sqrt{11}}$ dS cubic meters per second.

curl and paddle wheels

Let F be the velocity field of a fluid.
Look at a fixed point and curl F at the point.
Imagine a small paddle wheel at the point. The axis of the paddle wheel is a vector.

Fig 12 shows a typical paddle wheel with axis $\vec{a}$.
I want to find how fast the paddle wheel turns and which way it turns..

Imagine a little loop at the point going around the axis and containing surface area dS (Fig 13). The circ on the curve turns the paddle wheel. By (4),

$$\text{circ on loop in a righthanded direction around } \vec{a} = \frac{\text{curl } F \cdot \vec{a}}{\|\vec{a}\|} \, dS \text{ meters}^2/\text{sec}$$

The units on circ are meter^2/sec and the units on dS are meter^2 so

$\boxed{\frac{\text{curl } F \cdot \vec{a}}{\|\vec{a}\|} \text{ by itself (with no dS) has units "per second"}}$

which you can think of as "radians per second," units of angular rotation.
So I choose to think like this:

Let F be a velocity field.

Look at a paddle wheel at a point with vector axis $\vec{a}$ (Fig 12).
Evaluate curl F at that point.

(5) Righthanded turning rate of the paddle wheel around its axis

$$= \frac{\text{curl } F \cdot \vec{a}}{\|\vec{a}\|} \text{ radians per sec} \quad \text{(component of curl F in axis direction)}$$

If the component of curl F in the axis direction is *positive* (i.e., if curl F makes an acute angle with the axis vector) then the paddle wheel turns *righthanded* about its axis. (Fig 13), which means counterclockwise as viewed from the head of the axis.

If the component of curl F in the axis direction is *negative* (i.e., if curl F makes an obtuse angle with the axis) then the paddle wheel turns *lefthanded* about its axis which means clockwise as viewed from the head of the axis.

> If the axis of the paddle wheel is parallel to curl F (Fig 14) then the wheel turns at the rate of $\|\text{curl } F\|$ radians/sec and that's the most turning you can get at the point. The turning is righthanded around curl F.
>
> If the axis of the paddle wheel is perpendicular to curl F (Fig 15) so that the component of curl F in the axis direction is 0 then the wheel doesn't turn. (There are many perps to curl F so there are many ways to align the wheel to get no turning.)

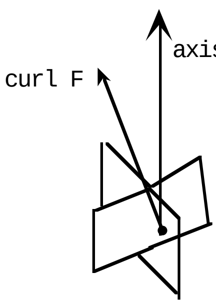
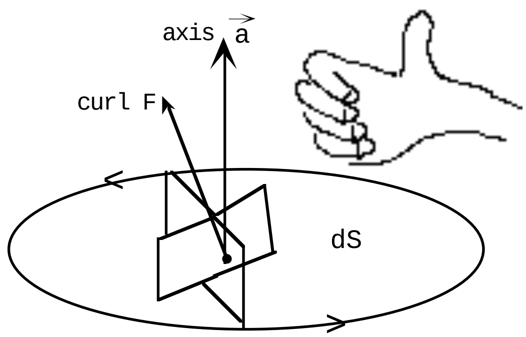
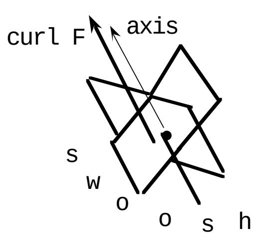


FIG 12

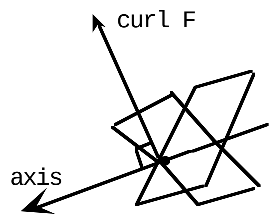


righthanded turning about $\vec{a}$

FIG 13



Max turning
FIG 14



No turning
FIG 15

example 3 continued

(f) Find the turning rate of a paddle wheel at point B with axis $\vec{u}$.

(g) How should a paddle wheel at B be aligned to get the most turning. And what is that max turning rate.

(h) How should a paddle wheel at B be aligned so that the wheel doesn't turn.

solution

(f) The wheel turns at the rate of $20/\sqrt{11}$ radians per second, lefthanded around u.

(g) $\|\text{curl } F\| = \sqrt{40}$

The axis of the paddle wheel should be parallel to curl F (as in Fig 14) in which case the paddle wheel turns at the rate of $\sqrt{40}$ radians/sec, ccl as seen from the head of curl F (i.e., righthanded around curl F).

(h) Tilt the paddle wheel so that its axis is perp to curl F.

warning (circulation versus turning)

Circulation and turning are NOT the same thing.

The *circulation* along a small loop includes the factor dS. And the units are square meters/second.

The *turning rate* of a paddle wheel does not include the factor dS. And the units are "per sec".

example 4

Let $F(x,y,z) = (x^2y,\ x + y + z,\ xyz)$.

Let $A = (3,4,1)$ and $B = (0,1,1)$.

(a) Fig 16 shows a paddle wheel at point B with axis along line AB. How fast does it turn and in what direction.

(b) Find the circulation on a small loop at point B going around line AB (Fig 17). In what direction is the circulation.

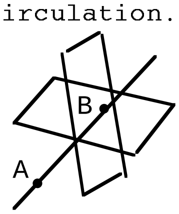


FIG 16

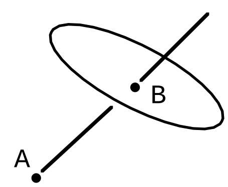


FIG 17

solution

(a) Curl $F = (xz - 1, -yz, 1-x^2)$.

Curl F at $B = (-1,-1,1)$.

The paddle wheel has axis $\overrightarrow{AB} = B - A = 3\vec{i} - 3\vec{j} + 0\vec{k}$

(You can use $\overrightarrow{BA}$ as the axis also as long as you are consistent.)

$$\frac{\text{curl } F\cdot\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{6}{\sqrt{18}}$$

The paddle wheel turns righthanded around $\overrightarrow{AB}$ (clockwise as viewed from A) at $6/\sqrt{18}$ radians per sec. (Note that there is no dS in the answer here.)

(b) If the loop contains surface area dS then

$$\text{circ} = \frac{6}{\sqrt{18}}\ dS \quad \text{(note the dS here)}.$$

The circ is directed righthanded around $\overrightarrow{AB}$ (Fig 17A).

The units are square meters/sec.

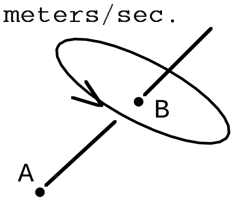


FIG 17A

clarification (about describing the direction of circulation on a loop in 3-space)

When you find circulation you should include a direction as well as a value.

On a small almost-straight curve the direction will be one of the two tangent directions.

On a small *loop* you can describe the direction as righthanded or lefthanded *around a vector*. The circ in Fig 18 is righthanded around $\vec{v}$ but lefthanded around $\vec{u}$. So it is amabiguous to describe circ as righthanded or lefthanded unless you say *around what vector*.

On a small loop you can also describe the direction as clockwise or counterclockwise as long as you clearly say where the viewer is. In Fig 18, s*he* sees *counterclockwise* circ. But *he* sees *clockwise* circ. So you have to say *where the viewer is*. The most helpful way to describe the direction of circ (or of turning) is to *draw a picture*.

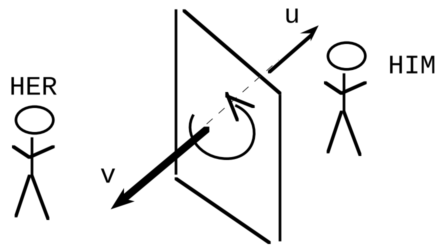


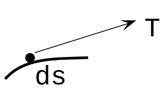
FIG 18

It is meaningless to call circulation
- righthanded around point P
- righthanded along a curve
- righthanded as viewed from point P
- righthanded

summary of through and along (this is your last chance)

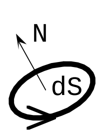
Let F be a velocity field (a volume flux density) in 3-space.

(A) Circulation along a small almost-straight curve with length ds and unit tangent T (Fig 19)	F·T ds units are m^2/sec
(B) Circulation along a small *closed* curve (loop) around a unit vector N and enclosing area dS (Fig 20)	curl F·N dS units are m^2/sec
(C) Flux through a small almost-flat surface (a window) with unit normal N and area dS (Fig 21)	F·N dS units are m^3/sec
(D) Flux out of a small *closed* surface (a box) enclosing vol dV (Fig 22)	div F dV units are m^3/sec
(E) Turning rate of a paddle wheel with axis $\vec{a}$ (Fig 23)	$\dfrac{\text{curl } F\cdot\vec{a}}{\lVert\vec{a}\rVert}$ units are radians/sec



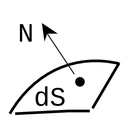
Circ along is F·T ds

FIG 19



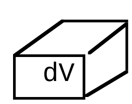
Circ along is curl F·N dS

FIG 20



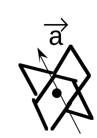
Flow through is F·N dS

FIG 21



Flow out is div F dV

FIG 22



Turning rate is $\dfrac{\text{curl } F\cdot\vec{a}}{\lVert\vec{a}\rVert}$

FIG 23

You'll see how to get circulation along a *not*-small curve and flux across a *not*-small surface in Chapter 3, using line and surface integrals.

Note that there is a ds [arc length] in (A) and a dS [surface area] in (B) and (C) and a dV [volume] in (D) and *nothing* extra in (E).

summary of units

Suppose F is a *mass flux density*.
F has units kg/sec per square meter
Div F has units kg/sec per cubic meter

Suppose F is the *velocity field* of a fluid.
F has units meters/sec. It also has the units of a *volume flux density* which are cubic meters/sec per square meter.
Div F has units cubic meters [of *fluid*]/sec per cubic meter[*of space*]
Curl F has units [radians] per second
Turning of a paddle wheel has units [radians] per second
Circulation of F has units square meters/sec

N and T

N and T in the table above (and throughout the notes) are a *unit* normal and *unit* tangent. If in (C), for instance, you have a *non–unit* normal n then the flow through the window is $\frac{F\cdot n}{\|n\|}\,dS$.

estimating curl Fat a point from a picture of F in 3-space

I'll illustrate with an example.
Suppose the F vectors all point in the k direction but get longer as y increases (Fig 24). I'll estimate curl F from the picture.
Curl F is a vector so there are three components to estimate. The first component is curl F·i. If you think of F as a velocity field then at a fixed point,

$$\text{curl } F\cdot i = \frac{\text{righthanded circ on a small loop around } i}{dS}$$

If the circ of F on the loop is righthanded (resp. lefthanded) around i then the first component of curl F is positive (resp. negative).
And similarly for the other two components of curl F.

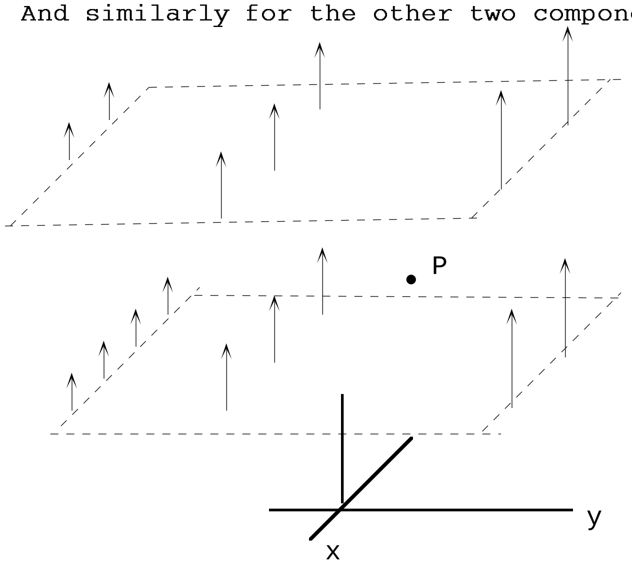


FIG 24

Remember that circ on a little almost−straight curve is F·T ds.
Fig 25 shows a small loop at point P around i.

On sides DC and AB there is no circ (F is perp to T).
On side AD there is lefthanded circ around i (ccl as viewed from the front).
On side BC the circ is righthanded around i.
The circ on BC is larger than the circ on AD since F is longer on BC.
Overall, there is righthanded circ around i. So far, curl F = (+, ?, ?).

Fig 26 shows a small loop at point P around j.
On sides HK and EG there is no circ.
On side EK there is lefthanded circ around j; on side GH there is the same amount of circ but it's around j.
Total circ is 0. So far, curl F = (+, 0, ?).

Fig 27 shows a small loop at point P around k. On each side the circ is 0. So the total circulation around k is 0.

So at point P, curl F = (positive, 0, 0). Same at every point.

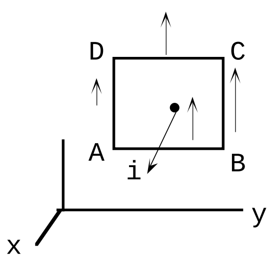


FIG 25

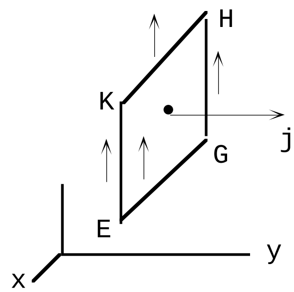


FIG 26

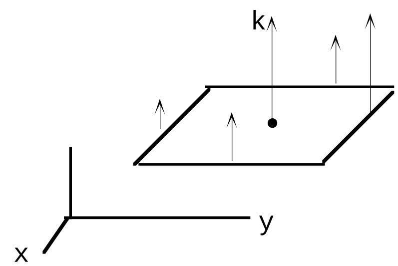


FIG 27

estimating curl Fat a point from a picture of a 2-dim F

Fig 28 shows a 2-dim vector field F = $p(x,y)\vec{i} + q(x,y)\vec{j}$.

You can think of F as $p(x,y)\vec{i} + q(x,y)\vec{j} + 0\vec{k}$ in a righthanded coord system in 3-space (Fig 29) where the x,y axes are in their usual position on the page and the z-axis comes out of the page towards you. The picture in Fig 28 is just a slice of the 3-dim field.

I want to estimate curl F at point P in Fig 28.

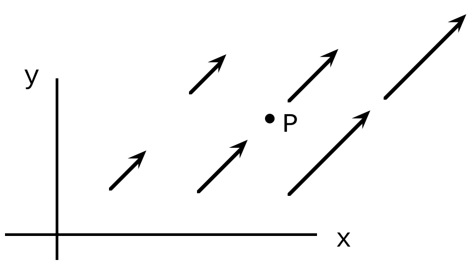


FIG 28

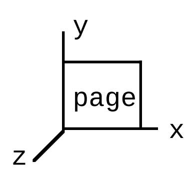


FIG 29

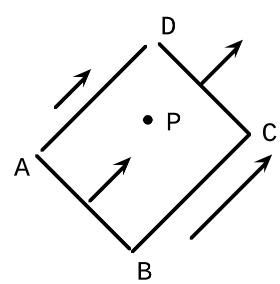


FIG 30

By (2), the first two components of curl F are 0 so curl F points either into the page or out of the page. So all that's left is to estimate the z-component of curl F (just decide if it is positive or negative). To do this look at the circulation at P around $\vec{k}$.

(6)

If there is righthanded circ around k then curl F = (0, 0, +) and curl F comes out of the page at you. If there is lefthanded circ around k then curl F = (0, 0, −) and curl F goes back into the page.

To make the decision in Fig 28 put a convenient small loop around the point (Fig 30). Remember that circ on a little almost–straight curve is $F \cdot T\, ds$.
On sides AB and DC, F is perp to T so there is no circ.

On side AD there is lefthanded circ around k (ccl as viewed from the front).
On side BC the circ is righthanded around k.
The circ on BC is larger than the circ on AD since F is longer on BC.
Overall the circ on ABCD is righthanded around k.
So at P, curl F = (0, 0, positive); curl F points out of the page.

mathematical catechism (you should know the answers to these questions)
Question What is the physical significance of the direction of curl F at a point.
Answer 1 It's the direction you'd want the axis of your paddle wheel to be parallel to get it to turn fastest.
Answer 2 It is the direction around which a little loop gets the most circulation.

Question What is the physical significance of the norm of curl F at a point and what units does it have.
Answer $\|$curl F$\|$ is the turning rate of a paddle wheel whose axis is parallel to curl F. It's the largest turning rate you can get at the point. The units are "per second" (or you can say "radians per sec").

clarification
Question Circ is $F \cdot T\, ds$ in (A). How did it suddenly become curl $F \cdot N\, dS$ in (B)
Answer The circulation on a small rectangular loop is the sum of four $F \cdot T\, ds$'s, one for each side. After some algebra, the sum came out to be curl $F \cdot N\, dS$.
[I did it for a convenient small loop in "why (4) works" in this section.]

PROBLEMS FOR SECTION 1.6

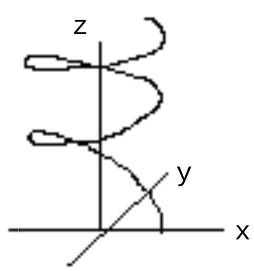
1. Find curl F.

(a) $F(x,y,z) = (y,\ yz,\ xyz)$ (b) $F(x,y,z) = \sin x\ \vec{i} + z\vec{j} + x^2 y^3 z^4\ \vec{k}$

(c) $F(x,y) = e^{xy}\ \vec{i} + x \sin xy\ \vec{j}$ (d) $F(x,y,z) = \left(x + y + z,\ xyz,\ \frac{xy}{z}\right)$

2. Let $g(x,y,z) = x^2 y + e^{xyz}$ and let $G(x,y,z) = (x^2 z,\ e^{xyz} + z,\ 0)$.
Find div, curl, gradient and Laplacian of whatever you can.

3. Let $F(x,y,z) = (x+y+z)\vec{j} + xz\vec{k}$ be the velocity field of a fluid. Look at the curve in the diagram (a helix) with parametric equations $x = \cos t$, $y = \sin t$, $z = t$. Find the circulation along a little piece of the curve at the point where $t = \pi$.



4. Let $F(x,y,z) = (xyz,\ x^2z,\ x + y + z)$ be the velocity field of a fluid.
Let point $A = (3,2,1)$.

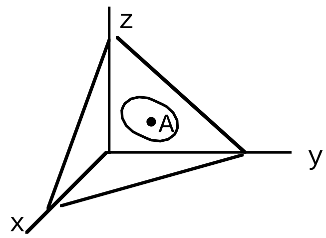
(a) In what direction is the fluid at point A flowing and at what speed.

(b) Find the flow in/out of a small box at A.

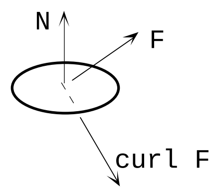
(c) The diagram shows a small window at A lying on the plane $4x + 6y + 2z = 26$.

(i) In which direction and at what rate does the fluid flow through the window.

(ii) Find the circulation (and its direction) around the boundary of the window.



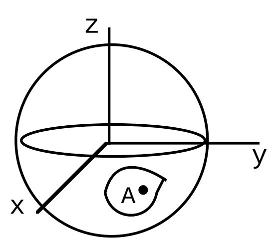
5. The diagram shows a small loop going around N in a fluid with velocity field F. In which direction is the circulation of F along the loop.



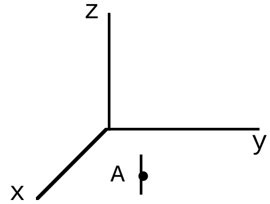
6. Point $A = (2,3,-1)$ is on a sphere centered at the origin in a fluid with velocity field $F(x,y,z) = (z - x^2)\mathbf{i} + (x + 2y)\mathbf{j} + xz\ \mathbf{k}$

(a) Find the circulation (including its direction and units) around a small loop lying on the sphere at point A.

(b) Find the flow *through* the loop in part (a) (including its direction and units).



(c) Find the circ on a small vertical (parallel to the z-axis) segment through A (include units and direction).



7. Let $F(x,y,z) = x\,\vec{i} + xy^3\,\vec{j} + xyz\,\vec{k}$ be the velocity field of a fluid.
Let A = (1, 1, −4) and B = (1, 3, 1).

(a) A paddle wheel is at point A. Its axis goes through point B. How fast does the wheel turn and in what direction as viewed from B.

(b) A small loop at point B goes around line BA.

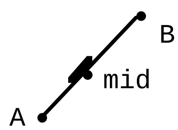
(i) Find the circulation along the loop (including appropriate units). Describe its direction as seen from point A.

(ii) Find the flow through the loop.

(iii) If you must keep the loop at point B, but can change its tilt, how would you tilt it to get the most circulation.
And what is the value of that max circ

(iv) If you must keep the loop at point B, but can change its tilt, how would you tilt it to get no circulation.

(c) Find the circulation on a little piece of segment AB at its midpoint.



8. Suppose F is a velocity field and at point P. What does it signify physically (specify units) if

(a) $\|F\| = 4$
(b) $\|\text{curl } F\| = 5$.

9. Let $F(x,y,z) = xy\vec{i} + xz\vec{j} + xyz\vec{k}$ be a velocity field. Put paddle wheels at points A = (1,2,3) and B = (1,−1,2) with axes pointing toward the origin.
You are sitting at the origin.
In what direction do you see each wheel turn.
Which wheel turns faster?

10. Let $F(x,y,z) = (xyz,\ x^2z,\ x + y + z)$ be the velocity field of a fluid.
A paddle wheel is at point A = (3,2,1). The wheel stays at A but you can point the axis in any direction you want.

(a) There is one axis line that will make the wheel turn fastest. Find its parametric equations (see (6) in Section 1.0).

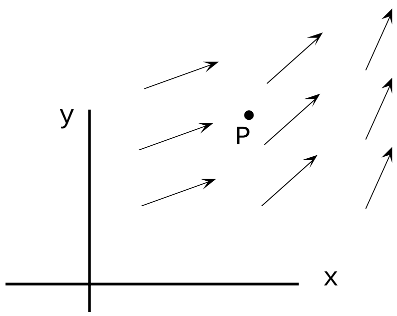
(b) There are many axis lines that will make the wheel not turn at all. In fact they all lie in a plane. Find the equation of the plane and find equations of any two of the lines.

11. Let $\vec{F}(x,y,z)$ be a velocity field.
I have a paddle wheel at point P and I align its axis with curl F at point P.
Your paddle wheel is at point Q and its axis is not aligned with curl F at point Q.
Does my paddle wheel turn faster than yours.

12. Suppose $\vec{u}$ is a fixed vector, F is the velocity field of a fluid and I compute curl $\vec{F}$ at point P.

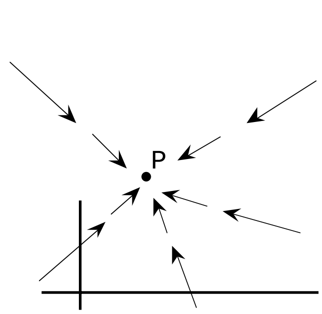
What is the physical significance of $\dfrac{\text{curl } \vec{F}\cdot\vec{u}}{\|\vec{u}\|}$. What units does it have?

13. The diagrams show 2-dim vector fields F. In each case, if possible find the direction of curl F and the sign of div F at point P.

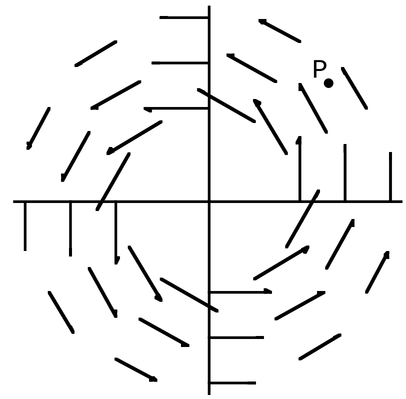


vectors have same length

(a)



(b)



Arrows get shorter
as you go away from
the origin

(c)

SECTION 1.7 SOME PRODUCT RULES AND IDENTITIES

Let f and g be scalar fields.

Let $\vec{F}$ and $\vec{G}$ be vector fields.
Let α be a scalar.

some product rules

(1) $\nabla\ fg = f\ \nabla g + g\ \nabla f$

(2) $\text{div}\ f\vec{G} = f\ \text{div}\ \vec{G} + \nabla f \cdot \vec{G}$

(3) $\text{curl}\ f\vec{G} = f\ \text{curl}\ \vec{G} + \nabla f \times \vec{G}$

(4) $\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl}\ \vec{F} - F \cdot \text{curl}\ \vec{G}$

DON'T MEMORIZE

some identities

(5) $\text{curl curl}\ \vec{F} = \text{grad div}\ \vec{F} - \text{Lapl}\ \vec{F}$

(6) $\text{grad div}\ \vec{F} = \text{curl curl}\ \vec{F} + \text{Lapl}\ \vec{F}$

DON'T MEMORIZE

(7) $\text{curl}\ \nabla f = \vec{0}$

(8) $\text{div curl}\ \vec{F} = 0$

YES, REMEMBER THESE
proofs are in problem #3

footnote

The identities in (5) and (6) involve Lapl $\vec{F}$.

I never defined the Laplacian of a *vector field* $\vec{F}$. In fact I probably said somewhere that you only take the Laplacian of a *scalar* field.
But as a matter of convenience people define the Lapl of a vector field like this. Let

$$\vec{F}(x,y,z) = p(x,y,z)\,i + q(x,y,z)\,j + r(x,y,z)\,k$$

First note that p, q, r are *scalar* fields so *their* Laplacians have already been defined. Then define

$$\text{Lapl}\ F = \text{Lapl}\ p\ \vec{i} + \text{Lapl}\ q\ \vec{j} + \text{Lapl}\ r\ \vec{k}$$

In other words, take the Laplacian of $\vec{F}$ componentwise.
One reason you might want to allow the Laplacian of a vector field is to make the identities in (5) and (6) come out nice.

And you might see it in a physics course, again for convenience. For example, each of the (scalar) components of an electric field in a charge free region has zero Laplacian. So in a charge free region, if $\vec{E} = E_1\vec{i} + E_2\vec{j} + E_3\vec{k}$ then

$$\text{Lapl}\ E_1 = 0$$
$$\text{Lapl}\ E_2 = 0$$
$$\text{Lapl}\ E_3 = 0$$

Instead of writing 3 separate equations, physicists write Lapl $\vec{E} = \vec{0}$.

sum rules and mult-by-a-constant rules

Let α be a fixed scalar (i.e., a constant).

(9) $\nabla(f + g) = \nabla f + \nabla g$

(10) $\text{div}(\vec{F} + \vec{G}) = \text{div}\ \vec{F} + \text{div}\ \vec{G}$ (proof in problem #1

(11) $\text{Lapl}(f + g) = \text{Lapl}\, f + \text{Lapl}\, g$

(12) $\text{curl}(\vec{F} + \vec{G}) = \text{curl}\ \vec{F} + \text{curl}\ \vec{G}$

(13) $\nabla(\alpha f) = \alpha \nabla f$

(14) $\text{div}(\alpha\vec{F}) = \alpha\ \text{div}\ \vec{F}$

(15) $\text{Lapl}(\alpha f) = \alpha\ \text{Lapl}\, f$

(16) $\text{curl}(\alpha\vec{F}) = \alpha\ \text{curl}\ \vec{F}$

easy to remember

proof of (2)

Let $G(x,y) = p(x,y)\vec{i} + q(x,y)\vec{j}$. Then $f\vec{G} = fp\,\vec{i} + fq\,\vec{j}$ and

$$(*)\quad \text{div}\ f\vec{G} = \frac{\partial(fp)}{\partial x} + \frac{\partial(fq)}{\partial y} \qquad \text{definition of divergence}$$

$$= f\frac{\partial p}{\partial x} + p\frac{\partial f}{\partial x} + f\frac{\partial q}{\partial y} + q\frac{\partial f}{\partial y} \qquad \text{product rule for derivatives}$$

$$= f\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) + \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(p, q) \qquad \text{collect terms}$$

$$= f\ \text{div}\ G + \nabla f\cdot G \qquad \text{voila}$$

warning

When you differentiate fp with respect to x in (*), you need a product rule since both f and p are functions of x (and y). Similarly for differentiating fq w.r.t. y.

review of notation

Given scalar fields $f(x,y)$ and $g(x,y)$.
Here are some valid notations.

$\frac{\partial f}{\partial x}$, f_x derivative of f w.r.t. x (a first order partial)

$\frac{\partial^2 f}{\partial x^2}$, f_{xx} second derivative of f w.r.t. x (a second order partial)

$\frac{\partial^2 f}{\partial x\ \partial y}$, f_{yx} deriv of f first w.r.t. y and then w.r.t. x
(mixed second order partial)

$\frac{\partial^2 f}{\partial y\ \partial x}$, f_{xy} deriv of f first w.r.t. x and then w.r.t.
(the other mixed second order partial)
(the two mixed partials are equal)

$\frac{\partial^3 f}{\partial x\ \partial y^2}$ deriv of f first w.r.t. y twice and then w.r.t. x

$\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ product of two derivatives (*not the same as the mixed partial* $\frac{\partial^2 f}{\partial x\ \partial y}$)

$\frac{\partial(fg)}{\partial x}$ deriv of the product fg w.r.t. x

$\frac{\partial f}{\partial x}\frac{\partial f}{\partial x}$ product of two derivatives (not the same as the 2nd order partial $\frac{\partial^2 f}{\partial x^2}$)

warning

Here are some *meaningless* notations.

$$\frac{\partial f^2}{\partial x^2} \qquad \text{(maybe you mean the second order partial } \frac{\partial^2 f}{\partial x^2}\text{)}$$

$$\frac{\partial f \ \partial g}{\partial x^2} \qquad \text{(did you mean the product } \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\text{) (who knows!!!)}$$

warning about style

To show that div $f\vec{G}$ = f div G + $\nabla f\cdot G$, it is neither good style nor good logic to write like this:

Don't write like this

$$\operatorname{div} f\vec{G} = f \operatorname{div} G + \nabla f\cdot G$$

$$\frac{\partial(fp)}{\partial x} + \frac{\partial(fq)}{\partial y} = f\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) + \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(p,q)$$

$$f\frac{\partial p}{\partial x} + p\frac{\partial f}{\partial x} + f\frac{\partial q}{\partial y} + q\frac{\partial f}{\partial y} = f\frac{\partial p}{\partial x} + f\frac{\partial q}{\partial y} + p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y}$$

$$0 = 0$$

TRUE!

Don't write like this

Any "proof" in mathematics that *begins* with what you want to prove and *ends* with TRUE is at best badly written and at worst incorrect and *drives me crazy*.

With this "method" I can prove that 3 = 4 as follows:

3 = 4 (start with what you want to prove)
4 = 3 (just reverse the preceding line)
7 = 7 (add the two preceding lines)
TRUE
So conclude that 3 = 4 (??????)

What you *should* do to prove that div $f\vec{G}$ equals f div G + $\nabla f\cdot G$ is calculate one until it turns into the other (which is what I did above in (*)) or calculate each one *separately* until they turn into the same thing. Then are you entitled to write that they are equal. But don't write div fG = f div G + $\nabla f\cdot G$ as your *first* line. It should be your *last* line.

In general, to show that a first thing equals a second thing, either work on one of them until it turns into the other or work on them both separately until they turn into the same thing. Your argument could have this form:

```
  first thing = ...
              = ...
              = xyz

 second thing = ...
              = ...
              = xyz

So
  first thing = second thing.
```

Or it could have this form:

```
 first thing = ...
             = ...
             = xyz

second thing = ...
             = ...
             = xyz

So
 first thing = second thing.
```

But *not* this form:

UGH

```
first thing = second thing

 ⋮

 1 = 1
 TRUE
```

UGH

warning about abstraction

You can't prove identities (that hold in general) unless you work with arbitrary (abstract, general) scalar fields and vector fields.

An arbitrary 3-dim scalar field is named f(x,y,z).

An arbitrary 3-dim vector field F is named $p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k}$ where p, q, r themselves are arbitrary scalar fields.

PROBLEMS FOR SECTION 1.7

1. Derive the sum rule for div(F+G) for 2-dim F and G.

2. Show that if F and G are gradients then div(F $\times$ G) = 0

   (a) by quoting some standard identities from the list at the beginning of the section.
   (b) starting from scratch.

3. First, punctuate the following identities by putting arrows over those 0's that deserve them. Then prove each one.
   (a) div(curl F) = 0 for any vector field F.
   (b) curl ∇f = 0 for any scalar field f.

4. There's a product rule for curl $f\vec{G}$ in (3) where f is an arbitrary scalar field and $\vec{G}$ is an arbitrary vector field. For practice, try deriving it as if you didn't know what the final result is supposed to be.

5. Let $\vec{u}$ be a constant vector, let $\vec{r}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ and let $r = \|\vec{r}\|$.

   Show that curl $r^2\vec{u} = 2(\vec{r} \times \vec{u})$.

6. Let $f(x,y) = xe^y$ and $G(x,y) = x\vec{i} + xy\,\vec{j}$
   Check that div fG done directly does come out to be what the product rule in (2) says it should.

7. Write the identities in (4) and (8) in ∇ notation.

REVIEW PROBLEMS FOR CHAPTER 1

1. Let

$$f(x,y,z) = 3x^2y^3 + z^4 \quad \text{and} \quad \vec{G}(x,y,z) = (e^x + e^{2y} + e^{3z}, \frac{xy}{z}, 3z).$$

Find div, curl, gradient, Laplacian of whatever you can.

2. Let $F(x,y,z) = (x^2y, x + y + z, xyz)$ be the velocity field of a fluid.
Let $B = (1,2,3)$ and $C = (-1,2,3)$.
Which turns faster, a paddle wheel at B aligned optimally or a paddle wheel at C aligned optimally.

3. Let f be a scalar field in 2-space.
Simplify $\operatorname{div}(f\nabla f)$ in the special case that f is harmonic. Do it
(a) directly and (b) using some of the identities in the preceding section.

4 Let f be a scalar field and let $\vec{G}$ be a vector field.
Which of the following don't make sense.

(a) $\nabla f + \vec{G}$ (b) $\operatorname{curl}\vec{G} + \operatorname{div}\vec{G}$ (c) $\nabla f\vec{G}$

(d) $\nabla \cdot f\vec{G}$ (e) ∇ div curl ∇g (f) curl div $\vec{G}$

5. Let $f(x,y,z) = x - x^2y^2 - z$ be the temperature at point (x,y,z). Let $C = (1,2,1)$.
(a) Find the temp at point C.
(b) In what direction would a calorie at point C move.
(c) Do calories go in or out of a little box around point C and at what rate (include units).
(d) At what rate do calories flow through a little window at point C lying on the sphere $x^2 + y^2 + z^2 = 6$. And in what direction do they flow?
(e) If you walked from point C toward point $D = (0,2,4)$, how would you feel temperature change initially (include units).

6. Let $F(x,y,z) = x\vec{i} + xy\vec{j} + xyz\vec{k}$ be the velocity field of a flow.
Let $P = (1,-4,2)$.
(a) Find the direction and speed of the drop at point P.
(b) Find the flux in or out of a small box around P.
(c) Given a small loop at point P lying in the plane $2x + 3z = 8$.
Find the circulation on the loop. In particular fill in the following blanks.
The circulation has numerical value ____________ .
The units are ___________

The circulation is directed ___________

Sketch the plane and show the direction of the circulation in the picture.

(d) Fill in the blanks
The flux through the loop in part (c) is in the direction of arrow _________.
Its numerical value is _______________ .
It has units _______________.

7. Let $\vec{r}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$.
(a) Sketch the vector field.
(b) Find $\operatorname{Lapl}(\vec{r} \cdot \vec{r})$.

8. Find by inspection (a) $\operatorname{div}\operatorname{curl}(f\nabla g)$ (b) $\operatorname{curl}(\nabla f \times \nabla f)$ (c) $(\operatorname{curl}\nabla f) \times \nabla f$

9. (a) Sketch the level sets of $\ln(x^2 + y^2)$.
(b) Continue from part (a) and find the gradient field, describe it very clearly in words and sketch it.

10. A uniform vector field is one where all the arrows have the same direction and the same length.
True or False. If true, prove it; if false, give a counterexample.
(a) If F is a uniform vector field then div F = 0.
(b) If div F = 0 then F is a uniform field.

CHAPTER 2 COORDINATE SYSTEMS

SECTION 2.0 REVIEW

the velocity vector

Suppose the position at time t of a particle moving in 2-space is given by the parametric equations

$$x = x(t),\ y = y(t).$$

The *position vector*

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

is drawn with its tail at the origin and its head at the point $(x(t), y(t))$ on the path (Fig 1).

The *velocity vector*

$$\vec{v}(t) = x'(t)\vec{i} + y'(t)\vec{j}$$

is drawn with its tail at the point $(x(t),y(t))$ on the path (Fig 1). It points in the instantaneous direction of motion (in the direction of increasing t) so it's tangent to the curve. The instantaneous speed of the particle at time t is $\|\vec{v}(t)\|$.

the differentials ds and $\vec{ds}$

Look at the path $x=x(t)$, $y=y(t)$.

Change t by dt so that a little piece of path is traced out. Let ds be the length of the little piece (Fig 2). Then

(1) $$\boxed{ds = \|v\|\ dt}$$ (because distance = speed × time)

In other words,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

I'd like to get an arrow that approximates the little piece of path (Fig 3). The arrow should be tangent to the little piece, point in the direction of increasing t, and have the same length as the little piece (i.e., have length ds). The arrow v(t) itself has the right direction but its length is wrong. Its length is how far the particle travels in *one* second. If the particle only travels say $\frac{1}{10}$ of a second then the approximating arrow should be $\frac{1}{10}$ v(t). Here, the particle travels dt seconds. So, in general:

(2)

> If $x = x(t)$, $y = y(t)$ and t changes by dt (Fig 3) the piece of path traced out can be approximated by the arrow
>
> $$\vec{v}(t)\ dt = \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right) dt$$
>
> This arrow is variously called $\vec{dr}$, $\vec{ds}$, $\vec{d\ell}$.

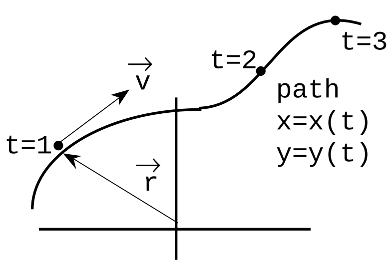


FIG 1

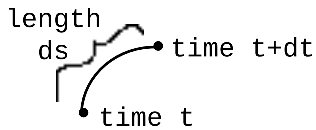


FIG 2

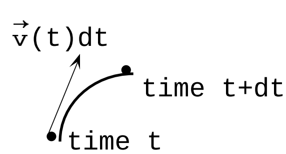


FIG 3

graphing parametric equations

I'll get the graph of the parametric equations $x = t$, $y = t^2$ (which I think of as the position of a particle at time t).

Eliminate the parameter by inspection and get $y = x^2$.
The graph is a parabola (Fig 1). The detailed path of the particle (including direction and timing) is in Fig 2.

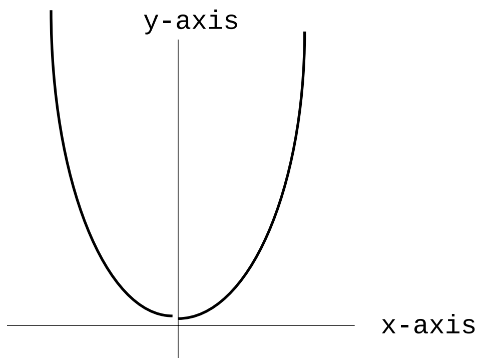


FIG 1

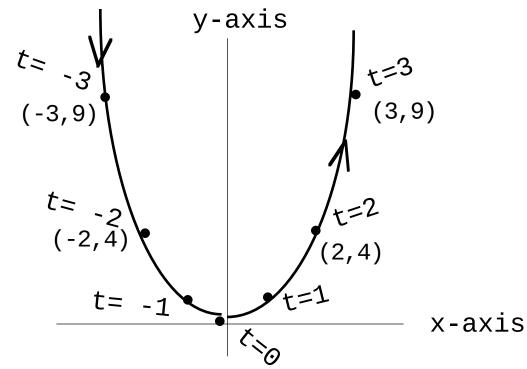


FIG 2

But it's not always that straightforward.
Look at the parametric equations $x = t^2$, $y = t^4$.

Again, $y = x^2$. But this time $x \geq 0$ (since $x = t^2$) and the graph is only half a parabola (Fig 3)
In particular, if the parametric equations describe the position of a particle at time t then the particle moves down the right half of the parabola and then turns around and goes back up again (Fig 4)
(In Fig 4, I drew the down and up paths as if they were different curves so you could see the motion better. But it's all supposed to be happening on the one parabola $y = x^2$.)

NOTICE. Even though I eliminated the parameter to get $y = x^2$, the graph of the parametric equations is *not* the entire parabola $y = x^2$. It's just the right half of the parabola.

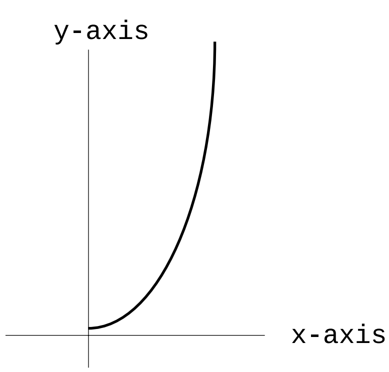


FIG 3

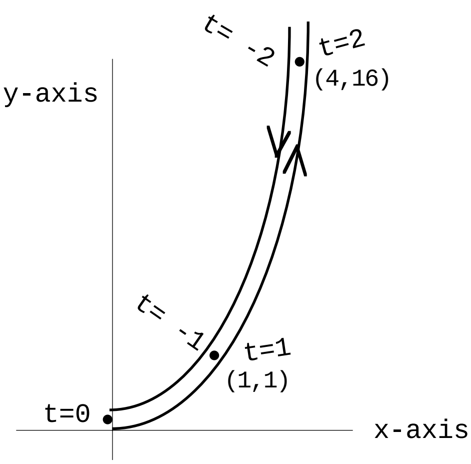


FIG 4

example 1

Sketch the graph of the parametric equations $x = \sin t$, $y = \sin^2 t$.

solution

Again $y = x^2$ but this time $-1 \le x \le 1$ and $0 \le y \le 1$ so the graph is only a portion of the parabola (Fig 5)

If you think of the equations as the path of a particle then the particle moves back and forth forever on a piece of the parabola (Fig 6). For instance, as t goes from $-\pi/2$ to $\pi/2$ the particle moves along the parabola from point $(-1,1)$ to point $(1,1)$; as t goes from $\pi/2$ t0 $3\pi/2$ the particle moves down the parabola from point $(1,1)$ back to $(-1,1)$.

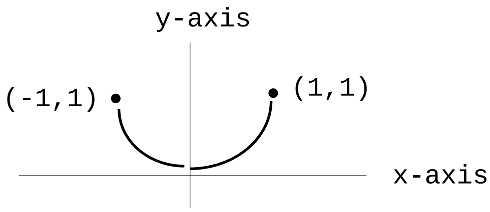


FIG 5

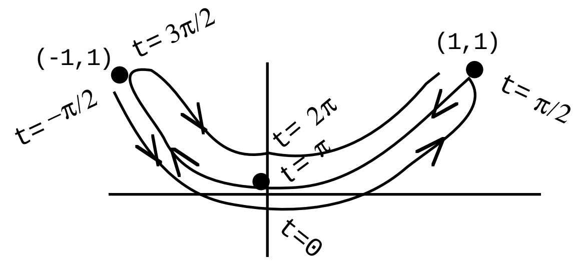


FIG 6

In general, suppose you want the graph of the parametric equations

$$x = x(t)$$
$$y = y(t)$$

and you eliminate the parameter to get a single equation in x and y.
The graph of the parametric equations *might be only part of* the graph of that single equation.
You should look carefully at the parametric equations themselves to decide on the extent of the parametric graph.

Eliminating the parameter does not necessarily tell the whole story.

example 2

Sketch the graph of the parametric equations

$$x = e^t$$
$$y = 3$$

solution

The graph lies on the line $y = 3$ (it takes no effort to "eliminate" the parameter). But the graph of the parametric equations is only part of line $y = 3$, the part where $x > 0$ (Fig 7) because e^t is always > 0.

Fig 8 shows the details.
The particle "starts" at point (0,3) when $t = -\infty$ and moves to the right forever.

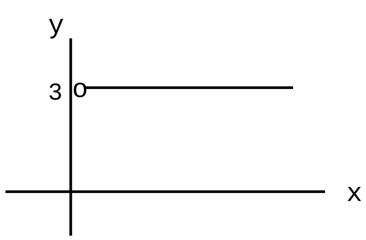


FIG 7

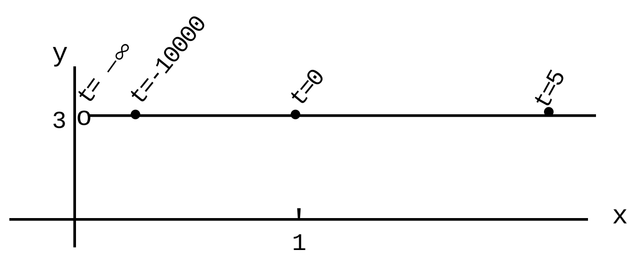


FIG 8

two famous sets of parametric equations

Let a and b be positive numbers.
Look at the parametric equations

$$x = a\cos t$$
$$y = a\sin t \quad \text{(the parameter is t)}$$

Then

$$\begin{aligned} x^2 + y^2 &= a^2\cos^2 t + a^2\sin^2 t \\ &= a^2(\cos^2 t + \sin^2 t) \\ &= a^2 \end{aligned}$$

The graph of the parametric equations is the circle with plain equation $x^2 + y^2 = a^2$. (The particle travels counterclockwise and t is the usual angle θ.)

Look at the parametric equations

$$x = a\cos t$$
$$y = b\sin t \quad \text{(the parameter is t)} \quad (\text{where } a \neq b)$$

Then

$$\frac{x}{a} = \cos t$$
$$\frac{y}{b} = \sin t$$
$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \cos^2 t + \sin^2 t \\ &= 1 \end{aligned}$$

The graph of the parametric equations is the ellipse with plain equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Fig 9). (The particle travels counterclockwise and t is *not* the usual angle θ except at the "corners".)

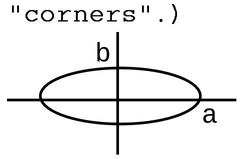


FIG 9

SECTION 2.1 INTRODUCTION

defining a 2-dim coordinate system

Equations of the form

$$x = x(u,v), \quad y = y(u,v)$$

can be thought of as relating a u,v coordinate system and an x,y Cartesian coordinate system, both in the same plane.

A curve of the form $u = u_0$ is called a *v-curve*; only v is changing on it.

A curve of the form $v = v_0$ is called a *u-curve*; only u is changing on it.

example 1 (parabolic coordinates)

The parabolic coordinate system is defined by

(1) $$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad v \ge 0.$$

I want to draw some u-curves and v-curves.

The v-curve u=2

The curve has parametric equations

(2) $$\begin{aligned} x &= \frac{1}{2}(4 - v^2) \\ y &= 2v \\ v &\ge 0 \quad \text{(the parameter is v).} \end{aligned}$$

To help identify the curve, eliminate the parameter (one way to do it is to solve the second equation for v and substitute in the first equation) to get the plain x,y equation

$$x = \frac{1}{2}(4 - \frac{1}{4}y^2),$$

which you should recognize as a parabola. But notice from (2) that $v \ge 0$ and $y = 2v$ so $y \ge 0$. The v-curve u=2 is just the upper half of the parabola (Fig 1).

The v-curve u = -2

The curve has parametric equations

(3) $$\begin{aligned} x &= \frac{1}{2}(4 - v^2) \\ y &= -2v \\ v &\ge 0 \end{aligned}$$

Eliminate the parameter to get the plain equation

$$x = \frac{1}{2}(4 - \frac{1}{4}y^2),$$

the same parabola as before. Notice from (2) that $y \le 0$ this time since $v \ge 0$ so the curve $u = -2$ is the lower half of the u=2 parabola.

The u-curve v=1

The curve has parametric equations

$$\begin{aligned} x &= \frac{1}{2}(u^2 - 1) \\ y &= u \quad \text{(the parameter is u).} \end{aligned}$$

It's a parabola with the plain equation

$$x = \frac{1}{2}(y^2 - 1)$$

Fig 1 shows a piece of parabolic coordinate paper.
The parabolic coordinates u and v don't have geometric significance like Cartesian coordinates x and y (over and up). In Fig 1, I plotted point P with coordinates u=2, v=1 by finding the intersection of the curves u=2, v=1.

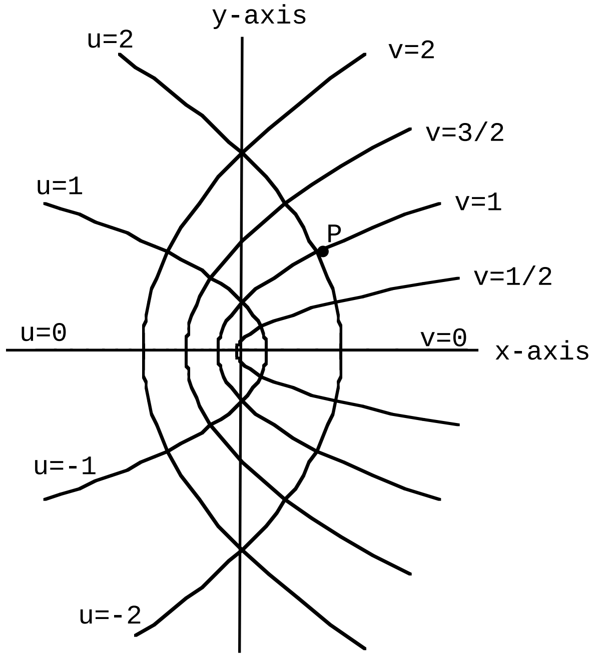


FIG 1 parabolic coordinate paper

polar coordinates

The polar coordinate system is defined by the equations

$$x = r\cos\theta,\quad y = r\sin\theta,\ r \ge 0$$

The coordinates r and θ have the physical significance shown in Fig 2.

r is the distance to the origin; $r = \sqrt{x^2+y^2}$.

θ is the angle measured counterclockwise from the positive x-axis.

Fig 3 shows the θ-curve r=3 (a circle) and the r-curve θ=π/4 (a half-line).

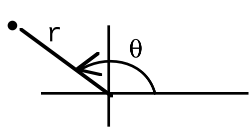


FIG 2

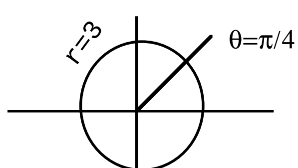


FIG 3

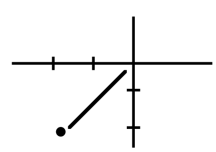


FIG 4

θ versus Arctan[x,y] versus Arctan y/x

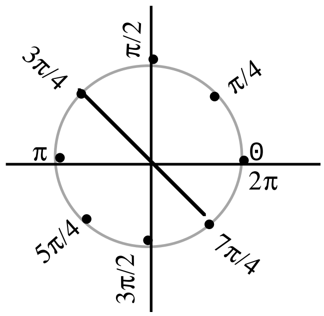
Look at point (−2, −2) (Fig 4). One person could say that it has angle 5π/4. Another person could say it has angle −3π/4. Another could say that it has angle 13π/4 etc. The coordinate θ is not unique.

Fig 5 show some points with θ chosen between 0 and 2π.

Fig 6 shows the same points but θ chosen between −π and π.

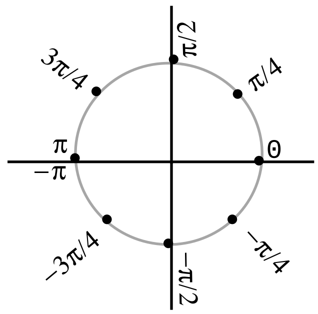
The computer program Mathematica chooses to measure the polar coordinate θ of point (x,y) in the −π to π range and uses the notation ArcTan[x,y] to stand for that unique value of θ (this ArcTan is a function of *two* variables).

```
In[3]:=
ArcTan[-2, -2]
Out[3]=
-3 Pi
-----
  4
```



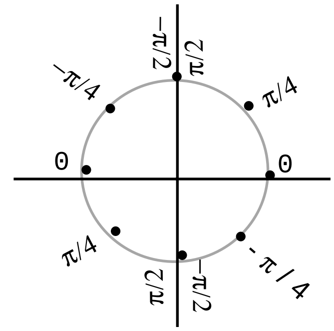
one set of θ values

FIG 5



values of ArcTan[x,y]
(another set of θ values)

FIG 6



values of ArcTan y/x

FIG 7

ArcTan[y/x] is another story (this ArcTan is a function of *one* variable).
Too often, students use the notation Arctan y/x for the polar coordinate θ of point (x,y). *Not right*. Don't do it. Here's what Arctan y/x actually means.

Arctan y/x is the angle *between* $-\pi/2$ *and* $\pi/2$ whose tangent is y/x

Fig 7 shows values of ArcTan y/x at various points.
For instance, for the point (−2, −2), a value of θ is $5\pi/4$. Another is $-3\pi/4$ etc.
You won't get any of these using ArcTan y/x:

y/x = −2/−2 = 1.
There are many angles whose tangent is 1, including $\pi/4$, $-3\pi/4$, $-7\pi/4$ etc.
ArcTan picks the one between $-\pi/2$ and $\pi/2$, namely $\pi/4$ which is not an angle for the original point at all.

```
In[23]:=
ArcTan[-2/-2]
Out[23]=
Pi
---
 4
```

warning ArcTan[x,y] is *one* of the θ values of the point (x,y), the one between $-\pi$ and π.
ArcTan[y/x] may not even go with point (x,y) at all.
It will if the point is in quadrants I or IV but not otherwise.
So, in general, *do not write that* θ = arctan y/x

3-dim u,v,w coordinate systems

Equations of the form

$$x = x(u,v,w),\quad y = y(u,v,w),\quad z = z(u,v,w)$$

can be thought of as relating a u,v,w coordinate system, with the usual x,y,z system.

The graph of $u = u_0$ is a surface, called a coordinate surface.

Similarly, other coordinate surfaces are $v = v_0$ and $w = w_0$.

A u-curve is a curve of the form $v=v_0$, $w=w_0$ (only u changes on it).

A v-curve is a curve of the form $u=u_0$, $w=w_0$ (only v changes on it).

A w-curve is a curve of the form $u=u_0$, $v=v_0$ (only w changes on it).

cylindrical coordinates

The equations

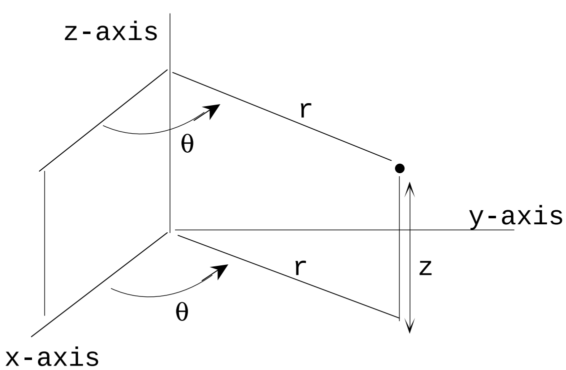
$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z,\quad r \ge 0$$

(i.e., z together with polar coordinates) define cylindrical coordinates.

Fig 8 shows the physical significance of the coordinates:

r is the distance to the z-axis; $r = \sqrt{x^2+y^2}$.
θ is the angle around from the x,z plane.
z measures up and down from the x,y plane as usual.

Fig 9 shows some coordinate surfaces and coordinate curves:
The coord surface $r=3$ is a cylinder.
The coord surface $\theta=\pi/4$ is a half plane.
The coord surface $z=4$ is a plane.
The θ-curve $r=3$, $z=4$ is a circle.
The r-curve $\theta=\pi/4$, $z=4$ is a half-line.
The z-curve $r=3$, $\theta=\pi/4$ is a line.



cylindrical coords r,θ,z

FIG 8

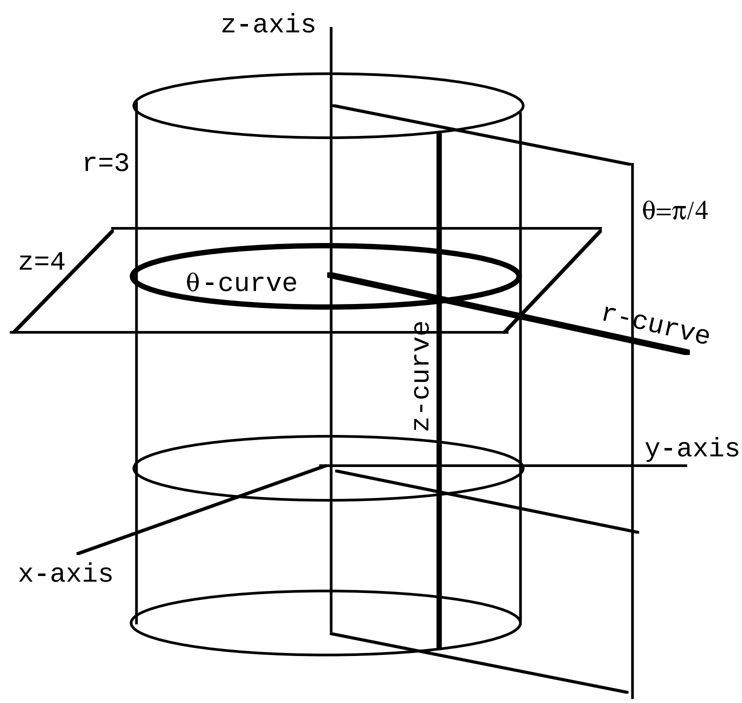


FIG 9

spherical coordinates

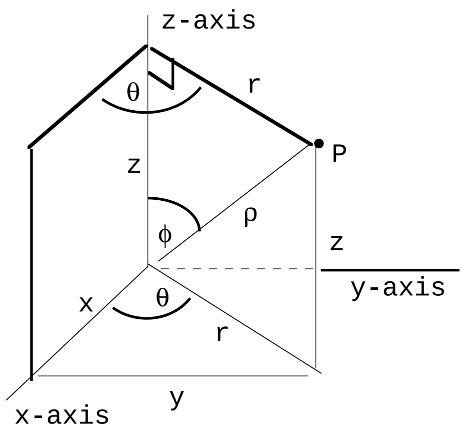
The spherical coord system is defined by the equations

$$(4)\qquad \begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \\ 0 \le \phi &\le \pi,\ \rho \ge 0 \end{aligned}$$

Fig 10 shows spherical (along with cylindrical and Cartesian) coordinates:

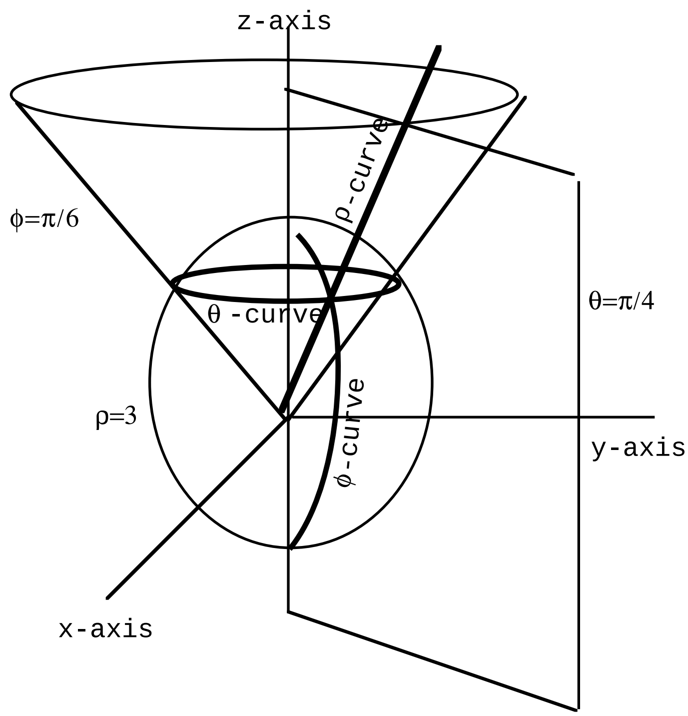
ρ is the distance to the origin; $\rho = \sqrt{x^2+y^2+z^2}$.
θ is the angle around from the x,z plane (same as the cylindrical coord θ).
ϕ is the angle down from the positive z-axis.

If you remember Fig 10 you can figure out the various relations among the three sets of coordinates.



cylindrical coords
spherical coords
Cartesian coords

FIG 10



ρ, ϕ, θ curves

FIG 11

Fig 11 shows the coordinate surfaces and coordinate curves:
The coordinate surface $\rho=3$ is a sphere.
The coordinate surface $\phi=\pi/6$ is a cone.
The coordinate surface $\theta=\pi/4$ is a half-plane.
The ρ-curve $\theta=\pi/4$, $\phi=\pi/6$ is the ray of intersection of a half-plane and a cone.
The ϕ-curve $\rho=3$, $\theta=\pi/4$ is the great semicircle of intersection of a sphere and a half-plane.
The θ-curve $\rho=3$, $\phi=\pi/6$ is the circle of intersection of a sphere and a cone.

Remember Figs 10 and 11

Fig 12 gives a cartographer's view of spherical coordinates. If $\rho=\rho_o$ then the point lies on a sphere centered at the origin with radius ρ_o, an "earth".
On the earth, ϕ measures "down" from the north pole N. On the great circle NASBN, ϕ goes from 0^o at N, to 90^o at A, to 180^o at S, then back to 90^o at B and finally back to 0^o at N. Each parallels of latitude (including the equator) is a circle on which ϕ and ρ are fixed (a θ-curve).
On the earth, θ measures "around" from the prime meridian in the x,z plane. Each meridian of longitude is a great semicircle on which θ and ρ are fixed (a ϕ-curve).

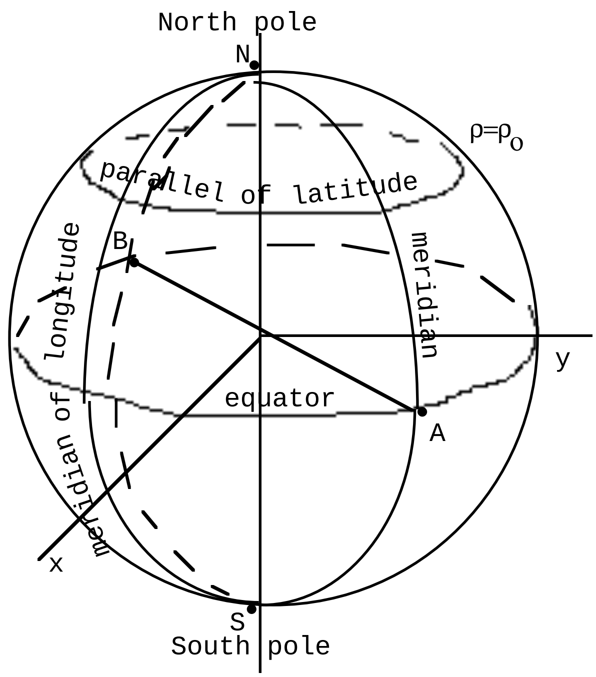


FIG 12

warning

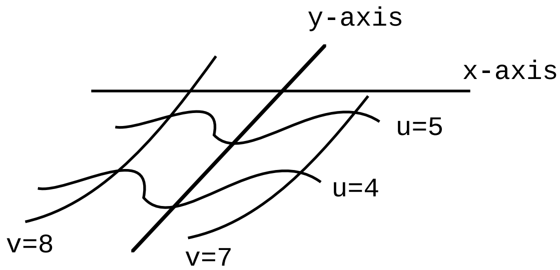
In a u,v system the curve v=2 is *not* a v-curve. It's a u-curve (because u is varying on it).
In a u,v,w system in 3-space, v=2 is *not* a v-curve. It's not a curve at all. It's a *surface.* A v-curve is a curve of the form $u=u_0$, $w=w_0$ on which only v varies.

u,v,z cylindrical coordinates

Any 2-dim u,v coordinate system can be turned into a 3-dim u,v,z in a natural way as follows.
Suppose x = x(u,v), y = y(u,v) define the zippo coord system in 2-space with the u-curves and v-curves shown in Fig 13. Then the equations

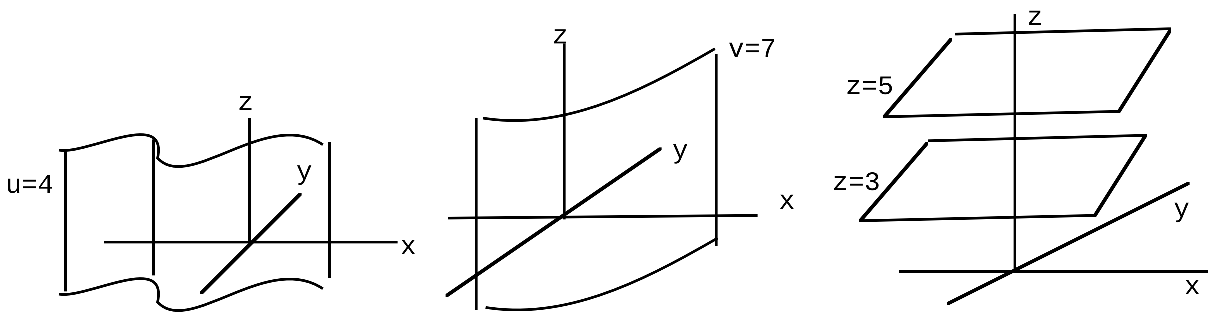
$$x = x(u,v),\ y = y(u,v),\ z = z$$

define the *zippo cylindrical* coord system. Fig 14 show some coordinate surfaces, called cylinders (I couldn't fit them all into one picture).
To draw the picture of the surface u=4 in the lefthand diagram in Fig 14, I copied the curve u=4 from Fig 9 and pasted it above the original curve and drew some connecting vertical lines.
Similarly, the surface v=7 in the center diagram in Fig 14 is gotten by sliding the v=7 curve from Fig 9 up and down in the z direction. The surfaces z=3 and z=5 in the righthand diagram in Fig 10 are planes parallel to the x,y plane.
The u-curve v=7, z=5 in Fig 15 is like the curve v=7 from Fig 3 but raised up 5; it's the intersection of the surface v=7 with the plane z=6. The z-curve u=4,v=7 in Fig 15 is the vertical line of intersection of the surface u=4 and the surface v=7.



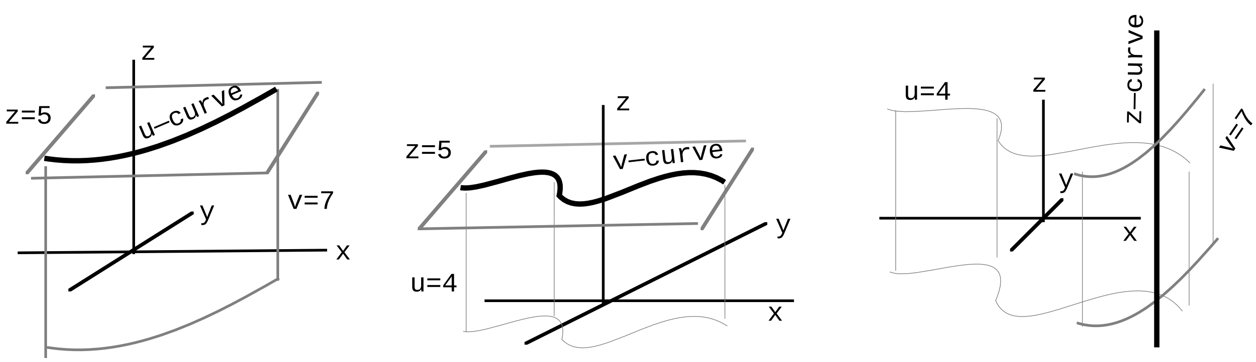
The 2-dim zippo coord system

FIG 13



Some coordinate surfaces in the 3-dim *zippo cylindrical* coordinate system

FIG 14



A u-curve, v-curve, z-curve in the *zippo cylindrical* coord system

FIG 15

PROBLEMS FOR SECTION 2.1

1. Let $x = 2u + 3v$, $y = u - v$ define a u,v coordinate system.
(a) Identify the u-curve that goes through the point $x = 0$, $y = 5$ (find parametric equations for it and then a plain equation).
(b) Identify the u-curve $v = 1$ (find parametric equations for it and then a plain equation).
(c) Identify the v-curve $u = 0$.
(d) Identify the v-curve $u = u_0$ and the u-curve $v = v_0$ in general and then sketch a piece of u,v coordinate paper.

2. Let $x = e^r \cos\theta$, $y = e^r \sin\theta$ define a new coordinate system.
(a) Identify the θ–curve $r=3$.
(b) Identify the r-curve $\theta = \pi/7$.
(c) Sketch a piece of coordinate paper with enough curves on it to make the pattern clear.

3. Suppose the restriction $v \ge 0$ is removed from the parabolic coordinate system in example 1.
(a) How does the v-curve $u=2$ change.
(b) Draw a piece of the new coordinate paper.
(c) What is there about the new coordinate paper that led to the decision to make $v \ge 0$.

4. Sketch in spherical coords
(a) the coordinate surface $\phi = 120^\circ$
(b) the θ-curve $\phi = 120^\circ$, $\rho = 4$
(c) the coordinate surface $\theta = 80^\circ$
(d) the ϕ-curve $\rho = 2$, $\theta = -20^\circ$

5. (a) Sketch the r-curve $\theta = -\pi/2$, $z = 5$ in cylindrical coordinates.
(b) Sketch the θ-curve $r = 4$, $z = 1$ in cylindrical coordinates.
(c) sketch the ρ-curve $\theta = -\pi/2$, $\phi = 30^\circ$ in spherical coordinates.

6. Parabolic cylindrical coordinates are defined by

$$x = \frac{1}{2}(u^2 - v^2),\ y = uv,\ z = z,\ v \ge 0 \quad \text{(parabolic coords plus z)}.$$

Sketch
(a) the coordinate surface $v=1$
(b) the coordinate surface $u=2$
(c) the coordinate surface $z=3$
(d) the u-curve, v-curve, and z-curve through the point P where $u=2, v=1, z=3$

SECTION 2.2 BASIS VECTORS

the vectors $\vec{e}_u$ and $\vec{e}_v$

In a Cartesian coordinate system, vectors are expressed in terms of the basis vectors $\vec{i}$ and $\vec{j}$. Here are the basis vectors used at the point $u=u_0$, $v=v_0$ in a coordinate system defined by $x = x(u,v)$, $y= y(u,v)$ (Fig 1):

$\vec{e}_u$ is a unit vector tangent to the u-curve $v=v_0$ and of the two tangent directions it points toward increasing u.

$\vec{e}_v$ is a unit vector tangent to the v-curve $u=u_0$ and of the two tangent directions it points toward increasing v.

Fig 1 shows e_u and e_v at the point u=1, v=3 in a hypothetical u,v coord system.

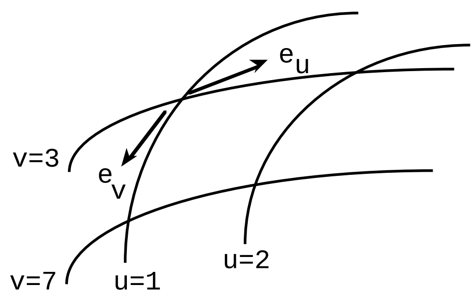


FIG 1

Here's how to find e_u and e_v algebraically. The u-curve $v=v_0$ has parametric equations

$$x = x(u,v_0), \quad y = y(u,v_0) \qquad \text{(the parameter is u)}$$

If you think of u as "time" then the velocity vector is

$$\text{vel}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$$

Then vel_u is tangent to the curve and points in the direction of increasing u so

(1)
$$\boxed{\begin{aligned} \vec{e}_u &= (\text{vel}_u)_{\text{unit}} = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}\right)_{\text{unit}} \\ \vec{e}_v &= (\text{vel}_v)_{\text{unit}} = \left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j}\right)_{\text{unit}} \end{aligned}}$$

Similarly for a u,v,w coordinate systems in 3-space,

(2)
$$\begin{aligned} \vec{e}_u &= (\text{vel}_u)_{\text{unit}} = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}\right)_{\text{unit}} \\ \vec{e}_v &= (\text{vel}_v)_{\text{unit}} = \left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}\right)_{\text{unit}} \\ \vec{e}_w &= (\text{vel}_w)_{\text{unit}} = \left(\frac{\partial x}{\partial w}\vec{i} + \frac{\partial y}{\partial w}\vec{j} + \frac{\partial z}{\partial w}\vec{k}\right)_{\text{unit}} \end{aligned}$$

basis vectors in polar, cylindrical and spherical coordinates

In polar coordinates (Fig 2), e_r points away from the origin (along an r-curve) and e_θ points "around" counterclockwise.

Note that unlike $\vec{i}$ and $\vec{j}$, the basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ vary from point to point.

In cylindrical coordinates (Fig 3), e_r points away from the z-axis, tangent to an r-curve; e_θ points "around" counterclockwise as seen from above, tangent to a θ-curve; $e_z = k$, tangent to a z-curve.

The cylinder in Fig 3 is $r =$ constant. At points on that cylinder,

e_r is perp to the cylinder
e_z lies on the cylinder
e_θ is tangent to the cylinder.

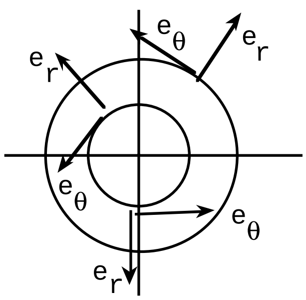


FIG 2 polar coords

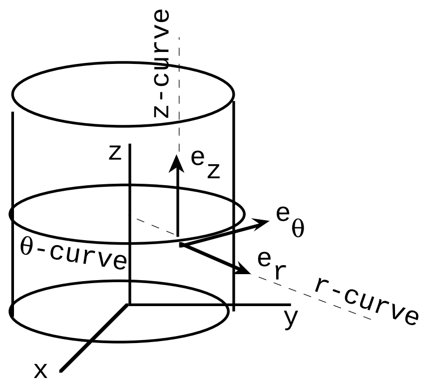


FIG 3 cylindrical coords

In spherical coordinates (Fig 4), e_ρ points away from the origin, tangent to a ρ-curve; e_ϕ points "down" tangent to a circle of longitude, a ϕ-curve; e_θ points "around" counterclockwise as seen from above (as in polar and cylindrical coords), tangent to a θ-curve..

In Fig 4, the sphere is $\rho =$ constant. At points on the sphere,

e_ρ is perp to the sphere
e_θ is tangent to the sphere
e_ϕ is tangent to the sphere

In Fig 4, the first cone is $\phi = \phi_o$ where $0 < \phi_o < \pi/2$; the second cone is $\phi = \phi_o$ where $\pi/2 < \phi_o < \pi$. For each cone, at any point,

e_ρ lies on the cone
e_θ is tan to the cone
e_ϕ is perp to the cone

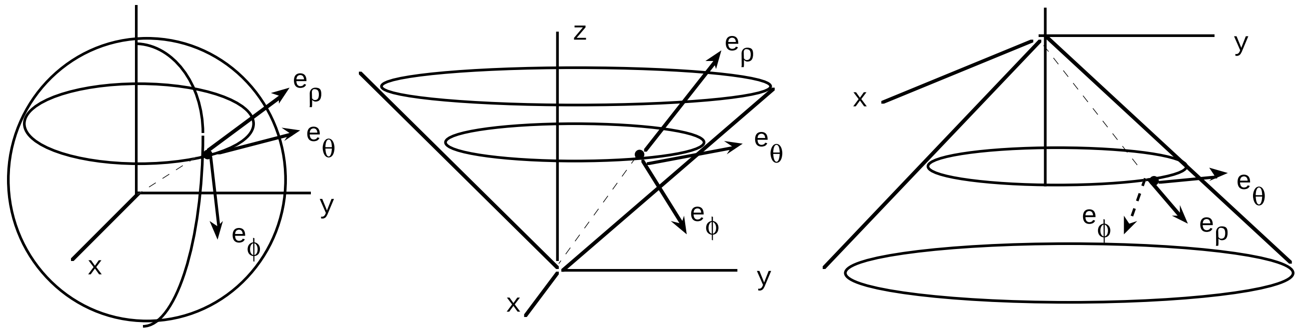


FIG 4 spherical coords

Now I want to find the components (w.r.t. $\vec{i}, \vec{j}, \vec{k}$) of these basis vectors. It's easier for e_r, e_θ, e_z and e_ρ because it's easy to describe their directions (out, around, up, out).

(3a) e_r in polar and cylindrical $= (xi + yj)_{\text{unit}}$

(this is away from the origin in 2-space and away from the z-axis in 3-space)

$$= \frac{x}{\sqrt{x^2+y^2}}\, i + \frac{y}{\sqrt{x^2+y^2}}\, j$$

$$= \frac{r\cos\theta}{r}\, i + \frac{r\sin\theta}{r}\, j$$

$$= \cos\theta\, i + \sin\theta\, j$$

(3b) e_θ in polar, cylindrical and spherical

$$= e_{r_{\text{left turn}}} = (-yi + xj)_{\text{unit}} = -\sin\theta\, i + \cos\theta\, j$$

(3c) $e_z = k$ (by inspection)

(3d) $e_\rho = (xi + yj + zk)_{\text{unit}}$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\, i + \frac{y}{\sqrt{x^2+y^2+z^2}}\, j + \frac{z}{\sqrt{x^2+y^2+z^2}}\, k$$

$$= \sin\phi\cos\theta\, \vec{i} + \sin\phi\sin\theta\, \vec{j} + \cos\phi\, \vec{k}$$

To find e_ϕ, I'll use (2):

$$e_\phi = \left(\frac{\partial x}{\partial\phi}\, i + \frac{\partial y}{\partial\phi}\, j + \frac{\partial z}{\partial\phi}\, k\right)_{\text{unit}}$$

$$= (\rho\cos\phi\cos\theta\, i + \rho\cos\phi\sin\theta\, j - \rho\sin\phi\, k)_{\text{unit}}$$

$$\text{Normalizing factor} = \sqrt{\rho^2\cos^2\phi\cos^2\theta + \rho^2\cos^2\phi\sin^2\theta + \rho^2\sin^2\phi}$$

$$= \rho\sqrt{\cos^2\phi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \sin^2\phi}$$

$$= \rho\sqrt{\cos^2\phi + \sin^2\phi}$$

$$= \rho$$

So

(3e) $e_\phi = \cos\phi \cos\theta\, i + \cos\phi \sin\theta\, j - \sin\phi\, k$

These are all on the reference page (but an exam could ask you to explain where they came from).

example 1 (away from the origin and around the origin)

Let $F(x,y) = x\vec{i} + y\vec{j}$ (Fig 5). The F vector attached to point (x,y) points away from the origin and has length $\sqrt{x^2 + y^2} = r$ (the polar coordinate r). So

$$F = x\vec{i} + y\vec{j} = r\vec{e}_r \quad \text{(Fig 5)}$$

Let $G(x,y) = -y\vec{i} + x\vec{j}$ (Fig 6). Then G is a left turn from F and has the same length as F, so

$$G = -y\vec{i} + x\vec{j} = r\vec{e}_\theta \quad \text{(Fig 6)}$$

Let $H(x,y,z) = xi + yj + zk$. The H vector attached to point (x,y,z) points away from the origin and has length $\sqrt{x^2 + y^2 + z^2} = \rho$. So

$$H = xi + yj + zk = \rho\vec{e}_\rho$$

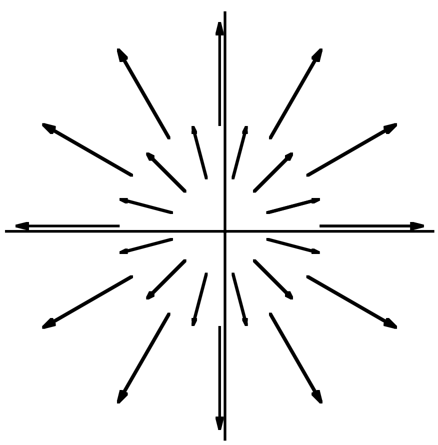


FIG 5 $xi + yj = re_r$

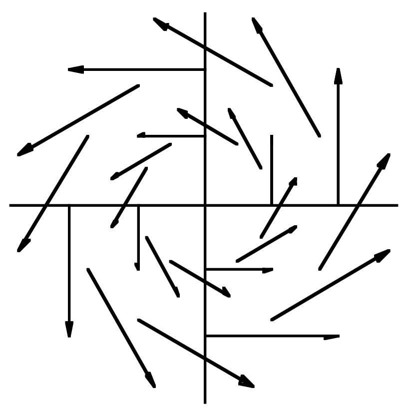


FIG 6 $-yi + xj = re_\theta$

famous electric force fields (do not memorize formulas)

Let F be *the electric force field (force per unit charge) in 3-space due to a unit charge at the origin*. The vectors point away from the origin and at a point which is distance d from the origin the vector has length $1/d^2$ (Fig 7). In Cartesian coords,

$$F(x,y,z) = \frac{1}{x^2 + y^2 + z^2}\,(x,y,z)_{\text{unit}} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\,(x\vec{i} + y\vec{j} + z\vec{k})$$

and in spherical coords,

$$F(\rho,\phi,\theta) = \frac{1}{\rho^2}\, e_\rho$$

Let F be the *electric force field in 3-space due to a line of unit charge density along the z-axis*. The vectors point away from the z-axis and at a point which is distance d from the z-axis the vector has length 1/d (Fig 8). The distance from (x,y,z) to the z-axis is $\sqrt{x^2+y^2}$ so in Cartesian coordinates,

$$F(x,y,z) = \frac{1}{\sqrt{x^2 + y^2}}\,(x,y,0)_{\text{unit}} = \frac{x}{x^2+y^2}\,\vec{i} + \frac{y}{x^2+y^2}\,\vec{j}$$

and in cylindrical coordinates,

$$F(r,\theta,z) = \frac{1}{r}\, e_r.$$

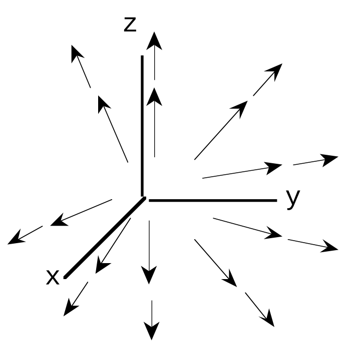
Let F be the *electric force field in 2-space due to a unit charge at the origin*. The vectors point away from the origin and at a point which is distance d from the origin the vector has length $1/d^2$ (Fig 9) In Cartesian coordinates,

$$F(x,y) = \frac{1}{(x^2 + y^2)}\,(x,y)_{\text{unit}} = \frac{x}{(x^2+y^2)^{3/2}}\,\vec{i} + \frac{y}{(x^2+y^2)^{3/2}}\,\vec{j}$$

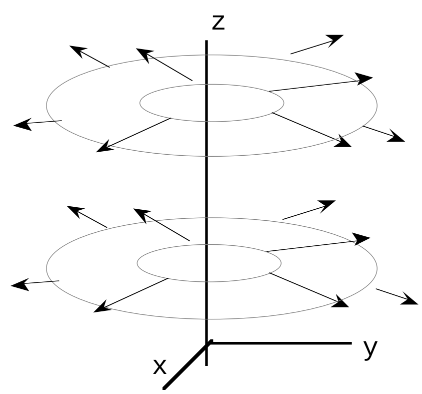
and in polar coords,

$$F(r,\theta) = \frac{1}{r^2}\, e_r$$

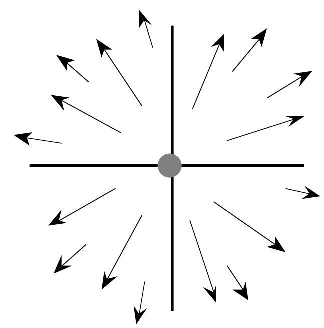
(The field in Fig 9 is a cross section of the field in Fig 7; i.e., the electric field in 2–space due to a unit charge at the origin is part of the electric field in 3-space due to a unit charge at the origin.)



3-dim electric field due to a point charge
FIG 7



electric field due to a line of charge
FIG 8



2-dim electric field due to a point charge
FIG 9

example 2

Let

$$G(x,y,z) = x\sqrt{x^2 + y^2 + z^2}\;\vec{i} + y\sqrt{x^2 + y^2 + z^2}\;\vec{j} + z\sqrt{x^2 + y^2 + z^2}\;\vec{k}$$

Write G in spherical coords.

solution $G = \rho(xi + yj + zk) = \rho\cdot\rho e_\rho = \rho^2\, e_\rho$

example 3

Parabolic coordinates (example 1, Section 2.1) are defined by

$$x = \frac{1}{2}(u^2 - v^2),\quad y = uv,\quad v \ge 0$$

By (1), the basis vectors are

(4)
$$\vec{e}_u = (u\vec{i} + v\vec{j})_{\text{unit}} = \frac{u}{\sqrt{u^2 + v^2}}\,\vec{i} + \frac{v}{\sqrt{u^2 + v^2}}\,\vec{j}$$

$$\vec{e}_v = (-v\vec{i} + u\vec{j})_{\text{unit}} = \frac{-v}{\sqrt{u^2 + v^2}}\,\vec{i} + \frac{u}{\sqrt{u^2 + v^2}}\,\vec{j}$$

At point u = 2, v = 1 for instance,

(5) $\vec{e}_u = \frac{2}{\sqrt{5}}\,\vec{i} + \frac{1}{\sqrt{5}}\,\vec{j}$ and $\vec{e}_v = -\frac{1}{\sqrt{5}}\,\vec{i} + \frac{2}{\sqrt{5}}\,\vec{j}$ (Fig 10)

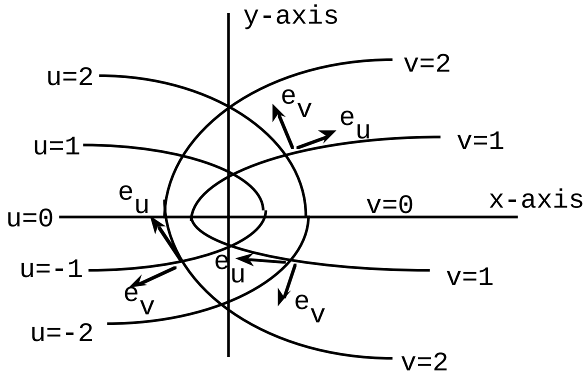


FIG 10

warning

1. Remember that e_u and e_v are *unit* vectors.

After finding $\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$ you have to *normalize* it to get e_u.

2. If a point has coordinates u=2, v=1 don't call the point (2,1). This notation is reserved for the point x=2, y=1 in the usual x,y system. You can use ((2,1)) or $(2,1)_{u,v}$ but you must distinguish it from the usual (2,1).

If a vector p is $3e_u + 7e_v$, don't wrote p = (3,7) because that notation is reserved for 3i + 7j.

orthogonal coordinate systems

A 2-dim coordinate system is called *orthogonal* if e_u and e_v are perpendicular at each point, i.e., if $e_u \cdot e_v = 0$, i.e., if the u-curves and v-curves intersect perpendicularly.

Similarly a 3-dim coordinate system is orthogonal if e_u, e_v, e_w are pairwise perpendicular at each point, i.e., if the u-curves, v-curves, w-curves intersect perpendicularly, i.e., if the coordinate surfaces intersect perpendicularly.

Polar coordinates, cylindrical coordinates and spherical coordinates are orthogonal coordinate systems (I can tell from the geometry –– see problem 14 for an algebraic argument). So is the less famous parabolic coord system (see problem 13).

To test for orthogonality, you can test vel_u, vel_v and vel_w instead of the unit vectors e_u, e_v, e_w since they have the same directions respectively.

warning

The definition of e_u and e_v says *nothing* about perpendicularity. They are not necessarily perp to each other or perp to u-curves and v-curves. It *may* happen, but not necessarily.

example 4

Let $x = 2u + 3v^2$, $y = u - v$.
Is the u,v coordinate system orthogonal.

solution

$vel_u = 2i + j$, $vel_v = 6vi - j$.

$vel_u \cdot vel_v = 12v - 1$. This is not 0 at every point (it is only 0 at points where $v = 1/12$). So the coord system is not orthogonal.

physical significance of the scalars a and b when $p = ae_u + be_v$ in an orthogonal u,v coord system

First of all, remember that the phrase "component of $\vec{p}$ in direction e_u" has a precise meaning. It means "signed projection onto the e_u line" (see Section 1.0)

Look at Fig 11, where $\vec{p} = 3e_u - 2e_v$.
The scalar 3 is the component of p in the direction of e_u (i.e., signed projection) (see (4) in Section 1.0), and the scalar -2 is the component of p in the direction of e_v. This worked because e_u and e_v in Fig 11 are orthogonal.

It doesn't work when e_u and e_v are not orthog (which is one of the many reasons why we usually prefer to use *orthog* coord systems). In Fig 12, $p = 3e_u + 2e_v$. But the component of p in direction e_u is 5, not 3, and the component of p in direction e_v is 4, not 2. If p is a force vector and you want to know "how many pounds of force push you in the e_u direction", the answer is 5 pounds, not 3 pounds.

So what *would* you call the scalars 3 and 2 in Fig 12? You don't have to call them anything. Or call them the first coord and second coord of p in the u,v coord system. Or even the first component and second component of p. But don't call them "the components of p in the e_u and e_v directions" because they are not signed projections.

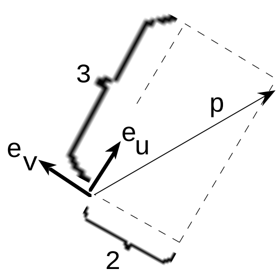


FIG 11

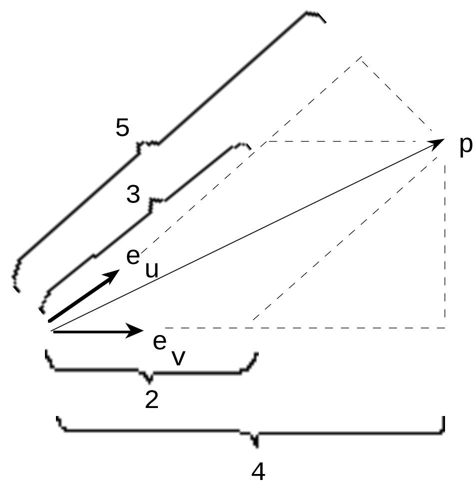


FIG 12

how to convert a vector from Cartesian coords to u,v coordinates

As just decided, in an *orthogonal* coordinate system, when $\vec{p}$ is written as $ae_u + be_v$, the scalars a and b are the signed projections of p onto e_u and e_v.

By the formulas for signed projections ((4) in Section 1.0),

(6) $$p = (p \cdot e_u)\, e_u + (p \cdot e_v)\, e_v \qquad \text{(Fig 13)}$$

The formula in (6) doesn't work if the coordinate system is not orthogonal.

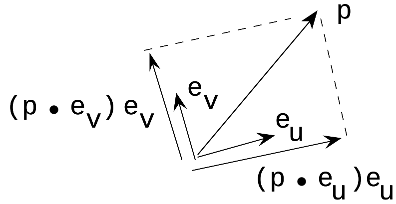


FIG 13

example 5

Let q = 3i + 2j.
Switch to the parabolic coordinate system (an orthogonal system).

(a) Write q in terms of the basis vectors at point u=2, v=1
(b) Write q in terms of the basis vectors at point u = 3, v=1.

solution

(a) I found the basis vectors in (5): At the point where u=2, v=1,

$$\vec{e}_u = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}, \qquad \vec{e}_v = -\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$$

By (6),

$$q = (q\cdot e_u)e_u + (q\cdot e_v)e_v = \frac{8}{\sqrt{5}}e_u + \frac{1}{\sqrt{5}}e_v$$

(b) The basis vectors in general are in (4). The basis vectors at point u = 3, v=1 are

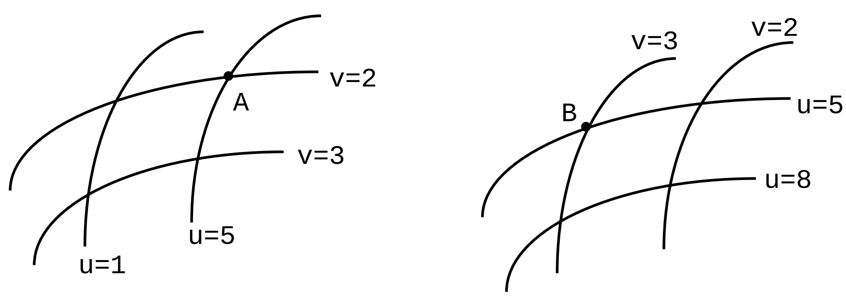
$$e_u = \frac{3}{\sqrt{10}}i + \frac{1}{\sqrt{10}}j, \quad e_v = \frac{-1}{\sqrt{10}}i + \frac{3}{\sqrt{10}}j.$$

By (6),

$$q = \frac{11}{\sqrt{10}}e_u + \frac{3}{\sqrt{10}}e_v$$

PROBLEMS FOR SECTION 2.2

1. Sketch e_u and e_v at point A and point B in the diagrams.



2. Sketch in cylindrical coordinates
   (a) the coordinate surface r=3
   (b) the coordinate surface $\theta = 2\pi/3$
   (c) the coordinate surface z = 1
   (d) the r-curve, θ-curve and z-curve through the point r=3, $\theta = 2\pi/3$, z=1 and the basis vectors at the point.

3. Sketch in spherical coordinates
   (a) the coordinate surface ρ=2
   (b) the coordinate surface $\phi = 2\pi/3$
   (c) the coordinate surface $\theta=2\pi/3$
   (d) the ϕ-curve, ρ-curve, θ-curve through the point $\phi=\pi/6$, $\theta = -\pi/3$, $\rho = 1$ and the basis vectors at the point.

4. I found e_r and e_θ (polar coords) in (3), mostly with geometry. Use the formula in (1) to find them again and check that they are perpendicular.

5. Sketch the vector field $\cos\theta\ e_r$ in 2-space (polar coords).

6. Sketch the vector field ve_u in parabolic coordinates.

7. Plot these vector fields in 3-space. Draw enough arrows to make the pattern clear and also write a brief description.
 A picture in 3-space usually looks clearer if there are "props" in the picture, like circles, spheres, cones, planes etc.
(a) re_θ (cylindrical coords)
(b) ρe_θ (spherical coords)
(c) ze_r (cylindrical coords)
(d) $\ell n\ r\ \vec{k}$ (cylindrical coords)
(e) $\sin\phi\ e_\rho$ (spherical coords)

8. Convert these vector fields to polar or spherical coordinates.
(a) $\dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j}$ (b) $-\dfrac{x}{x^2+y^2}\vec{i} - \dfrac{y}{x^2+y^2}\vec{j}$ (c) $x\vec{i} + y\vec{j} + z\vec{k}$

9.(a) Find $k \cdot e_\phi$ (a) algebraically and (b) geometrically.

10. Parabolic cylindrical coordinates (parabolic coords plus z) are defined by

$$x = \frac{1}{2}(u^2 - v^2),\ y = uv,\ z = w,\ v \ge 0$$

Sketch the basis vectors at the point P where u=2, v=1, z=3.

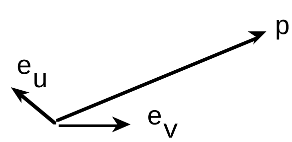
11. Verify that the parabolic coord system is orthogonal.

12. Let $x = x(u,v)$, $y = y(u,v)$ define a u,v coordinate system.
Find whichever of the following are possible with just this info:
$e_v \cdot e_v$, $e_u \cdot e_v$, $\|e_u\|$

13. I thought it was clear from geometry that the spherical coordinate basis vectors were orthogonal. Use (3b), (3(d), (3e) to check it algebraically.

14. Let $F = x\vec{i}$. Convert F to the parabolic coordinate system.

15. The diagram shows e_u, e_v and p.
(a) Estimate a and b so that $p = ae_u + be_v$.
(b) Estimate the component of p in the direction of e_u (i.e., signed projection) and the component of p in the direction of e_v.



16. Let $x = u^2 + uv$, $y = u - v$.
(a) Is the coordinate system orthogonal
(b) Find e_u and e_v.

17. Let $F = \vec{k}$. Convert F to spherical coordinates.

18. The formulas in (1) express the basis vectors e_u and e_v in terms of *i and j*. Now write e_u and e_v in terms of e_u *and* e_v.

19. The formulas in (3a) and (3b) express e_r and e_θ in terms of i and j:

$$e_r = \cos\theta\ i + \sin\theta\ j$$
$$e_\theta = -\sin\theta\ i + \cos\theta\ j$$

Now write i and j in terms of e_r and e_θ.

SECTION 2.3 DOTS, NORMS AND CROSS PRODUCTS IN ORTHOGONAL COORDINATE SYSTEMS

norms and dot products in an orthogonal coord system

Start with an *orthogonal* coordinate system with basis vectors $\vec{e}_u$, $\vec{e}_v$, $\vec{e}_w$. Norms and dots are done "as usual". In particular, if

$$\vec{a} = a_u\vec{e}_u + a_v\vec{e}_v + a_w\vec{e}_w$$

$$\vec{b} = b_u\vec{e}_u + b_v\vec{e}_v + b_w\vec{e}_w$$

then

(1) $$\|a\| = \sqrt{a_u^2 + a_v^2 + a_w^2}$$

(2) $$\vec{a}\cdot\vec{b} = a_u b_u + a_v b_v + a_w b_w$$

geometric proof of (1) in 2-space

Since the coord system is orthogonal, a_u is the signed projection of $\vec{a}$ onto e_u and similarly a_v is the signed projection of $\vec{a}$ onto e_v (Fig 1). So, by the Pythagorean theorem, $\|\vec{a}\|^2 = a_u^2 + a_v^2$.

algebraic proof of (2) in 2-space

$$\begin{aligned} a\cdot b &= (a_u\vec{e}_u + a_v\vec{e}_v)\cdot(b_u\vec{e}_u + b_v\vec{e}_v) \\ &= a_u b_u\, e_u\cdot e_u + a_u b_v\, e_u\cdot e_v + a_v b_u\, e_u\cdot e_v + a_v b_v\, e_v\cdot e_v \quad \text{by dot properties} \end{aligned}$$

But $e_u\cdot e_v = 0$ since the system is orthogonal. And $e_u\cdot e_u = 1$, $e_v\cdot e_v = 1$ since e_u and e_v are unit vectors (see property (g) of dot products in §1.0). So $a\cdot b = a_u b_u + a_v b_v$.

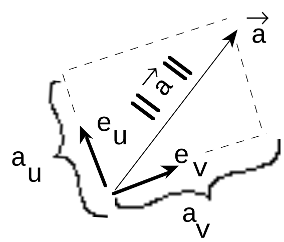


FIG 1

example 1

Let p = 2i + 8j.

(a) Express p in terms of e_r and e_θ at point $r=2$, $\theta=\pi/3$, in polar coordinates.

(b) Find $\|p\|$, using p = 2i + 8j

(c) Find $\|p\|$ again, but this time use your p in terms of e_r and e_θ from part (a).

solution (a) From §2.2 (and the reference page),

$e_r = \cos\theta\ i + \sin\theta\ j$, $e_\theta = -\sin\theta\ i + \cos\theta\ j$

At point $r=2$, $\theta=\pi/3$, $e_r = \frac{1}{2}\,i + \frac{1}{2}\sqrt{3}\,j$, $e_\theta = -\frac{1}{2}\sqrt{3}\,i + \frac{1}{2}\,j$.

Since the polar coord system is orthogonal (*this wouldn't work otherwise*), you can use (6) in the preceding section to write p in terms of e_r and e_θ:

$$p = (p\cdot e_r)e_r + (p\cdot e_\theta)e_\theta = (1 + 4\sqrt{3})e_r + (-\sqrt{3} + 4)e_\theta$$

(b) $\|p\| = \sqrt{4 + 64} = \sqrt{68}$.

(c) Since e_r and e_θ are orthogonal (*this wouldn't work otherwise*),

$$\|p\| = \sqrt{(1+4\sqrt{3})^2 + (-\sqrt{3}+4)^2}$$

$$= \sqrt{1 + 48 + 8\sqrt{3} + 3 - 8\sqrt{3} + 16}$$

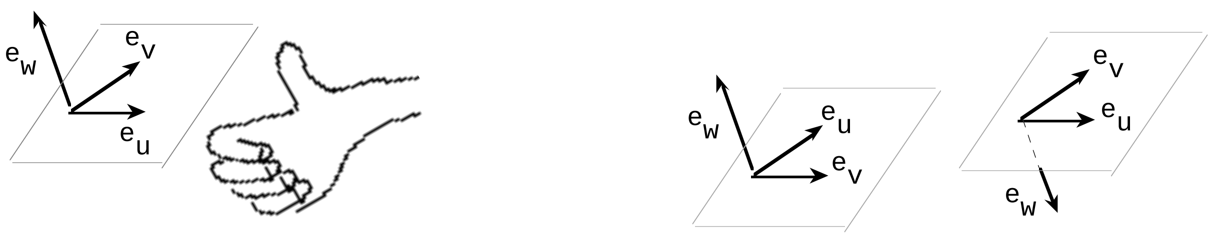
$$= \sqrt{68} \quad \text{(same as in part (b), as it should be)}$$

righthanded sets of basis vectors

A set of basis vectors e_u, e_v, e_w, *in that order*, is called *righthanded* if your thumb points more or less like (makes an acute angle with) e_w when the fingers of your right hand curl like e_u turning into e_v (Fig 2); if e_u, e_v, e_w are orthogonal as well as righthanded then your thumb would point *exactly* like e_w.

For example, Fig 3 shows several sets of axes for which the vectors i, j, k, in that order are righthanded.

For example, in Fig 4 (where the underlying i,j,k coord system is righthanded), the basis vectors e_ρ, e_ϕ, e_θ, in that order, are righthanded; if your fingers curl like e_ρ turning into e_ϕ your thumb will point like e_θ. Also in Fig 4, e_θ, e_ρ, e_ϕ are righthanded and so are e_ϕ, e_θ, e_ρ.



righthanded e_u, e_v, e_w lefthanded e_u, e_v, e_w

FIG 2

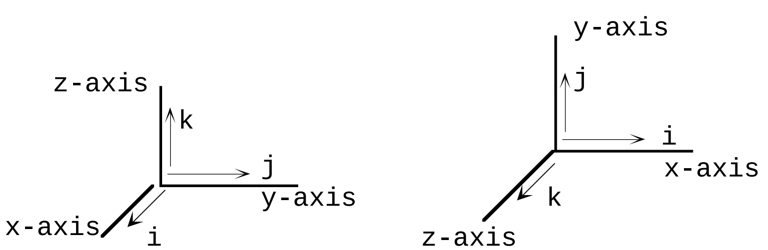


FIG 3 righthanded i, j, k

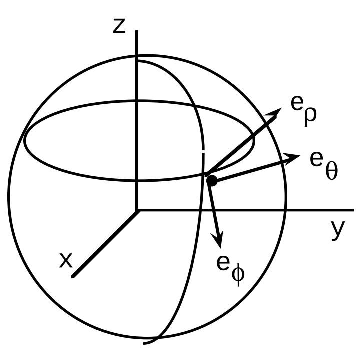


FIG 4

cross products using orthogonal *righthanded* basis vectors

$\vec{a} \times \vec{b}$ is defined as follows, independent of any coordinate system:

length $\quad \|a \times b\| = \|a\|\ \|b\| \sin\theta$ where θ is the angle between a and b.

direction $\quad$ a × b is perp to both a and b and furthermore, of the two perp directions, it points like your thumb when the fingers of your righthand curl like a turning into b.

In a previous calculus course you were shown that if you use righthandeded i, j, k as in Fig 3 then the cross product can be computed like this:

If

$$a = 2i + 3j + 4k$$
$$b = i - j + 5k$$

then

$$(3) \qquad a \times b = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 19i - 6j - 5k$$

Suppose you continue with any of the coordinate systems in Fig 3 but rewrite a and b like this (same vectors but write the components in a different order):

$$a = 4k + 2i + 3j$$
$$b = 5k + i - j$$

When you compute

$$\begin{vmatrix} k & i & j \\ 4 & 2 & 3 \\ 5 & 1 & -1 \end{vmatrix}$$

you get $-5k + 19i - 6j$, same as before; i.e., you still get a × b. Notice that k,i,j in Fig 3 are still righthanded.

But suppose you rewrite a and b like this:

$$a = 3j + 2i + 4k$$
$$b = -j + i + 5k$$

When you compute

$$\begin{vmatrix} j & i & k \\ 3 & 2 & 4 \\ -1 & 1 & 5 \end{vmatrix}$$

it turns out to be $6j - 19i + 5k$. But this is the $-(a \times b)$. Notice that j, i, k in Fig 3 is a lefthanded triple

Here is what mathematicians say to straighten this all out (proofs omitted).

In an orthogonal coordinate system where the basis vectors e_u, e_v, e_w are righthanded, to get the cross product (with the length and direction described above) use this rule:

If $\vec{a} = a_u \vec{e}_u + a_v \vec{e}_v + a_w \vec{e}_w$ and $\vec{b} = b_u \vec{e}_u + b_v \vec{e}_v + b_w \vec{e}_w$ then

(4) $$a \times b = \begin{vmatrix} e_u & e_v & e_w \\ a_u & a_v & a_w \\ b_u & b_v & b_w \end{vmatrix}$$

If e_u, e_v, e_w are *lefthanded* and you use the determinant in (4) you will end up with $-(a \times b)$.

example 2

I'll use the spherical coordinate system in Fig 4. Suppose

$$a = 2e_\rho$$
$$b = 3e_\phi + 4e_\theta$$

Choose a righthanded order, like e_ρ, e_ϕ, e_θ. Then

$$a \times b = \begin{vmatrix} e_\rho & e_\phi & e_\theta \\ 2 & 0 & 0 \\ 0 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{2} & 0 & 0 \\ \cancel{0} & 3 & 4 \end{vmatrix} e_\rho \quad - \begin{vmatrix} 2 & \cancel{0} & 0 \\ 0 & \cancel{3} & 4 \end{vmatrix} e_\phi \quad + \begin{vmatrix} 2 & 0 & \cancel{0} \\ 0 & 3 & \cancel{4} \end{vmatrix} e_\theta$$

$$= -8e_\phi + 6e_\theta$$

Since e_θ, e_ρ, e_ϕ is also a righthanded triple you can find $a \times b$ using

$$\begin{vmatrix} e_\theta & e_\rho & e_\phi \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{vmatrix}$$

But e_ρ, e_θ, e_ϕ are lefthanded so *do not use*

$$\begin{vmatrix} e_\rho & e_\theta & e_\phi \\ 2 & 0 & 0 \\ 0 & 4 & 3 \end{vmatrix}$$

(or if you do, remember that this is $-(a \times b)$).

example 3

Let

$$a = 2e_u + 3e_v - e_w$$
$$b = 5e_u + 2e_v + 4e_w$$

in a u,v,w coordinate system. If the system is orthogonal then

$$a \cdot b = 10 + 6 - 4 = 12$$
$$\|a\| = \sqrt{14}$$

And if e_u, e_v, e_w are righthanded as well as orthogonal then

$$a \times b = \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} e_u - \begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix} e_v + \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} e_w$$

$$= 14e_u - 13e_v - 11e_w$$

PROBLEMS FOR SECTION 2.3

1. If $a = 2e_\rho - 3e_\phi + e_\theta$ and $b = 4e_\rho + 2e_\phi - 5e_\theta$ (spherical coordinates) find $a \cdot b$ and $a \times b$.

2. Let $p = i + 4j$, $q = 5i - j$.
(a) Express p and q in terms of e_u and e_v at point u=2, v=3 in parabolic coordinates.
(b) Find $p \cdot q$ and $\|p\|$, using the Cartesian coordinates of p and q.

(c) Find $p \cdot q$ and $\|p\|$ again, but this time using the representation of p and q in terms of e_u and e_v from part (b).

3. Let $\vec{a} = e_\rho$ (spherical coordinates)

$\vec{b} = e_\theta$ (spherical coordinates)

(a) Find $\vec{a} \times \vec{b}$ without any algebra at all, just by looking at Fig 4 and thinking about the length and direction of the cross product.
(b) Find $\vec{a} \times \vec{b}$ again using their coordinates in the spherical coordinate system.

(c) In Cartesian coordinates,

$$a = (xi + yj + zk)_{\text{unit}} \qquad \text{(away from the origin)}$$

$$b = (-yi + xj)_{\text{unit}}$$

Find $\vec{a} \times \vec{b}$ again using the Cartesian coordinates of $\vec{a}$ and $\vec{b}$ and keep going until you see that it agrees with (a) and (b).

4. Suppose that at a point, $e_u = (2i+k)_{\text{unit}}$, $e_v = (2i+j+3k)_{\text{unit}}$, $e_w = (i+4j+k)_{\text{unit}}$
Are e_u, e_v, e_w (in that order) a righthanded triple?

5. The vector $\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$ is a unit vector and points sort of northeast.

Look at $\frac{3}{5} e_u + \frac{4}{5} e_v$ in a u,v coord system.
(a) Is it a unit vector.
(b) Does it point sort of northeast.

SECTION 2.4 SCALE FACTORS

the scale factors h_u and h_v

Start with a u,v coordinate system given by

$$x = x(u,v), \quad y = y(u,v)$$

Let ds_u be the arc length traced out when u starts at u_0 and changes by du and v stays fixed at v_0 (Fig 1). The little piece of u-curve traced out has parametric equations

$$x = x(u,v_0), \quad y = y(u,v_0), \quad u_0 \le u \le u_0 + du \qquad \text{(the parameter is u)}$$

If you think of the parameter u as time then there is a velocity vector

$$\text{vel}_u = \frac{\partial x}{\partial u}\, \mathbf{i} + \frac{\partial y}{\partial u}\, \mathbf{j}.$$

By (1) in §2.0,

$$ds_u = \|\text{vel}_u\|\, du \qquad \text{(distance = speed} \times \text{time).}$$

The factor $\|\text{vel}_u\|$ is called the *arc length magnification factor* and is denoted by h_u. So all in all,

(1)
$$\boxed{ds_u = h_u\, du}$$

where

(2)
$$\boxed{h_u = \|\text{vel}_u\| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}}$$

You can think of h_u as speed if you think of u as time. The actually units on h_u are meters/u-unit.

Similarly if u is fixed at u_0 and v changes by dv then arc length ds_v is traced out on the v-curve $u=u_0$ where

$$ds_v = h_v\, dv$$

$$h_v = \|\text{vel}_v\| = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2}$$

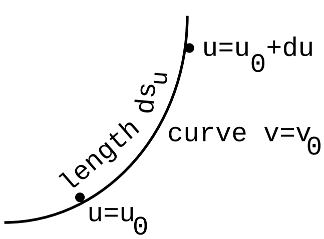


FIG 1

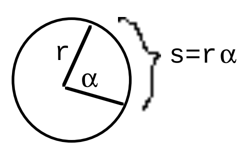


FIG 2

review of a geometry formula

Let s be the arc length cut off by a central angle of α radians (Fig 2). Then $s = r\alpha$.

scale factors in polar coordinates

$$\boxed{\begin{aligned} h_r &= 1 \\ h_\theta &= r \end{aligned}} \tag{3}$$

You can derive these scale factors algebraically using (1) (see problem 1) but you can also see them geometrically as follows:

Fig 3 shows r changing by dr. ds_r *is* dr so $h_r = 1$.

Fig 4 shows θ changing by dθ. By the rule in Fig 2, the arc length traced out is $ds_\theta = r\,d\theta$. So $h_\theta = r$. This means that for the same dθ (Fig 5) you get larger ds_θ when r is larger: If you start at point A in Fig 5 and change θ by dθ you trace out arc length AB. But if you start at point C, where r is larger, and change θ by the same dθ you trace out the larger arc length CD.

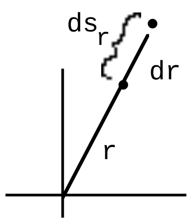


FIG 3

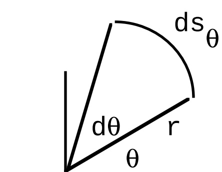


FIG 4

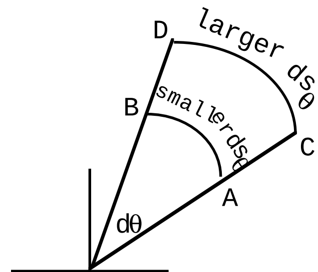


FIG 5

scale factors in cylindrical coordinates

$$\boxed{\begin{aligned} h_r &= 1 \\ h_\theta &= r \\ h_z &= 1 \end{aligned}} \tag{4}$$

scale factors in a u,v,z coordinate system (see the zippo cylindrical coordinate system in Section 2.1)

h_z is always 1 because when you change z by dz,you get arclength dz.

example 1 (scale factors in parabolic coordinates)

In the parabolic system,

$$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad v \ge 0.$$

So

$$\text{vel}_u = u\mathbf{i} + v\mathbf{j}$$

$$\text{vel}_v = -v\mathbf{i} + u\mathbf{j}$$

$$h_u = \|\text{vel}_u\| = \sqrt{u^2 + v^2} \quad \text{(by (1))}$$

$$h_v = \|\text{vel}_v\| = \sqrt{u^2 + v^2}$$

If u=3 and v=2 then $h_v = \sqrt{13}$. A particle starting at point u=3, v=2 and moving so that u stays fixed while v goes up by dv will move distance $ds_v = \sqrt{13}\,dv$ (Fig 6).

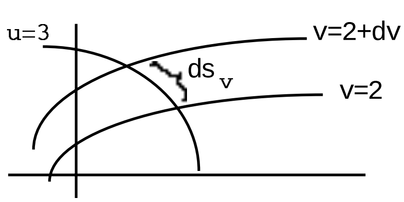


FIG 6

scale factors in spherical coordinates

(5)
$$\boxed{\begin{aligned} h_\rho &= 1 \\ h_\phi &= \rho \\ h_\theta &= \rho \sin\phi \end{aligned}}$$

Here's the geometric reason for these scale factors.
Fig 7 shows a ρ-curve where ϕ and θ are fixed and ρ changes by $d\rho$ sweeping out arc length ds_ρ. By inspection, $ds_\rho = d\rho$ so $h_\rho = 1$

Fig 8 shows a ϕ-curve where ρ and θ are fixed and ϕ changes by $d\phi$ sweeping out arc length ds_ϕ. By the formula in Fig 2, $ds_\phi = \rho\, d\phi$ so $h_\phi = \rho$.

Fig 9 shows a θ-curve where ρ and ϕ are fixed and θ changes by $d\theta$, sweeping out arc length ds_θ. Angle A in triangle ABC is 90° so $AB = \rho \sin\phi$. Then, by the formula in Fig 2, $ds_\theta = AB\, d\theta = \rho \sin\phi\, d\theta$. So $h_\theta = \rho \sin\phi$.

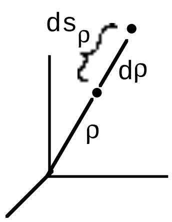


FIG 7

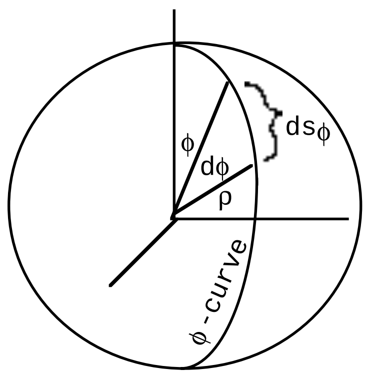


FIG 8

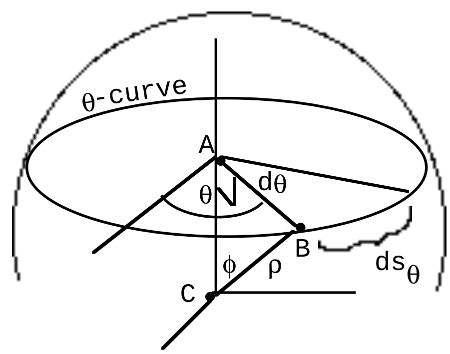


FIG 9

connection between e_u and h_u

The normalizing factor that turns the tangent vector vel_u into the *unit* vector e_u is $\|vel_u\|$ which happens to be h_u. So

(6)
$$\boxed{\vec{e}_u = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u} j\right)_{unit} = \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}\right)}$$

For example, in spherical coordinates,

$$\vec{e}_\phi = \frac{1}{h_\phi}\left(\frac{\partial x}{\partial \phi}\vec{i} + \frac{\partial y}{\partial \phi}\vec{j} + \frac{\partial z}{\partial \phi}\vec{k}\right)$$

$$= \frac{1}{\rho}\left(\rho \cos\phi \cos\theta\, \vec{i} + \rho\cos\phi\sin\theta\, \vec{j} - \rho\sin\phi\, \vec{k}\right)$$

(7)
$$= \cos\phi\cos\theta\,\vec{i} + \cos\phi\sin\theta\,\vec{j} - \sin\phi\,\vec{k}$$

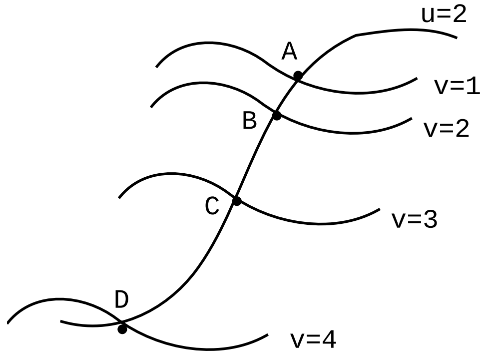
PROBLEMS FOR SECTION 2.4

1. Find the scale factors in polar coordinates algebraically.

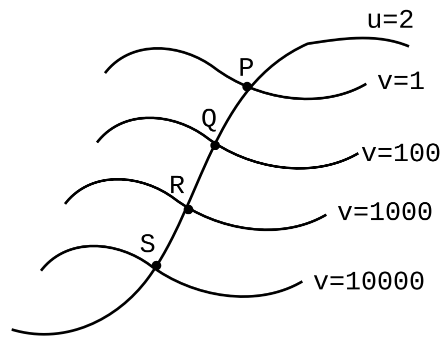
2. Find the spherical coordinate scale factors algebraically.

3. Define a u,v coordinate system with the equations $x = u^2 v^3$, $y = 3u + 2v$. A particle moves from point $u = 2$, $v = 1$ so that v stays fixed and u changes by du. How far does it go.

4. The diagrams show some u,v coordinate paper.
(a) Which of points A,B,C,D has the largest h_v.
(b) Which of points P,Q,R,S has the largest h_v.

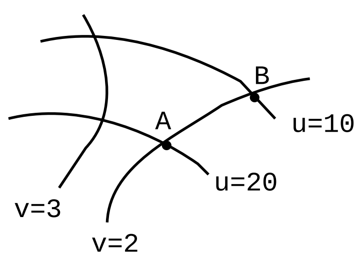


(a)



(b)

5. Look at the diagram below.
(a) Is h_v larger at A or at B.
(b) Which is larger at B, h_u or h_v.



6. Let $x = u^2 + v$, $y = u + v$. Which creates more arc length, fixing u and changing v by a little bit or fixing v and changing u by the same little bit.

7. Find the scale factors in parabolic cylindrical coordinates.

SECTION 2.5 CALCULUS IN ORTHOGONAL COORDS

repeated from Section 2.2

(1) In an *orthogonal* coordinate system, when $\vec{p} = ae_u + be_v$, the scalars a and b are the signed projections of p onto e_u and e_v.

gradient

Suppose temperature f is expressed as a function of u and v in a u,v *orthogonal* coordinate system. I want to express ∇f in terms of e_u and e_v; i.e., I want to find a and b so that

$$\nabla f = ae_u + be_v.$$

Since the u,v system is orthogonal,

$$a = \text{signed projection of } \nabla f \text{ onto } e_u \quad \text{(by (1) above)}$$

But by (A) in Section 1.2, the signed projection of ∇f in a direction is the rate of change of temp as you walk in that direction. So

$$a = \text{degrees/meter-in-the-u-direction}$$

Now let's see how to compute it.

$\partial f/\partial u$ is degrees per u-unit (i.e., rate of change of f with respect to u)

h_u (i.e., $\|vel_u\|$) is meters per u-unit.

Here is how a, $\partial f/\partial u$ and h_u are related

$$\underbrace{a}_{\frac{\text{degrees}}{\text{meter}}} \quad \textit{times} \quad \underbrace{h_u}_{\frac{\text{meters}}{\text{u-unit}}} \quad = \quad \underbrace{\frac{\partial f}{\partial u}}_{\frac{\text{degrees}}{\text{u-unit}}}$$

So

$$a = \frac{1}{h_u} \frac{\partial f}{\partial u}$$

Similarly for b. So

(2)
$$\boxed{\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u} e_u + \frac{1}{h_v}\frac{\partial f}{\partial v} e_v}$$

And similarly, in 3-space,

$$\boxed{\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u} e_u + \frac{1}{h_v}\frac{\partial f}{\partial v} e_v + \frac{1}{h_w}\frac{\partial f}{\partial w} e_w}$$

example 1

If $f(r,\theta) = r^2 + \cos\theta$ in polar coordinates then

$$\nabla f = \frac{1}{h_r}\, 2r\, e_r - \frac{1}{h_\theta} \sin\theta\, e_\theta = 2re_r - \frac{1}{r} \sin\theta \quad e_\theta$$

At point $r = 2$, $\theta = \pi/6$ for instance, $\nabla f = 4e_r - \frac{1}{4}\, e_\theta$.

gradients in some special cases

Suppose the temperature at a point in 2-space depends only on r where r is the usual polar coord, and not on θ. Or suppose the temperature at a point in 3-space depends only on r where r is the usual cylindrical coordinate (distance to the z-axis). Then

$$\nabla \text{ temp} = \frac{d(\text{temp})}{dr}\, e_r \tag{3}$$

Similarly, if the temperature at a point in 3-space depends only on the spherical coordinate ρ, and not on ϕ or θ, then

$$\nabla \text{ temp} = \frac{d(\text{temp})}{d\rho}\, e_\rho \tag{4}$$

proof of (3) in 2-space

$$\nabla\text{temp} = \frac{1}{h_r}\frac{\partial\text{temp}}{\partial r}\, e_r + \frac{1}{h_\theta}\frac{\partial\text{temp}}{\partial\theta}\, e_\theta$$

But $\partial\text{temp}/\partial\theta = 0$ here, and $h_r = 1$, so ∇temp turns into just $\dfrac{d(\text{temp})}{dr}\, e_r$

example 2

If the temperature in 2-space is 1/r then

$$\nabla\text{temp} = -\frac{1}{r^2}\, e_r$$

Fig 1 shows some level sets and gradients.

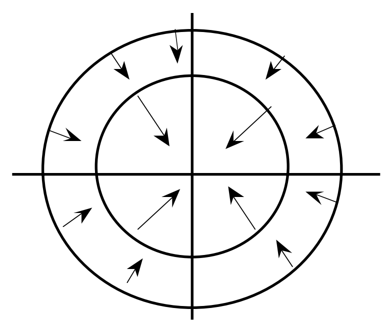


FIG 1

antigradients of some special fields

The results in (3) and (4) can be read backwards to get antigradients for certain vector fields:

If $F = r^3\, e_r$ (in polar or cylindrical coordinates) then $F = \nabla \frac{1}{4} r^4$; i.e., $\frac{1}{4} r^4$ is an antigradient of F.

If $F = 2\rho\, e_\rho$ (in spherical coords) then $F = \nabla\rho^2$, i.e., ρ^2 is an antigrad of F.

> If $\vec{F}$ = r-stuff $\vec{e}_r$ in polar or cyl coords then $\vec{F} = \nabla$(antideriv of r-stuff).
>
> If $\vec{F}$ = ρ-stuff $\vec{e}_\rho$ in spherical coords then $F = \nabla$(antideriv of ρ-stuff).

divergence

(5a)

If the u,v,w coordinate system is orthogonal and

$$F(u,v,w) = p(u,v,w)\ e_u + q(u,v,w)\ e_v + r(u,v,w)\ e_w$$

then

$$\text{div } F = \frac{1}{h_u h_v h_w}\left(\frac{\partial(h_v h_w p)}{\partial u} + \frac{\partial(h_u h_w q)}{\partial v} + \frac{\partial(h_u h_v r)}{\partial w}\right)$$

Note that to compute $\frac{\partial(h_v h_w p)}{\partial u}$, multiply p by $h_v h_w$ *first* and then differentiate the product w.r.t. u (probably with the derivative product rule).The factors h_v and h_w usually can't be pulled out of the derivative because they are usually not constants.

(5b)

If a 2-dim u,v coord system is orthogonal and

$F = p(u,v)\ e_u + q(u,v)\ e_v$ then

$$\text{div } F = \frac{1}{h_u h_v}\left(\frac{\partial(h_v p)}{\partial u} + \frac{\partial(h_u q)}{\partial v}\right)$$

why (5a) works

If F is a mass flux density in 3-space then div F dV is the net mass/sec leaving a little box with volume dV. So

$$\text{div } F = \frac{\text{net mass/sec out}}{\text{volume dV}}$$

I'll use this to find div F in the u,v,w coordinate system. I'm going to use the box in Fig 2 swept out by starting at point A with coordinates u, v, w and changing u by du, v by dv and w by dw.

To find the mass/sec out of the box, I'll find the flux out of all six faces.

Flux out the left face
= comp of F in −u direction × area of left face
= −p evaluated at a point on the left face, say A × area of left face (*)
= −p at A × $ds_v\ ds_w$ at left face (*)
= −p at A × $h_v\ h_w$ dv dw at left face
= ($-ph_v h_w$ at point A) dv dw
= $-ph_v h_w(u, v, w)$ dv dw

Similarly,

flux out the right face
= comp of F in the u direction × area of right face
= ($ph_v h_w$ at point B) dv dw
= $ph_u h_v(u+du, v, w)$ dv dw

So

net mass/sec out of the left and right faces

$$= ph_uh_v(u+du,v,w)\ dv\ dw\ - ph_vh_w(u,v,w)\ dv\ dw$$

$$= \Big[ph_uh_v(u+du,v,w) - ph_vh_w(u,v,w)\Big]\ dv\ dw \qquad \text{(factor)}$$

$$= \frac{ph_vh_w(u+du,v,w) - ph_vh_w\ (u,v,w)}{du}\ du\ dv\ dw \qquad \text{(multiply and divide by du)}$$

$$= \frac{\partial(ph_vh_w)}{\partial u}\ du\ dv\ dw$$

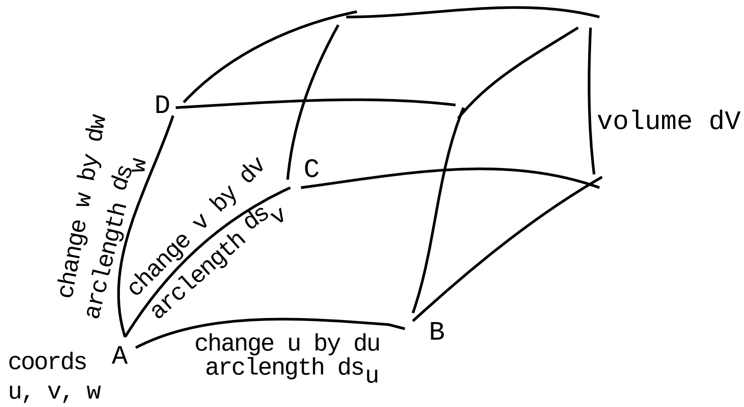


FIG 2

Similarly for the mass/sec out the top and bottom and the mass/sec out the front and back. So

net mass/sec out of box

$$= \left[\ \frac{\partial(h_v\,h_w\,p)}{\partial u} + \frac{\partial(h_u\,h_w\,q)}{\partial v} + \frac{\partial(h_u\,h_v\,r)}{\partial w}\ \right] du\ dv\ dw \tag{6}$$

And

$$dV = ds_u\,ds_v\,ds_w \qquad (*)$$

$$= h_uh_vh_w\ du\ dv\ dw \tag{7}$$

so

$$\text{div } F = \frac{\text{net mass/sec out}}{\text{volume dV}}$$

$$= \frac{(6)}{(7)}$$

$$= \frac{1}{h_u\,h_v\,h_w}\left(\ \frac{\partial(h_v\,h_w\,p)}{\partial u} + \frac{\partial(h_u\,h_w\,q)}{\partial v} + \frac{\partial(h_u\,h_v\,r)}{\partial w}\ \right) \qquad \text{QED.}$$

(*) These steps wouldn't work in a non-orthogonal coordinate system

why (5b) works

The vector field $F(u,v) = p(u,v)\ e_u + q(u,v)\ e_v$ in 2-space in a coordinate system where

$$x = x(u,v)$$
$$y = y(u,v)$$

can be considered to be

$$F(u,v,z) = p(u,v)\ e_u + q(u,v)\ e_v + 0\ k$$

in the u,v,z cylindrical coordinate system where

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z$$

Then (5a) becomes

$$\text{div } F = \frac{1}{h_u h_v h_z}\left(\frac{\partial(h_v h_z p)}{\partial u} + \frac{\partial(h_u h_z q)}{\partial v} + \frac{\partial(h_u h_v 0)}{\partial z} \right)$$

But $h_z = 1$ in a u,v,z cylindrical coordinate system and $\dfrac{\partial(h_u h_v 0)}{\partial z}$ is 0 so (5a) collapses to (5b).

example 3

Let $F(r,\theta) = r \sin\theta\ e_r + r\ e_\theta$ in polar coords. Then $h_r = 1$, $h_\theta = r$,

$$\text{div } F = \frac{1}{r}\left(\frac{\partial(r^2 \sin\theta)}{\partial r} + \frac{\partial r}{\partial \theta} \right)$$
$$= \frac{1}{r}(2r \sin\theta + 0)$$
$$= 2 \sin\theta$$

warning

In example 2, div F is *not* $\dfrac{\partial(r \sin\theta)}{\partial r} + \dfrac{\partial r}{\partial \theta}$.
Don't forget to stick scale factors in all the right places,

example 4

Let $f(\rho,\phi,\theta) = \rho^2 \sin\theta$ (spherical coords). Find Lapl f.

solution

Lapl f = div ∇f so first take gradient and then take divergence.

$h_\rho = 1$, $h_\phi = \rho$, $h_\theta = \rho \sin\phi$

$$\nabla f = \frac{1}{h_\rho}\frac{\partial f}{\partial \rho} e_\rho + \frac{1}{h_\theta}\frac{\partial f}{\partial \theta} e_\theta$$
$$= 2\rho \sin\theta\ e_\rho + \frac{1}{\rho \sin\theta}\rho^2 \cos\theta\ e_\theta$$
$$= 2\rho \sin\theta\ e_\rho + \rho\frac{\cos\theta}{\sin\theta}\ e_\theta$$

$$\text{Lapl } f = \text{div } \nabla f = \frac{1}{h_\rho h_\phi h_\theta}\left[\frac{\partial(2\rho \sin\theta \cdot \rho^2 \sin\phi)}{\partial\rho} + \frac{\partial\left(\rho\,\dfrac{\cos\theta}{\sin\phi}\cdot\rho\right)}{\partial\theta}\right]$$

$$= \frac{1}{\rho^2 \sin\phi}\left[6\rho^2 \sin\theta \sin\phi - \frac{\rho^2 \sin\theta}{\sin\phi}\right]$$

$$= 6 \sin\theta - \frac{\sin\theta}{\sin^2\phi}$$

curl

Suppose the u,v,w coord system is orthogonal and e_u, e_v, e_w in that order are righthanded. Let

$$\vec{F}(u,v,w) = p(u,v,w)\,\vec{e}_u + q(u,v,w)\,\vec{e}_v + r(u,v,w)\,\vec{e}_w$$

Then

(9)
$$\text{curl } \vec{F} = \begin{vmatrix} \dfrac{1}{h_v h_w} e_u & \dfrac{1}{h_u h_w} e_v & \dfrac{1}{h_u h_v} e_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ ph_u & qh_v & rh_w \end{vmatrix}$$

The determinant in (9) is fake but it helps you remember the formula. It works only if you expand the determinant across row 1. For instance, start like this

$$\begin{vmatrix} \dfrac{1}{h_v h_w} e_u & \dfrac{1}{h_u h_w} e_v & \dfrac{1}{h_u h_v} e_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ ph_u & qh_v & rh_w \end{vmatrix}$$

to get $\dfrac{1}{h_u h_v} e_u$ times $\left[\dfrac{\partial(rh_w)}{\partial v} - \dfrac{\partial(qh_v)}{\partial w}\right]$.

Here's the whole curl formula written out.

$$\text{curl } F = \left[\frac{\partial(rh_w)}{\partial v} - \frac{\partial(qh_v)}{\partial w}\right]\frac{1}{h_v h_w}\,e_u$$

$$- \left[\frac{\partial(rh_w)}{\partial u} - \frac{\partial(ph_u)}{\partial v}\right]\frac{1}{h_u h_w}\,e_v$$

$$+ \left[\frac{\partial(qh_v)}{\partial u} - \frac{\partial(ph_u)}{\partial v}\right]\frac{1}{h_u u_v}\,e_w$$

example 5

Let $F(\rho,\phi,\theta) = \cos\theta\,\vec{e}_\rho + \rho\,\vec{e}_\phi + 2\vec{e}_\theta$ in spherical coordinates.
Find curl F.

solution

e_ρ, e_ϕ, e_θ in that order are righthanded, and $h_\rho = 1$, $h_\phi = \rho$, $h_\theta = \rho\sin\phi$ so

$$\text{curl } F = \begin{vmatrix} \frac{1}{h_\phi h_\theta} e_\rho & \frac{1}{h_\rho h_\theta} e_\phi & \frac{1}{h_\rho h_\phi} e_\theta \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ h_\rho \cos\theta & h_\phi\,\rho & 2h_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\rho^2\sin\phi} e_\rho & \frac{1}{\rho\sin\phi} e_\phi & \frac{1}{\rho} e_\theta \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ \cos\theta & \rho^2 & 2\rho\sin\phi \end{vmatrix}$$

$$= \frac{1}{\rho^2\sin\phi}\, e_\rho \left[\frac{\partial(2\rho\sin\phi)}{\partial\phi} - \frac{\partial\rho^2}{\partial\theta} \right]$$

$$- \frac{1}{\rho\sin\phi}\, e_\phi \left[\frac{\partial(2\rho\sin\phi)}{\partial\rho} - \frac{\partial\cos\theta}{\partial\theta} \right]$$

$$+ \frac{1}{\rho}\, e_\theta \left[\frac{\partial\rho^2}{\partial\rho} - \frac{\partial\cos\theta}{\partial\phi} \right]$$

$$= (2\rho\cos\phi - 0)\frac{1}{\rho^2\sin\phi}\, e_\rho - (2\sin\phi + \sin\theta)\frac{1}{\rho\sin\phi}\, e_\phi + (2\rho - 0)\frac{1}{\rho}\, e_\theta$$

$$= \frac{2\cos\phi}{\rho\sin\phi}\, e_\rho - \left[\frac{2}{\rho} + \frac{\sin\theta}{\rho\sin\phi}\right] e_\phi + 2e_\theta$$

warning

When you use the curl formula in (9), you must use a righthanded set of basis vectors. For spherical coordinates that means e_ρ, e_ϕ, e_θ *in that order* (or any cyclic permutation thereof) in row 1. But *not* e_ρ, e_θ, e_ϕ which is a lefthanded triple.

why (9) works

Let

$$F(u,v,w) = p(u,v,w)\, e_u + q(u,v,w)\, e_v + r(u,v,w)\, e_w$$

I want to express curl F in terms of e_u, e_v and e_w; i.e., I want to find a, b and c so that

$$\text{curl } F = ae_u + be_v + ce_w.$$

I'll work on c. Since the u,v,w system is orthogonal, by (1) above, c is the signed projection of curl F onto e_w, i.e., the component of curl F in direction e_w.

But by (4) in §1.5,

(comp of curl F in direction e_w) dS
= righthanded circ on a little loop around e_w, with area dS

So

$$(10) \qquad c = \frac{\text{righthanded circ on a little loop around } e_w}{dS}$$

To find c, first I need a loop around e_w.

In Fig 3, I started at point A with coordinates u,v,w and drew the righthanded orthogonal basis vectors e_u, e_v, e_w. For my small closed curve around e_w, I'll choose the "rectangle" ABCD: Side AB came from changing u by du while v stayed fixed. Side AD came from changing v by dv while u stayed fixed.

Let dS be the surface area inside the curve.

The righthanded direction on the curve w.r.t. e_w is A to B to C to D to A.

The circ along the B-to-C segment for instance is the tangential component of F on the segment, namely q, times length BC. Similarly for the circs along the other three sides, so

$$\begin{aligned} \text{righthanded circ} = {} & p \text{ on AB} \times \text{length AB} \\ & + \ q \text{ on BC} \times \text{length BC} \\ & + -p \text{ on CD} \times \text{length CD} \\ & + -q \text{ on DA} \times \text{length DA} \end{aligned}$$

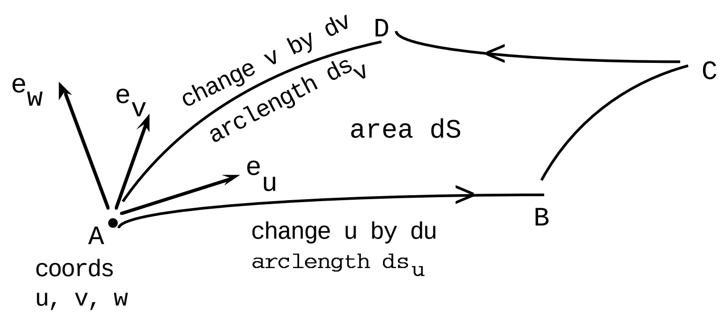


FIG 3

Length AB = ds_u = (h_u on AB) du, and similarly for the other sides. So

$$\begin{aligned} \text{righthanded circ} = {} & (ph_u \text{ on AB, say at point A}) \ du \\ & + (qh_v \text{ on BC, say at point B}) \ dv \\ & - (ph_u \text{ on CD, say at point D}) \ du \\ & - (qh_v \text{ on DA, say at point A}) \ dv \end{aligned}$$

Rearrange to get

$$\text{righthanded circ} = \frac{qh_v(u+du,v,w) - qh_v(u,v,w)}{du} \ du \ dv$$

$$- \frac{ph_u(u,v+dv,w) - ph_u(u,v,w)}{dv} \ du \ dv$$

$$(11) \qquad = \left[\frac{\partial(qh_v)}{\partial u} - \frac{\partial(ph_u)}{\partial v} \right] du \ dv$$

Since the coord system is orthogonal,

$$dS = ds_u \, ds_v$$

(12) $$= h_u du \; h_v dv.$$

Then, by (10),

$$c = \frac{(11)}{(12)} = \frac{1}{h_u \, h_v}\left[\frac{\partial (q h_v)}{\partial u} - \frac{\partial (p h_u)}{\partial v}\right].$$

Similarly for a and b. The three formulas can be combined into the single rule in (9). QED

APPENDIX LAPLACIAN IN POLAR, CYLINDRICAL AND SPHERICAL COORDINATES

These are just special cases of the general formulas for div and grad but you might want to quote them directly in another course.

polar coordinates

$$\text{Lapl}[\ f(r,\theta)\ ] = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

cylindrical coordinates

$$\text{Lapl}\ [f(r,\theta,z)] = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial z^2}$$

spherical coordinates

$$\text{Lapl}[\ f(\rho,\phi,\theta)] = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cos\phi}{\rho^2 \sin\phi}\frac{\partial f}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

PROBLEMS FOR SECTION 2.5

1. Find the gradients of the following scalar fields in spherical coordinates.
 (a) $\sin\phi$ (b) $\rho \sin\phi$

2. Find the gradient of an arbitrary $f(r,\theta)$ in polar coordinates.

3. Find ∇temp using the most appropriate coordinate system. In each case, sketch the temp level sets, the gradient field and describe them in words.
 (a) Temperature at a point in 2-space is the cube of the distance from the point to the origin.
 (b) Temp at a point in 3-space is 1/(distance to z-axis).

4. The arrows of a vector field in 2-space point away from the origin and the length of the arrow at a point is 1/(distance to origin).
 Find an antigradient for F in the most appropriate coordinate system.

5. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Find $\nabla(\|\vec{r}\|^3)$
   (a) by sticking to the x,y,z coord system
   (b) by switching to a new coord system and check that the two answers agree

6. Let $F = ue_v$ in parabolic coordinates. Find div F.

7. Define an r,θ coordinate system with the equations $x = e^r \cos\theta$, $y = e^r \sin\theta$.
(a) Show that the coordinate system is orthogonal.

(b) If $F = r\vec{e}_r + \cos\theta\, \vec{e}_\theta$ find div F.

(c) Find the gradient of r^3.

(d) Find Lapl of r^3

8. Find Lapl $\dfrac{1}{x^2+y^2+z^2}$. The answer should be in the same coordinate system as the original problem but along the way you might want to use another coord system.

9. Let $v(r,\theta)$ be an arbitrary scalar field in polar coords. Find Lapl v in polar coordinates.

10. Find div and curl of $2z\vec{e}_r + 3z\vec{e}_\theta$ (cylindrical coords).

11. Find an *anti*gradient, div and curl of (a) $\dfrac{1}{\rho^2}\, e_\rho$ (b) $\dfrac{1}{r^2}\, e_r$

12. Let $F = e_u$ in parabolic coordinates. Find curl F.

13. Let $F = \rho e_\theta$ (spherical coordinates).
Find curl F.

14. Let $F = \dfrac{x}{x^2+y^2+z^2}\, i + \dfrac{y}{x^2+y^2+z^2}\, j + \dfrac{z}{x^2+y^2+z^2}\, k$

(a) Find div F (using Cartesian coordinates)
(b) Find F using a non-Cartesian coordinate system.
And check that your answers to (a) and (b) agree.
(c) Find curl F.

15. Let $F = \cos\theta\, e_\theta$ (polar coordinates). Find curl F.

REVIEW PROBLEMS FOR CHAPTER 2

1. Find Lapl($\rho \sin\theta$) (spherical coords)

2. In the parabolic coord system, $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $v \ge 0$.

Suppose the temperature at a point is $u^2 + 2v$. If you walk from point $u = 1$, $v = 2$ in the direction of arrow $2e_u - 3e_v$, what rate of change of temperature per foot do you feel instantaneously.

Where exactly will you use the fact that the parabolic coord system is orthogonal.

3. Fig 1 in Section 2.1 showed parabolic coordinate paper. Where are the u-axis and v-axis ?????

4. If $F(x,y) = 2\vec{i} + 3\vec{j}$ then F is a uniform field (all arrows have the same length and same direction). Suppose $F(u,v) = 2e_u - 3e_v$ in some u,v coordinate system. Is F uniform.

5. Let $F(x,y) = \dfrac{x}{x^2 + y^2}\, i + \dfrac{y}{x^2 + y^2}\, j$.

(a) Write F in polar coordinates.

(b) Find div F twice, using the x,y system and again using polar coordinates.
(c) Find curl F.

6. (a) Sketch the vector field e_ϕ (spherical coordinates).
(b) Find curl e_ϕ.

7. Let $x = 2r\cos\theta$, $y = 4r\sin\theta$, $r \ge 0$ define a coordinate system.
(a) Identify all the θ-curves and sketch some.
(b) Identify the r-curve $\theta = \pi/4$ and add it to the picture.
(c) Is the coord system is orthogonal
(d) Find the basis vectors at point $r=2$, $\theta = \pi/4$ and sketch them in the picture.

(e) Find h_r at point $r=2$, $\theta = \pi/4$ and explain what it means physically.

8. Let $F = 2i + 4j + 6k$. Express F in spherical coordinates at the point where $\rho = 117$, $\theta = \pi/2$, $\phi = \pi/4$.

9. Let $F = p\vec{i} + q\vec{j}$ where $p = \dfrac{y^3}{(x^2 + y^2)^2}$ and $q = \dfrac{-xy^2}{(x^2 + y^2)^2}$

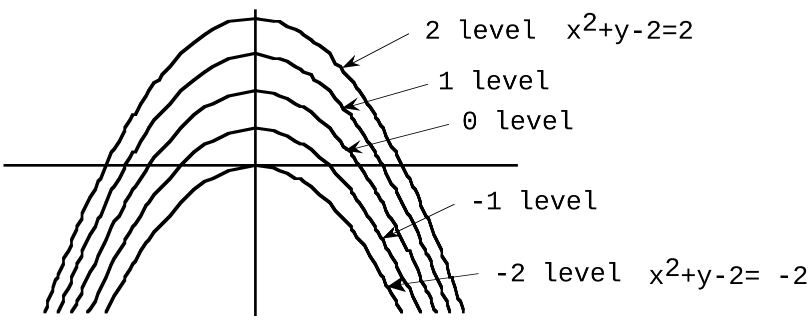
Write F in polar coordinates.

APPENDIX TO CHAPTERS 1 AND 2

using Mathematica to plot level sets

Here's how to get the −2, −1, 0, 1, 2 level sets of $x^2 + y - 2$ (Fig 1 in §1.1 but I added the labels separately in a drawing program).

```
ContourPlot[x^2 + y - 2,{x,-4,4},{y,-4,4}, ContourShading->False,
     Frame->False, Axes->True, PlotPoints->25, Contours->{0,-1,2,1,-2},
     AxesOrigin->{0,0},Ticks->{{-4,4},{-4,4}},AspectRatio->Automatic];
```



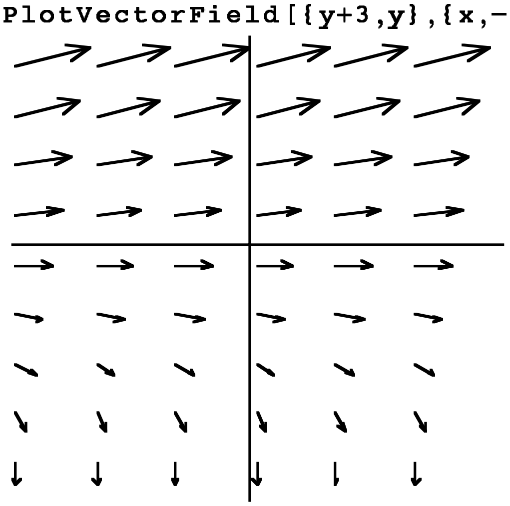
using Mathematica to plot vector fields

First load the PlotField package.

```
<< Graphics`PlotField`
```

Here's how to plot the vector field $\vec{F}(x,y) = (y+3)\vec{i} + y\vec{j}$ for $-2 \le x \le 2$, $-3 \le y \le 3$ so that the arrows along a row occur at intervals of .7 and the arrows along a column also occur at intervals of .7 (Fig 4 in §1.1). If you leave out the two .7's in the instruction, you get the default spacing which would give you more arrows than I got, spaced closer together.

```
PlotVectorField[{y+3,y},{x,-2,2,.7},{y,-3,3,.7},Axes->True,Ticks->None];
```



using Mathematica to find div, curl, gradient in various coordinate systems

First, load the package with all the vector operations in it.

```
<<Calculus`VectorAnalysis`
```

Here's how to find the divergence of $x^2 y^2 \vec{i} + (2x + 3y)\vec{j}$. The program will only take the divergence of a 3-dim vector field so use third component 0. And you have to tell Mathematica which coordinate system to use.

```
SetCoordinates[Cartesian[x, y, z]]

Div[{x^2 y^2,2x + 3y, 0}]
```

$3 + 2 x y^2$

```
Curl[{x^2 y^2, x y z, 2x + 3y + 4z}]
```

$\{3 - x y, -2, -2 x^2 y + y z\}$

```
Grad[x^2 y^3 z + 3z]
```

$\{2 x y^3 z, 3 x^2 y^2 z, 3 + x^2 y^3\}$

Instead of specifying a default coord system, or if you want to override your default system, you can specify each time what coord system to use.

```
Div[{u^2 v^2,2u + 3v, 0}, Cartesian[u,v,w]]
```

$3 + 2 u v^2$

```
Div[{rho^3,0,0}, Spherical[rho,phi,theta]]
```

$5 \, rho^2$

```
Laplacian[v[r, theta], Cylindrical[r,theta,z]]//Together
```

$$\frac{v^{(0,2)}[r, theta] + r\, v^{(1,0)}[r, theta] + r^2\, v^{(2,0)}[r, theta]}{r^2}$$

The notation $v^{(2,0)}$ means the second derivative of v w.r.t. its first variable, i.e., $\frac{\partial^2 v}{\partial r^2}$

CHAPTER 3 COMPUTING LINE INTEGRALS AND SURFACE INTEGRALS

SECTION 3.0 REVIEW

computing double integrals

There are two ways to set up $\int f(x,y)\ dA$ over a region R.

$$\int_R f(x,y)\ dA = \int_{\text{lowest y in R}}^{\text{highest y in R}} \int_{\text{x on left bdry}}^{\text{x on right bdry}} f(x,y)\ dx\ dy$$

$$\int_R f(x,y)\ dA = \int_{\text{leftmost x in R}}^{\text{rightmost x in R}} \int_{\text{y on lower bdry}}^{\text{y on upper bdry}} f(x,y)\ dy\ dx$$

example 1

I'll set up $\int_R f(x,y)\ dA$ where R is the triangular region in Fig 1.

method 1

$$\int_R f(x,y)\ dA = \int_{x=0}^{3} \int_{y=2}^{y=8-2x} f(x,y)\ dy\ dx$$

method 2

$$\int_R f(x,y)\ dA = \int_{y=2}^{8} \int_{x=0}^{4-y/2} f(x,y)\ dx\ dy$$

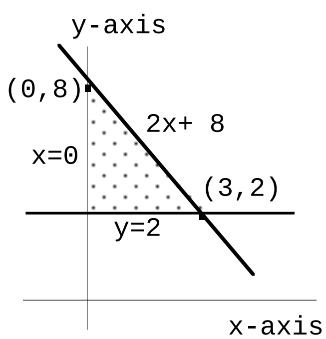


FIG 1

warning

The limits in Fig 1 are *not* $\int_{x=0}^{2} \int_{y=2}^{8}$.

These limits go with a rectangle, not the triangle.

warning

The inner limits are *boundary* values while the outer limits are *extreme* values.

The inner limits will *contain the other variable* unless the boundary in question is a vertical or horizontal line.

The limits on the outer integral are *always constants*.

double integrating on a region with a two-curve boundary

Set up $\int_R f(x,y)\ dA$ where R is the region bounded by the line $x + y = 6$ and the parabola $x = y^2$ (Fig 2)

method 1

$$\int_R f(x,y)\ dA = \int_{y=-3}^{2} \int_{x=y^2}^{x=6-y} f(x,y)\ dx\ dy$$

method 2 (Fig 3)

The lower boundary is the parabola $x = y^2$; solve the equation for y to get $y = -\sqrt{x}$ (choose the negative square root because y is negative on the lower part of the parabola). But the upper boundary consists of two curves, the parabola and the line. To continue with this order of integration divide the region into the two indicated parts, I and II:

$$\int_R f(x,y)\ dA = \int_I f(x,y)\ dA + \int_{II} f(x,y)\ dA$$

$$= \int_{x=0}^{4} \int_{y=-\sqrt{x}}^{\sqrt{x}} f(x,y)\ dy\ dx + \int_{x=4}^{9} \int_{y=-\sqrt{x}}^{y=6-x} f(x,y)\ dy\ dx$$

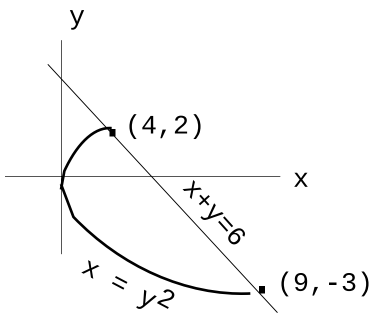


FIG 2

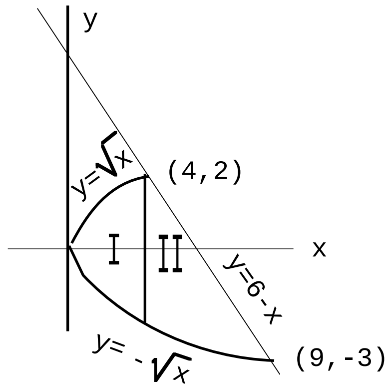


FIG 3

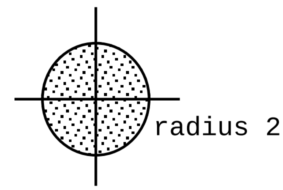


FIG 4

warning

Examine the boundaries carefully to catch the ones consisting of more than one curve.

double integrating in polar coordinates

To find $\int f(x,y)\ dA$ on a region, let

$x = r \cos\theta$

$y = r \sin\theta$

$r^2 = x^2 + y^2$

$dA = r\ dr\ d\theta$

and use the limits of integration $\int_{\text{smallest }\theta}^{\text{largest }\theta} \int_{r \text{ on inner bdry}}^{r \text{ on outer bdry}}$.

double integrating on a disk centered at the origin

I'll set up $\int x^2 y^4\ dA$ on the disk in Fig 4, center at the origin and radius 3.

method 1 (polar coordinates ----- the best method)

$$\int x^2 y^4\ dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (r\cos\theta)^2\ (r\sin\theta)^4\ \ r\ dr\ d\theta$$

method 2

The circle has equation $x^2 + y^2 = 9$. So

$$\int x^2 y^4\ dA = \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2 y^4\ dy\ dx$$

method 3

$$\int x^2 y^4 \, dA = \int_{y=-3}^{3} \int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y^4 \, dx \, dy$$

warning

The limits are *not* $\int_{y=-3}^{3} \int_{x=-3}^{3}$

Those limits go with a square region, not a disk.

example 2

The limits of integration in polar coordinates that go with just the lower half-disk in Fig 4 are $\int_{\theta=\pi}^{2\pi} \int_{r=0}^{4}$

warning

Don't forget that in polar coordinates it's $dA = \boxed{r} \, dr \, d\theta$. Remember the extra r.

shortcut

$\int x \, dA$ and $\int y \, dA$ are each 0 on a disk centered around the origin.

In fact, if n is odd, $\int x^n \, dA$ and $\int y^n \, dA$ are each 0 on the disk.

area

$$\int_{\text{region R}} 1 \, dA = \text{area of region R}$$

SECTION 3.1 THE LINE INTEGRAL $\int$ F · T ds

unit tangents

T represents a unit tangent to a curve (in 2-space or 3-space) (Fig 1). For any curve, there are two possible T directions so a problem must specify which to use.

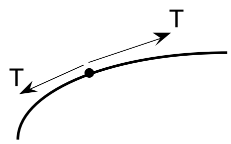


FIG 1

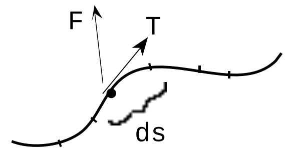


FIG 2

definition of the line integral $\int$ F · T ds

Start with a vector field F and a curve, and choose a direction for the unit tangent T; i.e., start with a *directed* curve.

Divide the curve into small pieces and let a typical piece have length ds (Fig 2).

For each piece, compute F·T ds, the component of F in the T direction, times ds. The line integral is the sum of all the F·T ds's from the pieces, i.e.,

$$\boxed{\int_{\text{curve}} F\cdot T\ ds = \sum F\cdot T\ ds}$$

application to circulation

Suppose F is the velocity field of a fluid. Look at Fig 2 again.

For each small piece of the curve, F·T ds is a *circulation* (see (3) in §1.6).

The line integral on the entire curve adds all the F·T ds's from the small pieces, i.e., adds all the little circulations, so

$$\boxed{\text{circulation along a curve in the T direction} = \int_{\text{curve}} F\cdot T\ ds}$$

Most people refer to $\int_{\text{curve}} F\cdot T\ ds$ as the *circulation along the curve in the T direction* even if F was not specifically designated as the velocity field of a fluid.

application to work

Suppose F is a force field.

Physicists say that if F doesn't change and you walk in a straight line then

(*) work done by F = component of F in the direction of motion × distance moved

Look at Fig 2 again.

Suppose you move along the entire curve. Then you are not moving in a straight line and F doesn't necessarily stay constant so you can't use (*) directly.

But if you look at a small almost-straight piece of the curve with length ds, then you *can* use (*): the work done by F as you traverse the small piece is F·T ds.

The line integral on the entire curve adds all the F·T ds's from the small pieces, i.e., adds all the little works, so

$$\boxed{\text{total work done by F as you traverse the curve} = \int_{\text{curve}} F\cdot T\ ds}$$

The work is positive if F helps the particle move and is negative if F hinders.

computing $\int F \cdot T \, ds$

Given a vector field $\vec{F}(x,y,z)$ and a directed curve, i.e., a curve with a chosen T direction, i.e., a curve with an initial point and a final point (Fig 4b). To find $\int F \cdot T \, ds$ on the curve, parametrize the curve with equations of the form

(1) $$x = x(t), \; y = y(t), \; z = z(t), \; t_{low} \le t \le t_{high}$$

Then

(2) $$\int F \cdot T \, ds = \int_{t_{low}}^{t_{high}} F \cdot \pm v \, dt$$

where

$$v = x'(t) \, \vec{i} + y'(t) \, \vec{j} + z'(t) \, \vec{k} \quad \text{(the usual velocity vector)}$$

and you must choose whichever of v and −v points like T and use (1) to convert x's and y's to t's.

In 2-space, drop all the z stuff.

why (2) works

In Fig 3, change t by dt which in turn traces out a little piece of the curve with length ds in Fig 4.

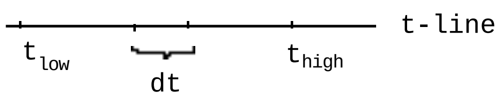


FIG 3 t world

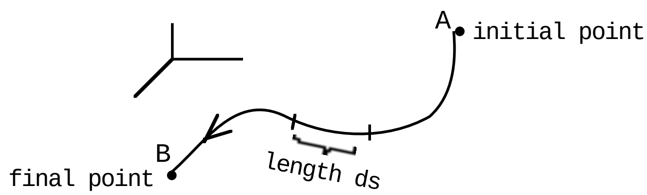


FIG 4 x,y,z world

I want to express T and ds from the x,y,z world in terms of t and dt from the t world. From (1) in Section 2.0,

(3) $$\boxed{ds = \|\vec{v}\| \, dt}$$

$\|v\|$ is called the *arc length magnification factor* of the parametrization in (1).

Furthermore v is tangent to the curve so T is either v_{unit} or $-v_{unit}$; i.e.,

(4) $$T = \frac{v}{\|v\|} \text{ or } \frac{-v}{\|v\|}$$

Lines (3) and (4) show that v serves a dual purpose, it's a *smart* tangent vector: it points either like T or opposite to T and its length is the arc length magnification factor. Put (3) and (4) together to get

$$T \, ds = \pm v \, dt$$

Finally, the sum of $F \cdot T$ ds's in Fig 4b equals the sum of $F \cdot \pm v$ dt's in Fig 4a which is what (2) says. QED

clarification

In (2), it is *not* that T turns into ±v and ds turns into dt (although it seems like that for practical purposes). Instead, T turns into either $\dfrac{v}{\|v\|}$ or $\dfrac{-v}{\|v\|}$ and ds turns into $\|v\|$ dt and T *times* ds cancels down to ±v dt.

Here's another way to look at it. When you replace T by ±v you are actually replacing T by $\pm\|v\|T$ which is a way of inserting the mag factor, $\|v\|$, automatically:

$$F \bullet T\ ds = F \bullet \pm \underbrace{||v||\ T\ dt}_{ds}$$

computing $\int$ F · T ds continued

There's a way to do it so that you don't have to stop and think about whether you should use v or −v.

Suppose you want to line integrate on a directed curve (say in 2-space) from A to B. Suppose the curve has parametrization x=x(t), y=y(t) where $3 \le t \le 7$. There are two possibilities (Fig 5):

(i) t=3 goes with A and t=7 goes with B
(ii) t=3 goes with B and t=7 goes with A

Remember that v always points in the direction in which t *increases*.

If (i) holds then v points like T so, by (2),

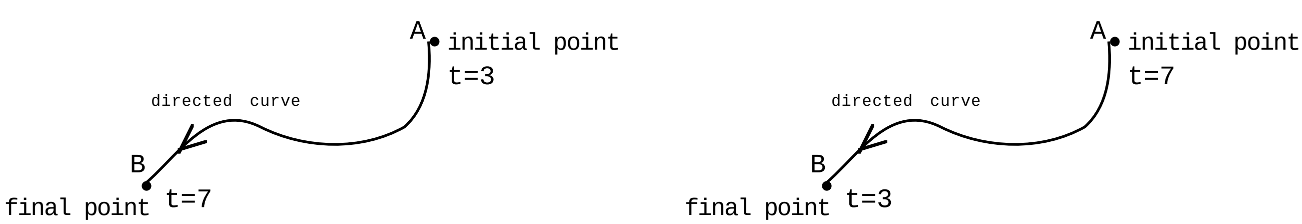
$$\int F\cdot T\ ds = \int_{t=3}^{7} F\cdot v\ dt = \int_{t \text{ at } A}^{t \text{ at } B} F\cdot v\ dt$$

If (ii) holds then −v points like T so, by (2),

$$\begin{aligned}\int F\cdot T\ ds &= \int_{t=3}^{7} F\cdot -v\ dt \\ &= \int_{t=7}^{3} F\cdot v\ dt \qquad \left(\begin{array}{l}\text{deleting the minus sign and reversing the}\\ \text{limits of integration cancel each other out}\end{array}\right) \\ &= \int_{t \text{ at } A}^{t \text{ at } B} F\cdot v\ dt\end{aligned}$$

So here's the one rule that covers *both* cases, i.e., works all the time:

$$\boxed{\int F\cdot T\ ds = \int_{t \text{ at initial point on curve}}^{t \text{ at final point on curve}} F\cdot v\ dt} \tag{5}$$



(i) (ii)

FIG 5

And here's some popular notation which takes (5) one step further.

> If $F = p\vec{i} + q\vec{j}$ then $\int F \cdot T\, ds$ is often written as $\int p\, dx + q\, dy$.
> In accordance with (5) it can be computed by using
>
> $$dx = x'(t)\, dt,\ dy = y'(t)\, dt$$
>
> with the limits as in (5). All in all,
>
> (6) $$\int F \cdot T\, ds = \int p\, dx + q\, dy = \int_{t \text{ at initial point on curve}}^{t \text{ at final point on curve}} p\, x'(t)\, dt + q\, y'(t)\, dt$$

You can use any of (2), (5), (6) to compute $\int F \cdot T\, ds$. I like (6).
The same ideas work in 3-space.

clarification

The limits in (5) and (6) can come out "backwards", that is, t at the initial point might be larger than t at the final point. When this happens, it compensates for your using v when T actually points like −v.

In all straightforward integral applications (when you are adding f(x) dx's or f(x,y) dA's etc.), the limits go from smaller to larger, as in (2). They can come out backwards only in devious instances when you deliberately reverse them to compensate for a sign change elsewhere.

How you replace T	Limits
Carefully, with whichever of v and −v points like T	$\int_{\text{low t}}^{\text{high t}}$
Automatically with v as in (5) and (6)	$\int_{\text{initial t}}^{\text{final t}}$

By the way, if you replace T by a *unit* tangent vector (rather than v) then you have lost the length magnification factor and ds has to be replaced by $\|v\|\, dt$.

example 1

Let $F(x,y) = xy\,\vec{i} + (x+y)\vec{j}$.
Find $\int F \cdot T\, ds$ on curve $y = x^2$ directed from point A = (1,1) to point B = (0,0) (Fig 6).

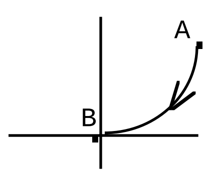


FIG 6

solution *method 1* (using (2))
Parametrize the curve. One possibility is

$$x = t,\ y = t^2,\ 0 \le t \le 1$$

Then $t_A = 1$, $t_B = 0$, $v = (1, 2t)$. But v points in the B to A direction (in the direction of increasing t) so use −v.

$$F \cdot -v = -xy - 2t(x+y) = -t^3 - 2t(t + t^2) = -3t^3 - 2t^2$$

$$\int F \cdot T\, ds = \int_{t_{low}}^{t_{high}} F \cdot -v\, dt = \int_{t=0}^{1} (-3t^3 - 2t^2)\, dt = -17/12$$

method 2 (using (6))

Use the same parametrization as in method 1. Then

$$t_{\text{initial point}} = t_A = 1 \text{ (Fig 6)}$$

$$t_{\text{final point}} = t_B = 0$$

$$dx = dt,\ dy = 2t\ dt$$
$$p = xy,\ q = x+y$$

$$\int \mathbf{F}\cdot\mathbf{T}\ ds = \int_{\text{initial } t}^{\text{final } t} \underbrace{t^3}_{p}\ \underbrace{dt}_{dx} + \underbrace{(t+t^2)}_{q}\ \underbrace{2t\ dt}_{dy}$$

$$= \int_{t=1}^{0} (3t^3 + 2t^2)\ dt = -17/12$$

If F is a force field, then it does negative work to a particle which moves on the curve from A to B.

If F is the velocity field of a fluid then the circulation along the curve from A to B is −17/12 square meters/sec.

example 2 (line integrating on a circle)

Find $\int y^3\ dx + x^2\ dy$ on the circle $x^2 + y^2 = 4$ directed clockwise.

solution

The circle has parametric equations

$$x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi.$$

To go with the *clockwise* direction use

$$t_{\text{initial}} = 2\pi \text{ and } t_{\text{final}} = 0.$$

Or you could any of these combinations

initial t	final t
4π	2π
0	-2π
$\pi/2$	$-3\pi/2$

As long as the final t is 2π *less than* the initial t.

Then

$$\int y^3\ dx + x^2\ dy = \int_{t=2\pi}^{0} 8\sin^3 t\cdot -2\sin t\ dt + 4\cos^2 t\cdot 2\cos t\ dt$$

$$= \int_{2\pi}^{0} (-16\sin^4 t + 8\cos^3 t)\ dt$$

footnote

By inspection, $\int_{2\pi}^{0} \cos^3 t\ dt = 0$ (just as much area above as below). So this boils down to $\int_{2\pi}^{0} -16\sin^4 t\ \ dt$

example 2 continued ("consecutive" t's)

Continue with the parametric equations $x = 2\cos t$, $y = 2\sin t$. They describe a particle moving ccl around and around a circle (Fig 6A)

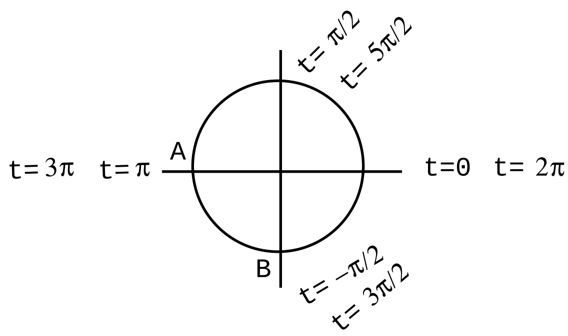


Fig 6A

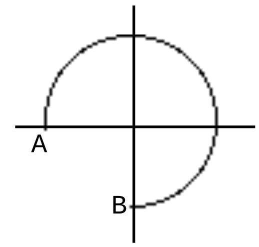


FIG 6B

Suppose you want to line integrate on the piece of the circle in Fig 6B. Then you can use

$$t_B = -\pi/2, \quad t_A = \pi$$

or

$$t_B = 3\pi/2, \quad t_A = 3\pi$$

etc.

But *not* $t_B = 3\pi/2$, $t_A = \pi$ since these are not "consecutive times" from the point of view of the traveling particle.

Then whether you choose initial $t = t_A$ and final $t = t_B$ or vice versa depends, as usual, on whether the curve is directed clockwise or counterclockwise.

notation

The symbol $\oint$ is often used when the line integral is done on a *loop*.

other notation

Physicists might write $\overrightarrow{ds}$ (or $\overrightarrow{d\ell}$ or $\overrightarrow{dr}$) for the vector T ds. In other words their $\overrightarrow{ds}$ is a tangent vector whose length is $ds = \|v\|\,dt$.

If $\vec{F} = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$, the following all mean the same line integral:

$$\int \vec{F}\cdot\vec{T}\,ds$$

$$\int \vec{F}\cdot\overrightarrow{ds}$$

$$\int \vec{F}\cdot\overrightarrow{d\ell}$$

$$\int \vec{F}\cdot\overrightarrow{dr}$$

$$\int \vec{F}_T\,ds$$

$$\int p\,dx + q\,dy + r\,dz$$

reversing the direction on a curve

$\int F\cdot T\,ds$ changes sign if the direction of the curve is reversed.

when you end up with $\int$ ds

Suppose that $F \cdot T$ is a *constant*, say K, so that $\int F \cdot T \, ds$ turns into $\int K \, ds$ which is $K \int ds$.

To find $\int ds$ you can use

$$\boxed{\int_{\text{curve}} ds = \text{length of curve}}$$

(this helps *provided* you know the curve length from geometry).

This works because $\int_{\text{curve}} ds$ adds all the small ds's so it's the total length.

example 3

Suppose $F = r e_\theta$ (Fig 7).

Find $\oint F \cdot T \, ds$ on a counterclockwise circle with radius 4 centered at the origin.

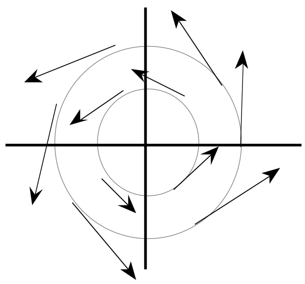


FIG 7

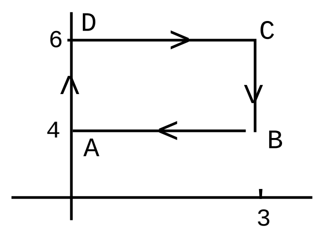


FIG 8

solution On the circle, $F = 4e_\theta$, ccl $T = e_\theta$, $F \cdot T = 4$ so

$$\oint F \cdot T \, ds = \oint 4 \, ds$$
$$= 4 \oint ds$$
$$= 4 \times \text{length of circle}$$
$$= 4 \times 8\pi$$
$$= 32\pi.$$

example 4

Let $F = x^2 \vec{j}$. Find $\oint F \cdot T \, ds$ on the loop in Fig 8.

solution

$\int F \cdot T \, ds$ on the loop in Fig 8

$= \int F \cdot T \, ds$ on A-to-D $+ \int F \cdot T \, ds$ on D-to-C

$+ \int F \cdot T \, ds$ on C-to-B $+ \int F \cdot T \, ds$ on B-to-A

On the B-to-A path, $T = -i$, $F = x^2 j$, $F \cdot T = 0$, $\int F \cdot T \, ds = 0$

On the D-to-C path, $T = i$, $F \cdot T = 0$, $\int F \cdot T \, ds = 0$

On the A-to-D, $x = 0$, $F = \vec{0}$, $F \cdot T = 0$, $\int F \cdot T \, ds = 0$

So all that's left to find is $\int F \cdot T \, ds$ on C-to-B

method 1

Line BC has parametric equations $x=3$, $y=y$, $4 \le y \le 6$.

$$\int F \cdot T \, ds = \int 0 \, dx + x^2 \, dy = \int_{\text{initial } y}^{\text{final } y} 9 \, dy = \int_{y=6}^{4} 9 \, dy = -18$$

method 2

On BC, $x = 3$, $T = -j$, $F = 9j$, $F \cdot T = -9$

$$\int F \cdot T \, ds = \int -9 \, ds = -9 \times \text{length of BC} = -9 \times 2 = -18$$

Final answer for $\int F \cdot T \, ds$ on the path in Fig 8 is -18.

review (parametrizing curves) (see Section 2.0 also)

- The line through point (x_0, y_0, z_0) and parallel to arrow $a\vec{i} + b\vec{j} + c\vec{k}$ has parametric equations

(7)
$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

- The circle with center at the origin and radius 4 has parametric equations

(8)
$$x = 4 \cos t$$
$$y = 4 \sin t$$

- The circle with center at (3,7) and radius 4 has parametric equations

$$x = 3 + 4 \cos t$$
$$y = 7 + 4 \sin t$$

- The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has parametric equations

(9)
$$x = a \cos t$$
$$y = b \sin t$$

For example, the equation $3x^2 + y^2 = 2$ can be rewritten as $\dfrac{x^2}{2/3} + \dfrac{y^2}{2} = 1$ so the ellipse $3x^2 + y^2 = 2$ has parametric equations

(10)
$$x = \sqrt{2/3} \cos t$$
$$y = \sqrt{2} \sin t$$

footnote

In (8), the parameter t is also the polar coord θ. For example, if $t = \pi/4$ in (8) then the point (x,y) has polar coordinate $\theta = \pi/4$ (and polar coord r=4).

But that's not the case in (9) and (10). If $t = \pi/4$ in (10), the point (x,y) does not have polar coordinate $\theta = \pi/4$. But it *is* true in (9) and (10) that if $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$ then the correspoinding polar coordinate θ is also $0, \pi/2, \pi, 3\pi/2, 2\pi$.

- The curve with equation $y = x^3 \sin x$ has parametric equations

$$x = t$$
$$y = t^3 \sin t$$

or equivalently

$$x = x$$
$$y = x^3 \sin x \quad \text{(the parameter is x)}$$

- The curve with equation $x = y^4 + 2y$ has parametric equations

$$x = t^4 + 2t$$
$$y = t$$

or equivalently

$$x = y^4$$
$$y = y \quad \text{(the parameter is y)}$$

- The line through points (1,5) and (3,9) has plain equation $y = 2x+3$ so it has parametric equations

$$x = t$$
$$y = 2t + 3$$

or equivalently

$$x = x$$
$$y = 2x + 3$$

It also has parametric equations

$$x = \frac{3-t}{2}$$
$$y = t$$

You can also use a 2-dim version of (7) with say parallel vector $2\vec{i}+4\vec{j}$ and point (1,5) to get parametric equations

$$x = 1 + 2t$$
$$y = 5 + 4t$$

- The vertical line $x=7$ in 2-space has parametric equations

$$x = 7$$
$$y = t \quad \text{(the parameter is t)}$$

or equivalently

$$x = 7$$
$$y = y \quad \text{(the parameter is y)}$$

- The horizontal line $y=3$ in 2-space has parametric equations

$$x = t$$
$$y = 3$$

or equivalently

$$x = x$$
$$y = 3 \quad \text{(the parameter is x)}$$

- The curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = x^2y^3$ (Fig 8) has parametric equations

$$x = 2\cos t$$
$$y = 2\sin t$$
$$z = (2\cos t)^2(2\sin t)^3$$
$$0 \le t \le 2\pi$$

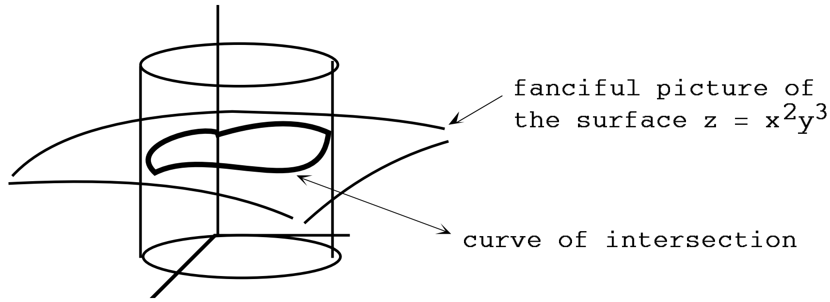


FIG 8

mathematical catechism (you should know the answers to these questions)

Question If $x = x(t)$, $y = y(t)$ what does ds stand for.

Answer ds is the arclength you get on the parametrized curve when the parameter t changes by dt.

Question And how would you compute it.
Answer $ds = \|v\|\,dt$ where $v = x'(t)\,i + y'(t)\,j$

PROBLEMS FOR SECTION 3.1

1. Let $F(x,y) = 2x\vec{i} + xy\vec{j}$.
   A curve has parametric equations $x = 3t^2$, $y = 2t$.
   Let $A = (3,-2)$, $B = (12,4)$

   (a) Check that A and B are both on the curve.
   (b) Find $\int F\cdot T\,ds$ on the curve directed from B to A.
   Set it up twice, using (6) and then again using (2).
   (c) Find T.

2. (a) Set up $\int xy\,dx + y\,dy$ on the line segment from $A = (1,1)$ to $B = (-1,5)$
   (i) using x as the parameter
   (ii) using y as the parameter
   (iii) using the 2-dim version of the parametrization in (7)

   (b) The line integral is the work done by ________ to _________ ?

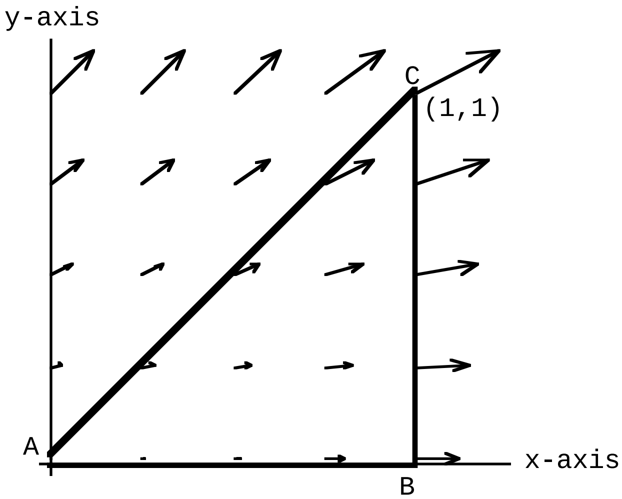
3. Let $F(x,y) = y\,\vec{i} + y\,\vec{j}$.
   Look at the circle centered at the origin with radius R.
   Set up $\int F\cdot T\,ds$
   (a) on the entire circle, directed clockwise
   (b) on the bottom right quarter of the circle directed from point $(0, -R)$ to point $(R,0)$.

4. The diagram shows the vector field $(x^3 + y)\vec{i} + y^2 \vec{j}$.

Look at $\oint (x^3 + y)\, dx + y^2\, dy$ on the ccl loop ABCA.

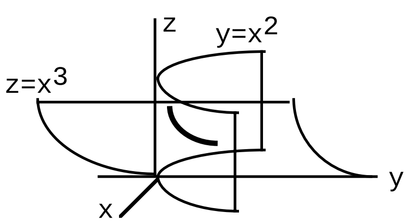
(a) Use the diagram to predict the sign of the line integral on the A-to-B part. On the B-to-C part. On the C-to-A part.

(b) Compute the line integral on ABCA (and see if your prediction came true)



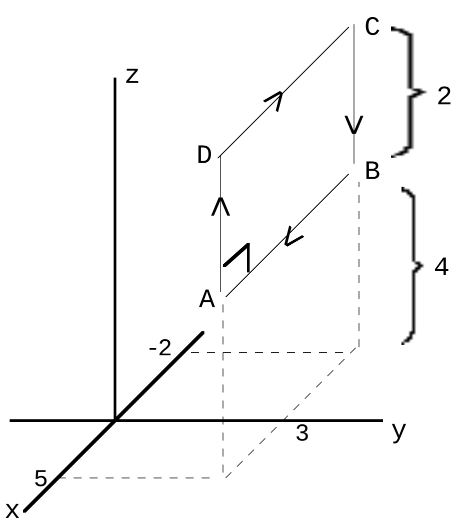
5. Let $F(x,y,z) = (y,z,x)$. Find the work done by F to a particle which moves on a line from point A = (1,2,3) to B = (2,3,5).

6. Let $F(x,y,z) = xy\,\vec{i} + xz\,\vec{j} + x\,\vec{k}$. Find the circulation on the curve of intersection (see the diagram) of surfaces $y = x^2$ and $z = x^3$ directed from point A = (2,4,8) to point B = (1,1,1).

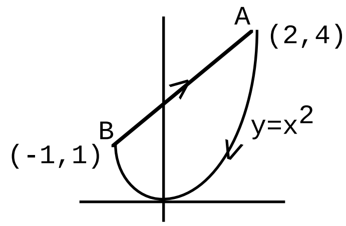


7. Let $F = z\vec{i}$. Find $\int F \cdot T\, ds$ on the rectangle ADCBA in the diagram in plane y = 3.

Do it with very little effort.

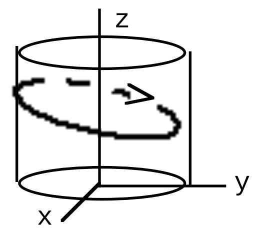


8\. Let $\vec{F} = x^2 y\,\vec{i} + (y + 3)\vec{j}$. Set up $\oint \vec{F}\cdot T\,ds$ around the loop in the diagram.



9\. Set up $\oint (x^2 + y^4)\,dx - 2\,dy$ around the ellipse $3x^2 + y^2 = 3$ directed ccl.

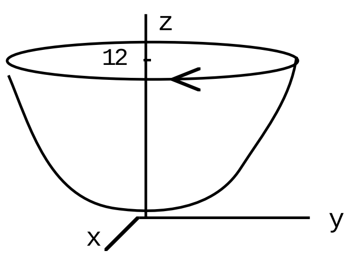
10\. Set up $\int x^2\,\vec{k}\cdot\vec{T}\,ds$ on the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $4x + y + z = 16$, directed clockwise as seen from above.



11\. Let $F = 4\vec{e}_r$ and let $B = (3,6)$. Find $\int F\cdot T\,ds$ *easily* on the line segment from B to the origin.

12\. Let $F = yz\,i + x\,k$
The diagram shows the cup $z = 2x^2 + y^2$, $0 \le z \le 12$.

(a) Find parametric equations for the rim of the cup.
(b) Set up $\int F\cdot T\,ds$ on the rim directed as in the diagram.



SECTION 3.2 THE LINE INTEGRAL $\int F\cdot N\ ds$ IN 2-SPACE

Everything in the last section about $\int F\cdot T\ ds$ holds for 2-space *and* 3-space. But the line integral $\int F\cdot N\ ds$ in this section is defined only for 2-space.

unit normals to a curve in 2-space

The notation $\vec{N}$ is used for a unit normal to a curve in 2-space (Fig 1). There are two possible N directions so a problem must specify which one it wants.

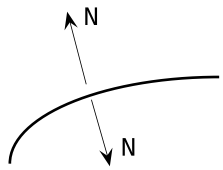


FIG 1

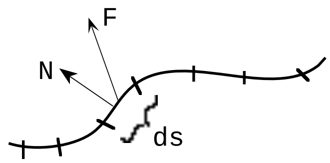


FIG 2

definition of $\int F\cdot N\ ds$

Start with a 2-dim vector field F, a curve in the plane and a direction for the unit normal N (Fig 2).

Divide the curve into many pieces and let a typical piece have length ds.

For each piece, compute F·N, the component of F in the N direction, and multiply by ds. The line integral is the sum of all the F·N ds's from the pieces, i.e.,

$$\int_{\text{curve}} F\cdot N\ ds = \sum F\cdot N\ ds$$

application to flux

If some 2-dim stuff is flowing in the plane with flux density F then

F·N ds = flux across a small piece of curve with length ds

The line integral adds all the little fluxes so

$$\boxed{\int_{\text{curve}} F\cdot N\ ds \text{ is the flux across the curve}}$$

This only makes sense in 2-space. There is no such thing as "across a curve" in 3-space.

computing $\int F\cdot N\ ds$

Here's how to turn the line integral $\int F\cdot N\ ds$ into a line integral of the form $\int G\cdot T\ ds$ which you already know how to compute (preceding section)

(1)

Suppose $F(x,y) = p(x,y)\vec{i} + q(x,y)\vec{j}$

Given a curve and an N.

Direct the curve (i.e., choose a T) so that as you walk from the initial point to the final point, N points to your right (Fig 3); i.e., choose the T that is 90° "ahead" of N.

In particular, if the curve is *closed*, then for an *outer* N use a *ccl* direction on the curve (Fig 4) and for an *inner* N use a *clockwise* direction.

Then

$$\int F\cdot N\ ds = \int_{\text{directed curve}} -q\ dx + p\ dy$$

Now parametrize the curve and keep going as in the preceding section.

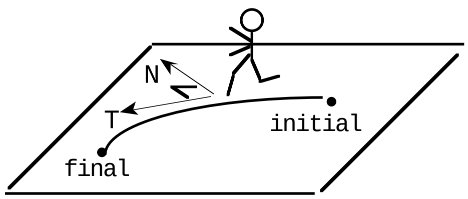


FIG 3

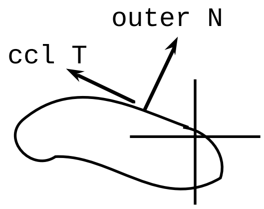


FIG 4

why (1) works

The key idea is that if you swivel both F and N, their dot product $F \cdot N$ doesn't change because the signed projection of F on N is not changed by the swiveling.

In particular if you swivel F and N each 90° ccl (each makes a left turn) (Fig 5) then

$$F \cdot N = F_{\text{leftTurn}} \cdot N_{\text{leftTurn}}$$

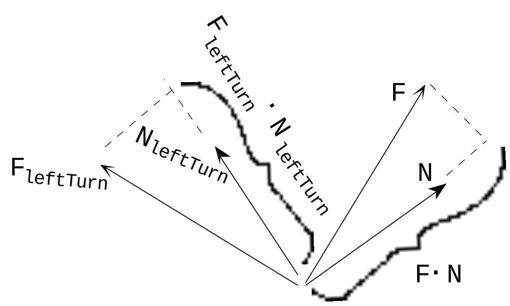


FIG 5

Furthermore,

$$N_{\text{leftTurn}} = T_{90^\circ \text{ ahead of } N} \quad \text{as in Figs 3, 4}$$

$$F_{\text{leftTurn}} = -q\vec{i} + p\vec{j} \quad \text{(see §1.0)}.$$

So

$$\int F \cdot N \, ds = \int F_{\text{leftTurn}} \cdot T_{90^\circ \text{ ahead of } N} \, ds = \int -q \, dx + p \, dy \qquad \text{QED}$$

example 1

Fig 6 shows the flux density

$$F(x,y) = xy\,\vec{i} + (x+y)\,\vec{j}$$

and the parabola $y = x^2$ between points A = (0,0) and B = (2,4). Find the flux across the curve in the indicated N direction.

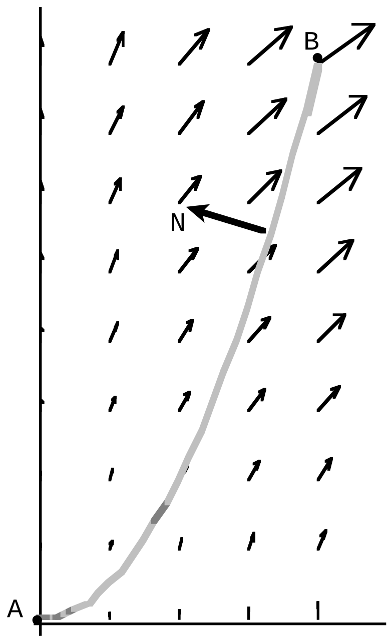


FIG 6

solution The parabola has parametric equations

$$x = x$$
$$y = x^2$$

Direct it from B (where x=2) to A (where x=0) so that N is to your right as you walk along the curve. Then

$$\text{flux across curve in the N direction} = \int F\cdot N\,ds$$

$$= \int -(x+y)\,dx + xy\,dy \text{ on the B-to-A curve}$$

$$= \int_{x=2}^{0} -(x + x^2)\,dx + x^3\,2x\,dx$$

$$= -122/15 \text{ units of stuff per sec}$$

warning

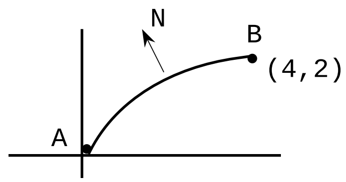
When you switch from $\int F\cdot N\,ds$ to $\int -q\,dx + p\,dy$ you have to decide which T direction to choose on the curve. On an exam I would like to see evidence that a decision took place.

reversing the normal direction

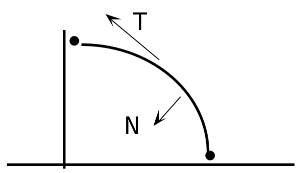
$\int F\cdot N\,ds$ changes sign if you reverse the direction of N.

PROBLEMS FOR SECTION 3.2

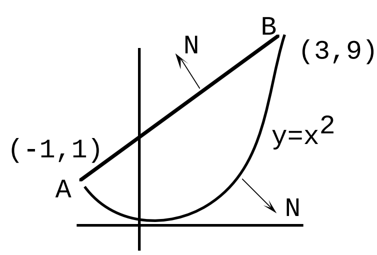
1. Let $F(x,y) = 3x\vec{i} + y\vec{j}$. Find the flux flowing across the piece of the parabola $x = y^2$ in the diagram in the direction indicated.



2. Let $F(x,y) = (3y, 4x)$.
The curve in the diagram is a quarter of a circle centered at the origin with radius 2.
(a) Find $\int F\cdot T\, ds$ where T is directed as in the diagram.
(b) Find $\int F\cdot N\, ds$ where N is directed as in the diagram.



3. Let $F = x^2 y\, \vec{i} + y\vec{j}$. The diagram shows a loop consisting of a line segment and a piece of a parabola.
(a) Compute the flux flowing out of the loop.
(b) Find the circulation on the loop. Is the circ clockwise or counterclockwise.



4. Let $F = \vec{i} + 3\vec{j}$.
Draw pictures and predict the flux into the unit circle centered at the origin. Then find it with a line integral.

SECTION 3.3 THE SURFACE INTEGRAL $\int F\cdot N\ dS$ PART I

unit normals to a surface in 3-space

In the rest of this chapter, N is a unit normal to a surface in 3-space (Fig 1). There are two possible N directions so a problem must specify which one to use.

definition of $\int F\cdot N\ dS$

Start with a vector field F(x,y,z) and a surface in space, and choose a direction for the unit normal N (Fig 2).

Divide the surface into many pieces and let a typical piece have surface area dS.

For each piece, compute F·N, the component of F in the N direction, and multiply by dS. The surface integral is the sum of all the F·N dS's from the pieces; i.e.,

$$\boxed{\int_{\text{surface}} F\cdot N\ dS = \sum F\cdot N\ dS}$$

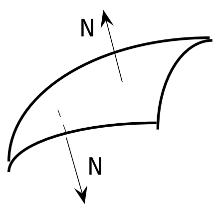


FIG 1

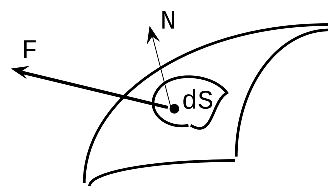


FIG 2

application to flux across a surface

Suppose F is the flux density of a fluid in three-space. Look at Fig 2 again.

For each small piece of the surface, F·N dS is the flux through the piece (see (4) in Section 1.3).

The surface integral adds all F·N dS's from the small pieces, i.e., adds all the little fluxes, so

$$\boxed{\text{flux across (i.e., through) the surface in the N direction} = \int_{\text{surface}} F\cdot N\ dS}$$

In particular:

> If F is the velocity field of a fluid (a volume flux density) then $\int F\cdot N\ dS$ is the cubic meters of fluid/sec flowing through the surface in the N direction.
>
> If F is a mass flux density then $\int F\cdot N\ dS$ is the kg/sec flowing through the surface in the N direction.

Most people refer to $\int_{\text{surface}} F\cdot N\ dS$ as the *flux through the surface in the N direction* even if F was not specifically designated as a flux density.

computing $\int F\cdot N\ dS$ using a smart n (the most general method)

Start with a vector field F, a surface in 3-space and specify one of the two normal directions to the surface. To find $\int F\cdot N\ dS$, parametrize the surface with equations of the form

$$x = x(u,v)$$
$$y = y(u,v) \qquad (1)$$
$$z = z(u,v)$$

On the surface, sweep out a patch, called a *surface area element*, with area dS by starting at a point (P in Fig 3) with parameter values u and v and letting u change by du and v change by dv. (The patch is PEDC.) I want to express dS and N in terms of u, v, du and dv.

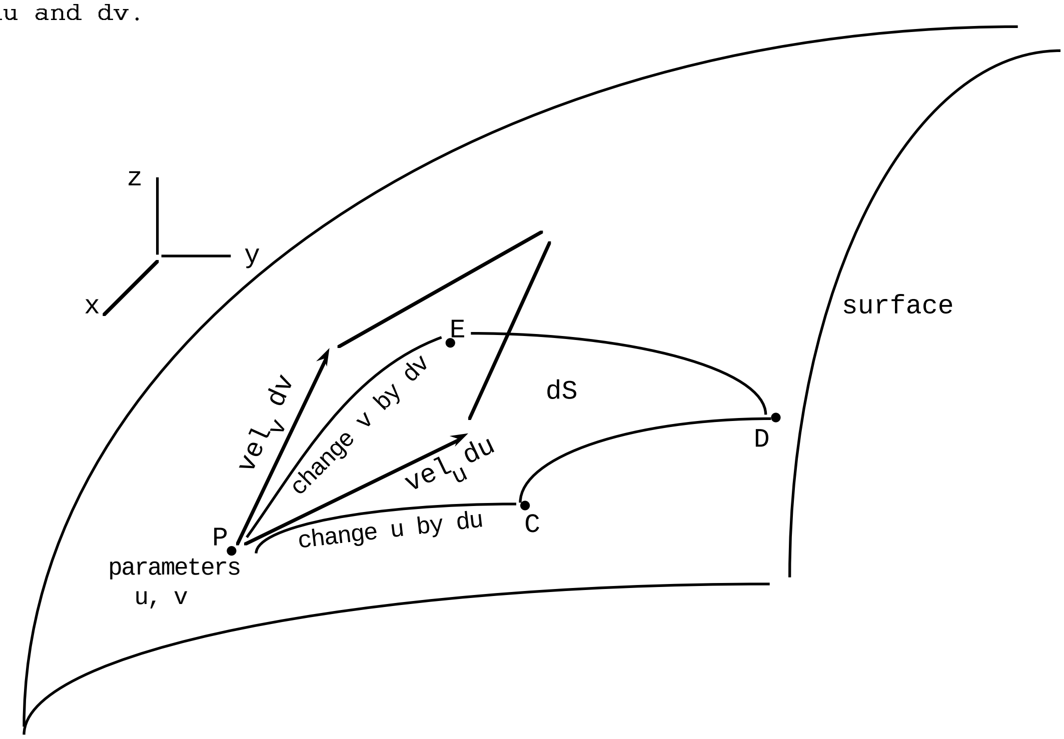


FIG 3

If PC is the curve on the surface created by changing u by du while v stayed fixed then the curve PC has parametric equations

$$x = x(u, v_0)$$
$$y = y(u, v_0)$$
$$z = z(u, v_0)$$

The "velocity" vector

$$\text{vel}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

is tangent to that edge, and the vector vel_u du approximates the side PC (tangent direction, same length) (see (2) in Section 2.0). Similarly, let

$$\text{vel}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

Then vel_v dv is an arrow approximating edge PE in Fig 3.

The two vectors vel_u du and vel_v dv determine a parallelogram (lying in the plane tangent to the surface at point P) whose area is approximately dS. So

$$dS = \|\text{vel}_u\, du \times \text{vel}_v\, dv\| \qquad \text{(property (a) of cross products from §1.0)}$$

$$= \|\, du\, dv\, \text{vel}_u \times \text{vel}_v \,\| \qquad \text{(property (g) of cross products from §1.0)}$$

(2) $$= \|\, \text{vel}_u \times \text{vel}_v \,\|\, du\, dv \qquad \text{(property (a) of norms from §1.0)}$$

Let

(3) $$\boxed{n = \text{vel}_u \times \text{vel}_v = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \right) \times \left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k} \right)}$$

Then (2) can be written as

(4) $$\boxed{dS = \underbrace{\|n\|}_{\substack{\textit{surface area} \\ \textit{magnification factor}}} \; du\, dv}$$

Furthermore, n is perp to vel_u and vel_v (the cross product is perp to each of its factors), so n is perp to the plane of the parallelogram and is a normal to the original surface. So N is either n_{unit} or $-n_{\text{unit}}$; i.e.,

(5) $$N = \frac{n}{\|n\|} \quad \text{or} \quad \frac{-n}{\|n\|}$$

Lines (4) and (5) show that n serves a dual purpose; it's a *smart* n: it points like N or like −N and its length is the surface area mag factor. Put (4) and (5) together to get

$$N\, dS = \frac{\pm n}{\|n\|} \; \|n\|\, du\, dv \; = \pm n\, du\, dv$$

In other words, $F\cdot N\, dS$ is either $F\cdot n\, du\, dv$ or $F\cdot -n\, du\, dv$.
All in all:

> To find $\int_{\text{surface}} F\cdot N\, dS$, parametrize the surface with equations of the form
>
> $x = x(u,v)$
> $y = y(u,v)$
> $z = z(u,v)$
>
> Let $n = \text{vel}_u \times \text{vel}_v$.
> Choose whichever of n and −n points like N.
> Then
>
> (6) $$\int_{\text{surface}} F\cdot N\, dS = \int_{\text{u, v parameter world}} F\cdot \pm n\, du\, dv$$
>
> Express everything in terms of u and v, and double integrate over the parameter world, i.e., put in u and v limits of integration that sweep out the surface.
> There's a review of *double* integrals in Section 3.0.

clarification

In (6), it is *not* that N turns into n or −n and dS turns into du dv (although it seems like that for practical purposes). Instead, N turns into $\frac{n}{\|n\|}$ or $\frac{-n}{\|n\|}$ and dS turns into $\|n\|\, du\, dv$ and N *times* dS cancels down to $\pm n\, du\, dv$.

Here's another way to look at it. When you replace N by $\pm n$ you are actually replacing N by $\pm\|n\|N$ which is a way of inserting the mag factor, $\|n\|$, automatically:

$$F \bullet N\, dS = F \bullet \pm \underbrace{\|n\|\; N\; du\, dv}_{dS}$$

example 1 (parametrizing a surface given its x,y,z equation)

Fig 4 shows a piece of plane $6x + 4y + 2z = 5$.

Let $F(x,y,z) = z\vec{k}$. Find the flux flowing up through the surface.

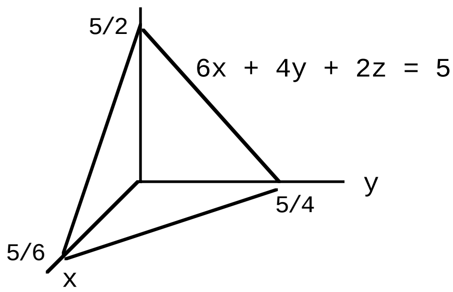


FIG 4

solution

The surface has parametric equations

$$x = x$$
$$y = y$$
$$z = \frac{1}{2}(5 - 6x - 4y)$$

The parameter world is the projection of the surface in the x,y plane (Fig 5).

> If x and y are your parameters, the parameter world is the projection of the surface in the x,y plane.
> If x and z are your parameters, the parameter world is the projection of the surface in the x,z plane
> etc.

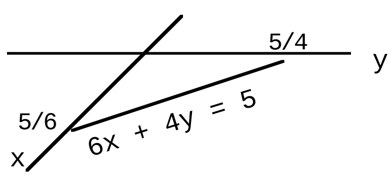


FIG 5

$$n = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) \times \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) = (1, 0, -3) \times (0, 1, -2) = (3, 2, 1)$$

n has a positive z component so n_{upper} is n rather than $-n$.

$$\text{Flux up through surface} = \int F \cdot \text{upper N } dS$$

(7)
$$= \int_{x,\, y \text{ projection}} F \cdot n_{\text{upper}}\, dx\, dy$$

$$= \int_{x,\, y \text{ projection}} z\, dx\, dy$$

$$= \int_{x,\, y \text{ projection}} \frac{1}{2}(5 - 6x - 4y)\, dx\, dy$$

$$= \int_{x=0}^{5/6} \int_{y=0}^{(5-6x)/4} \frac{1}{2}(5 - 6x - 4y)\, dy\, dx \quad \left[= \frac{125}{288} \right]$$

If F is a mass flux density then the flux up through the surface is 125/288 kg/sec. If F is a velocity field then the flow up through the surface is 6125/288 cubic meters/sec.

warning

The limits of integration are *not* $\int_{x=0}^{5/6} \int_{y=0}^{5/4}$ Those limits go with a rectangular region, not the triangular region in Fig 5.

Question

I can look at the equation of the plane $6x + 4y + 2z = 5$ and see by inspection that $6i + 4j + 2k$ is a normal (page 7 in Section 1.0). Or I can get normal $6i + 4j + 2k$ by take the gradient of $6x + 4y + 2z$.

Can I use this as my n in (7) instead of (3,2,1).

Answer

No. The n that you should use when $F \cdot N\, dS$ turns into $F \cdot \pm n\, du\, dv$ (which in this case is $F \cdot \pm n\, dx\, dy$) has to do more than be a normal. It has to be a *smart* normal meaning that its length has to be the magnification factor that turns du dv into dS. That n is $vel_u \times vel_v$. You can't use any old n.

Question

OK, so I can't use $6i + 4j + 2k$ as the smart n. Suppose I normalize it to get the *unit* normal $\frac{1}{\sqrt{56}}(6i + 4j + 2k)$. Can I use this as my N.

answer

Yes. But if you replace N by precisely what it equals then you have to replace dS by precisely what *it* equals, which in this example is not dx dy. You have to find the surface area mag factor (namely $\|n\|$) and use $dS = \|n\|\, dx\, dy$. In other words, you still have to find the smart n and extract its norm. So finding N by inspection didn't help (it will help later in this section -- see lids coming up).

parametrizing a surface of revolution

Here's an example to illustrate the idea.

Suppose the graph of $z = x^5 - 2$, $0 \le x \le 4$, in the x,z plane (Fig 6) is revolved around the z-axis. The surface of revolution (Fig 7) has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r^5 - 2$$

warning *Not* $z = x^5 - 2 = (r\cos\theta)^5 - 2$

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 4$$

where r and θ are the usual cylindrical coordinates.

To see why the z equation is $z = r^5 - 2$ look at Fig 7. At a typical point P on the original curve in the x,z plane, it is given that $z = x^5 - 2$. In the x,z plane, x and r are identical. So at P, z is also $r^5 - 2$. As P revolves (look at point Q for instance), its r coordinate and z coordinate stay the same so at Q, we still have $z = r^5 - 2$.

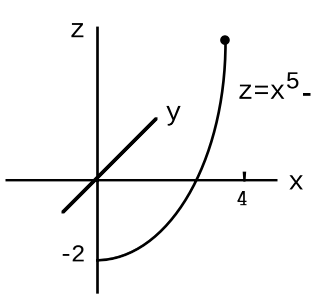


FIG 6

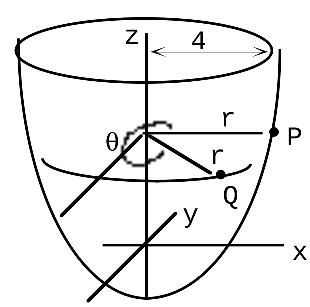


FIG 7

If the curve z = x-stuff, $x \ge 0$, in quadrants I and/or IV in the x,z plane is revolved around the z-axis then the surface of revolution has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r\text{-stuff}$$

where r and θ are the usual cylindrical coordinates, i.e., r is the distance to the z-axis and θ is the usual "around from the x,z plane" angle.

This only works if the curve in the x,z plane that you are revolving is in quadrants I or IV where $x \ge 0$ since the argument involved swapping the cylindrical coord r for x in the x,z plane and the swap doesn't work unless x is positive It's a slightly different rule if you revolve something in quadrants II or III.

Any surface whose cross sections in planes parallel to the x,y plane are circles centered on the z−axis is a surface of revolution. Here are some examples of how to go from their equations in x,y,z to parametric equations with parameters r and θ.

$z = \ln\sqrt{x^2+y^2}$ $x = r\cos\theta,\ y = r\sin\theta,\ z = \ln r$

$z = e^{x^2+y^2}$ $x = r\cos\theta,\ y = r\sin\theta,\ z = e^{r^2}$

$z = (x^2+y^2)^3$ $x = r\cos\theta,\ y = r\sin\theta,\ z = r^6$

$z = \sqrt{2x^2+2y^2}$ $x = r\cos\theta,\ y = r\sin\theta,\ z = \sqrt{2}\,r$

example 2

Let $F = x^2 z\vec{k}$.

I'll find $\int F\cdot\text{inner }\vec{N}\,dS$ over the cone in Fig 8 with radius 2 and height 6, using the cylindrical coords r and θ as parameters.

The cone is the surface of revolution you get by revolving segment AB in Fig 9 around the z-axis. Line AB has equation $z = 3x$ in the x,z plane. So the cone has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 3r \qquad \textbf{warning } \textit{Not } z = 3x = 3r\cos\theta$$
$$0 \le \theta \le 2\pi,\ 0 \le r \le 2 \text{ (the parameter world)}$$

footnote You can also see that $z = 3r$ using the similar triangles APQ and ABC in Fig 10:

$$\frac{z}{r} = \frac{6}{2}$$

Then

$$n = \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) \times \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right)$$

$$= (\cos\theta, \sin\theta, 3) \times (-r\sin\theta, r\cos\theta, 0)$$
$$= (-3r\cos\theta, -3r\sin\theta, r)$$

Now decide whether to use n or −n. The inner normal N to the cone points up rather than down (just look at Fig 8 to see this). My n points up (its third component is positive) so that's the one I want, i.e., $n_{\text{inner}} = n$, not $-n$.

warning When you use (6) you have to decide whether to use n or −n. On an exam I would like to see evidence like this that a decision took place.

(8) $$\int \vec{F}\cdot\text{inner } \vec{N}\ dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} x^2 z\vec{k}\cdot\vec{n}_{\text{inner}}\ )\ dr\ d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} x^2 zr\ \ dr\ d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (r\cos\theta)^2\ 3r\cdot r\ \ dr\ d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^4\cos^2\theta\ dr\ d\theta \qquad [\ = 96\pi/5\ ]$$

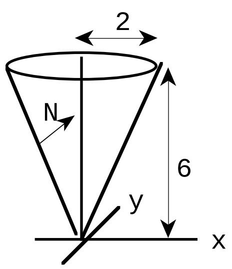


FIG 8

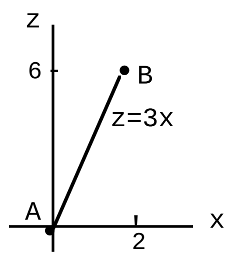


FIG 9

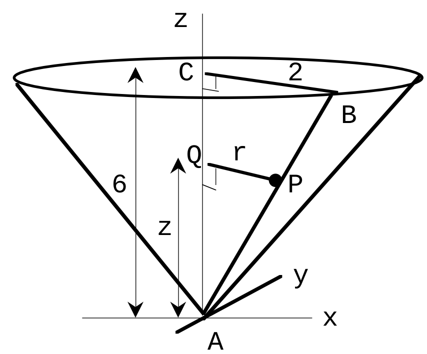


FIG 10

example 2 continued

(a) Sketch the surface area element corresponding to this parametrization (the patch traced out on the cone if you start at some point A with parameters r and θ, and change r by dr and θ by dθ).

(b) Find dS (the area of the surface area element from part (a)).

(c) Find the surface area mag factor.

solution

(a) To get a picture, start at point A on the cone in Fig 11 with parameters r, θ. If you change θ by dθ while r stays fixed the little curve AB is traced out. If you start at A and change r by dr while θ stays fixed; in order to stay on the cone you must move up and out to point D which is dr further from the z-axis. Fig 11 shows the surface area element ABCD.

(b) $dS = \|n\|\ dr\ d\theta = \sqrt{9r^2\cos^2\theta + 9r^2\sin^2\theta + r^2}\ \ dr\ d\theta = \sqrt{10}\ r\ dr\ d\theta$

(c) The surface area mag factor for this parametrization is $\sqrt{10}\,r$.

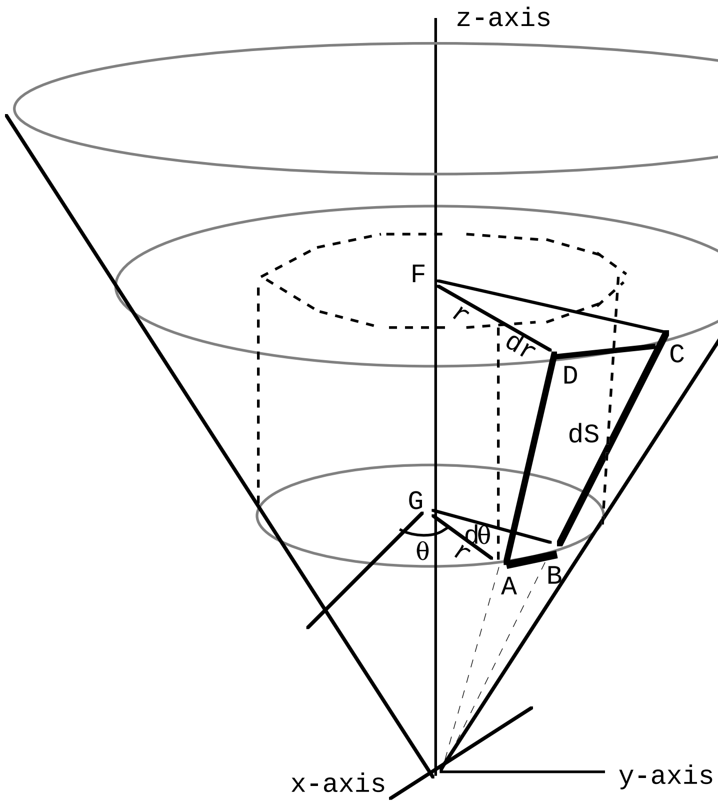


FIG 11

Question

If $dS = \sqrt{10}$ r dr dθ in example 2, then how come in (8) it looks like dS turned into plain dr dθ. What happened to the $\sqrt{10}$ r.

Answer 1

It *was* there but it canceled out before you could even see it. In (8), dS got replaced by $\|n\|$ dr dθ $= \sqrt{10}$ r dr dθ and N was replaced by $\dfrac{n}{\|n\|} = \dfrac{n}{\sqrt{10}\ r}$ and the $\sqrt{10}$ r's canceled out. The net result was that F·N dS got replaced by F·n dr dθ.

Answer 2 It *is* there, carried by the n. The unit vector N was replaced by n which is actually $\sqrt{10}$ rN.

contexts

question

I was told in Calculus that dA = r dr dθ.
So how come in (b), you got $dS = \sqrt{10}$ r dr dθ.
And what's the difference between dA and dS anyway.

answer

I use dA for the area of a small patch in 2-space. I use dS for the area of a small patch on a *surface* in 3-space.
Here's the context in which dA = r dr dθ.
Start with the equations

$x = r \cos\theta$
$y = r \sin\theta$ (r and θ are polar coordinates in 2-space)

Sweep out a little patch in 2-space by changing r by dr, θ by dθ (Fig 12).
Then dA = r dr dθ.

Here's a context in which $dS = r\,dr\,d\theta$.
Start with the equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z \qquad (r, \theta, z \text{ are cylindrical coordinates in 3-space})$$

Sweep out a little patch in 3-space by changing r by dr, θ by $d\theta$ (while z stays fixed) (Fig 13). The little patch is on a plane where z = constant and its area is $dS = r\,dr\,d\theta$.

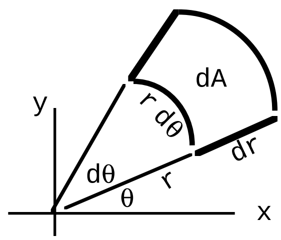


FIG 12

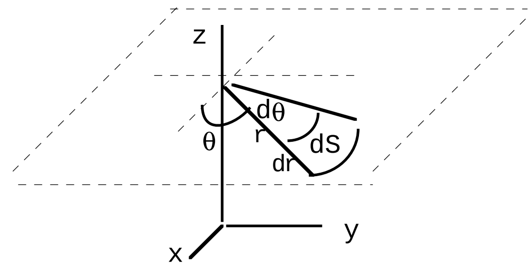


FIG 13

But the context in example 2, where dS where $dS = \sqrt{10}\, r\,dr\,d\theta$, is

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 3r \qquad (\text{not } z = z)$$

When r changes by dr and θ changes by $d\theta$, z changes also since z is 3r and the patch in Fig 11 is traced out on a *cone* not on a *plane* (Fig 13). The two contexts are different, the two patches are different and the two dS's are different.

example 1 continued

The surface back in Fig 4 has parametric equations

$$x = x,\ y = y,\ z = \frac{1}{2}(5 - 6x - 4y)$$

(a) Sketch the surface area element swept out on the surface if you start at a point P with parameters x and y and change x by dx and y by dy.
(b) Find dS.
(c) Find the surface area mag factor.

solution

(a) If I start at point P in Fig 14 and change x by dx while y stays fixed and $z = \frac{1}{2}(5 - 6x - 4y)$ (which forces me to stay on the plane), I have to come forward and down and the segment PA is traced out on the plane. If I start at P and change y by dy while x stays fixed and $z = \frac{1}{2}(5 - 6x - 4y)$ (which keeps me on the plane) I have to move to the right and down and the segment PC is traced out. The surface area element is PABC. It's the projection onto the surface of the little box EFGH in the parameter world.

(b) $dS = \|n\|\,dx\,dy = \sqrt{14}\,dx\,dy$

(c) The surface area mag factor is $\sqrt{14}$.

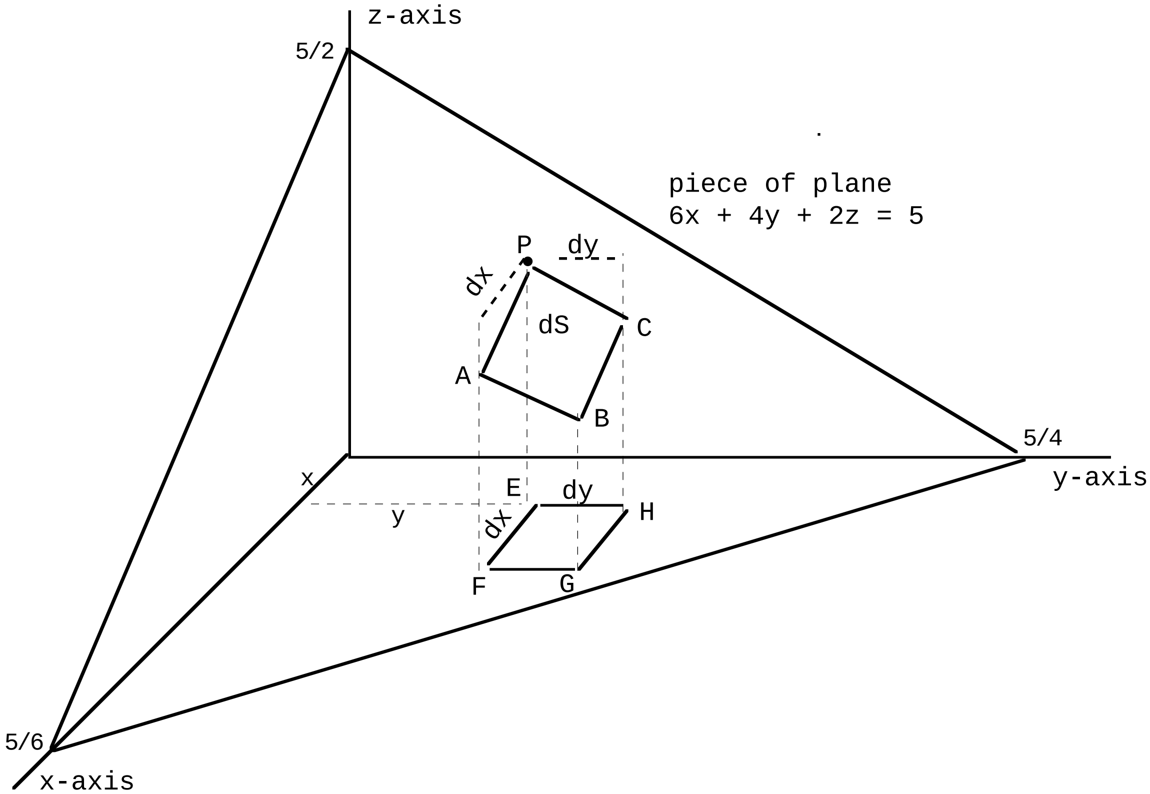


FIG 14

parametrizing a "cylinder"

The graph of $y = x^3$, $-1 \le x \le 3$ in *two*-space is the curve in Fig 15.
The graph of $y = x^3$, $-1 \le x \le 3$, $0 \le z \le 2$ in *three*-space is the cylinder in Fig 16 (see page 8 in Section 1.0).
The surface has parametric equations

$x = x$
$y = x^3$
$z = z$

The parameter world is the projection of the surface in the x,z plane where $-1 \le x \le 3$, $0 \le z \le 2$

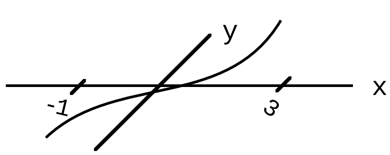


FIG 15

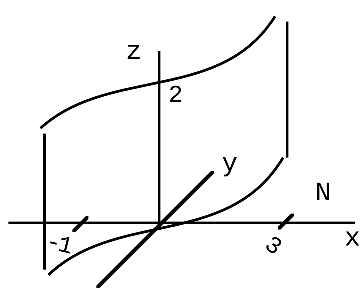


FIG 16

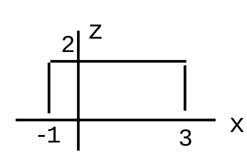


FIG 17

example 4

Fig 18 shows a piece of a plane parallel to the z-axis.
Find parametric equations for it.

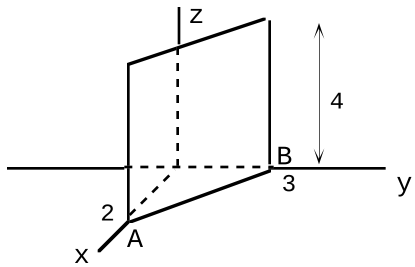


FIG 18

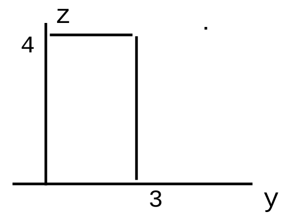


FIG 19

solution

Line AB in the x,y plane has equation $3x + 2y = 6$.

The graph of $3x + 2y = 6$ in 3-space is the whole plane in Fig 18 (a plane cylinder). One set of parametric equations for the plane is

$$x = (6 - 2y)/3$$
$$y = y$$
$$z = z$$

The parameter world is the projection of the surface in the y,z plane (Fig 19) where $0 \le y \le 3$, $0 \le z \le 4$.

notation The symbol $\oint$ is often used when the surface integral is done on a *closed surface*, like a box or an eggshell.

other notation

Let $\vec{F}(x,y,z) = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$.
These notations all mean the same surface integral:

$$\int \vec{F}\cdot\vec{N}\; dS$$

$$\int \vec{F}\cdot\overrightarrow{dS}$$

$$\int p\; dy\, dz + q\; dz\, dx + r\; dx\, dy$$

reversing N

$\int F\cdot N\; dS$ changes sign if the direction of N is reversed.

when you end up with ∫ dS

Suppose that by inspection you can see that $F\cdot N$ is a *constant*, say K, so that $\int F\cdot N\; dS$ turns into $\int K\; dS$ which is $K\int dS$.

To find $\int dS$ you can use

$$\boxed{\int_{\text{surface}} dS = \text{area of surface}}$$

(provided you know the surface area from geometry).

This works because $\int_{\text{surface}} dS$ adds all the small dS's to get a total surface area.

example 5

Suppose mass flows perpendicularly away from a line, and the mass flux density F at distance r from the line has length r^2. Find the mass/sec flowing out of the cylinder in Fig 20 with radius R, height h and with the line as its axis.

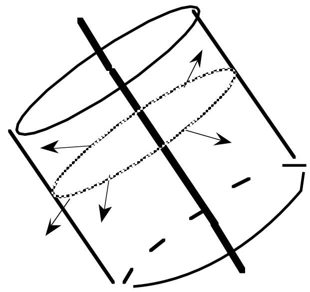


FIG 20

solution

The F arrows on the cylinder point in the outward normal direction and have length R^2 so

$$F \cdot \text{outer } N \text{ on the cylinder} = \text{length of } F = R^2$$

$$\begin{aligned} \text{mass/sec out} &= \int F \cdot \text{outer } N \, dS \\ &= \int R^2 \, dS \\ &= R^2 \int dS \\ &= R^2 \times \text{surface area of cylinder} \\ &= R^2 \times 2\pi R h \\ &= 2\pi R^3 h \end{aligned}$$

special case of $\int F \cdot N \, dS$ on a "lid" (a surface in a plane parallel to a coordinate plane)

I'll illustrate the idea by continuing from example 2 where $F = x^2 z \vec{k}$ and I found the flux into a cone.

Suppose I want the flux into the closed surface consisting of the cone *plus* the lid (Fig 21).

I already know from example 2 that the flux into the cone surface is $96\pi/5$. Now I want to find the flux flowing in through the lid, i.e., I need $\int_{\text{lid}} F \cdot \text{lower } N \, dS$.

I don't need a smart n to do this.

The lid has parametric equations

$x = x$
$y = y$
$z = 6$

The x,y parameter world is the disk in Fig 22 with radius 2.

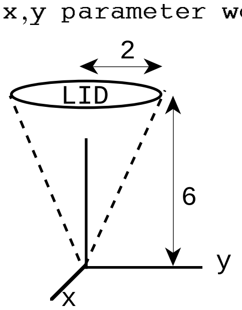


FIG 21

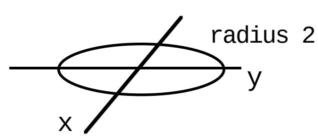


FIG 22

On the lid

inner N = $-\vec{k}$
z = 6

F = $6x^2\ \vec{k}$
F·inner N = $-6x^2$

warning
On the *cone*, $z = r^2$.
But on the *lid* it's a new game where z = 6.

dS = dx dy (when x changes by dx and y changes by dy, the surface area swept out on plane z=6 is dx dy)

$$\int_{\text{lid}} F\cdot\text{inner N dS} = \int_{\text{parameter world}} -6x^2\ dx\ dy$$

This last integral is a double integral over the disk inside the circle $x^2 + y^2 = 4$ in Fig 22. You can do it in Cartesian coordinates like this:

$$\int_{\text{lid}} F\cdot\text{lower N dS} = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -6x^2\ dy\ dx \quad [\ = -24\pi]$$

Or better still, in polar coordinates like this:

$$\int_{\text{lid}} F\cdot\text{lower N dS} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -6\ (r\cos\theta)^2\ \ r\ dr\ d\theta \quad [\ = -24\pi]$$

warning
In polar coords $dA = r\ dr\ d\theta$.
Remember the extra r.

Either way, the flux down through the lid is -24π.
The total flux *into* the covered cone is $\frac{96}{5}\pi - 24\pi = -\frac{24}{5}\pi$.

(8)

Here's the quick way to do $\int$ F·N dS on a surface lying in the x,y plane or the x,z plane or the y,z plane or in a plane parallel to one of them, i.e., on a lid. Get N by inspection (it's ±k or ±j or ±i). Find F·N on the lid. Change dS to dA. And double integrate over the lid as if it were in 2-space.

mathematical catechism (You should know the answers to these questions.)

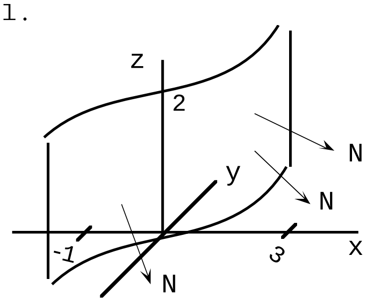
Question Suppose I have parametric equations for a surface (parameters are u and v) What is a surface area element.
Answer It's the little patch swept out on the surface when u changes by du and v changes by dv.

Question Suppose I have parametric equations for a surface (parameters are u and v). What does dS stand for.
Answer It's the surface area of the little patch swept out on the surface when u changes by du and v changes by dv.

Question And how would you compute dS.
Answer dS = $\|n\|$ du dv where n = $\text{vel}_u \times \text{vel}_v$

PROBLEMS FOR SECTION 3.3

1. Continue with the surface in Fig 16 with equation $y = x^3$, $-1 \le x \le 3$. Let $F = y\vec{i}$. Find the flux flowing through the surface in the N direction indicated below. Just set up the integral.



2. Revolve the graph of $z = x^3$ between points (0,0) and (2,8) in the x,z plane around the z-axis to get a surface of revolution.

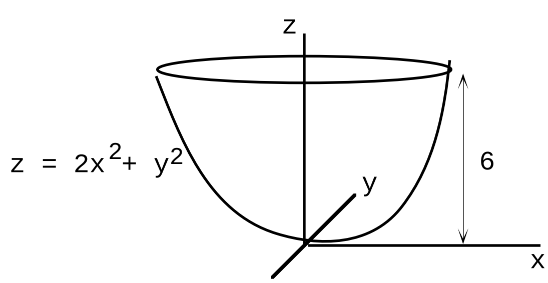
(a) Find parametric equations for the surface.

(b) Let $F = x\vec{i} + x\vec{j} + z^2\vec{k}$. Set up $\int F \cdot \text{outer } N \, dS$ on the surface.

(c) Sketch the surface area element traced out by starting at a point P and changing θ by $d\theta$ and r by dr (and label P, θ, r, $d\theta$ and dr in the diagram).

(d) Find the surface area of the patch you drew in (c).

3. The diagram shows the cup $z = 2x^2 + y^2$ with height 6



(a) Parametrize the cup and identify the parameter world.

(b) Let $F = yz\,\vec{j}$. Set up an integral to find the flux out of the cup.
(c) For your parametrization in part (a), sketch the surface area element and find dS.
(d) Set up an integral that finds the surface area of the cup.

(e) For each of the following vector fields, decide by inspection if

$\int_{\text{cup}} F \cdot \text{outer } N \, dS$ is positive, negative or zero.

(i) $F = x^2\,\vec{i}$

(ii) $F = x\vec{i}$

(iii) $F = y\vec{k}$

4. The diagram shows a cone frustrum.

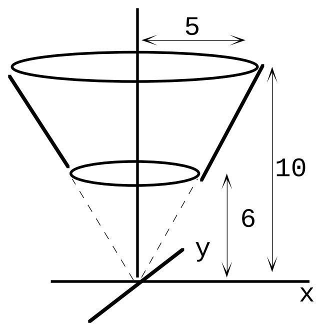
Let $F(x,y,z) = x\vec{i}$.

(a) Parametrize the frustrum and describe the parameter world.

(b) Find the flux out of the frustrum.

(c) Look at the closed surface consisting of the frustrum plus the top lid plus the bottom lid. Find the flux out.

(d) Find the dS that goes with your parametrization of the frustrum. And find the surface area mag factor.



5. (a) Sketch the surface in 3-space with equation $y = x^2$, $-2 \le x \le 2$, $0 \le z \le 3$.

(b) Let $F = y\vec{j}$. Find the flux through the surface. In what direction is it flowing.

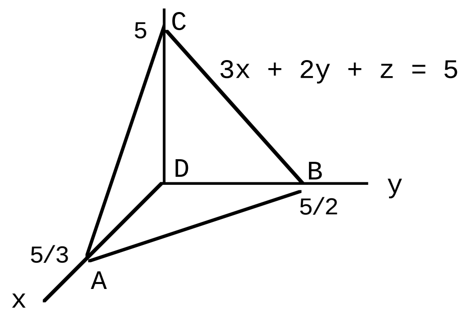
(c) Continue with the parametrization in my solution to part (b). Sketch the surface area element traced out by starting at a point P on the surface and changing x by dx and z by dz.

(d) Find the surface are of the patch you drew in part (c)

6. The diagram shows a tetrahedron,. One of its faces lies in plane $3x + 2y + z = 5$ and the other three faces lie in the coordinate planes.

Let $F(x,y,z) = (y + z)j$

Find the flux out of the tetrahedron



7. What is ambiguous about this problem:
Given the surface $2x + 3y + 4z = 5$. Find dS.

8. What do these equations parametrize. Draw a picture.

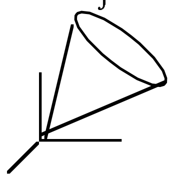
$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = e^r$$
$$r \ge 0, \quad 0 \le \theta \le 2\pi$$

9. What does it mean to say that something is a surface area magnification factor.

10. Suppose $\mathbf{F} = \rho^2\, \mathbf{e}_\rho$.

(a) Find (easily) $\int$ $\mathbf{F} \cdot$ outer $\mathbf{N}\, dS$ on the sphere with center at the origin and radius R.

(b) Find (by inspection) $\int$ $\mathbf{F} \cdot$ outer $\mathbf{N}\, dS$ on the cone in the diagram.

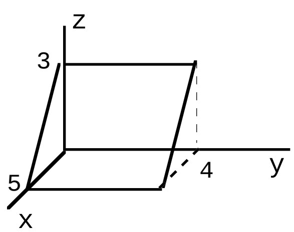


11. Suppose you use one parametrization for a surface and I use a different parametrization for the same surface. We each find n using (3). Will our n's come out the same?

12. Problem #2(a) found the parametric equations of the surface of revolution swept out when the graph of $z = x^3$ between points (0,0) and (2,8) in the x,z plane is revolved around the z-axis.

Suppose you start with the graph of $z = y^3$ between points (0,0) and (2,8) in the y,z plane and revolved it around the z-axis. What are the parametric equations of the surface of revolution.

13. Parametrize this piece of a plane (perp to the x,z plane).



14. Parametrize the plane determined by the points

A = (1, 2, 3)

B = (−7, 1, 3)

C = (4, 2, 1)

SECTION 3.4 THE SURFACE INTEGRAL $\int F\cdot N\ dS$ PART II

the area secant principle

I'll get a general rule for projecting area from a horizontal plane up to a plane inclined at an acute angle γ. I need this before getting back to surface integrals.

Fig 1 shows a rectangle ABCD in the floor with dimensions a, b and area ab. The projection up is rectangle AEFD. Look at the right triangle ABE to see that $AE = a\sec\gamma$. So

$$\text{projection area AEFD} = AD\cdot AE = ab\sec\gamma;$$

i.e.,

(1) $$\boxed{\text{projected area} = \sec\gamma \cdot \text{old area}}$$

I'm going to jump to the conclusion that this holds for an old area of any kind, not just the special old rectangle ABCD in Fig 1.

Furthermore, the angle γ between the two planes in Fig 1, namely angle EAB, happens to be the same as the angle between an upper normal to the inclined plane and $\vec{k}$ (because if you swivel the pair of vectors $\overrightarrow{AE}$ and $\overrightarrow{AB}$ around 90°, $\overrightarrow{AB}$ will point like $\vec{k}$ and $\overrightarrow{AE}$ will point like the upper normal).

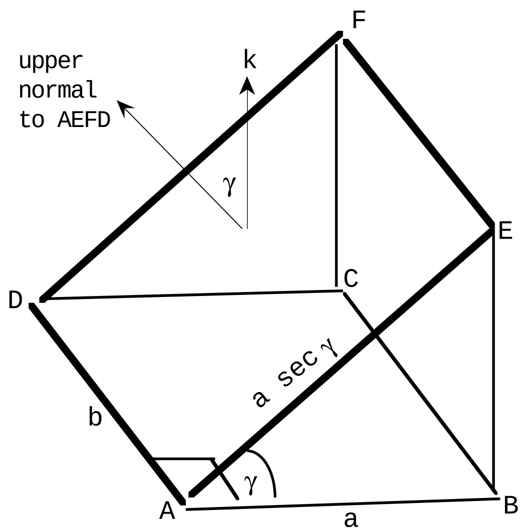


FIG 1

computing $\int F\cdot N\ dS$ in the special case where the parameters are x and y

Suppose a surface has an equation in x,y,z, i.e., an equation of the form $g(x,y,z) = K$, and you can solve that equation for z to get parametric equations

$$x = x$$
$$y = y$$
$$z = z(x,y)$$

This is the rule we already have from the preceding section:

(2) $$\boxed{\int F\cdot N\ dS = \int_{\text{projection in x, y plane}} F\cdot \pm n\ dx\ dy \quad \text{where} \quad n = \text{vel}_x \times \text{vel}_y.}$$

The rule worked because the n in (2) has two properties: It points like N or like $-N$ and $dS = \|n\|\ dx\ dy$.

I'll find another way to get n in this special case where the parameters are x and y. (Jump down to the big box below if you're impatient.)

Fig 2 shows the hypothetical surface and the surface area element swept out by starting at point P and changing x by dx and y by dy. The surface area element is the projection up of the little rectangle in the x,y plane. So, as in (1),

$$dS = \sec\gamma \, dx \, dy$$

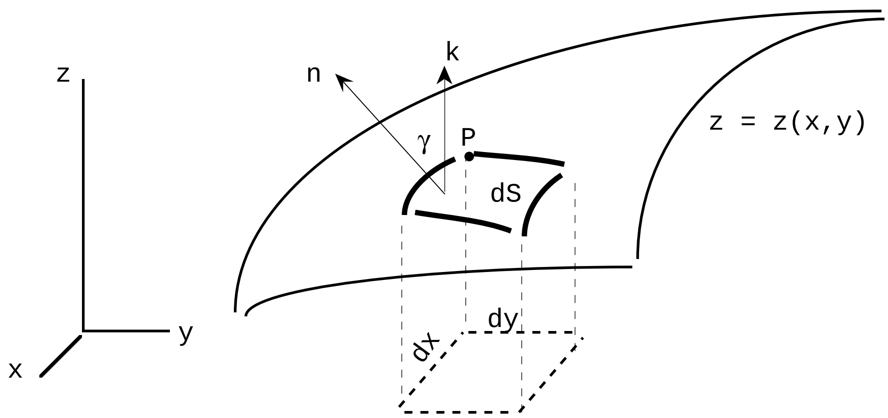


FIG 2

footnote In Fig 1, I'm projecting up to a *plane* surface and in Fig 2, I'm projecting up to a wiggly surface. But if what you're projecting up is small, you can think of the wiggly surface near P as almost a plane (the tangent plane at P).

Now let's actually find sec γ.
γ is the angle between k and an upper normal to the surface.
∇g is a normal to the surface but we can't say whether it is upper or lower. The best we can say is that the upper normal is one of $\pm\nabla g$. So (temporarily)

$$\cos\gamma = \frac{\pm\nabla g \cdot \mathbf{k}}{\|\nabla g\| \; \|\mathbf{k}\|}$$

$$= \frac{\pm\partial g/\partial z}{\|\nabla g\|}$$

I want to make cos γ comes out *positive* because γ is an acute angle. The best way to make that happen is to use

$$\cos\gamma = \frac{|\partial g/\partial z|}{\|\nabla g\|} \qquad \text{(absolute value in numerator, norm in denominator)}$$

Then

$$\sec\gamma = \frac{\|\nabla g\|}{|\partial g/\partial z|} \qquad \text{(absolute value in denominator, norm in numerator)}$$

$$(3) \qquad = \Big\| \underbrace{\frac{\nabla g}{\partial g/\partial z}}_{\text{call this n}} \Big\| \qquad \text{by the vector algebra rule } \|a\vec{u}\| = |a| \; \|\vec{u}\|$$

The vector n in (3) has two nice properties:

1. It is a normal to the surface because it's a multiple of ∇g.
   In fact it is always an *upper* normal because its third component is 1).

2. Its norm is sec γ, the surface area mag factor, i.e., $dS = \|n\| \, dx \, dy$.

So it's a smart n, just as smart as $\text{vel}_x \times \text{vel}_y$. In fact it's *is* one of $\pm\text{vel}_x \times \text{vel}_y$.

All in all, suppose a surface has an equation in x,y,z which can be solved for z so that the surface has parametric equations of the form

$$x = x, \quad y = y, \quad z = z(x,y).$$

where the parameter world is the projection of the surface in the x,y plane.

Rewrite the equation as

x,y,z stuff = constant.

Let g be the x,y,z stuff.

Then you can find n in either of two ways:

$n = \text{vel}_x \times \text{vel}_y$ (could be an upper or a lower normal)

$$n_{\text{upper}} = \frac{\nabla g}{\partial g/\partial z}$$

Decide whether you want n or $-n$ and use

$$\int F\cdot N\, dS = \int_{\text{projection in x, y plane}} F\cdot \pm n\, dx\, dy \quad \text{(this is a double integral)}$$

(There's a review of double integrals in Section 3.0)

A similar version holds if you want to use y and z as parameters. Just swap letters: Use

$$n = \frac{\nabla g}{\partial g/\partial x} \quad \text{or} \quad n = \text{vel}_y \times \text{vel}_z.$$

$$\int F\cdot N\, dS = \int_{\text{projection in y, z plane}} F\cdot \pm n\, dx\, dy$$

Similarly if you use x and z as parameters.

example 1

The graph of $z^2 = 9 + x^2 + y^2$ is shown in Fig 3a (a hyperbolic paraboloid)
The surface in this problem is that part of the top branch shown in Fig 3b.

Let $F = z\vec{k}$.
I'll set up $\int F\cdot\text{outer } N\, dS$ on the surface using x and y as parameters.

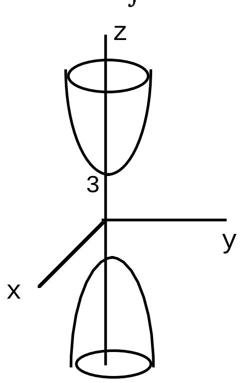


FIG 3a

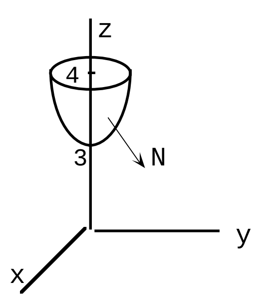


FIG 3b

The surface has parametric equations

$$x = x$$
$$y = y$$
$$z = \sqrt{9+x^2+y^2}$$

(choose the *positive* square root to go with the *upper* branch of the surface where $z > 0$)

The parameter world is in the projection of the surface in the x,y plane, the disk in Fig 4.

method 1 for finding an n The equation of the surface can be written as $z^2-x^2-y^2 = 9$. Let $g = z^2-x^2-y^2$. Then

$$n = \frac{\nabla g}{\partial g/\partial z} = \frac{(-2x,\ -2y,\ 2z)}{2z} = \left(-\frac{x}{z},\ -\frac{y}{z},\ 1\right)$$

This is an *upper* normal (the n from this method is always upper).

method 2 for getting an n

$$n = \left(\frac{\partial x}{\partial x},\ \frac{\partial y}{\partial x},\ \frac{\partial z}{\partial x}\right) \times \left(\frac{\partial x}{\partial y},\ \frac{\partial y}{\partial y},\ \frac{\partial z}{\partial y}\right) = \left(1,\ 0,\ \frac{x}{\sqrt{9+x^2+y^2}}\right) \times \left(0,\ 1,\ \frac{y}{\sqrt{9+x^2+y^2}}\right)$$
$$= \left(\frac{-x}{\sqrt{9+x^2+y^2}},\ \frac{-y}{\sqrt{9+x^2+y^2}},\ 1\right)$$

This n happens to be upper also. But it's not automatic. With this method for finding a normal you have to examine the n each time to see what you've got.

In Fig 3b you can see that the outer N is lower, not upper, so use $n_{outer} = -n$.

The circular cross section at z=4 is $x^2 + y^2 + 9 = 16$, $x^2 + y^2 = 7$ so the projection of the surface in the x,y plane is a disk centered at the origin with radius $\sqrt{7}$ (Fig 4).

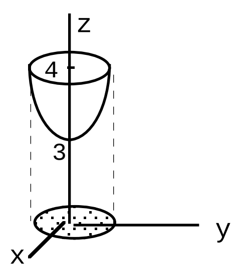


FIG 4

$$\int F\cdot \text{outer } N\ dS = \int_{x,y\ proj} F\cdot n_{outer}\ dx\ dy$$
$$= -\int_{x,y\ proj} z\ dx\ dy$$
$$= -\int_{x,y\ proj} \sqrt{9+x^2+y^2}\ dx\ dy$$

warning

Don't forget to replace z with $\sqrt{9+x^2+y^2}$ before you do any integrating. You should have only parameter letters in the integrand.

If you continue the integration in Cartesian coordinates then the limits are

$$\int_{x=-\sqrt{7}}^{\sqrt{7}} \int_{y=-\sqrt{7-x^2}}^{\sqrt{7-x^2}} \qquad \text{or} \qquad \int_{y=-\sqrt{7}}^{\sqrt{7}} \int_{x=-\sqrt{7-y^2}}^{\sqrt{7-y^2}}$$

warning Don't use $\int_{x=-\sqrt{7}}^{\sqrt{7}} \int_{y=-\sqrt{7}}^{\sqrt{7}}$

Those limits go with a square, not a circle.

It's better to do the double integral in polar coordinates:

(8) $\int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{7}} \sqrt{9+r^2} \quad r\, dr\, d\theta$ **warning** Remember the extra r.

inner integral = $\frac{1}{3}\ (9+r^2)^{3/2}\Big|_{r=0}^{\sqrt{7}} = \frac{37}{3}$

outer integral = $-2\pi\cdot\frac{37}{3}$

example 1 continued

(a) Sketch the surface area element traced out if you start at a point P on the surface with parameter values x and y and change x by dx and y by dy.

Show P, x, y, dx, dy in the picture.

(b) Find dS.

solution

(a) If you start at point P in Fig 5 and change x by dx while y is fixed and you stay on the surface, curve PA is traced out (if you didn't have to stay on the surface, segment PE would be traced out and you'd end up in front of the cup).

If you start at P and change y by dy while x is fixed and you stay on the surface, curve PB is traced out.

The surface area element is PACB.

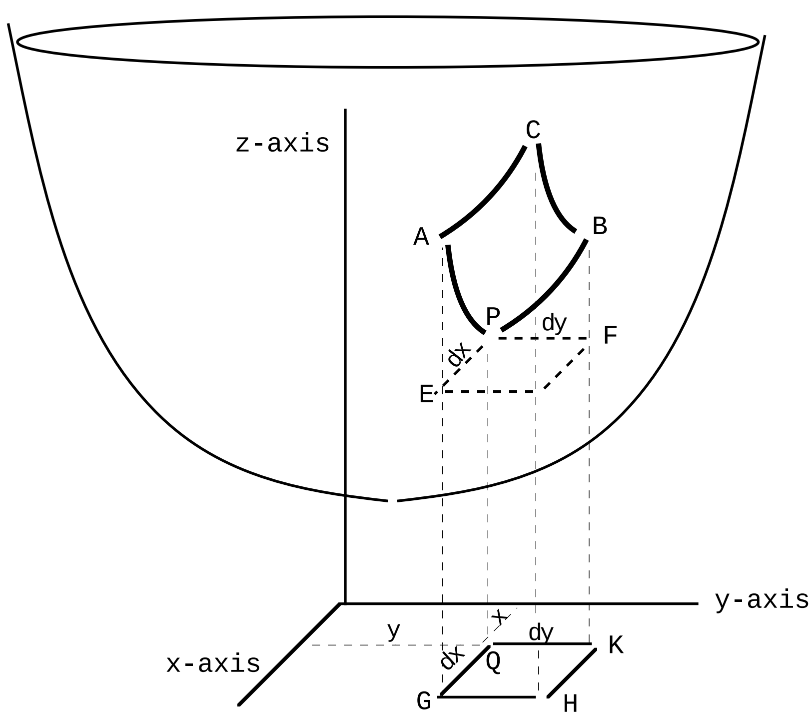


FIG 5

(b) $\|n\| = \sqrt{\dfrac{x^2+y^2}{9+x^2+y^2} + 1} = \sqrt{\dfrac{2x^2+2y^2+9}{x^2+y^2+9}}$

$$dS = \|n\|\, dx\, dy = \sqrt{\frac{2x^2+2y^2+9}{x^2+y^2+9}}\, dx\, dy$$

warning There shouldn't be any z's in your answer.

example 1 continued some more

The cross sections of the surface in Fig 4 are circles (if $z = z_o$ then the cross section is $x^2 + y^2 = z_0^2 - 9$). That means that the surface is a surface of revolution and it can be nicely parametrized using r and θ:

$x = r \cos \theta$
$y = r \sin \theta$
$z = \sqrt{9 + r^2}$
$0 \le r \le \sqrt{7},\ 0 \le \theta \le 2\pi$

The parameter world is a rectangle in a plane with an r,θ Cartesian coord system (Fig 6). Since r and θ also have geometric significance you can see them again in Fig 7.

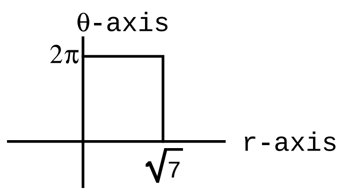


FIG 6

(a) Find $\int$ F·outer N again using the new parametrization (F is still $z\vec{k}$).
(b) Sketch the surface area element swept out when you start at a point P with parameters r and θ and change r by dr and θ by dθ. Show P, r, θ, dr, dθ in the picture.
(c) Find dS.

solution

(a) $n = \left(\cos\theta,\ \sin\theta,\ \dfrac{r}{\sqrt{9+r^2}}\right) \times (-r\sin\theta,\ r\cos\theta,\ 0)$

$= \left(\dfrac{-r^2\cos\theta}{\sqrt{9+r^2}},\ \dfrac{-r^2\sin\theta}{\sqrt{9+r^2}},\ r\right)$ (this is an upper normal since $r \ge 0$)

$n_{outer} = n_{lower} = -n$

$F\cdot -n = zr = r\sqrt{9+r^2}$

$$\int F\cdot \text{outer } N\, dS = \int F\cdot n_{outer}\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{7}} r\sqrt{9+r^2}\, dr\, d\theta$$

This agrees with the result in (8) but the spirit was different.

(b) If you start at P in Fig 7 and change θ by $d\theta$ while r is fixed and you stay on the surface, curve PB is traced out. If you start at P and change r by dr while θ stays fixed and you stay on the surface, curve PC is traced out.
The surface element is PBDC.

(c) $$\|n\| = \sqrt{\frac{r^4 \cos^2\theta + r^4 \sin^2\theta}{9 + r^2} + r^2} = r\sqrt{\frac{2r^2+9}{r^2+9}}$$

$$dS = \|n\|\, dr\, d\theta = r\sqrt{\frac{2r^2+9}{r^2+9}}\, dr\, d\theta$$

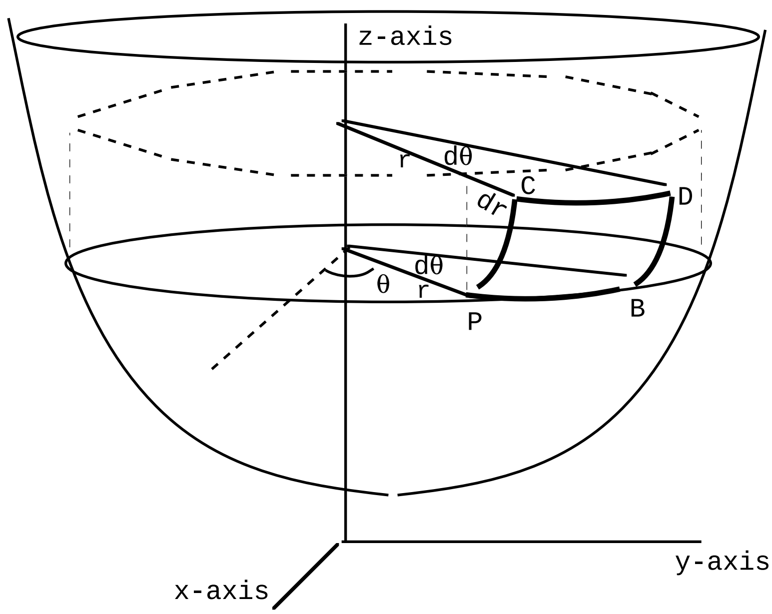
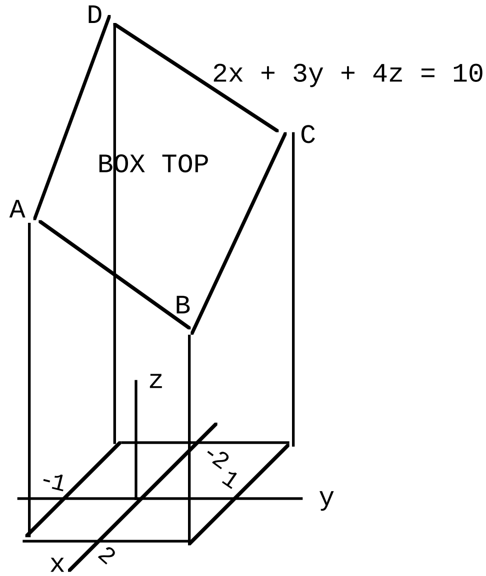


FIG 7

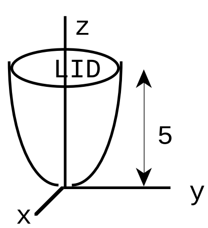
PROBLEMS FOR SECTION 3.4

1. Let $F = z\vec{j}$.
   The box top in the diagram lies in the plane $2x + 3y + 4z = 10$.

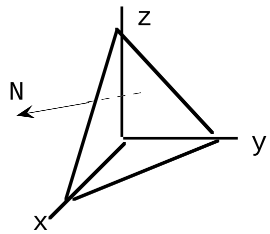


(a) Find $\int F\cdot$upper N dS over the box top
(b) Find the surface area of the box top.
(c) Find the flux out of the entire box.

2. Let $F = 3\vec{i} + z\vec{k}$ be a mass flux density.
 The diagram shows a closed surface consisting of the paraboloid $z = x^2 + y^2$ with a lid in plane z=5.
(a) Find the flux (what units does it have) into the closed surface.
 When you do the flux into the paraboloid try it twice, once with x and y as parameters and again with r and θ as parameter.
(b) Find just that much of the flux that flows into the closed surface through the front half of the paraboloid. Use y and z as parameters.



3. The surface in the diagram is the first octant portion of the plane $x + 2y + 3z = 1$.

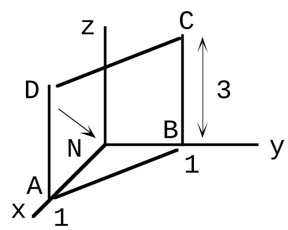


(a) Let N point back as shown in the diagram. Let $F = x\vec{j}$.
Find $\int F \cdot N \, dS$

(i) using x and y as parameters
(ii) using y and z as parameters

(b) If x changes by dx and y changes by dy how much surface area dS is traced out on the plane $x + 2y + 3z = 1$.
(c) If y changes by dy and z changes by dz how much surface area dS is traced out on the plane $x + 2y + 3z = 1$.

4. Let $F = x\vec{i} + xy^2\,\vec{j}$.
The plane in the diagram is parallel to the z-axis.
Set up $\int F \cdot \text{forward } N \, dS$ on the piece of the plane shown.



SECTION 3.5 THE SURFACE INTEGRAL $\int$ F·N dS PART III

In this section, I'll integrate on some special surfaces (spheres, cylinders, cones) where you can find N and dS separately instead of switching from N dS as a whole to n du dv.

from the reference page

cylindrical coords	*spherical coords*
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\phi$
$e_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$	$e_\rho = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$
$e_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$	$e_\phi = \cos\phi\cos\theta\,\vec{i} + \cos\phi\sin\theta\,\vec{j} - \sin\phi\,\vec{k}$
$h_r = 1$	$e_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$
$h_\theta = r$	$h_\rho = 1$
$h_z = 1$	$h_\phi = \rho$
	$h_\theta = \rho\sin\phi$

computing $\int$ F·N dS on the sphere $\rho=\rho_0$ using ϕ and θ as parameters

Suppose a sphere is centered at the origin with radius ρ_o.

The sphere has parametric equations

$$\boxed{\begin{aligned} x &= \rho_o \sin\phi\cos\theta \\ y &= \rho_o\sin\phi\sin\theta \\ z &= \rho_o\cos\phi \end{aligned}}$$

The parameter world for the whole sphere is $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.
By inspection,

$$\boxed{\text{outer } \vec{N} = \vec{e}_\rho}$$

The surface area element on the sphere (Fig 1) is swept out by starting at a point P on the sphere (Fig 1) and changing ϕ by $d\phi$ and θ by $d\theta$ (while ρ stays fixed at ρ_o). The θ-curve PA is perp to the ϕ-curve PB so the patch is a "rectangle" and (using h_ϕ and h_θ from column 2 above)

$$\boxed{dS = ds_\phi\, ds_\theta = h_\phi\, h_\theta\, d\phi\, d\theta = \rho_o^2 \sin\phi\, d\phi\, d\theta}$$

Then

$$\int \text{F}\cdot\text{outer N dS on sphere} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \text{F}\cdot e_\rho\, \rho_o^2 \sin\phi\, d\phi\, d\theta$$

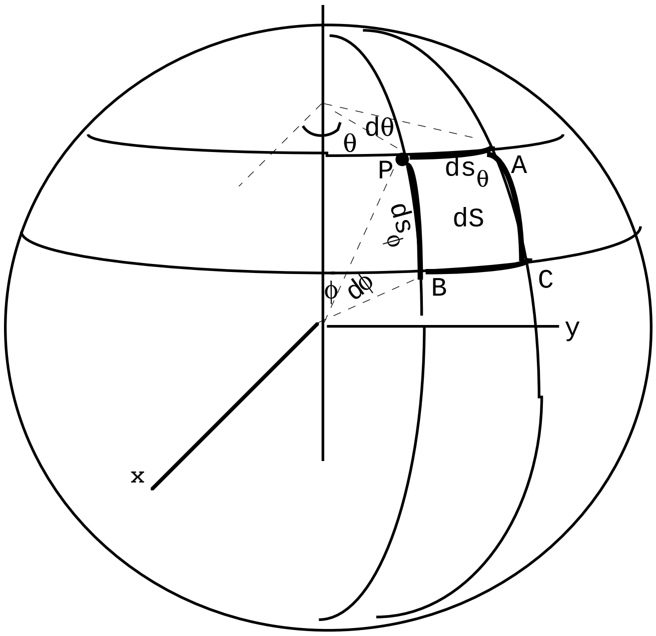


FIG 1

example 1

Let $F = \vec{j}$. Look at the first octant piece of the sphere with center at the origin and radius 3 (Fig 2).

(a) Find the flux out of this one-eighth sphere using parameters ϕ, θ.

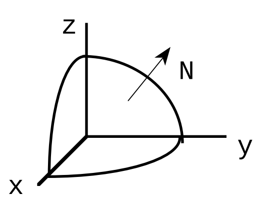


FIG 2

solution

(a) The surface has parametric equations

$$x = 3 \sin\phi \cos\theta$$
$$y = 3 \sin\phi \sin\theta$$
$$z = 3 \cos\phi$$
$$0 \le \phi \le \pi/2, \quad 0 \le \theta \le \pi/2$$

Outer $N = e_\rho = \sin\phi \cos\theta \, \vec{i} + \sin\phi \sin\theta \, \vec{j} + \cos\phi \, \vec{k}$

$dS = h_\phi \, h_\theta \, d\phi \, d\theta = 9 \sin\phi \, d\phi \, d\theta$

$F \cdot \text{outer } N = \sin\phi \sin\theta$

$$\text{Flux out} = \int_{\text{surface}} F \cdot \text{outer } N \, dS$$

(1) $$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin\phi \sin\theta \cdot 9 \sin\phi \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} 9 \sin^2\phi \sin\theta \, d\phi \, d\theta \qquad [\ = \tfrac{9}{4} \pi\ ]$$

footnote

You could also use $\int_{\text{param world}} F\cdot\pm n\ d\phi\ d\theta$ where $n = vel_\phi \times vel_\theta$, the smart n from Section 3.3. You'd end up with the same integral as above but you will have to do a lot of algebra to find (and simplify) n.

example 1 continued

(b) Find $\int F\cdot \text{outer } N\ dS$ again using parameters r and θ.

(c) Draw the surface area element and find dS for the r,θ parametrization.

solution

(b) The surface has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{9-r^2}$$
$$0 \le \theta \le \pi/2,\quad 0 \le r \le 3$$

warning

You can't use r and θ to parametrize the *entire* sphere in one shot. You would have to use

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{9-r^2}$$
$$0 \le \theta \le 2\pi,\quad 0 \le r \le 3$$

for the *top* hemisphere and

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \text{MINUS } \sqrt{9-r^2}$$
$$0 \le \theta \le 2\pi,\quad 0 \le r \le 3$$

With φ and θ you can catch a whole sphere in one shot.

$$n = \left(\cos\theta,\ \sin\theta,\ \frac{-r}{\sqrt{9-r^2}}\right) \times (-r\sin\theta,\ r\cos\theta,\ 0)$$

$$= \left(\frac{r^2\cos\theta}{\sqrt{9-r^2}},\ \frac{r^2\sin\theta}{\sqrt{9-r^2}},\ r\right) \quad \text{(this is an upper outer normal)}$$

$$\text{Flux out} = \int_{\text{surface}} F\cdot\text{outer } N\ dS$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} F\cdot n_{\text{outer}}\ dr\ d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} \frac{r^2\sin\theta}{\sqrt{9-r^2}}\ dr\ d\theta \quad \left[= \frac{9}{4}\pi\right]$$

(c) Fig 3 shows the surface area element for the r,θ parametrization. (When I started at point P and changed θ by dθ, arc PB was swept out. When I started at point P and changed r by dr, arc PC was swept out.)

$$dS = \|n\|\ dr\ d\theta = \sqrt{\frac{r^4}{9-r^2} + r^2}\ dr\ d\theta = \frac{3r}{\sqrt{9-r^2}}\ dr\ d\theta$$

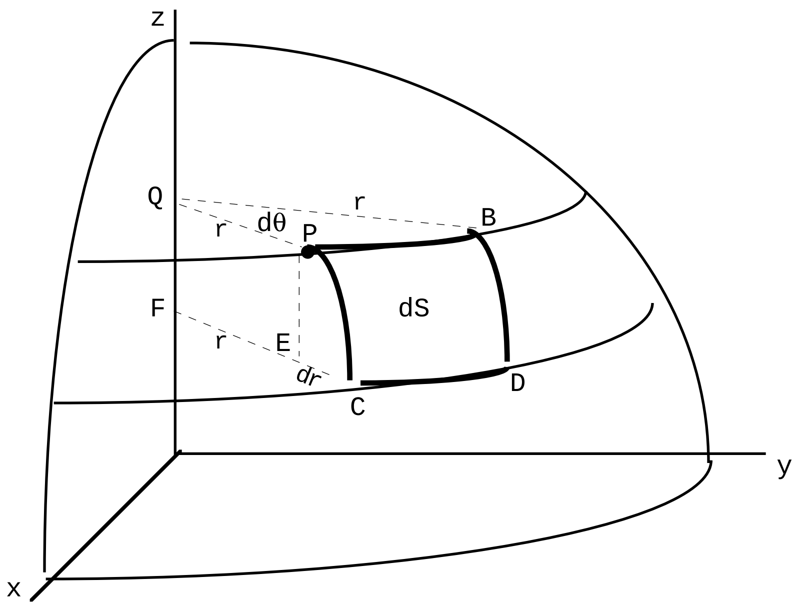


FIG 3

computing $\int$ F·N dS on the cylinder $r = r_o$ using z and θ as parameters

Look at the cylinder standing on the x,y plane in Fig 4, going around the z-axis with radius r_o and height H.

The cylinder has parametric equations

$$\boxed{\begin{aligned} x &= r_o \cos\theta \\ y &= r_o \sin\theta \\ z &= z \\ 0 \le \theta &\le 2\pi,\ 0 \le z \le H \end{aligned}}$$

By inspection,

$$\boxed{\text{outer } N = e_r}$$

Create a patch on the cylinder with area dS by starting at a point P and changing θ by dθ and z by dz. The θ-curve PA is perp to the z—curve PC so the patch is a "rectangle" and

$$\boxed{dS = ds_z\, ds_\theta = h_z\, h_\theta\, dz\, d\theta = r_o\, dz\, d\theta}$$

Then

$$\int F\cdot \text{outer } N \, dS \text{ on cylinder} = \int_{\theta=0}^{2\pi} \int_{z=0}^{H} F\cdot e_r \, r_o \, dz\, d\theta$$

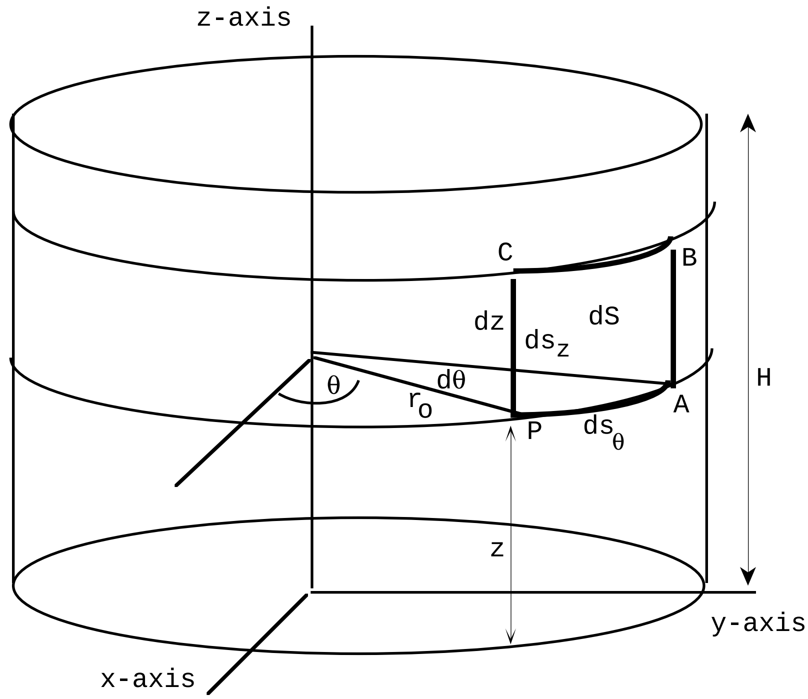


FIG 4

example 2

Let $F = y\vec{j} + 2\vec{k}$.

Find $\int$ F·outer N dS on the cylindrical surface in Fig 5.

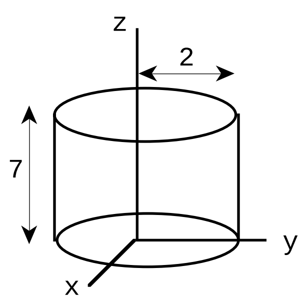


FIG 5

solution 1

The surface has parametric equations

$x = 2 \cos\theta$

$y = 2 \sin\theta$

$z = z$

$0 \le \theta \le 2\pi, \ 0 \le z \le 7$

Outer $N = e_r = \cos\theta \, \vec{i} + \sin\theta \, \vec{j}$

F·outer $N = y \sin\theta = 2 \sin^2\theta$

$dS = h_\theta \, h_z \, d\theta \, dz = 2 \, d\theta \, dz$

$$\int \text{F}\cdot\text{outer N dS} = \int_{\theta=0}^{2\pi} \int_{z=0}^{7} 2 \sin^2\theta \cdot 2 \, d\theta \, dz$$

$$= = \int_{\theta=0}^{2\pi} \int_{z=0}^{7} 4 \sin^2\theta \, dz \, d\theta \quad [= 28\pi]$$

solution 2

Use the same parametrization and find the smart n.

$$n = (-2 \sin\theta, 2\cos\theta, 0) \times (0, 0, 1)$$
$$= (2\cos\theta, 2\sin\theta, 0)$$

This n happens to be outer (it actually is $2e_r$) so $n_{outer} = n$, not $-n$.

$$F\cdot n = y\cdot 2\sin\theta = 2\sin\theta \; 2\sin\theta = 4\sin^2\theta$$

$$\int F\cdot \text{outer } N\, dS = \int_{\theta,z \text{ world}} F\cdot n_{outer}\, dz\, d\theta$$
$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{7} 4\sin^2\theta\, dz\, d\theta$$

Notice that in solution 1, the mag factor 2 is introduced when I replaced dS by $2\, d\theta\, dz$. In solution 2, the mag factor 2 was carried by the n.

finding $\int F\cdot N\, dS$ on the cone $\phi = \phi_o$ using parameters ρ and θ

Look at the cone in Fig 6.

The cone has parametric equations

$$\boxed{\begin{aligned} x &= \rho\sin\phi_o\cos\theta \\ y &= \rho\sin\phi_o\sin\theta \\ z &= \rho\cos\phi_o \end{aligned}}$$

The parameter world is $0 \le \theta \le 2\pi$, $0 \le \rho \le \rho_o$ (Fig 7). You can see from Fig 7 that $\rho_o = H/\cos\phi_o$.

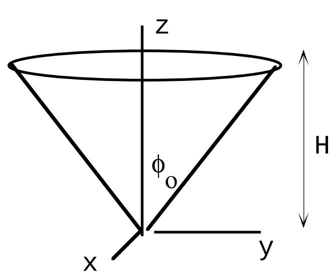


FIG 6

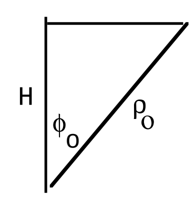


FIG 7

By inspection,

$$\boxed{\text{outer } N = e_\phi}$$

Fig 8 shows the surface area element PBCD swept out by starting at a point P on the cone and changing θ by $d\theta$, ρ by $d\rho$. The θ-curve PB is perp to the ρ-curve PD so the patch is a "rectangle" and

$$\boxed{dS = ds_\theta\, ds_\rho\, d\theta\, d\rho = h_\theta\, h_\rho\, d\theta\, d\rho = \rho\sin\phi_o\, d\rho\, d\theta}$$

Then

$$\int F\cdot\text{outer } N\, dS \text{ on cone} = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{\rho_o} F\cdot e_\phi\; \rho\sin\phi_o\, d\rho\, d\theta$$

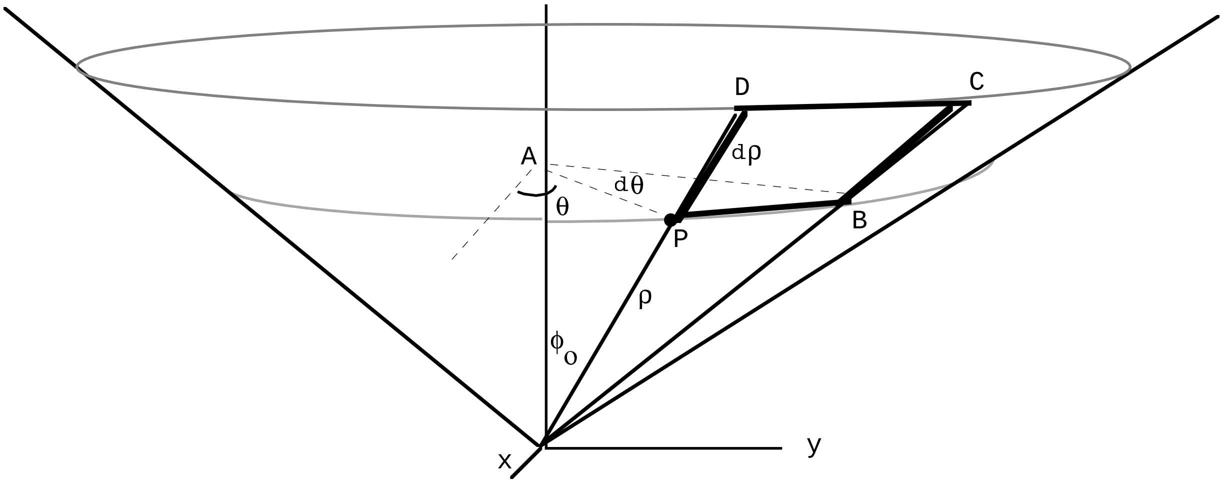


FIG 8

example 3

Let $F = z\,\vec{k}$. Look at the cone in Fig 9 with cone angle $\pi/6$ and height 4. Find the flux out of the cone (the cone does not include the lid).

(a) Use ρ and θ as parameters
(b) Use r and θ as parameters

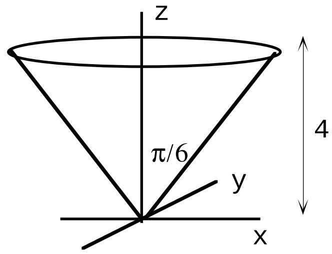


FIG 9

solution
(a) The cone has parametric equations

$$x = \rho \sin \pi/6 \cos\theta = \frac{1}{2}\cos\theta$$

$$y = \rho \sin \pi/6 \sin\theta = \frac{1}{2}\sin\theta$$

$$z = \rho\cos \pi/6 = \frac{1}{2}\sqrt{3}\;\rho$$

outer $N = e_\phi = \ \dots\ \vec{i} + \ \dots\ \vec{j} - \sin\phi\ \vec{k}$ where $\phi = \pi/6$

$F \cdot N = -\frac{1}{2}\,z = -\frac{1}{4}\sqrt{3}\;\rho$

$dS = h_\rho\, h_\theta\, d\rho\, d\theta = \rho \sin \pi/6\, d\rho\, d\theta = \frac{1}{2}\rho\, d\rho\, d\theta$

The parameter world is $0 \le \theta \le 2\pi,\ 0 \le \rho \le \ 8/\sqrt{3}$ (Fig 10).

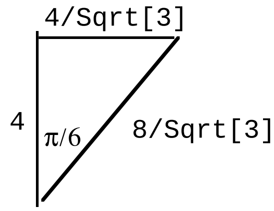


FIG 10

$$\int \text{F}\cdot\text{outer N dS} = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{8/\sqrt{3}} -\frac{1}{4}\sqrt{3}\,\rho \quad \frac{1}{2}\,\rho \; d\rho\; d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\rho=0}^{8/\sqrt{3}} -\frac{1}{8}\sqrt{3}\,\rho^2 \; d\rho\; d\theta \qquad \left[\, = -\frac{128}{9}\,\pi \,\right]$$

(b) The cone is the surface of revolution swept out by revolving the line segment in Fig 11 around the z-axis.

The line segment has equation $z = \sqrt{3}\,x$, $0 \le x \le 4/\sqrt{3}$

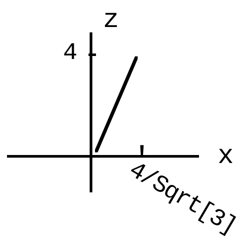


FIG 11

The cone has parametric equations

$x = r\cos\theta$

$y = r\sin\theta$

$z = \sqrt{3}\,r$

$0 \le \theta \le 2\pi,\; 0 \le r \le 4/\sqrt{3}$

Then

$$n = (\cos\theta,\ \sin\theta,\ \sqrt{3}\,) \times (-r\sin\theta,\ r\cos\theta,\ 0)$$
$$= (\,-r\sqrt{3}\cos\theta,\ -r\sqrt{3}\sin\theta,\ r)$$

n is an inner normal since it points down so $n_{\text{outer}} = -n$.

$F\cdot -n = -zr = -\sqrt{3}\,r^2$

$$\int \text{F}\cdot\text{outer N dS on cone} = \int_{\text{parameter world}} F\cdot n_{\text{outer}}\; dr\; d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4/\sqrt{3}} -\sqrt{3}\,r^2 \;\; dr\; d\theta$$

warning

In part (b), even though the parameters are r and θ, dS is *not* $h_r h_\theta\, dr\, d\theta = r\, dr\, d\theta$.

Here's the context in which dS *does* equal $r\, dr\, d\theta$. Start with

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

Change r by dr and θ by dθ while z stays fixed. Then a surface area element is

traced out on a *plane* (where z is constant) (Fig 12) and $dS = h_r\ h_\theta\ dr\ d\theta = r\ dr\ d\theta$.
But the context in part (b) is

$$x = r \cos\theta, \quad y = r \sin\theta, \quad z = \sqrt{3}\ r$$

Different contexts, different patches (Fig 12 vs. Fig 13), different dS's.

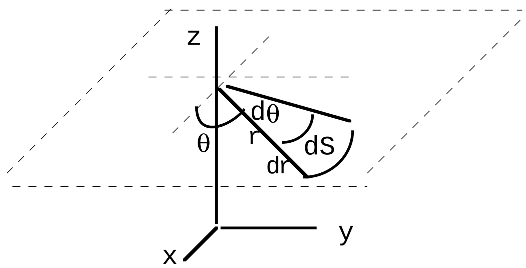


FIG 12

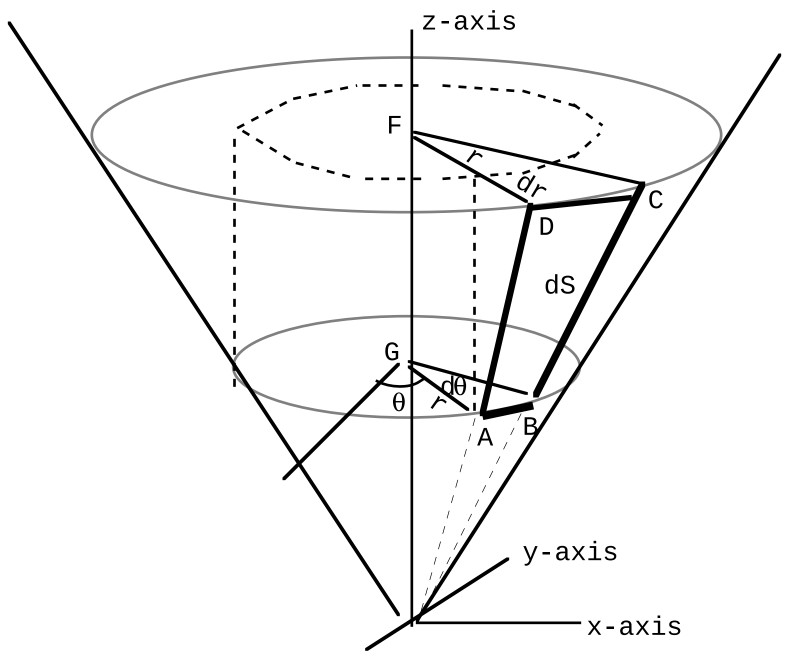


FIG 13

summary (this is your last chance)

When you find $\int \mathbf{F}\cdot\mathbf{N}\ dS$, you have to parametrize the surface and find N dS.

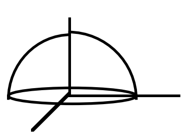
The general method (that always works) is to find a smart n and replace N dS all at once by n du dv.
This section says that in certain instances it might be more efficient to find N and dS separately.
Don't mix up these two methods. If you choose (or are coerced) to actually find N then you must also carefully find dS (it isn't necessarily plain du dv). On the other hand if you use n then your mag factor is built in automatically and N dS becomes n times plain du dv.

PROBLEMS FOR SECTION 3.5

1. Let $F = \rho \cos\phi\ \vec{e}_\rho$. The hemisphere in the diagram has center at the origin and radius 3.

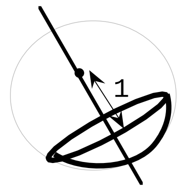
(a) Describe the F vectors on the hemisphere and sketch them.

(b) Find $\int F\cdot\text{outer } N\ dS$.



2. The diagram shows a spherical cap on a sphere of radius 3 cut off by a plane at distance 1 from the center

Find $\int$ dS to get the surface area of the cap (get a numerical answer).



3. Let $F = \frac{1}{r}\, e_r$.

Find the flux out of the sphere centered at the origin with radius R.
Try it twice
(a) Use ϕ and θ as parameters.
(b) Use r and θ as parameters

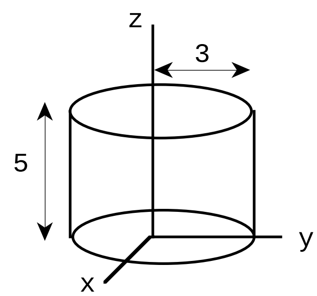
4. Let $F = \frac{1}{\rho^2}\, e_\rho$.

Find the flux out of the sphere centered at the origin with radius R.

5. Find $\int F \cdot \text{outer } N \, dS$ on the cylinder in the diagram if

(a) $F = x\vec{i}$

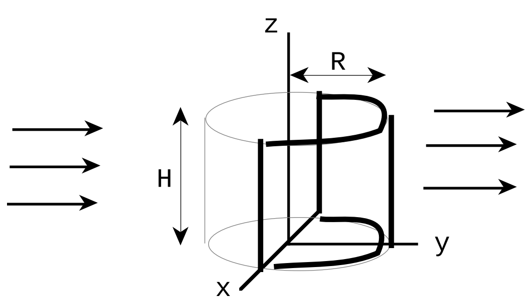
(b) $F = z\vec{j}$ (do this by inspection)



6. The diagram shows a cylinder with radius R and height H.

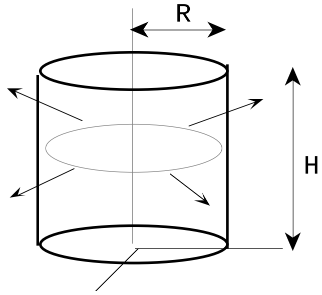
Let $F = \vec{j}$

Find the flux flowing out of the right half of the cylinder.



7. Let $F = \frac{1}{r}\, e_r$, the electric field due to a line of charge on the z-axis.

Find the flux out of the tin can in the diagram (the cylindrical surface plus flat top and bottom pieces).



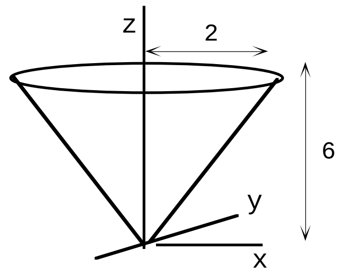
8. Your friend in another course is given the equations

$$x = 3 \sin\phi \cos\theta$$
$$y = 3 \sin\phi \sin\theta$$
$$z = 3 \cos\phi$$

and told to compute $\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) \times \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right)$

You look over her shoulder and get the answer immediately (well almost immediately), without computing any derivatives and without doing any crossing. How do you do it and what answer do you get.

9. Let $F = z\vec{k}$. Find the flux into the cone in the diagram
(a) using ρ and θ as parameters
(b) using r and θ as parameters



REVIEW PROBLEMS FOR CHAPTER 3

When you compute a line integral or a surface integral directly (as opposed to using some of the theory in the next chapter) you have to parametrize the curve or the surface. For your answer to be intelligible, you have to include your parametric equations.

reminder

In 2-space, the graph of an equation in x and/or y is a curve.
In 3-space, the graph of an equation in x and/or y and/or z is a surface, *not* a curve. For example, the graph of the equation $x = y^2$ in 3-space is a *surface*.

A curve in 2-space or 3-space has parametric equations with *one* parameter.

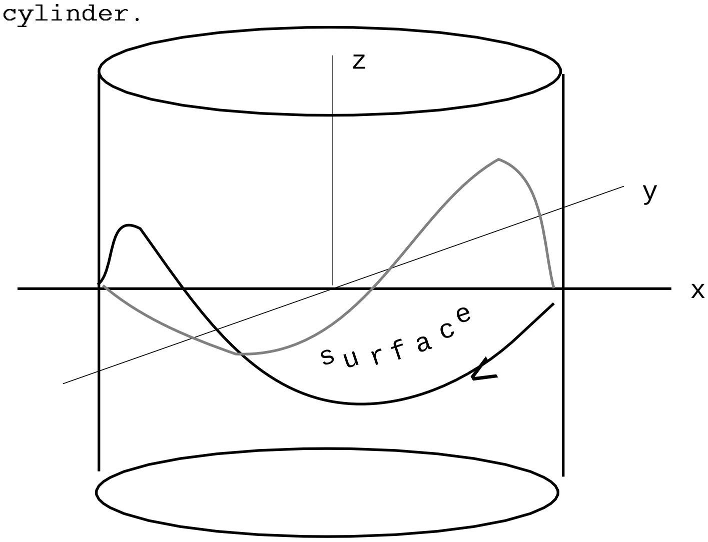
A surface in 3-space has parametric equations with *two* parameters.

1. Let $F = xy\,\vec{i} + (x^2 + 2)\vec{j}$.
   Let A = (1,1), C = (2,3).
   (a) Find the circulation of F along the line segment from A to C.
   (b) Set up the integral for the flux across segment AC in the upper N direction.

2. The diagram shows the intersection of the cylinder $x^2 + y^2 = 9$ and the surface $z = xy$.

Let $F = y\vec{i} + z\vec{j} + x\vec{k}$

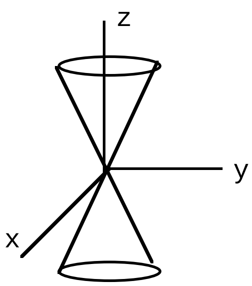
(a) Find the circulation of F around the curve of intersection in the indicated direction.
(b) Find the flux up through the piece of surface $z = xy$ trapped inside the cylinder.

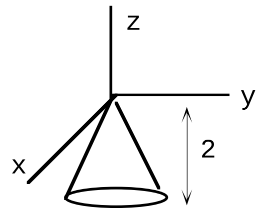


3. Let F = i + k.

The graph of $z^2 = x^2 + 4y^2$ is an elliptic cone.

Find the flux out of the closed surface in the righthand diagram consisting of a piece of the cone plus a lid.



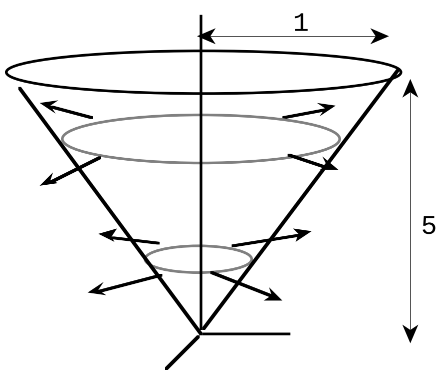


4. If x = x(u,v), y = y(u,v), z = z(u,v) what does dS mean and how do you compute it.

5. Let $F = \frac{1}{r}\, e_r$, the electric field due to a line of charge on the z-axis.

The surface in the diagram consists of a cone plus its lid.

Find the flux out of the surface.

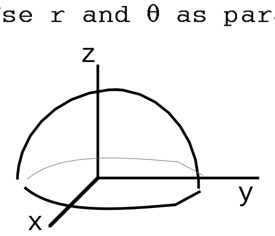


6. Let F = k. Find the flux out of the hemisphere in the diagram with center at the origin and radius 6. Do it three ways.

(a) Use ϕ and θ as parameters.

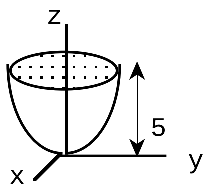
(b) Use x and y as parameters

(c) Use r and θ as parameters.



7. Let $F = e_\rho$ (spherical coord basis vector).

The diagram shows the cup $z = x^2 + y^2$, $0 \le z \le 5$.



(a) Find the flux flowing out through the cover (just the cover, not the cup itself)
Suggestion: First convert F to Cartesian coords.
(b) Find the circulation around the rim of the lid directed ccl as viewed from above.

8. What geometric thing does each of the following parametrize/describe. Draw pictures.

(a) $x = 2\cos\theta,\ y = 2\sin\theta,\ z = 4,\ 0 \le \theta \le 2\pi$

(b) $x = 2\cos\theta,\ y = 2\sin\theta,\ z = z,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 5$

(c) $x = 2\cos\theta,\ y = 2\sin\theta,\ z = \theta,\ 0 \le \theta \le 4\pi$

(d) $x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ 0 \le z \le 5$

(e) $x = r\cos\theta,\ y = r\sin\theta,\ z = 4,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 2$

(f) $x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ 0 \le \theta \le 2\pi,\ r \ge 0$

(g) $x = r\cos\theta,\ y = r\sin\theta,\ z = r^2,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 4$

(h) $x = r\cos\theta,\ y = r\sin\theta,\ z = 7 - r,\ 0 \le \theta \le 2\pi,\ 1 \le r \le 2$

9. Here are some integrals. In each case, *without doing any computing*, decide if the integral is positive, negative or zero. Explain briefly.

(i) $\int y\vec{i}\cdot T\, ds$ on the curve in Fig A directed from A to B

(ii) $\int e^y\, dV$ on the spherical region in Fig B (centered at the origin)

(iii) $\int x^3\, dA$ on the circular region in Fig C (centered at the origin)

(iv) $\int x^3\, dA$ on the circular region in Fig D (not centered at the origin)

(v) $\int y\vec{k}\cdot \text{outer } N\, dS$ on the top hemisphere in Fig B

(vi) $\int dA$ on the region in Fig E

(vii) $\int ds$ on the curve in Fig E

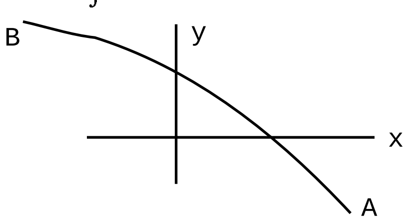


FIG A

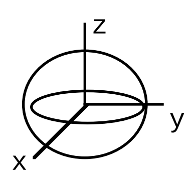


FIG B

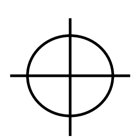


FIG C

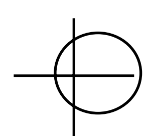


FIG D

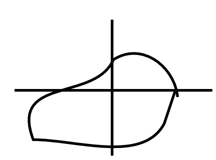


FIG E

10. Fill in the blanks (the first blank should be filled in with something like "arc length", "area", "volume", "surface area" etc.

(a) If $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ and u changes by du and v changes by dv then the _____ traced out is ____.

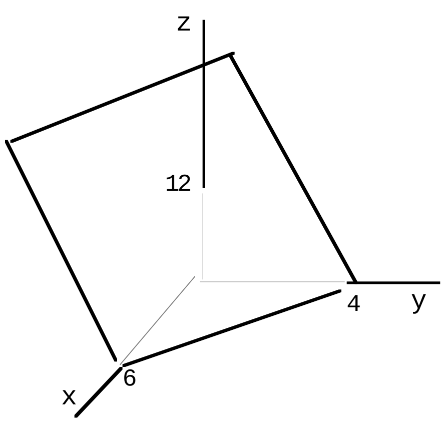
(b) If $x = x(u,v)$, $y = y(u,v)$ and u changes by du then the ______ traced out is ____

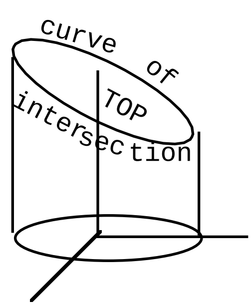
(c) If $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$ and ρ changes by $d\rho$, θ changes by $d\theta$ then the ______ traced out on a ______ is ______ . Draw a picture.

(d) If $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$ and ρ changes by $d\rho$ and ϕ changes by $d\phi$ then the _______ traced out on a ________ is _______ . Draw a picture.

(e) If $x = r \cos\theta$, $y = r \sin\theta$, $z = z$ and θ changes by $d\theta$ and z changes by dz then the _____ traced out on a _____ is _____ . Draw a picture.

11. The lefthand diagram shows the plane $2x + 3y + z = 12$.
The righthand diagram shows part of the cylinder $x^2 + y^2 = 4$ and its intersection with the plane.
Find parametric equations, *including the parameter world*, for
(a) the curve of intersection
(b) the cylindrical surface in the diagram (above the x,y plane and under the plane)
(c) the slanted top





CHAPTER 4 INTEGRAL THEORY

SECTION 4.0 REVIEW

computing triple integrals

Here's one way to set up $\int f(x,y,z)\, dV$ over a solid region R in 3-space.

Identify the lower and upper boundary surfaces of R (Fig 1). Solve the equation of each boundary for z to get the inner limits of integration.

Use the projection of the solid in the x,y plane to get x and y limits for the middle and outer integrals (as if you were doing a double integral over the projection):

$$\underbrace{\int \qquad \int}_{\substack{\text{double integrate over projection} \\ \text{of R in the x,y plane}}} \qquad \int_{z \text{ on lower bdry}}^{z \text{ on upper bdry}}$$

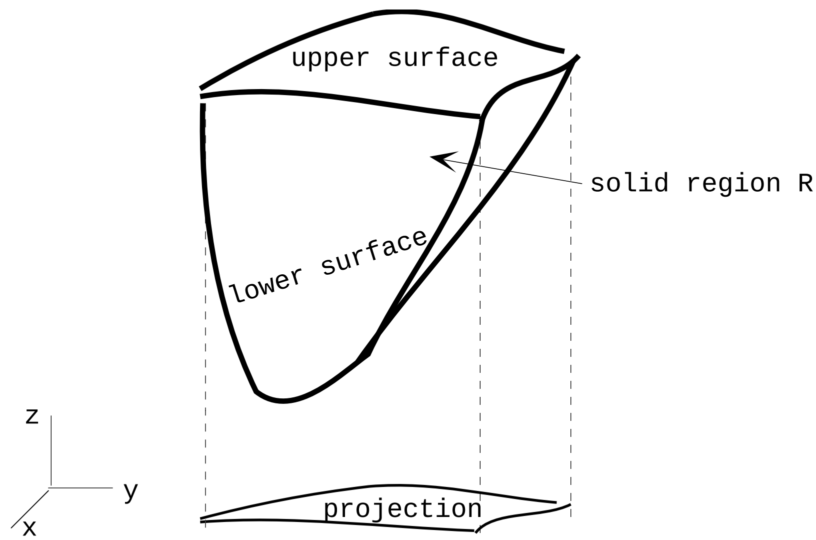


FIG 1

Similarly you can integrate first w.r.t. x. In this case, identify the rear and forward boundary surfaces of the solid R. Solve the equation of each boundary for x to get the inner limits of integration and use the projection of the solid in the y,z plane for the middle and outer limits:

$$\underbrace{\int \qquad \int}_{\substack{\text{double integrate over projection} \\ \text{of R in the y,z plane}}} \qquad \int_{x \text{ on rear bdry}}^{x \text{ on forward bdry}}$$

Similarly you can use

$$\underbrace{\int \qquad \int}_{\substack{\text{double integrate over projection} \\ \text{of R in the x,z plane}}} \qquad \int_{y \text{ on left bdry}}^{y \text{ on right bdry}}$$

example 1

Set up $\int f(x,y,z)\ dV$ over the solid tetrahedron in Fig 2, where plane ABC has equation $6x + 4y + 3z = 12$.

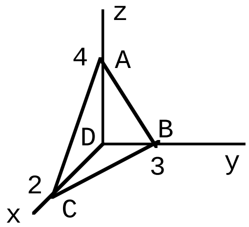


FIG 2

solution The lower boundary is $z = 0$, the upper boundary is $z = \frac{1}{3}(12 - 6x - 4y)$. The projection in the x,y plane is the triangular region CBD where line BC has equation $6x + 4y = 12$, $3x + 2y = 6$.

$$\int f(x,y,z)\ dV = \int_{x=0}^{2} \int_{y=0}^{(6-3x)/2} \int_{z=0}^{(12-6x-4y)/3} f(x,y,z)\ dz\ dy\ dx$$

method 2 Rear bdry is plane $x = 0$, forward is plane ABC where $x = \frac{1}{6}(12 - 4y - 3z)$. The projection in the y,z plane is the triangular region ADB where line AB has equation $4y + 3z = 12$.

$$\int f(x,y,z)\ dV = \int_{y=0}^{3} \int_{z=0}^{(12-4y)/3} \int_{x=0}^{(12-4y-3z)/6} f(x,y,z)\ dx\ dz\ dy$$

integrating in cylindrical coordinates (polar coords plus z)

Substitute

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$ (i.e., leave z alone)

$x^2 + y^2 = r^2$

$dV = r\ dz\ dr\ d\theta$

The inner limits should be $\int_{z \text{ on lower boundary surface}}^{z \text{ on upper boundary surface}}$.

The middle and outer limits are the same as if you were using polar coordinates to double integrate over the projection of the solid in the x,y plane.

example 2

Set up $\int y^2\ dV$ over the cylinder in Fig 3 with radius 3 and height 7.

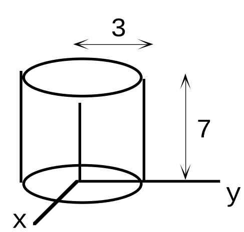


FIG 3

solution The lower boundary is plane z=0, upper is plane z=3.
The projection in the x,y plane is a circular region with radius 3.

$$\int y^2\ dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{z=0}^{7} (r\sin\theta)^2\ r\ dz\ dr\ d\theta$$

example 3

Set up $\int y^2\, dV$ over the solid inside the surface $z = 2x^2 + 2y^2$ and under plane z=r (Fig 4).

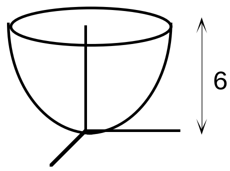


FIG 4

solution

The lower boundary is the surface $z = 2x^2 + 2y^2$ which in cylindrical coordinates is $z = 2r^2$. The upper boundary is the plane z=6.

The cross section of the solid in the plane z=6 is the inside of circle $2x^2 + 2y^2 = 6$. So the projection of the solid in the x,y plane is a circular region with radius $\sqrt{3}$. All in all,

$$\int y^2\, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \int_{z=2r^2}^{6} (r \sin\theta)^2\, r\, dz\, dr\, d\theta$$

triple integration in spherical coordinates

To evaluate $\int_R f(x,y,z)\, dV$ in spherical coords, substitute

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$dV = ds_\rho\, ds_\phi\, ds_\theta = h_\rho\, h_\phi\, h_\theta\, d\rho\, d\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

And, for the regions for which spherical coords are advisable, switch to three single integrals with limits as follows.

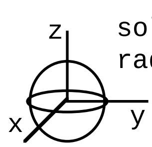
$$\int_{\text{smallest } \theta}^{\text{largest } \theta} \int_{\text{smallest } \phi}^{\text{largest } \phi} \int_{\rho \text{ on inner bdry}}^{\rho \text{ on outer bdry}}$$

The limits on the inner integral will contain θ and/or ϕ unless the inner and outer boundaries are spheres centered at the origin.
Theoretically the middle limits may contain θ but in practice they will be constants.
The outer limits are always constants.

example 4

Here are some common regions with spherical boundaries with the corresponding limits on a triple integral.

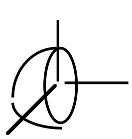
FIG 5



solid sphere
radius a

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a}$$

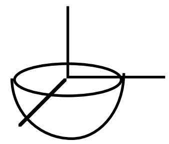
FIG 6



left solid
hemisphere
radius a

$$\int_{\theta=\pi}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a}$$

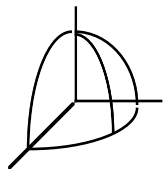
FIG 7



lower solid hemisphere radius a

$$\int_{\theta=0}^{2\pi} \int_{\phi=\pi/2}^{\pi} \int_{\rho=0}^{a}$$

FIG 8



first octant part of solid sphere radius a

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a}$$

volume

$$\int_{\text{solid region R}} 1 \, dV = \text{volume of the region R}$$

SECTION 4.1 ANTIGRADIENTS AND THE ZERO-CURL RULE

notation

The symbol $\oint$ is used for a line integral on a *loop*, i.e., a closed curve which ends up back where it started (e.g., an entire circle).
The symbol $\int$ is still used for any kind of path, loops and non-loops alike.

line integrating $F \cdot T$ when F is a gradient

Suppose F is a gradient, say $F = \nabla f$.
In other words, suppose F has an antigradient, f. Then

$$\int F \cdot T \, ds \text{ on any path from A to B} = f(B) - f(A) = \text{change in antigrad of } F.$$

why f(B) – f(A) works

Suppose F is a gradient, say of f. Then $F = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$ and

$$\int F \cdot T \, ds = \int \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \text{sum of } \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\text{'s}$$

From calculus, $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ is the change, called df, in f(x,y) as x changes by dx and y changes by dy. As you walk from A to B in Fig 1, there is a change df in (temperature) f(x,y) as you walk from A to A_1, another change df as you walk from A_1 to A_2 etc. The line integral adds the df's. So the line integral is the *total* change in f(x,y) from beginning to end, which is f(B) − f(A) no matter what the path from A to B.

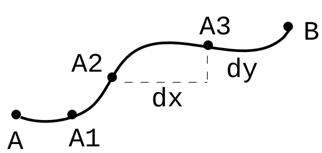


FIG 1

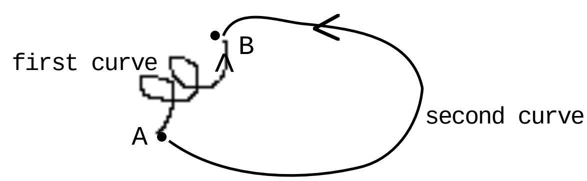


FIG 2

corollary

Suppose F has an antigradient, f. Then

(A) $\int F \cdot T \, ds$ is *independent of path* meaning that if two curves have the same initial and final points (Fig 2) then $\int F \cdot T \, ds$ has the same value on the curves [that value is f(final point) − f(initial point)]

(B) On every loop, $\oint F \cdot T \, ds$ is 0.

Because if the loop starts and ends say at point P then the line integral is f(P) − f(P) = 0

terminology

Let $F = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k}$.
Suppose further that $F = \nabla f$.
Mathematicians call f a *potential* (or a primitive) for F (physicists call −f the potential).
The differential p dx + q dy + r dz is called *exact*.
If a particle moves in force field F, then physicists say that −f(x,y,z) is the *potential energy* of the particle at point (x,y,z). The work done by F to the particle depends only on where the particle starts and ends and not on the actual path.
F is called *conservative* because it turns out that the total energy (kinetic plus potential) is conserved.

computing antigradients

Let

(3) $$F(x,y) = (2xy^3 + 3x^2)\vec{i} + (3x^2y^2 + 4y)\vec{j}$$

Here's how to find an antigrad for F if possible (most vector fields don't have antigrads). Let

(4) $$p = 2xy^3 + 3x^2, \quad q = 3x^2y^2 + 4y.$$

To find an f so that $\nabla f = F$ you need

$$\frac{\partial f}{\partial x} = p \quad \text{and} \quad \frac{\partial f}{\partial y} = q.$$

Begin by antidifferentiating p w.r.t. x to get the terms

(5) $$x^2y^3 + x^3$$

The derivative w.r.t. y of this tentative antigradient is $3x^2y^2$. Compare this with q in (4). Since it's missing the term 4y, fix up the tentative answer in (5) by adding $2y^2$ to get

(6) $$x^2y^3 + x^3 + 2y^2.$$

Now it has the correct partial w.r.t. y. Note that fixing up the answer like this does not change its partial w.r.t. x since the additional term didn't contain the variable x.

So (6) is an antigradient (a potential) for F.

example 1

Continue with the F in (3) and find $\int F\cdot T\,ds$ on the D-to-C curve in Fig 3.

solution I just found that

$$F = \nabla\underbrace{(x^2y^3 + x^3 + 2y^2)}_{f}$$

so

$$\int F\cdot T\,ds \text{ on } \textit{any} \text{ D-to-C path} = f(C) - f(D) = (x^2y^3 + x^3 + 2y^2)\Big|_{(1,0)}^{(-1,0)} = -2$$

example 2

I'll find $\int dx + (2y+3)\,dy$ on the path in Fig 4, directed from P to Q.

The vector field $\vec{i} + (2y+3)\vec{j}$ is the gradient of $x + y^2 + 3y$ so

$$\int dx + (2y+3)\,dy = (x+y^2+3y)\Big|_{(1,2)}^{(-1,-1)} = -14$$

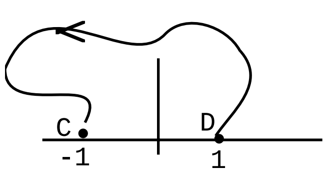


FIG 3

FIG 4

example 3

Let

$$F(x,y) = (xz^2 + z)\,\vec{i} + 3y^2\,\vec{j} + (x^2z + x + 2z)\vec{k}$$

Here's how to find an antigradient for F. Let

(7) $$p = xz^2 + z, \quad q = 3y^2\ ,\ r = x^2z + x + 2z$$

To find an f so that $\nabla f = F$ you need

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = r$$

Begin by antidifferentiating p w.r.t. x to get the terms

(8) $$\frac{1}{2}x^2z^2 + xz$$

The derivative w.r.t. y of this tentative antigradient is 0. But you want it to be $3y^2$. So fix up the tentative answer in (8) by adding y^3 to get

(9) $$\frac{1}{2}x^2z^2 + xz + y^3.$$

Now it has the correct partial w.r.t. y. Note that fixing up the answer like this does not change its partial w.r.t. x since the additional term didn't contain the variable x.

The derivative w.r.t. z of this tentative answer in (9) is $x^2z + x$. Compare this to r in (7). Since it's missing the term 2z, fix up the tentative answer by adding z^2 to get

(10) $$\frac{1}{2}x^2z^2 + xz + y^3 + z^2.$$

Now it has the correct partial w.r.t. z. Note that fixing up the answer like this does not change its partial w.r.t. x or y since the additional term didn't contain an x or a y.

So (10) is an antigradient for F, i.e.,

$$F = \nabla\left(\frac{1}{2}x^2z^2 + xz + y^3 + z^2\right).$$

some easy antigradients

Remember (Section 2.5) that fields of the form r-stuff $\vec{e}_r$ and ρ-stuff $\vec{e}_\rho$ have easy antigradients. For example,

$$3r^2\,\vec{e}_r = \nabla r^3.$$

So if $F = 3r^2\,\vec{e}_r$ then F has antigradient r^3, any $\oint F\cdot T\,ds$ is 0 and $\int F\cdot T\,ds$ is independent of path; for instance,

$$\int F\cdot T\,ds \text{ on the P-to-Q-curve in Fig 4} = r^3\Big|_P^Q = r^3\Big|_{r=\sqrt{5}}^{r=\sqrt{2}} = 2\sqrt{2} - 5\sqrt{5}$$

the zero-curl rule

Let F be a vector field.

These four statements about F (called the zero curl list) go together. Either they all hold or none of them holds; i.e., if any one of them holds, all the others hold and if any one of them fails then all of the others fail.

(But there are exceptions coming up in the next section.)

When the four statements hold, F is called *irrotational* or *conservative*.

(1) $\oint$ F·T ds = 0 on every loop.

(2) $\int$ F·T ds is independent of path meaning that if two paths have the same initial and final points then $\int$ F·T ds has the same value on the paths.

(3) F is a gradient; i.e., F has an antigradient
In this case you can use f(B) − f(A) to compute $\int$ F·T ds on a path from A to B.

(4) Curl F = $\vec{0}$.

Remember that in 2-space, if F = p(x,y)$\vec{i}$ + q(x,y)$\vec{j}$ then
curl F = $(0,0,\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y})$ so in 2-space, (4) amounts to $\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$.

Here's a way to use this rule to test an F to see if it has an antigradient:
Find curl F.

If curl F = $\vec{0}$ then F has an antigradient.

If curl F ≠ $\vec{0}$ then F does not have an antigradient.

Here's the version in the 2-dim case where F = p(x,y)$\vec{i}$ + q(x,y)$\vec{j}$.
Find $\partial q/\partial x$ and $\partial p/\partial y$.

If $\partial q/\partial x = \partial p/\partial y$ then F has an antigradient.
If $\partial q/\partial x \neq \partial p/\partial y$ then F doesn't have an antigradient.

To prove the zero-curl rule I'll show that (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4) and (4) ⇒ (1). This is enough to show that every statement on the list implies every other statement on the list.

warning

Statement (4) on the list says curl F = $\vec{0}$, not curl F = 0. Curl is a *vector*.

why the zero-curl rule works: (1) ⇒ (2)

Assume every $\oint$ F·T ds is 0. Let P and Q be any two points and draw any two paths from Q to P (Fig 5). I want to show that $\oint$ F·T ds is the same on the two paths.

$\int$F·T ds on path QAP + $\int$F·T ds on path PBQ

= $\oint$ F·T ds on the loop QAPBQ
= 0 by hypothesis.

So $\int$ F·T ds on path QAP = $-\int$ F·T ds on path PBQ
= $\int$ F·T ds on path QBP QED

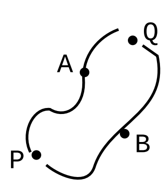


FIG 5

why the zero-curl rule works: (2) ⇒ (3) (harder)

Assume that $\int F\cdot T\,ds$ is independent of path.

I'll show that F is a gradient by defining a function w(x,y,z) which will turn out to be an antigradient for F.

Let Q be a fixed point.

A particle starts at point Q. Let

w(x,y,z) = work done by F to the particle when it moves to point (x,y,z).

The function w makes sense because the work (i.e., the line integral $\int_Q^{(x,y,z)} F\cdot T\,ds$) depends only on point (x,y,z) and not on the particular path *from* Q *to* (x,y,z).

Every point has a w value.

Think of the particle as wearing a w—meter which registers cumulative work done to it by F.

For example, if w = 7 at point A then F does 7 units of work as the particle moves from Q to A. If w = 9 at point B then as the particle continues from A to B, F does an additional 2 units of work.

To see why $F = \nabla w$ look at the particle at a point and think about which way it should go so that w increases most rapidly. Since w is the (cumulative) work done by F,

(11) direction of steepest increase of w is the F direction

because that's the direction in which the particle gets the biggest boost from F.

Remember that work is force × distance. So when the particle does move in the F direction, F does $\|F\|$ work-units per meter on it. In other words:

(12) max rate of change of w is $\|F\|$ work-units per meter

But from Section 1.2:

(13) Direction of steepest increase of w is the ∇w direction.

(14) Max rate of change of w is $\|\nabla w\|$ work-units per meter.

Compare (11) and (13), (12) and (14) to see that ∇w and F point the same way and have the same length. So $F = \nabla w$.

why the zero-curl rule works: (3) ⇒ (4)

This follows from the identity curl $\nabla f = \vec{0}$ in Section 1.7.

adjacent loop law (cute trick needed for the next proof and later stuff)

Let F be a vector field.

Remember that the circulation of F on a curve is "the tendency of F to be flowing along a curve". It's $F\cdot T\,ds$ on a small almost-straight curve and $\int F\cdot T\,ds$ on a not-small curve.

Fig 6(a) shows two loops, I and II, that share boundary AB. I drew then slightly apart just so that you could see them better. Then

ccl circulation on I + ccl circulation on II = ccl circulation on III in Fig 6(b) because the circulations along AB cancel out in the sum.

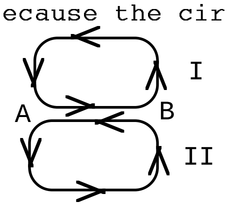


FIG 6(a)

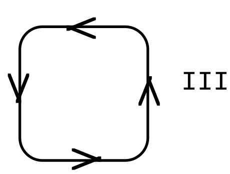
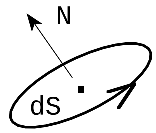


FIG 6(b)

repeat of (4) in Section 1.6 (needed for the next argument and later stuff)

(15)

Circulation of F on this small loop = curl F·N dS



(Amended version coming in the next section.)

why the zero-curl rule works: (4) ⇒ (1)

Suppose curl F = $\vec{0}$. Pick any loop. I want to show that $\oint$ F·T ds is 0 on the loop.

For the 2-dim case, fill the inside of the loop with little (point-size) loops (Fig 7). For the 3-dim case, pick a surface bounded by (i.e., "inside") the loop and fill the surface with little (point-size) curves (Fig 8). Then

sum of small circs in Figs 7 or 8

= circulation on big outer loop by the adjacent loop law

= $\oint$ F·T ds on outer loop   (directions indicated in the diagrams)

(*) Each small circ is curl F·N dS by (15).

But curl F = $\vec{0}$ so each small circ is 0 and the sum of the small circs is 0.
So $\oint$ F·T ds on big loop = 0 QED.

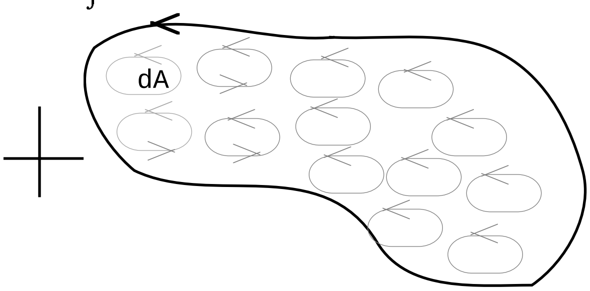


FIG 7

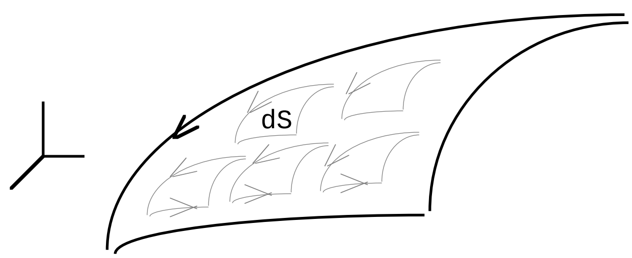


FIG 8

example 4 (an F that isn't a gradient)

Let

$$F(x,y) = \underbrace{x^2 y}_{p}\ \vec{i} + \underbrace{x^3 y^2}_{q}\ \vec{j}$$

Then $\partial q/\partial x = 3x^2y^2$, $\partial p/\partial y = x^2$.
The partials aren't equal so F doesn't have an antigradient.
Here's what happens if you skip the test and try to find an antigrad.

You need f so that

$$\frac{\partial f}{\partial x} = p \quad \text{and} \quad \frac{\partial f}{\partial y} = q.$$

Antidifferentiate p w.r.t. x to get the temporary answer $\frac{1}{3}x^3 y$. The derivative w.r.t. y of this temporary answer is $\frac{1}{3}x^3$. You want this to be q and there's no hope. So F doesn't have an antigradient.

example 5

Find $\int (3y + e^x \sin x)\, dx + (3x + 7 \sin^4 y)\, dy$ on the ellipse $5x^2 + 6y^2 = 7$ directed clockwise.

solution

Let $p = 3y + e^x \sin x$, $q = 3x + 7 \sin^4 y$. Notice that $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ [= 3].

The ellipse is a *loop* so the line integral is 0 by the zero-curl rule.

mathematical catechism (you should know the answers to these questions)

Question Given a vector field F.
What does it mean to say that $\int F\cdot T\, ds$ is independent of path.

Answer 1 It means that $\int F\cdot T\, ds$ on a (directed) curve depends on the initial and final points of the curve but not on the curve itself.

Answer 2 It means that for any two points A and B, $\int F\cdot T\, ds$ is the same on any curve from A to B.

Question What does it mean to say that $\int F\cdot T\, ds$ is *not* independent of path.

Answer It means that there exist two paths with the same initial and final points on which $\int F\cdot T\, ds$ has two different values.

PROBLEMS FOR SECTION 4.1

1. Find $\int \frac{3x^2}{y}\, dx - \frac{x^3}{y^2}\, dy$ along the curve $y = x^2\sqrt{x+1}$ from point (3,18) to point $(2, 4\sqrt{3})$.

2. If $F = \nabla(xy \sin x)$, find $\int F\cdot T\, ds$ on the line segment from (1,2) to (2,3).

3. Let A = (0,1,2) and B = (4,1,3).
Find $\int yz\, dx + xz\, dy + xy\, dz$ on the straight line A-to-B path. Do it twice
(a) directly (b) with integral theory

4. Let A = (0,1,2), B = (4,1,3), P = (6,7,8).
(a) Let $F = \frac{1}{r}\, e_r$. Find the work done by F to a particle which moves from A to B.
(b) Let $F = \frac{1}{\rho^2}\, e_\rho$. Find the work done by F to a particle which moves from A to P to B.

5. Test to see if F is a gradient. If so, of what.

(a) $F(x,y) = (2xy^3 + 3x^2)\vec{i} + (3x^2y^2 + 4y)\vec{j}$

(b) $F(x,y) = x^2 y\, \vec{i} + x^3 y^2\, \vec{j}$

(c) $F(x,y,z) = 2xy \sin z\, \vec{i} + x^2 \sin z\, \vec{j} + (x^2 y \cos z + 7)\vec{k}$

(d) $F(x,y,z) = xyz\, \vec{i} + (2x + y)\vec{j} + 3x^2\, \vec{k}$

6. Given that curl $\vec{F} = \vec{0}$ and $\int F \cdot T\, ds$ on path A B C D is 5 (no diagram necessary).

Find $\int F \cdot T\, ds$ on
(a) path D C B A
(b) path A X Y Z D
(c) path A B C D A

7. Find q if possible so that the line integral is independent of path.

(a) $\int y\, dx + q\, dy + 3y^2\, dz$

(b) $\int yz\, dx + q\, dy + 3y^2\, dz$

8. Do the following vector fields have antigradients?

(a) $\dfrac{x}{(x^2 + y^2)^2}\vec{i} + \dfrac{y}{(x^2 + y^2)^2}\vec{j}$

(b) $f(x)\vec{i} + g(y)\vec{j}$

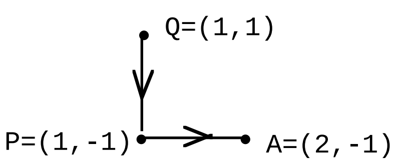
(c) $x e^{\sqrt{x^2+y^2}}\, \vec{i} + y e^{\sqrt{x^2+y^2}}\, \vec{j}$

9. What if anything can you conclude about the following line integrals on an anonymous loop.

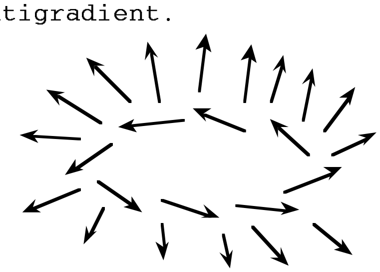
(a) $\oint 2xy\, dx + x^2\, dy$

(b) $\oint xy\, dx + dy$

10. Use some integral theory to find $\int 3x^2y^2\, dx + (2x^3y + 3)\, dy$ on the path QPA in the diagram.



11. The diagram shows a vector field F. Explain why it looks like F doesn't have an antigradient.



12. Given vector field F.
What if anything can you conclude about curl F if

(a) $\int$ F·T ds on A B C Q A = 0

(b) $\int$ F·T ds on A B C Q A = 7

(c) $\int$ F·T ds on A B C Z = 7 and $\int$ F·T ds on A Q P R Z = 9

(d) $\int$ F·T ds on A B C Z = 7 and $\int$ F·T ds on A Q P R Z = 7

13. You know that sin $\pi/2$ = 1.
But there are *other* numbers besides $\pi/2$ whose sine is 1.

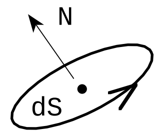
Similarly, you know that the curl of a gradient is $\vec{0}$ (I got this in Section 1.7).
Are there *other* scalar fields besides gradients that also have zero curl.

SECTION 4.2 EXCEPTIONS TO THE ZERO CURL RULE

amended version of (4) from Section 1.6

(1)

Circulation of F on this small loop = curl F·N dS



But not if F blows up at the point!

If F blows up at the point then neither F nor curl F is defined at the point so there is no curl F·N dS

retracting part of the zero curl rule in TWO-space when F blows up at a point

Let F = $p\vec{i} + q\vec{j}$. Suppose $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ (i.e., curl $\vec{F} = \vec{0}$).

The zero-curl rule in the last section said that in this case every $\oint$ F·T ds is 0.

Here's a repeat of the proof from the previous section.

Suppose curl F = $\vec{0}$. Pick any loop in 2-space.
Fill the inside of the curve with little loops (Fig 1). Then
sum of small circs = ccl circ on big loop
= $\oint$ F·T ds on big loop ccl

(A) Each small circ is curl F·N dS.

But curl F = $\vec{0}$ so each small circ is 0 and the sum of the small circs is 0

So $\oint$ F·T ds on big curve ccl (and clockwise) is 0. QED

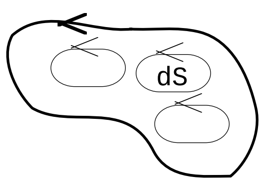


FIG 1

Now suppose further that F blows up at a point.

If a loop does not enclose the blowup point (Fig 2) then the argument in the box still works and it's *still* true that $\oint$ F·T ds = 0.

If the blowup is *inside* the loop (Fig 3) we have no conclusion about $\oint$ F·T ds.

In this case, step (A) in the argument no longer holds: The circulation on the small loop going around the blowup (Fig 3A) is *not* curl F·N dS (there isn't even a curl F value at the blowup point.) The argument breaks down here and does not lead to any conclusion about $\oint$ F·T ds on the big loop.

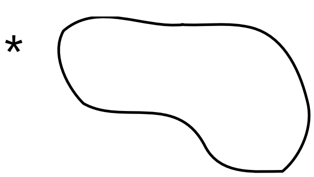


FIG 2 circ = 0

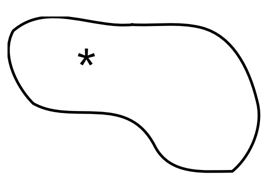


FIG 3 circ = ?

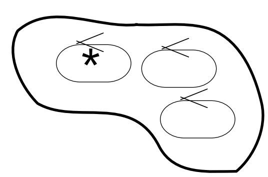


FIG 3A

The next rule says that while you can't conclude that $\oint \mathbf{F}\cdot\mathbf{T}\,ds = 0$ on loops around the blowup, at least you will get the same value for $\oint \mathbf{F}\cdot\mathbf{T}\,ds$ no matter what loop you use going around the blowup.

deformation of path principle in TWO-space for the line integral $\oint \mathbf{F}\cdot\mathbf{T}\,ds$

> Let $\mathbf{F}(x,y) = p(x,y)\vec{i} + q(x,y)\vec{j}$.
>
> Suppose $\dfrac{\partial q}{\partial x} = \dfrac{\partial p}{\partial y}$ but F blows up at a point.
>
> Then $\oint \mathbf{F}\cdot\mathbf{T}\,ds$ has the same value on every ccl loop around the blowup (and the opposite value on every *clock*wise loop). In Fig 4,
>
> $$\oint \mathbf{F}\cdot\mathbf{T}\,ds \text{ on outer ccl loop} = \oint \mathbf{F}\cdot\mathbf{T}\,ds \text{ on inner ccl loop}$$

You can refer unambiguously to the line integral ccl around the blowup since there is only one possible value.

Here's why.

Look at the two loops in Fig 4, an oval and a rectangle, going around the blowup.
I want to show that $\int \mathbf{F}\cdot\mathbf{T}\,ds$ is the same on the oval and the rectangle as long as they are directed the same way.
Fill the space *between* the oval and rectangle with little loops, directed say ccl (Fig 5). Then

sum of the small circs = *ccl* circ on rectangle + *clockwise* circ on oval

Each small circ is $\operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS$ (actually N=k here). This holds *despite* the blowup since all the little loops in Fig 5 are *between* the oval and the rectangle where F does *not* blow up. And $\operatorname{curl}\mathbf{F} = \vec{0}$ so each small circ is 0. So

ccl circ on rectangle + clockwise circ on oval = 0

ccl circ on rectangle = ccl circ on oval

$$\oint \mathbf{F}\cdot\mathbf{T}\,ds \text{ on rectangle ccl} = \oint \mathbf{F}\cdot\mathbf{T}\,ds \text{ on oval ccl} \quad \text{QED}$$

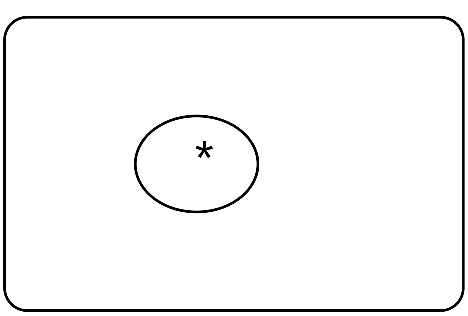


FIG 4

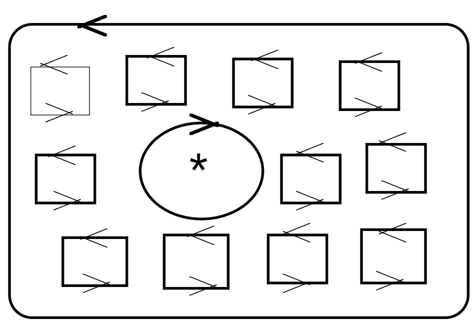


FIG 5

example 1 (a counterexample)

Let

$$F = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j} = \frac{1}{r}\,e_\theta$$

Then curl $F = \vec{0}$ (except at the origin where it isn't defined).

Here's one way to find curl F:

$$\frac{\partial q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \text{ (quotient rule) } = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial p}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

So $\partial q/\partial x = \partial p/\partial y$, curl $F = \vec{0}$

Another way is to use the rule for curl in cylindrical coords:

$$\text{curl } \vec{F} = \begin{vmatrix} \frac{1}{r}\,e_r & e_\theta & \frac{1}{r}\,k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \frac{1}{r}\,r & 0 \end{vmatrix} = \vec{0}$$

But F blows up at the origin because $1/r \to \infty$ as $r \to 0+$.

So $\oint F\cdot T\, ds$ is 0 on any loop *not* going around the origin.

And so far we have no conclusion about $\oint F\cdot T\, ds$ on loops going around the origin.

I'll get a conclusion for this specific F by testing $\oint F\cdot T\, ds$ on one such loop. By the deformation principle, I'll get the same answer no matter what loop I use.

I'll use the circle $x^2 + y^2 = 1$ because on that circle F simplifies to $-yi + xj$ and the computing will be easier.

The circle has parametric equations

$$x = \cos t$$
$$y = \sin t$$

Then

$$\oint F\cdot T\, ds \text{ on the circle directed say ccl} = \int -y\, dx + x\, dy \quad \text{ccl}$$

$$= \int -\sin t\cdot -\sin t\, dt + \cos t\cdot \cos t\, dt$$

$$= \int_{t=0}^{2\pi} \sin^2 t\, dt + \cos^2 t\, dt$$

$$= \int_{t=0}^{2\pi} dt$$

$$= 2\pi$$

By the deformation principle, $\oint F\cdot T\, ds$ is 2π on *every* ccl loop around the origin and is -2π on every clockwise loop around the origin.

footnote Here's another way to do the testing. I can find $\int F \cdot T\, ds$ on a loop around the origin by first finding an antigradient for F. The formula for *taking* the gradient in polar coordinates is

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta$$

With a little trial and error you can go backwards to see that

$$F = \nabla \theta$$

But the antigradient θ is tricky because each point has many θ values. It's still true that

$$\oint F \cdot T\, ds = \text{antigrad(final point)} - \text{antigrad(initial point)}$$
$$= \text{final } \theta - \text{initial } \theta$$

but you must interpret initial and final θ's appropriately. No matter where you start on the loop in Fig 6, as you move around ccl back to the start again θ has increased so that it is considered to be 2π larger when you return to the start (e.g., if you started at $\theta = \pi/7$ we would say that $\theta = 15\pi/7$ when you return). So

$$\oint F \cdot T\, ds \text{ on the loop in Fig 6} = \theta \Big|_{\text{start}}^{\text{end}} = \text{change in } \theta = 2\pi$$

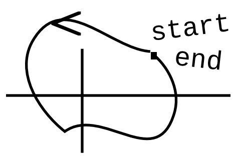


FIG 6

Because of this counterexample we know that the following statement is *false*:

FALSE For *all* 2-dim F, if curl $F = \vec{0}$ then $\oint F \cdot T\, ds = 0$ on all loops FALSE

If F is a 2-dimensional vector field and curl $\vec{F} = \vec{0}$ but F blows up at a point then $\oint F \cdot T\, ds$ on a loop around the blowup *might not be* 0.

Whether it is or not depends on the particular F and the only way to tell is to do a computation. (It might still come out to be 0 anyway -- next example) By the deformation principle, you get to choose which loop around the blowup to test.

example 2 (where the blowup does NOT make a difference)

Let $F = \frac{1}{r^3} e_r$ (polar coords)

F blows up at the origin because $1/r^3 \to \infty$ as $r \to 0+$.

Curl $F = \vec{0}$ (except at the origin where it is not defined)

One way to find curl F is to use the rule for curl in cylindrical coords from §2.5:

$$\text{curl } \vec{F} = \begin{vmatrix} \frac{1}{r}\, e_r & e_\theta & \frac{1}{r}\, k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 1/r^3 & 0 & 0 \end{vmatrix} = 0\,\frac{1}{r}\, e_r + 0\, e_\theta + \frac{1}{r}\, k = \vec{0}$$

A faster way is to see that F is a gradient since it's of the form r-stuff e_r (see "easy potentials" in Section 4.1). Then curl F = $\vec{0}$ because curl of any gradient is $\vec{0}$ (part of the zero curl rule).

So $\oint$ F·T ds is 0 on any loop *not* going around the origin.

And $\oint$ F·T ds has the same value on every ccl loop around the origin.

But we don't know yet what that value is. To find out, I'll compute the line integral on a convenient loop around the origin. I can choose whatever around-the-origin-loop I like since $\oint$ F·T ds is the same on all of them by the deformation principle.

I'll line integrate on the circle $x^2 + y^2 = 1$, i.e., on the circle r = 1.

question Should you always choose a *circle*.

answer No. You should choose a loop that makes the algebra come out easy. See problem #1

method 1 The circle has parametric equations

$x = \cos t$

$y = \sin t$.

On the unit circle, r = 1, $F = e_r = (x\vec{i} + y\vec{j})_{\text{unit}}$. But on the unit circle $x\vec{i} + y\vec{j}$ is already a unit vector so $F = x\vec{i} + y\vec{j}$ and

$$\oint_{\text{ccl}} F\cdot T\, ds = \oint_{\text{ccl}} x\, dx + y\, dy$$

$$= \int_{t=0}^{2\pi} \cos t \cdot -\sin t\, dt + \sin t \cdot \cos t\, dt$$

$$= \int_{t=0}^{2\pi} 0\, dt$$

$$= 0$$

method 2 On the unit circle around the origin, T is perp to e_r so F·T = 0 and $\oint$ F·T ds = 0. QED

footnote

Here's another way to do the testing. I can find $\oint$ F·T ds on a loop around the origin by first finding an antigradient for F. F is of the form r−stuff e_r so it has an easy antigradient (preceding section):

$$F = \nabla\left(\frac{-1}{2r^2}\right)$$

No matter where you start on a loop around the origin, (Fig 6) you end up back at the same point with the *same* r:

$$\oint F\cdot T\, ds = \frac{-1}{2r^2} = \Big|_{r \text{ at start}}^{r \text{ at end}} = 0$$

So despite the blowup, $\oint$ F·T ds is 0 on all loops, whether they go around the blowup or not.

retracting another part of the zero curl rule in TWO-space when F blows up at a point

The zero-curl rule in the last section said that if curl F = $\vec{0}$ then $\int$ F·T ds is independent of path. The argument used the fact that if curl F is zero then every $\oint$ F·T ds is 0. Here's a repeat of the argument from the preceding section that $\oint$ F·T ds = 0 $\Rightarrow$ ind of path.

In Fig 7,

A–to–B on I + B–to-A on II = clockwise loop

So

A-to-B on I = A-to-B on II + clockwise loop

If the line integral on the clockwise loop is 0 then the two A-to-B's are equal

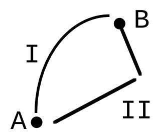


FIG 7

Now suppose that F = $p\vec{i} + q\vec{j}$ and $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ (i.e., curl $\vec{F} = \vec{0}$).
And suppose further that F blows up at a point.

If two A-to-B paths do *not* trap the blowup between them (Fig 8), it's *still* true that $\int$ F·T ds is the same on the two paths (because $\oint$ F·T ds = 0 on the loop they form so the argument in the box above still works.)
Suppose two A-to-B paths *do* trap the blowup between them (Fig 9).

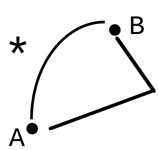


FIG 8

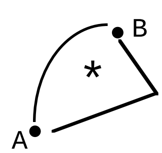


FIG 9

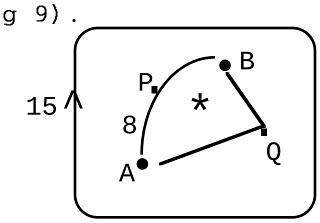


FIG 10

For a particular F, $\oint$ F·T ds around the blowup might turn out to be 0 anyway (to find out, test it on a convenient loop). In that case, it's still true that $\int$ F·T ds is the same on the two A-to-B paths in Fig 9.

For a particular F, $\oint$ F·T ds around the blowup point may turn out to be non-zero. In that case, $\int$ F·T ds will *not* be the same on the two A-to-B paths in Fig 9.
For example, suppose $\oint$ F·T ds on a clockwise loop around the blowup turns out to be 15 (Fig 10) and suppose $\int$ F·T ds = 8 on the APB path.

abuse of notation Instead of writing $\int$ F·T ds = 8 on the APB path, I'll shorten it to APB = 8.

Here's how to find AQB.

APB + BQA = clockwise loop around blowup = 15

So

BQA = 15 − APB = 15 − 8 = 7

AQB = −7 (*different* from APB)

(Remember that the hypothesis in all this, aside from the blowup, is curl $F = \vec{0}$.)

example 3

Look at Fig 11. Suppose

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

p and/or q blow up at point Z

$\oint$ p dx + q dy = 7 on a ccl loop around Z

$\int$ p dx + q dy = 6 on segment A to B.

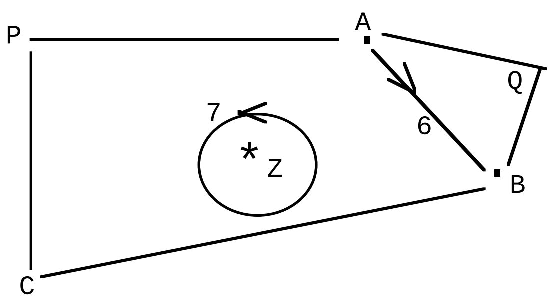


FIG 11

Find

(a) AQB

(b) APCB

solution (a) AQB = 6, same as the straight line AB path since the two paths don't enclose the blowup point between them.

(b) APCB + BA = one trip ccl around the blowup = 7

so

APCB = 7 − BA = 7 + AB = 7 + 6 = 13

example 4

Look at Fig 12.

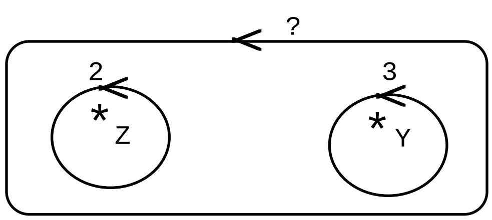


FIG 12

Suppose

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

p and/or q blow up at points Y and Z

$\oint$ p dx + q dy = 2 on a ccl loop around Z

$\oint$ p dx + q dy = 3 on a ccl loop around Y

Find

$\oint$p dx + q dy on the ccl rectangular loop that goes around both Z and Y.

solution 1

Fill the in—between region with small loops directed as in Fig 13.

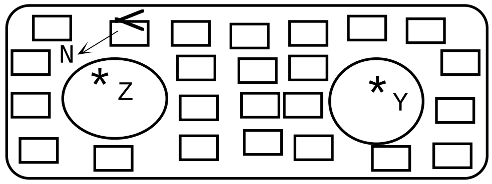


FIG 13

On the one hand,

sum of the small circs = rectangle circ ccl + Z-loop clockwise + Y-loop clockwise

On the other hand, each small circ is curl F·N dS (this is safe to use since F does not blow up inside any small loop) which is 0 since curl F = $\vec{0}$

So

rectangle ccl + Z-loop clockwise + Y-loop clockwise = 0

rectangle ccl = Z-loop ccl + Y-loop ccl = 2 + 3 = 5   QED

solution 2

Add a two-way cut (the dotted lines in Fig 14) to the outer rectangle. (The two dotted segments are supposed to be one on top of the other –– I drew a little space between them just for clarity). This doesn't change the line integral (i.e., rectangle-plus-cuts = rectangle) but it conveniently splits the large rectangle into the two smaller ones, I and II.

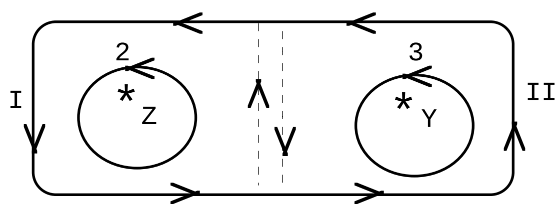


FIG 14

Then

ccl circ on big rectangle in Fig 12
= ccl circ on I + ccl circ on II in Fig 14
= 2 + 3   (deformation principle)
= 5 QED

warning

You can't play the deformation game unless curl F is zero to begin with. And there is no reason to play it in 2-space unless F blows up at a point.

why the zero curl rule still works in THREE-space if F blows up at a POINT

Suppose curl F = $\vec{0}$.

The zero-curl rule in the last section said that in this case every $\oint$ F·T ds is 0.

This still holds in 3-space even if F blows up at a point (i.e., a point blowup is trouble in 2-space but not in 3-space). (Very interesting!) Here's why.

Suppose curl F = $\vec{0}$ but F blows up at a point in 3-space.
Look at the loop in Fig 15 that seems to go "around" the blowup.

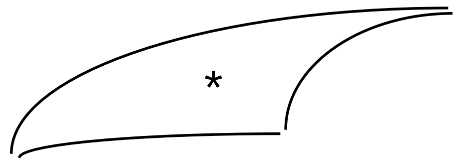


FIG 15

There are *many* surfaces with that loop as its boundary. (That's the key difference between 2-space and 3-space.) Fig 16 shows five of them -- think of the loop as a hat brim and the surfaces in Fig 16 as some of many hats with that brim.

The first two hats in Fig 16 go through the blowup point, the last three don't.

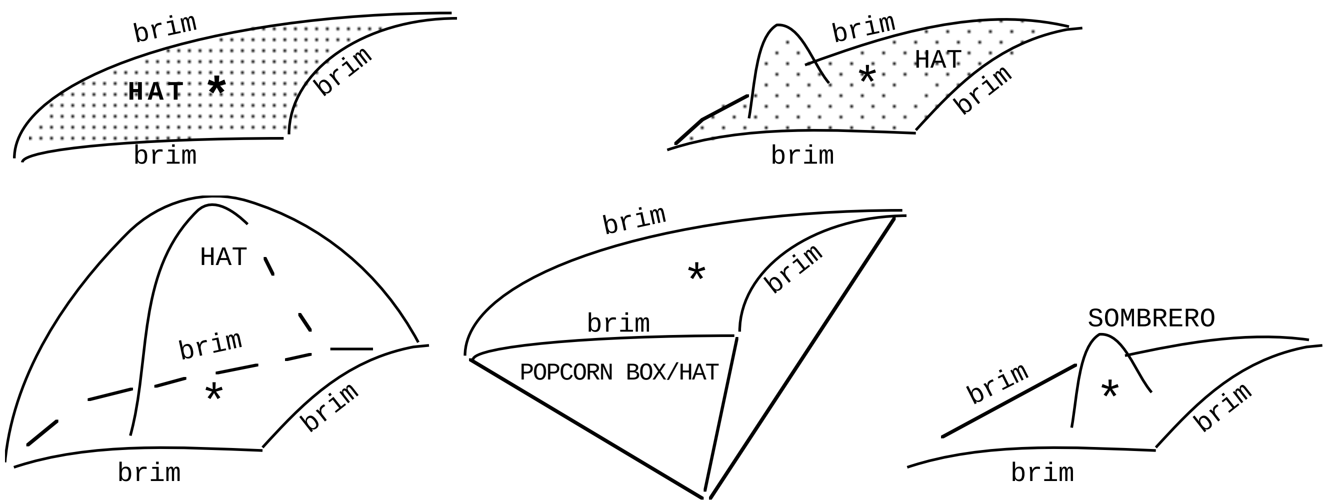


FIG 16

Pick any hat that does *not* go through the blowup point. On *that* hat draw little loops (Fig 17). (In Fig 17, I only drew a few loops, on the front of the hat. They're meant to go all over the hat.) The blowup point is *not* on the hat. It's inside the hat, where your head would be.

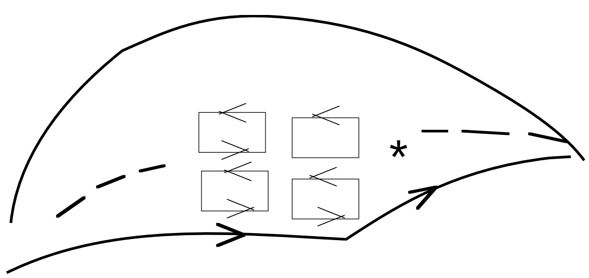


FIG 17

Then continue with the original argument.

sum of small circs in Fig 17
= circulation on big loop (the brim)
= $\oint$ F·T ds on big loop directed as in Fig 17

(A) Each small circ is curl F·N dS.

But curl F = $\vec{0}$ so each small circ is 0.

So $\oint$ F·T ds on big loop is 0. QED

The argument works this time. Step (A) is OK because the loops are drawn on the hat surface and the blowup point is *not* on the hat surface. QED

Subtle point If I try to show that $\oint$ F·T ds is 0 on the loop in Fig 15 using the hat in Fig 18 below, which *does* go through the blowup point, the argument doesn't work.

One of the little loops in Fig 18 encloses the blowup point, and step (A) in the box fails. It isn't that I get *different* conclusions about $\oint$ F·T ds using the surface in Fig 18 vs. Fig 17. It's that using the surface in Fig 18 doesn't lead to any conclusion at all about $\oint$F·T ds on the loop/brim. It's a dead end. But that doesn't undermine or contradict the conclusion (that the line integral is 0) I got with the hat in Fig 17. Some of you may think this is sneaky but it really isn't.

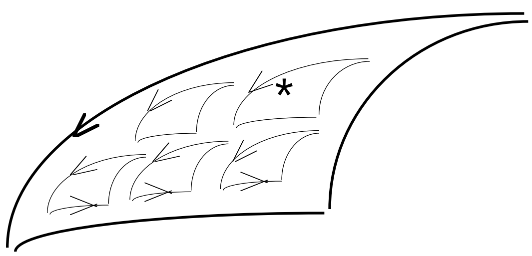


FIG 18

retracting part of the zero curl rule in THREE-space when F blows up on a LINE

Suppose curl $\vec{F} = \vec{0}$.

The zero-curl rule in the last section said that in this case every $\oint$ F·T ds is 0.

As just shown, the rule still holds in 3-space even if F blows up at a point.
But suppose *F blows up along an entire line* in 3-space.

footnote

I really mean that curl F is $\vec{0}$ except on the blowup line where F and curl F are not defined.

If a loop does *not* go around the blowup line (Fig 19), it's still true that $\oint$ F·T ds = 0. (Pick the flattish hat with the loop as the brim, draw small loops on it and use the argument in the box above.)

But if a loop goes around the blowup line (Fig 20) then I have no conclusions about $\oint$ F·T ds on the loop.

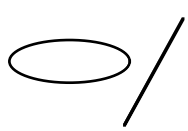


FIG 19

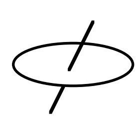


FIG 20

Here's why.
For the loop going around the blowup *line* in Fig 20 any hat I draw with the loop as its brim will be pierced by the blowup line (Fig 21). The original argument in the box above breaks down at line (A) because one of the small loops on the hat (Fig 22) will go around the blowup point and the circ on that loop will not be curl F·N dS.

So the original argument does not lead to any conclusion about $\oint$ F·T ds on the loop in Fig 20.

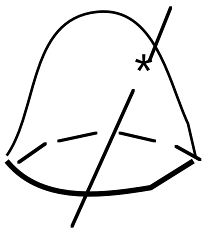


FIG 21

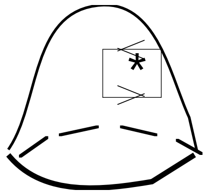


FIG 22

The next rule says that at least you will get the same value for $\oint$F·T ds no matter what loop you use going around the blowup line.

deformation principle in THREE-space for the line integral $\oint$ F·T ds

Suppose curl F = $\vec{0}$ but F blows up on a *line*. Then $\oint$ F·T ds has the same value on all loops with a common direction around the blowup line. In Fig 23, $\oint$ F·T ds is the same on each loop if you direct them similarly.

So you can refer unambiguously to the line integral around the blowup line (once you've picked a direction for the loop)

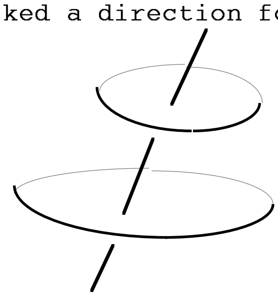


FIG 23

Here's why.
Look at two loops going around the blowup line (Fig 23).
Make a lampshade with those two loops as its rims (Fig 24) and fill the lampshade with little loops. View directions say from above.

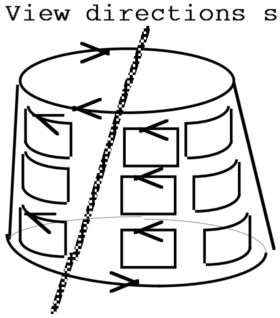


FIG 24

Sum of all the small circs directed as in Fig 24
= *clockwise* circ on top rim + *ccl* circ on lower rim.

(B) Each small circ is curl F·N dS.

But curl F = $\vec{0}$ so each small circ is 0. So

circ *clockwise* on top rim + circ *ccl* on lower rim = 0
circ *clockwise* on top rim = circ *clockwise* on lower rim

So in Fig 24,

$$\oint \text{F·T ds on top loop clockwise} = \oint \text{F·T ds on lower loop clockwise} \qquad \text{QED}$$

This argument holds despite the blowup line since all the little loops in Fig 24 are on the lampshade where F does *not* blow up and it's safe to say in line (B) that a small circ is curl F·N dS.

warning

There is no deformation principle for line integrating around a blowup line unless curl F = $\vec{0}$ to begin with.

example 5 (a counterexample)

Let F = $\frac{1}{r}\, e_\theta$. This is the field from example 1 but now consider it to be a 3–dim vector field. In that case it blows up on the z–axis where r = 0. From example 1, we have curl F = $\vec{0}$.

So $\oint$ F·T ds is 0 on any loop *not* going around the z-axis.

And so far we have no conclusion about $\oint$ F·T ds on loops going around the z-axis.

From example 1 we know that $\oint$ F·T ds = 2π on the unit ccl circle in the x,y plane centered at the origin. By the deformation principle, it is 2π on every similarly directed loop around the z-axis.

Because of this counterexample we know that the following statement is *false*:

FALSE For *all* 3–dim F, if curl F = $\vec{0}$ then $\oint$ F·T ds = 0 on all loops FALSE

If F is a 3–dimensional vector field and curl $\vec{F}$ = $\vec{0}$ but F blows up at a point then $\oint$ F·T ds on a loop around the blowup *might not be* 0.

Whether it is or not depends on the particular F and the only way to tell is to do a computation. (It might still come out to be 0 anyway –– next example) By the deformation principle, you get to choose which loop around the blowup to test.

example 6 (where the blowup does NOT make a difference)

Let F = $\frac{1}{r^3}\, e_r$ (polar coords)

This is the field from example 2 but now consider it to be a 3–dim vector field. In that case it blows up on the z–axis where r = 0. As found in example 2, its curl is $\vec{0}$. So $\oint$ F·T ds is 0 on any loop *not* going around the z-axis.

And from example 2 we know that $\oint$ F·T ds is 0 on the unit ccl circle in the x,y plane centered at the origin. By the deformation principle, it is 0 on every loop around the z-axis.

So $\oint$ F·T ds is 0 on all loops, whether they go around the blowup line or not.

retracting another part of the zero curl rule in THREE-space when F blows up on a LINE

Suppose curl $\vec{F} = \vec{0}$ but F blows up on a line

If two A-to-B paths do *not* encircle the blowup line (Fig 25), it's still true that $\int F \cdot T\, ds$ is the same on the two paths.

Suppose two A-to-B paths *do* encircle the blowup line (Fig 26).

For a particular F, $\oint F \cdot T\, ds$ around the blowup line might turn out to be 0 anyway (to find out, test it on a convenient loop). In that case, it's still true that $\int F \cdot T\, ds$ is the same on the two A-to-B paths in Fig 26.

For a particular F, $\oint F \cdot T\, ds$ around the blowup point may turn out to be non-zero. In that case, $\int F \cdot T\, ds$ will *be different* on the two A-to-B paths in Fig 26.

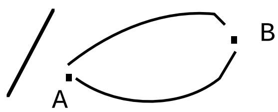


FIG 25

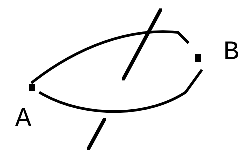


FIG 26

PROBLEMS FOR SECTION 4.2

1. Let $F(x,y) = \dfrac{-y}{x^2 + 4y^2}\, i + \dfrac{x}{x^2 + 4y^2}\, j.$

(a) Show that curl $F = \vec{0}$.

(b) Take advantage of integral theory to find $\oint F \cdot T\, ds$ on the ccl unit circle with center (5,17).

(c) Take advantage of integral theory to find $\oint F \cdot T\, ds$ on the ccl ellipse $7x^2 + 8y^2 = 9$.

2. Repeat the preceding problem but with $F(r,\theta) = \dfrac{1}{r^4}\, e_r$ (polar coordinates).

3. Suppose

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

p and q blow up at the indicated point (the asterisk) in Fig A.

$\oint p\,dx + q\,dy = 10$ on a ccl loop around the blowup point.

$\int p\,dx + q\,dy = 7$ on the A-to-P-to-B path in Fig A.

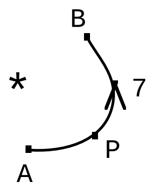


FIG A

Find $\oint p\,dx + q\,dy$ on the loops in Figs B and C and on the A-to-B paths in Figs D—I.

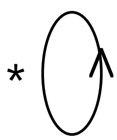


FIG B

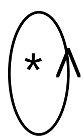
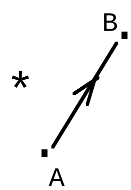


FIG C



FIB D

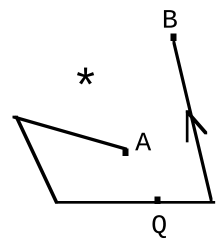


FIG E

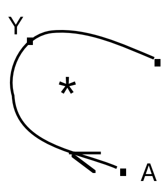


FIG F

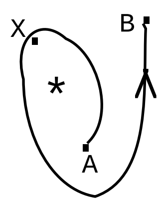


FIG G

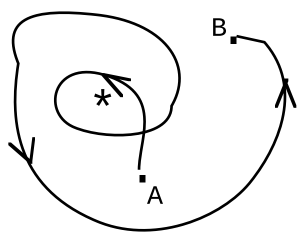


FIG H

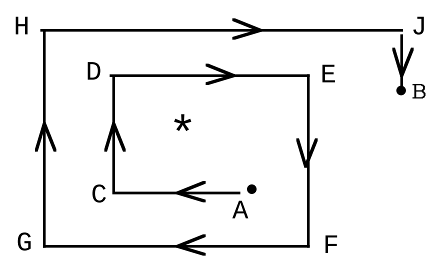


FIG I

4. Let $F = p\vec{i} + q\vec{j}$.

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

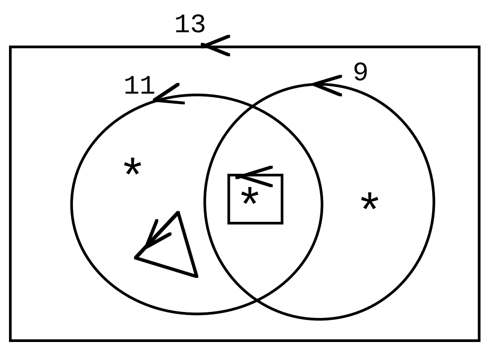
F blows up at the three points in the diagram

$\oint F\cdot T\, ds = 13$ on the big ccl rectangle

$\oint F\cdot T\, ds = 11$ on the left ccl circle

$\oint F\cdot T\, ds = 9$ on the right ccl circle

Find $\oint_{ccl} F\cdot T\, ds$ on (a) the triangle (b) the smaller rectangle



5. Here are three vector fields in 3-space.

(a) $F(\rho,\phi,\theta) = \frac{1}{\rho^2}\, e_\rho$, the electric field F in 3-space due to a unit charge at the origin.

(b) $F(r,\theta,z) = \frac{1}{r}\, e_r$ the electric field in 3-space due to a line of unit charge density along the z-axis.

(c) $F(x,y,z) = = \frac{-y}{2x^2 + 3y^2}\, i + \frac{x}{2x^2 + 3y^2}\, j$ (a totally unimportant field)

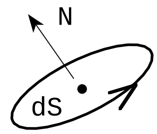
For each field, Look at all loops in 3-space and find $\oint F\cdot T\, ds$ on each one.

SECTION 4.3 GREEN'S THEOREM AND STOKES' THEOREM

repeated amended version of (4) from Section 1.6

(1)

Circulation of F on this small loop = curl F·N dS (as per §1.6)



But not if F blows up at the point!

Green's theorem for the line integral $\oint$ **F·T ds** on a LOOP in TWO-space

Suppose p and q do not blow up on or inside a loop in 2-space. Then

$$\underbrace{\oint p\,dx + q\,dy \text{ around a } \textit{counterclockwise} \text{ loop}}_{\text{line integral}} = \underbrace{\int \left[\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right] dA \text{ over the inside}}_{\text{double integral}} \text{ (Fig 1)}$$

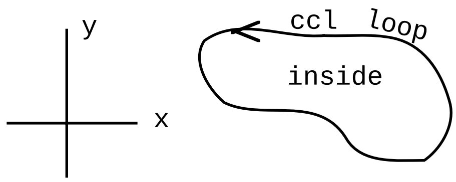


FIG 1

For $\oint$ p dx + q dy on a *clockwise* loop, put a *minus sign* in front of the double integral.
But you can't use Green's theorem if p and/or q blow up inside the closed curve.

There's a review of double integrals in Section 3.0.

why Green's theorem works

Let $F = p\,\vec{i} + q\,\vec{j}$.
Pick any loop in 2-space (i.e., in the x,y plane in 3-space).
Fill the inside of the loop with little loops (Fig 2).

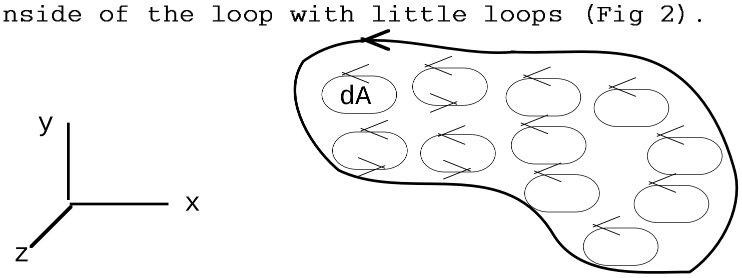


FIG 2

Sum of the small circulations directed say ccl
= ccl circ on big loop
= $\oint$ p dx + q dy on the big ccl loop in Fig 2

Each small circ is a curl F·righthanded N dA
The righthanded N is k and curl F·k = $\dfrac{\partial q}{\partial x} - \dfrac{\partial p}{\partial y}$.
So

$\oint$ p dx + q dy on the big ccl loop in Fig 2

$$= \text{sum of } \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dA\text{'s}$$

$$= \int_{\text{inside loop}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dA$$

example 1

The top part of the loop in Fig 3 is a semicircle with center at the origin and radius a. You can find $\oint_{ccl} x^2 y\, dx - xy^2\, dy$ around the loop directly by parametrizing the two pieces of the loop but you can also use Green's theorem like this:

$$\oint_{ccl} x^2 y\, dx - xy^2\, dy = \int (-y^2 - x^2)\, dA$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a} -r^2\, r\, dr\, d\theta = -\frac{\pi a^4}{4}$$

warning If you integrate in polar coords, remember that it's r dr dθ, not plain dr dθ.

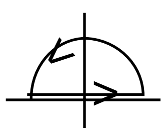


FIG 3

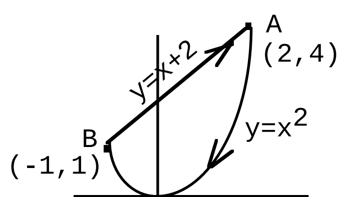


FIG 4

example 2

Find $\oint_{clock} x^2 y\, dx + (y+3)\, dy$ on the loop in Fig 4.

method 1 (directly)

The segment BA has parametric equations $x=x, y=x+2$, $-1 \le x \le 2$.

The parabola piece has parametric equations $x=x$, $y=x^2$, $-1 \le x \le 2$.

$$\oint_{clock} = \int_{\text{B to A on line}} + \int_{\text{A to B on parabola}}$$

$$= \int_{-1}^{2} x^2(x+2)\, dx + (x+2+3)\, dx + \int_{2}^{-1} x^4\, dx + (x^2+3)\, 2x\, dx$$

$$= \int_{-1}^{2} (x^3 + 2x^2 + x + 5)\, dx + \int_{2}^{-1} (x^4 + 2x^3 + 6x)\, dx = 63/20$$

method 2 (Green's theorem)

$\oint x^2 y\, dx + (y+3)\, dy$ on the CLOCKWISE loop in Fig 4

$$= \text{MINUS} \int -x^2\, dA \text{ over the pinside of the loop in Fig 4}$$

$$= \int_{x=-1}^{2} \int_{y=x^2}^{x+2} x^2\, dy\, dx$$

$$= \int_{x=-1}^{2} x^2(x+2-x^2)\,dx$$

$$= \int_{x=-1}^{2} (-x^4 + x^3 + 2x^2)\,dx = 63/20$$

warning 1 The limits on the double integral are *not* $\int_{x=-1}^{2} \int_{y=0}^{4}$. Those limits go with a rectangle, not the region in Fig 4.

warning 2 The limits are *not* $\int_{x=-1}^{2} \int_{y=0}^{x^2}$. Those limits go with the region in Fig 4A.

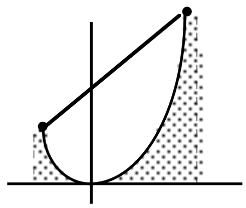


FIG 4A

oriented hat brim and hats

Start with a directed loop in 3-space (Fig 5)

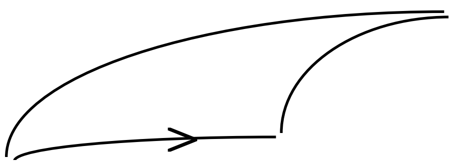


FIG 5

describing directions on loops in 3-space The loop in Fig 5 is meant to be 3-dimensional. The direction can be called ccl when viewed as if it were a 2-dim picture. Or ccl as viewed from above. Or clockwise as viewed from below. The best way to describe it is to just draw the picture.

There are many surfaces with that loop as its boundary -- think of the loop as a hat brim and the surfaces as hats (Fig 6).

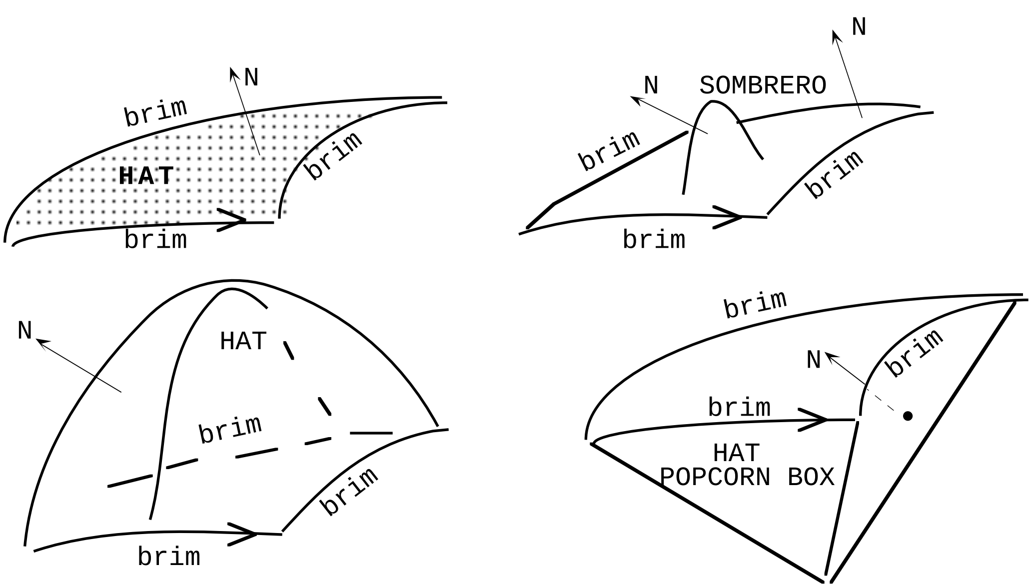


FIG 6

The brim has two T directions and each hat has two N directions. Fig 6 shows the N that is oriented righthandedly w.r.t. the indicated T direction.

You can also decide which is the righthanded N like this. Fig 7 shows a hat and brim. Draw a little loop on the hat right near the brim (Fig 8). Give it the same direction as the brim on their common boundary. If you curl the fingers of your right hand around the little loop your thumb points out of the hat. So for this hat, the N that is righthanded w.r.t. the direction on the brim is the *outer* N.

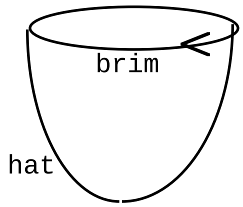


FIG 7

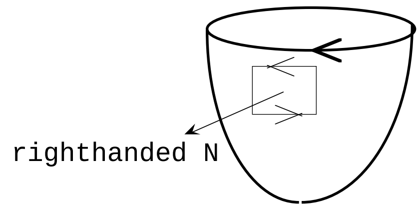


Fig 8

Stokes' theorem for the line integral $\oint$ F · T ds on a LOOP in THREE-space

Start with a loop (hat brim) in 3-space (Fig 5).
Pick any hat with that brim (Fig 6).
Let T (unit tangent to the brim) and N (unit normal to the hat) be oriented righthandedly (Fig 6)
Let F be a vector field that does not blow up on the hat.
Then

$$\underbrace{\oint F\cdot T\,ds \text{ on the brim}}_{\text{line integral}} = \underbrace{\int \text{curl } F\cdot N\,dS \text{ on the hat}}_{\text{surface integral}}$$

why Stokes' theorem works

Fill the hat with little loops directed as shown in Fig 9.

Sum of small circs in Fig 9 = brim circ

$$= \oint F\cdot T\,ds \text{ on the brim directed as in Fig 9}$$

Each small circ is a curl F·N dS (by (1)) where N has the direction in Fig 9.
Their sum is $\int_{\text{hat}} \text{curl } F\cdot N\,dS$

So

$$\oint F\cdot T\,ds \text{ on the brim} = \int_{\text{hat}} \text{curl } F\cdot N\,dS$$

where T and N have the directions in Fig 9.

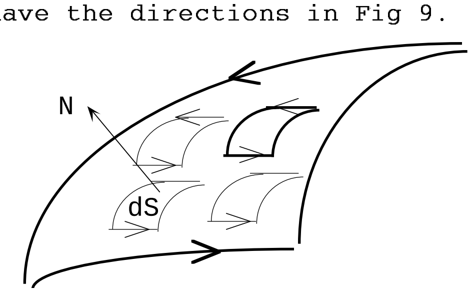


FIG 9

connection between Green and Stokes

Suppose the hat brim in 3-space lies in the x,y plane and you choose as your hat the beanie also lying in the x,y plane. If T is ccl then the righthanded N is k, curl $F\cdot N = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$ and the surface integral becomes a double integral over the inside of the curve.

In other words, Green's theorem is the 2-dim version of Stokes' theorem.

example 4

Use integral theory to find $\oint y\,dx + z\,dy + x\,dz$ around the curve of intersection of the saddle $z = xy$ and cylinder $x^2 + y^2 = 9$, with the indicated direction (Fig 10). (One of the review problems for Chapter 4 did this directly.)

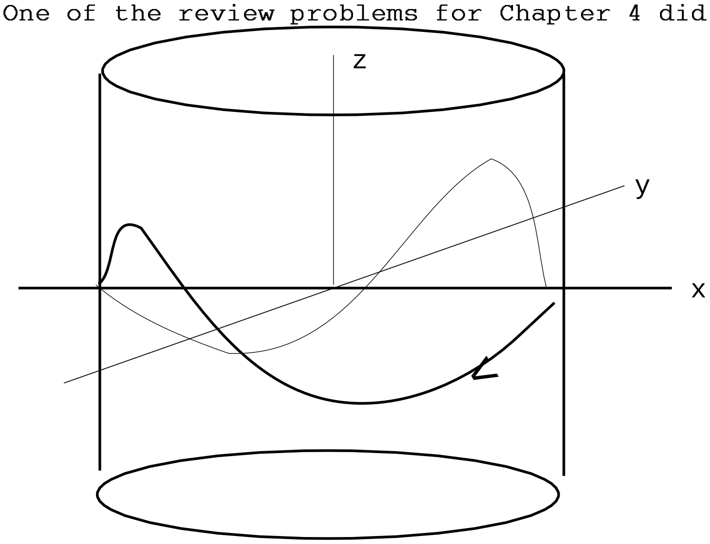


FIG 10

solution 1 Choose a hat with the curve as its brim. Fig 11 shows the wavy hat $z = xy$.

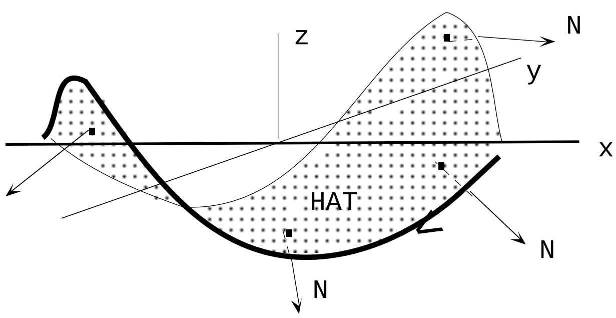


FIG 11

The N that is righthanded with respect to T is the one in Fig 11 with a negative z-component. By Stokes' theorem,

$\oint \mathbf{F}\cdot\mathbf{T}\,ds$ on the curve of intersection $= \int \text{curl}\,\mathbf{F}\cdot\mathbf{N}\,dS$ on the hat in Fig 11

warning

Do not just write $\oint \mathbf{F}\cdot\mathbf{T}\,ds = \int \text{curl}\,\mathbf{F}\cdot\mathbf{N}\,dS$

That statement is meaningless unless you say *what loop/brim* you are line integrating on and *what hat* you are surface integrating on.

Now do the surface integral.

$F = yi + zj + xk$

Curl $F = (-1, -1, -1)$.

The hat has parametric equations

$x=x$

$y=y$

$z=xy$

where the parameter world is the projection of the hat in the x,y plane, a disk with center at the origin and radius 3 (Fig 12)

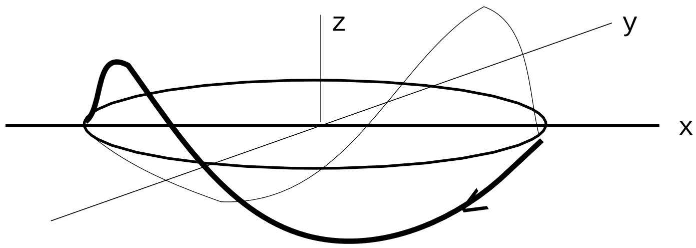


FIG 12

method 1 for getting a smart n $n = (1,0,y) \times (0,1,x) = (-y,-x,1)$.

method 2 for getting a smart n Write the surface as $z - xy = 0$. Let $g(x,y,z) = z - xy$.

Then $n = \dfrac{\nabla g}{\partial g/\partial z} = (-y,-x,1)$

The N in Fig 11 has a negative z-component so use $-n$ instead of n. Then

$$\int \text{curl } F\cdot N \, dS \text{ on hat} = \int_{x,y \text{ proj}} \text{curl } F\cdot -n \, dx\, dy$$

$$= \int_{x,y \text{ proj}} (-y-x+1)\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (\,-r \sin\theta - r\cos\theta + 1])\; r\, dr\, d\theta$$

$$= \ldots = 9\pi$$

footnote Better still, by inspection, $\int -y\, dA$ and $\int -x\, dA$ are zero and $\int 1\, dA$ = projection area $= 9\pi$

So $\oint_{\text{curve}} F\cdot T\, ds = 9\pi$

solution 2 I'll use the top hat in Fig 13 with lid in plane $z = H$ where all that matters about H is that it be large enough so that the lid is above the brim.

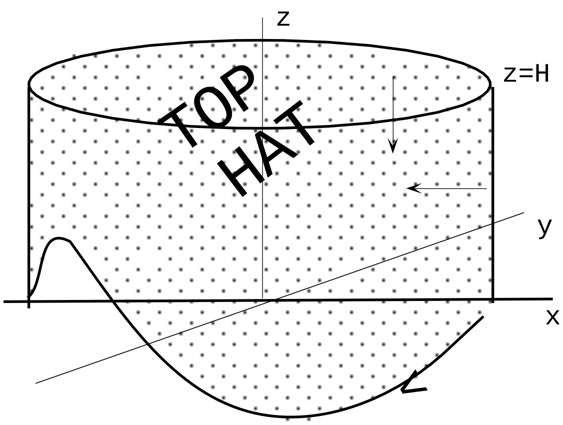


FIG 13

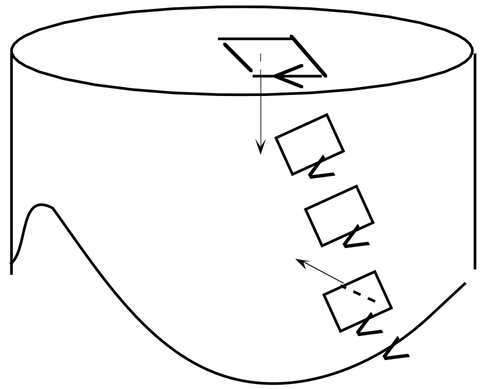


FIG 13A

The N on the top hat that is righthanded w.r.t. T is *inner*. I decided this by drawing little loops on the hat (Fig 13A) and curling my fingers around them.

By Stokes' theorem,

$\oint$ F·T ds on the curve of intersection = $\int$ curl F·inner N dS on top hat in Fig 13.

The top hat has two parts, the cylindrical part and the lid.

LID

On the lid, the righthanded N is $-$k, curl F·$-$k = 1.

$\int$ curl F·N dS on lid = $\int$ 1 dS on lid = area of lid = 9π

CYLINDER

The cylindrical surface has parametric equations

$x = 3\cos\theta$
$y = 3\sin\theta$
$z = z$
$z_{\text{on brim}} \le z \le H$ where $z_{\text{brim}} = xy = 9\cos\theta\sin\theta$

$0 \le \theta \le 2\pi$

On the cylinder,
$r = 3$
$N = -e_r = -(\cos\theta, \sin\theta, 0)$
$dS = 3\,dz\,d\theta$ (page 4 in Section 3.5)
curl F·N = $\cos\theta + \sin\theta$

$\int$ curl F·N dS on cylinder part of hat

$$= \int_{\theta=0}^{2\pi} \int_{z=9\cos\theta\sin\theta}^{H} (\cos\theta + \sin\theta)\, 3\,dz\,d\theta = \ldots = 0$$

Final answer = $9\pi + 0 = 9\pi$ (didn't matter what H was).

solution 3 I could also use an upside down top hat (Fig 14) but enough is enough.

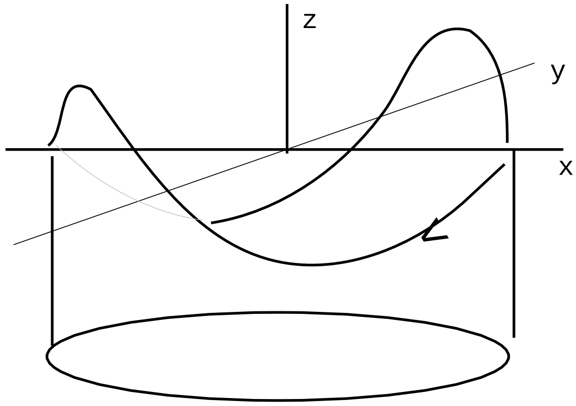


FIG 14

warning

1. Green's theorem and Stokes' theorem are just for the line integral $\oint \mathbf{F}\cdot\mathbf{T}\,ds$, i.e., the line integral $\int \mathbf{F}\cdot\mathbf{T}\,ds$ on a *loop*.

2. You have to choose the righthanded N when you use Stokes' theorem. On an exam, I would like to see evidence that a decision took place. And indicate exactly what the righthanded N *is* (inner? outer? upper? lower?). Better still, draw a picture.

3. His name was Stokes, not Stoke so his theorem is Stokes' theorem not Stoke's theorem.

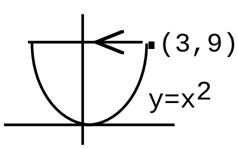
PROBLEMS FOR SECTION 4.3

1. Use integral theory instead of doing these directly.

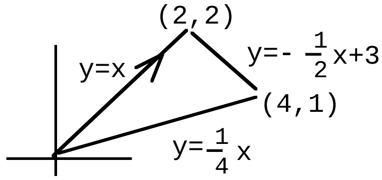
(a) Find $\oint x \sin y\,dx + xy^3\,dy$ on the directed loop in the diagram.

(b) Find $\oint (x^2 - y^2)\,dx + x\,dy$ clockwise around the circle $x^2 + y^2 = 3$.

(c) Let $\mathbf{F} = x^3\,\vec{i} + x^2\,\vec{j}$. Find the circulation of F on the directed loop in the diagram.



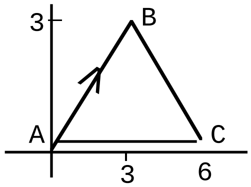
(a)



(c)

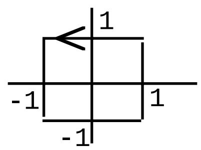
2. Find $\oint y^2\,dx + 3x\,dy$ on the loop in the diagram

(a) directly (b) with integral theory



3. Let f be an arbitrary scalar field in 2-space.
Use Green's theorem on $\oint_{\text{ccl loop}} -\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial x}\,dy$ and see where it leads.

4. Let $p = \frac{-y}{x^2+y^2}$, $q = \frac{x}{x^2+y^2}$.
You want to compute $\oint p\,dx + q\,dy$ on the square in the diagram.



Turns out that

$$\frac{\partial q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial p}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

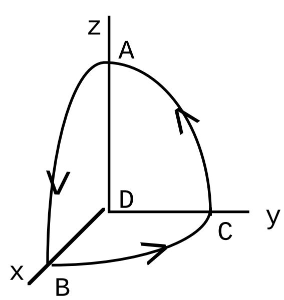
So $\partial q/\partial x = \partial p/\partial y$ (so far so good).

(a) What's wrong with using Green's theorem now to get

$$\oint p\,dx + q\,dy = \int_{\text{inside square}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dA = \int 0\,dA = 0.$$

(b) What's the right answer.

5. The diagram shows some of the traces in the coordinate planes of a sphere (coming out of the page) with center at the origin and radius 6.
Find $\oint (x+z)\,dx + (x+z)\,dy + (x+y)\,dz$ on the loop ABCA.
(a) Do it directly
(b) Do it again with integral theory twice, using two different hats.

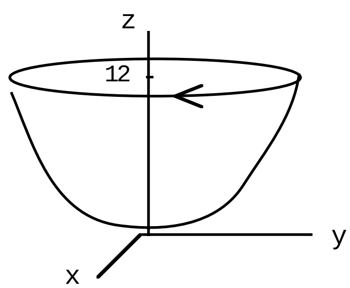


6. Use integral theory on $\frac{1}{2}\oint_{\text{ccl}} -y\,dx + x\,dy$ around an arbitrary loop in the x,y plane and keep going until you get something nice.

7. Let F = yz i + x k

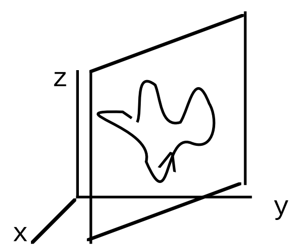
The diagram shows the cup $z = 2x^2 + y^2$, $0 \le z \le 12$.

Use integral theory to find $\int$ F·T ds on the rim directed as in the diagram.



8. A loop lies in a plane perpendicular to the x,y plane.

Find $\oint$ z dx + (x + y) dy + x dz around the loop.



9. The diagram shows a closed surface (like a potato skin) with a directed loop on it (like a wobbly equator).

F is a vector field that does not blow up on the potato skin (no guarantees off the potato skin).

Quote some integral theory and finish the sentence with an appropriate new integral:

$\oint$F·T ds on the loop = ?

10. The original Stokes theorem was about brims and hats.
Fig A shows the Lone Ranger's mask. There are three loops in the diagram, an outer boundary loop and two eye-loops.

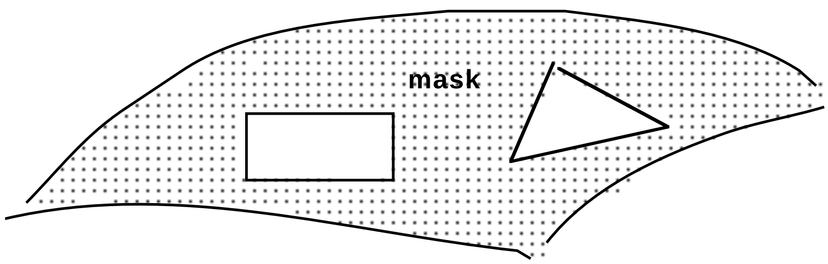


FIG A

Let F be a vector field that does not blow up on the mask. I don't guarantee how F behaves off the mask. For instance, F could blow up on the lines in Fig B

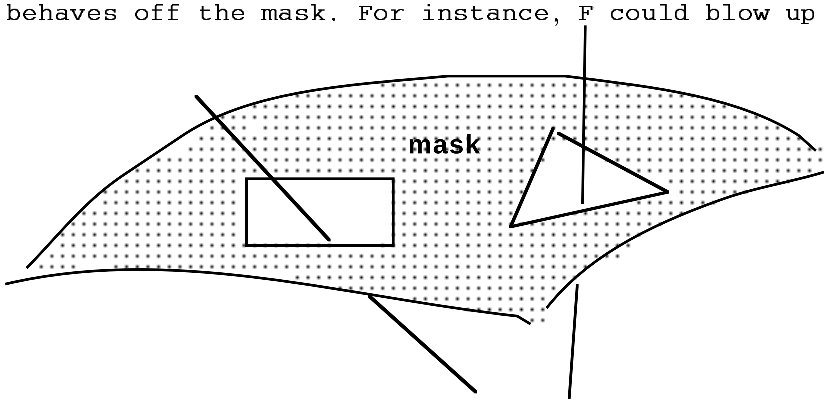


FIG B

Find a connection between line integrating on the three loops and surface integrating on the mask. In other words, make up a Stokesian theorem for masks. And show how you get it.

Do it by drawing little loops. It worked for hats in Fig 9 -- it will work for masks.

SECTION 4.4 THE ZERO-DIVERGENCE RULE AND EXCEPTIONS

closed surfaces (egg shells, footballs)

A surface that encloses a volume is called a *closed* surface. For example, the peel of an orange, the skin of a potato, an egg shell, a tin can-plus-lids, a cardboard box are all closed surfaces. An umbrella, a hat, a lid are *not* closed surfaces.

The symbol $\oint$ is used for a surface integral on a closed surface.

The symbol $\int$ is still used for a surface integral on any kind of surface, closed and not closed alike.

similarly oriented N's on hats with the same brim

Fig 1 shows two hats with the same brim and similarly oriented normals N1 and N2.

Fig 2 shows two more hats with the same brim and similarly oriented normals N1 and N2.

Fig 2A shows a hat and a flat hat with the same brim and similarly oriented normals N1 and N2

In each picture if you reversed both the N1 and N2, naturally they would still be called similarly oriented.

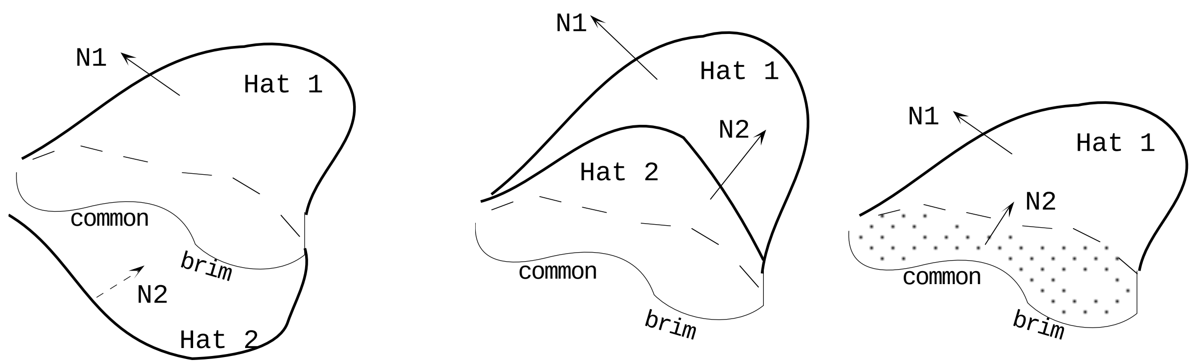


FIG 1 FIG 2 FIG 3

Note that in Fig 1 where the two hats are on "opposite sides" of the brim, one of the N's is out its hat and the other is into its hat.

In Fig 2, where the two hats are on the "same side" of the brim, similarly oriented N's are either both out of their hats or both into their hats.

Here are some equivalent official definitions of "similarly oriented" N's on a pair of hats (or just remember Figs 1 and 2).

(1) Fix a direction on the hat brim (either direction is OK as long as you are consistent). Normals to the two hats are oriented alike if they are both righthanded or are both lefthanded w.r.t. to the direction on the brim.

In Fig 1, N1 and N2 are both righthanded w.r.t. a counterclockwise-as—viewed—from—above direction on the brim.

(2) Picture the *closed* surface (football) made up of the two hats in Fig 1 put together (Fig 1A) or the two hats in Fig 2 put together (Fig 2A). Then a pair of similarly oriented N's will have one N into the football and the other N out of the football.

Note that in Fig 2A "out of a hat" and "out of the football" are not necessarily the same. N2 is out of its hat but into the dented football. So you can't use the words "outer" and "inner" unless the context is clear.

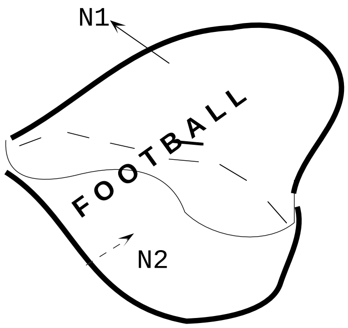


FIG 1A

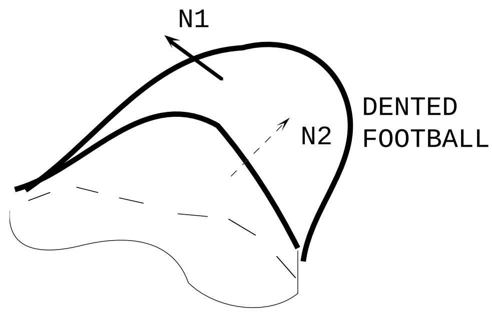


FIG 2A

zero-divergence rule

Let F be a vector field.

These four statements about F go together. Either they all hold or none of them holds; i.e., if any one of them holds, all the others hold and if any one of them fails then all of the others fail.
(But there are exceptions coming up later in this section.)

(1) $\oint$ F·N dS = 0 on every closed surface.

(2) $\int$ F·N dS is independent of surface meaning that if several hats have the same brim (Figs 1, 2, 3) then $\int$ F·N dS has the same value on all the hats provided that the N's are similarly oriented.

(3) F is a curl (i.e., F has an anticurl).

(4) Div F = 0.

When these four statements hold, F is called *solenoidal* and its anticurl is called a *vector potential* for F.

The proof of the zero-div rule will try to show that (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4) and (4) ⇒ (1). This shows that every statement on the list implies every other statement on the list.

why the zero-div rule works: (1) ⇒ (2)

Assume every $\oint$ F·N dS is 0.

Look at the two hats with the same brim and similarly oriented normals N1 and N2 in Fig 4.

I want to show that $\int$ F·N1 dS on Hat 1 = $\int$ F·N2 dS on Hat 2.

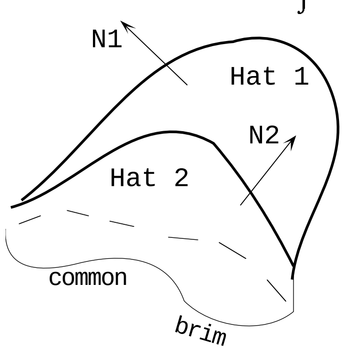


FIG 4

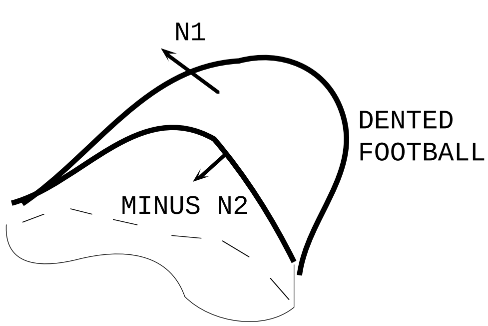


FIG 5

$\int$ F·N1 dS on Hat 1 + $\int$ F·MINUS N2 dS on Hat 2

= $\oint$ F·outer N dS on dented football (a closed surface) in Fig 5

= 0 by hypothesis

So

$$\int F\cdot N1 \, dS \text{ on Hat 1} = \int F\cdot N2 \, dS \text{ on Hat 2} \qquad \text{QED}$$

why the zero-div rule works: (2) ⇒ (3)

I don't have a good argument.

why the zero-div rule: (3) ⇒ (4)

If F = curl G then div F = div curl G = 0 by the identity in §1.6.

adjacent box law (cute trick needed for the next proof and later stuff)

Let F be any vector field.

Fig 6 shows two boxes (closed surfaces I and II, that share a boundary face (bottom of I and top of II). I drew them slightly apart just so that you see them better. Then

flux out of I + flux out of II = flux out of merged box in Fig 7

because the flux back and forth across the common face cancels out.

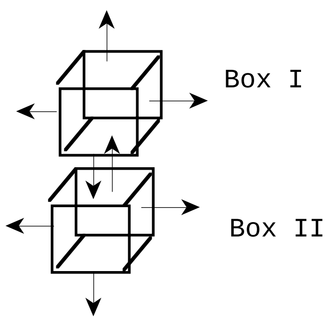


FIG 6

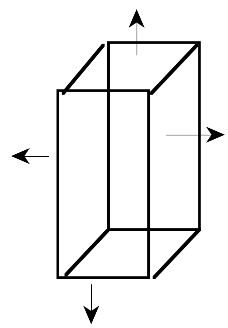
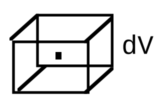


FIG 7

repeat of (3) in Section 1.4 (needed for the next proof)

Net flux out of this little box = div F dV



(Amended version coming later in this section)

why the zero-div rule works: (4) ⇒ (1)

Suppose div F = 0.

I want to show that $\oint$ F·N dS is 0 on any football (closed surface).

Fill the inside of the football with little boxes (Fig 8).

(1) Sum of small fluxes out in Fig 8 = flux out of football

$$= \oint \text{F}\cdot\text{outer N dS on football.}$$

(A) Each small flux is div F dV.

But div F = 0 so each small flux is 0 and the sum of the small fluxes is 0.

So $\oint$ F·outer N dS on the football is 0. QED

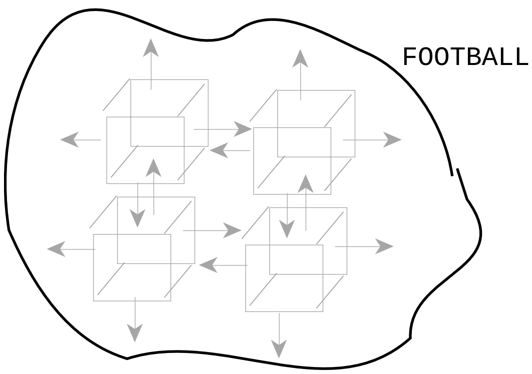


FIG 8

footnote

Here's another way to see why (4) ⇒ (1).

Look at any closed surface. Since div F = 0, F is the flux density of an incompressible flow with no sources or sinks. No flux is swallowed by a sink; whatever flows into the football also flows out. No flux is created by a source. Whatever flows out of the football also flowed in. So the net flux across the surface of the football is 0.

So $\oint$ F·N dS on the football is 0.

example 1

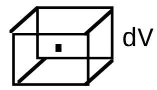
Let $F = y\vec{i} + z\vec{j} + x\vec{k}$. Find the flux out of the ellipsoid $x^2 + 2y^2 + 3z^2 = 7$.

solution

Div F = 0, the ellipsoid is a closed surface so $\oint$ F·outer N dS = 0 by the zero-div rule.

amended version of (3) in Section 1.4

Net flux out of this little box = div F dV



But not if F blows up at the point!

If F blows up at the point then neither F nor div F is defined at the point so there is no div F dV.

retracting part of the zero divergence rule when F blows up at a point in 3-space

Suppose div F = 0.

The zero-divergence rule says that in this case every $\oint$ F·N dS is 0.

But there are exceptions.

Suppose div F = 0 but F blows up at a point.

footnote
I really mean that div F is 0 except at the blowup point where F and div F are not defined.

If a closed surface does *not* enclose the blowup (Fig 9) the argument in (1) still works and it's still true that $\oint$ F·N dS is 0.

If a closed surface *does* enclose the blowup (Fig 10) we have no conclusion about $\oint$ F·N dS. In this case, step (A) in (1) no longer holds. The flux out of the small box around the blowup point (Fig 11) is *not* div F dV (there's isn't a div F value at all at the blowup point). The argument breaks down here and does not lead to any conclusion about $\oint$ F·N dS. on the big surface.

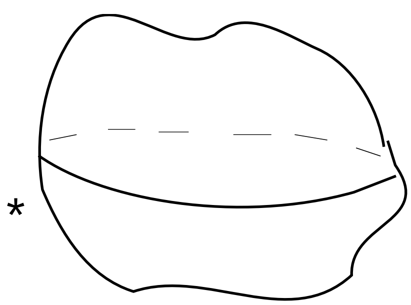


FIG 9

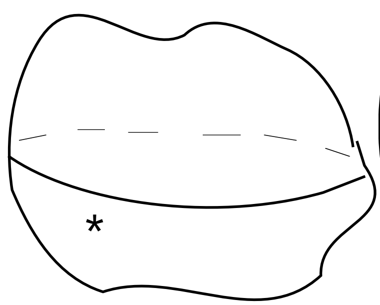


FIG 10

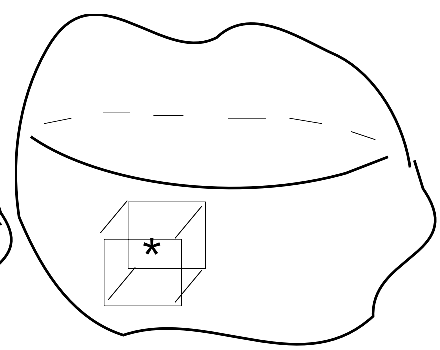


FIG 11

The next rule says that while you can't conclude that $\oint$ F·N dS = 0 on surfaces enclosing the blowup, at least you will get the same value for $\oint$ F·N dS no matter what surface you use enclosing the blowup.

deformation of surface principle for the surface integral $\oint$ F·N dS

> Suppose div F = 0 but F blows up at a point.
> Then $\oint$ F·outer N dS has the same value on all closed surfaces with the blowup inside (and the opposite value if you use inner N's). In Fig 12,
>
> $\oint$ F·outer N dS on spherical surface = $\oint$ F·outer N dS on box surface

You can refer unambiguously to $\oint$ F·outer N dS on a closed surface enclosing the blowup since there is only one possible value.

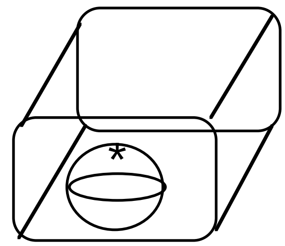


FIG 12

Here's why.
I want to show that $\oint$ F·outer N dS is the same on the box and the sphere in Fig 12. Fill the space *between* them with little boxes (Fig 13). Then

(2) sum of all the little fluxes out = flux *out* of large box + flux *into* sphere

Each little flux is div F dV. This holds *despite* the blowup since all the little boxes in Fig 13 are *between* the inner sphere and the outer box where F does *not* blow up.

And div F = 0 so each little flux out is 0. So the lefthand side of (2) is 0 and

flux *out* of large box + flux *into* sphere = 0

flux *out* of large box = flux *out* of sphere.

$\oint$ F·outer N dS on spherical surface = $\oint$ F·outer N dS on box surface QED

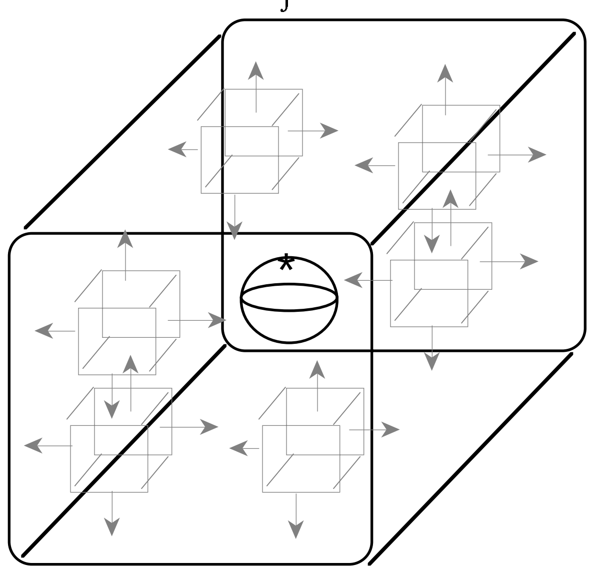


FIG 13

footnote

You can also look at it like this. In the region *between* the spherical surface and the box surface F does *not* blow up. It is safe to say that the flow *between* the two surfaces is incompressible with no sources and sinks. Whatever flows out of the sphere continues undiminished and unenhanced out of the box. So

$$\oint F\cdot \text{outer } N\, dS \text{ on sphere} = \oint F\cdot \text{outer } N\, dS \text{ on box}$$

example 2 (a famous counterexample)

Let

$$F = \frac{1}{\rho^2}\, e_\rho \qquad \text{(the electric field due to a unit charge at the origin)}$$

Then

$$\begin{aligned} \operatorname{div} F &= \frac{1}{\rho^2 \sin\phi}\,\frac{\partial}{\partial\rho}\left(\rho^2 \sin\phi\ \frac{1}{\rho^2}\right) \qquad \text{(see Section 2.5)} \\ &= \frac{1}{\rho^2 \sin\phi}\,\frac{\partial}{\partial\rho}(\sin\phi\,) \\ &= 0 \end{aligned}$$

but F blows up at the origin.

$\oint F\cdot N\, dS = 0$ on any closed surface *not* enclosing the origin.

And so far we have no conclusion about $\oint F\cdot T\, ds$ on surfaces enclosing the origin.

I'll get a conclusion for this specific F by testing $\oint F\cdot T\, ds$ on one such surface.

By the deformation principle, I'll get the same answer no matter what surface I use.

I'll surface integrate on the unit sphere with center at the origin because on that sphere F simplifies to e_ρ and the computing will be easier.. On that sphere,

$$\rho = 1, \quad F = e_\rho, \quad \text{outer } N = e_\rho, \quad F\cdot\text{outer } N = 1$$

$$\begin{aligned} \oint_{\text{sphere}} F\cdot\text{outer } N\, dS &= \oint 1\, dS \\ &= \text{surface area of the unit sphere} \\ &= 4\pi \end{aligned}$$

By the deformation principle, $\oint F\cdot\text{outer } N\, dS$ is 4π on *any* closed surface enclosing the origin.

retracting another part of the zero div rule when F blows up at a point

The zero-div rule in the last section said that if div F = 0 then $\int F\cdot N\, dS$ is independent of surface. The argument used the fact that if div F = 0 then every $\oint F\cdot N\, dS$ is 0.

Now suppose div F = 0 but F blows up at a point.

If two hats with a common brim and similarly oriented N's do *not* enclose the blowup between them (Fig 14) it's still true that $\int F\cdot N1\, dS$ on Hat 1 = $\int F\cdot N2\, dS$ on Hat 2. (because the football consisting of the two hats put together is a closed surface not containing the blowup and the original argument that (1) ⇒ (2) still works).

Suppose two hats with a common brim *do* trap the blowup between them (Fig 15).

For a particular F, $\oint F\cdot\text{outer } N\, dS$ on a surface enclosing the blowup might turn out to be 0 anyway (to find out, test it on a convenient surface). In that case, it's still true that $\int F\cdot N\, dS$ is the same on the hats provided the N's are directed similarly.

For a particular F, $\oint$ F·outer N dS on a surface enclosing the blowup point may turn out to be non-zero. In that case, $\int$ F·N dS will *not* be the same on the two hats in Fig 15.

(Remember that the hypothesis in all this, aside from the blowup, is div F = 0.)

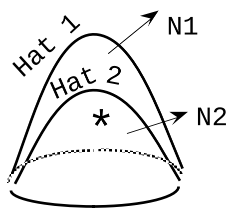


FIG 14

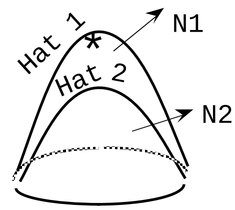
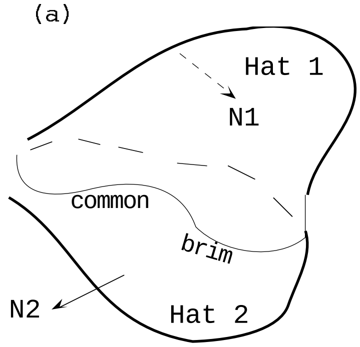


FIG 15

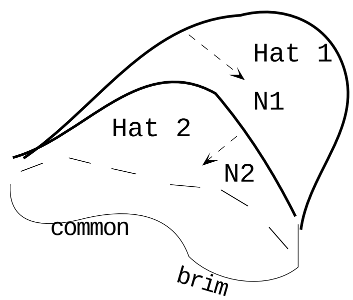
PROBLEMS FOR SECTION 4.4

1. Are N1 and N2 similarly oriented?

(a)



(b)



2. Let $F = x^2y\,\vec{i} + z\,\vec{j} - 2xyz\,\vec{k}$.
Find $\oint$ F·inner N dS on the surface of an apple.

3. Let $F = xyz\ i + e^y \cos z\ j + \sin z \cos z\ k$.
Find $\oint$ curl F·outer N dS on a sphere. Any sphere.
Note that the integrand is *curl* F·outer N, not F·outer N.

4. Suppose curl a = b, grad c = d, div f = g, anticurl h = m, antigrad n = p, antidivergence q = r.
(a) Which functions are vector fields and which are scalar fields.
(b) Find all the potentials (both scalar and vector) possible.

5. Let $F = x^3y\,\vec{i} + q(x,y,z)\,\vec{j} + xyz^2\,\vec{k}$.
Find all possible q's so that F has an anticurl.

6. Let

$$F(x,y,z) = \frac{x}{x^2+y^2+z^2}\vec{i} + \frac{y}{x^2+y^2+z^2}\vec{j} + \frac{z}{x^2+y^2+z^2}\vec{k}$$

(a) Convert F to spherical coordinates.
(b) Does F has a (scalar) potential.
(c) Does F have a vector potential.

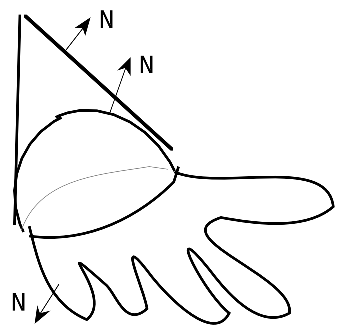
7. What if anything can you conclude about the numerical value of $\oint$ F·outer N dS on a particular football if
(a) F = yi + zj + xk
(b) F = x^2 i + y^2 j + z^2 k

8. What if anything can you conclude about the numerical value of $\int$ F·upper N dS on a particular umbrella if
(a) div F is 0 everywhere (b) div F is not 0 everywhere

9. What if anything can you conclude about div F if

(a) $\oint$ F·N dS = 0 on a specific football
(b) $\oint$ F·N dS = 6 on a specific football
(c) $\int$ F·N dS = π on a specific umbrella

10. Let F(x,y,z) = x^2 i − 2xz k.
If $\int$ F·outer N dS is 7 on the cone in the diagram, find $\int$ F·outer N dS on
(a) the hemisphere
(b) the glove.



11. Look at ellipsoid $2x^2 + 2y^2 + 5z^2 = 6$

Use integral theory (instead of direct computation) to find $\int$ k·outer N dS on
(a) the top half
(b) the bottom half
(c) the whole ellipsoid

SECTION 4.5 THE DIVERGENCE THEOREM

divergence theorem (Gauss' theorem) for the surface integral $\oint \mathbf{F} \cdot \mathbf{N}\, dS$ on a *closed* surface

The divergence theorem is about surface integrating on a closed surface (e.g., an egg shell) (Fig 1) versus triple integrating on the solid region inside the surface (the egg).

> Suppose F does not blow up on or inside a closed surface. Then
>
> $$\underbrace{\oint_{\text{closed surface}} F\cdot \text{outer } N\, dS}_{\text{surface integral on egg shell}} = \underbrace{\int_{\text{inside}} \text{div } F\, dV}_{\text{triple integral on egg}}$$

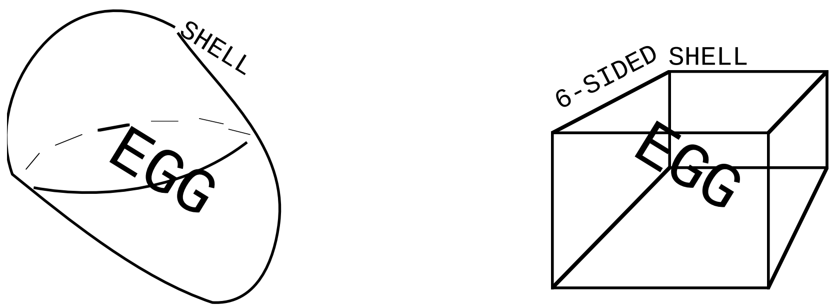


FIG 1

There's a review of triple integrals in Section 4.0.

why the divergence theorem works

Fill the egg with little boxes (Fig 2).

Sum of the small fluxes out
 = flux out of shell (by the adjacent box law in the preceding section)
 = $\oint F\cdot \text{outer } N\, dS$ on the egg shell in Fig 2

Each small flux is a div F dV (since there are no blowups inside the egg. So

$\oint F\cdot \text{outer } N\, dS$ on the egg shell = sum of div F dV's

 = $\int$ div F dV on the egg QED

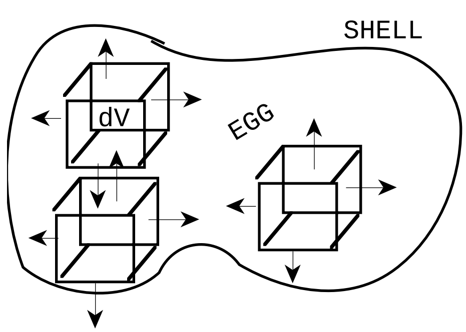


FIG 2

example 1

Let $F = \sin^2\theta\ e_r$ (cylindrical coordinates).

Fig 3 shows a closed surface (tin can) consisting of a cylindrical surface with radius R and height H *plus the top and bottom lids.*

Find $\oint F\cdot\text{outer } N\ dS$ on the tin can.

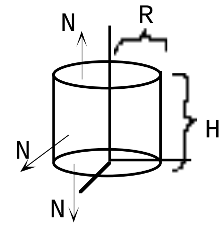


FIG 3

solution method 1 (directly)

$$\oint F\cdot N\ dS = \int_{\text{top}} F\cdot N\ dS + \int_{\text{bottom}} F\cdot N\ dS + \int_{\text{cylinder part}} F\cdot N\ dS$$

LIDS

On the top and bottom lids, the outer N's are k and $-k$ respectively, $e_r\cdot\pm k = 0$ so the surface integrals are 0 on the two lids.

CYLINDER

The cylinder has parametric equations

$x = R\cos\theta$

$y = R\sin\theta$

$z = z$

$0 \le \theta \le 2\pi,\ 0 \le z \le H.$

On the cylinder, $r = R$, outer $N = e_r$, $F\cdot\text{outer } N = \sin^2\theta$, $dS = R\ dz\ d\theta$ (see §3.5).

$$\int_{\text{cylinder}} F\cdot\ \text{outer } N\ dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{H} \sin^2\theta\ \ R\ dz\ d\theta$$

$$= \pi RH \qquad \text{(use integral tables on the reference page)}$$

ANSWER $= 0 + 0 + \pi RH = \pi RH$

method 2 (divergence theorem)

$$\text{div } F = \frac{1}{r}\frac{\partial(r\sin^2\theta)}{\partial r} = \frac{1}{r}\cdot\sin^2\theta \quad \text{(use the formula for div in u,v,w coords)}$$

$$\oint F\cdot\text{outer } N\ dS \text{ on tin can} = \int_{\text{solid can}} \text{div } F\ dV$$

$$= \int \frac{1}{r}\cdot\sin^2\theta\ \ dV$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R}\int_{z=0}^{H} \frac{1}{r}\cdot\sin^2\theta \quad r\ dz\ dr\ d\theta$$

(in cylindrical coords, dV is $r\ dz\ dr\ d\theta$)

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R}\int_{z=0}^{H} \sin^2\theta\ dz\ dr\ d\theta$$

$$= \pi RH \qquad \text{QED}$$

warning

In example 1, method 1, when you *surface* integrate on the cylinder, r is a constant, namely R. In method 2, when you *triple* integrate on the solid cylinder, r is *not* R; r is one of the variables of integration and it ranges from 0 *to* R.

example 2

Let $F = z^2\ \vec{k}$. Find $\oint F\cdot\text{outer } N\ dS$ over the surface of the tetrahedron in Fig 4 where three faces lie in the coordinate plates and the fourth face is in the plane $3x + 2y + z = 4$. Do it (a) directly (b) with integral theory.

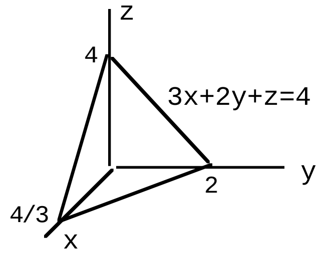


FIG 4

solution (a) On the back face and the left face, F is perp to N so $F\cdot N = 0$ and $\int F\cdot N\ dS = 0$.

On the bottom face, z = 0 so $F = \vec{0}$ so $\int F\cdot N\ dS = 0$.

The slanted face has parametric equations x=x, y=y, z=4−3x−2y.
Let g = z+3x+2y−4 Then

$$n = \frac{\nabla g}{\partial g/\partial z} = (3,2,1)$$

$$F\cdot n = z^2 = (4-3x-2y)^2$$

$$\int_{\text{slanted face}} F\cdot N\ dS = \int F\cdot n\ dx\ dy \text{ over the x,y projection in Fig 5}$$

$$= \int z^2\ dx\ dy \text{ over the x,y projection}$$

(1) $$= \int_{x=0}^{4/3}\int_{y=0}^{2-\frac{3}{2}x} (4-3x-2y)^2\ dy\ dx \qquad [\ = \frac{32}{9}\ ]$$

Final answer is $\frac{32}{9} + 0 + 0 + 0 = \frac{32}{9}$

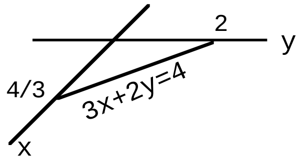


FIG 5

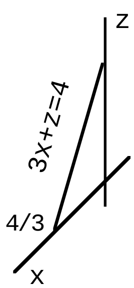


FIG 6

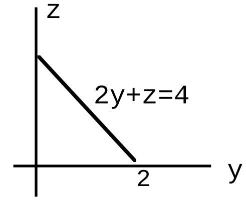


FIG 7

(b) $\oint$ F·outer N dS on tetrahedron surface = $\int_{\text{inside}}$ div F dV

There are several ways to put in limits on the triple integral.

version 1

z goes from lower to upper boundary and then the x,y limits come from the projection of the solid in the x,y plane (Fig 5):

$$\int_{x=0}^{4/3} \int_{y=0}^{2-\frac{3}{2}x} \int_{z=0}^{4-3x-2y} 2z \, dz \, dy \, dx$$

Question

How come back in (1), the z^2 in the integrand got replaced by (4−3x−2y) but here you are leaving the 2z alone and not replacing it with 2(4−3x−2y)

Answer

In (1), I was *surface* integrating on the plane 3x + 2y + z = 4 using x and y as parameters, i.e., with x and y as the variables of integration. On that plane z *is* 4−3x−2y. But here I am *triple* integrating on the solid tetrahedron and z is not 4−3x−2y in the solid tetrahedron. In fact z goes from 0 to 4−3x−2y in the tet.

version 2

$$\int_{x=0}^{4/3} \int_{z=0}^{4-3x} \int_{y=0}^{(4-z-3x)/2} 2z \, dy \, dz \, dx$$

y goes from left to right boundary and then the x,z limits come from the projection of the solid in the x,z plane (Fig 6)

version 3

$$\int_{y=0}^{2} \int_{z=0}^{4-2y} \int_{x=0}^{(4-z-2y)/3} 2z \, dx \, dz \, dy$$

x goes from rear to forward boundary and then the y,z limits come from the projection of the solid in the y,z plane (Fig 7)

warning

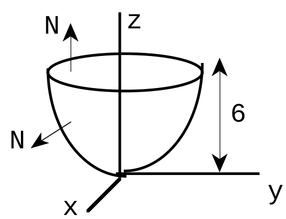
The divergence theorem is only for a surface integral on a *closed* surface. In other words, it's for egg shells, not hats.

some pretty analogies

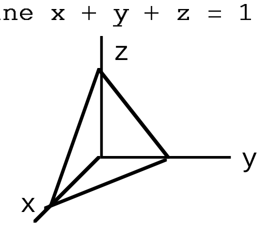
line integrals in 3-space	*surface integrals in 3-space*
• F is a gradient iff curl F = $\vec{0}$	• F is a curl iff div F = 0
• If curl F = $\vec{0}$ and F stays finite then (i) $\int$ F·T ds is independent of path (ii) $\oint$ F·T ds = 0 (iii) $\int$F·T ds on a curve = antigrad(end) − antigrad(start)	• If div F = 0 and F is finite then (i) $\int$ F·N dS is ind of surface (ii) $\oint$ F·N dS = 0 (iii) $\int_{\text{hat}}$ F·N dS on a surface = $\oint_{\text{brim}}$ anticurl F·righthand T ds
• If curl F = $\vec{0}$ but F blows up along a line then $\oint$ F·T ds has the same value on every closed curve around the blowup line provided you use similarly directed T's.	• If div F = 0 but F blows up at a point then $\oint$ F·N dS has the same value on all closed surfaces enclosing the blowup provided you use similarly directed N's.
• If F doesn't blow up then $\oint$ F·T ds = $\int$ curl F· righthand N dS on surface inside	• If F doesn't blow up then $\oint$ F·outer N dS = $\int$ div F dV on the solid inside

PROBLEMS FOR SECTION 4.5

1. Let F = y $\vec{i}$ + y^2z^2 $\vec{k}$. Find $\int$ F·outer N dS over the paraboloid-plus-lid in the diagram. The paraboloid has equation $z = x^2 + y^2$. Do it (a) directly and (b) again with integral theory.

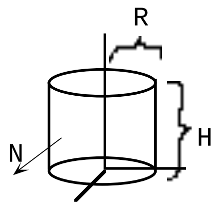


2. The flux density is F(x,y,z) = xy i + yz j + zx k.
Use integral theory to find the flux out of the tetrahedron in the diagram bounded by the plane x + y + z = 1 and the coordinate planes.



3. Let F(x,y,z) = (xz, 0, z^2).
Use integral theory to find $\oint$ F·inner N dS over the sphere $x^2 + y^2 + z^2 = 1$

4. Let $F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$.
The problem is to find $\int F \cdot \text{outer } N \, dS$ on the cylindrical surface in the diagram, *not* including the lids. One possibility is to just do it directly. Instead, do it a little indirectly by finding $\oint F \cdot \text{outer } N \, dS$ on the cylinder-plus-lids (using integral theory) and subtracting $\int F \cdot \text{outer } N \, dS$ on the lids.



5. (a) If $g(x,y,z)$ is a scalar field and N is a vector, what does the notation $\partial g/\partial N$ mean and how do you find it.
(b) If $g(x,y,z)$ is a scalar field what does the notation $\nabla^2 g$ mean.
(c) The divergence product rule from (2) in Section 1.7 is

$$\text{div } f\vec{G} = f \, \nabla \cdot \vec{G} + \nabla f \cdot \vec{G}$$

Use it to show that for any closed surface and any scalar fields f and g,

$$\oint_{\text{surface}} f \, \frac{\partial g}{\partial \text{ outer } N} \, dS = \int_{\text{inside}} (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV \quad \textit{(Green's identity)}$$

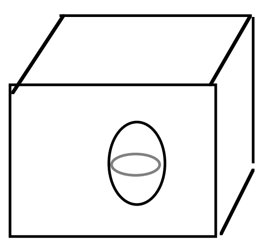
(d) Let f be a harmonic scalar field.
Use Green's identity from part (c) to fill in the blank:

$$\oint_{\text{surface}} f \, \frac{\partial f}{\partial \text{ outer } N} \, dS = \int_{\text{inside}} \underline{\hspace{5cm}}$$

6. The diagram shows a box with an eggshell inside.
Let F be a vector field that stays finite on the box, on the egg shell and in the region between them (inside the box but outside the egg).
(F might blow up inside the egg shell so don't go there.)

Find a connection between
surface integrating on the box
surface integrating on the egg shell
triple integrating

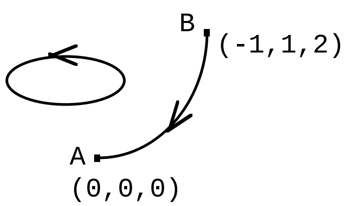
Do it by using appropriate small boxes.



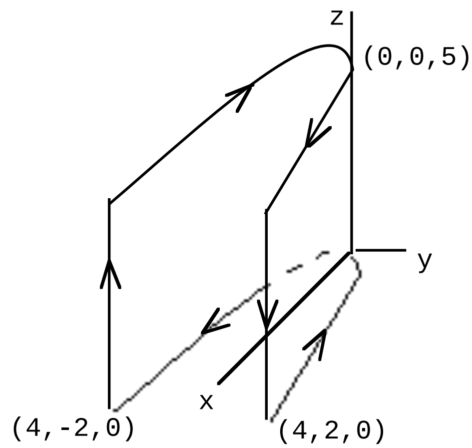
REVIEW PROBLEMS FOR CHAPTER 4

When you line integrate or surface integrate or double integrate or triple integrate , especially if you are switching from one to the other, say what you are integrating *on* (what curve? what surface? what region? etc). And if N's and/or T's are involved indicate which way they go.

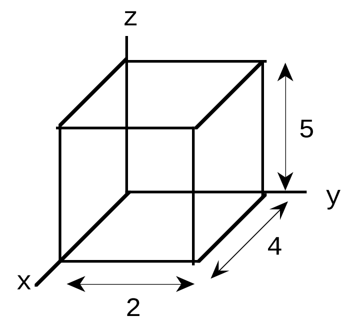
1 Let $F = re_r$. Find $\int F \cdot T \, ds$ on each of the two curves in the diagram.



Problem 1



Problem 2



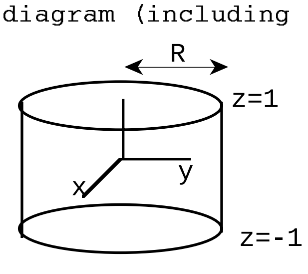
Problem 3

2. Find $\oint 2y\,dx - z\,dy + x^2\,dz$ on the loop in the diagram, lying on cylinder $x = y^2$.

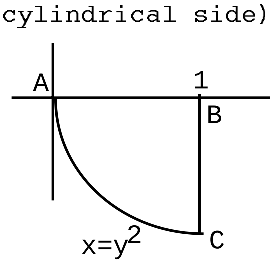
Do it (a) directly and (b) again with integral theory

3. Let $F = x\,\vec{i} + 2y\,\vec{j}$.
Find $\oint F \cdot \text{outer } N \, dS$ on the box in the diagram (a) directly and (b) with integral theory.

4. Let $F = \frac{1}{\rho^2} e_\rho$. Use integral theory to find the flux out of the tin can in the diagram (including top, bottom and the cylindrical side).



Problem 4



Problem 5

5. Find each of the following with integral theory if possible. Forget about it otherwise.

(a) $\int xy\,dx + x^2\,dy$ on the C to A path in the diagram.

(b) $\oint xy\,dx + x^2\,dy$ on the clockwise loop CABC

(c) $\int 2xy\,dx + x^2\,dy$ on the C to A path

(d) $\oint 2xy\,dx + x^2\,dy$ on the clockwise loop CABC

6. Given $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, p and q blow up at the indicated points inside the triangle and the square in the diagram (but otherwise are finite).

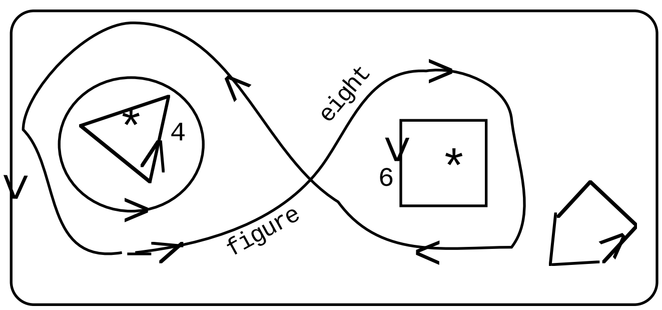
Also given

$\oint$ p dx + q dy ccl around triangle = 4

$\oint$ p dx + q dy ccl around square = 6

Find $\oint$ p dx + q dy around

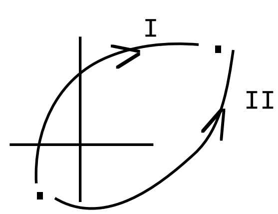
(a) the ccl circle
(b) the ccl pentagon
(c) the figure eight as directed in the diagram
(d) the ccl rectangle



7. Let $F(x,y,z) = x^2\,\vec{i} - x\vec{j} - 2xz\vec{k}$.
(a) Test to see if F has an antigradient (just decide yes or no without trying to actually find the antigradient).
(b) Test to see if F has an anticurl.
(c) Does F have an antidivergence??????????

8. Decide if possible whether $\int_{I} F\cdot T\,ds$ is equal or unequal to $\int_{II} F\cdot T\,ds$ (see the diagram) if

(a) $F = 3xy\,\vec{i} + x^2\,\vec{j}$ (b) $F = 2x\,\vec{i} + 2\vec{j}$



9. What does it mean to say
(a) $\int F\cdot T\,ds$ is independent of path
(b) $\int F\cdot N\,dS$ is independent of surface

10. Can you decide if curl F is zero or nonzero if

(a) $\int$ F·T ds = 0 on curve I

(b) $\int$ F·T ds = 7 on curve I

(c) $\int$ F·T ds = 0 on curve II

(d) $\int$ F·T ds = 7 on curve II

11. Let $F = x^2 \vec{k}$.

The diagram shows three surfaces and respective N's:

(i) the plane surface inside a circle of radius 3 in the x,y plane

(ii) a hemisphere with radius 3

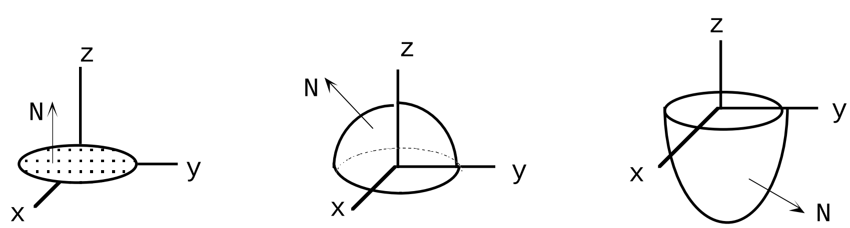
(iii) a piece of paraboloid $z = x^2 + y^2 - 9$ (the part below the x,y plane)

(a) Find the value of $\int$ F·indicated N dS on each surface, as efficiently as possible.

(b) Now you're going to find $\int$ F·N dS on each surface again in a different but still efficient manner.

First use a little trial and error to get an anticurl for F (why is it guaranteed that there *is* an anticurl).

Then use your anticurl to find $\int$ F·N dS on each surface.



(i) (ii) (iii)

12. Given a loop and scalar fields f and g.

Here's the gradient product rule from Section 1.7:

$$\nabla(fg) = f\nabla g + g\nabla f$$

Use it to show that

$$\oint f\nabla g\cdot T \text{ ds on the loop } = \oint g\nabla f\cdot \text{opposite } T \text{ ds on the loop}$$

SUMMARY OF ALONG AND THROUGH (from Chapters 1, 3 and 4)

Let F be a vector field in 3-space (that doesn't blow up).

Circulation of F on a *small almost-straight curve* (Fig 1) is $F\cdot T\,ds$ (§1.6) (definition).

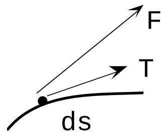


FIG 1

Circulation of F on a *small loop* (Fig 2) is a sum of a *few* $F\cdot T\,ds$'s which turned out to be curl $F\cdot N\,dS$ (§1.6).

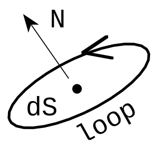


FIG 2

Circulation of F on a (not-small) curve (Figs 3, 4) is a sum of *many* $F\cdot T\,ds$'s.
The sum is called the line integral $\int F\cdot T\,ds$ (§3.1).

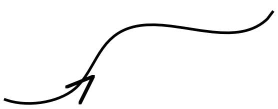


FIG 3

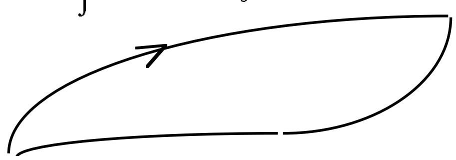


FIG 4

Flux through a *small almost-flat surface* (Fig 5) is $F\cdot N\,dS$ (§1.3) (definition).

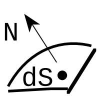


FIG 5

Flux out of a *small closed surface* (Figs 6, 7) is a sum of a *few* $F\cdot N\,dS$'s which turned out to be div $F\,dV$ (§1.4)

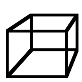


FIG 6

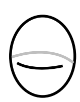


FIG 7

Flux through a (not-small) surface (Figs 8, 9) is a sum of *many* $F\cdot N\,dS$'s.
The sum is called the surface integral $\int F\cdot N\,dS$ (§3.3).

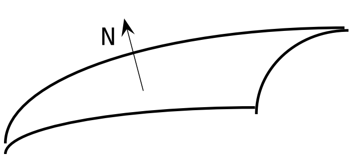


FIG 8

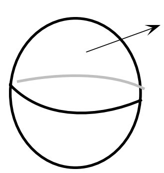


FIG 9

The circulation of F on a hat brim is a sum of circs on many small loops on the hat (Fig 10)

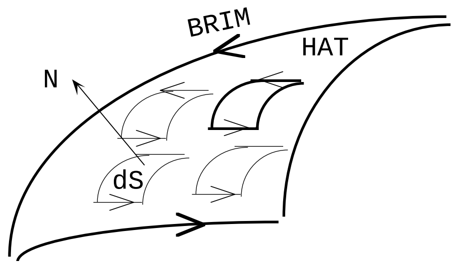


FIG 10

which leads to Stokes' theorem (§4.3):

$\oint$ F·T ds on a hat brim = $\int$ curl F·righthanded N dS on hat

The flux through a *closed* surface is a sum of fluxes out of many small boxes inside (Fig 11)

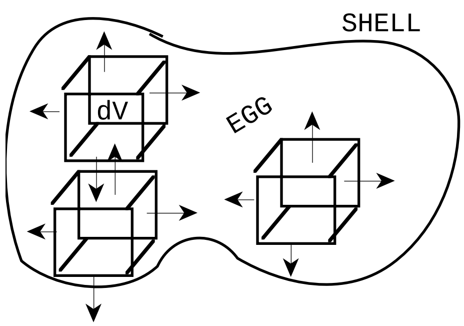


FIG 11

which leads to the divergence theorem (§4.5):

$\oint$ F·outer N dS on a closed surface = $\int$ div F dV on the inside

CHAPTER 5 COORDINATE SYSTEMS CONTINUED

SECTION 5.0 REVIEW

2×2 determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

computing an $n \times n$ determinant by expansion by minors

First you have to know what minors and cofactors are.

Every entry in a square array has a *minor* and a *cofactor*, defined as follows.

The *minor* of an entry is the determinant you get by deleting the row and col of that entry .

The *cofactor* of an entry is the minor with a sign attached according to the entry's location in the checkerboard pattern in Fig 1.

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

FIG 1

For example if you start with

$$\begin{matrix} 1 & 2 & 7 & 8 \\ 4 & 3 & 1 & 9 \\ 5 & 2 & 9 & 6 \\ 8 & 1 & 3 & 8 \end{matrix}$$

then to get the minor of the 7 in row 1, col 3, form a determinant by deleting row 1 and col 3 :

$$\text{minor of the 7 in row 1, col 3} = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{7} & \cancel{8} \\ 4 & 3 & \cancel{1} & 9 \\ 5 & 2 & \cancel{9} & 6 \\ 8 & 1 & \cancel{3} & 8 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 9 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix}$$

To get the cofactor, choose the sign in row 1 col 3 of the checkerboard in Fig 1.

$$\text{cofactor of the 7 in row 1, col 3} = + \begin{vmatrix} 4 & 3 & 9 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix}$$

(in this case the minor and the cofactor are the same)

Similarly,

$$\text{minor of the 1 in row 2, col 3} = \begin{vmatrix} 1 & 2 & 8 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix}$$

$$\text{cofactor of the 1 in row 2, col 3} = - \begin{vmatrix} 1 & 2 & 8 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix}$$

Here's how to compute say a 3×3 det (same idea for an $n \times n$).

Pick any row.

det = 1st entry in the row × its cofactor
+ 2nd entry in the row × its cofactor
+ 3rd entry in the row × its cofactor

You can also find the det by picking any col. Then

det = 1st entry in the col × its cofactor
+ 2nd entry in the col × its cofactor
+ 3rd entry in the col × its cofactor

Here's a determinant expanded down column 2:

$$\begin{vmatrix} 10 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2\underbrace{\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}_{-6} + 5\underbrace{\begin{vmatrix} 10 & 3 \\ 7 & 9 \end{vmatrix}}_{69} - 8\underbrace{\begin{vmatrix} 10 & 3 \\ 4 & 6 \end{vmatrix}}_{48} = -27$$

example 1

Here's an expansion down column 4, a good column to use because it has some zero entries:

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & -1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & 1 & 4 \end{vmatrix}$$

Now work on each 3 × 3 det. I'll expand the first one down col 1 and the second across row 1:

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & -1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 4 & 0 \end{vmatrix} = 2\left(1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \right) \quad 3\left(1\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \right)$$

$$= 2(2 + 2) - 3(-7 + 4 + 9)$$
$$= -10$$

some properties of determinants

1. Interchanging two rows (or two cols) changes the sign of the determinant.
For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \\ 7 & 7 & 7 \end{vmatrix} = -\begin{vmatrix} 7 & 7 & 7 \\ 4 & 5 & 5 \\ 1 & 2 & 3 \end{vmatrix} \quad \text{(rows 1 and 3 were switched)}$$

2. Multiplying a row (or col) by a number will multiply the entire det by that number.
If say row 2 is doubled then new det = 2 × old det:

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

In other words, a common factor can be pulled out of a row or a col.

For example,

$$\begin{vmatrix} 6 & 3 & 9 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 3 & 3 & 9 \\ 1 & 1 & 5 \\ 1 & 3 & 1 \end{vmatrix}}_{\substack{\text{pull 2 out} \\ \text{of col 1}}} = 6\underbrace{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 5 \\ 1 & 3 & 1 \end{vmatrix}}_{\substack{\text{pull 3 out} \\ \text{of row 1}}}$$

If M is 7 × 7 and *every* entry of M is quadrupled, then you can pull a 4 out of each of the 7 rows (or equivalently out of each of the 7 columns) so the determinant of M is multiplied by 4^7.

3. If you take the transpose, i.e., turn the rows into cols and the cols into rows, the determinant doesn't change.

For example

$$\begin{vmatrix} a1 & a2 & a3 \\ b1 & b2 & b3 \\ c1 & c2 & c3 \end{vmatrix} = \begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}$$

the scalar triple product

If u,v,w are 3-dim vectors then $u \cdot v \times w$ is called a scalar triple product.

If

$$u = (u_1, u_2, u_3)$$
$$v = (v_1, v_2, v_3)$$
$$w = (w_1, w_2, w_3)$$

then

$$u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1}$$

For example, if

$$u = (1,2,3),\ v = (4,6,-1),\ w = (0,3,2)$$

then

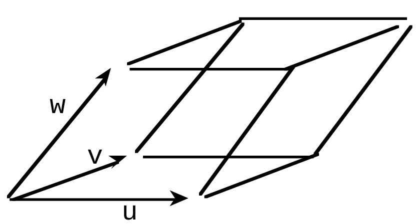
$$u \cdot v \times w = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & -1 \\ 0 & 3 & 2 \end{vmatrix} = \ldots = 35$$

or equivalently

$$v \times w = (15,-8,12)$$

$$u \cdot v \times w = (1,2,3) \cdot (15,-8,12) = 35$$

The absolute value of $u \cdot v \times w$ is the volume of the parallelepiped determined by the vectors u,v,w (Fig 2).



volume is $|u \cdot v \times w|$

FIG 2

For non-zero vectors u,v,w, $u \cdot v \times w = 0$ iff the arrows u,v,w are coplanar when attached to a common tail.

cyclic permutations

Start with say

a b c d e

and rearrange (permute) in the following special way: Picture the letters as beads on a bracelet and slide one of the end letters around (Fig 3) (I'm sliding clockwise but you can also slide counterclockwise).

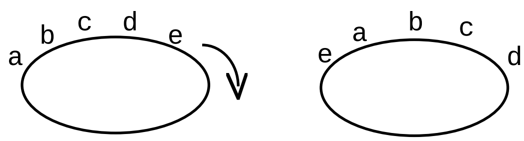


FIG 3

The new permutation is

e a b c d.

It's called a *cyclic permutation* of a b c d e.
You can do it again three more times getting

d e a b c
c d e a b
b c d e a

Once more and you're back to the original a b c d e.

cyclic permutation and scalar triple products

There are six scalar triple products involving factors p, q, r
Three of the products have the same value. The other three have the opposite value (e.g., if one of the scalar triple products is 7 then two more are 7 and the other three are −7).
Here's the rule for telling which ones go together.
If you permute the letters cyclically (leaving the dot and cross alone) then the value of the scalar triple product doesn't change. If you permute non−cyclically then the sign changes.

$$\left.\begin{array}{l} p \bullet q \times r \\ r \bullet p \times q \\ q \bullet r \times p \end{array}\right\}\text{same value}$$

$$\left.\begin{array}{l} q \bullet p \times r \\ r \bullet q \times p \\ p \bullet r \times q \end{array}\right\}\text{same value}$$

opposite value

SECTION 5.1 dA and dV

area elements

Start with a u,v coordinate system in the plane.
Sweep out a patch, called an *area element*, by starting at a point P and changing u by du and v by dv. Its area is called dA.

Fig 1 shows the patch in polar coords and Fig 2 shows it in parabolic coords.

Similarly, in a u,v,w coordinate system in 3-space, a *volume element* is the little region swept out by changing u by du, v by dv, w by dw. Its volume is called dV.

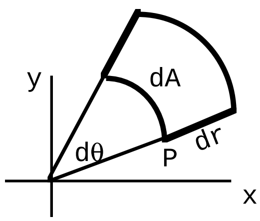


FIG 1

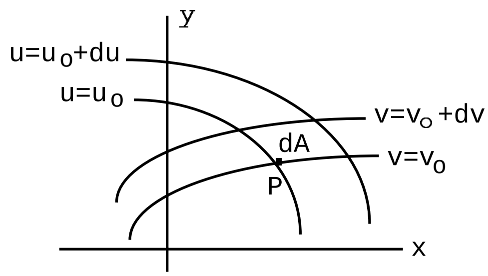


FIG 2

finding dA and dV in orthogonal coordinate systems

If the u,v coordinate system is *orthogonal* then the area element (Fig 3) is (almost) a rectangle with sides $ds_u = h_u\, du$ and $ds_v = h_v\, dv$.

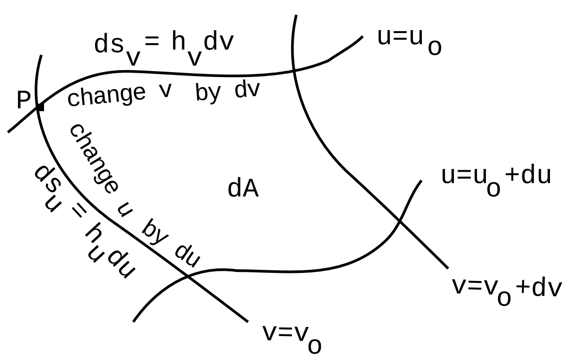


FIG 3

The area of the rectangle is the product of the sides so

(1) $$\boxed{dA = \underbrace{h_u h_v}_{\text{area magnification factor}}\ du\, dv}$$

Similarly, if a u,v,w coordinate system is *orthogonal* then

(2) $$\boxed{dV = \underbrace{h_u h_v h_w}_{\text{volume magnification factor}}\ du\, dv\, dw}$$

example 1 (dA in polar coords)

In polar coords (Fig 4), $h_r = 1$, $h_\theta = r$ so $dA = h_r\, h_\theta\, dr\, d\theta = r\, dr\, d\theta$.

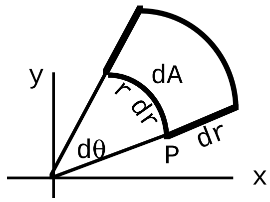


FIG 4

warning

The area and vol mag factors are products of the scale factors *only* in an orthog coord system. Otherwise, use the more general formulas coming up in (5) and (7).

example 2 (dV in spherical coordinates)

Fig 5 shows a volume element in spherical coordinates swept out starting at point A with coordinates ρ, ϕ, θ.

The faces of the box are coordinate surfaces:

The face ABFE lies on a cone with cone angle ϕ.

The opposite face DCGH lies on a cone with cone angle $\phi + d\phi$.

The face ADCB lies on a sphere with radius ρ.

The opposite face EHGF lies on a sphere with radius $\rho + d\rho$.

The face ADHE lies on a half-plane, hinged along the z-axis at angle θ.

The opposite face BCGF lies on a half-plane with angle $\theta + d\theta$.

The edges of the box are ρ-curves, ϕ-curves and θ-curves. For instance, the curve AD is on a ϕ-curve; it was traced out by changing ϕ by $d\phi$ while ρ and θ are fixed.

In spherical coordinates, $h_\rho = 1$, $h_\phi = \rho$, $h_\theta = \rho \sin\phi$.

Length $AE = ds_\rho = h_\rho\, d\rho = d\rho$

Length $AD = ds_\phi = h_\phi\, d\phi = \rho\, d\phi$

Length $AB = ds_\theta = h_\theta\, d\theta = \rho \sin\phi\, d\theta$

$dV = h_\rho\, h_\phi\, h_\theta\, d\rho\, d\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

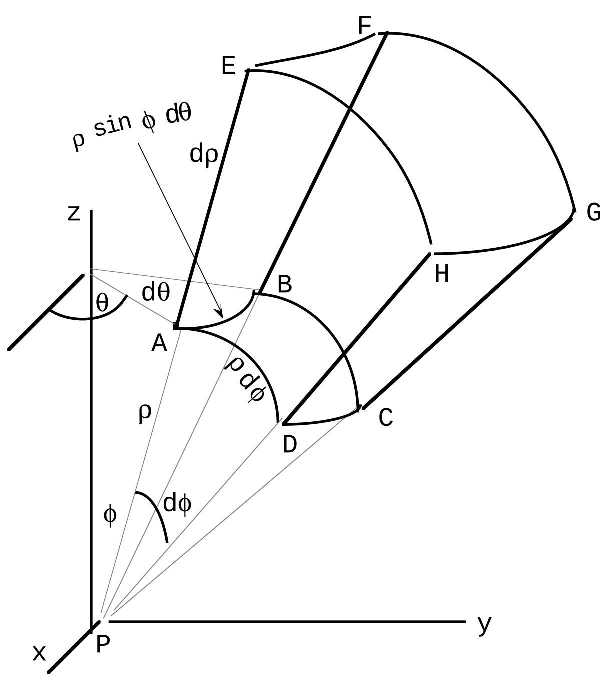


FIG 5

definition of the Jacobian determinant

Given equations $x = x(u,v)$, $y = y(u,v)$. The Jacobian determinant is defined like this:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{3}$$

You can also put the partials w.r.t. u down the first column instead of across the row, and the partials w.r.t. v down the second column instead of across the first row and still get the same determinant. In other words, we also have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Similarly, given equations $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{4}$$

And so on.

example 3

If

$$x = 3v^2 + u + 5w$$
$$y = 2uv$$
$$z = 3vw$$

then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 2v & 0 \\ 6v & 2u & 3w \\ 5 & 0 & 3v \end{vmatrix}$$

Note It doesn't matter whether you start by putting 1 2v 0 across the first row, as I did, or down the first column. You'll get the same answer either way.

I'll expand across row 1 (to take advantage of one of the zero entries):

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 1 \begin{vmatrix} 2u & 3w \\ 0 & 3v \end{vmatrix} - 2v \begin{vmatrix} 6v & 3w \\ 5 & 3v \end{vmatrix}$$

$$= 6uv - 2v(18v^2 - 15vw)$$

$$= 6uv - 36v^3 + 30vw$$

The Jacobian is a *scalar field*. There is a Jacobian value at each point.

dA in a *not necessarily orthogonal* u,v coordinate system

Fig 6 shows an area element in an arbitrary u, v coordinate system.

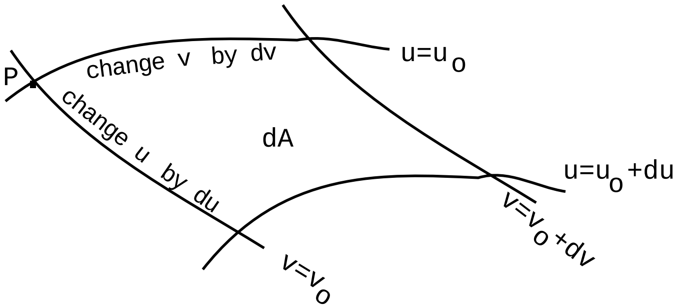


FIG 6

(5) $$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv \qquad \text{(Fig 6)}$$

area magnification factor

letters here match the denom letters

The vertical bars in (5) mean the absolute value of the determinant inside.
The formula in (5) is more general than the formula in (1) because (5) holds whether the coord system is orthogonal or not.

why (5) works

Look at the left edge of the area element in Fig 6. It has parametric equations

$$x = x(u,v_0)$$
$$y = y(u,v_0)$$
$$u_0 \le u \le u_0 + du \qquad \text{(the parameter is u)}$$

By (2) in Section 2.0, the left edge is approximated by arrow

$$du\left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}\right) \qquad \text{(Fig 7)}$$

Similarly the upper edge of the patch is approximated by the arrow

$$dv\left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j}\right)$$

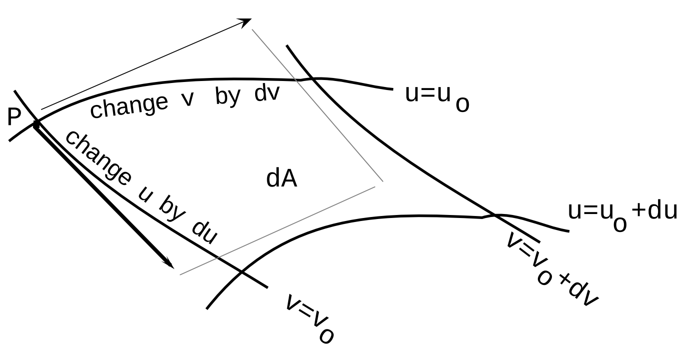


FIG 7

Now you can think of the patch as almost the parallelogram in Fig 7 determined by the two arrows in which case

$$dA = \|\text{cross product of arrows}\| \qquad \text{(the double bars mean norm)}$$

where

$$\text{cross product} = du\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0\right) \times dv\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0\right)$$

$$= du\,dv \left(0,\ 0,\ \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}\right) \quad \text{(vertical bars mean determinant)}$$

$$= du\,dv\ \left(0,\ 0,\ \frac{\partial(x,y)}{\partial(u,v)}\right)$$

The norm of a vector of the form (0,0,k) is $|k|$ (vert bars mean abs value) so

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad \text{(vertical bars mean absolute value)} \quad \text{QED}$$

example 4 (dA in polar coords again)

If $x = r\cos\theta$, $y = r\sin\theta$ then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= |r|\ dr\,d\theta$$

$$= r\,dr\,d\theta \quad \text{(since the polar coord } r \text{ is always} \ge 0)$$

example 5

If a u,v coordinate system is defined by

$$x = u^2 + v, \quad y = 3u + 2v$$

then

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \left|\begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix}\right| du\,dv \quad \text{(outer bars are abs value, inner bars are det)}$$

$$= |4u - 3|\ du\,dv \quad \text{(bars are abs value signs)}$$

warning

Get the absolute value signs right. And be consistent.
Here are some *wrong* ways to write in example 5.

(a) $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix}$ *wrong*

It's wrong because the left side is the *absolute value* of the Jacobian determinant but the right side is the determinant written without absolute values. For a correct version of (a), either write

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} \quad \text{(absolute value signs on } \textit{neither} \text{ side)}$$

or write

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left\| \begin{matrix} 2u & 3 \\ 1 & 2 \end{matrix} \right\| \quad \text{(absolute value signs on } \textit{both} \text{ sides)}$$

(b) $dA = \dfrac{\partial(x,y)}{\partial(u,v)}\, du\, dv$ *wrong*

It's wrong because it leaves out the absolute value signs around the Jacobian. The correct version is in (5):

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

(c) $\begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} = |4u - 3|$ *wrong*

It's wrong because the left side is a determinant without absolute value signs and the right side suddenly includes absolute value signs.

taking absolute values correctly

If Jac $= -6u$ then it is not necessarily true that $|\text{Jac}| = 6u$; i.e., the absolute value of $-6u$ is not necessarily $6u$. If u is negative then $|6u|$ happens to be $-6u$. So unless you know whether u is positive or negative the best you can do is write $|\text{Jac}| = |-6u|$ (same as $|6u|$). On the other hand if Jac $= -6u^2$ then $|\text{Jac}| = 6u^2$ because $6u^2$ is always ≥ 0.

dV in a *not necessarily orthogonal* u,v,w coordinate system

(6)
$$\boxed{dV = \underbrace{\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|}_{\text{volume magnification factor}} du\, dv\, dw \quad \text{(Fig 7)}}$$

The vertical bars in (6) mean the absolute value of the determinant inside.
The formula in (6) is more general than the formula in (2) because (6) holds whether the coord system is orthogonal or not.

why (6) works

The volume element in Fig 8 is (almost) a parallelepiped determined by arrows

$$\text{vel}_u\ du = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}\right) du$$

$$\text{vel}_v\ dv = \left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}\right) dv$$

$$\text{vel}_w\ dw = \left(\frac{\partial x}{\partial w}\vec{i} + \frac{\partial y}{\partial w}\vec{j} + \frac{\partial z}{\partial w}\vec{k}\right) dw$$

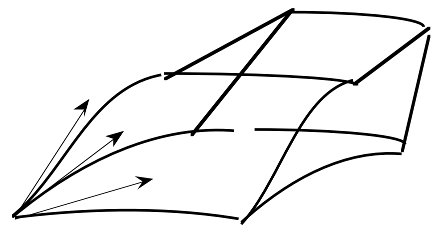


FIG 8

By the rule in Fig 2 in §5.0,

$$dV = |\, \text{vel}_u\, du \cdot \text{vel}_v\, dv \times \text{vel}_w\, dw| \qquad \text{(vert bars are abs value)}$$

$$= |\text{vel}_u \cdot \text{vel}_v \times \text{vel}_w|\ du\, dv\, dw \qquad \text{(pull out the positive scalars)}$$

$$= \left\| \begin{matrix} \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \\ \partial x/\partial w & \partial y/\partial w & \partial z/\partial w \end{matrix} \right\| du\, dv\, dw \qquad \text{by (1) in §5.0}$$

(outer bars are still abs value, inner bars are det signs)

$$= \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw \qquad \text{(vert bars are abs value)} \quad \text{QED}$$

sign of the Jacobian in 3-space

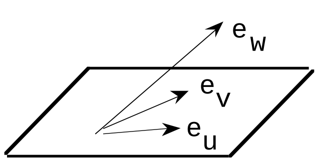
Let

$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$

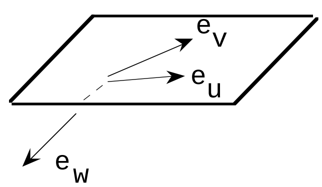
define a u,v,w coordinate system superimposed on the usual (righthanded) x,y,z coord system in 3-space.

Look at the Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

> If the Jacobian is positive at a point then at that point the vectors e_u, e_v, e_w *in that order* are a righthanded triple (Fig 9) meaning that if you curl the fingers of your right hand like e_u turning into e_v, your thumb points more or less like e_w (i.e., makes an acute angle with e_w).
>
> And if the Jac is negative then e_u, e_v, e_w are lefthanded (Fig 10) meaning e_w makes an obtuse angle with your thumb.



pos Jac
FIG 9



neg Jac
FIG 10

Here's why this works.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \\ \partial x/\partial w & \partial y/\partial w & \partial z/\partial w \end{vmatrix}$$

(7) $$= vel_u \cdot vel_v \times vel_w$$

Fig 11 shows a typical vel_v and vel_w and their cross product.
If the Jac is positive then the dot product in (7) is positive and vel_u makes an acute angle with the cross product $vel_v \times ve_w$. I drew a typical such vel_u in the picture.

Now just look at Fig 11 (and curl your fingers etc) to see that the following are all righthanded triples.

vel_v vel_w vel_u
vel_u vel_v vel_w (that's the one I was after)
vel_w vel_u vel_v

footnote
You can look at pictures to see that cylically permuting three vectors doesn't change "handedness". For instance, if p,q,r are a lefthanded triple then so are r,p,q and q,r,p.

So the vectors e_u, e_v, e_w (and cyclic permutations of them) are righthanded because they point like vel_u, vel_v, vel_w (the only difference is that the e's are unit vectors).

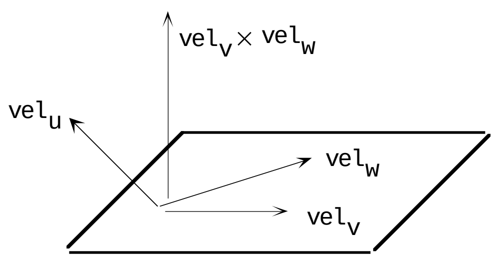


FIG 11

sign of the Jacobian in 2-space

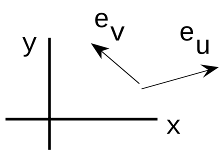
Let

$x = x(u,v)$
$y = y(u,v)$

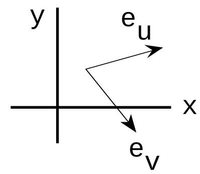
define a u,v coord system superimposed on the usual (righthanded) x,y coord system in 2-space.

Look at the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

> If the Jacobian is positive at a point (Fig 12) then as the fingers of your right hand curl like e_u turning into e_v, your thumb points out of the page.
> If the Jac is negative (Fig 13) as your fingers curl like e_u turning into e_v, your thumb points into of the page.



pos Jac
FIG 12



neg Jac
FIG 13

Here's why this works.
Think of e_u and e_v in Figs 12 and 13 as if they were in the x,y plane in a righthanded x,y,z coord system *in 3-space*. where k comes out of the page at you (Figs 14, 15)

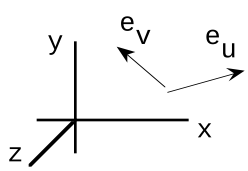


FIG 14

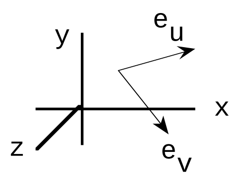


FIG 15

Algebraically this amounts to using the u,v,z cylindrical coord system where

$x = x(u,v)$
$y = y(u,v$
$z = z$

In Fig 14, the vectors e_u, e_v, k are righthanded so $\dfrac{\partial(x,y,z)}{\partial(u,v,z)}$ is positive.

In Fig 15, the vectors e_u, e_v, k are lefthanded so $\dfrac{\partial(x,y,z)}{\partial(u,v,z)}$ is negative.

But you will see in problem #7 that $\dfrac{\partial(x,y,z)}{\partial(u,v,z)} = \dfrac{\partial(x,y)}{\partial(u,v)}$.

So back in Fig 12, $\dfrac{\partial(x,y)}{\partial(u,v)}$ is positive and back in Fig 13, $\dfrac{\partial(x,y)}{\partial(u,v)}$ is negative.

warning

Order counts!

$$\frac{\partial(x,y,z)}{\partial(a,q,m)} \text{ means } \begin{vmatrix} \partial x/\partial a & \partial y/\partial a & \partial z/\partial a \\ \partial x/\partial q & \partial y/\partial q & \partial z/\partial q \\ \partial x/\partial m & \partial y/\partial m & \partial z/\partial m \end{vmatrix}.$$

The derivatives w.r.t. a are in the first row (or col), derivatives w.r.t. q are in the second row (or col) and derivatives w.r.t. m are in the third row (or col).
If this Jacobian is positive then arrows e_a, e_q, e_m *in that order* will be righthanded.

Jacobians that use different orders, such as

$$\frac{\partial(x,y,z)}{\partial(q,a,m)}, \quad \frac{\partial(x,y,z)}{\partial(m,q,a)}, \quad \frac{\partial(x,y,z)}{\partial(m,a,q)},$$

all have the same absolute value, and no matter what order you use it is still true that

$$dV = |\text{Jacobian}|\ da\ dq\ dm$$

but the signs of the Jacobians may differ, depending on how you scramble the rows or cols of the Jacobian determinant.

the Jacobian in an *orthogonal* coord system

Suppose the u,v coord system is *orthogonal*.
Then the area mag factors in (1) and (5) must agree. So

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = h_u\, h_v$$

Furthermore, the Jacobian is positive if the coord system is righthanded and negative if the coord system is lefthanded so all in all

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{cases} h_u h_v & \text{if the system is righthanded} \\ -h_u h_v & \text{if the system is lefthanded} \end{cases}$$

Similarly, suppose the u,v,w coord system is orthogonal.
Then the vol mag factors in (2) and (6) must agree. So

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = h_u\, h_v\, h_w$$

In particular,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{cases} h_u h_v\, h_w & \text{if the system is righthanded} \\ -h_u h_v\, h_w & \text{if the system is lefthanded} \end{cases}$$

PROBLEMS FOR SECTION 5.1

1. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ for the parabolic coordinate system two ways.

2. Let $x = u + v$, $y = u - v$.
(a) Sketch a piece of u,v coord paper.
(b) Find e_u and e_v.
(c) Is the u,v system orthogonal?
(d) Find $\frac{\partial(x,y)}{\partial(u,v)}$
(e) Sketch an area element and find its area.
(f) Is the u,v system lefthanded or righthanded.

3. Let $x = 3u^2 + 7v$, $y = 2v$.
(a) Is the u,v coord system orthogonal.
(b) Find dA in the u,v system.
(c) Is the u,v coord system lefthanded or righthanded.

4. Start at a point P with cylindrical coordinates r, θ, z.
(a) Sketch the volume element swept out by changing r by dr, θ by dθ and z by dz. Label P, r, θ, z, dr, dθ, dz in the picture and find dV.
(b) Suppose r changes by dr and z change by dz while θ stays fixed. What is traced out (draw a picture) and what is its length/area/volume.

5. (boring) Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = v^3w$, $y = 2u + 3v + 4w$, $z = uvw$.

6. Define a u,v coordinate system with the equations

$x = x(u,v)$
$y = y(u,v)$

Call it the blue coord system.
Then the blue *cylindrical* coord system is defined by

$x = x(u,v)$
$y = y(u,v)$
$z = z$

Start computing $\frac{\partial(x,y)}{\partial(u,v)}$ and $\frac{\partial(x,y,z)}{\partial(u,v,z)}$ and just go far enough to show that they are equal.

SECTION 5.2 MAPPINGS

mappings from a u,v plane to an x,y plane

I've been thinking of equations of the form

$$x = x(u,v), \quad y = y(u,v)$$

as relating the usual Cartesian x,y coordinate system with a new u,v coordinate system all in the same plane.

They can also be thought of as a mapping from a plane with a u,v Cartesian coordinate system to a plane with an x,y Cartesian coordinate system.

For example, from one point of view the equations

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad v \ge 0, \tag{1}$$

define parabolic coordinates. Now think mapping! Point u=2, v=1 (Fig 1a) maps to the image point $x = 3/2$, $y = 2$ (Fig 1b). The half-line $u = 2$, $v \ge 0$ (Fig 2a) maps to the curve in the x,y plane with parametric equations

$$x = \frac{1}{2}(4 - v^2), \; y = 2v, \; v \ge 0,$$

This curve has plain equation $x = \frac{1}{2}(4 - \frac{1}{4}y^2)$ (Fig 2b). It's the top half of a parabola. The half–parabola is called the *image* of the half–line; the half–line is called the *pre-image* of the half–parabola.

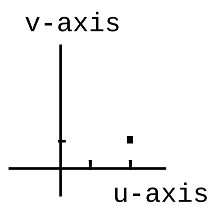


FIG 1a

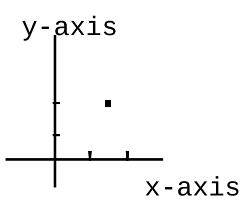


FIG 1b

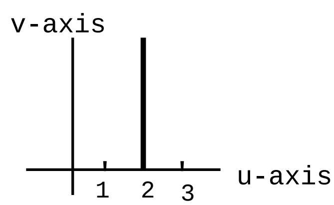


FIG 2a

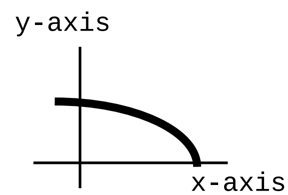


FIG 2b

area and volume magnification factors

This is just a restatement of ideas in the preceding section from a mapping point of view.

Think of

$$x = x(u,v)$$
$$y = y(u,v)$$

as a mapping from a u,v plane to an x,y plane.

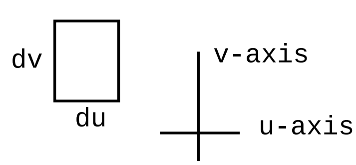
Create a rectangle in the u,v plane by changing u by du and v by dv (Fig 3a)

The rectangle has area du dv.

In the image world (Fig 3b), the image area dA is given by

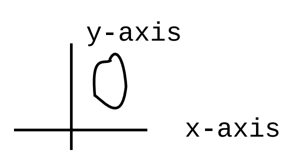
$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv$$

The value of the Jacobian changes from point to point so its absolute value is a "local" mag factor only. Two du dv patches in different parts of the old plane may be magnified differently when they are mapped to the new plane.



area is du dv

FIG 3a



$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv$$

FIG 3b

Similarly think of

$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$

as mapping from a u,v,w space to an x,y,z space

Make a little box in a u ,v, w world by making changes du, dv, dw. The box has volume du dv dw.

Its image in x,y,z space has volume dV where

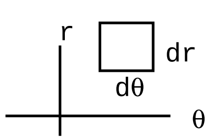
$$dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\ dv\ dw$$

example 1

The equations $x = r\cos\theta$, $y = r\sin\theta$ can be thought of as mapping from an r,θ plane to an x,y plane.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

The area mag factor is $|r| = r$. The little rectangle with area $dr\, d\theta$ in the old plane (Fig 4a) maps to a patch with area $r\, dr\, d\theta$ in the new plane (Fig 4b).



area $dr\, d\theta$

FIG 4a

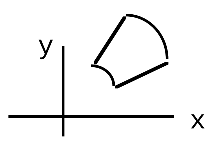


image area $r\, dr\, d\theta$

FIG 4b

example 2

Here's a clever way to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = R^2$. Let

$$u = \frac{x}{a}, \quad v = \frac{y}{b}$$

where a and b are the positive square roots of a^2 and b^2 respectively.

These equations map *from* an x,y, plane *to* a u,v plane. The ellipse in the x,y plane maps to the circle $u^2 + v^2 = R^2$ in the u,v plane.

Rewrite the equations as

$$x = au, \quad y = bv$$

which map *from* a u,v plane *to* an x,y plane. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |ab| = ab$$

The area mag factor is ab and it's *constant* (same magnification at every point) so *any* region in the u,v plane should be magnified by ab when it is mapped to the x,y plane.

Circle area in the u,v plane $\times$ ab = ellipse area in the x,y plane.

Ellipse area $= ab \times \pi R^2 = \pi ab R^2$

footnote

You can stick with the original mapping equations $u = x/a$, $v = y/b$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/a & 0 \\ 0 & 1/b \end{vmatrix} = \frac{1}{ab}$$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{1}{ab}\right| = \frac{1}{ab} \quad \text{(because a and b are positive)}$$

The Jacobian is constant so regions in the x,y plane are magnified by 1/ab when they are mapped to the u,v plane.

Ellipse area in x,y plane $\times \frac{1}{ab}$ = circle area in u,v plane

Ellipse area = ab $\times$ circle area = ab $\times \pi R^2 = \pi abR^2$

sign of the Jacobian of a mapping from a u,v plane to an x,y plane

Let

(2) $$x = x(u,v), \quad y = y(u,v)$$

define a mapping from a plane with a Cartesian u,v coord system to a plane with a Cartesian x,y coordinate system.

There are two ways to walk around a point in any plane –– clockwise and ccl.

Look at the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

> Let P be a point in the u,v plane and let P' be its image in the x,y plane. Walk on a small loop around P .
>
> If the Jacobian is positive at P then the image walker goes the same way around P' (i.e., either both walkers go clockwise or both go ccl). We say orientation is preserved.
>
> If the Jac is negative at P then the image walker goes the opposite way around P' (if one goes clockwise then the other goes ccl). We say orientation is reversed.

Here's why.

Suppose P has coordinates u_o, v_o in the u,v plane.

Look at the closed counterclockwise path PABCP in Fig 5.

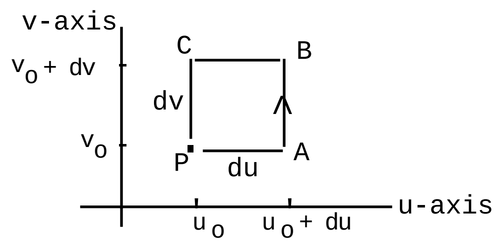


FIG 5

I'll find the image path in the x,y plane. Think of the equations in (2) as determining a u, v coordinate system in the x, y plane (Fig 6). The image point P' is the intersection of the v-curve $u = u_o$ and the u-curve $v = v_o$. I happened to draw P' in quadrant II –– doesn't matter where it is. I also drew the hypothetical v-curve $u = u_o + du$.

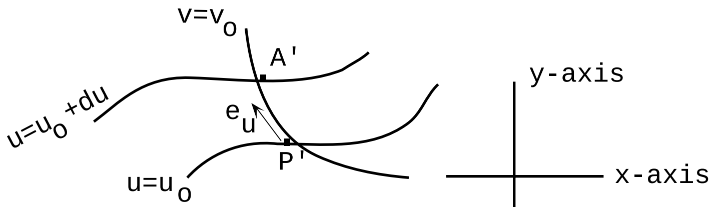
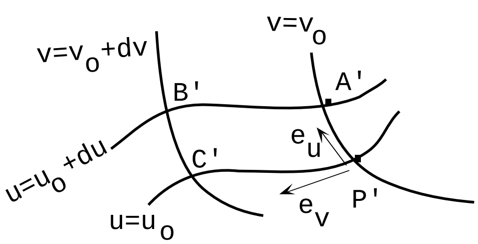


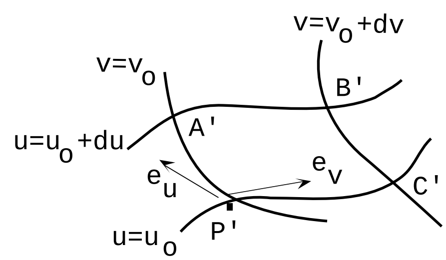
FIG 6

The image walker goes from P' to A' in Fig 6, in the direction of increasing u corresponding to the PA path in Fig 5, and then turns in the direction of increasing v corresponding to path AB in Fig 5. But does she turn left or right from A'.

Consider the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ at point $u=u_o$, $v=v_o$, i.e., at point P'. If it's positive then e_v must be ahead of e_u (see the preceding section), Fig 7 fleshes out to Fig 6 and the image walker turns left at A' to go in the increasing v direction If the Jac is negative at P' then e_u must be ahead of e_v, Fig 6 becomes Fig 8 and the image walker turns right at A' to go in the increasing v direction.



Positive Jac
FIG 7



Negative Jac
FIG 8

So if the Jac is positive the image path P'A'B'C'P' is also ccl (orientation is preserved)
But if the Jac is negative the image path P'A'B'C'P' is clockwise (orientation is reversed). QED

example 3

Let $x = u^2 + v^2$, $y = u^3 v$. This is a mapping from a u,v plane to an x,y plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 3u^2 v \\ 2v & u^3 \end{vmatrix} = 2u^4 - 6u^2 v^2.$$

If u=1, v=2 the Jacobian is −22. A small region around point u=1, v=2 in the u,v plane maps to a region around point x=5, y=2 with about 22 times as much area. And if a particle moves clockwise around the point u=1, v=2, its image moves ccl around the image point x=5, y=2 in the x,y plane.

inverse Jacobians

Suppose the equations

$$x = x(u,v), \quad y = y(u,v) \tag{3}$$

can be solved for u and v to get equations

$$u = u(x,y), \quad v = v(x,y). \tag{4}$$

Then

(5) $$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}}$$

why the inverse rule in (5) works

The equations in (3) define a mapping from a u,v plane to an x,y plane. The equations in (4) define the inverse mapping from the x,y plane back to the u,v plane.

Suppose $\frac{\partial(x,y)}{\partial(u,v)}$ is -7 at point (u_0, v_0). Then a region in the u,v plane around point (u_0,v_0) gets magnified sevenfold when it is mapped to a region around the image point (x_0,y_0) in the x,y plane, and orientations are reversed (e.g., a clockwise walk maps to a ccl walk). So the inverse mapping which takes a region around (x_0, y_0) back to a region around (u_0, v_0) should also reverse orientation (it maps a ccl walk back to a clockwise walk) and should shrink area to 1/7-th its original size. So at point (x_0, y_0) you should have $\frac{\partial(u,v)}{\partial(x,y)} = -\frac{1}{7}$

example 4

Find $\frac{\partial(x,y)}{\partial(u,v)}$ if $u = x^3 + 2y$, $v = x^3$.

method 1 First solve for x and y:

(6) $$x = \sqrt[3]{v}\ ,\ y = \tfrac{1}{2}(u - v)$$

Then find the Jacobian directly.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3}v^{-2/3} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{6}\,v^{-2/3}$$

method 2

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3x^2 & 3x^2 \\ 2 & 0 \end{vmatrix} = -6x^2$$

(7) $$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-6x^2}$$

The answers look different but if you replace the x in (7) by $\sqrt[3]{v}$ from (6) you'll see that they agree.

PROBLEMS FOR SECTION 5.2

1. Let $u = 2x + y$, $v = 3x - 4y$. Find $\frac{\partial(x,y)}{\partial(u,v)}$ twice.

(a) Without solving for x and y explicitly
(b) By solving for x and y explicitly

2. Let $u = x^2 - y^2$, $v = x^2 + y^2$, $x \geq 0$, $y \geq 0$.

(a) Find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

(b) Solve the equations for x and y to get the equations for the inverse mapping. Then use the inverse equations to find $\dfrac{\partial(x,y)}{\partial(u,v)}$ directly to see if it really is the reciprocal of the Jacobian in part (a).

3. Let $x = u + v - 2w^3$, $y = u + 2v$, $z = v$.

(a) Find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

(b) Use part (a) to find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at point $x=61$, $y=12$, $z=5$.

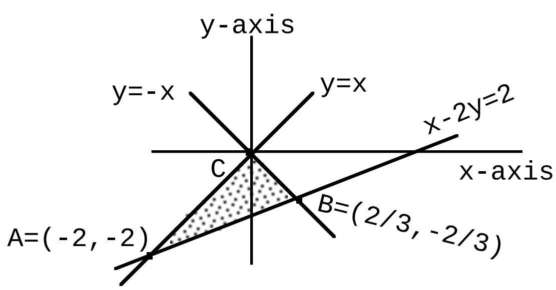
4. The equations $u = 2x - y$, $v = x - 2y$ define a mapping from the x,y plane to a u,v plane.

The diagram shows a triangular region ABC in the x,y plane.

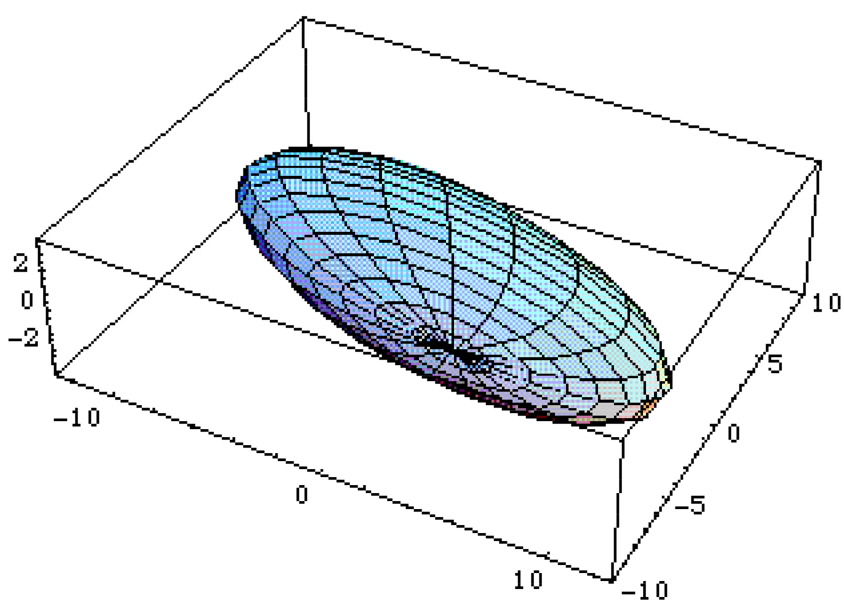
(a) Find the area of the region ABC

(b) Use Jacobians to find the area of the image region in the u,v plane.

(c) Actually find the image of the region and then find its area directly to see if the answer agrees with part (b).



5. Find the volume inside the surface $(x + y + z)^2 + (2y + 5z)^2 + 9z^2 = R^2$ by choosing a mapping that turns the surface into a sphere.



6. For each of these mappings, predict the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ by thinking about magnification and orientation and then compute the Jacobian to confirm your prediction.

(a) $x = u + 2$, $y = v - 1$

(b) $x = -u$, $y = v$

7. When you compute $\frac{\partial(p,q)}{\partial(r,s)}$ directly what letters do you expect to find in your answer; i.e., is the Jacobian a function of p and q or is it a function of r and s. Or all four?

SECTION 5.3 DOUBLE AND TRIPLE INTEGRALS IN A NEW COORDINATE SYSTEM

double integration in a (not necessarily orthogonal) u,v coordinate system

Here's how to find $\int_{x,y\text{ , region}} f(x,y)\; dA$ using a new u,v coordinate system.

(I) Use the equations relating x and y with u and v to replace all x's and y's in the integrand with u's and v's.

(II) Instead of using dA = dx dy, use the area element created by changing u by du and v by dv:

$$\boxed{dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\; dv}$$

In the special orthog case, $dA = h_u\, h_v\; du\; dv$.

(III) Put in u,v limits that sweep out the region; i.e.,

$$\int_{\text{smallest v}}^{\text{largest v}} \int_{\text{entering u boundary}}^{\text{exiting u boundary}} \qquad \text{or} \qquad \int_{\text{smallest u}}^{\text{largest u}} \int_{\text{entering v boundary}}^{\text{exiting v boundary}}$$

Integrating in u,v coordinates is also referred to as making a *change of variables* or *substitution* .
A similar idea works for triple integrals.

example 1

I'll find $\int_{\text{region in Fig 1}} x^2\, y^2\; dA$.

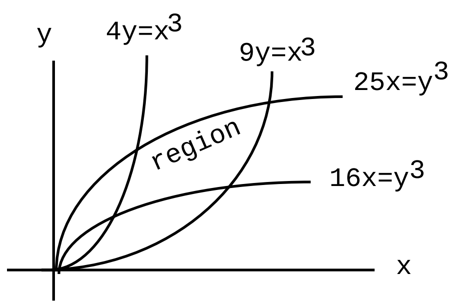


FIG 1

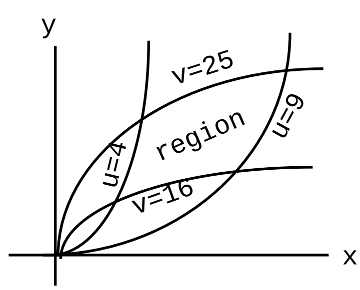


FIG 2

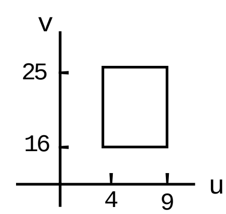


FIG 3

The region of integration is messy in x,y coordinates because the lower and upper boundaries each consist of two curves, and the left and right boundaries each consist of two curves. Another method is to use a u,v coordinate system defined by

$$(1) \qquad u = \frac{x^3}{y}\,, \quad v = \frac{y^3}{x}$$

because in this system the region of integration is the "rectangle" where

$$4 \le u \le 9, \quad 16 \le v \le 25 \text{ (Fig 2)}.$$

In fact, instead of thinking of a u,v coordinate system you can think of the equations in (1) as mapping from an x,y plane to a u,v plane. The image of the region in Fig 1 is literally the rectangle in Fig 3.

Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3x^2/y & -y^3/x^2 \\ -x^3/y^2 & 3y^2/x \end{vmatrix} = 8xy$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy} \quad \text{by the inverse rule}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{8xy}\right| = \frac{1}{8xy} \quad \text{since } 8xy \text{ is positive in the region}$$

$$\int_{\text{region}} x^2\, y^2 \, dA = \int_{u=4}^{9} \int_{v=16}^{25} x^2 y^2 \, \frac{1}{8xy} \, dv\, du$$

$$= \int_{u=4}^{9} \int_{v=16}^{25} \frac{1}{8}\, xy \, dv\, du$$

From (1), $uv = x^2 y^2$ so $xy = \sqrt{uv}$ (take the *positive* square root since $xy \geq 0$ in the region). So

$$\int_{\text{region}} x^2 y^2 \, dA = \frac{1}{8} \int_{u=4}^{9} \int_{v=16}^{25} \sqrt{uv}\, dv\, du$$

$$\text{inner integral} = \sqrt{u}\, \frac{2}{3}\, v^{3/2} \Big|_{16}^{25} = \frac{2}{3}\, 61\sqrt{u}$$

$$\text{outer integral} = \frac{1}{8}\, \frac{2}{3}\, 61\, \frac{2}{3}\, u^{3/2} \Big|_{4}^{9} = \frac{1159}{18}$$

warning

1. When you integrate in a u,v coord system, dA is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ du dv, not $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ du dv. One way to remember the correct version is to picture it "canceling" like this:

$$\left|\frac{\partial(x,y)}{\cancel{\partial(u,v)}}\right| \cancel{du\, dv}$$

2. And don't forget the absolute value signs around the Jacobian.

3. When you put in u,v limits of integration, a lower limit of integration should always be smaller than the corresponding upper limit.

For instance you should *not* have $\int_{v=6}^{4}$.

4. In example 1, the substitution $u=x^3$ would not be useful.
The substitution $u = x^3/y$ was useful because two of the bounding curves of the region have equations of the form x^3/y = constant.
One way to spot a potentially useful substitution is to rewrite the equations of the bounding curves so that all the letters are on one side.

5. $\int_{\text{region in Fig 1}} x^2\, y^2\, dA$. is *not* the area of the region.

Only $\int_{\text{region in Fig 1}} 1\, dA$ is the area of the region.

So *when you compute a double integral,. don't refer to it as an "area".*

footnote So what good *is* the integral in example 1.
If you insist on a geometric interpretation the integral is the volume in 3-space under the graph of $z = x^2y^2$ and over the region in Fig 1. (Well, it's actually the volume above the x,y plane minus the volume below the x,y plane but in this case the graph happens to lie entirely above the x,y plane so the integral really is a plain volume.)

The integral is also for instance the total mass of the region if its density is x^2y^2 grams per square cm.

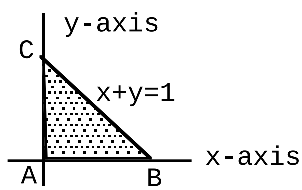
PROBLEMS FOR SECTION 5.3

1. Switch to a good u,v coord system and find $\int (x^2 - y^2)\, dA$ over the region in quadrant III bounded by $y = x + 3$, $y = x - 3$, $xy = 16$, $xy = 4$.

2. Switch to a useful u,v coord system and set up $\int x^2\, dA$ over the region bounded by lines $y = 2x + 3$, $y = 2x - 5$, $y = -x + 1$, $y = -x + 6$.
(If you do the integral directly, you have to divide the region of integration into three parts.)

3. Make a substitution (switch to u's and v's) and find $\int \sin xy\, dA$ over the region in quadrant I bounded by the curves $xy = 2$, $xy = 1$, $y = x$ and $y = 4x$.

4. Make a substitution to set up $\int e^{(y-x)/(y+x)}\, dA$ over the region ABC in the diagram.



footnote Mathematica can do the original integral directly so a substitution is not necessary. We're just practicing.

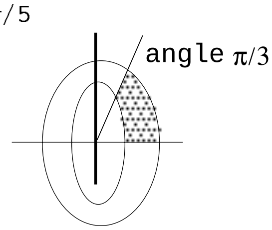
```
In[24]:=
Integrate[ Exp[(y - x)/(y + x)], {x, 0, 1}, {y, 0, 1 - x}]
Out[24]=
      2
-1 + E
-------
   4 E
```

5. Set up $\int xy\, dA$ over the region in the diagram between the ellipses

$$\frac{x^2}{4} + \frac{y^2}{25} = 1 \quad \text{and} \quad \frac{x^2}{4} + \frac{y^2}{25} = 2$$

(a) using the coordinate system defined by
$x = 2r\cos\theta$, $y = 5r\sin\theta$, $r \ge 0$, $0 \le \theta \le 2\pi$

(b) using the substitution $u = x/2$, $v = y/5$

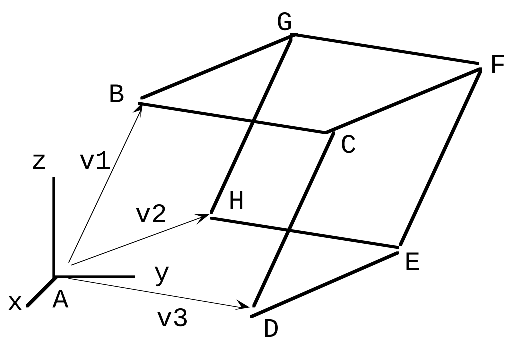


6. Look at $\int \frac{1}{x+2y}\, dA$ over the region bounded by the axes and lines $x + 2y = 4$, $x + 2y = 1$.
(a) Set it up in the x,y system.
(b) Set it up in some nice u,v coordinate system

7. The diagram shows the parallelepiped determined by the arrows

$$\vec{v}_1 = (-1,0,4), \quad \vec{v}_2 = (1,4,-2), \quad \vec{v}_3 = (-2,1,3)$$

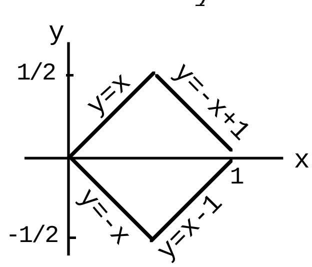
attached to the origin. Switch to a convenient coord system and set up $\int y^2\, dV$ over the solid parallelepiped.
Remember (Section 1.0) that if a plane has normal vector (a,b,c) and goes through point (x_0,y_0,z_0) then the plane has equation $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.



8. Let N be a positive integer.
Look at $\int (x^2 - y^2)^N\, dA$ over the region in the diagram. It can be set up in x,y coordinates if you divide up the region of integration, say into a top and bottom half:

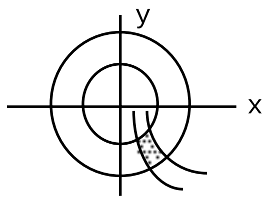
$$\int (x^2 - y^2)^N\, dA = \int_{y=-1/2}^{0} \int_{x=-y}^{y+1} (x^2 - y^2)^N\, dx\, dy + \int_{y=0}^{1/2} \int_{x=y}^{1-y} (x^2 - y^2)^N\, dx\, dy$$

But when I tried to do these integrals with Mathematica I found that it could do the integration for any *specific* value of N, but not for the abstract N. So find some new u,v coord system in which you can actually compute the integral.



REVIEW PROBLEMS FOR CHAPTER 5

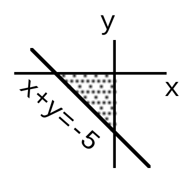
1. The region in the diagram is bounded by circles $x^2 + y^2 = 10$, $x^2 + y^2 = 11$ and hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 2$. Switch to a convenient coordinate system to set up $\int x \, dA$ over the region so that you can avoid having to split up the region of integration.



2. Let n and m be positive integers. Then

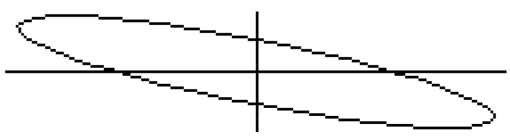
$\int (x + y)^n (3x - y)^m \, dA$ over the region in the diagram

$$= \int_{x=-3}^{0} \int_{y=-3-x}^{0} (x + y)^n (3x - y)^m \, dy \, dx$$



But Mathematica couldn't do this integration.
Set it up with a substitution.

3. Find the area enclosed by the curve $(x+y)^2 + (x+6y)^2 = 100$

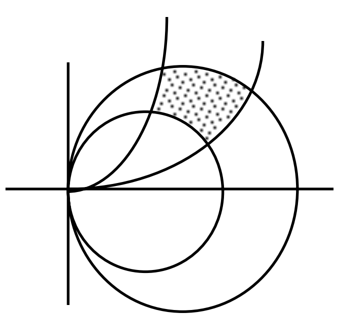


4. Look at the region R in the diagram, bounded by parabolas $y=5x^2$, $y=x^2$, the circle with radius 2 and center (2,0) and the circle with radius 1 and center (1,0).

(a) Find a u,v coordinate system that turns the region into a "rectangle".

(b) Make up an integrand so that your class (of dopes) can do $\int_R \ldots \, dA$ over the region easily with the substitution in part (a).

(c) Find some more rigged integrands so that $\int_R \ldots \, dA$ can be done easily.



5. Let $x = u^2 - v$, $y = uv$ define a new u,v coordinate system.
If u starts at 0 and changes by du, v starts at -10 and changes by dv, how much area is swept out in the x, y plane.

6. Since the spherical coord system is righthanded, $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$ should come out to be $h_\rho\, h_\phi\, h_\theta$. Compute the Jacobian to check.

7. Fill in the blanks.

(a) I have an x,y Cartesian coordinate system as usual.
If $x = x(u,v)$, $y = y(u,v)$ and u changes by du and v changes by dv then the _____ traced out is ______.

(b) If $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ and u changes by du and v changes by dv and w changes by dw then the _____ traced out is ______ .

SUMMARY OF MAG FACTORS FROM CHAPTERS 2,3,5 (REVIEW THIS FOR THE FINAL EXAM)

- Let
  $x = x(u,v)$
  $y = y(u,v)$

These equation can be thought of as mapping from a plane with a u,v Cartesian coordinate system to a plane with an x,y Cartesian coord system.
They can also be thought of defining a u,v coord system on top of the usual Cartesian coord system in an x,y plane.

If u changes by du and v changes by dv then an area element is swept out in the x,y plane. Its area is given by

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Special case: If the u,v coord system is orthogonal then $dA = h_u h_v \, du\, dv$

- Let
  $x = x(u,v,w)$
  $y = y(u,v,w)$
  $z = z(u,v,w)$

These equation can be thought of as mapping from a space with a u,v,w Cartesian coordinate system to a space with an x,y,z Cartesian coord system.
They can also be thought of defining a u,v,w coord system on top of the usual Cartesian coord system in an x,y,z space.

If u changes by du, v changes by dv and w changes by dw then a volume element is swept out in the x,y,z space. Its volume is given by

$$dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

Special case: If the u,v,w coord system is orthog then $dV = h_u h_v h_w \, du\, dv\, dw$

- Let
  $x = x(t)$
  $y = y(t)$

These are parametric equations of a curve in 2-space.
If t changes by dt then a little curve you could call an arclength element is swept out. Its arclength is given by

$$ds = \|v\|\, dt \text{ where } v = (dx/dt,\ dy/dt)$$

- Let
  $x = x(t)$
  $y = y(t)$
  $z = z(t)$

These are parametric equations of a curve in 3-space.
If t changes by dt then an arclength element is swept out. Its arclength is given by

$$ds = \|v\|\, dt \text{ where } v = (dx/dt,\ dy/dt,\ dz/dt)$$

- Let

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

These are parametric equations of a *surface* in 3-space (there are *three* variables, x, y, z and *two* parameters u,v). If u changes by du and v changes by dv then a surface area element is traced out *on* the surface. Its surface area is given by

$$dS = \|n\|\ du\ dv \text{ where } n = vel_u \times vel_v$$

Special cases: See Sections 4.3 and 4.4

PROBLEMS

1. Let

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

If u changes by du while v is fixed what geometric thing is traced out? Where? And what is the size of the geometric thing?

2. Let

$$x = x(u,v)$$
$$y = y(u,v)$$

If u changes by du while v is fixed then what geometric thing is traced out? Where? And what is the size of the geometric thing?

3. Let

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

If ρ changes by $d\rho$, θ changes by $d\theta$, ϕ changes by $d\phi$ then what geometric thing is traced out. Draw a picture of it and find its size.

4. Let

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

If ρ change by $d\rho$ and θ changes by $d\theta$, what geometric thing is traced out. Where? Draw a picture of it and find its size.

5. Let

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

If ρ changes by $d\rho$ and ϕ changes by $d\phi$ what geometric thing is traced out. Where? Draw a picture of it and find its size.

6. Let

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

If θ changes by $d\theta$ and z changes by dz then what geometric thing is traced out. Where? Draw a picture of it and find its size.

SOLUTIONS Section 1.0

1. (a) $(-11, -12, 27)$ (b) $-17\vec{i} + 13\vec{j} + \vec{k}$ (c) $(0,0,21)$
2. -35

3. Let $\vec{u}$ and $\vec{v}$ be unit vectors. By one of the properties of cross products,

$$\|u \times v\| = \|u\|\ \|v\|\ \sin\theta$$

But $\|u\| = \|v\| = 1$ since u and v are unit vectors.
And $\theta = 90^{\circ}$ since u and v are perp so $\sin\theta = 1$.
So $\|u \times v\| = 1\cdot 1\cdot 1 = 1$. QED

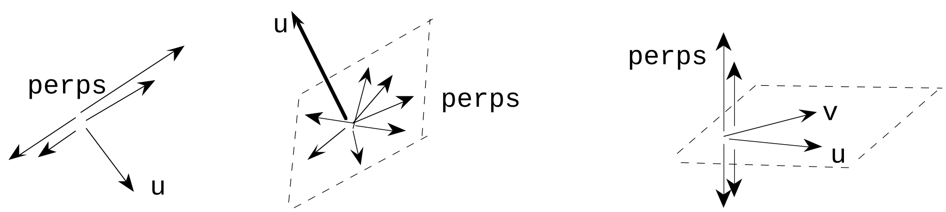
4. (a) Anything which dots with u to give 0 will be a perp.

The two vectors $u_{\text{left turn}} = (7,2)$ and $u_{\text{right turn}}$ $(-7,-2)$ are perps. So are any multiples of them, such as $(14,4)$, $(-7\pi,-\pi)$ etc. There are infinitely many possibilities but all the possibilities are multiples of one another so in a sense there is only one basic answer. All the perps to u point along the same line in 2-space.

(b) Anything which dots with u to give 0 will be a perp, e.g., $(7,2,0)$ and $(-7,-2,0)$. But there are many more possibilities. For instance you could choose $x=5$, $y=-13$ and find the corresponding z like this: $2\cdot 5 - 7\cdot -13 + 3z = 0$, $z = -101/3$, so a perp is $5i-13j - \frac{101}{3}k$.

There are infinitely many possibilities (any arrow on the floor is perp to a person standing on the floor).

(c) One possibility is $u \times v = -13i + 5j - 3k$. There are infinitely many other possibilities but they are just all the multiples of $-13i+5j-3k$.



4(a) 4(b) 4(c)

5. $\overrightarrow{AB} = 3i + j$

u points like $\overrightarrow{AB}$ but has length 2 so $u = 2(3i+j)_{\text{unit}} = \frac{6}{\sqrt{10}}\, i + \frac{2}{\sqrt{10}}\, j$.

v has the same length as u so $v = u_{\text{left turn}} = -\frac{2}{\sqrt{10}}\, i + \frac{6}{\sqrt{10}}\, j$.

There are several ways to get w.
The direction of w is opposite to v. Equivalently the direction of w is $u_{\text{right turn}}$.
And w has length 7. So
$w = 7(i-3j)_{\text{unit}} = \frac{7}{\sqrt{10}}\, i - \frac{21}{\sqrt{10}}\, j$

6. (a) $x=2$, $y=3$, $z=t$ (i.e., z = anything)
(b) $z = 4$

7. (a) The old p has length $\sqrt{2}$ so the new p also has length $\sqrt{2}$. And the new p points like q. By (3),

$$\text{new } p = \sqrt{2}\, q_{\text{unit}} = \sqrt{2}\left(\frac{2}{\sqrt{40}}\, i + \frac{6}{\sqrt{40}}\, j\right) = \frac{2}{\sqrt{20}}\, i + \frac{6}{\sqrt{20}}\, j.$$

(b) The component of p in direction of q is $\frac{p \cdot q}{\| q \|} = \frac{-4}{\sqrt{40}}$ so the vector projection has length $4/\sqrt{40}$. And the vector projection points like $-q$. By (3),

$$\text{vector projection} = \frac{4}{\sqrt{40}}\,(-q)_{\text{unit}} = \frac{4}{\sqrt{40}}\left(-\frac{2}{\sqrt{40}}\,\vec{i} - \frac{6}{\sqrt{40}}\,j\right) = -\frac{8}{40}\,i - \frac{24}{40}\,j$$

8. (a) The "signed projection" of v onto itself is just $\|v\|$ which is $\sqrt{14}$.

(b) Comp of v in direction of v $= \frac{v \cdot v}{\| v \|} = \frac{\|v\|^2}{\| v \|}$ (dot property) $= \|v\|$

SOLUTIONS Section 1.1

1. (a) Each non-zero level set is a hyperbola (with two branches)

The 0 level set is $x^2 - y^2 = 0$, $y = \pm x$, a pair of lines (the asymptotes of all the hyperbolas).

(b) The 100 level is $\dfrac{1}{x+y} = 100$, the line $x + y = \dfrac{1}{100}$

The $-\frac{1}{7}$ level is $\dfrac{1}{x+y} = -\dfrac{1}{7}$, the line $x + y = -7$ etc.

Can't have $\dfrac{1}{x+y} = 0$ so there is no 0 level set.

(c) e^{x+y} is always positive so there are no zero or negative level sets.

If $C > 0$, the level set $e^{x+y} = C$ is the line $x + y = \ln C$ (slope -1 and y-intercept $\ln C$).

(d) x^2+y^2 is always ≥ 0, $e^{x^2+y^2}$ is always ≥ 1 so the levels begin at 1.

If $C \geq 1$, the level set $e^{x^2+y^2} = C$ is the circle $x^2+y^2 = \ln C$, center at the origin and radius $\sqrt{\ln C}$.

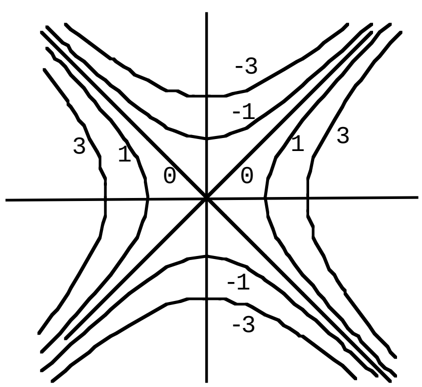
(e) If $C \neq 0$ then $3xy = C$ is a hyperbola. The zero level set is $3xy = 0$, the x-axis and y-axis combined.

(e) The 0 level set is $y^2 = 0$, the x-axis.

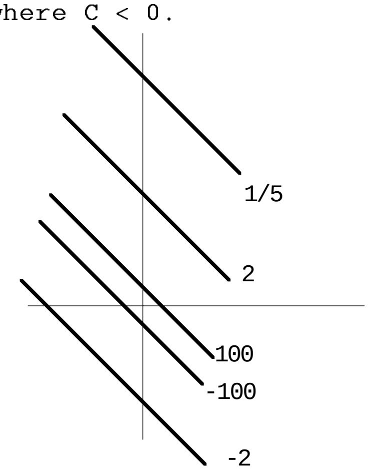
The level set $y^2 = C$ for $C > 0$ is the pair of lines $y = \pm\sqrt{C}$.

For example, the 8 level set is the pair of lines $y = \pm\sqrt{8}/$

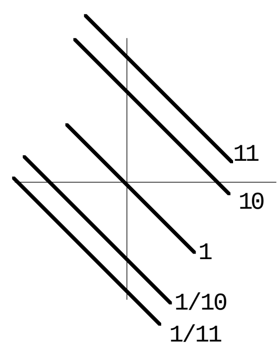
There are no level sets $y^2 = C$ where $C < 0$.



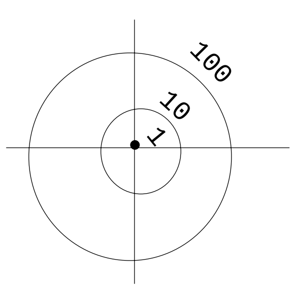
(a)



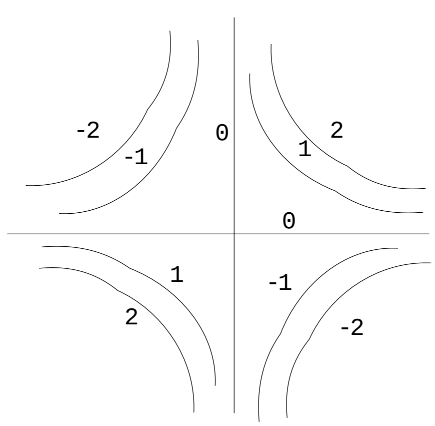
(b)



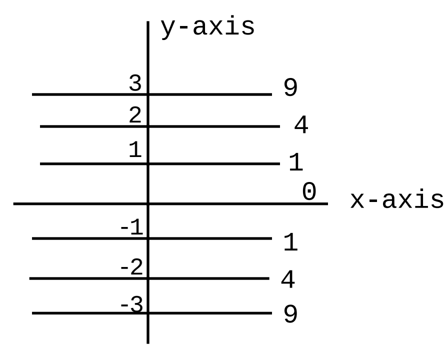
(c)



(d)



(e)



(f)

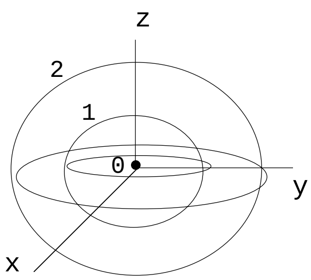
2. (a) The C level set is a sphere with center at the origin and radius $\sqrt{C}$ if $C > 0$, a point if $C = 0$, and empty if $C < 0$.

(b) The 6 level set for instance is $y^2 - x = 6$.

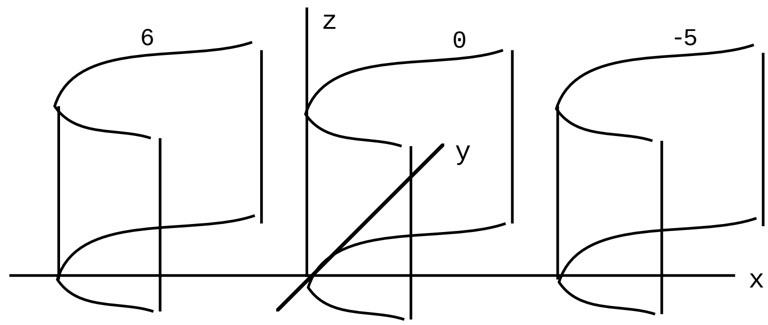
Do you remember that the graph in 3-space of an equation missing one letter (z, in this case) is a cylinder. Draw the curve $y^2 - x = 6$ in the x,y plane (it's a parabola) and move it up and down to generate the whole surface, a parabolic cylinder.

Similarly, all the level sets are parabolic cylinders.

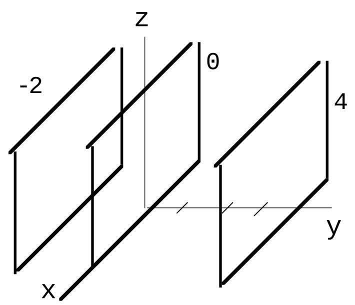
(c) Planes parallel to the x,z plane



(a)



(b)



(c)

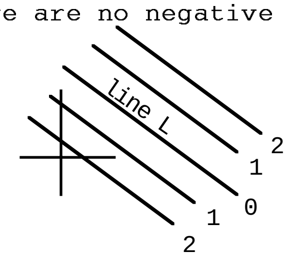
3. $f(2,6) = 12$ so the point is on the 12 level set, the hyperbola $xy = 12$.

4. Line L itself is the 0 level set.

The 2 level set for instance is the set of points at distance 2 from line L. It's a pair of parallel lines.

Similarly for all the C level sets, $C > 0$.

There are no negative level sets since distance is never neg.



5. The 6 level is

$$\frac{2}{\sqrt{(x+2)^2 + (y-1)^2 + (z-3)^2}} = 6$$

$$(x+2)^2 + (y-1)^2 + (z-3)^2 = 1/9$$

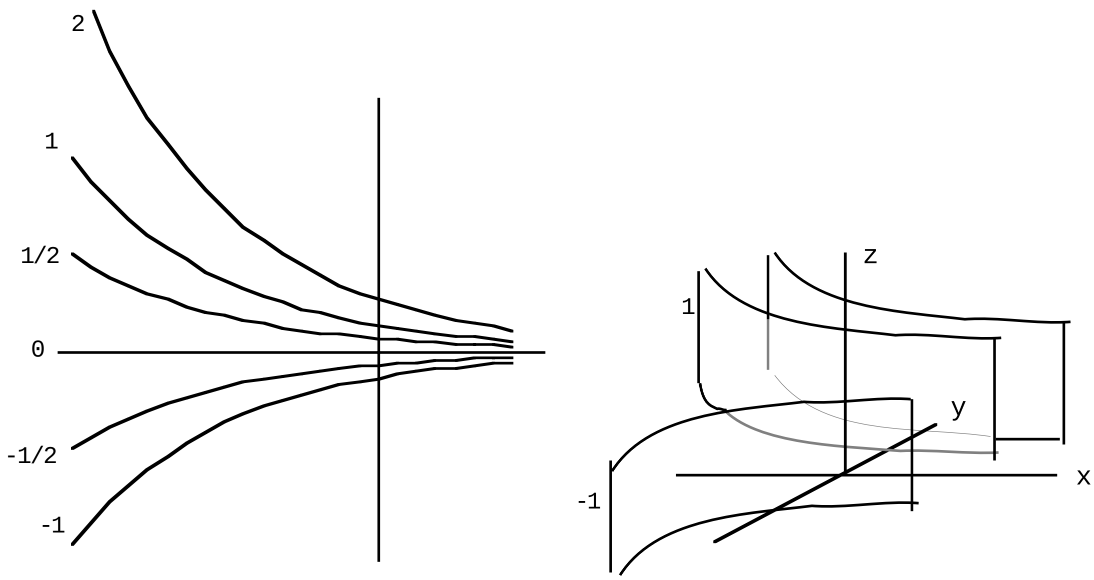
It's a sphere with center at $(-2, 1, 3)$ and radius $1/3$.

Similarly, the 7 level is a sphere with center $(-2,1,3)$ and radius $2/7$.
The higher the level, the smaller the sphere.
The 0 level set and negative level sets are empty.
You could say that the ∞ level is the point sphere $(-2,1,3.)$

6. Can't intersect. If say the 3 level and 4 level intersected then the intersection point would have temperature 3 *and* 4 which is impossible.

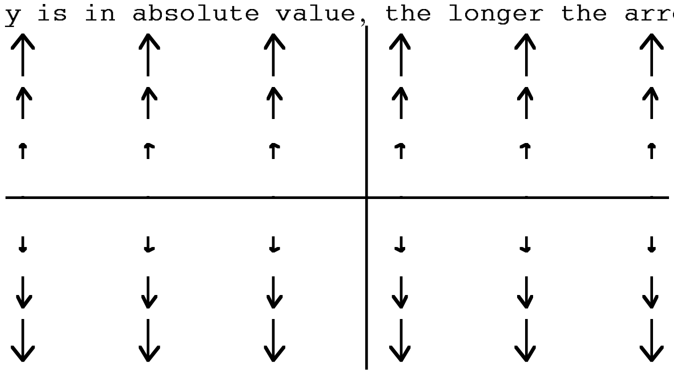
7. The entire plane is the 6 level set. All other levels are empty.

8. (a) If $C \neq 0$, the C level set is $ye^x = C$, the exponential curve $y = Ce^{-x}$.
The zero level set is $ye^x = 0$. If $ye^x = 0$ then either $e^x = 0$ (impossible) or $y = 0$. So the zero level set is the x-axis.
(b) The level set $ye^x = C$ is a cylindrical surface in 3-space. For instance to sketch the graph of $y = 2e^{-x}$ in 3-space, draw the curve $y = 2e^{-x}$ in the x,y plane and move it up and down to generate the whole surface, a cylinder.

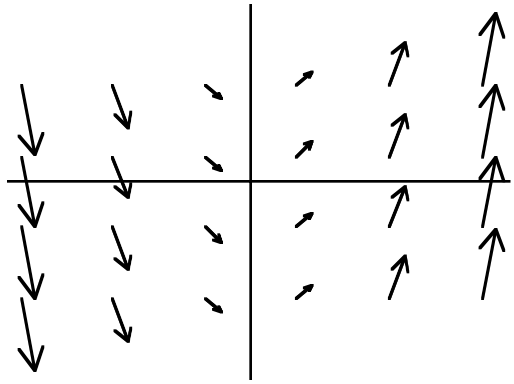


(a) (b)

9. (a) The arrows are all parallel to $\vec{j}$. If $y > 0$ they point like j. If $y < 0$ they point opposite to j They are longer when the absolute value of y is large.
The larger y is in absolute value, the longer the arrow.



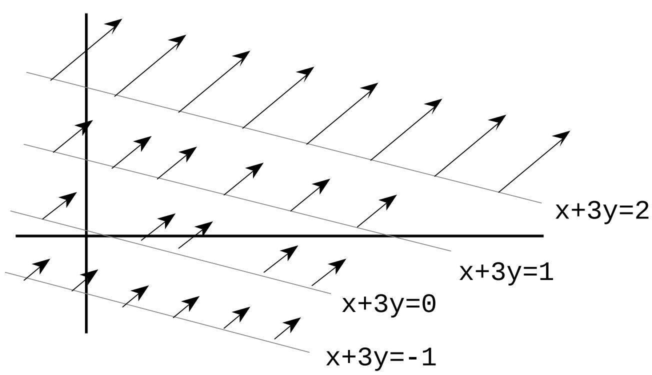
(b) All the arrows along a vertical line are the same since F is independent of y. The arrows are longer and steeper when x is larger in absolute value.



(c) Instead of trying to plot one arrow at a time notice that on the line x+3y=2 for instance, the arrows are all $e^2\,\vec{i} + e^2\,\vec{j}$. I drew the line as a guide. The arrows at points on this line all point northeast and have length $e^2\sqrt{2}$.

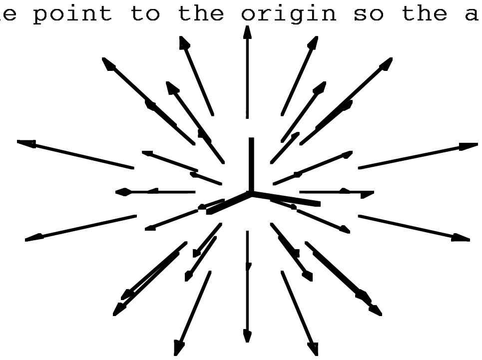
In general, all the arrows point northeast and at points on the line x+3y = k the arrows have length $e^k\sqrt{2}$.

In my diagram, the arrows are much shorter than they should be. I didn't have room to draw them full length.



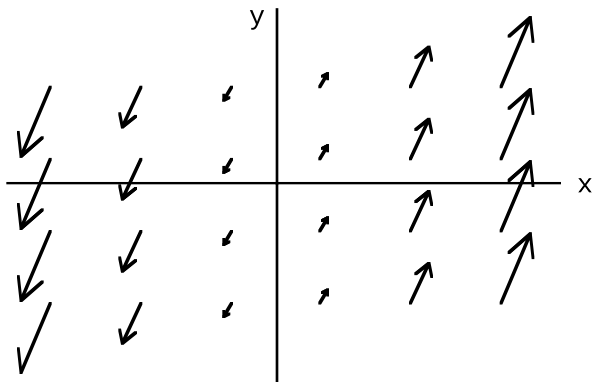
10. Use (3) from §1.0. $\text{NewF}(x,y) = 2\,\text{OldF}(x,y)_{\text{unit}} = \dfrac{2(y+3)}{\sqrt{(y+3)^2+y^2}}\,i + \dfrac{2y}{\sqrt{(y+3)^2+y^2}}\,j$

11. (a) It's an away-from-the-origin field in 3−space. The arrow at point (x,y,z) points away from the origin. The length of the arrow at a point is the distance from the point to the origin so the arrows get longer further from the origin.

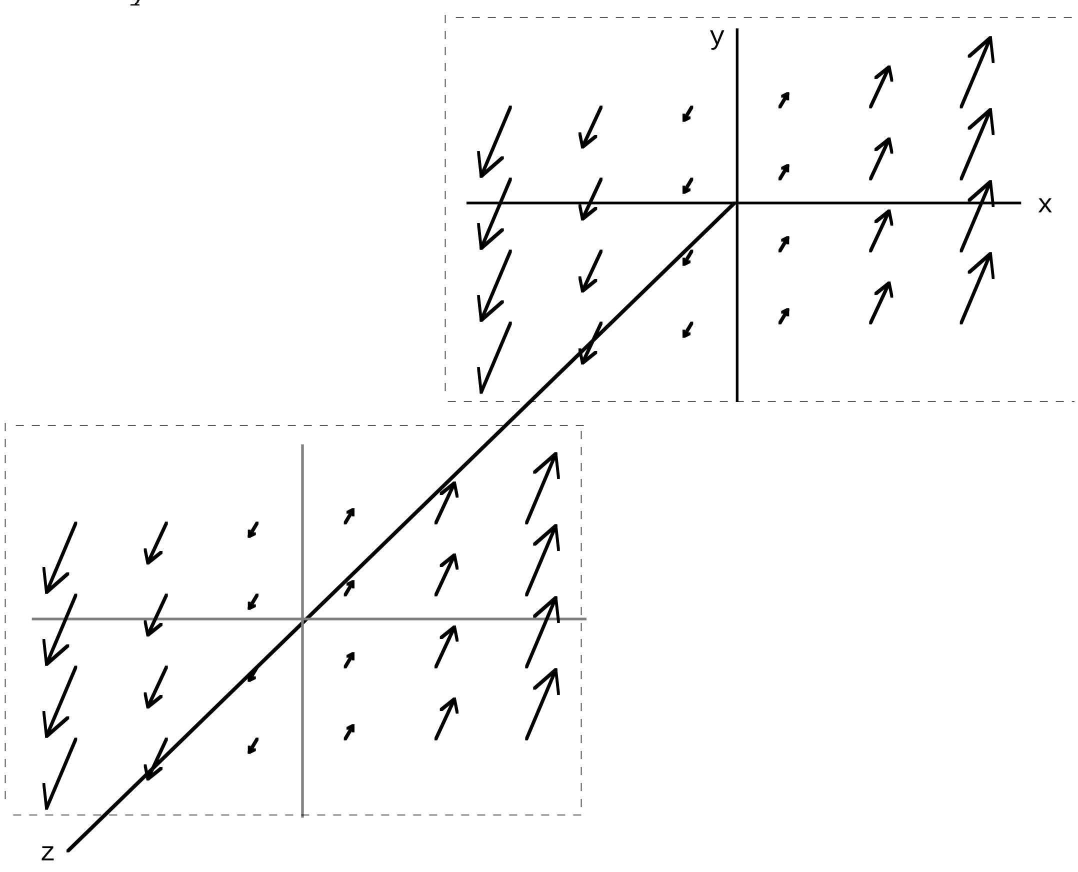


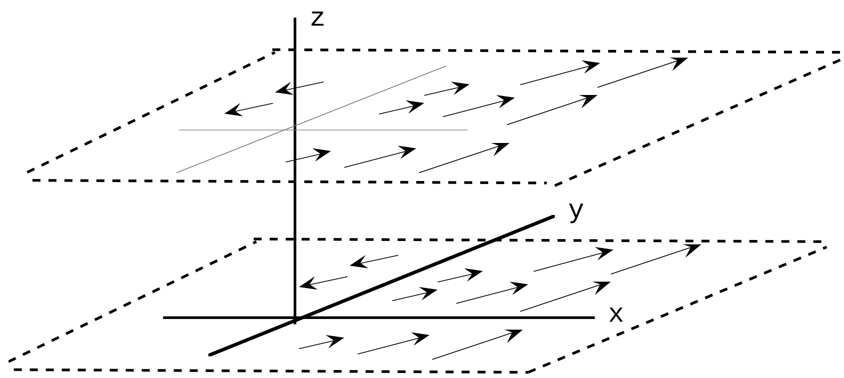
(b) It's the away-from-the-z-axis field in Fig 9.

12.(a) The arrows are x(1,3) so all the arrows are parallel to i+3j. When x > 0 the arrows point like i+3j. When x < 0 the arrows are opposite to i+3j.
The length F(x,y) is $|x|\sqrt{10}$ so the arrows get longer as you move away from the y-axis.



(b) Since the F formula is independent of z, the arrows in the plane $z=z_0$ look the same as the arrows in the x,y plane. So the vector field is layer after layer of arrows that look like the ones from part (a). I drew the picture twice, first with the z-axis coming forward and then again with the z-axis going up. Each time I only drew two layers.





13. (a) The distance from point (x,y,z) to the origin is $\sqrt{x^2+y^2+z^2}$

The arrow $x\vec{i} + y\vec{j} + z\vec{k}$ with its tail at point (x,y,z) points *away* from the origin so you want a vector that points like $-x\vec{i} - y\vec{j} - z\vec{k}$. And you want it to have length $1/(x^2+y^2+z^2)$. Use (3) of Section 1.0 to get

$$F(x,y,z) = \frac{1}{x^2+y^2+z^2} \; (-x,\ -y,\ -z)_{\text{unit}}$$

$$= \frac{1}{x^2+y^2+z^2} \left(\frac{-x}{\sqrt{x^2+y^2+z^2}} , \frac{-y}{\sqrt{x^2+y^2+z^2}} , \frac{-z}{\sqrt{x^2+y^2+z^2}} \right)$$

$$= \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}} , \frac{-y}{(x^2+y^2+z^2)^{3/2}} , \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right)$$

(b) The distance from point (x,y) to the origin is $\sqrt{x^2+y^2}$.

$$G(x,y) = \frac{1}{x^2+y^2} \; (-x,\ -y)_{\text{unit}}$$

$$= \frac{-x}{(x^2+y^2)^{3/2}} \, i + \frac{-y}{(x^2+y^2)^{3/2}} \, j$$

14. (a) A vector at point (x,y) which points away from the origin is xi+yj.

The one pointing ccl is the left turn vector −yi+xj. Normalize it to get a unit vector and multiply by 3 to get length 3. So $F(x,y) = \frac{-3y}{\sqrt{x^2+y^2}} \, i + \frac{3x}{\sqrt{x^2+y^2}} \, j$

(b) All the arrows point like i + 3j.

The field is independent of y since all the vectors on a vertical line are the same. To get roughly appropriate lengths, multiply by a positive scalar (which can contain an x but not a y) which decreases as x increases. One possibility is

$$F(x,y) = e^{-x} \, (i+3j) = e^{-x} \, i + 3e^{-x} \, j$$

(because e^{-x} goes from ∞ to 0 as x goes from −∞ to ∞).

15. Here's one way to do it.

Start with $\dfrac{1}{\sqrt{p^2(x,y,z) + q^2(x,y,z) + r^2(x,y,z)}} \; \left(p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k} \right)$

This is an arbitrary vector field with unit length arrows.
Then let

$$F = C \frac{1}{\sqrt{p^2(x,y,z) + q^2(x,y,z) + r^2(x,y,z)}} \left(p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k} \right)$$

where C is a constant. Then F is an arbitrary vector field with same-length arrows; in particular, all the field arrows have length $|C|$ (the absolute value of C).

(Or just use positive constants C to begin with; you don't lose any vector fields with constant length by doing that.)

SOLUTIONS Section 1.2

1. ∇temp = $(y^2 + 6,\ 2xy)$. At point (1,2), ∇temp = (10,4)

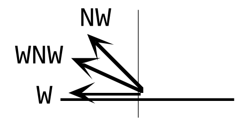
(a) $\overrightarrow{SW} = (-1,-1)$, $\dfrac{\nabla \text{temp}\cdot \text{SW}}{\|\text{SW}\|} = -14/\sqrt{2}$.

Temp is dropping by $14/\sqrt{2}$ degrees per meter.

(b) $\overrightarrow{PQ} = (2,-6)$, $\dfrac{\nabla \text{temp}\cdot \overrightarrow{PQ}}{\|\text{PQ}\|} = -4/\sqrt{40}$. Temp is dropping by $4/\sqrt{40}$ deg/meter.

(c) $\dfrac{\nabla \text{temp}\cdot -\vec{j}}{\|-\text{j}\|} = -4$. Temp is dropping by 4° per meter.

(d) $\|\nabla\text{temp}\| = \sqrt{116}$. Temp is increasing by $\sqrt{116}$ deg/meter.



Problem 1(e)

(e) WNW makes an angle of $7\pi/8$ with the positive x-axis.
A unit WNW vector is $(\cos 7\pi/8,\ \sin 7\pi/8)$

$$\frac{df}{d\,\text{WNW}} = \nabla\text{temp}\cdot\text{WNW} = 10\ \cos 1\ 7\pi/8 + 4\ \sin\ 7\pi/8$$

$10\cos 7\pi/8 + 4\sin 7\pi/8$ is negative (it's about -7.7) so temp is dropping by $-(10\cos 7\pi/8 + 4\sin 7\pi/8)$ degrees/meter.

(f) A vector at P pointing away from the origin is u = i + 2j.

$\dfrac{\nabla\text{temp}\cdot \text{u}}{\|\text{u}\|} = \dfrac{18}{\sqrt{5}}$. Temp is increasing by $\dfrac{18}{\sqrt{5}}$ degrees/meter

(g) Temp is dropping by $18/\sqrt{5}$ degrees/meter

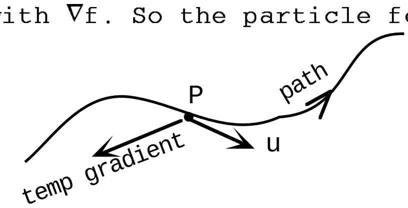
2. ∇temp = $(2xy,\ x^2)$, ∇temp(2,3) = (12,4), NE = (1,1)

$$\frac{d\text{temp}}{d\text{NE}} = \frac{\nabla\text{temp}\cdot\text{NE}}{\|\text{NE}\|} = 16/\sqrt{2}$$

Both runners feel temp increase by $16/\sqrt{2}$ degrees per meter.

Question What happens if you use (2,2) for NE instead of (1,1)
Answer You get $32/\sqrt{8}$ which simplifies to $16/\sqrt{2}$.

3. At point P the tangent vector u in the direction of motion makes an obtuse angle with ∇f. So the particle feels temp dropping.



4. ∇temp = $(2xy,\ x^2)$. At point (2,3), ∇temp = $12\vec{i} + 4\vec{j}$.

(a) Direction of steepest ascent is $12\vec{i} + 4\vec{j}$.

(b) $\|\nabla\text{temp}\| = \sqrt{160}$ so temp; is increasing by $\sqrt{160}$ degrees/meter in that direction.

(c) (i) The temp at point (2,3) is $4\cdot 3 = 12$.
The height of the temp hill is 12.

(ii) See part (b). It's as if you were walking up a temp hill which has slope $\sqrt{160}$.

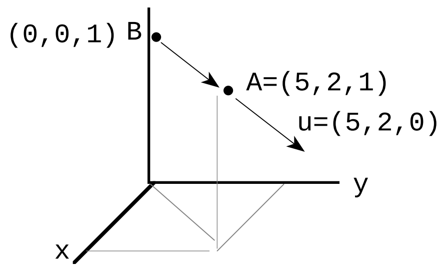
(iii) NW direction is $-\vec{i} + \vec{j}$.

$$\frac{\nabla\text{temp}\cdot\text{NW}}{\|\text{NW}\|} = \frac{-8}{\sqrt{2}}$$

As you walk NW, temp is dropping by $8/\sqrt{2}$ degrees/meter.
It's as if you are walking down a temp hill which has slope $8/\sqrt{2}$.

5. $\nabla f = (y, x-2y, 1)$, $\nabla f(A) = (2,1,1)$.
(a) A vector at point A going perpendicularly away from the z-axis is 5i + 2j. Call it u.

Here's how I got it.
The foot of the perpendicular from A to the z-axis has the same height as A.
So the foot, call it B, is (0, 0, 1).
Arrow BA = (5,2,0).
If that arrow is planted at point A, it points away from the z-axis.



$$\frac{\nabla f \cdot u}{\|u\|} = \frac{12}{\sqrt{29}}$$

f is going up by $12/\sqrt{29}$ f-units/meter.

(b) Any directions perp to ∇f such as (−1,2,0), (5,−8,−2) etc.

(c) (i) $\nabla f(A) = (2,1,1)$, $\nabla f(B) = (4,-2,1)$, $\overrightarrow{AB} = (1,2,1)$, $\overrightarrow{BA} = (-1,-2,-1)$.

$$\frac{\nabla f(A) \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{5}{\sqrt{6}}, \qquad \frac{\nabla f(B) \cdot \overrightarrow{BA}}{\|BA\|} = -\frac{1}{\sqrt{6}}.$$

Initially, A feels temp increasing by $5/\sqrt{6}$ degrees per meter and B feels temp dropping by $1/\sqrt{6}$ degrees per meter.

(ii) the midway point is $C = (\frac{11}{2}, 3, \frac{3}{2})$, $\nabla f(C) = (3, -\frac{1}{2}, 1)$, $\dfrac{\nabla f(C) \cdot \overrightarrow{AB}}{\|AB\|} = \dfrac{3}{\sqrt{6}}$.

As the two particles pass one another at point C, the first particle (heading from A to B) feels temp increasing by $3/\sqrt{6}$ degrees per meter. The second particle, heading in the opposite direction, must feel temp *dropping* by $3/\sqrt{6}$ degrees per meter.
(d) Moves in the direction of $-\nabla f$, i.e., in direction of −2i − j − k.
$\|\nabla f\| = \sqrt{6}$ so pressure decreases by $\sqrt{6}$ air pressure units per meter.

6. (a) First I'm going to find ∇f at point P.
At P, ∇f points like 3i + 2j so $\nabla f = c(3i + 2j)$ where $c > 0$.
And $\|\nabla f\| = 2$ so $\sqrt{9c^2 + 4c^2} = 2$, $13c^2 = 4$, $c = 2/\sqrt{13}$.
At P, $\nabla f = \dfrac{6}{\sqrt{13}}\, i + \dfrac{4}{\sqrt{13}}\, j$.

Now I can find any rate of change I like.
A NE vector is i + j

$$\frac{\nabla f \cdot NE}{\|NE\|} = \frac{10/\sqrt{13}}{\sqrt{2}}$$

So as you move NE, f increases by $10/\sqrt{26}$ f-units per meter.

(b) The rate of change of f in the north direction is just $\partial f/\partial y$, the second component of ∇f.

So as you move north, f increases by $4/\sqrt{13}$ f-units per meter.

7. Direction from A toward (1,1) is $u = -j$. Direction toward (7,10) is $v = 6i + 8j$. Let $\nabla f(A) = (a,b)$. Then

$$\frac{\nabla f(A)\cdot u}{\|u\|} = 2, \qquad \frac{\nabla f(A)\cdot v}{\|v\|} = -4$$

$$-b = 2, \quad \frac{1}{10}(6a + 8b) = -4$$

$$b = -2, \; a = -4$$

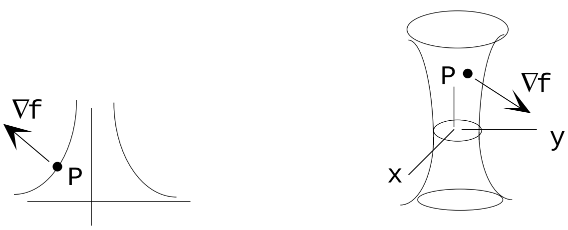
$$\nabla f(A) = (-4,-2)$$

Temp rises maximally in direction of $-4i - 2j$. Max rate is $\sqrt{20}$ degrees/meter.

8. (a) $f(-1,2) = 2$ so P is on the 2 level set which is $x^2y = 2$, $y = 2/x^2$.
$\nabla f = (2xy, x^2)$, $\nabla f(P) = (-4,1)$.

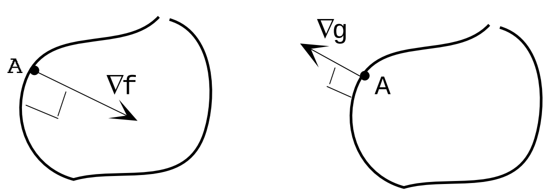
(b) $f(1,2,1) = 12$ so P is on the 12 level set $x^2 + 2y^2 - z^2 + 4 = 12$, i.e., $x^2 + 2y^2 - z^2 = 8$.

$\nabla f = (2x, 4y, -2z)$, $\nabla f(P) = (2,8,-2)$

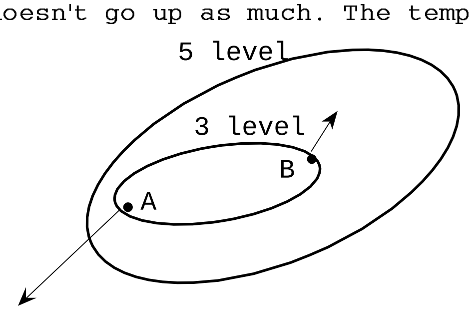


(a) (b)

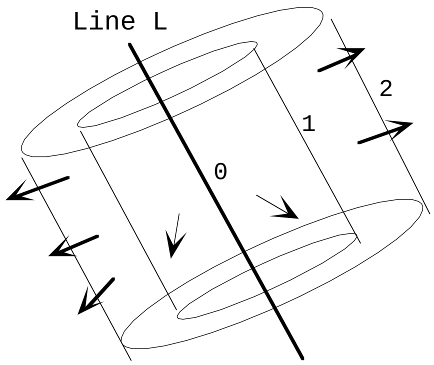
9. The gradients are perp to level sets and point to higher levels (see diagram). Imagine walking from A in the ∇f direction. Look at the size of the step it takes to go up 10 f-degrees, from the 50 to the 60 level. As you walk from A in the ∇g direction, for the same size step it looks like you go up roughly only 1 or 2 g-degrees. At A, the f temperature hill is steeper than the g temp hill. So ∇f is longer than ∇g.



10. At each point, ∇f is perp to the level set and points toward the higher level. Look at how far you have to you walk in the ∇f direction from A to get the temp to go up 2 degrees. If you walk that same distance in the ∇f direction from B, temp doesn't go up as much. The temp hill leaving A is steeper. So ∇f is longer at A.



11. Distance is never neg so there are no neg level sets.
Line L itself is the 0 level set.
The 2 level set (consisting of points at distance 2 from the line) is a cylinder with axis L, radius 2 etc.
Gradients are perp to the level sets and point to higher levels so they point out of each cylinder.
Each ∇f has length 1 because if you walk away from L in the direction of ∇f (i.e., perpendicularly away from L), f increases by 1 meter for each meter you walk (since f *is* distance to L).



12. $\nabla(f+g) = \left(\frac{\partial(f+g)}{\partial x}, \frac{\partial(f+g)}{\partial y}\right)$ by definition of the gradient

$= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)$ by the sum rule for derivatives

$= \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{\nabla f} + \underbrace{\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)}_{\nabla g}$ rearrange

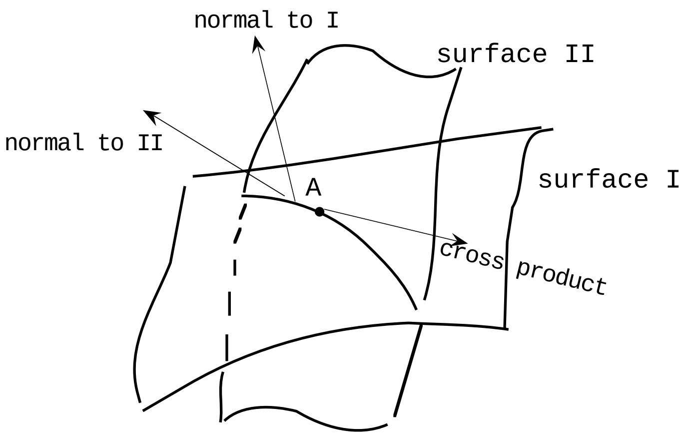
13.(a) If $x = 1$ and $y = -1$ then $3x^2 - 2y^2$ is 1 which does equal the z coord of P. So the point satisfies the equation.

(b) Surface is $3x^2 - 2y^2 - z = 0$, $\nabla(3x^2 - 2y^2 - z) = (6x, -4y, -1)$.
At P this is $6i + 4j - k$; this is a normal to the surface.

14. Let $f(x,y,z) = x^4 + y^4 + z^4$.
Let $g(x,y,z) = z - xy$.

Then $\nabla f = 4x^3\ i + 4y^3\ j + 4z^3\ k$ and $\nabla g = -yi - xj + k$.
At point A, $\nabla f = 4i + 32j + 32k$ and $\nabla g = -2i - j + k$.

∇f is normal to surface I, so it is perp to the curve of intersection.
∇g is normal to surface II, so it is perp to the curve of intersection.
(The two normals in the diagram are not necessarily perp to each *other*.)
So $\nabla f \times \nabla g$ is tangent to the curve. An answer is $64i - 68j + 60k$.
A simpler tangent vector is $16i - 17j + 15k$.



15. ∇temp = (2x, 3). ∇temp at P = (6,3)
Now I need a walk vector, i.e., I need an outer perp to the oval.
Let $g(x,y) = x^2 + y^3$. The oval is the 10 level set of g.
$\nabla g = (2x, 4y^3)$. ∇g at P = (6,4). This is a vector perp to the oval. And I can see from the picture than an *outer* perp at P has positive components, an *inner* perp at P would have negative components. So my $6i + 4j$ is an outer perp. I'll use the simpler normal $3i + 2j$ as my walk vector

$$\frac{\nabla \text{temp} \cdot \text{walk}}{\|\text{walk}\|} = \frac{24}{\sqrt{13}}$$

Temp is increasing by $24/\sqrt{13}$ degrees/meter as you initially start to walk.

SOLUTIONS Section 1.3

1. (a) It flows in the direction of $F(Q) = -i + 9j + 6k$

(b) $\dfrac{F(Q)\cdot n}{\|n\|} = \dfrac{-4}{\sqrt{3}}$

The flux through the window is $\dfrac{4}{\sqrt{3}}$ dS (where dS is the area of the window).

The direction is $-n$.
The units are kg/sec

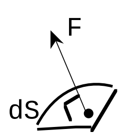
(c) kg/sec per square meter

2. Need a normal to the window.
The paraboloid is $z - x^2 - y^2 = 0$
$\nabla(z-x^2-y^2) = (-2x,-2y,1)$.
Plug in B to get a normal n to the surface at point B: $n = -6i-2j+ k$.
n points up, and for the paraboloid up happens to be *inner*.
At B, $F = 10i + 3j + k$.
$$\frac{F\cdot n}{\|n\|} = \frac{-65}{\sqrt{41}}$$
The mass flux is *out* of the window at the rate of $\dfrac{65}{\sqrt{41}}$ dS kg/sec.

3. The field arrows are perp to the sphere surface so at every point on the sphere, $F\cdot N = \|F\| = 2$ and the fluid flows out at the rate of 2 kg/sec per square meter. The key idea is that this is constant, the same at all points on the sphere.
The surface area of the sphere is $4\pi r^2 = 144\pi$ square meters.
So all in all, the flow out of the sphere is

2 kg/sec per sq meter $\times$ 144π sq meters = 288π kg/sec

4. The units on $\|F\|$ are kg/sec per square meter.
If a window through P is normal to F then at point P the flow through the window ,in the F direction, is 4 kg/sec per square meter.
Equivalently, if a *small almost-flat* window at point P is normal to F and has area dS then the flux through the window is 4 dS kg/sec



5. (a) $F(Q) = 6i + 5j + 3k$
$$\frac{F\cdot n}{\|n\|} = \frac{4}{\sqrt{6}}$$
If the window has area dS then $\dfrac{4}{\sqrt{6}}$ dS cubic meters/sec flow through the window in the n direction.
(b) The component of F is largest in the direction of F itself.
So tilt the window so that it is perp to F.
$F(Q) = \sqrt{70}$ so the max flux would be $\sqrt{70}$ dS cubic meters/sec

6. $\nabla f = (2 + 6x + z)i + xk$
At point P,

$$\nabla f = -2i - k$$

$$\text{Heat Flux Density} = -\nabla f = 2i + k$$

(a) The calorie at point P is moving in the direction of the HFD 2i + k

(b) At point P, $\|HFD\| = \sqrt{5}$.

The flux through the window is $\sqrt{5}$ dS cal/sec where dS is the area of the little window. The flux is in the direction of the HFD 2i + k

(c) The window has normal vector i. The component of the HFD in the i direction is 2. So the heat flux through the window is 2 dS cal/sec in the i direction (from left to right in the diagram).

(d) The plane has normal n = i − j − 10k (at every point).

$$\frac{HFD \cdot n}{\|n\|} = \frac{-8}{\sqrt{102}}$$

Calories flow through the plane at P in the −n direction at the rate of $8/\sqrt{102}$ cal/sec *per square meter*.

SOLUTIONS Section 1.4

1. (a) div $F = y + 2y + xy = 3y + xy$
   (b) div $G = 0 + z + y = y + z$
   (c) $\nabla(x \sin y) = (\sin y, x \cos y)$
   div $\nabla(x \sin y) = -x \sin y$
   (d) div $F = y \cos xy + 3y^2$

2. div $F = 1 + 1 + 1 = 3$.
It's the same at every point.
Mass flows out of the box at the rate of 3 dV kg/sec where dV is the volume of the little box.

3. Div $F = xye^x + ye^x + xe^x + 2y$.
At point $(-1,2,3)$, div $F = 4 - 1/e$. This is positive.
If the little box has volume dV then the flux out is $(4 - \frac{1}{e})$dV kg/sec.

4. $\|F\|$ has units kg/sec per square meter.
If a window through P is perp to F then mass flows through the window in the F direction at the rate of 4 kg/sec per square meter.
Div F has units kg/sec per cubic meter.
Mass flows into the region around point P at the rate of 5 kg/sec per cubic meter.

5. Div $F = 2xz^3 + 6xy^2z$. At point P, div $F = 28$.
If the box at P has volume dV then 28 dV calories per sec flow out of the box.

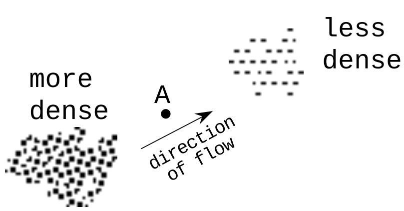
6. (a) The mass flows in the direction of $F(-1,2,2) = 2i + 2j + 4k$.
(b) At Q, $\|F\| = \sqrt{24}$. The mass flows through the window in the F direction at the rate of $\sqrt{24}$ dS kg/sec.
(c) A normal to the plane is $n = 2i-3j-5k$.

$$\frac{F\cdot n}{\|n\|} = \frac{(2)(2) + 2(-3) + 4(-5)}{\sqrt{38}} = \frac{-22}{\sqrt{38}}$$

If the window has area dS then the flux goes through the window in the $-n$ direction at $\frac{22}{\sqrt{38}}$ dS kg/sec.

(d) Div $F = 2xy + x^2 + y$. At point Q, div $F = -1$.
If the box has volume dV then the flux into the box is 1 dV kg/sec.

(e) The point is a sink because the net flow is *in*.
(f) Can't tell. The net flow into the point may be due to a change in density (the fluid flowing through the point could be getting denser as time goes on) rather than to a sink.



7. (a) $F(P) = 2i + 6j + 8k$

At P, $\dfrac{F \cdot u}{\|u\|} = \dfrac{-12}{\sqrt{3}}$

The flux through the window is $\dfrac{12}{\sqrt{3}}$ dS cubic meters/sec in the $-u$ direction.

(b) Div $F = yz + 2y + 3xz^2$; at $(1,1,2)$, div $F = 16$.
Flux goes out of the box at 16 dV cubic meters/sec.
(c) Source

8. Mass is kilograms; mass flux is kilograms *per second*;
mass flux density is kg/sec per square meter

9. Call the temperature $f(x,y,z)$.
(a) $\nabla f = (2xy^3 + 2z^3)i + 3x^2y^2\, j + 6xz^2\, k$
At point P, $\nabla f = 14i + 3j - 24k$.
Heat flux density $= -\nabla f$.
The flow is in the direction of HFD $= -14i - 3j + 24k$.
(b) (i) At point P, $\|HFD\| = \sqrt{781}$.
The flow through the window is $\sqrt{781}$ dS cal/sec where dS is the area of the window.
The flow is in the direction of $-14i - 3j + 24k$.

(ii) The window has normal vector $\vec{j}$.
$HFD \cdot j = -3$
The flow is 3 dS cal/sec through the window in the $-j$ direction (from right to left in the diagram).

(iii) A vector normal to the plane is $u = i - j - k$.
$\dfrac{HFD \cdot u}{\|u\|} = \dfrac{-35}{\sqrt{3}}$.

The flux through the window is $\dfrac{35}{\sqrt{3}}$ dS in the direction of $-u$. The units are cal/sec.

(c) Heat Flux Density $= -\nabla f = -(2xy^3 + 2z^3)i - 3x^2y^2\, j - 6xz^2\, k$
$\text{div}(HFD) = -2y^3 - 6x^2y - 12xz$
At point P, div(HFD) = 16.
If the box has volume dV then 16 dV cal/sec flow *out* of the region.

The flux is *out* of the box at the rate of 4 dV cal/sec .

10. (a) $\nabla\text{temp} = y^2z^3\, i + 2xyz^3\, j + 3xy^2z^2\, k$
$\nabla\text{temp}(Q) = 8i - 16j - 12k$

$\overrightarrow{QP} = (2, 0, -1)$

$$\frac{\nabla\text{temp} \cdot \overrightarrow{QP}}{\|\overrightarrow{QP}\|} = \frac{28}{\sqrt{5}}$$

You feel temp going up initially by $28/\sqrt{5}$ degrees per meter.
(b) At Q, Heat Flux Density $= -\nabla\text{temp} = -8i + 16j + 12k$. That's the direction in which calories flow through point Q.
$\|HFD\| = \sqrt{464}$
Calories flow through the window at the rate of $\sqrt{464}$ dS cal/sec .

(c) heat flux density = $-y^2z^3\, i - 2xyz^3\, j - 3xy^2z^2\, k$

div(HFD) = $-2xz^3 - 6xy^2z$

At Q, div(HFD) = 28

If the box has volume dV then 28 dV calories/sec flow *out* of the box.

(d) $\dfrac{\text{HFD}\cdot n}{\|n\|} = \dfrac{20}{\sqrt{14}}$

$\dfrac{20}{\sqrt{14}}$ calories/sec per square meter go through the window in the direction of n.

11. Draw a small box at point P.

method 1 Use the box IJKL in the lefthand diagram, whose sides are perp to the field.

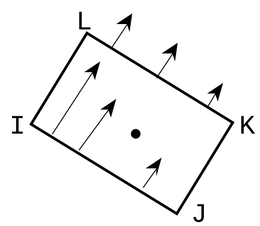
No mass/sec enters or leaves through sides IL and JK since the flow is tangent to those sides.

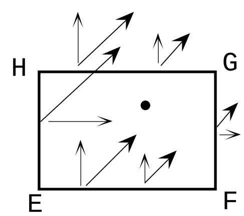
The flow out through side LK is less than the flow into side IJ because the sides have the same length but the arrows are longer on IJ. So the net flow is *in*.

Div F should be negative at the point.

method 2 Try the box EFGH in the righthand diagram, whose sides are parallel to he axes. Use the components of the field normal to the sides of the box. The flow in through side EF is the same as the flow out through HG. But the flow in through EH is larger than the flow out through FG. So there is a net flow *into* the box.

Div F should be negative at the point.

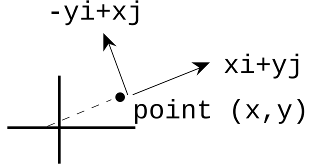


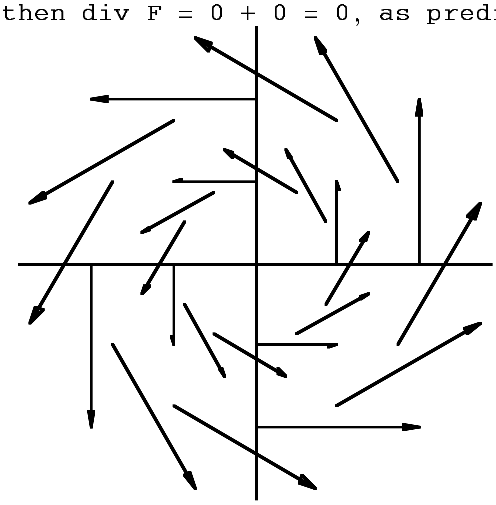


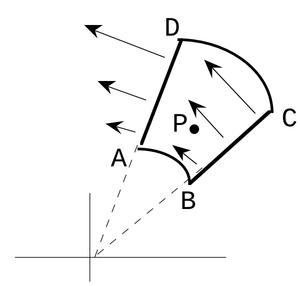
12. (a) Remember that the vector (x,y) with its tail at point (x,y) points away from the origin. The vector (−y,x) with its tail at point (x,y) is a *left* turn from the away-from-the-origin direction (look at the diagram on the left below). The vector field is in the middle diagram.

(b) The diagram on the right shows box ABCD at point (5,6) made up of radial lines and arcs. Nothing enters or leaves through sides AB and CD (the field is tangent to them). As much enters through BC as leaves through AD. So the net flux out of the box is 0 and div F = 0 at the point (and at every point for that matter).

(c) If F(x,y) = −yi + xj then div F = 0 + 0 = 0, as predicted in (b).







SOLUTIONS Section 1.5

1. (a) $\text{Lapl} = \frac{\partial^2 \; x^2 \sin y}{\partial x^2} + \frac{\partial^2 \; x^2 \sin y}{\partial y^2} = 2 \sin y - x^2 \sin y$

(b) $\text{Lapl} = 6yz^3 + 12xy + 18x^2 yz$

(c) $\text{Lapl} = y^2 e^{xy} + 6x + x^2 e^{xy}$

(d) $\text{Lapl} = xy^2 e^{xy} + 2y\, e^{xy} + 2y^3 + x^3 e^{xy} + 6x^2 y$

2. $\text{div } F = \frac{3x^2}{y} + x^3$, $\text{Lapl div } F = \frac{6}{y} + 6x + \frac{6x^2}{y^3}$

3. $\text{Lapl } f = e^x \sin y - e^x \sin y = 0$. So f is harmonic.

4. $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = -12z^2 + 12y^3$, $\frac{\partial^2 f}{\partial z^2} = 12z^3 - 12y^2$

$\text{Lapl } f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ so f is harmonic.

5. (a) $\nabla\text{temp} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla\text{temp}(Q) = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

You are walking in direction of $\vec{u} = P - Q = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\frac{\nabla\text{temp}\cdot u}{\|u\|} = \frac{-12}{3} = -4$

As you start to walk, temp is dropping by 4 degrees per meter.

(b) The heat flux density is $-\nabla\text{temp}$

$\text{HFD}(Q) = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

Look at a window through Q perp to $2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

At point Q, calories flow through the window in the direction of $2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ at the rate of $\sqrt{24}$ calories/sec per square meter.

(c) $\frac{\text{HFD}\cdot n}{\|n\|} = \frac{6}{\sqrt{14}}$

Flow through the window is $\frac{6}{\sqrt{14}}\, dS$ cal/sec

(d) *method 1*

$\text{div(HFD)} = \text{div}(-2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}) = -6$

div(HFD) at point Q = -6

Calories flow INTO the box at the rate of 6 dV cal/sec

method 2

$\text{Lapl temp} = \frac{\partial^2\text{temp}}{\partial x^2} + \frac{\partial^2\text{temp}}{\partial y^2} + \frac{\partial^2\text{temp}}{\partial z^2} = 6$

Lapl temp (Q) = 6

Calories flow INTO the box at the rate of 6 dV cal/sec

6. *method 1*

$\nabla\text{temp} = y \sin z\, \mathbf{i} + x \sin z\, \mathbf{j} + xy \cos z\, \mathbf{k}$

Heat flux density = $-\nabla\text{temp}$

Div(HFD) = $xy \sin z$

At point Q, div(HFD) = 2

Heat flows out at the rate of 2 dV calories/sec.

method 2

Lapl temp = $0 + 0 - xy \sin z$

Lapl temp (Q) = -2

Heat flows *out* at the rate of 2 dV cal/sec

7. (a) Can't find this. I need the heat flux density to find the flow across a window and I don't have it. I have $-$div(HFD) but not HFD itself.

(b) Lapl temp at Q = 20

Heat flows into the box at the rate of 20 dV cal/sec.

SOLUTIONS Section 1.6

1. (a) Here's my scratch work for finding curl F:

$$\begin{array}{ccc} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & yz & xyz \end{array}$$

Curl $F = (xz - y, -yz, -1)$

(b) Curl $F = (3x^2 y^2 z^4 - 1)\vec{i} - 2xy^3 z^4 \vec{j}$

(c) Curl $F = (0, 0, xy\cos xy + \sin xy - xe^{xy})$

(d) Curl $F = \left(\frac{x}{z} - xy, 1 - \frac{y}{z}, yz - 1\right)$

2. $\nabla g = (2xy + yze^{xyz}, x^2 + xze^{xyz}, xye^{xyz})$

Lapl $g = 2y + y^2z^2 e^{xyz} + x^2z^2 e^{xyz} + x^2y^2 e^{xyz}$

div $G = 2xz + xz\, e^{xyz}$

curl $G = (-xy\, e^{xyz} - 1, x^2, yz\, e^{xyz})$

3. The $t=\pi$ point is $(-1,0,\pi)$.

The velocity vector $\vec{v}(t) = (-\sin t, \cos t, 1)$ is tangent to the curve at the t-point. A tangent at the $t=\pi$ point is $(0,-1,1)$.

$F(-1,0,\pi) = (-1+\pi)j - \pi k$.

Component of F in the v direction $= \frac{F \cdot v}{\|v\|} = \frac{1-2\pi}{\sqrt{2}}$. This is negative.

If the little piece of curve has length ds then the circ on it is $\frac{2\pi-1}{\sqrt{2}}$ ds in the direction of $-v$. The units are square meters/sec.

4.(a) $F(A) = 6i+9j+6k$, $\|F(A)\| = \sqrt{153}$.

Fluid flows in the direction of arrow $6i+9j+6k$ at $\sqrt{153}$ meters/sec

(b) Div $F = yz + 1$. At point A, div $F = 3$.

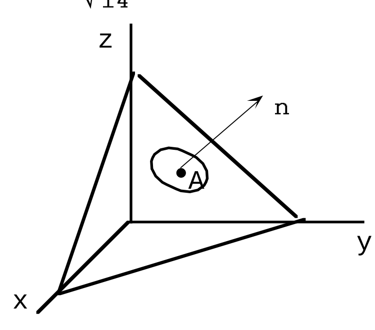
If the box has volume dV then 3 dV cubic meters of fluid per second flow out.

(c) You can get a normal to the window by inspection because plane $ax + by + cz = d$ has normal $ai + bj + ck$. Or you can use $\nabla(4x + 6y + 2z)$. Either way, a normal to the plane at A, and at every point on the plane as well, is $4i + 6j + 2k$. I'm going to use the simpler normal $n = 2i + 3j + k$

(i) At A, $F = 6i+9j+6k$.

$$\frac{F \cdot n}{\|n\|} = \frac{45}{\sqrt{14}}$$

If the window has area dS then the flow through the window in the n direction (see the diagram) is $\frac{45}{\sqrt{14}}$ dS m^3 per sec



(ii) curl F = $(1-x^2,\ xy-1,\ xz)$.
At point A, curl F = $-8i + 5j + 3k$.

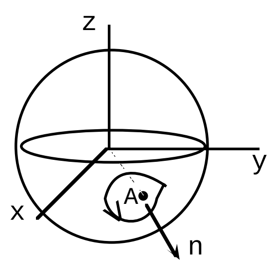
$$\frac{\text{curl } F \cdot n}{\|n\|} = \frac{2}{\sqrt{14}}$$

If the curve encloses area dS then the circ is $\frac{2}{\sqrt{14}}$ dS. It is righthanded around n (counterclockwise as viewed from above).

> *Question* In (c), I used (2,3,1) as my normal instead of (4,6,2). That's OK because both vectors are normal to the surface.
> The F vector at A came out to be (6,9,6). Would it be OK to replace F by (2,3,2) in part (c).
> *Answer* No. If you divide F by 2, then you'll get only half as much flow across the window in (i) and half as much circulation in (ii).

5. The F arrow itself is irrelevant. It looks like there is an obtuse angle between curl F and N. So curl F has a negative component in the N direction. The circ is lefthanded around N or equivalently clockwise as viewed from the head of N or equivalently righthanded around −N.

6. (a) At any point on a sphere with center at the origin, the outer normal points away from the origin (i.e., like a radius of the sphere). The arrow from the origin to point A is $2i + 3j - k$ so a normal n at point A is $2i + 3j - k$.



Curl F = $(1-z)j + k$.
At point A, curl F = $2j + k$

$$\frac{\text{curl } F \cdot n}{\|n\|} = \frac{5}{\sqrt{14}}$$

The circ around the loop is $\frac{5}{\sqrt{14}}$ dS where dS is the surface area on the sphere inside the loop. It is directed righthanded around n (counterclockwise as viewed from the outside of the sphere).

(b) At point A, F = $-5i + 8j - 2k$

$$\frac{F \cdot n}{\|n\|} = \frac{16}{\sqrt{14}}$$

The fluid flows through the loop (i.e., through the little window on the sphere) at the rate of $\frac{16}{\sqrt{14}}$ dS cubic meters/sec in the direction of n, i.e., out of the sphere.

(c) Circ is $F \cdot T$ ds.
F(A) = $-5i + 8j - 2k$, T = k, $F \cdot T = -2$
The circ is 2 ds square meters/sec in the −k direction.

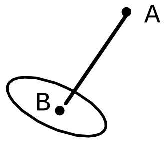
7. (a) curl F = $xz\, i - yz\, j + y^3\, k$
At point A, curl F = $-4i + 4j + k$.

$\overrightarrow{AB} = 2j + 5k$

$$\frac{\text{curl } F \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{13}{\sqrt{29}}$$

The paddle wheels turns at the rate of $13/\sqrt{29}$ radians/sec, righthanded around $\overrightarrow{AB}$, ccl as viewed from B.

(b) Let dS be the area enclosed by the curve.



(i) At B, curl F = (1,−3,27).

$$\frac{\text{curl } F \cdot \overrightarrow{BA}}{\|\overrightarrow{BA}\|} = \frac{-129}{\sqrt{29}}$$

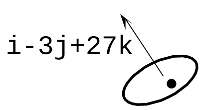
The circ is $\frac{129}{\sqrt{29}}$ dS m^2/sec lefthanded around $\overrightarrow{BA}$, clockwise as viewed from A.

(ii) F is a volume flux density.
At B, F = (1,27,3).

$$\frac{F \cdot \overrightarrow{BA}}{\|\overrightarrow{BA}\|} = \frac{-69}{\sqrt{29}}$$

The flow is $\frac{69}{\sqrt{29}}$ dS m^3 of fluid/sec in the A to B direction.

(iii) Tilt the loop so that it goes around curl F (so that curl F is perp to the surface inside the loop -- see the diagram).



$\|$curl F$\|$ = $\sqrt{739}$. So with the tilt in the diagram, the circulation on the loop is $\sqrt{739}$ dS square meters/sec (righthanded around curl F).

(iv) Tilt the loop so that it goes around a vector perp to curl F. Equivalently, tilt the loop so that the loop and curl F lie in the same plane. There are infinitely many such vectors. For instance 9j + k is perp to i−3j+27k so one way to get no circulation is to tilt the curve at B so that it goes around 9j + k.

(c) Midpoint = (1,2, −3/2)

$\overrightarrow{AB}$ = (0,2,5)

F(mid) = i + 8j − 3k

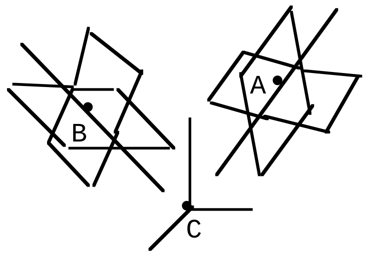
$$\frac{F \cdot AB}{\|AB\|} = \frac{16-15}{\sqrt{29}} = \frac{1}{\sqrt{29}}$$

Circ on the little piece of the segment is $\frac{1}{\sqrt{29}}$ ds m^2/sec in the A to B direction.

8. (a) *answer 1* The units are meters/sec.
The drop of fluid at point P is moving at 4 meters/sec
answer 2 The units cubic meters/sec per square meter.
The flow at P across a window through P normal to F is 4 cubic meters/sec per square meter.

(b) The units are "per second". A paddle wheel at point P with its axis parallel to curl F will turn at the rate of 5 radians per second [and that's the fastest that a paddle wheel at P can turn].

9. Curl F = (xz−x)i − yzj + (z−x)k.
Curl F at A is 2i − 6j + 2k.
Curl F at B is i + 2j + k.
Let the origin be called C



At A, $\dfrac{\text{curl } F\cdot\overrightarrow{AC}}{\|\overrightarrow{AC}\|} = \dfrac{(2,-6,2)\cdot(-1,-2,-3)}{\sqrt{14}} = \dfrac{4}{\sqrt{14}}$.

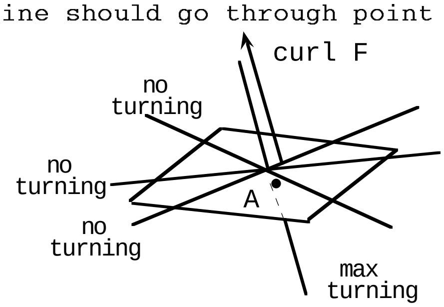
At B, $\dfrac{\text{curl } F\cdot\overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \dfrac{-1}{\sqrt{6}}$.

The wheel at A turns righthanded around $\overrightarrow{AC}$. You see the wheel at A turn ccl.

The wheel at B turns lefthanded around $\overrightarrow{BC}$. You see the wheel at B turn clockwise.

The wheel at A turns faster because $4/\sqrt{14}$ is larger than $1/\sqrt{6}$ (only absolute values matter here).

10. (a) Curl F = $(1-x^2$, xy−1, xz).
At point A, curl F = −8i + 5j + 3k .
The line should go through point A and be parallel to curl F.



The line has parametric equations
x = 3 − 8t
y = 2 + 5t
z = 1 + 3t (see (8) in Section 1.0)

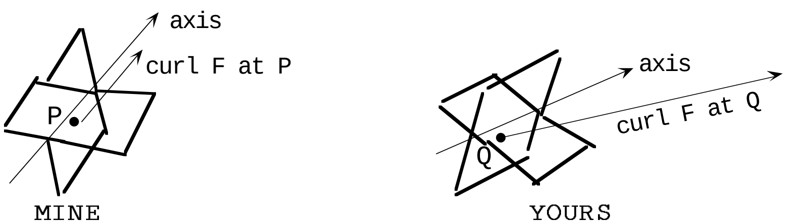
(b) The plane goes through point A and a normal vector is curl F = −8i + 5j + 3k.
The plane has equation

−8(x−3) + 5(y−2) + 3(z−1) = 0 (see (9) in Section 1.0)
8x − 5y − 3z = 11

The lines must go through point A and each must have a parallel vector that is perp to curl F. One such vector is i + j + k (because it dots with curl F to give 0). So one such line is x = 3 + t, y = 2 + t, z = 1 + t.
Another possible parallel vector is 3i + 8k so another line is
x = 3 + 3t, y = 2, z = 1 + 8t.

11. Not necessarily.
Look at the diagrams below.
There is more turning at Q than at P because the component of curl F at Q in the axis direction is larger than the component of curl F at P in the axis direction. Even though I did the best I could *at point P*, I might not beat your paddle wheel *at point Q*. Life is not fair.

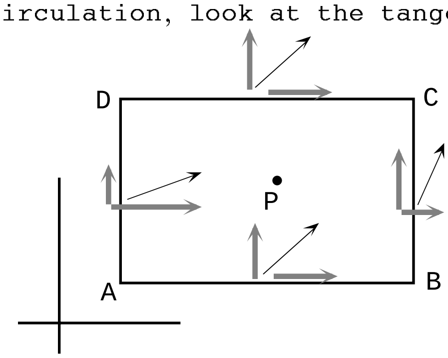
> *Question* But I thought you get the most turning when the paddle wheel's axis is aligned with curl F.
>
> *Answer* Of all paddle wheels *at a fixed point*, the one that turns fastest is the one with its axis parallel to curl F. But in this problem the two competing paddle wheels are at different points.



12. It is the turning rate (righthanded around v) of a small paddle wheel at point P with its axis pointing like $\vec{v}$..
  The units are "per sec" or radians/sec.

footnote. On the other hand, $\dfrac{\text{curl } \vec{F}\cdot\vec{u}}{\|\vec{u}\|}$ *times dS* would be the circulation directed righthanded around u along a small loop going around $\vec{u}$ with area dS, and would have units square meters/sec.

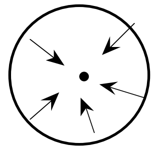
13.(a) I drew a loop around P and pictured each F as the sum of a tangential and normal vector. For circulation, look at the tangential components.



On AD, the circ is up the segment (clockwise on the loop)
On BC, the circ is also up the segment (ccl on the loop).
  But the component of F in the T direction is larger on BC than on AD.
On DC and AB the circs are the same numerical value but oppositely directed.
All in all, the circ is ccl.
So at point P, curl F = (0, 0, pos); curl F points out of the page.

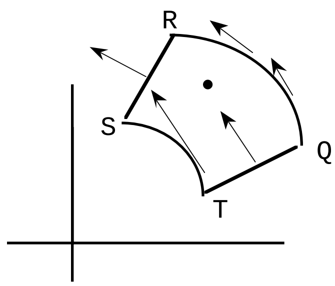
For divergence, look at the normal components.
The flow in across AB equals the flow out across DC.
The flow in across AD beats the flow out across BC.
Net flow is in. Div F is negative at P.

(b) The circulation on a small circle around the point looks like 0 since any $F \cdot T\, ds$ is 0.



So curl F = (0,0,0) at point P (curl is the *vector* $\vec{0}$, not the number 0).
Div F is negative at P since the net flux is *into* the circular box.

(c) Put a small loop around the point as shown in the diagram, made up of radial sides and circular sides.



No circ on sides TQ and SR.

Circ on ST is directed from T to S which counts as *clockwise* as far as the loop is concerned. Circ on QR is directed from Q to R which is counterclockwise along the loop..

ST is shorter than QR but arrows on ST are longer than arrows on QR

Can't tell who wins. Can't tell whether curl F points out or into the page.

I would need more information about exactly how the arrow lengths varied with distance to the origin.

For div F, look at the flux in and out of the box.
Nothing flows in through sides ST and QR.
Just as much goes in through TQ as flows out through RS.
Net flux out of the box is 0 so div F = 0 at the point.

SOLUTIONS Section 1.7

1. Let

$$F = F_1(x,y)\,\vec{i} + F_2(x,y)\,\vec{j}$$

$$G = G_1(x,y)\,\vec{i} + G_2(x,y)\,\vec{j}$$

Then

$$F + G = (F_1 + G_1)\vec{i} + (F_2 + G_2)\vec{j}$$

$$\operatorname{div}(F + G) = \frac{\partial(F_1 + G_1)}{\partial x} + \frac{\partial(F_2 + G_2)}{\partial y} \quad \text{definition of divergence}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial G_2}{\partial y} \quad \text{sum rule for derivatives}$$

$$= \underbrace{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}}_{\text{div } F} + \underbrace{\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}}_{\text{div } G} \quad \text{rearrange}$$

2. Given that $F = \nabla f$ and $G = \nabla g$ where f and g are arbitrary scalar fields. Want to show that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

(a) $\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{curl}\nabla f - \nabla f \cdot \operatorname{curl}\nabla g$ by (4)

$= \nabla g \cdot \vec{0} - \nabla f \cdot \vec{0}$ by (7)

$= 0 + 0 = 0$

Note where I used 0 and where I used $\vec{0}$.

(b) $$\nabla f \times \nabla g = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \times \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$$

$$= (f_y g_z - f_z g_y,\ f_z g_x - f_x g_z,\ f_x g_y - f_y g_x)$$

$$\operatorname{div}(\nabla f \times \nabla g) = D_x(f_y g_z - f_z g_y) + D_y(f_z g_x - f_x g_z) + D_z(f_x g_y - f_y g_x)$$

$$= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} + f_z g_{xy} + f_{zy} g_x - f_{xy} g_z - f_x g_{zy}$$

$$+ f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} \quad \text{by a lot of product rule}$$

$= 0$ (it all cancels out)

3. The first identity is OK. Div is a scalar so the zero is the number 0.

The second identity should be $\operatorname{curl} \nabla f = \vec{0}$ because curl is a vector.

(a) Let $F(x,y,z) = p(x,y,z)\,\vec{i} + q(x,y,z)\,\vec{j} + r(x,y,z)\,\vec{k}$. Then

$$\operatorname{curl} F = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z},\ \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x},\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{curl} F) = \frac{\partial^2 r}{\partial x\,\partial y} - \frac{\partial^2 q}{\partial x\,\partial z} + \frac{\partial^2 p}{\partial y\,\partial z} - \frac{\partial^2 r}{\partial y\,\partial x} + \frac{\partial^2 q}{\partial z\,\partial x} - \frac{\partial^2 p}{\partial z\,\partial y}$$

Remember that mixed partials are equal, e.g., $\frac{\partial^2 r}{\partial x\,\partial y} = \frac{\partial^2 r}{\partial y\,\partial x}$, so it all cancels out and you get $\operatorname{div}(\operatorname{curl} F) = 0$.

(b) $\operatorname{curl} \nabla f = \operatorname{curl}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial^2 f}{\partial z\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial z}, \text{etc.}, \text{etc.}\right) = (0,0,0) = \vec{0}$ since mixed partials are equal.

4. Let $G(x,y,z) = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k}$

Then $fG = fp\,\vec{i} + fq\,\vec{j} + fr\,\vec{k}$

$$\text{curl } fG = \left(\frac{\partial(fr)}{\partial y} - \frac{\partial(fq)}{\partial z}\right)\vec{i} + \left(\frac{\partial(fp)}{\partial z} - \frac{\partial(fr)}{\partial x}\right)\vec{j} + \left(\frac{\partial(fq)}{\partial x} - \frac{\partial(fp)}{\partial y}\right)\vec{k}$$

$$= \left(f\frac{\partial r}{\partial y} + r\frac{\partial f}{\partial y} - f\frac{\partial q}{\partial z} - q\frac{\partial f}{\partial z}\right)\vec{i} + \left(f\frac{\partial p}{\partial z} + p\frac{\partial f}{\partial z} - f\frac{\partial r}{\partial x} - r\frac{\partial f}{\partial x}\right)\vec{j}$$

$$+ \left(f\frac{\partial q}{\partial x} + q\frac{\partial f}{\partial x} - f\frac{\partial p}{\partial y} + p\frac{\partial f}{\partial y}\right)\vec{k}$$

warning Don't forget the product rule when you differentiate the products fp, fq and fr.

Rearrange to get

$$\text{curl } fG = f\,\underbrace{\left[\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)i + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)j + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)k\right]}_{\text{this is curl } G}$$

$$+ \underbrace{\left(r\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z}\right)\vec{i} + \left(p\frac{\partial f}{\partial z} - r\frac{\partial f}{\partial x}\right)\vec{j} + \left(q\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)\vec{k}}_{\text{this happens to be } \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \times G}$$

So the identity is $\text{curl}\, fG = f \text{ curl } G + \nabla f \times G$

5. $r = \sqrt{x^2 + y^2 + z^2}$. Let $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$. Then

$$r^2\vec{u} = a(x^2 + y^2 + z^2)\vec{i} + b(x^2 + y^2 + z^2)\vec{j} + c(x^2 + y^2 + z^2)\vec{k}$$

$$\text{curl } r^2\vec{u} = (2cy - 2bz)\vec{i} + (2az - 2cx)\vec{j} + (2bx - 2ay)\vec{k}$$

$$\vec{r} \times \vec{u} = (x,y,z) \times (a,b,c) = (cy - bz)\vec{i} + (az - cx)\vec{j} + (bx - ay)\vec{k}$$

So $2(\vec{r} \times \vec{u})$ *is* curl $r^2\vec{u}$. QED

6. *directly*

$fG = x^2 e^y\, i + x^2ye^y\, j$

div $fG = 2xe^y + x^2\,(ye^y + e^y)$ [product rule] $= 2xe^y + x^2ye^y + x^2\,e^y$ (*)

what the product rule says it should be

div $G = 1 + x$

f div $G = xe^y + x^2e^y$

$\nabla f = e^y\, i + xe^y\, j$

$\nabla f \cdot G = xe^y + x^2ye^y$

$f \text{ div } G + \nabla f \cdot G = xe^y + x^2e^y + xe^y + x^2ye^y$

$= 2xe^y + x^2e^y + x^2ye^y$ (**)

Compare (*) and (**). Div fG does equal f div G + $\nabla f \cdot G$. QED

7. (4) $\nabla \cdot (F \times G) = G \cdot \nabla \times F - F \cdot \nabla \times G$

(8) $\nabla \cdot \nabla \times F = 0$

SOLUTIONS review problems for Chapter 1

1. $\text{div } G = e^x + \frac{x}{z} + 3$

$\text{curl } G = \left(\frac{xy}{z^2},\ 3e^{3z},\ \frac{y}{z} - 2e^{2y} \right)$

$\nabla f = (6xy^3,\ 9x^2y^2,\ 4z^3)$

$\text{Lapl } f = 6y^3 + 18x^2y + 12z^2$

2. $\text{Curl } F = (xz - 1,\ -yz,\ 1-x^2)$

$\text{Curl } F \text{ at } B = 2\vec{i} - 6\vec{j}$.

$\text{Curl } F \text{ at } C = -4\vec{i} - 6\vec{j}$.

An optimally aligned paddle wheel in a velocity field F turns at $\|\text{curl } F\|$ radians per second (and has its axis parallel to curl F at that point).

$\|\text{curl } F \text{ at } C\| > \|\text{curl } F \text{ at } B\|$ so the optimally-aligned paddle wheel at C turns faster.

3. (a) $f\nabla f = f\frac{\partial f}{\partial x}\vec{i} + f\frac{\partial f}{\partial y}\vec{j}$

$\text{div } f\nabla f = \frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right)$ by the definition of div

$= f\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}$ product rule used twice

$= f\underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{\text{This is Lapl } f \text{ which is 0 since } f \text{ is harmonic}} + \underbrace{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}_{\nabla f\cdot\nabla f}$ rearrange

$= \nabla f\cdot\nabla f$ (which you can call $\|\nabla f\|^2$ if you like) QED

(b) Use (2) from §1.7 with $G = \nabla f$.

$\text{div } fG = f \text{ div } G + \nabla f\cdot G$ so

$\text{div } f\nabla f = f \text{ div } \nabla f + \nabla f\cdot\nabla f$

$= f \text{ Lapl } f + \|\nabla f\|^2$

$= \|\nabla f\|^2$ (Lapl f = 0 since f is harmonic)

4. (a) OK. ∇f and G are vector fields and you can add two vector fields.
(b) No good. Curl G is a vector field and div G is a scalar field and you can't add a vector and a scalar.
(c) No good. fG is a vector field and you can't take the gradient of a vector field.
(d) OK. $\nabla\cdot fG$ means div of fG. fG is a vector field and you can take div of a vector field.
(e) OK. ∇g is a vector field. You can take curl of a vector field and you get a vector field. You can take div of that vector field to get a scalar field which you can take the gradient of.
(f) No good. Div G is a scalar field and you can't take curl of a scalar field.

5. (a) $f(1,2,1) = -4$. So the temp at point C is -4°.

(b) $\nabla f = (1-2xy^2)\,i - 2x^2y\,j - \vec{k}$, $\nabla f(C) = -7i-4j-k$.
A calorie at point C moves in the direction of $-\nabla f(C) = 7i+4j+k$.

(c) *method 1*

$\text{Lapl } f = -2y^2 - 2x^2$

At point C, Lapl f = -10.

If the box has volume dV then calories go *out* at the rate of 10 dV cal/sec.

method 2

The Heat Flux Density is $-\nabla f = (2xy^2 - 1)i + 2x^2yj + k$

div HFD $= 2y^2 + 2x^2$

div HFD (C) = 10

Calories go out of the box at the rate of 10 dV cal/sec.

(d) One way to get a normal to the sphere at point (x,y,z) is to use $\nabla(x^2 + y^2 + z^2) = (2x,2y,2z)$. It happens to be an outer normal (plot and see).

Another way is to remember (Section 1.1) that at point (x,y,z) an away-from-the-origin-vector is xi + yj + zk.

So an outer normal to the window at point C is 2i + 4j + 2k.

I'll use the simpler outer normal n = i + 2j + k.

$-\nabla f$ is a heat flux density. From part (b), $-\nabla f$ at point C is 7i + 4j + k

$$\frac{-\nabla f \cdot n}{\|n\|} = \frac{16}{\sqrt{6}}$$

If the window has area dS then $\frac{16}{\sqrt{6}}$ dS calories/sec go through the window in the n direction, i.e., *out* of the sphere.

(e) You are walking in direction $\overrightarrow{CD} = -i + 3k$.

$$\frac{(-7i-4j-k)\cdot\overrightarrow{CD}}{\|\overrightarrow{CD}\|} = \frac{4}{\sqrt{10}}$$

As you leave C you feel the temp going up instantaneously by $4/\sqrt{10}$ degrees/meter.

6.(a) At P the direction of the flow is F = i−4j−8k; speed is $\|F\| = 9$ meters/sec.

(b) Div F = 1 + x + xy. At P, div F = −2.

2 dV m^3 per sec flow in.

(c) curl F = (xz, −yz, y); at point P , curl F = (2,8,−4).

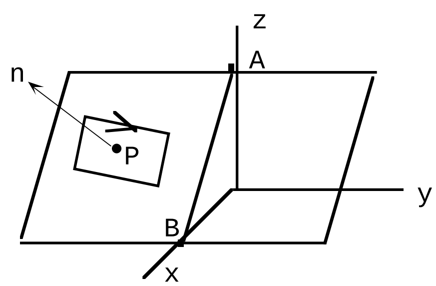
A normal to the plane is n = (2,0,3).

$$\frac{\text{curl } F \cdot n}{\|n\|} = -\frac{8}{\sqrt{13}}$$

The circ on the loop has numerical value $\frac{8}{\sqrt{13}}$ dS (where dS is the area enclosed).

The units are m^2/sec.

The circulation is lefthanded around n (or equivalently, righthanded around −n) as indicated in the diagram.



The technique for drawing the graph of 2x+3z=8 (where the letter y is missing) is to draw the line 2x+3z=8 in the x,z plane (that's line AB) and then move it left and right.

(d) F is a volume flux density. At P, F=i−4j−8k.

A normal to the plane is n = (2,0,3)

$$\frac{F\cdot n}{\|n\|} = \frac{-22}{\sqrt{13}}$$

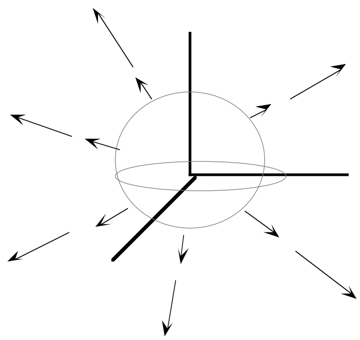
The flux through the loop in part (c) is in the direction of arrow $-n$

Its numerical value is $\frac{22}{\sqrt{13}}$ dS [where dS is the area enclosed by the loop].

The units are m^3/sec.

7.(a) At any point P, the arrow points away from the origin; its length is the distance from the origin to P (see the diagram below).

(b) $r \cdot r = x^2 + y^2 + z^2$, Lapl $(r \cdot r) = 6$



8.(a) div curl F = 0 for any vector field F. So answer is 0.

(b) $\nabla f \times \nabla f = \vec{0}$ since the cross product of any vector with itself is $\vec{0}$.

So $\text{curl}(\nabla f \times \nabla f) = \text{curl}\ \vec{0} = \vec{0}$ (the answer is $\vec{0}$, not 0).

(c) $\text{curl}\ \nabla f = \vec{0}$ because the curl of any gradient is $\vec{0}$.

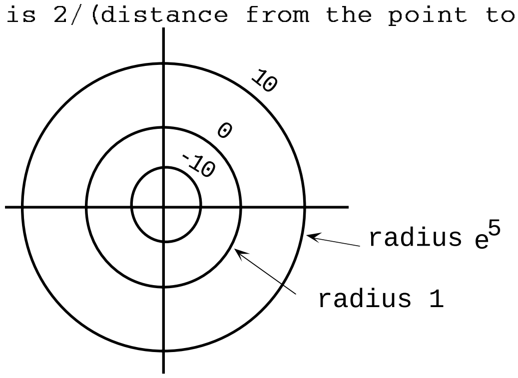
So $(\text{curl}\ \nabla f) \times \nabla f = \vec{0} \times \nabla f = \vec{0}$.

9. (a) $\ln(x^2 + y^2) = k$ iff $x^2 + y^2 = e^k$. So the k level set is a circle with center at the origin and radius $\sqrt{e^k} = e^{k/2}$. For instance, the 0 level set has radius 1, the -10 level set has radius e^{-5}, the origin is the $-\infty$ level set, the 10 level set has radius e^5.

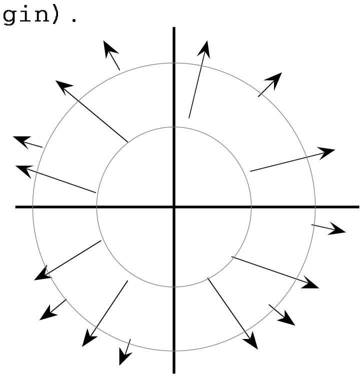
(b) $$\nabla \ln(x^2 + y^2) = \frac{2x}{x^2+y^2}\, i + \frac{2y}{x^2+y^2}\, j = \frac{2}{\sqrt{x^2+y^2}} \left[\frac{x}{\sqrt{x^2+y^2}}\, i + \frac{x}{\sqrt{x^2+y^2}}\, j \right]$$

$$= \frac{2}{\sqrt{x^2+y^2}}\, (xi + yj)_{\text{unit}}$$

The gradient vector at point (x,y) points away from the origin and its length is 2/(distance from the point to the origin).



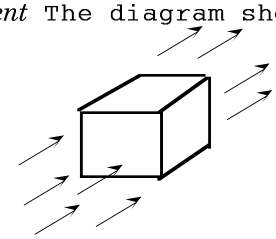
(a)



(b)

10. (a) True

physical argument The diagram shows a uniform field and a convenient box.



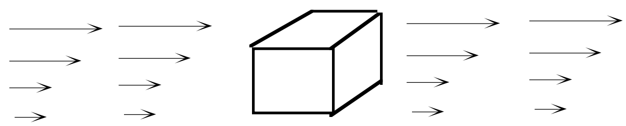
The net mass/sec out is 0 (as much goes in as comes out) so div F = 0.

mathematical argument F(x,y,z) = (a, b, c) where a,b,c are constants. So

$$\text{div } F = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0 + 0 + 0 = 0$$

(b) False

physical argument Look at the field F and the box in the diagram.



The net mass/out of the box is 0 so div F = 0 at every point. But F is not uniform.

mathematical argument Here's a counterexample.

Let $F(x,y,z) = (y^2 z,\ x^3 + z^2,\ x^3 y)$.

I made up F to be of the form (no x's, no y's, no z's).

Div F = 0 but F is not uniform.

Question How could you make up many more subtle counterexamples, i.e., ones that aren't of the form (no x's, no y's, no z's).

Answer Start with any F and take its curl. Then div of your curl F will be 0 because div of *any* curl is 0.

SOLUTIONS Section 2.1

1. (a) If $x=0$, $y=5$ then $0 = 2u + 3v$, $5 = u - v$ so the point has u,v coordinates $u = 3$, $v = -2$.

The u-curve through the point is named $v = -2$. It has parametric equations

$$x = 2u - 6$$
$$y = u+2.$$

Eliminate the parameter to get the plain equation $x - 2y = -10$.
The u-curve is a line.

(b) It has parametric equations $x = 2u + 3$, $y = u-1$.
Eliminate the parameter to get $x - 2y = 5$. It's a line.

(c) It has parametric equations $x = 3v$, $y = -v$. It's the line $x = -3y$.

(d) The v-curve $u = u_0$ has parametric equations

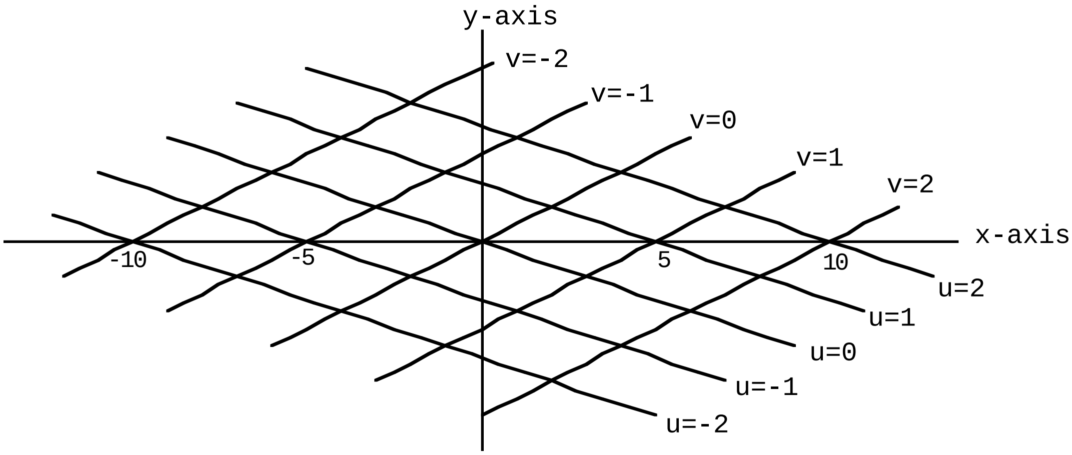
$$x = 2u_0 + 3v$$
$$y = u_0 - v$$

Eliminate the parameter to get $x + 3y = 5u_0$. The v-curves are parallel lines.

The u-curve $v = v_0$ has parametric equations

$$x = 2u + 3v_0$$
$$y = u - v_0$$

Eliminate the parameter to get $x - 2y = 5v_0$. The u-curves are parallel lines.



2. (a) The θ-curve $r=3$ has parametric equations

$$x = e^3 \cos \theta$$
$$y = e^3 \sin \theta.$$

I know this is a circle with center at the origin and radius e^3 (see "famous parametric equations" in Section 2.0).

(b) The r-curve $\theta = \pi/7$ has parametric equations

$$x = e^r \cos \pi/7$$
$$y = e^r \sin \pi/7$$

Note that $e^r > 0$ for all r, $\cos \pi/7$ and $\sin \pi/7$ are positive so $x > 0$ and $y > 0$.

To eliminate the parameter, divide the two equations to get

$$y/x = \tan \pi/7, \quad y = x \tan \pi/7.$$

This is a line through the origin inclined at angle $\pi/7$. But since $x > 0$, $y > 0$, the curve is only the half of the line in quadrant I.

(c) As in part (a):

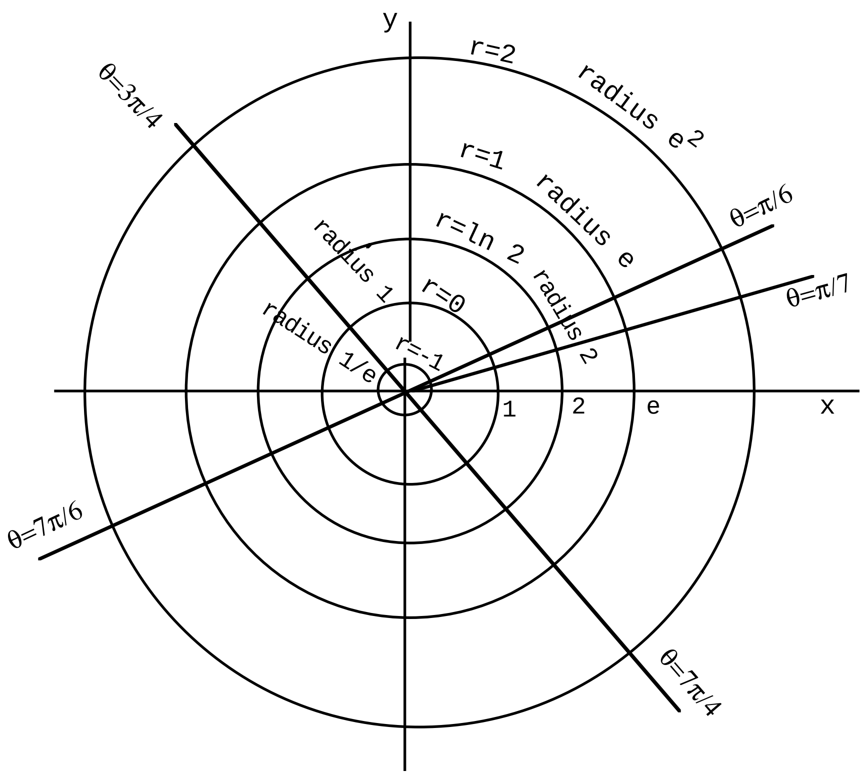
The θ-curve r=2 is a circle centered at the origin with radius e^2.
The θ-curve r=0 is a circle with radius $e^0 = 1$.
The θ-curve r=−1 is a circle with radius $e^{-1} = 1/e$.
The circle with radius 2 is the θ-curve $r = \ln 2$.

As in part (b), the r-curve $\theta = \theta_o$ is a half line through the origin swung around at angle θ_o.

The coord paper is like polar coord paper except that the θ-curve $r=r_o$ is a circle with radius e^{r_o} instead of radius r_o. And negative r's are allowed.

θ still has the usual geometric meaning but r is no longer distance to the origin. A point with r-coord r_o is distance e^{r_o} from the origin.

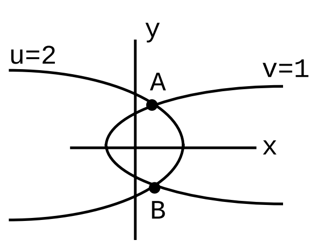
At the origin $r=-\infty$.



3. (a) You still get parametric equations

$$x = \frac{1}{2}(4 - v^2)$$
$$y = 2v$$

and when you eliminate the parameter you still get $x = \frac{1}{2}(4 - \frac{1}{4}y^2)$ but now you don't have the restriction $y \ge 0$ so the v−curve u=2 is a whole parabola, not just the top half.



The coordinate paper is now ambiguous. For instance it gives the *false* impression that points A and B both have coords u=2, v=1. But the coordinate system itself is fine:

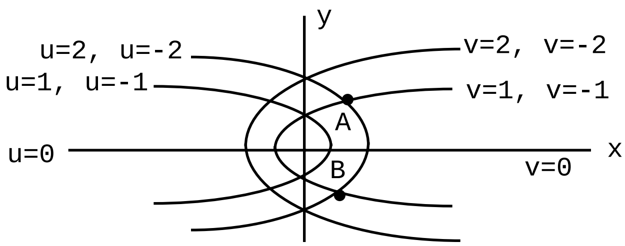
Question Where would you go if someone said to move to the point where u=2, v=1.

Answer From the coordinate paper you can't tell if it's A or B but plug u=2, v=1 into the original equations in (1) to see that x and y are positive so it's point A, not point B.

(b) Now there's a u–curve named v = –2 and it happens to be the same as the v=2 curve.

And the v–curve u = –2 is a whole parabola not just the bottom half, and is the same parabola as the u=2 curve.

The coordinate paper includes the same parabolas as before but each u–curve and v–curve is a whole parabola now. Here's a little piece of it.



(c) The coordinate paper makes it *seem* as if point A has four sets of coordinates:

u=2, v=1; u=2, v=–1; u=–2, v=1; u=–2, v=–1.

And it looks like B also has these same four sets of coordinates.

If this were true, it would be a disaster. Where would you go (A or B?) if someone said "go to the point with coordinates u=2, v=1".

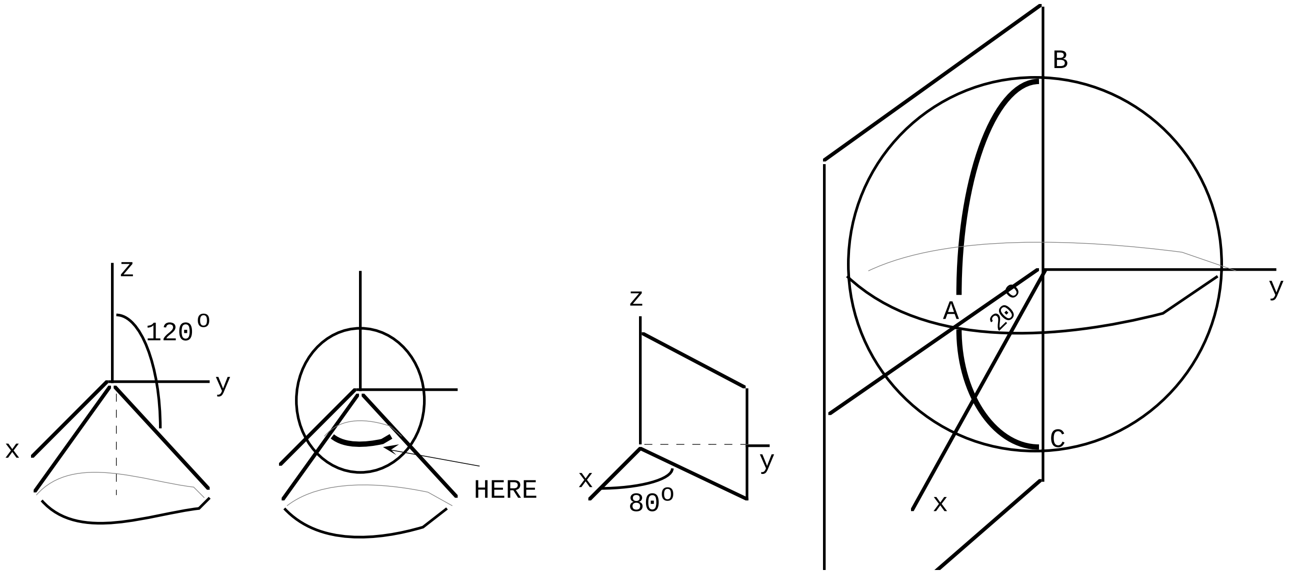
It is unacceptable for one set of coordinates to correspond to two different points.

But that is not happening here even if the paper makes it seem so. Plug into the equations in (1) and you'll see that u=2, v=1 and u = –2, v = –1 go with point A while u = –2, v=1 and u=2, v= –1 go with B.

That's OK, one point can have two sets of coordinates.

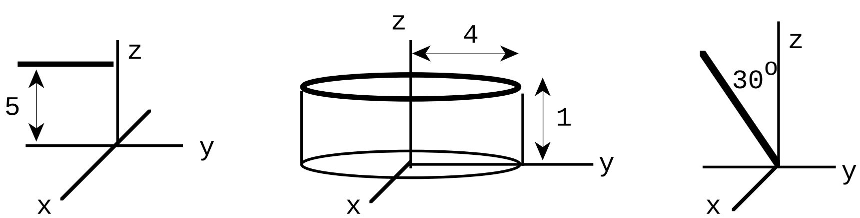
Conclusion: Without the restriction the coordinate system is OK but the coordinate paper becomes misleading. It's better with the condition v ≥ 0.

4. (a) a cone (see the diagram)
 (b) circle of intersection of cone and sphere
 (c) a half-plane
 (d) semicircle of intersection of a sphere and a half-plane. See BAC in the diagram.



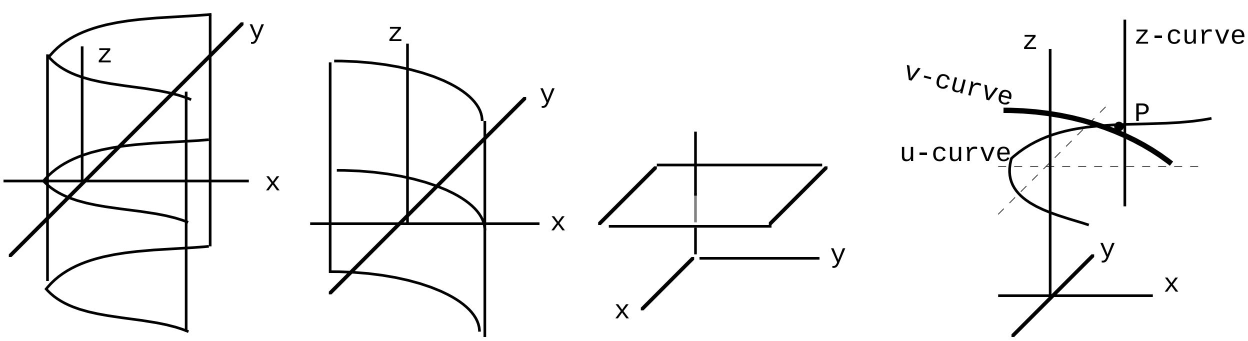
(a) (b) (c) (d)

5. (a) a half-line in the y,z plane (see the diagram)
   (b) a circle
   (c) a half-line in the y,z plane



(a) (b) (c)

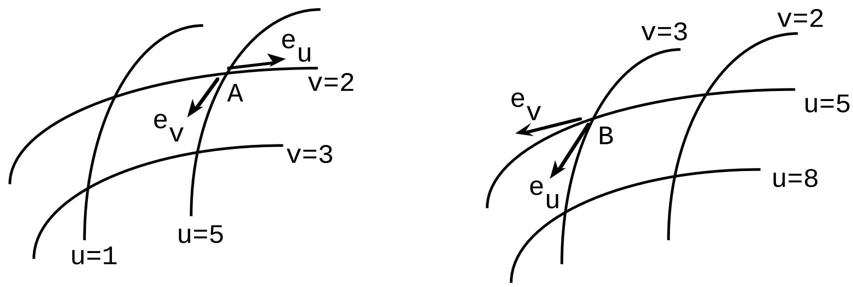
6. Parabolic cylindrical coordinates are just parabolic coordinates plus the coordinate z .
(a) Take the curve v=1 from 2-dim parabolic coordinates and move it up and down (i.e., in the z-direction) to sweep out a surface, called a parabolic cylinder. In practice you can draw the picture by drawing the parabola v=1 in the x,y plane, then drawing duplicates of it above and below, and connecting them all with lines parallel to the z-axis.
(b) Do the same as in part (a) but with the half-parabola u=2 from parabolic coords.
(c) The ordinary plane z=3.
(d) The u-curve is a half-parabola, as in 2–dim parabolic coords, but up at height 3.
The v-curve is a parabola, as in 2–dim parabolic coords, but up at height 3.
The z-curve is a curve on which z alone is changing, a vertical line.



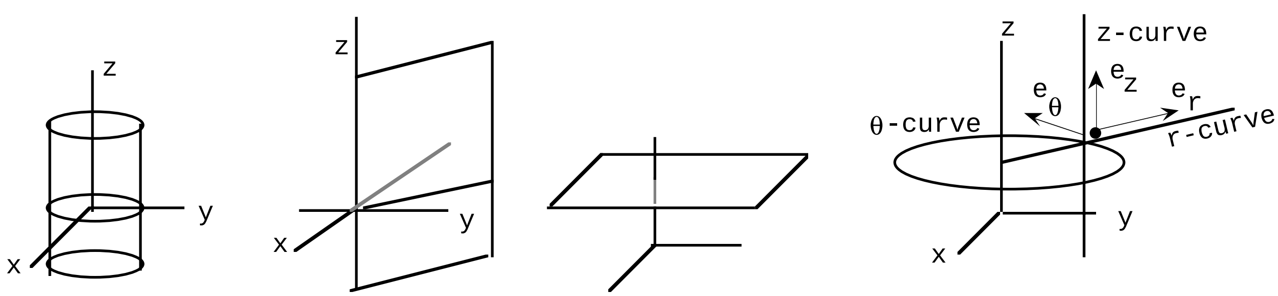
(a) (b) (c) (d)

SOLUTIONS Section 2.2

1. For e_u at A, draw a vector with unit length, tangent to the u-curve v=2 and pointing in the direction of increasing u.

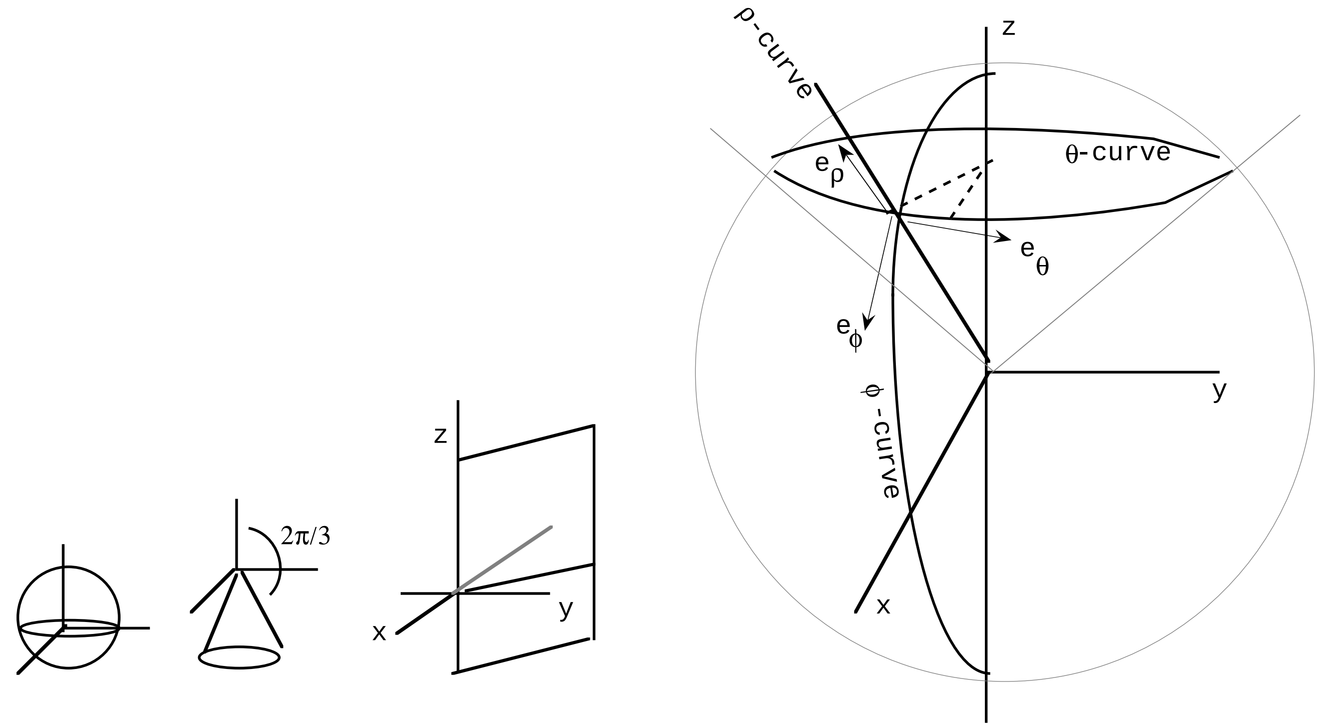


2. (a) r is distance to the z-axis.
r=3 is a cylinder around the z-axis with radius 3.
(b) θ is "around" from the x,z plane.
θ=2π/3 is a half plane around at angle 2π/3.
(c) z in cylindrical is the same as z in Cartesian. So z=1 is a plane at height 1.
(d) The r-curve is a ray going through the point and perpendicular to the z-axis. It's the intersection of the half-plane from part (b) and the whole plane from (c).
The θ-curve is a circle of radius 3 around the z-axis, up at height 1. It's the intersection of the cylinder from part (a) and the plane from (c)
The z-curve is a vertical line, the intersection of the cylinder from part (a) and the half-plane from part (b).
e_θ points ccl as seen from above, around the circle in the diagram.
e_z points straight up (e_z = k).
e_r points away from the z-axis.



(a) (b) (c) (d)

3. (a) A sphere around the origin with radius 2 (see the diagram).
(b) A cone with "outside" cone angle 2π/3.
(c) A half-plane around at angle 2π/3
(d) The ρ-curve is a ray through the origin and the point. It's the intersection of a cone and a half-plane.
The ϕ-curve is a great circle on the sphere ρ=1, in the half-plane swung around at angle −π/3.
The θ-curve is a circle around the z-axis. It's the intersection of a cone with angle π/6 and a sphere of radius 1.
e_ρ points away from the origin.
e_ϕ points downish, tangent to the ϕ-curve.
e_θ points around, ccl as seen from above.



(a) (b) (c) (d)

4. $x = r\cos\theta,\ y = r\sin\theta$

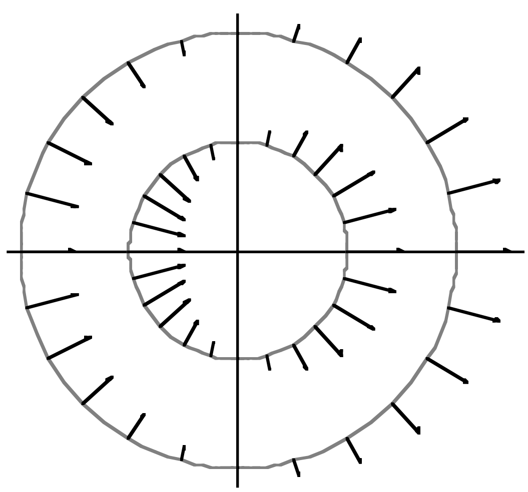
$e_r = (\text{vel}_r)_{\text{unit}} = (\cos\theta\,\vec{i} + \sin\theta\,\vec{j})_{\text{unit}} = \cos\theta\,\vec{i} + \sin\theta\, j$

$e_\theta = (\text{vel}_\theta)_{\text{unit}} = (-r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j})_{\text{unit}} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

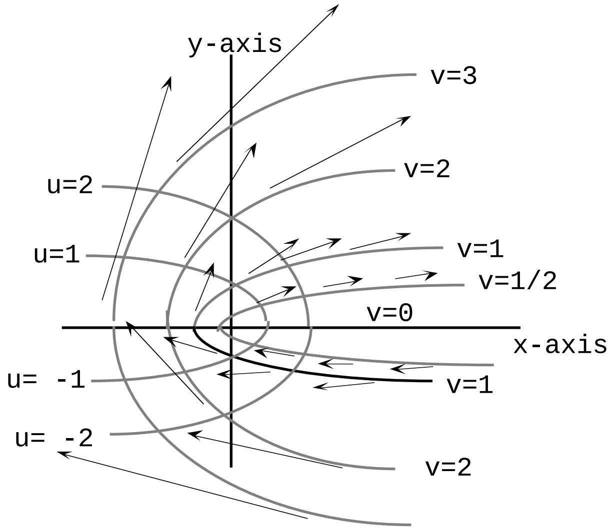
$e_r \cdot e_\theta = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$ so e_r and e_θ are perp at every point.

5. The arrows are independent of r, so the arrows on one circle around the origin look the same as on any other circle.

All the arrows point away from or toward the origin. They point toward the origin when $\cos\theta$ is negative (when $\pi/2 \le \theta \le 3\pi/2$) and away from the origin when $\cos\theta$ is positive. The arrow lengths change as you go around a circle because the absolute value of $\cos\theta$ goes from 1 to 0 to 1 to 0 to 1.

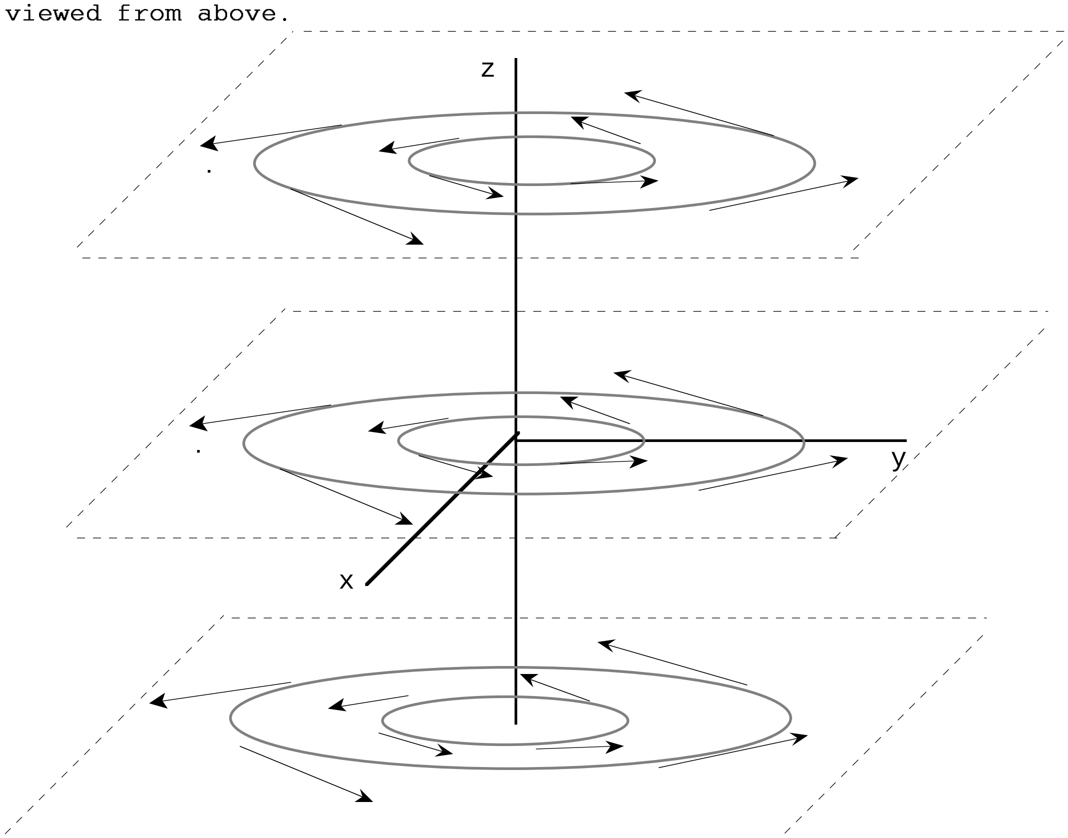


6. The arrow at a point is tangent to the u-curve $v=v_0$ through that point and points in the direction of increasing u. The arrows at points on the u-curve $v=v_0$ all have the same length, namely length v_0.

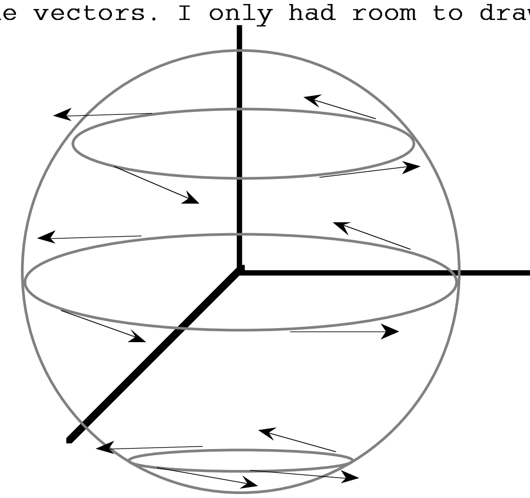


7. (a) The arrows at points in plane z = 3 for instance are the same as the arrows at corresponding points in the plane z = 4. So you can draw one set of arrows and then copy and paste to get the rest.

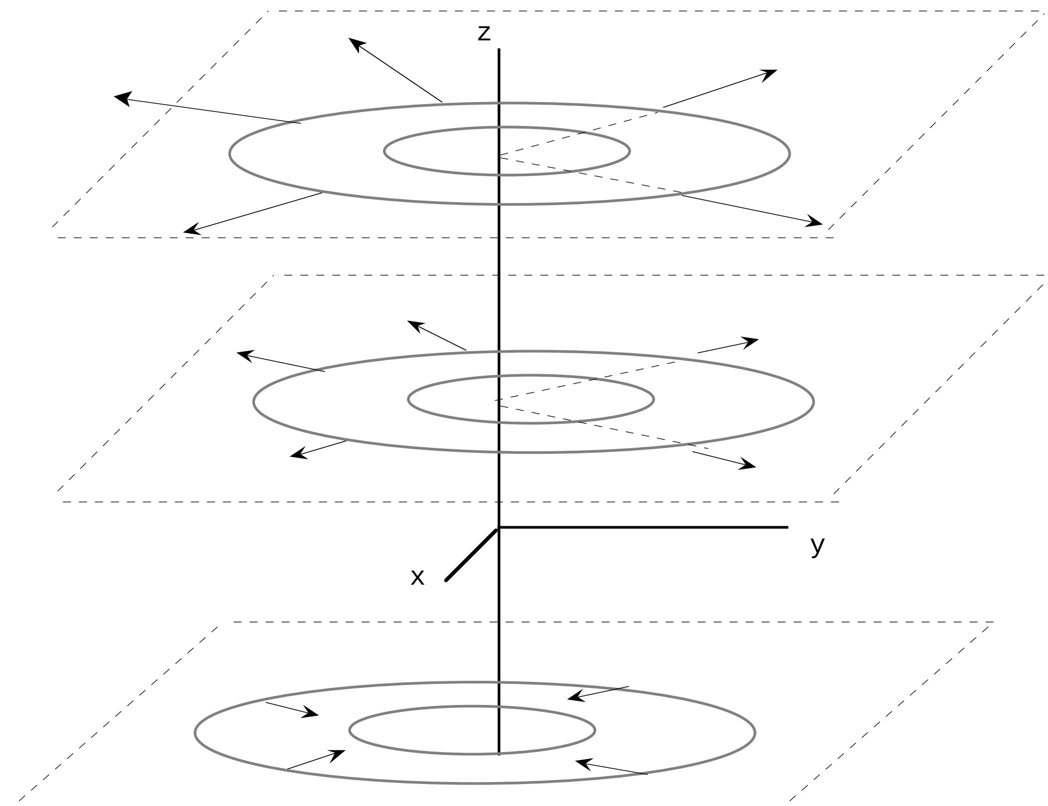
In any horizontal plane, at points on a circle around the z–axis with radius r_o the arrows have length r_o and point "around", tangent to the circle and ccl as viewed from above.



(b) On the sphere $\rho=\rho_o$, the field vectors point "around" (tangent to θ-curves) and all have the same length, namely each length is ρ_o. The larger the sphere, the longer the vectors. I only had room to draw one sphere.



(c) All the arrows in the plane $z = z_o$ have the same length.
If $z_o > 0$ (i.e., in a horizontal plane above the x,y plane), the arrows all have length z_o and point away from the z-axis, perp to a circle going around the z-axis.
If $z_o < 0$, the arrows have length $|z_o|$ and point clockwise as viewed from above.
The arrows at points in the plane $z = 0$ are all the zero vector.
The arrows get longer as you move further up or down from the x,y plane.



(d) The diagram shows a bunch of arrows attached to points in plane $z = z_o$. The arrows in any other horizontal plane are the same so I only drew two .

The length of the arrow is the absolute value of $\ln r$.

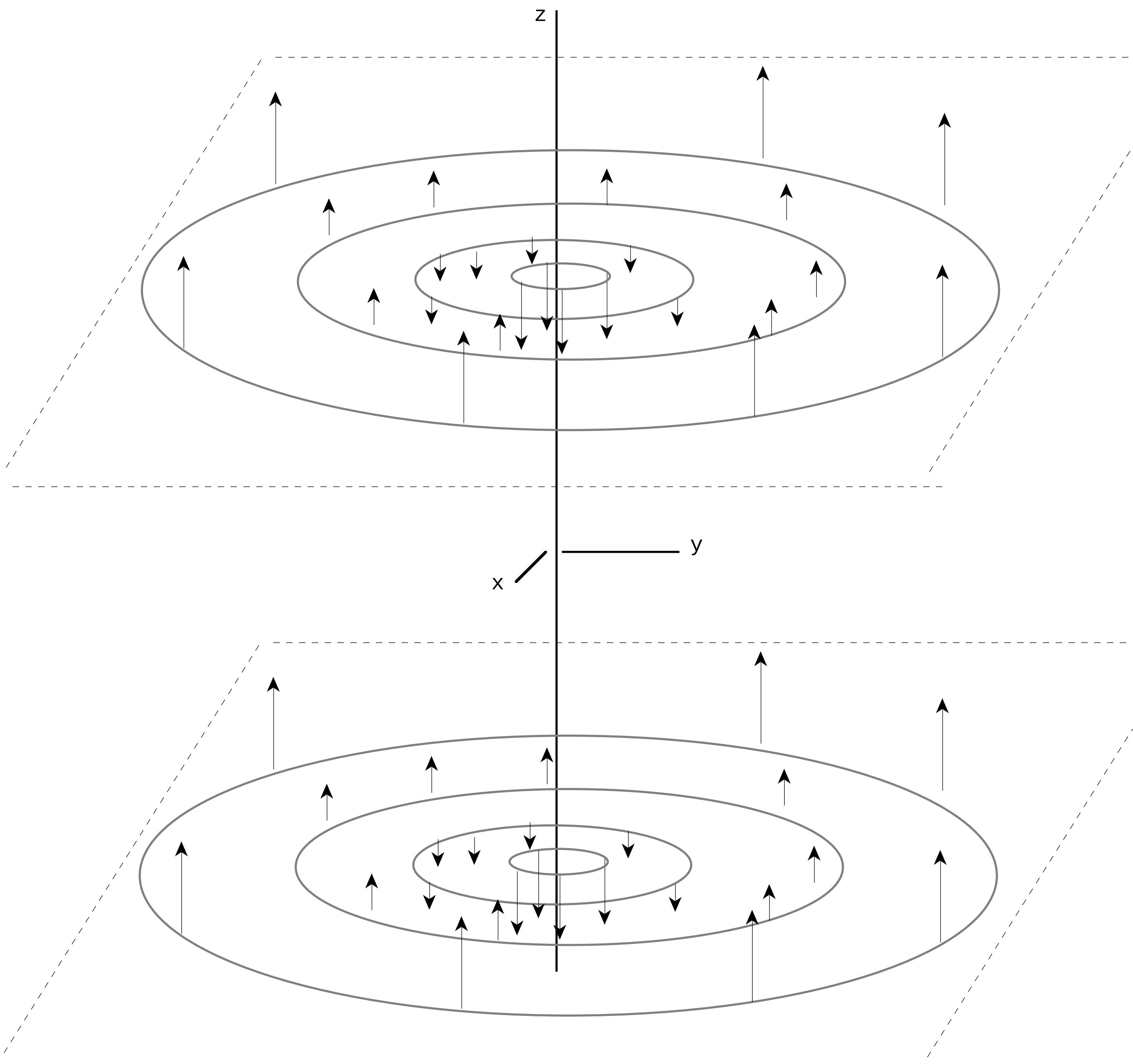
If $r > 1$ $\ln r > 0$. Furthermore, $\ln r \to \infty$ as $r \to \infty$.

On a circle around the z-axis with radius > 1, the vectors point up and get very long as the circle gets larger.

If $r = 1$ then $\ln r = \ln 1 = 0$. So on the circle with radius 1 the vectors have zero length.

If $0 < r < 1$ then $\ln r < 0$. Furthermore, $\ln r \to -\infty$ as $r \to 0+$

On a circle around the z-axis with radius < 1, the vectors point down and get very long as the circle gets smaller.

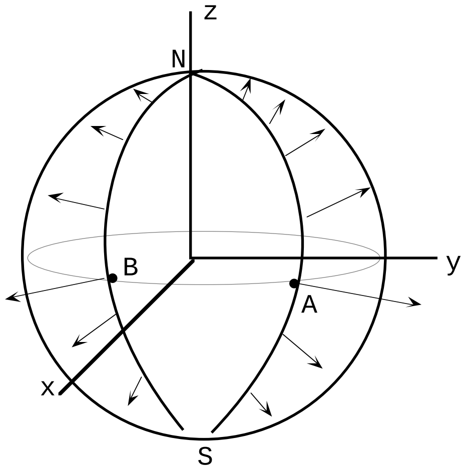


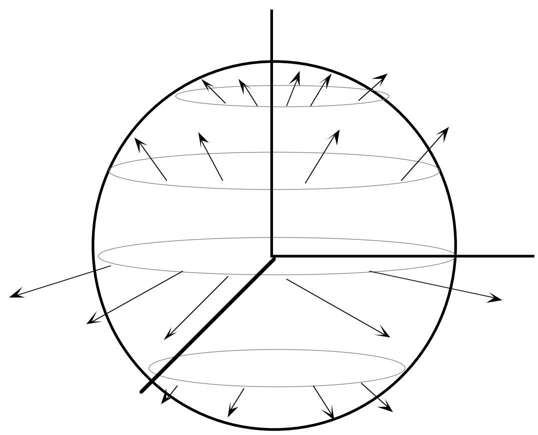
(e) The arrows point away from the origin. As ϕ goes from 0 to π, sin ϕ goes from 0 to 1 and back to 0 again. I drew three versions of the vector field. (The spheres and cones in the various diagrams are there just to help make the picture clearer. The vector field itself is just a bunch of arrows.)

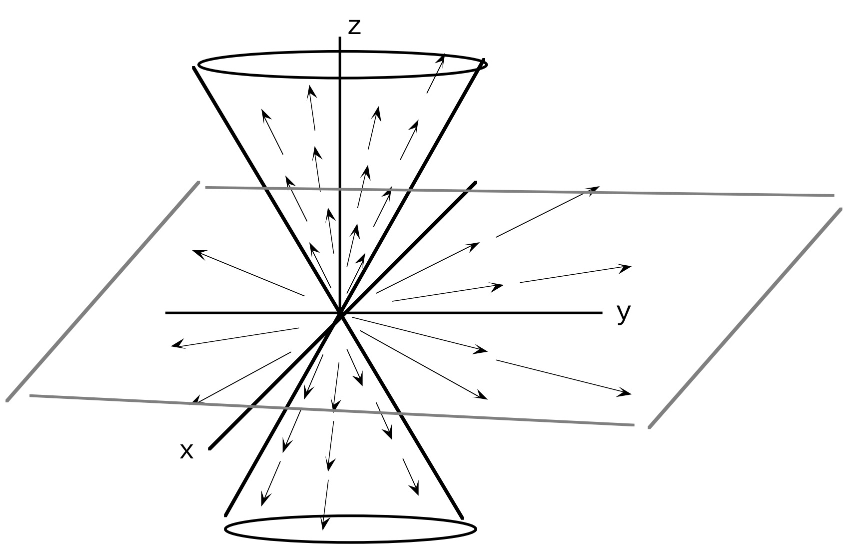
The first diagram shows a sphere with two ϕ-curves, NAS and NBS. The arrows always point perpendicularly away from the sphere. As you go down a ϕ-curve, the arrow lengths go from 0 to 1 (at a point in the x,y plane where $\phi = 90^{\circ}$) and back to 0 again. The arrow are the same on any other ϕ-curve on this sphere and the same on any other sphere.

The second diagram shows a sphere with several θ-curves. On each curve, ϕ is fixed so sinϕ is fixed. The arrows all point perpendicularly away from the sphere again.

The third version shows three cones, say $\phi = 30^0$, $\phi = 90^0$ (the x,y plane) and $\phi = 150^0$. At a point on a cone, the arrow points away from the origin and actually lies in the cone surface. All arrows at points on a cone have the same length (because ϕ is constant on the cone). If the cone opens up wider, the arrows are longer until ϕ is $\pi/2$ and the arrow lengths are 1, as long as they get. If ϕ is 150° then the arrows still point away from the origin and have the same length as they did on the cone where $\phi = 30^{\circ}$







8. (a) $-yi + xj$ is re_θ and $x^2 + y^2$ is r^2 so the field is $\frac{1}{r}\, e_\theta$

(b) $xi + yj$ is re_r so the field is $-\frac{1}{r}\, e_r$

(c) $\rho\, e_\rho$

9. (a) $k \cdot e_\phi = (0,0,1) \cdot (\cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi) = -\sin\phi$

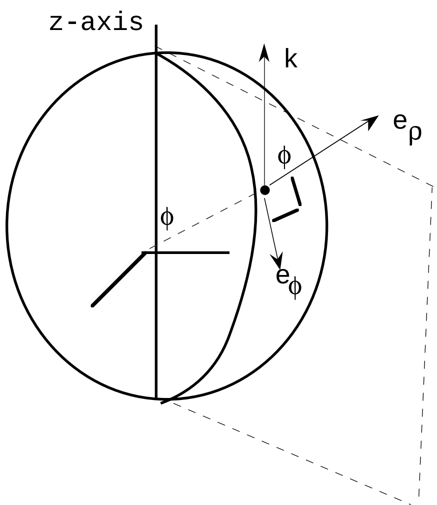
(b) $k \cdot e_\phi = \|k\|\|e_\phi\| \cos$ of angle between k and e_ϕ

= cos of angle since k and e_ϕ are unit vectors.

Now let's find that angle. At a fixed point, k, e_ρ and e_ϕ all lie in a common plane, along with the z-axis, since all three vectors are perp to e_θ.

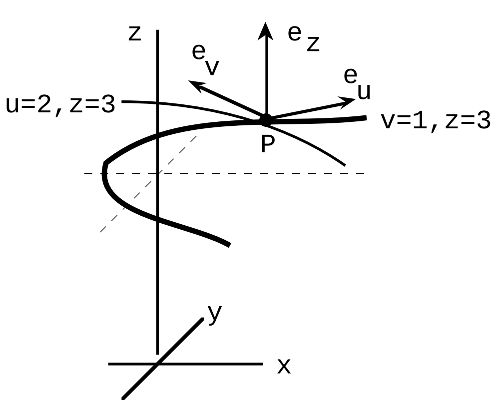
The angle between k and e_ϕ is $\phi + \pi/2$ (k flops down through angle ϕ to line up with e_ρ and then would have to go another $\pi/2$ radians to line up with e_ϕ.

So $k \cdot e_\phi = \cos(\phi + \pi/2) = \cos\phi \cos \pi/2 - \sin\phi \sin \pi/2 = -\sin\phi$.



10. See Fig 10 for a picture of 2-dim parabolic coordinate paper, including the basis vectors at the point where u=2, v=1.

Move the curves and arrows up to height 3 to get e_u and e_v at point P. The vector e_z is a unit vector pointing in the direction of increasing z so it's just k.



11. $\text{vel}_u = u\vec{i} + v\vec{j}$

$\text{vel}_v = -v\vec{i} + u\vec{j}$

$\text{vel}_u \cdot \text{vel}_v = 0$. So the parabolic system is orthogonal.

Note that vel_u and vel_v are not e_u and e_v because the vels are not *unit* vectors. But they have the same directions as e_u and e_v so for the purposes of testing for orthogonality they are sufficient.

12. No matter what coordinate system you're in, e_u and e_v are unit vectors so $e_v \cdot e_v = 1$ (dot product of a unit vector with itself is always 1) and $\|e_u\| = 1$. Can't say anything about $e_u \cdot e_v$.

13. One version of e_θ is $(-yi + xj)_{\text{unit}}$

And one version of e_ρ is $(xi + yj + zk)_{\text{unit}}$

$(-yi + xj) \cdot (xi + yj + zk) = -yx + xy = 0$.

So e_θ and e_ρ are perp.

$e_\phi = \cos\phi \cos\theta\, i + \cos\phi \sin\theta\, j - \sin\phi\, k$

$e_\rho = \sin\phi \cos\theta\, \vec{i} + \sin\phi \sin\theta\, \vec{j} + \cos\phi\, \vec{k}$

$$e_\phi \cdot e_\rho = \cos\phi \sin\phi \cos^2\theta + \cos\phi \sin\phi \sin^2\theta - \sin\phi \cos\phi$$

$$= \cos\phi \sin\phi\, \underbrace{[\cos^2\theta + \sin^2\theta]}_{1} - \sin\phi \cos\phi = 0$$

So e_ϕ and e_ρ are perp.

Finally,

$e_\phi = \cos\phi \cos\theta\, i + \cos\phi \sin\theta\, j - \sin\phi\, k$

$e_\theta = -\sin\theta\, i + \cos\theta\, j$

$e_\phi \cdot e_\theta = -\cos\phi \cos\theta \sin\theta + \cos\phi \sin\theta \cos\theta = 0$

So e_ϕ and e_θ are perp. QED

14. $e_u = (u\vec{i} + v\vec{j})_{\text{unit}} = \dfrac{u}{\sqrt{u^2+v^2}}\,\vec{i} + \dfrac{v}{\sqrt{u^2+v^2}}\,\vec{j}$ (as in example 3)

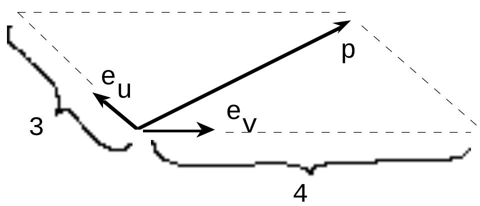
$$e_v = \frac{-v}{\sqrt{u^2+v^2}}\,\vec{i} + \frac{u}{\sqrt{u^2+v^2}}\,\vec{j}$$

$$F \cdot e_u = \frac{xu}{\sqrt{u^2+v^2}} = \frac{\frac{1}{2}(u^2-v^2)u}{\sqrt{u^2+v^2}}$$

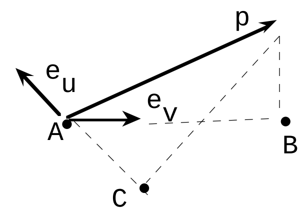
$$F \cdot e_v = \frac{-xv}{\sqrt{u^2+v^2}} = \frac{-\frac{1}{2}(u^2-v^2)v}{\sqrt{u^2+v^2}}$$

By (6), $F = \dfrac{\frac{1}{2}(u^2-v^2)u}{\sqrt{u^2+v^2}}\, e_u - \dfrac{\frac{1}{2}(u^2-v^2)v}{\sqrt{u^2+v^2}}\, e_v$.

15. (a) See diagram. Looks like p is approximately $3e_u + 4e_v$ so $a \approx 3$, $b \approx 4$.
(b) The component of p in the direction of e_u (signed projection) is $-AC$, about -1.5.
The component of p in the direction of e_v is AB, about 3.



(a)



(b)

16. (a) $vel_u = (2u+v)i + j$, $vel_v = ui - j$.

$vel_u \cdot vel_v$ is not always 0 so the coord system is not orthog.

(b) $$e_u = (vel_u)_{unit} = \frac{2u+v}{\sqrt{(2u+v)^2+1}}\, i + \frac{1}{\sqrt{(2u+v)^2+1}}\, j$$

$$e_v = \frac{u}{\sqrt{u^2+1}}\, i - \frac{1}{\sqrt{u^2+1}}\, j$$

17. $$e_\rho = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$$
$$F \cdot e_\rho = \cos\phi$$
$$e_\phi = \cos\phi\cos\theta\,\vec{i} + \cos\phi\sin\theta\,\vec{j} - \sin\phi\,\vec{k}$$
$$F \cdot e_\phi = -\sin\phi$$
$$e_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$
$$F \cdot e_\theta = 0$$

By (6), $F = \cos\phi\, e_\rho - \sin\phi\, e_\phi$

18. By inspection, $e_u = 1e_u + 0e_v$ and $e_v = 0e_u + 1e_v$.

19. The polar coord system is orthog.

(*) $$e_r = \cos\theta\, i + \sin\theta\, j$$
$$e_\theta = -\sin\theta\, i + \cos\theta\, j$$

By (6),
$$i = (i \cdot e_r)e_r + (i \cdot e_\theta)e_\theta = \cos\theta\, e_r - \sin\theta\, e_\theta$$
$$j = (j \cdot e_r)e_r + (j \cdot e_\theta)e_\theta = \sin\theta\, e_r + \cos\theta\, e_\theta$$

You could also get this by solving the system of equations in (*) for i and j but this way was faster.

SOLUTIONS Section 2.3

1. The spherical coordinate system is orthogonal so $a \cdot b = 8-6-5 = -3$.
And e_ρ, e_ϕ, e_θ, *in that order*, are righthanded so by (4), $a \times b = 13e_\rho + 14e_\phi + 16e_\theta$.

2. (a) First, find e_u and e_v at point u=2, v=1.

$$e_u = (\text{vel}_u)_{\text{unit}} = \frac{u}{\sqrt{u^2+v^2}}\,i + \frac{v}{\sqrt{u^2+v^2}}\,j,$$

$$e_v = (\text{vel}_v)_{\text{unit}} = \frac{-v}{\sqrt{u^2+v^2}}\,i + \frac{u}{\sqrt{u^2+v^2}}\,j.$$

At point u=2, v=3, $e_u = \frac{2}{\sqrt{13}}\,i + \frac{3}{\sqrt{13}}\,j$, $e_v = \frac{-3}{\sqrt{13}}\,i + \frac{2}{\sqrt{13}}\,j$.

Since the parabolic coord system is orthogonal, you can use (6) in the preceding section to write p and q in terms of e_u and e_v.

$$p = (p\cdot e_u)e_u + (p\cdot e_v)e_v = \frac{14}{\sqrt{13}}\,e_u + \frac{5}{\sqrt{13}}\,e_v$$

$$q = (q\cdot e_u)e_u + (q\cdot e_v)e_v = \frac{7}{\sqrt{13}}\,e_u - \frac{17}{\sqrt{13}}\,e_v$$

(b) $p\cdot q = (1)(5) + (4)(-1) = 1$

$\|p\| = \sqrt{1+16} = \sqrt{17}$

(c) Since e_u and e_v are orthogonal,

$$p\cdot q = \frac{14}{\sqrt{13}}\;\frac{7}{\sqrt{13}} + \frac{5}{\sqrt{13}}\left(-\frac{17}{\sqrt{13}}\right) = \frac{98}{13} - \frac{85}{13} = 1$$

$$\|p\| = \sqrt{\frac{196}{13} + \frac{25}{13}} = \sqrt{\frac{221}{13}} = \sqrt{17}$$

The answers from (c) agree with the answers from (b).

3.(a) The cross product should point like my thumb if my fingers curl like e_ρ turning into e_θ. My thumb is pointing like $-e_\phi$. And

length of the cross product

$$= \underbrace{\|e_\rho\|}_{1}\ \underbrace{\|e_\theta\|}_{1}\ \underbrace{\text{sin of angle between } e_\rho \text{ and } e_\theta}_{\sin 90^\circ} = 1$$

So the cross product points like $-e_\phi$ and has length 1. So it *is* $-e_\phi$

(b) e_ρ, e_ϕ, e_θ [in that order] are righthanded so I'll write a and b using this order.

$$a = 1\ e_\rho + 0\ e_\phi + 0\ e_\theta$$

$$b = 0\ e_\rho + 0\ e_\phi + 1\ e_\theta$$

$$a \times b = 0\ e_\rho - 1\ e_\phi + 0\ e_\theta = -e_\phi$$

(c) $a \times b = \dfrac{-xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\ i - \dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\ j + \dfrac{x^2+y^2}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\ k$

Doesn't look the same as (b) yet. Make these replacements:

$\sqrt{x^2+y^2} = r$ (the cylindrical coordinate, distance to z–axis).

$\sqrt{x^2+y^2+z^2} = \rho$

$x = r \cos \theta$
$y = r \sin \theta$
$z = \rho \cos \phi$
$\cos \phi = r/\rho$ (see Fig 6 in Section 1.1)

and you'll end up with $-\cos \phi \cos \theta\ \vec{i} - \cos \phi \sin \theta\ \vec{j} + \sin \phi\ \vec{k}$ which *is* $-e_\phi$.

4. Since all of this involves only directions of vectors, and not their lengths, instead of using the *unit* vectors e_u, e_v, e_w, it's OK to use the non-unit vectors $a = 2i+k$, $b = 2i+j+3k$, $c = i+4j+k$.

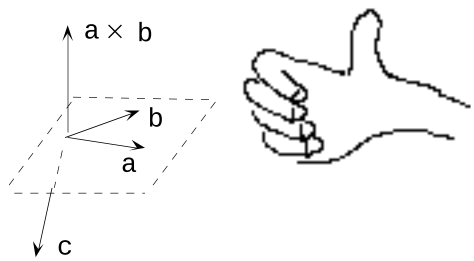
I want to test to see if my thumb points like c when the fingers of my right hand curl like a turning into b.

First find $a \times b$ which definitely points like your thumb. Then see if c makes an acute angle with $a \times b$.

$$a \times b = -i - 4j + 2k$$

$$c \cdot (a \times b) = -1 - 16 + 2 = -15$$

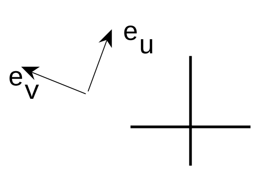
Since the dot product is negative, c makes an *obtuse* angle with $a \times b$.
So a,b,c *in that order* are lefthanded. So e_u, e_v, e_w is a lefthanded triple.



5. (a) Yes *if* the u,v system is orthogonal; in that case you take norms in the "usual" way and the vector has norm $\sqrt{(3/5)^2 + (4/5)^2} = 1$.

No *if* the u,v system is not orthogonal. In this case, the vector is the diagonal of a parallelogram that is *not* a rectangle, with sides 3/5 and 4/5. Its length is not 1.

(b) Not necessarily whether or not the system is orthogonal. For instance with the basis vectors in the diagram, $\frac{3}{5} e_u + \frac{4}{5} e_v$ would point sort of northwest.



SOLUTIONS Section 2.4

1. $h_r = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$h_\theta = \sqrt{r^2 \sin^2\theta + r^2 \cos^2\theta} = \sqrt{r^2} = r$ (since r is positive)

2. $h_\rho = \sqrt{(\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi}$

$= \sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\theta} = \sqrt{\sin^2\phi + \cos^2\phi} = 1$

$h_\phi = \sqrt{(\rho\cos\phi\cos\theta)^2 + (\rho\cos\phi\sin\theta)^2 + (-\rho\sin\phi)^2} = \rho$

$h_\theta = \sqrt{(-\rho\sin\phi\sin\theta)^2 + (\rho\sin\phi\cos\theta)^2 + 0} = \rho\sin\phi$

3. $\frac{\partial x}{\partial u} = 2uv^3$, $\frac{\partial y}{\partial u} = 3$, $h_u = \sqrt{4u^2v^6 + 9}$

If $u = 2$ and $v = 1$ then $h_u = 5$ so the particle moves distance $ds_u = 5\,du$.

4. Remember that $h_v = \|vel_v\|$, your "speed" if v is time and you walk with u fixed.

(a) As you walk on the u=2 curve from A to B to C to D, your "speed" is increasing (you cover more distance between times v=2 and v=3 than you did between times v=1 and v=2). So $\|vel_v\|$ is increasing as v increases. So h_v is largest at D.

(b) As you walk along PQRS, your "speed" is decreasing: The distance from R to S is about the same as the distance from Q to R but it takes much more "time" (change in v) to get from R to S than it took to go from Q to R. So it looks like $h_{v \text{ at } P} > h_{v \text{ at } Q} > h_{v \text{ at } R} > h_{v \text{ at } S}$.

5. (a) As you walk from point B with u fixed and v increasing, in the one "second" between "time" v=2 and "time" v=3 you traverse BD.

As you walk from A with u fixed and v increasing, in the one second from v=2 to v=3 you traverse AC.

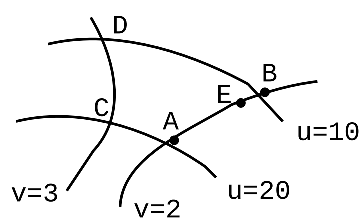
You get *further* in one second on the first path; i.e., you are traveling faster on the first path.

So h_v at B is larger than h_v at A.

(b) If you start at B, fix v and let u change then in one "second" you get to E.

If you start at B, fix u and let v change, then in one "second" you get to D.

You get further in one "second" if v does the changing; i.e., you go faster if v does the changing. So at B, $h_v > h_u$.



6. $h_u = \sqrt{4u^2 + 1}$, $h_v = \sqrt{2}$.

$h_u = h_v$ iff $4u^2 + 1 = 2$, $u = \pm 1/2$

So if $u = \pm 1/2$ you get the same arc length from a u-change as from a v-change.
If $u > 1/2$ or $u < -1/2$ then $h_u > h_v$ and you get more arc length from a u-change.

If $-1/2 < u < 1/2$ then $h_v > h_u$ and you get more arc length from a v-change.

7. The parabolic cylindrical coord system is defined by

$x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$, $v \geq 0$.

The scale factors h_u and h_v are the same as in parabolic coords; each is $\sqrt{u^2+v^2}$.

And $h_z = 1$.

SOLUTIONS Section 2.5

1. (a) $\nabla \sin\phi = \dfrac{\partial(\sin\phi)}{\partial\rho}\, e_\rho + \dfrac{1}{\rho}\dfrac{\partial(\sin\phi)}{\partial\phi}\, e_\phi + \dfrac{1}{\rho\sin\phi}\dfrac{\partial(\sin\phi)}{\partial\theta}\, e_\theta$

$= \dfrac{1}{\rho}\cos\phi\, e_\phi$

(b) $\nabla(\rho\sin\phi) = \dfrac{\partial(\rho\sin\phi)}{\partial\rho}\, e_\rho + \dfrac{1}{\rho}\dfrac{\partial(\rho\sin\phi)}{\partial\phi}\, e_\phi + \dfrac{1}{\rho\sin\phi}\dfrac{\partial(\rho\sin\phi)}{\partial\theta}\, e_\theta$

$= \sin\phi\, e_\rho + \cos\phi\, e_\phi$

2. $\nabla f = \dfrac{\partial f}{\partial r}\, e_r + \dfrac{1}{r}\dfrac{\partial f}{\partial\theta}\, e_\theta$

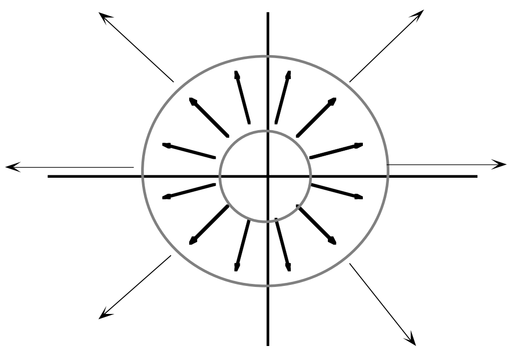
3.(a) In polar coords, temp $= r^3$ so ∇temp $= 3r^2\, e_r$.

Temp level sets are circles. The K level set (where $K \ge 0$) is a circle with center at the origin and radius $\sqrt[3]{K}$. The gradient arrows all point away from the origin and the length of the arrow at a point is 3 times the square of the distance from the point to the origin.

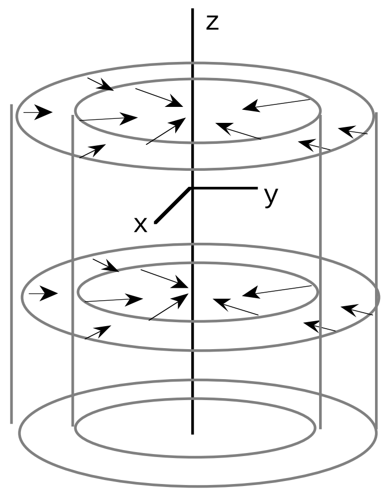
(b) In cylindrical coords, temp $= \dfrac{1}{r}$ so ∇temp $= -\dfrac{1}{r^2}\, e_r$.

The level surfaces are cylinders. The K level set (where $K \ge 0$) is the cylinder whose axis is the z-axis and with radius $1/K$. The gradient arrows all point toward the z-axis and the length of the arrow at a point is $1/(\text{dist to z-axis})^2$.

In each case, the gradient arrows are perp to the level sets.



(a)



(b)

4. $F = \dfrac{1}{r}\, e_r$ so $F = \nabla \ln r$, i.e., an antigradient is $\ln r$.

5. (a) $\|\vec{r}\|^3 = (x^2+y^2+z^2)^{3/2}$

$\nabla\|\vec{r}\|^3 = \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x\vec{i} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2y\vec{j} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2z\vec{k}$

$= 3\sqrt{x^2+y^2+z^2}\,(x\vec{i} + y\vec{j} + z\vec{k})$

(b) $\|\vec{r}\|$ = distance from point (x,y,z) to the origin so $\|\vec{r}\| = \rho$ in spherical coords,

$\|\vec{r}\|^3 = \rho^3$,

$\nabla\|\vec{r}\|^3 = \nabla\rho^3 = 3\rho^2\, e_\rho$

This agrees with part (a) because

$$3\rho^2\ e_\rho = 3(x^2+y^2+z^2)\ \frac{1}{\sqrt{x^2+y^2+z^2}}\ (x\vec{i}+y\vec{j}+z\vec{k})$$

$$= 3\sqrt{x^2+y^2+z^2}\,(xi+yj+zk)$$

6. $h_u = \|vel_u\| = \sqrt{(\partial x/\partial u)^2 + (\partial y/\partial u)^2} = \sqrt{u^2+v^2}$

$h_v = \|vel_v\| = \sqrt{u^2+v^2}$

$$\text{div } F = \frac{1}{u^2+v^2}\ \frac{\partial(u\sqrt{u^2+v^2})}{\partial v}$$

$$= \frac{1}{u^2+v^2}\cdot u\,\frac{v}{\sqrt{u^2+v^2}}$$

$$= \frac{uv}{(u^2+v^2)^{3/2}}$$

7.(a) $vel_r = e^r \cos\theta\ i + e^r \sin\theta\ j$, $vel_\theta = -e^r\sin\theta\ i + e^r\cos\theta\ j$

$vel_r \cdot vel_\theta = 0$ so coord system is orthogonal.

(b) $h_r = \|vel_r\| = e^r$, $h_\theta = \|vel_\theta\| = e^r$

$$\text{div } F = \frac{1}{e^{2r}}\left(\frac{\partial(re^r)}{\partial r} + \frac{\partial(e^r\cos\theta)}{\partial\theta}\right)$$

$$= e^{-2r}\,(re^r + e^r - e^r\sin\theta)$$

$$= e^{-r}(r+1-\sin\theta)$$

(c) $$\nabla r^3 = \frac{1}{e^r}\ \frac{\partial(r^3)}{\partial r}\ \vec{e}_r$$

$$= 3r^2\ e^{-r}\ \vec{e}_r$$

(d) Lapl r^3 = div ∇r^3 = div of $3r^2\ e^{-r}\ \vec{e}_r$ from part (c)

$$= \frac{1}{e^{2r}}\ \frac{d(e^r\ 3r^2\ e^{-r})}{dr}$$

$$= \frac{1}{e^{2r}}\ \frac{d(3r^2)}{dr}$$

$$= 6re^{-2r}$$

8. $\dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$ in spherical coordinates

$h_\rho = 1$, $h_\phi = \rho$, $h_\theta = \rho\sin\phi$

$$\text{Lapl}\ \frac{1}{\rho^2} = \text{div}\ \nabla\ \frac{1}{\rho^2}$$

$$\nabla\ \frac{1}{\rho^2} = -\ \frac{2}{\rho^3}\ \vec{e}_\rho$$

$$\text{div}\ \nabla \frac{1}{\rho^2} = \text{div}\left(-\frac{2}{\rho^3}\vec{e}_\rho\right)$$

$$= -\frac{1}{\rho^2 \sin\phi}\ \frac{\partial\left(\frac{2}{\rho^3}\ \rho^2 \sin\phi\right)}{\partial\rho}$$

$$= -\frac{1}{\rho^2 \sin\phi}\ \frac{\partial\left(\frac{1}{\rho}\ 2 \sin\phi\right)}{\partial\rho}$$

$$= -\frac{1}{\rho^2 \sin\phi}\ \cdot 2 \sin\phi \cdot -\frac{1}{\rho^2}$$

$$= \frac{2}{\rho^4}$$

So back in the original coordinate system,

$$\text{Lapl}\ \frac{1}{x^2+y^2+z^2} = \frac{2}{(x^2+y^2+z^2)^2}$$

9. $\nabla v = \frac{\partial v}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial v}{\partial\theta}$

Lapl v = div ∇v

$$= \frac{1}{r}\left[\frac{\partial\left(r\frac{\partial v}{\partial r}\right)}{\partial r} + \frac{\partial\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right)}{\partial\theta}\right]$$

$$= \frac{1}{r}\left[r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial^2 v}{\partial\theta^2}\right] \quad \text{(deriv product rule in here)}$$

$$= \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}$$

Note

It didn't turn up in this problem, but you would also need a product rule to find $\frac{\partial\left(r\frac{\partial v}{\partial\theta}\right)}{\partial r}$ because $\partial v/\partial\theta$ is a function of θ *and* of r.

The derivative would be $r\frac{\partial^2 v}{\partial r\ \partial\theta} + \frac{\partial v}{\partial r}$.

10. $h_r = 1$, $h_\theta = r$, $h_z = 1$

(a) $\text{div} = \frac{1}{r}\left[\frac{\partial(2rz)}{\partial r} + \frac{\partial(3z)}{\partial\theta}\right]$

$$= \frac{2z}{r}$$

(b) curl = $\begin{vmatrix} \frac{1}{r}\, e_r & e_\theta & \frac{1}{r}\, e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 2z & 3rz & 0 \end{vmatrix}$

$$= -3r\,\frac{1}{r}\,e_r + 2\cdot e_\theta + 3z\,\frac{1}{r}\,e_z$$

$$= -3e_r + 2e_\theta + \frac{3z}{r}\,e_z$$

11. (a) $F = \frac{1}{\rho^2}\,e_\rho$, $h_\rho = 1$, $h_\phi = \rho$, $h_\theta = \rho \sin\phi$

$F = \nabla(-\frac{1}{\rho})$, i.e., an antigradient is $-1/\rho$

$$\text{div } F = \frac{1}{h_\rho h_\phi h_\theta}\;\frac{\partial(\rho^2 \sin\phi \cdot \frac{1}{\rho^2})}{\partial\rho} = \frac{1}{\text{who cares}}\;\frac{\partial \sin\phi}{\partial\rho} = 0$$

curl $F = \vec{0}$ because F is a gradient and curl of a gradient is $\vec{0}$.

If you used (9) to get the curl, you should get the answer $\vec{0}$ but that's the long way.

(b) $h_r = 1$, $h_\theta = r$, $h_z = 1$

$F = \nabla(-\frac{1}{r})$, i.e., an antigradient is $-1/r$.

$$\text{div } F = \frac{1}{h_r h_\theta h_z}\;\frac{\partial(r\cdot\frac{1}{r^2})}{\partial r} = \frac{1}{r}\;\frac{\partial(1/r)}{\partial r} = \frac{1}{r}\cdot -\frac{1}{r^2} = -\frac{1}{r^3}$$

curl $F = \vec{0}$ because F is a gradient.

12. First you have to think of F as being in parabolic cylindrical coordinates where $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$, $v \ge 0$.

Then $h_u = h_v = = \sqrt{u^2 + v^2}$ (see problem 6), $h_z = 1$.

$$\text{Curl } F = \begin{vmatrix} \frac{1}{\sqrt{u^2+v^2}}\,e_u & \frac{1}{\sqrt{u^2+v^2}}\,e_v & \frac{1}{u^2+v^2}\,e_z \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ 1\cdot\sqrt{u^2+v^2} & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{u^2+v^2}\,e_z \cdot \frac{\partial(\sqrt{u^2+v^2}\,)}{\partial v}$$

$$= \frac{1}{u^2+v^2}\cdot\frac{-v}{\sqrt{u^2+v^2}}\;\vec{k}$$

Note that $e_z = k$.

13. curl F =

$$\begin{vmatrix} \frac{1}{\rho^2 \sin\phi}\, e_\rho & \frac{1}{\rho \sin\phi}\, e_\phi & \frac{1}{\rho}\, e_\theta \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 0 & 0 & \rho\cdot\rho \sin\phi \end{vmatrix}$$

$$= \rho^2 \cos\phi \cdot \frac{1}{\rho^2 \sin\phi}\, e_\rho - 2\rho \sin\phi \cdot \frac{1}{\rho \sin\phi}\, e_\phi$$

$$= \frac{\cos\phi}{\sin\phi}\, e_\rho - 2\, e_\phi$$

14. (a) $$\text{div } F = \frac{x^2+y^2+z^2 - x\cdot 2x}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - y\cdot 2y}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - z\cdot 2z}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

(b) $xi + yj + zk$ is ρe_ρ.

$$F = \frac{1}{\rho^2}\, \rho e_\rho = \frac{1}{\rho}\, e_\rho$$

$$\text{Div } F = \frac{1}{\rho^2 \sin\phi}\; \frac{\partial(\rho^2 \sin\phi \cdot \frac{1}{\rho})}{\partial\rho}$$

$$= \frac{1}{\rho^2 \sin\phi}\; \frac{\partial\, \rho \sin\phi}{\partial\rho}$$

$$= \frac{1}{\rho^2 \sin\phi} \cdot \sin\phi = \frac{1}{\rho^2}$$

This agrees with part (a) since $x^2+y^2+z^2$ *is* ρ^2

(c) curl F = $\vec{0}$ because F is a gradient (it's of the form ρ-stuff e_ρ) and curl of any gradient is $\vec{0}$.

15. To be able to take curl, you have to think of F as being in cylindrical coordinates where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
The basis vectors e_r, e_θ, k [$= e_z$] in that order are righthanded.

$$\text{Curl } F = \begin{vmatrix} \frac{1}{r}\, e_r & e_\theta & \frac{1}{r}\, k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \cos\theta\cdot r & 0 \end{vmatrix}$$

$$= \cos\theta\; \frac{1}{r}\, k$$

SOLUTIONS review problems for Chapter 2

1. $\nabla\ \rho \sin\theta = \sin\theta\ e_\rho + \dfrac{1}{\rho \sin\phi}\ \rho \cos\theta\ e_\theta$

$$= \sin\theta\ e_\rho + \frac{1}{\sin\phi} \cos\theta\ e_\theta$$

Lapl $\rho \sin\phi = \text{div}\ (\sin\theta\ e_\rho + \dfrac{1}{\sin\phi}\cos\theta\ e_\theta)$

$$= \frac{1}{\rho^2 \sin\phi}\left(\frac{\partial(\rho^2 \sin\phi \sin\theta)}{\partial\rho} + \frac{\partial(\rho \frac{1}{\sin\phi}\cos\theta)}{\partial\theta} \right)$$

$$= \frac{1}{\rho^2 \sin\phi}\left[2\rho \sin\phi \sin\theta - \frac{\rho}{\sin\phi}\sin\theta\right]$$

$$= \frac{2}{\rho}\sin\theta - \frac{\sin\theta}{\rho \sin^2\phi}$$

2. $h_u = h_v = \sqrt{u^2 + v^2}$

(*) $\nabla(u^2 + 2v) = \dfrac{1}{\sqrt{u^2+v^2}}\, 2u e_u + \dfrac{1}{\sqrt{u^2+v^2}}\, 2e_v$ (by (2) in §2.5)

At point u=1, v=2, the gradient is $\dfrac{2}{\sqrt{5}}\, e_u + \dfrac{2}{\sqrt{5}}\, e_v$.

(**) $\dfrac{d\text{temp}}{ds} = \dfrac{\nabla\text{temp}\cdot(2e_u - 3e_v)}{\|2e_u - 3e_v\|} = -\dfrac{2}{\sqrt{65}}$ degrees per meter.

I used the fact that the system is orthog at step (*) because the formula in §2.5 for taking the gradient in a u,v coord system is only for an *orthog* system.
I also used the orthogonality at the last step in line (**) when I computed dots and norms in the "usual" manner (Section 2.3). Can't do that in a non-orthog system.

3. There is no u-axis or v-axis, just like there is no r-axis and no θ-axis on a piece of polar coordinate paper.

4. Not necessarily. It's uniform if e_u and e_v don't change direction from point to point, but not otherwise. For instance $2e_r - 3e_\theta$ (polar coords) is not a uniform field.

5. (a) $F = \dfrac{1}{r^2}(xi + yj) = \dfrac{1}{r^2}\, r e_r = \dfrac{1}{r}\vec{e}_r$

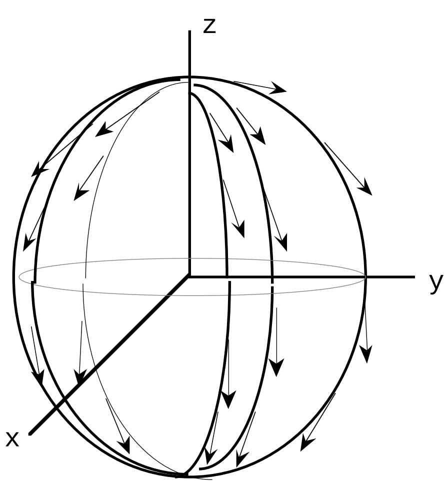
(b) *Cartesian coords* $\text{div } F = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} = \dfrac{x^2 + y^2 - x(2x)}{(x^2+y^2)^2} + \dfrac{x^2 + y^2 - y(2y)}{(x^2+y^2)^2} = 0$

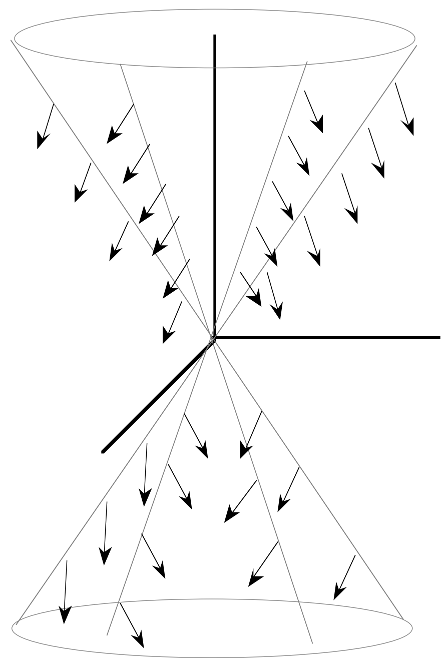
polar coords $\text{div } F = \dfrac{1}{r}\dfrac{\partial(h_\theta \frac{1}{r})}{\partial r} = \dfrac{1}{r}\dfrac{\partial 1}{\partial r} = 0$

(c) $F = \dfrac{1}{r}\vec{e}_r = \nabla(-1/r^2)$. Curl $F = \vec{0}$ because the curl of any gradient is $\vec{0}$.

6. (a) Look at a sphere around the origin. The field arrows are tangent to the great circles on the sphere through the North and South poles. The arrows all point "down" the great circle from North pole toward South pole. All the arrows have length 1. The arrows look similar on every size sphere,

Another way to visualize the field is to look at a cone. The diagram shows the cones $\phi = 30^{\circ}$ and $\phi = 150^{\circ}$. The field arrows are perpendicular to the cones and point downish. On the top cone the arrows point out. On the bottom cone the arrows point in. At points on the x,y plane (the "cone" $\phi = 90^{\circ}$) the field arrows are all $-\vec{k}$





$$\text{(b) curl } e_{\phi} = \begin{vmatrix} \dfrac{1}{\rho^2 \sin\phi}\, e_{\rho} & \dfrac{1}{\rho \sin\phi}\, e_{\phi} & \dfrac{1}{\rho}\, e_{\theta} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ 0 & 1 \cdot \rho & 0 \end{vmatrix} = \frac{1}{\rho}\, e_{\theta}$$

7. (a) The θ-curve $r=r_0$ has parametric equations

$$x = 2r_0 \cos\theta$$

$$y = 4r_0 \sin\theta$$

If you don't recognize immediately what curve this parametrizes then eliminate the parameter θ like this:

$$\frac{x}{2} = r_0 \cos\theta$$

$$\frac{y}{2} = r_0 \sin\theta$$

so

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = r_0^2 \cos^2\theta + r_0^2 \sin^2\theta = r_0^2(\cos^2\theta + \sin^2\theta) = r_0^2 .$$

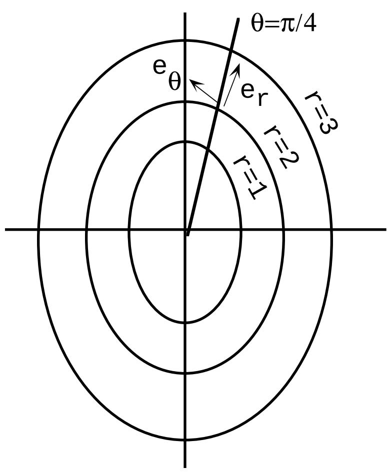
The θ-curve is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = r_0^2$.

(b) The curve $\theta=\pi/4$ has parametric equations

$$x = r\sqrt{2}$$
$$y = 2r\sqrt{2}$$
$$r \ge 0.$$

Divide the equations to eliminate the parameter r and get $y = 2x$. And note that since $r \ge 0$ you also have $x \ge 0$, $y \ge 0$. So the graph is not the whole line $y = 2x$, it's just half the line, a ray through the origin with slope 2.
The ray is *not* at angle $\pi/4$; it's inclined at an angle larger than $\pi/4$.

footnote The coordinates r and θ do not have geometric significance here. They are not the nice geometric r and θ from polar coordinates.
r is not the distance from a point to the origin. If a point has coordinate $r = 7$, it is not distance 7 from the origin. It just means that the point is on the ellipse $x^2/4 + y^2/9 = 49$.
And θ is not the "around angle" (except at points on the axes).



(c) $\text{vel}_r = 2\cos\theta\, i + 4\sin\theta\, j$, $\text{vel}_\theta = -2r\sin\theta\, i + 4r\cos\theta\, j$

$\text{vel}_r \cdot \text{vel}_\theta = 12r\cos\theta\sin\theta.$

Dot product is not always 0. Coord system is not orthogonal.

(d) At point $r=2$, $\theta=\pi/4$,

$$\text{vel}_r = \sqrt{2}\, i + 2\sqrt{2}\, j$$

$$\text{vel}_\theta = -2\sqrt{2}\, i + 4\sqrt{2}\, j$$

$$e_r = (\text{vel}_r)_{\text{unit}} = \frac{1}{\sqrt{5}}\, i + \frac{2}{\sqrt{5}}\, j$$

(tangent to the r-curve and pointing in direction of increasing r)

$$e_\theta = -\frac{1}{\sqrt{5}}\, i + \frac{2}{\sqrt{5}}\, j$$

(tangent to the θ-curve and pointing in direction of increasing θ)

(e) $h_r = \|\text{vel}_r\| = \sqrt{10}$.

If you start at point $r=2$, $\theta=\pi/4$ and change r by dr then you walk $\sqrt{10}\,dr$ meters (in the direction of e_r).

8. Use the rule in (6) in Section 2.2.

$$e_\rho = \sin\phi \cos\theta \,\vec{i} + \sin\phi \sin\theta \,\vec{j} + \cos\phi \,\vec{k}$$

$$e_\phi = \cos\phi \cos\theta \,\vec{i} + \cos\phi \sin\theta \,\vec{j} - \sin\phi \,\vec{k}$$

$$e_\theta = -\sin\theta \,\vec{i} + \cos\theta \,\vec{j}$$

At the given point,

$$e_\rho = 0i + \tfrac{1}{2}\sqrt{2}\, j + \tfrac{1}{2}\sqrt{2}\, k$$
$$e_\phi = 0i + \tfrac{1}{2}\sqrt{2}\, j - \tfrac{1}{2}\sqrt{2}\, k$$
$$e_\theta = -i$$

$$F\cdot e_\rho = 5\sqrt{2}$$
$$F\cdot e_\phi = -\sqrt{2}$$
$$F\cdot e_\theta = -2$$
$$F = 5\sqrt{2}\, e_\rho - \sqrt{2}\, e_\phi - 2e_\theta$$

9. Do some rearranging. And use the fact that $-yi + xj$ is re_θ

$$F = \frac{y^3}{(x^2+y^2)^2}\, i + \frac{-xy^2}{(x^2+y^2)^2}\, j$$

$$= \frac{-y^2}{(x^2+y^2)^2}\,(-yi + xj) \quad \text{rearrange}$$

$$= \frac{-r^2 \sin^2\theta}{r^4}\, re_\theta$$

$$= \frac{-\sin^2\theta}{r}\, e_\theta$$

footnote You could also do this like I did #8 (by finding $F\cdot e_r$ and $F\cdot e_\theta$) (they should come out to be 0 and $\dfrac{-\sin^2\theta}{r}$ respectively) but it's longer and messier and uncool that way.

SOLUTIONS Section 3.1

1. (a) Point A goes with $t = -1$ and point B goes with $t = 2$.

(b) *method 1* (using (6)) $\int F\cdot T\, ds = \int 2x\, dx + xy\, dy$

$$= \int_{\text{initial } t}^{\text{final } t} 2\cdot 3t^2\cdot 6t\, dt + 3t^2\cdot 2t\cdot 2\, dt = \int_2^{-1} 48t^3\, dt \quad [= -180]$$

method 2 (using (2))

$v = (6t, 2)$

v points in the B to A direction (in the direction of increasing t) so it's $-v$ that points like T.

$$(*) \quad \int F\cdot T\, ds = \int_{t=-1}^{2} F\cdot -v\, dt = \int_{t=-1}^{2} (-36t^3 - 12t^3)\, dt = \int_{t=-1}^{2} -48t^3\, dt \quad [= -180]$$

(c) $T = -v_{\text{unit}} = \dfrac{-6t}{\sqrt{36t^2+4}}\, i - \dfrac{2}{\sqrt{36t^2+4}}\, j$

Question So why doesn't $\sqrt{36t^2+4}$ turn up in the calculations in (*) when T got replaced.

Answer It is there but it canceled out before you could even see it.

T is $\dfrac{-v}{\sqrt{36t^2+4}}$ and ds is $\sqrt{36t^2+4}\, dt$ and when you find $F\cdot T\, ds$ the square root cancels out and all you end up with is $F\cdot -v\, dt$.

2. (a) (i) The equation of the line is $y = -2x + 3$.

The segment has parametric equations $x=x$, $y = -2x+3$, $-1 \le x \le 1$.

$$\int xy\, dx + y\, dy = \int_{x=1}^{-1} x(-2x+3)dx + (-2x+3)\cdot -2\, dx \quad \left[= \frac{40}{3}\right]$$

(ii) The segment has parametric equations $x = \frac{1}{2}(3-y)$, $y=y$, $1 \le y \le 5$.

$$\int xy\, dx + y\, dy = \int_{y=1}^{5} \frac{1}{2}(3-y)\, y \cdot -\frac{1}{2}\, dy + y\, dy \quad \left[= \frac{40}{3}\right]$$

(iii) I'll use A as (x_0, y_0) and $\overrightarrow{AB} = -2i + 4j$ as the parallel vector.

Line AB has parametric equations $x = 1-2t$, $y = 1 + 4t$.

Point A is the $t=0$ point. Point B is the $t=1$ point.

$$\int xy\, dx + y\, dy = \int_{t=0}^{1} (1-2t)(1+4t)\cdot -2\, dt + (1+4t)\, 4\, dt \quad \left[= \frac{40}{3}\right]$$

(b) It's the work done by $F = xy\,\vec{i} + y\vec{j}$ to a particle which moves on the segment from A to B.

3. (a) The entire circle has parametric equations
$x = R\cos t,\ y = R\sin t,\ 0 \le t \le 2\pi$.

$$\int \mathbf{F}\cdot\mathbf{T}\,ds = \int y\,dx + y\,dy = \int_{t=2\pi}^{0} R\sin t\cdot -R\sin t\,dt + R\sin t\cdot R\cos t\,dt$$

(the limits go from 2π to 0 to go with clockwise)

$$= R^2\int_{t=2\pi}^{0} (-\sin^2 t + \sin t\cos t)\,dt \quad [\ = \pi R^2\ ]$$

(b) Like part (a) except now the limits of integration are $\int_{3\pi/2}^{2\pi}$ or $\int_{-\pi/2}^{0}$

4. (a) Call the vector field F.
$\mathbf{F}\cdot\mathbf{T}$ is positive on the A-to-B piece, positive on the B-to-C piece and negative on the C-to-A piece .
So the ccl line integral will be pos on A-to-B, pos on B-to-C and neg on C-to-A.

(b) Line AB has parametric equations $x=x,\ y=0$.
Line BC has parametric equations $x=1,\ y=y$.
Line CA has parametric equations $x=x,\ y=x$.

$$\oint_{\text{ccl}} = \int_{A \text{ to } B} + \int_{B \text{ to } C} + \int_{C \text{ to } A}$$

$$= \underbrace{\int_{x=0}^{1} (x^3+0)\,dx + 0\,dx}_{1/4 \quad \text{(pos)}} + \underbrace{\int_{y=0}^{1} (1+y)\,0\,dy + y^2\,dy}_{1/3 \quad \text{(pos)}} + \underbrace{\int_{x=1}^{0} (x^3+x)\,dx + x^2\,dx}_{-13/12 \quad \text{(neg)}}$$

(aha! predictions worked out)

$$= -1/2$$

5. One parametrization for the line is $x = 1 + t,\ y = 2 + t,\ z = 3 + 2t$.
Then $t_A = 0,\ t_B = 1$ and

$$\text{work} = \int \mathbf{F}\cdot\mathbf{T}\,ds = \int y\,dx + z\,dy + x\,dz$$

$$= \int_{t=0}^{1} (2+t)\,dt + (3+2t)\,dt + (1+t)\,2\,dt = \int_{t=0}^{1} (7+5t)\,dt \quad [\ = 19/2\ ]$$

6. The curve has parametric equations $x = x,\ y = x^2,\ z = x^3$

$$\text{circ} = \int xy\,dx + xz\,dy + x\,dz = \int_{2}^{1} x\,x^2\,dx + x\,x^3\,2x\,dx + x\,3x^2\,dx$$

$$= \int_{2}^{1} (4x^3 + 2x^5)\,dx \quad [\ = -36]$$

7. On AD, $\mathbf{T} = \mathbf{k}$, $\mathbf{F}\cdot\mathbf{T} = 0$, $\int \mathbf{F}\cdot\mathbf{T}\,ds = 0$

On BC, $\mathbf{T} = -\mathbf{k}$, $\mathbf{F}\cdot\mathbf{T} = 0$, $\int \mathbf{F}\cdot\mathbf{T}\,ds = 0$

On AB, $z = 4$, $\mathbf{F} = 4\mathbf{i}$, $\mathbf{T} = \mathbf{i}$, $\mathbf{F}\cdot\mathbf{T} = 4$

$$\int \mathbf{F}\cdot\mathbf{T}\,ds = \int 4\,ds = 4 \times \text{length AB} = 28$$

On CD, $z = 6$, $\mathbf{F} = 6\mathbf{i}$, $\mathbf{T} = -\mathbf{i}$, $\mathbf{F}\cdot\mathbf{T} = -6$

$$\int \mathbf{F}\cdot\mathbf{T}\,ds = \int -6\,ds = -6 \times \text{length CD} = -42$$

Final answer is $28 - 42 = -14$.

8. The segment BA has parametric equations $x=x, y=x+2, -1 \le x \le 2$.
The parabola part of the path has parametric equations $x=x,\ y=x^2,\ -1 \le x \le 2$.

$$\int \mathbf{F}\cdot\mathbf{T}\,ds \text{ on the loop} = \int_{\text{B to A on line}} + \int_{\text{A to B on parabola}}$$

$$= \int_{-1}^{2} x^2(x+2)\,dx + (x+2+3)\,dx + \int_{2}^{-1} x^4\,dx + (x^2+3)\,2x\,dx$$

$$= \int_{-1}^{2} (x^3 + 2x^2 + x + 5)\,dx + \int_{2}^{-1} (x^4 + 2x^3 + 6x)\,dx \quad [\, = 63/20]$$

9. The ellipse is $x^2 + \frac{1}{3}y^2 = 1$

It has parametric equations $x = \cos t,\ y = \sqrt{3}\sin t,\ 0 \le t \le 2\pi$

$$\oint (x^2 + y^4)\,dx - 2\,dy \text{ on the ellipse ccl}$$

$$= \int_{t=0}^{2\pi} (\cos^2 t + 9\sin^4 t)\cdot -\sin t\,dt - 2\sqrt{3}\cos t\,dt$$

$$= \int_{0}^{2\pi} (-\sin t\cos^2 t - 9\sin^5 t + 2\sqrt{3}\cos t)\,dt$$

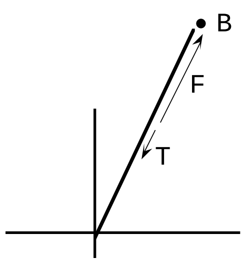
10. The curve has parametric equations
$x = 2\cos t,\ y = 2\sin t,\ z = 16 - 2\sin t - 8\cos t,\ 0 \le t \le 2\pi$

$$\int x^2\,\vec{k}\cdot\vec{T}\,ds = \int 0\,dx + 0\,dy + x^2\,dz = \int_{2\pi}^{0} 4\cos^2 t\,(-2\cos t + 8\sin t)\,dt \quad [= 0]$$

(The limits are 2π to 0 because the direction is clockwise.)

11. At every point on the line segment, F points away from the origin and T points toward the origin.

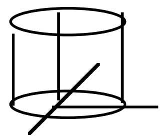
$T = -e_r,\ F\cdot T = -4,\ \int F\cdot T\,ds = \int -4\,ds = -4 \times \text{length of path} = -4\sqrt{45}$



12. (a) The rim is the ellipse $2x^2 + y^2 = 12$ (which can be written as $\frac{x^2}{6} + \frac{y^2}{12} = 1$) in the plane $z = 12$. It has parametric equations

$$x = \sqrt{6}\cos t$$
$$y = \sqrt{12}\sin t$$
$$z = 12$$
$$0 \le t \le 2\pi$$

warning The parametrization is not correct unless it includes z=12. If you leave out z=12 then it is understand that z=z and you have parametrized an elliptic *cylinder*, a *surface*, not a curve.



(b) $\int \mathbf{F}\cdot\mathbf{T}\,ds = \int yz\,dx + x\,dz$

$$= \int_{\text{initial } t}^{\text{final } t} \sqrt{12}\sin t \cdot 12\cdot -\sqrt{6}\sin t\,dt + 0\,dt$$

$$= \int_{t=2\pi}^{0} -72\sqrt{2}\sin^2 t\,dt \qquad [\ = -72\sqrt{2}\,\pi \text{ from the reference page}]$$

SOLUTIONS Section 3.2

1\. Flux across = $\int$ F·N ds = $\int$ −y dx + 3x dy on the parabola directed from B to A so that N is on the right as the curve is traversed.

The curve has parametric equations $x = y^2$, $y=y$.

$$\int F\cdot N\,dS = \int_{y=2}^{0} -y\cdot 2y\,dy + 3y^2\,dy = \int_{y=2}^{0} y^2\,dy \quad [\ = -8/3]$$

2\. The circle has parametric equations $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \pi/2$.

(a) $\int F\cdot T\,ds = \int 3y\,dx + 4x\,dy$

$$= \int_{t=0}^{\pi/2} 6\sin t\cdot -2\sin t\,dt + 8\cos t\cdot 2\cos t\,dt$$

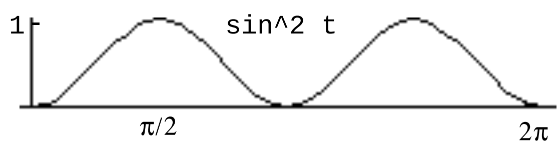
$$= \int_{t=0}^{\pi/2} (-12\sin^2 t + 16\cos^2 t)\,dt$$

footnote

Here's how to compute the integral.

$$\int_{t=0}^{\pi/2} \sin^2 t\,dt = \frac{1}{4}\int_{t=0}^{2\pi} \sin^2 t\,dt \quad \text{(look at areas under } \sin^2 t)$$

$$= \frac{1}{4}\pi \quad \text{(use the integral tables on the ref page)}$$



Similarly $\int_{t=0}^{\pi/2} \cos^2 t\,dt = \frac{1}{4}\pi$

So final answer is $-12\cdot\frac{1}{4}\pi + 16\cdot\frac{1}{4}\pi = \pi$

Question

Is it always true that $\int_{x=0}^{\pi/2} f(x)\,dx = \frac{1}{4}\int_{x=0}^{2\pi} f(x)\,dx$

Answer

No. It depends on f(x). It was true for $\sin^2 x$ but is not true for plain sin or for x^2 or x^3 etc.

(b) $\int F\cdot N\,ds = \int -4x\,dx + 3y\,dy$ on the curve *clockwise*

(so N is on your right as you walk)

$$= \int_{t=\pi/2}^{0} -8\cos t\cdot -2\sin t\,dt + 6\sin t\cdot 2\cos t\,dt$$

$$= \int_{t=\pi/2}^{0} 28\sin t\cos t\,dt \quad [\ = -14]$$

3. Line AB has parametric equations x=x, y=2x+3.
The parabola has parametric equations x=x, $y=x^2$.

(a) flux out $= \int$ F·outer N ds

$$= \oint_{ccl} -y\,dx + x^2 y\,dy \quad \text{(use } ccl \text{ so that N is to your right as you walk)}$$

$$= \int_{\text{segment B to A}} + \int_{\text{parabola A to B}}$$

$$= \int_{x=3}^{-1} -(2x+3)\,dx + x^2(2x+3)\ 2\,dx \ + \int_{x=-1}^{3} -x^2\,dx + x^2\cdot x^2\ 2x\,dx$$

$$\left[= -116 + \frac{700}{3} = \frac{352}{3} \right]$$

(b) I'll find the ccl circ.

$\int$ F·T ds ccl

$$= \int x^2 y\,dx + y\,dy \text{ on B to A line} + \int x^2 y\,dx + y\,dy \text{ on A to B parabola}$$

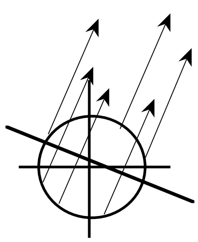
$$\text{Line} = \int_{x=3}^{-1} x^2(2x+3)\,dx + (2x+3)\ 2\,dx \quad [= -108]$$

$$\text{Parabola} = \int_{x=-1}^{3} x^2\ x^2\,dx + x^2\ 2x\,dx \quad \left[= \frac{444}{5} \right]$$

So the ccl circulation is $-108 + \frac{444}{5} = -\frac{96}{5}$

The circulation is actually 96/5 *clockwise*.

4. The field is uniform. I drew a diameter perp to the field arrows and compared the flow in across the lower semicircle with the flow out across the upper semicircle. They look the same to me so the net flux in is 0.



The circle has parametric equations x = cos t, y = sin t.

$$\text{Flux in} = \int \text{F}\cdot\text{inner N ds} = \oint_{\text{clockwise}} -3\,dx + dy = \int_{2\pi}^{0} -3\cdot -\sin t\,dt + \cos t\,dt = 0$$

SOLUTIONS Section 3.3

1. The surface has parametric equations

$x = x$
$y = x^3$
$z = z$
$-1 \le x \le 3,\ 0 \le z \le 2$

$n = \text{vel}_x \times \text{vel}_z = (1, 3x^2, 0) \times (0, 0, 1) = (3x^2, -1, 0)$

This n has a positive x component so that's the one I want, not −n.

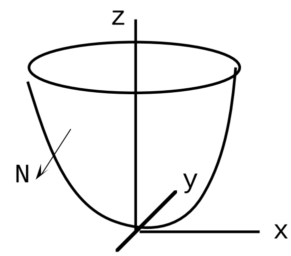
On the surface, $F \cdot n = 3x^2 y = 3x^2\ x^3 = 3x^5$

$$\text{Flux} = \int F \cdot N\, dS = \int_{z=0}^{2} \int_{x=-1}^{3} 3x^5\, dx\, dz \quad \{ = 3^6 - 1]$$

2. (a) $x = r\cos\theta,\ y = r\sin\theta,\ z = r^3,\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi$

(b) $n = (\cos\theta, \sin\theta, 3r^2) \times (-r\sin\theta, r\cos\theta, 0) = (-3r^3\cos\theta, -3r^3\sin\theta, r)$

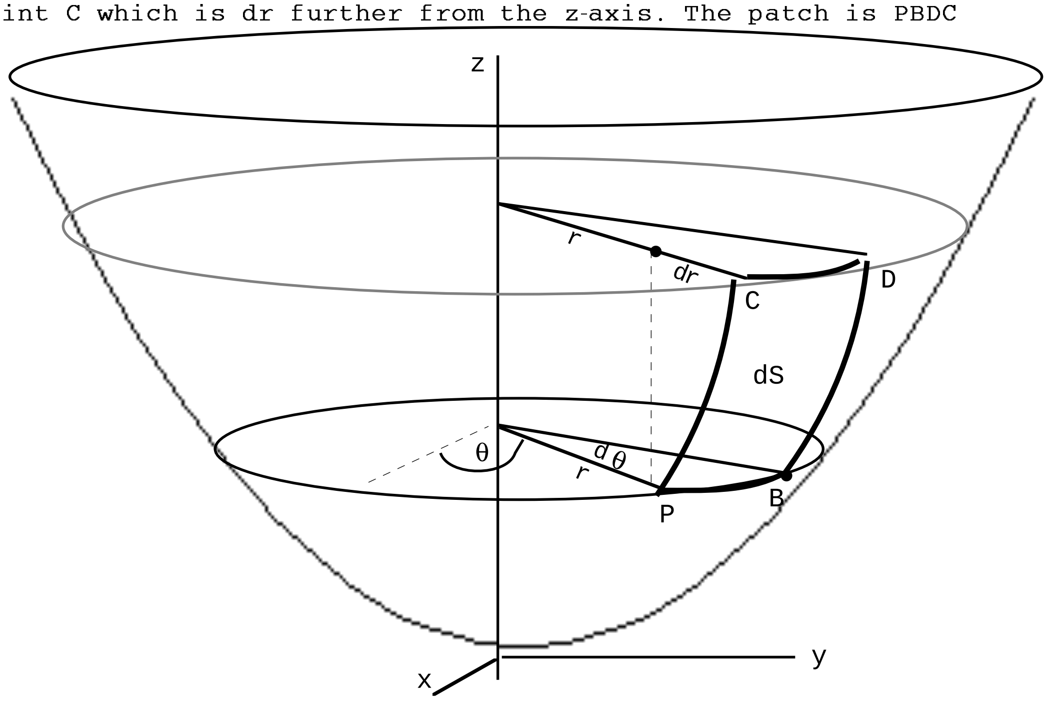
$n_{\text{outer}} = -n$ since the outer normal should have a neg z-component (look at the pic).



$$\int F \cdot \text{outer } N\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} F \cdot n_{\text{outer}}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (3r^4\cos^2\theta + 3r^4\cos\theta\sin\theta - r^7)\, dr\, d\theta$$

(c) If you start at P in the diagram below and change θ by dθ while r stays fixed, the little curve PB is traced out on the surface. If you start at P and change r by dr while θ stays fixed in order to stay on the surface, you must move up and out to point C which is dr further from the z-axis. The patch is PBDC



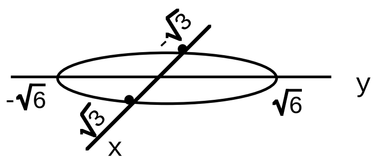
(d) $dS = \|n\|\, dr\, d\theta = r\sqrt{1+9r^4}\ dr\, d\theta$

3. (a) Here's one possibility:

$x = x$
$y = y$
$z = 2x^2 + y^2$

(It isn't a good idea to try to use r and θ as parameters since the cup does not have circular cross sections.)

The x,y parameter world is the projection of the surface in the x,y plane, the inside of ellipse $2x^2 + y^2 = 6$.



(b) $n = (1, 0, 4x) \times (0, 1, 2y) = (-4x, -2y, 1)$

This is an upper normal. which makes it point *into* the cup. So $n_{outer} = -n$.

On the surface, $F\cdot n_{outer} = 2y^2z = 2y^2(2x^2 + y^2)$

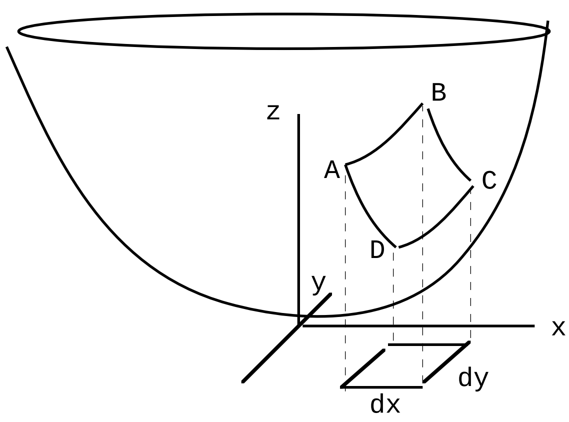
$$\text{Flux out} = \int_{\text{cup surface}} F\cdot n_{outer}\, dx\, dy$$

$$= \int_{\text{parameter world}} 2y^2(2x^2 + y^2)\, dx\, dy$$

$$= \int_{x=-\sqrt{3}}^{\sqrt{3}} \int_{y=-\sqrt{6-2x^2}}^{\sqrt{6-2x^2}} 2y^2(2x^2 + y^2)\, dy\, dx \quad [= 36\sqrt{2}\,\pi\ ]$$

(c) The surface area element is the patch swept out when x changes by dx and y changes by dy. In my diagram, I started at point A. When I changed x by dx with y fixed, (while z changes so that you stay on the cup) I went to D. When I started at A and changed y by dy, I went to B. The surface area element is ABCD. It's the projection onto the surface of a little dx by dy rectangle in the x,y plane.

$$dS = \|n\|\, dx\, dy = \sqrt{16x^2 + 4y^2 + 1}\ \ dx\, dy$$

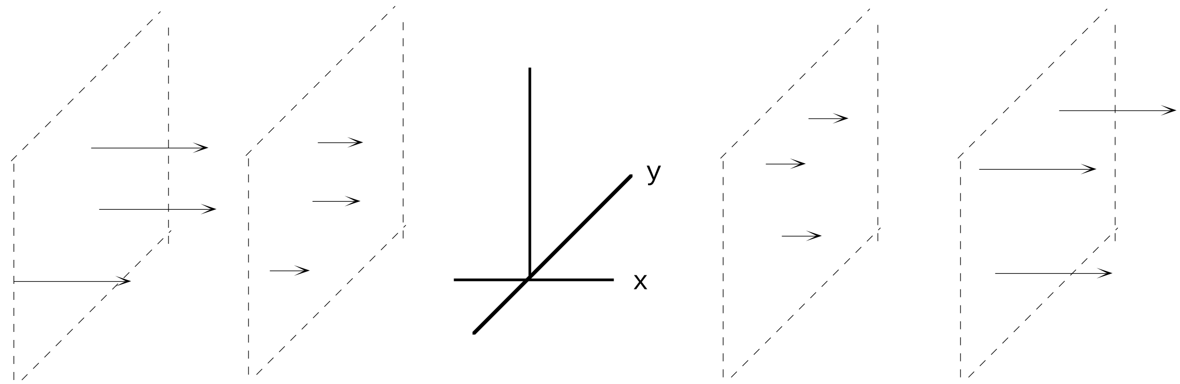


(d) $$\text{surface area} = \int_{\text{cup surface}} dS = \int_{\text{parameter world}} \|n\|\, dx\, dy$$

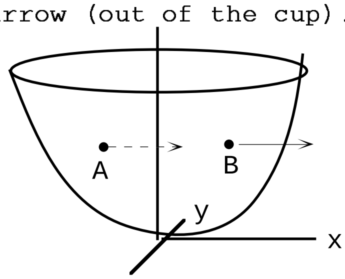
$$= \int_{x=-\sqrt{3}}^{\sqrt{3}} \int_{y=-\sqrt{6-2x^2}}^{\sqrt{6-2x^2}} \sqrt{16x^2 + 4y^2 + 1}\ \ dy\, dx$$

(Mathematica couldn't do this integration but it could numerically integrate to get the approximate answer 54.96)

(e) (i) Here's what the field looks like

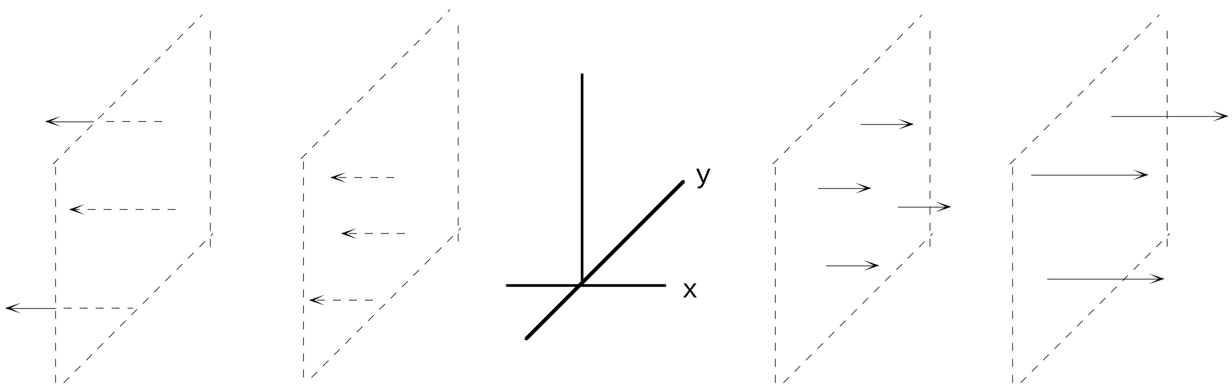


For every point (x_o, y_o, z_o) on the left side of the cup where the flux arrow points to the right (into the cup) there is symmetric point $(x_o, -y_o, z_o)$ on the right side with an identical flux arrow (out of the cup).

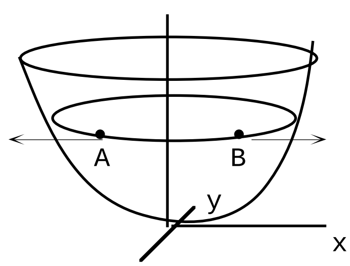


The flux in through the left side is the same as the flux out through the right side. Net flux out is zero.

(ii) Here's what the field looks like.

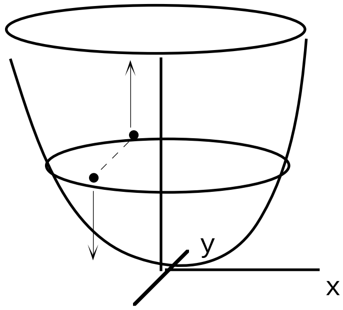


At every point on the left side of the cup, F points left (out of the cup). At every point on the right side of the cup, F points right (out of the cup).

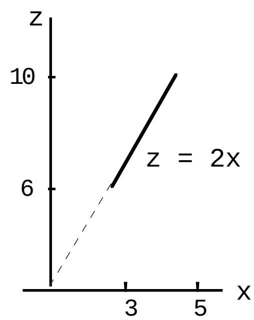


Flux flows out of the cup everywhere, Total flux out is positive.

(iii) For every point (x_o, y_o, z_o) on the back of the cup where the F arrow points up (into the cup) there is a symmetrically placed point $(x_o, -y_o, z_o)$ at the front of the cup where a same-length F arrow points down (out of the cup). The flux into the cup through the back half is the same as the flux out of the cup through the front half. Net flux out is zero.



4. (a) The radius at the bottom of the frustrum is 3 (similar triangles). The frustrum is swept out when the line segment $z = 2x$, $3 \le x \le 5$ is revolved around the z-axis.



The frustrum has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 2r$$
$$3 \le r \le 5,\ 0 \le \theta \le 2\pi$$

(b) $n = (\cos\theta, \sin\theta, 2) \times (-r\sin\theta, r\cos\theta, 0) = (-2r\cos\theta, -2r\sin\theta, r)$.
n is up and into the cone. so I'll use $n_{outer} = -n$.

$$\text{flux out of frustrum} = \int F\cdot \text{outer } N\, dS$$

$$= \int_{\text{parameter world}} F\cdot n_{outer}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=3}^{5} x\ 2r\cos\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=3}^{5} 2(r\cos\theta)^2\, dr\, d\theta \quad [\ = 196\pi/3\ ]$$

(c) By inspection, nothing flows out of either lid (the F field glides along the lids, i.e., $F\cdot N$ is 0 on each cover. So the answer here is the same as the answer to part (b).

(d) $dS = \|n\|\, dr\, d\theta = r\sqrt{5}\, dr\, d\theta$.

The surface area mag factor is $r\sqrt{5}$.

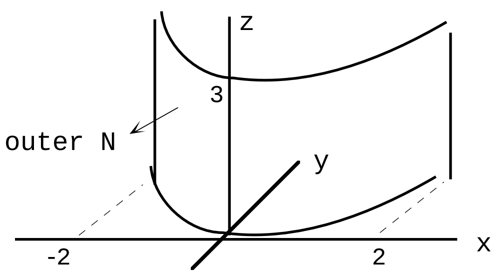
5. (a) The graph of $y = x^2$, $-2 \le x \le 2$ *in the x, y plane* is a piece of a parabola.
The graph of $y = x^2$, $-2 \le x \le 2$, $0 \le z \le 3$ in 3-space is a piece of a *parabolic cylinder*.

(b) I'll use parametrization $x = x$, $y = x^2$, $z = z$.
The parameter world is the projection of the surface in the x,z plane where
$-2 \le x \le 2$, $0 \le z \le 3$
$n = (1, 2x, 0) \times (0, 0, 1) = (2x, -1, 0)$.
This is an outer normal to the surface (because the y coord is negative).
On the surface, $F \cdot n_{outer} = -y = -x^2$.

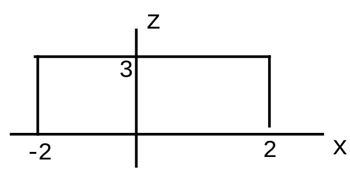
$$\text{flux out} = \int F \cdot \text{outer } N \, dS = \int_{x,\,z \text{ projection}} F \cdot n_{outer} \, dx\, dz$$

$$= \int_{x=-2}^{2} \int_{z=0}^{3} -x^2 \, dx\, dz \quad [\ = -16]$$

So the flux is really 16 stuff-units/sec *into* the surface (you can see from a picture that it's *in* since the F arrows all point "back" in my diagram and make obtuse angles with the outer N).

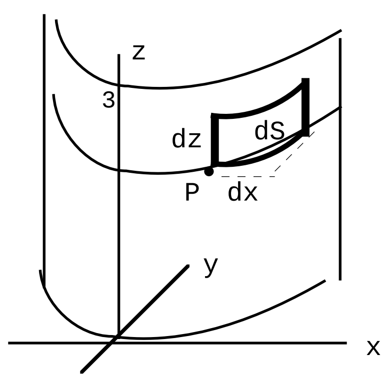


surface



projection in x,z plane

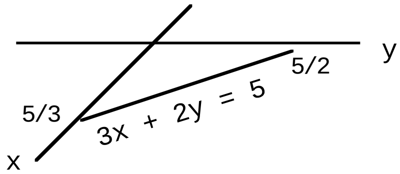
(c)



(c) $dS = \|n\| \, dx\, dz = \sqrt{4x^2 + 1} \, dx\, dz$

6. *flux out across face ABC*
The face has parametric equations
$x = x$
$y = y$
$z = 5 - 3x - 2y$

The parameter world is the projection of the surface in the x,y plane



$n = (1, 0, -3) \times (0, 1, -2) = (3, 2, 1)$

$n_{upper} = n$ rather than $-n$.

flux out through this face $= \int F \cdot \text{upper } N \, dS$

$$= \int_{x,\,y \text{ projection}} F \cdot n_{upper} \, dx\, dy$$

$$= \int_{x=0}^{5/3} \int_{y=0}^{(5-3x)/2} 2(5 - 3x - y)\, dy\, dx \quad \left[= \frac{125}{12} \right]$$

The other three faces are all lids.

flux out across bottom face

Flux is 0 since F glides across it, i.e., F is perp to N on the rear face.

flux out across rear face

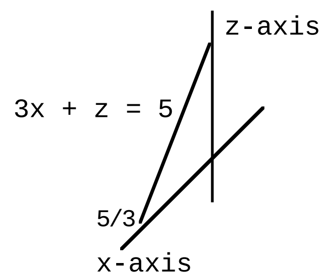
zero

flux out across left face

On the left face

$y = 0$, $F = z\mathbf{j}$, $N = -\mathbf{j}$, $F \cdot N = -z$, $dS = dA$

$$\text{flux out} = \int_{x=0}^{5/3} \int_{z=0}^{5-3x} -z \, dz\, dx \quad \left[= -\frac{125}{18} \right]$$



Final answer is the sum of the face fluxes $\left[= \frac{125}{36} \right]$

7. You can't ask someone to find dS unless you specify a parametrization. dS depends on how you parametrize the surface. The surface area element swept out when u changes by du and v changes by dv depends on the parametrization and so does its area dS.

For example, if one person parametrizes the plane $2x + 3y + 4z = 5$ with

$$x = u$$
$$y = v$$
$$z = \frac{1}{4}(5 - 2u - 3v)$$

and another uses

$$x = 3u$$
$$y = 4v$$
$$z = \frac{1}{4}(5 - 6u - 12v)$$

and another uses

$$x = \frac{1}{2}(5 - 3u - 4v)$$
$$y = u$$
$$z = v$$

they all get different dS's.

8. It's the surface of revolution (Fig B) swept out by revolving the curve $z = e^x$, $x \geq 0$ (Fig A) in the x,z plane around the z-axis.

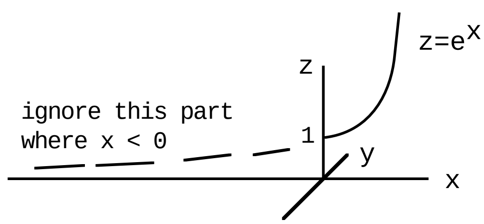


FIG A

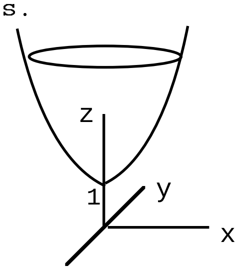


FIG B

footnote
If you revolve the *ignored* part in Fig A (the part in quadrant TWO) you get the surface of revolution in Fig C.

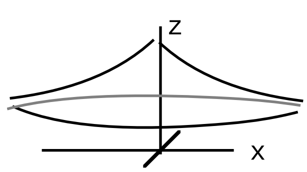


FIG C

The rule for the parametric equations of the surface of revolution is a little different here. The parametric equations turn out to be

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = e^{-r} \qquad \text{(note: exponent here is MINUS r)}$$
$$r \geq 0,\ 0 \leq \theta \leq 2\pi$$

That's because if a point is in quadrant II or III in the x,z plane then x is negative so it's $-x$ that equals r

9. It means that if the surface is parametrized, say with parameters u and v, and dS is the surface area traced out by changing u by du and v by dv then
dS = mag factor times du dv.

10. (a) On the sphere, $\rho = R$, $F = R^2 e_\rho$, outer $N = e_\rho$, $F \cdot$ outer $N = R^2$

$$\int F \cdot \text{outer } N \, dS = \int R^2 \, dS = R^2 \times \text{surface area of sphere} = R^2 \cdot 4\pi R^2 = 4\pi R^4$$

(b) F points radially away from the origin and the N's on the cone are always perp to radial lines so all $F \cdot N$'s on the cone are 0 and $\int F \cdot N \, dS = 0$. No flux goes out of the cone (the F arrows glide along the cone).

11. Not necessarily. Our n's will have the same or opposite direction (i.e., our n's will both be normals to the surface). But they might have different norms.

For example, suppose the surface is the plane $2x + 3y + 4z = 5$.

If I use the parametrization

$x = x$
$y = y$
$z = \frac{1}{4}(5 - 2x - 3y)$

then my n comes out to be $\frac{1}{2} i + \frac{3}{4} j + k$ if I used $vel_x \times vel_y$ or $-\frac{1}{2} i - \frac{3}{4} j - k$ if used $vel_y \times vel_x$

If you use

$x = 3u$
$y = 4v$
$z = \frac{1}{4}(5 - 6u - 12v)$

then your n comes out to be $6i + 9j + 12k$

If someone else uses

$x = \frac{1}{2}(5 - 3x - 4y)$
$y = y$
$z = z$

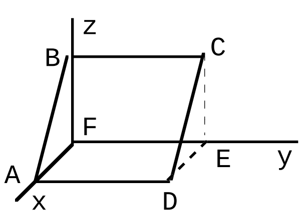
then her n comes out to be $i + \frac{3}{2} j + 2k$

All the n's have different norms because each parametrization has a different mag factor. When you change the parameters by a little bit and look at the surface area swept out on the surface, it's different for each parametrization.

12. It's the same surface whether you revolve $z = x^3$ from the x,z plane or revolve $z = y^3$ from the y,z plane. Same parametric equations:

$x = r \cos\theta$
$y = r \sin\theta$
$z = r^3$
$0 \le r \le 2, \ 0 \le \theta \le 2\pi$

13 The surface ABCD is a plane cylinder (move line AB to the right).



In the x,z plane, line AB has equation $z = -\frac{3}{5} x + 3$

So the surface has parametric equations

$x = x$
$y = y$
$z = -\frac{3}{5} x + 3$

The parameter world is $0 \le x \le 5$, $0 \le y \le 4$, the projection of the surface in the x,y plane, the rectangular region ADEF.

14. $\overrightarrow{AB} = (-8,-1,0)$

$\overrightarrow{AC} = (3, 0, -2)$

$AB \times AC = (2, -16, 3)$. This is a normal to the plane.
The plane has equation

$2(x - 1) - 16(y - 2) + 3(z - 3) = 0$

$2x - 16y + 3z = -21$

So a parametrization is

$x = x$

$y = y$

$z = (-21 - 2x + 16y)/3$

The parameter world is the entire x,y plane.

SOLUTIONS Section 3.4

1.(a) The box top has parametric equations

$x = x,\ y = y,\ z = \frac{1}{4}(10 - 2x - 3y),\ -2 \le x \le 2,\ -1 \le y \le 1$

method 1 for getting n

Let $g = 2x + 3y + 4z$. Then

$$n_{upper} = \frac{\nabla g}{\partial g/\partial z} = (1/2,\ 3/4,\ 1)$$

method 2 for getting n

$n = (1, 0, -1/2) \times (0, 1, -3/4) = (1/2, 3/4, 1)$, an upper n.

Then $F \cdot n_{upper} = \frac{3}{4} z$

$\int F \cdot \text{upper } N\ dS = \int F \cdot n_{upper}\ dx\ dy$ over projection in x,y plane

$$= \int_{x=-2}^{2} \int_{y=-1}^{1} \frac{3}{4} \cdot \frac{1}{4}(10 - 2x - 3y)\ dy\ dx \quad [\ = 15\ ]$$

(b) *fast way*

$\|n\| = \frac{1}{4}\sqrt{29}$

Since the mag factor is constant, the box top area is $\frac{1}{4}\sqrt{29}$ times the base area.

So box top surface area is $\frac{1}{4}\sqrt{29} \cdot 8$.

slow way

$$\text{surface area} = \int dS = \int_{x,\ y\ \text{world}} \|n\|\ dx\ dy$$

$$= \int_{x,\ y\ \text{world}} \frac{1}{4}\sqrt{29}\ dx\ dy$$

$$= \int_{y=-2}^{2} \int_{x=-1}^{1} \frac{1}{4}\sqrt{29}\ dx\ dy$$

(c) From part (a) we know that the flux out the top is 15. The other faces are lids.

out the bottom, back, front

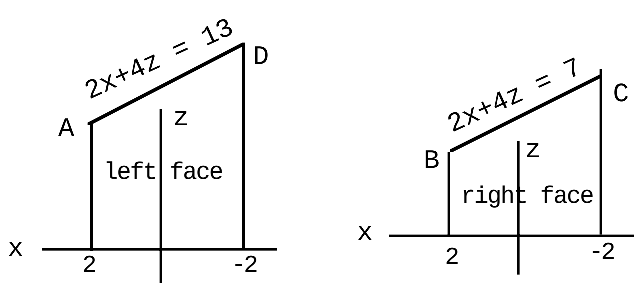
F glides along these faces (i.e., F is perp to N on these faces). No flux across.

out the left face

On the left face, $y = -1$, $N = -j$, $F \cdot N = -z$, $dS = dA$.

Line AD in plane $y = -1$ has equation $2x - 3 + 4z = 10$, $2x + 4z = 13$.

$$\text{flux out} = \int_{\text{left face}} -z\ dA \quad [\text{double integral}] = \int_{x=-2}^{2} \int_{z=0}^{(13-2x)/4} -z\ dz\ dx \quad [\ = -\frac{523}{24}\ ]$$



out the right face

On the right face, $y = 1$, $N = j$, $F \cdot N = z$, $dS = dA$

Line BC in plane $y = 1$ has equation $2x + 3 + 4z = 10$, $2x + 4z = 7$

$$\text{flux out} = \int_{\text{right face}} z\ dA = \int_{x=-2}^{2} \int_{z=0}^{(7-2x)/4} z\ dz\ dx \quad [\ = \frac{163}{24}\ ]$$

out the whole box

Sum of flux out of all the faces is $15 - \frac{523}{24} + \frac{163}{24}$ which happens to be 0.

2.(a)

The flux is $\oint F\cdot \text{inner } N\, dS = \int F\cdot \text{inner } N\, dS$ on paraboloid $+ \int F\cdot \text{down } N\, dS$ on lid

Units are kilograms/sec.

PARABOLOID PART

method 1 using x and y as parameters

The paraboloid part has parametric equations $x=x$, $y=y$, $z = x^2 + y^2$.

Let $g = x^2+y^2-z$.

Then $n_{\text{upper}} = \dfrac{\nabla g}{\partial g/\partial z} = (-2x, -2y, 1)$

You can also use

$n = (1, 0, 2x) \times (0, 1, 2y) = (-2x, -2y, 1)$ (an upper n)

The inner normal points uppish so $n_{\text{inner}} = (-2x, -2y, 1)$

The cross section at the top of the cup is the circle $x^2 + y^2 = 5$. So the projection in the x,y plane is a circular region with radius $\sqrt{5}$.

$$\int F\cdot \text{inner } N\, dS = \int_{x,y \text{ projection}} F\cdot n_{\text{inner}}\, dx\, dy$$

$$= \int_{x,y \text{ projection}} (-6x + z)\, dx\, dy$$

$$= \int_{x,y \text{ projection}} (-6x + [x^2+y^2]\ )\, dx\, dy$$

In Cartesian coordinates this is $\displaystyle\int_{x=-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{5-x^2}}^{\sqrt{5-x^2}} (-6x + x^2 + y^2)\, dy\, dx.$

In polar coordinates (better) this is

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{5}} (-6r \cos\theta + r^2)\, r\, dr\, d\theta \quad [\ = 25\pi/2\ ]$$

method 2 using r and θ as parameters

The paraboloid has parametric equations

$x = r \cos\theta$

$y = r \sin\theta$

$z = r^2$

$0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{5}$

$n = (\cos\theta, \sin\theta, 2r) \times (-r \sin\theta, r \cos\theta, 0)$

$= (-2r^2 \cos\theta, -2r^2 \sin\theta, r)$

Use n, not −n, to get an inner normal.

$F\cdot n = -6r^2 \cos\theta + rz = -6r^2 \cos\theta + r^3$

$$\int F\cdot \text{inner } N\, dS = \int F\cdot n_{\text{inner}}\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{5}} (-6r^2 \cos\theta + r^3)\, dr\, d\theta$$

LID

The lid has parametric equations $x = x$, $y = y$, $z = 5$.

warning On the lid, z is *not* $x^2 + y^2$, it's 5

inner $N = -k$

$F = i + 5k$

$F \cdot$ inner $N = -5$

$$\int F \cdot \text{ inner } N \, dS \text{ on lid} = \int -5 \, dS \text{ on lid} = -5 \times \text{area of lid}$$

The lid is a circular region with radius Sqrt[5] so flux in through the lid is -25 pi.

Add down-through-the-lid to into-the-paraboloid to get into-the-closed-surface.

footnote The flux into the paraboloid from the field $3i$ by itself is 0 by inspection (nothing flows through the lid and the flow into the back half of the paraboaloid cancels the flow out of the front half of the paraboloid).
So you could just use the zk field.

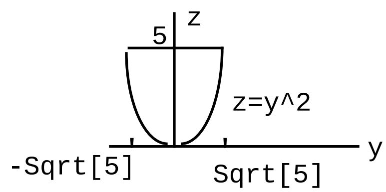
(b) The front half has parametric equations

$x = \sqrt{z-y^2}$ (use the positive square root for the front half and the negative square root for the back half)

$y = y$

$z = z$

The parameter world is the projection of the surface in the y,z plane



method 1 for getting the smart n $\quad n = \text{vel}_y \times \text{vel}_z$

method 2 for getting the smart n

Let $g = x^2 + y^2 - z$

Then $n = \dfrac{\nabla g}{\partial g/\partial x} = \dfrac{(2x,\ 2y,\ -1)}{2x} = (1,\ y/2x,\ -1/2x)$

This is a forward n (because the first component is positive).
To get an inner normal use $-n$.

$$F \cdot -n = -3 + z/\sqrt{z-y^2}$$

$$\text{Flux} = \int F \cdot \text{inner } N \, dS = \int_{y=-\text{Sqrt}[5]}^{\text{Sqrt}[5]} \int_{z=y^2}^{5} \left(-3 + \frac{z}{\sqrt{z-y^2}}\right) dz \, dy$$

3.(a)(i) The plane has parametric equations $x = x$, $y = y$, $z = \frac{1-x-2y}{3}$.

method 1 for getting n

Let $g = x+2y+3z$.

Then $n_{\text{upper}} = \frac{\nabla g}{\partial g/\partial z} = \frac{(1,2,3)}{3} = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$.

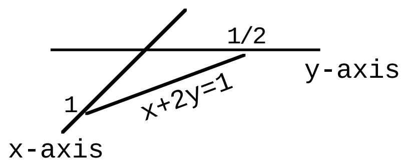
method 2 for getting n

$n = \left(1,0,-\frac{1}{3}\right) \times \left(0,1,-\frac{2}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$ (happens to be upper)

The backward N has a negative z-component so $n_{\text{backward}} = -n$ rather than n.

$F\cdot -n = -2x/3$

The parameter world is the projection of the surface in the x,y plane.



$$\int F\cdot \text{backward } N\, dS = \int_{x,y \text{ proj}} F\cdot n_{\text{backward}}\, dx\, dy = \int_{y=0}^{1/2}\int_{x=0}^{1-2y} -\frac{2x}{3}\, dx\, dy \quad \left[= -\frac{1}{18}\right]$$

Could also have limits $\int_{x=0}^{1}\int_{y=0}^{(1-x)/2}$

(ii) The plane has parametric equations $x=1-2y-3z$, $y=y$, $z=z$.

method 1 for getting n

Let $g = x+2y+3z$.

Then $n = \frac{\nabla g}{\partial g/\partial x} = (1,2,3)$

I used $\partial g/\partial x$ in the denominator here because it was x, not z, that I solved for to get the parametrization.

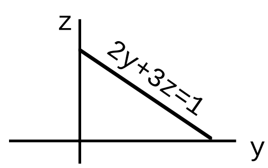
method 2 for getting n

$$n = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) \times \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (-2,1,0) \times (-3,0,1) = (1,2,3)$$

The backward N has a negative z-component so use $-n$ rather than n.

$F\cdot -n = -2x = -2\,(1-2y-3z)$

Get the y,z limits from the projection of the surface in the y,z plane.



$$\int F\cdot \text{backward } N\, dS = \int_{y,z \text{ projection}} F\cdot -n\, dy\, dz$$

$$= \int_{z=0}^{1/3}\int_{y=0}^{\frac{1}{2}(1-3z)} -2\,(1-2y-3z)\, dy\, dz \qquad \left[= -\frac{1}{18}\right]$$

(b) Continue with the parametrization from (a)(i). $dS = \|n\|\, dx\, dy = \frac{1}{3}\sqrt{14}\, dx\, dy$

(c) Continue with the parametrization from (a)(ii). $dS = \|n\|\, dy\, dz = \sqrt{14}\, dy\, dz$.

4. Line AB in the x,y plane has equation $x + y = 1$.
The plane is a cylinder with equation $x + y = 1$ in 3-space (see "cylinders in 3-space"in Section 1.0).
The plane has parametric equations

$$x = 1-y$$
$$y = y$$
$$z = z$$

(You can also use $x=x$, $y=1-x$, $z=z$. But there's no way to use x and y as parameters.)

method 1 for getting n

Let $g = x+y$. Then $n = \dfrac{\nabla g}{\partial g/\partial x} = (1, 1, 0)$

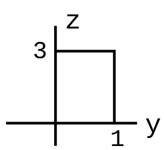
method 2 for getting n

$$n = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) \times \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (-1,1,0) \times (0,0,1) = (1, 1, 0).$$

The forward N has positive x and y components so $n_{\text{forward}} = (1, 1, 0)$

$F \cdot \text{forward } n = x + xy^2 = (1-y) + (1-y)y^2$

The parameter world is the projection of the surface in the y,z plane.



$$\int F \cdot \text{forward } N\, dS = \int_{y,z \text{ projection}} F \cdot (1,1,0)\, dy\, dz$$
$$= \int_{y=0}^{1} \int_{z=0}^{3} \Big[(1-y) + (1-y)y^2 \Big]\, dz\, dy \quad \left[= \frac{7}{4} \right]$$

SOLUTIONS Section 3.5

1. (a) Each F vector is perp to the sphere. Lengths depend on ϕ; they vary from 3 at the north pole to 0 at the equator. See the diagram below.

(b) The hemisphere has parametric equations

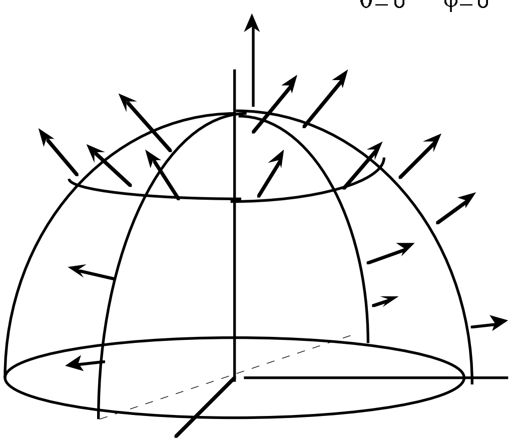
$x = 3 \sin\phi \cos\theta$
$y = 3 \sin\phi \sin\theta$
$z = 3 \cos\phi$
$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2$

outer $N = e_\rho$

$F \cdot \text{outer } N = 3 \cos\phi\ e_\rho \cdot e_\rho = 3 \cos\phi$

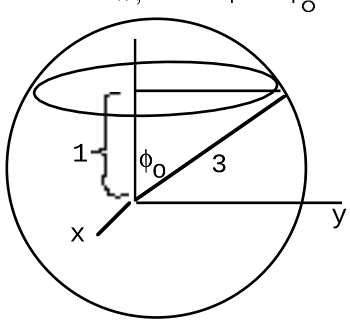
$dS = h_\phi h_\theta\, d\phi\, d\theta = 9 \sin\phi\, d\phi\, d\theta$

$$\int F \cdot \text{outer } N\, dS = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 27 \cos\phi \sin\phi\, d\phi\, d\theta \quad [= 27\pi]$$



2. I'll put in axes so that the center of the sphere is at the origin and the plane is z=1. Then the polar cap has parametric equations

$x = 3 \sin\phi \cos\theta$
$y = 3 \sin\phi \sin\theta$
$z = 3 \cos\phi$
$0 \le \theta \le 2\pi,\ 0 \le \phi \le \phi_o$ where ϕ_o is the angle in the diagram.



Then

$dS = h_\phi h_\theta\, d\phi\, d\theta = 9 \sin\phi\, d\phi\, d\theta$

$$\text{surface area} = \int_{\text{cap}} dS$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi_o} 9 \sin\phi\, d\phi\, d\theta$$

$$= 2\pi \cdot -9 \cos\phi \Big|_{\phi=0}^{\phi_o}$$

$$= 2\pi(-9\cos\phi_0 + 9)$$

$$= 12\pi \qquad \text{(read from the diagram that } \cos\phi_o = 1/3)$$

Notice that you don't have to actually find ϕ_o. All you need is $\cos\phi_o$.

3. (a) The sphere has parametric equations

$$x = R\sin\phi\cos\theta$$
$$y = R\sin\phi\sin\theta$$
$$z = R\cos\phi$$
$$0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

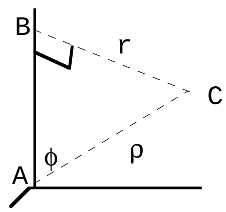
$$F = \frac{1}{r}\, e_r = \frac{1}{r}(\cos\theta, \sin\theta, 0)$$

$$\text{outer } N = e_\rho = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$dS = h_\phi\, h_\theta\, d\phi\, d\theta = R^2\sin\phi\, d\phi\, d\theta$$

$$F\cdot\text{outer } N = \frac{1}{r}(\sin\phi\cos^2\theta + \sin\phi\sin^2\theta) = \frac{1}{r}\sin\phi$$

Now you have to express r (distance to the z-axis) in terms of the parameters ϕ and θ. You can get it geometrically from triangle ABC which shows that in general $r = \rho\sin\phi$. On the sphere in this problem, $\rho = R$ so $r = R\sin\phi$.



$$\text{Flux out} = \int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \frac{1}{R\sin\phi}\ \sin\phi\ R^2\sin\phi\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} R\sin\phi\, d\phi\, d\theta \qquad [\ = 4\pi R\ ]$$

(b) Can't parametrize the whole sphere all at once. Must do the top and bottom hemispheres separately.

The top hemisphere has parametric equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{R^2 - r^2}$$
$$0 \le r \le R,\ 0 \le \theta \le 2\pi$$

Then

$$n = \left(\cos\theta, \sin\theta, \frac{-r}{\sqrt{R^2-r^2}}\right) \times (-r\sin\theta, r\cos\theta, 0)$$

$$= \left(\frac{r^2}{\sqrt{R^2-r^2}}\cos\theta,\ \frac{r^2}{\sqrt{R^2-r^2}}\sin\theta,\ r\right)$$

This is an upper n so on the top hemisphere it is outer.

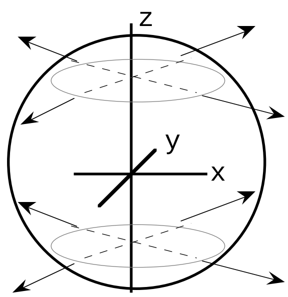
$$\text{Flux out of top half} = \int F\cdot\text{outer } N\, dS$$

$$= \int F\cdot n_{\text{outer}}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} \frac{r}{\sqrt{R^2-r^2}}\, dr\, d\theta$$

$$= 2\pi\cdot -\sqrt{R^2-r^2}\,\Big|_{r=0}^{R} = 2\pi R.$$

Look at a picture of the field F



to see that the flux out of the bottom half is the same as the flux out of the top. So the flux out of the whole sphere is $2 \cdot 2\pi R = 4\pi R$.

4. On the sphere, F points like the outer N and has length $1/R^2$.
So $F \cdot$ outer $N = 1/R^2$ and

$$\text{flux out} = \int F\cdot\text{outer } N\, dS = \int \frac{1}{R^2}\, dS = \frac{1}{R^2} \times \text{surface area} = \frac{1}{R^2} \times 4\pi R^2 = 4\pi$$

(same flux out no matter what the radius of the sphere).

5. (a) The cylinder has parametric equations

$x = 3\cos\theta$
$y = 3\sin\theta$
$z = z$
$0 \le \theta \le 2\pi,\ 0 \le z \le 5$

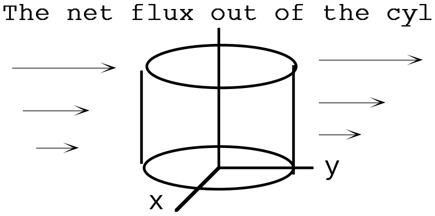
$N = e_r = \cos\theta\, \vec{i} + \sin\theta\, \vec{j}$

$dS = h_\theta\, h_z\, d\theta\, dz = 3\, d\theta\, dz$

$F\cdot N = x\cos\theta = 3\cos^2\theta$

$$\int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{2\pi} \int_{z=0}^{5} 9\cos^2\theta\, dz\, d\theta \quad [\ = 45\pi\ ]$$

(b) The surface integral is 0 because the arrows entering the cylinder match the arrows leaving. The net flux out of the cylinder is 0.



6. The half-cylinder has parametric equations

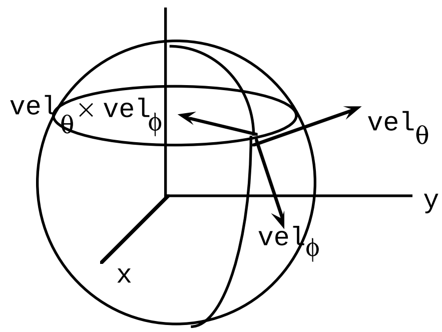
$x = R\cos\theta$
$y = R\sin\theta$
$z = z$
$0 \le \theta \le \pi,\ 0 \le z \le H.$
$N = e_r$
$F\cdot N = F\cdot e_r = j\cdot(\cos\theta, \sin\theta, 0) = \sin\theta$
$dS = h_z\, h_\theta\, dz\, d\theta = R\, dz\, d\theta$

$$\int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{\pi} \int_{z=0}^{H} R\sin\theta\, dz\, d\theta = 2RH$$

7. By inspection, nothing flows out of the top or bottom of the can.
On the cylinder, $F = \frac{1}{R} e_r$, outer $N = e_r$, $F\cdot$ outer $N = \frac{1}{R}$

$$\text{flux out} = \int F\cdot\text{outer } N\, dS = \int \frac{1}{R}\, dS = \frac{1}{R} \times \text{surface area of cylinder} = \frac{1}{R}\, 2\pi RH = 2\pi H$$

8. The equations parametrize a sphere with radius 3.
The cross product is the smart n which is normal to the sphere and whose length is the surface area mag factor.
So first of all, the cross product is a vector that points like e_ρ or $-e_\rho$.
vel_θ points in direction of increasing θ (see the diagram)
vel_ϕ points in the direction of increasing ϕ)
By the righthanded rule for cross products, $vel_\theta \times vel_\phi$ points into the sphere not out of the sphere. So the cross product points like $-e_\rho$.



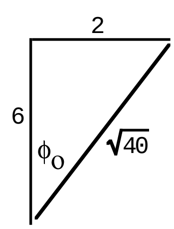
Second, the norm of the cross product is the mag factor $h_\phi\, h_\theta = 9 \sin\phi$.
So all in all, the cross product is $-9 \sin\phi\; e_\rho$

footnote
Here it is done directly to check.

$$n = (-3 \sin\phi \sin\theta,\ 3 \sin\phi\cos\theta,\ 0) \times (3\cos\phi\cos\theta,\ 3\cos\phi\sin\theta,\ -3\sin\theta)$$

$$= (-9 \sin^2\phi\cos\theta,\ -9\sin^2\phi\sin\theta,\ -9\cos\phi\sin\phi\cos^2\theta - 9\cos\phi\sin\phi\sin^2\theta)$$
$$= (-9\sin^2\phi\cos\theta,\ -9\sin^2\phi\sin\theta,\ -9\cos\phi\sin\phi)$$
$$= -9\sin\phi\,(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$
$$= -9\sin\phi\; e_\rho$$

9. (a) Let ϕ_o be the cone angle. I don't have to find ϕ_o itself because I only need $\sin\phi_o$ and $\cos\phi_o$.



The cone has parametric equations

$$x = \rho \sin\phi_o \cos\theta = \frac{2}{\sqrt{40}}\,\rho\cos\theta$$

$$y = \rho\sin\phi_o\sin\theta = \frac{2}{\sqrt{40}}\,\rho\sin\theta$$

$$z = \rho\cos\phi_o = \frac{6}{\sqrt{40}}\,\rho$$

$$0 \le \rho \le \sqrt{40},\quad 0 \le \theta \le 2\pi$$

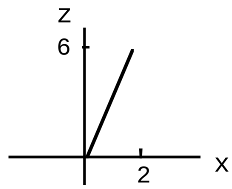
$$\text{inner } N = -e_\phi = -\left(\frac{6}{\sqrt{40}}\cos\theta\,\vec{i} + \frac{6}{\sqrt{40}}\sin\theta\,\vec{j} - \frac{2}{\sqrt{40}}\vec{k}\right)$$

$$F\cdot\text{inner } N = \frac{2}{\sqrt{40}}\,\frac{6}{\sqrt{40}}\,\rho = \frac{12}{40}\,\rho$$

$$dS = h_\theta\, h_\rho\, d\theta\, d\rho = \frac{2}{\sqrt{40}}\,\rho\, d\rho\, d\theta$$

$$\int \mathbf{F}\cdot \text{inner } \mathbf{N}\, dS = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\sqrt{40}} \frac{12}{40} \frac{2}{\sqrt{40}} \rho^2 \, d\rho \, d\theta \quad [\ = 16\pi\ ]$$

(b) The cone is the surface of revolution swept out when you revolve the line segment $z = 3x$, $0 \le x \le 2$, in the x,z plane around the z-axis.



The cone has parametric equations

$x = r \cos \theta$

$y = r \sin \theta$

$z = 3r$

$0 \le r \le 2, \ 0 \le \theta \le 2\pi$

$\mathbf{n} = (\cos\theta, \sin\theta, 3) \times (-r\sin\theta, r\cos\theta, 0) = (-3r\cos\theta, -3r\sin\theta, r)$

This an upper inner n so that's what I'll use.

$\mathbf{F}\cdot\mathbf{n} = zr = 3r^2$

$$\int \mathbf{F}\cdot \text{inner } \mathbf{N}\, dS \text{ on the cone} = \int_{\text{parameter world}} \mathbf{F}\cdot\mathbf{n}_{\text{inner}} \, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 3r^2 \, dr\, d\theta \qquad [\ = 16\pi]$$

SOLUTIONS review problems for Chapter 3

1. Segment AC has parametric equations $x=x$, $y=2x-1$, $1 \le x \le 2$

(a) circ $= \int F\cdot T\, ds = \int xy\, dx + (x^2+2)\, dy$

$$= \int_1^2 x(2x-1)\,dx + (x^2+2)\cdot 2\, dx \quad [= 71/6]$$

(b)

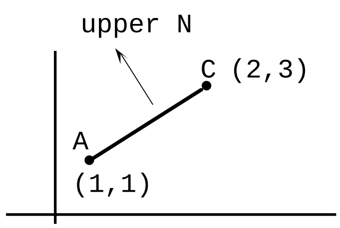
flux across $= \int F\cdot \text{upper } N\, ds$ on segment AC

$= \int (-x^2+2)\, dx + 2xy\, dy$ on the segment directed from C to A

(so that N is on your right as you walk)

$$= \int_{x=2}^{1} -(x^2+2)\,dx + x(2x-1)\cdot 2\, dx$$

$$= \int_{x=2}^{1} (3x^2 - 2x - 2)\, dx$$



2. The curve has parametric equations

$x = 3\cos\theta$

$y = 3\sin\theta$

$z\ [= xy] = 9\cos\theta\sin\theta$

The T direction in the diagram is one in which θ is *decreasing*.

$$\oint F\cdot T\, ds = \int y\, dx + z\, dy + x\, dz$$

$$= \int_{\theta=2\pi}^{0} -9\sin^2\theta\, d\theta + 27\cos^2\theta\sin\theta\, d\theta + 27\cos\theta\,(\cos^2\theta - \sin^2\theta)\, d\theta$$

$[= 9\pi]$

footnote

$\int_{2\pi}^{0} \cos^2\theta\sin\theta\, d\theta = 0$ by inspection since the graph of $\cos^2\theta\sin\theta$ is just as much above the θ-axis as below.

Similarly for $\int_{2\pi}^{0} \cos^3\theta\, d\theta$ and $\int_{2\pi}^{0} \cos\theta\sin^2\theta\, dt$.

So all that's left to do is $\int_{2\pi}^{0} -9\sin^2\theta\, d\theta$.

(b) The surface has parametric equations $x = x$, $y = y$, $z = xy$ where the parameter world is the projection of the surface in the x,y plane, a disk with center at the origin and radius 3.

(You can also use $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2\cos\theta\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 3$ but the algebra gets messy.)

method 1 for getting n

Let $g = z - xy$. Then

$$n_{\text{upper}} = \frac{\nabla g}{\partial g/\partial z} = (-y, -x, 1)$$

method 2 for getting n

$$n = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) \times \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) = (1, 0, y) \times (0, 1, x) = (-y, -x, 1)$$

To get an *upper* normal, use n rather than $-n$.

$$\int F\cdot\text{upper } N\, dS = \int_{x,y \text{ projection}} F\cdot n_{\text{upper}}\, dx\, dy = \int_{x,y \text{ projection}} (-y^2 - x^2y + x)\, dx\, dy$$

I'll set up the double integral in polar coords.

$$\int F\cdot\text{upper } N\, dS$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \left[\, -(r\sin\theta)^2 - (r\cos\theta)^2\, r\sin\theta + r\cos\theta\right]\; r\, dr\, d\theta \qquad \left[\, = \frac{81}{4}\pi\,\right]$$

footnote By inspection, $\int x^2y\, dA$ and $\int x\, dA$ are zero so you really only have to do $\int -y^2\, dA$.

footnote The cylinder determined the parameter world (it acted like a cookie cutter) but otherwise did not play a role in the computation. You are not surface integrating on the cylinder. You are surface integrating on the surface $z = xy$.

3. LID

Outer $N = -k$

$F\cdot$outer $N = -1$

$dS = dA$

$$\int F\cdot\text{outer } N\, dS \text{ on lid} = \int -1\, dS \text{ on lid}$$

version 1

$$\int -1\, dS \text{ on lid} = -\text{area of lid}$$

The lid is the inside of the ellipse $x^2 + 4y^2 = 4$.

The ellipse equation can be written a $\frac{x^2}{4} + y^2 = 1$.

The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

So the answer here is -2π.

version 2

$$\int -1\, dS \text{ on lid} = \int -1\, dA \text{ on the projection in the x,y plane}$$

$$= \int_{y=-1}^{1} \int_{x=-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} -1\, dx\, dy$$

CONE

The cone has parametric equations

$x = x$

$y = y$

$z = -\sqrt{x^2+4y^2}$

The parameter world is the projection in the x,y plane (inside of the ellipse $x^2 + 4y^2 = 4$)

(It is not a good idea to use the usual r and θ as parameters since this is not a circular cone.)

Let $g = z^2 - x^2 - 4y^2$

Then $n = \dfrac{\nabla g}{\partial g/\partial z} = (-x/z, -4y/z, 1)$ (upper and outer)

$F \cdot n_{\text{outer}} = -\dfrac{x}{z} + 1$

$$\int F\cdot \text{outer N } dS = \int_{\text{param world}} F\cdot n_{\text{outer}}\, dx\, dy$$

$$= \int_{y=-1}^{1} \int_{x=-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \left(\frac{x}{\sqrt{x^2+4y^2}} + 1 \right) dx\, dy \quad [= 2\pi]$$

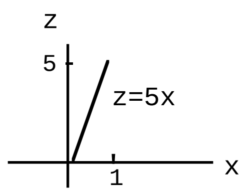
Total flux out is 0.

4. dS is the surface area swept out when u changes by du and v changes by dv.

$dS = \|n\|\, du\, dv$ where $n = \left(\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right) \times \left(\dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right)$

(There are shortcuts for getting dS in special cases but this is all you can say in general.)

5. By inspection, no flux comes out the top of the surface since F arrows glide along the top ($F\cdot N$ is 0 on the top)
The cone can be swept out by revolving the line $z = 5x$, $0 \le x \le 1$ in the y,z plane around the z-axis.



Problem 5(a)

The cone has parametric equations

$x = r\cos\theta$
$y = r\sin\theta$
$z = 5r$
$0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$n = (\cos\theta, \sin\theta, 5) \times (-r\sin\theta, r\cos\theta, 0)$
$= (-5r\cos\theta, -5r\sin\theta, r)$

An outer normal has a negative third component so $n_{\text{outer}} = -n$.

$F = \dfrac{1}{r}(\cos\theta, \sin\theta, 0)$

$F\cdot -n = 5\cos^2\theta + 5\sin^2\theta = 5$

flux out of cone $= \int F\cdot \text{outer N } dS$

$$= \int_{\text{param world}} F\cdot -n\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 5\, dr\, d\theta \quad [= 10\pi]$$

6. (a) The hemisphere has parametric equations

$x = 6\sin\phi\cos\theta$
$y = 6\sin\phi\sin\theta$
$z = 6\cos\phi$
$0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi$

outer $\vec{N} = \vec{e}_\rho = (\cdot,\ \cdot,\ \cos\phi)$

$dS = h_\phi\, h_\theta\, d\phi\, d\theta = 36\sin\phi\, d\phi\, d\theta$

$F\cdot$outer $N = k\cdot e_\rho = \cos\phi$

$$\text{flux out} = \int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \cos\phi\cdot 36\sin\phi\, d\phi\, d\theta$$

$$\left[\ = 36\cdot 2\pi\cdot \frac{1}{2}\sin^2\theta\Big|_{\theta=0}^{\pi/2} = 36\pi\ \right]$$

Question Now that you found the flux out of the hemisphere, what's the flux out of the entire sphere.

Answer Zero. The F arrows pointing up (and in) on the lower hemisphere match the F arrows pointing up (and out) on the top hemisphere. The flux going into the lower hemisphere is the same as the flux going out of the top. Net flux out is zero.

(b) The hemisphere has parametric equations

$x = x$
$y = y$
$z = \sqrt{36-x^2-y^2}$.

The parameter world is the projection in the x,y plane.

method 1 for getting n

The sphere has equation $x^2+y^2+z^2 = 3\ 6$. Let $g = x^2+y^2+z^2$.

$$n_{upper} = \frac{\nabla g}{\partial g/\partial z} = \frac{(2x,\ 2y,\ 2z)}{2z} = \left(\frac{x}{z},\ \frac{y}{z},\ 1\right)$$

method 2 for getting n

n = (1, 0, doesn't matter) × (0, 1, doesn't matter) = (·, ·, 1) (upper)

The hemisphere's outer normal has a positive z-component so $n_{outer} = n$, not $-n$.

$F\cdot$outer $n = 1$

$$\int F\cdot\text{outer } N\, dS = \int_{\text{projection}} 1\, dx\, dy$$

= area of projection in the x,y plane

$= 36\pi$

(c) The sphere has parametric equations

$x = r\cos\theta$
$y = r\sin\theta$
$z = \sqrt{36-r^2}$
$0 \le \theta \le 2\pi,\ 0 \le r \le 6$

$n = (\cos\theta,\ \sin\theta,\ \text{doesn't matter}) \times (-r\sin\theta,\ r\cos\theta,\ 0) = (\cdot,\ \cdot,\ r)$

This n is outer since it's upper.

$F\cdot$outer $n = r$

$$\int F\cdot\text{outer } N\, dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{6} F\cdot\text{outer } n\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{6} r\, dr\, d\theta\ \ [\ = 36\pi]$$

7. (a) To convert F, it is not good enough to say

$e_\rho = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$ since that leaves θ's and φ's in F.

I'm going to start fresh.

F(x,y,z) = unit vector pointing away from the origin

$$= (xi + yj + zk)_{\text{unit}} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}} \right)$$

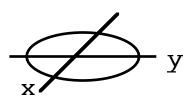
On the cover, outer N = k, z = 5, $F = \left(\cdot,\ \cdot,\ \frac{5}{\sqrt{x^2+y^2+25}}\right)$

$$F\cdot\text{outer } N = \frac{5}{\sqrt{x^2+y^2+25}}$$

dS = dA

$$\text{flux out of cover} = \int \frac{5}{\sqrt{x^2+y^2+25}}\, dA$$

where the double integral is over a disk with center at the origin and radius 5 in an x,y plane.



In Cartesian coords:

$$\text{flux out} = \int_{x=-5}^{5} \int_{y=-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \frac{5}{\sqrt{x^2+y^2+25}}\, dy\, dx$$

Mathematica had trouble doing this one.

In polar coords (better):

$$\text{flux out} = \int_{\theta=0}^{2\pi} \int_{r=0}^{5} \frac{5}{\sqrt{r^2+25}}\, r\, dr\, d\theta \quad [\ = 50\pi(-1+\sqrt{2})\ ]$$

(b) The rim is the circle $5 = x^2 + y^2$ in the plane z=5. It has parametric equations

$x = \sqrt{5}\cos t$
$y = \sqrt{5}\sin t$
$z = 5$
$0 \le t \le 2\pi$

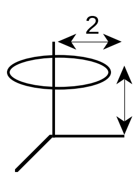
$$\int F\cdot T\, ds \text{ on ccl rim} = \int \frac{x}{\sqrt{x^2+y^2+z^2}}\, dx + \frac{y}{\sqrt{x^2+y^2+z^2}}\, dy + \frac{z}{\sqrt{x^2+y^2+z^2}}\, dz$$

$$= \int_{t=0}^{2\pi} \frac{\sqrt{5}\cos t}{\sqrt{5+25}} \cdot -\sqrt{5}\sin t\, dt + \frac{\sqrt{5}\sin t}{\sqrt{5+25}} \cdot \sqrt{5}\cos t\, dt + 0\, dt$$

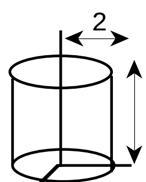
$$= \int_{t=0}^{2\pi} 0\, dt = 0.$$

(If you look at F·T's on the rim you can predict that the sum of F·T ds's cancels out to 0. There are just as many F's "with" T as "against" T.)

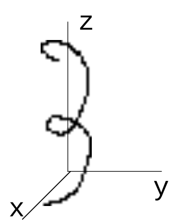
8. See the diagrams below.
(a) A circle in plane z=4 with center on the z-axis and radius 2.
(b) A cylinder standing on the x,y plane with the z-axis as its axis and height 5. The top and bottom lids are *not* included.
(c) A helix. If you think of the parametric equations as the path of a particle then the x,y part makes the particle circle around twice and the z part makes it rise from height 0 to height 4π.
(d) The difference between this and part (b) is that here r is not fixed at 2, it ranges between 0 and 2. So the equations describe a *solid* cylinder as opposed to part (b) which was a cylindrical *surface*.
(e) A disk in plane z = 4 with center on the z-axis and radius 2. The difference between this and part (a) is that here r is not fixed at 2.
(f) All of space; r,θ,z are the cylindrical coords of the point (x,y,z).
(g) The surface of revolution gotten by revolving the curve $z = x^2$, $0 \le x \le 4$ in the x,z plane around the z-axis.
(h) The surface of revolution traced out by revolving the line segment $z = 7 - y$, $1 \le y \le 2$ in the y,z plane around the z-axis. It's the frustrum of an upside down cone.



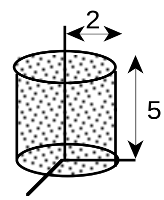
(a)



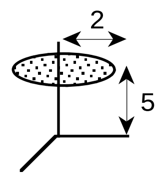
(b)



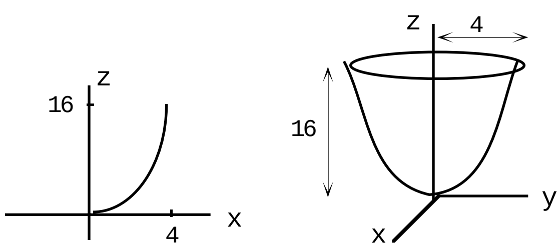
(c)



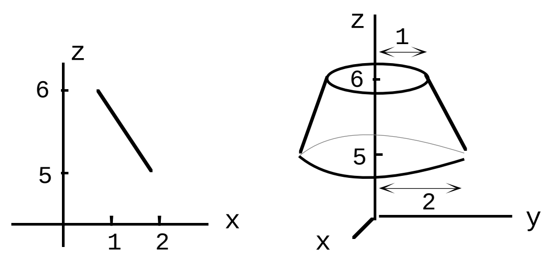
(d)



(e)

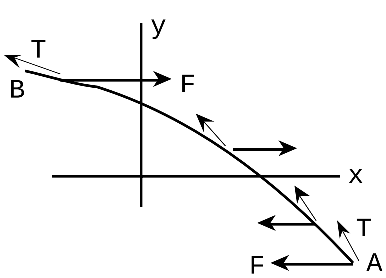


(g)

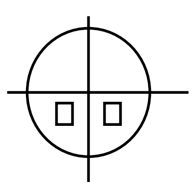


(h)

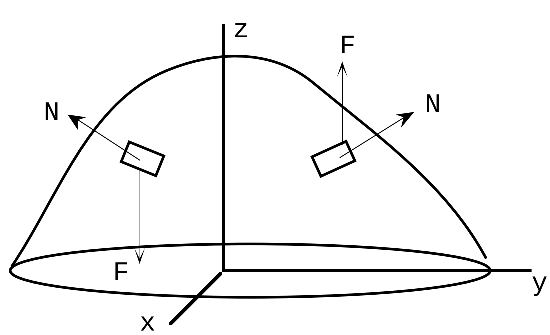
9. (i) Negative.
To see why, divide the curve into pieces and look at the sum of the F·T ds's. Some of the terms in the sum are negative (where F makes an obtuse angle with T) and some are positive but it looks like the negatives terms outweigh the positives so the integral is *negative*. (F does negative work to a particle which walks on the curve from A to B.)



(i)



(iii)



(v)

(ii) Positive. Each $e^y\,dV$ is positive since e^y is always positive. So the sum of $e^y\,dV$'s is positive.

(iii) Zero.
For every positive $x^3\,dA$ on the right side there is one on the left side with the opposite value. The sum of $x^3\,dA$'s is 0.
(iv) Positive. The sum of the $x^3\,dA$'s has a preponderance of positive terms and the sum is positive.

(v) Zero. For every positive $F \cdot N\,dS$ on the right side there is an $F \cdot N\,dS$ with the opposite value on the left side. The sum of the $F \cdot N\,dS$'s is 0.

(vi) Positive. The integral is the area of the region. Doesn't matter what the region is, the integral is always positive.

(vii) Positive. The line integral is just the length of the curve. Doesn't matter what curve you are integrating on, this line integral is always positive.

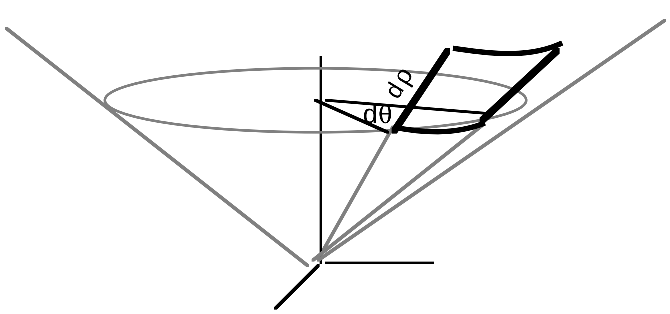
10. (a) surface area traced out is $dS = \|n\|\,du\,dv$ where

$$n = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

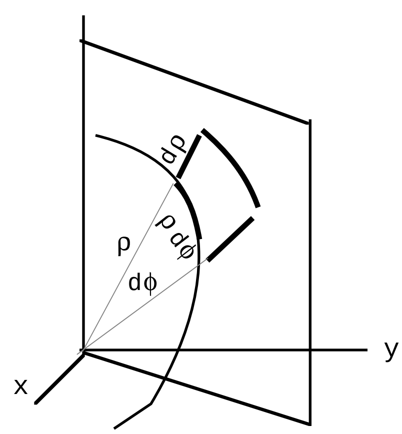
(the equations parametrize a surface in 3-space)

(b) arc length traced out (on a u-curve) is $ds_u = h_u\,du$ where $h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$
(the equations describe a u,v coordinate system in 2-space

(c) surface area traced out on a cone (where ϕ = constant) is
$dS = h_\rho\, h_\theta\, d\rho\, d\theta = \rho \sin\phi\, d\rho\, d\theta$
(d) surface area traced out on a half plane (where θ is constant) is
$dS = h_\rho\, h_\phi\, d\rho\, d\phi = \rho\, d\rho\, d\phi$

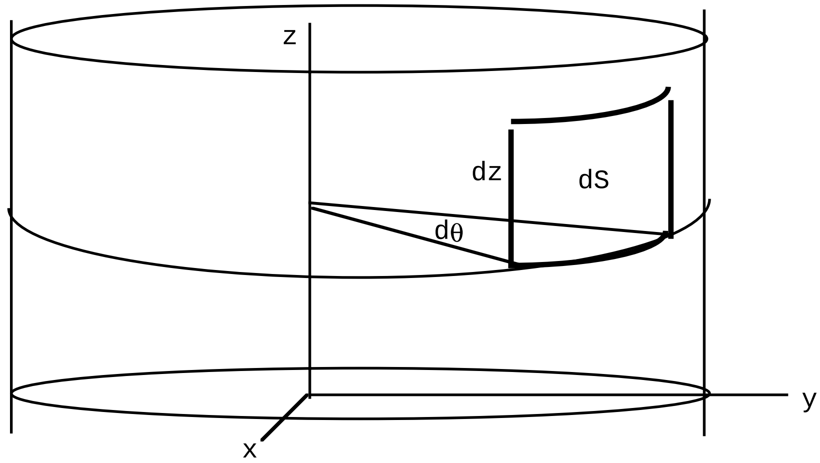


(c)



(d)

(e) surface area traced out on a cylinder (where r = constant) is
$dS = h_\theta \, h_z \; d\theta \, dz = r \, d\theta \, dz$



(e)

11. Remember that when you parametrize a curve there should be one parameter. When you parametrize a surface you need two parameters.

(a) $x = 2 \cos t$ (or use θ instead of t)
$y = 2 \sin t$
$z = 12 - 4 \cos t - 6 \sin t$
$0 \le t \le 2\pi$

(b) $x = 2 \cos \theta$
$y = 2 \sin \theta$
$z = z$
$0 \le \theta \le 2\pi$, $0 \le z \le 12 - 4 \cos \theta - 6 \sin \theta$

(c) *version 1*
$x = x$
$y = y$
$z = 12 - 2x - 3y$

The parameter world is the projection of the lid in the x,y plane, a disk with radius 2 where $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $-2 \le x \le 2$

version 2
$x = r \cos \theta$
$y = r \sin \theta$
$z = 12 - 4 \cos \theta - 6 \sin \theta$
$0 \le \theta \le 2\pi$, $0 \le r \le 2$

SOLUTIONS Section 4.1

1. The vector field $\frac{3x^2}{y}\, i - \frac{x^3}{y^2}\, j$ has antigradient $\frac{x^3}{y}$ so

$$\int \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy = \left.\frac{x^3}{y}\right|_{(3,18)}^{(2,4\sqrt{3})} = \frac{8}{4\sqrt{3}} - \frac{27}{18}$$

2. $$\left. xy \sin x \right|_{(1,2)}^{(2,3)} = 6 \sin 2 - 2 \sin 1$$

3. One set of parametric equations for line AB is
$x = 4t$
$y = 1$
$z = 2 + t$

I used (7) in Section 3.1 with A as the point (x_o, y_o, z_o) and $\overrightarrow{AB} = (4,0,1)$ as the parallel vector.

Point A corresponds to t=0, point B corresponds to t=1.

$$\int yz\,dx + xz\,dy + xy\,dz = \int_{t=0}^{1} (2+t)\cdot 4\,dt + 4t(2+t)\cdot 0\,dt + 4t\,dt$$

$$= \int_0^1 (8+8t)\,dt = 12$$

(b) $yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = \nabla(xyz)$ so the line integral is $\left. xyz \right|_A^B = 12$

4. (a) $\frac{1}{r}\, e_r = \nabla(\ell n\, r)$

$$\text{work} = \int F\cdot T\,ds = \left.\ell n\, r\right|_A^B = \left.\ell n\, r\right|_{r=1}^{r=\sqrt{17}} = \ell n\sqrt{17} - \ell n\, 1 = \ell n\sqrt{17}$$

(b) $\frac{1}{\rho^2}\, e_\rho = \nabla(-\frac{1}{\rho})$

$$\text{work} = \int F\cdot T\,ds = \left.-\frac{1}{\rho}\right|_A^B$$

$$= \left.-\frac{1}{\rho}\right|_{\rho=\sqrt{5}}^{\rho=\sqrt{26}} = -\frac{1}{\sqrt{26}} + \frac{1}{\sqrt{5}}$$

Note that point P is irrelevant here. That's the whole point of independence of path. Only the initial and final points matter.

5.(a) $p = 2xy^3 + 3x^2$, $q = 3x^2 y^2 + 4y$
$\partial q/\partial x = \partial p/\partial y =$ [6xy\S(2]
So F is a gradient. Antigradient is $x^2 y^3 + x^3 + 2y^2$.
(b) $p = x^2 y$, $q = x^3 y^2$, $\partial q/\partial x = 3x^2 y^2$, $\partial p/\partial y = x^2$.
The partials are not equal so F is not a gradient.

(c) Curl $F = \vec{0}$ so F is a gradient. $F = \nabla(x^2 y \sin z + 7z)$.

(d) Curl is not $\vec{0}$ so F is not a gradient.

6.(a) Reversing the direction changes the sign. Answer is −5. (It's irrelevant that curl $F = \vec{0}$).

(b) Curl $F = \vec{0}$ so the line integral is independent of path. It has the same value on every A to D path. Answer is 5.

(c) Curl $F = \vec{0}$ so the line integral on the *loop* ABCDA is 0.

7.(a) Let $F = yi + qj + 3y^2 k$. Need curl $F = \vec{0}$.

curl $F = \left(6y - \frac{\partial q}{\partial z},\ 0,\ \frac{\partial q}{\partial x} - 1\right)$

Need $\frac{\partial q}{\partial z} = 6y$, $\frac{\partial q}{\partial x} = 1$. Can get it with $q = 6yz + x$.

(b) Let $F = yzi + qj + 3y^2 k$. Then curl $F = \left(6y - \frac{\partial q}{\partial z},\ y,\ \frac{\partial q}{\partial x} - z\right)$

Need curl $F = \vec{0}$.

But the second component of curl F is y, not 0 (except in the x,z plane). There is no way to *make* it always be 0. So Can't make the line integral independent of path.

8.(a) *method 1* $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ $\left[= \frac{-4xy}{(x^2+y^2)^3} \right]$ Yes

method 2 By inspection, field is $\frac{1}{r^4}\, r\vec{e}_r = \frac{1}{r^3}\,\vec{e}_r$ = r-stuff e_r Yes

(antigradient is $-\frac{1}{2r^2}$)

(b) $p = f(x)$, $q = g(y)$, $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ [= 0]. Yes there is an antigrad.

(c) Field is $re^r\,\vec{e}_r$. = r-stuff e_r .Yes

9.(a) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ (each is 2x). The line integral is 0 no matter what the anonymous loop is.

(b) $\frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y}$ so item (1) on the zero curl list fails, i.e., it is not true that the line integral is 0 on *every* loop. There must be some loop on which the line integral is non-zero. But you don't which one(s). So you have no conclusion about the line integral on any one particular anonymous loop.

Note that you cannot conclude that the line integral is nonzero on every loop.

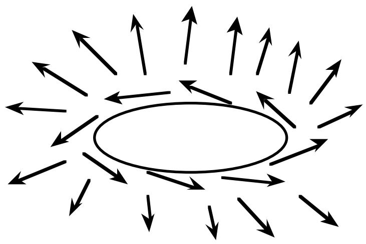
The logic here is that the negation of "all boxes are red" is *not* "no boxes are red". The correct negation is "at least one box is not red".

10. (a) This is $\int F\cdot T\,ds$ where $F = 3x^2y^2\,i + (2x^3y + 3)j = \nabla(x^3y^2 + 3y)$.

$$\int F\cdot T\,ds = (x^3y^2 + 3y)\Big|_Q^A = 5 - 4 = 1 \quad \text{(Not that P is irrelevant.)}$$

Another method which uses a little theory is to actually compute $\int F\cdot T\,ds$ on the direct Q-to-A path instead of Q-to-P-to-A but even doing that much integrating is not as neat as using $f(B) - f(A)$.

11. Look at the loop in the diagram below. It looks like $\int$F·T ds on the loop is not zero (I see ccl circulation). So item (1) on the zero-curl list fails. So (3) can't hold either. So F can't have an antigradient.



12. This is all logic.
(a) No conclusion. Having $\int$ F·T ds zero on one particular closed path isn't enough to trigger the zero curl rule. You would trigger it only if $\int$ F·T ds were zero on *every* closed path.
(b) Note that ABCQA is a *loop*. Item (1) from the zero-curl list doesn't hold so item (4) can't hold. Curl F $\neq \vec{0}$.
(c) The line integral has different values on two particular A to Z paths. So (2) on the zero curl list doesn't hold. So neither does (4). Curl F $\neq \vec{0}$.
(d) No conclusion. Having $\int$ F·T ds the same on two particular A to Z paths is not enough to trigger the zero curl list. It wouldn't be enough even to have $\int$ F·T ds the same on *all* A to Z paths. (It would be enough if you had $\int$ F·T ds the same on any two paths with the same starting and ending points -- not just A and Z).

13. No others. The fact that (4) implies (3) from the zero curl list means that *only* gradients have zero curls.

SOLUTIONS Section 4.2

1. (a) $\frac{\partial q}{\partial x} = \frac{(x^2 + 4y^2) - x\cdot 2x}{(x^2+4y^2)^2} = \frac{4y^2 - x^2}{(x^2 + 4y^2)^2}$

$$\frac{\partial p}{\partial y} = \frac{(x^2 + 4y^2)(-1) - -y\cdot 8y}{(x^2+4y^2)^2} = \frac{4y^2 - x^2}{(x^2 + 4y^2)^2}$$

So $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, curl $F = \vec{0}$

(b) We know that curl $F = \vec{0}$ but F flows up at the origin.
The circle does not go around the blowup. So $\oint F\cdot T\,ds = 0$.

(c) The ellipse goes around the blowup. We don't know yet if $\oint F\cdot T\,ds$ is zero or nonzero. But at least we know that by the deformation principle, $\oint p\,dx + q\,dy$ will be the same on every ccl loop around the blowup. Switch from the given ellipse to the more convenient ellipse $x^2 + 4y^2 = 1$. It's more convenient because on this new ellipse F is simpler, namely $F = -yi + xj$.
The ellipse has parametric equations

$x = \cos t$

$y = \frac{1}{2}\sin t$ (see (9) in Section 3.1 for parametrizing an ellipse)

Then

$\oint F\cdot T\,ds$ ccl on the old ellipse ccl

$$= \oint F\cdot T\,ds \text{ ccl on the new ellipse ccl}$$

$$= \int -y\,dx + x\,dy$$

$$= \int_{t=0}^{2\pi} -\frac{1}{2}\sin t\cdot -\sin t\,dt + \cos t\cdot\frac{1}{2}\cos t\,dt$$

$$= \int_{t=0}^{2\pi} \frac{1}{2}(\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi}\frac{1}{2}\,dt = \pi.$$

2. (a) *method 1*
F is a gradient since it is of the form r-stuff e_r. That makes its curl zero.

method 2

$$\text{curl } \vec{F} = \begin{vmatrix} \frac{1}{r}e_r & e_\theta & \frac{1}{r}k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 1/r^4 & 0 & 0 \end{vmatrix} = \vec{0}$$

(b) Curl $F = \vec{0}$ but F blows at the origin.
The circle doesn't go around the blowup so $\oint F\cdot T\,ds = 0$.

(c The ellipse goes around the blowup. We don't know yet if $\oint F\cdot T\,ds$ is zero or nonzero. But at least we know that by the deformation principle, $\oint p\,dx + q\,dy$ will have the same value on every ccl curve around the blowup.
Switch from the given ellipse to the more convenient unit circle around the origin. (It's more convenient because on the unit circle, $r = 1$ and F becomes just e_r).

method 1 for doing the line integral on the unit circle
The circle has parametric equations

$x = \cos t$

$y = \sin t$

On the unit circle, $r = 1$, $F = e_r = x\,i + y\,j$

$$\int F\cdot T\, ds = \int x\,dx + y\,dy$$
$$= \int_{t=0}^{2\pi} \cos t\cdot -\sin t\, dt + \sin t\cdot \cos t\, dt$$
$$= \int_{t=0}^{2\pi} 0\, dt = 0$$

method 2 for doing the line integral on the unit circle

On the unit circle, ccl $T = e_\theta$, $F = e_r$, $F\cdot T = 0$, $\oint F\cdot T\, ds = 0$.

footnote

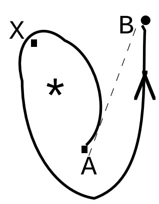
Here's another way to do the testing on an arbitrary loop around the blowup.
$F = \nabla\left(\frac{-1}{3r^3}\right)$ so on any curve going around the blowup, say from point Q back to point Q again, $\oint F\cdot T\, ds = \left.\frac{-1}{3r^3}\right|_{r\text{ at }Q}^{r\text{ at }Q} = 0$.

3. (B) 0
(C) 10
(D) AB = APB by ind of path since the blowup is not enclosed by the combined paths = 7
(E) 7 (same reasoning as in (D))

(F) AYB + BPA = once around the blowup clockwise = −10
So AYB = −10 + APB = −10 + 7 = −3

(G) *version 1*

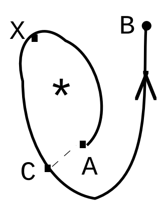
AXB + straight BA = once around the origin ccl = 10



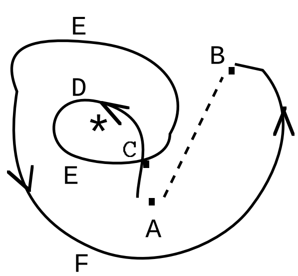
So AXB = 10 − straight BA = 10 + straight AB = 10 + 7 = 17

version 2

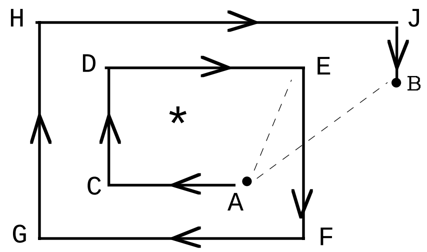
AXB = AXCA + ACB (there's an extra CA and an extra AC which cancel)
= 10 + 7 = 17



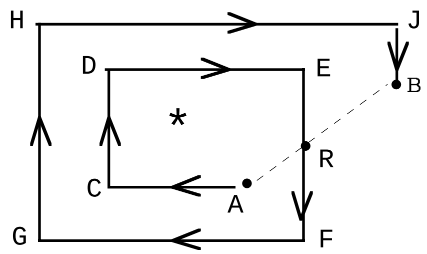
(H) Here's one method.
path = CDEC + ACEFBA + AB (the extra BA and AB cancel)
= twice around ccl + AB = 20 + 7 = 27



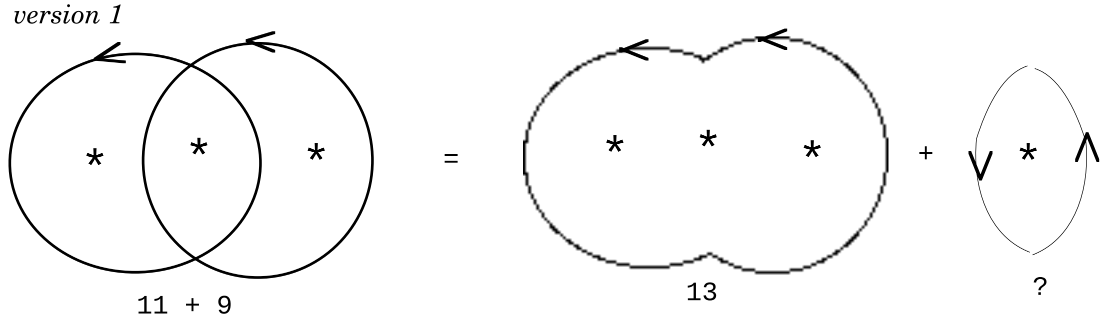
(I) *method 1*
path = ACDEA + AEFGHJBA [so far EA and AE cancel, there's an extra BA] − BA
= two clockwise loops around blowup + AB
= −10 + −10 + 7 = −13



method 2
path = AREDCA + RFGHJBR [so far there's an extra BRA] − BRA
= twice around the blowup clockwise + AB
= −20 + 7 = 13



4. (a) 0 since the triangle doesn't go around a blowup.
(b) *version 1*



So ? = 7
So smaller rectangle is also 7 by deformation again.
version 2
Let x be the value of $\int p\,dx + q\,dy$ ccl on a loop around just the left hand blowup
Let y be the value on a ccl loop just around the middle blowup.
Let z be the value on a ccl loop just around the righthand blowup.
Then (see example 4)
$x + y = 11$
$y + z = 9$
$x + y + z = 13$

Solve the system of equations to get y = 7 (and x = 4, z = 2)

5. (a) Curl F = $\vec{0}$ because F is of the form ρ-stuff e_ρ so it has an antigrad so its curl is zero. F is 3−dim and blows up only at the origin. Everything on the zero curl list continues to hold. So $\oint F\cdot T\,ds$ is 0 on every loop.

(b) Curl $F = \vec{0}$ because F is of the form r-stuff e_r.

F blows up on the z-axis where $r = 0$.

$\oint F \cdot T\, ds$ must be 0 on any loop *not* going around the z-axis.

And $\oint F \cdot T\, ds$ has the same value on all loops around the z-axis directed say ccl from above but we don't know yet what that value is. To find out, I'll compute the line integral on a convenient loop around the z-axis. I can choose whatever around-the-z-axis-loop I like since $\int F \cdot T\, ds$ is the same on all of them by the deformation principle.

I'll line integrate on the ccl unit circle lying in the x,y plane, with center at the origin. I chose this circle because on the circle, $r = 1$ and $F = e_r = xi + yj$

method 1

At points on this circle, the ccl T and F both lie in the x,y plane and are perpendicular (F is out and T is around) so $F \cdot T = 0$ so the line integral is 0.

method 2

The circle has parametric equations $x = \cos t$, $y = \sin t$, $z=0$.

$$\oint F \cdot T\, ds = \oint x\, dx + y\, dy + 0\, dz = \int_0^{2\pi} \cos t \cdot -\sin t\, dt + \sin t \cdot \cos t\, dt = 0$$

footnote

Another method

$F = \nabla \ln r$ so on any loop say from point Q to point Q, whether it goes around the z-axis or not, $\oint F \cdot T\, ds = \ln r \Big|_Q^Q = 0$.

All, in all, every $\oint F \cdot T\, ds$ is 0 despite the blowup.

(c) dq/dx turns out to equal dp/dy (each is $\dfrac{-2x^2 + 3y^2}{(2x^2 + 3y^2)^2}$ by the quotient rule)

so Curl $F = \vec{0}$

F blows up on the z-axis where $2x^2 + 3y^2 = 0$.

$\oint F \cdot T\, ds$ must be 0 on any loop *not* going around the z-axis.

And $\oint F \cdot T\, ds$ has the same value on all loops around the z-axis directed say ccl from above but we don't know yet what that value is. To find out, I'll compute the line integral on a convenient loop around the z-axis.

I choose the ellipse $2x^2 + 3y^2 = 1$ in the x,y plane because on this ellipse $F = -yi + xj$.

The ellipse's equation in the x,y plane is $\dfrac{x^2}{1/2} + \dfrac{y^2}{1/3} = 1$ so in 3-space it has parametric equations

$$x = \frac{1}{\sqrt{2}} \cos t$$

$$y = \frac{1}{\sqrt{3}} \sin t$$

$$z = 0$$

$\oint F \cdot T\, ds$ on the ellipse directed ccl as viewed from above

$$= \oint -y\, dx + x\, dy + 0\, dz$$

$$= \int_{t=0}^{2\pi} -\frac{1}{\sqrt{3}} \sin t \cdot -\frac{1}{\sqrt{2}} \sin t\, dt + \frac{1}{\sqrt{2}} \cos t \cdot \frac{1}{\sqrt{3}} \cos t\, dt$$

$$= \int_{t=0}^{2\pi} \frac{1}{\sqrt{6}} (\cos^2 t + \sin^2 t)\, dt$$

$$= \int_{t=0}^{2\pi} \frac{1}{\sqrt{6}}\, dt$$

$$= \frac{2\pi}{\sqrt{6}}$$

All in all, $\oint \mathbf{F}\cdot\mathbf{T}\, ds$ is 0 on loops not going around the z-axis and is $2\pi/\sqrt{6}$ on loops going around the z-axis directed ccl as viewed from above.

SOLUTIONS Section 4.3

1. Use Green's theorem

(a) $\oint x \sin y\, dx + xy^3\, dy$ COUNTERclockwise $= \int (y^3 - x \cos y)\, dA$ on inside.

Can set up the double integral in two ways:

version 1 $\displaystyle\int_{x=-3}^{3} \int_{y=x^2}^{9} (y^3 - x \cos y)\, dy\, dx$ [= 8748]

version 2 $\displaystyle\int_{y=0}^{9} \int_{x=-\sqrt{y}}^{\sqrt{y}} (y^3 - x \cos y)\, dx\, dy$

(b) $\oint (x^2-y^2)\,dx + x\, dy$ on CLOCKWISE circle = MINUS $\displaystyle\int_{\text{inside circle}} (1 + 2y)\, dA$

$$= -\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} (1 + 2r \sin\theta)\, r\, dr\, d\theta$$

footnote

$\int 1\, dA$ = area of circle $= 3\pi$ by inspection

$\int 2y\, dA = 0$ by inspection (when you add 2y dA's in the region, the negative ones cancel out the positive ones)

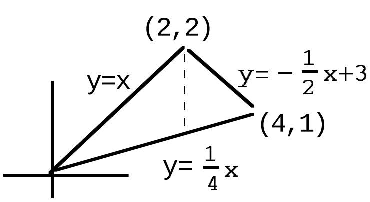
Final answer is -3π.

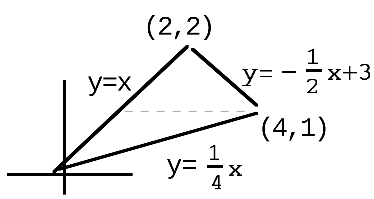
(c) circ $= \oint \mathbf{F}\cdot\mathbf{T}\, ds$ CLOCKWISE = MINUS $\displaystyle\int_{\text{inside}} 2x\, dA$.

To put in limits of integration you have to divide the region into two subregions.

answer 1 (lefthand diagram) $\displaystyle -\int_{x=0}^{2} \int_{y=x/4}^{x} 2x\, dy\, dx - \int_{x=2}^{4} \int_{y=x/4}^{-\frac{1}{2}x+3} 2x\, dy\, dx$ [= −12]

answer 2 (righthand diagram) $\displaystyle -\int_{y=0}^{1} \int_{x=y}^{4y} 2x\, dx\, dy - \int_{y=1}^{2} \int_{x=y}^{6-2y} 2x\, dx\, dy$





2. (a) Line AB has equation y=x, line BC has equation y = 6−x.
AB has parametric equations x=x, y=x
BC has parametric equations x=x, y=6−x
CA has parametric equations x=x, y=0

$$\oint_{\text{clock}} y^2\, dx + 3x\, dy = \int_{A \text{ to } B} + \int_{B \text{ to } C} + \int_{C \text{ to } A}$$

$$= \int_{x=0}^{3} x^2\, dx + 3x\, dx \quad + \int_{3}^{6} (6-x)^2\, dx + 3x\cdot -dx + \int_{x=6}^{0} 0\, dx + 3x\cdot 0\, dy$$

$$= \frac{45}{2} - \frac{63}{2} + 0 = -9$$

(b) $\oint y^2\, dx + 3x\, dy$ on CLOCKWISE circle = MINUS $\int (3 - 2y)\, dA$

$$= \int_{y=0}^{3} \int_{x=y}^{6-y} (-3 + 2y)\, dx\, dy \quad [= -9]$$

warning

If you want inner y limits you have to split the region up because the upper boundary consists of *two* curves, y=x and y=6−x. And don't use limits $\int_{y=0}^{3} \int_{x=0}^{6}$ because they go with a rectangle not the triangle.

3. $$\oint_{\text{ccl}} \underbrace{-\frac{\partial f}{\partial y}}_{p}\, dx + \underbrace{\frac{\partial f}{\partial x}}_{q}\, dy = \int_{\text{inside}} \Big(\underbrace{\frac{\partial^2 f}{\partial x^2}}_{\partial q/\partial x} \;-\; \underbrace{-\frac{\partial^2 f}{\partial y^2}}_{\partial p/\partial y} \Big)\, dA = \int_{\text{inside}} \text{Lapl } f\, dA$$

4.(a) Can't use Green's theorem because p and q blow up at the origin which is inside the square.

(b) You can do the line integral directly but better still, since $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ you can use the deformation principle to switch to any other ccl curve around the origin. The easiest is the unit circle where $x^2 + y^2 = 1$. On the unit circle, $p = -y$, $q = x$. The circle has parametric equations $x = \cos t$, $y = \sin t$.

$$\oint_{\text{ccl circle}} p\, dx + q\, dy = \oint -y\, dx + x\, dy = \int_{t=0}^{2\pi} -\sin t \cdot -\sin t\, dt + \cos t \cdot \cos t\, dt$$

$$= \int_{t=0}^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_{t=0}^{2\pi} dt = 2\pi.$$

5. (a) Parametrize each of the three pieces of the loop.
Curve AC is part of a circle in the y,z plane. It has parametric equations $x=0$, $y = 6 \cos t$, $z = 6 \sin t$, $0 \le t \le \pi/2$.

$\int (x+z)\, dx + (x+z)\, dy + (x+y)\, dz$ on the CA curve

$$= \int_{t=0}^{\pi/2} 0 + 6 \sin t \cdot -6 \sin t\, dt + 6 \cos t \cdot 6 \cos t\, dt = 0$$

Curve BC has parametric equations $x = 6 \cos t$, $y = 6 \sin t$, $z = 0$, $0 \le t \le \pi/2$.

$\int (x+z)\, dx + (x+z)\, dy + (x+y)\, dz$ on the BC curve

$$= \int_{t=0}^{\pi/2} 6 \cos t \cdot -6 \sin t\, dt + 6 \cos t \cdot 6 \cos t\, dt = 9\pi - 18$$

Curve AC has parametric equations $x = 6 \cos t$, $y = 0$, $z = 6 \sin t$, $0 \le t \le \pi/2$.
If you look at the x,z plane so that the positive x-axis is to your right, as usual, then the direction on the quarter circle AB is clockwise; $t_A = \pi/2$, $t_B = 0$.

$\int (x+z)\, dx + (x+z)\, dy + (x+y)\, dz$ on the AB curve

$$= \int_{t=\pi/2}^{0} (6 \cos t + 6 \sin t) \cdot -6 \sin t\, dt + 0 + 6 \cos t \cdot 6 \cos t\, dt = 18$$

Final answer is $0 + 9\pi - 18 + 18 = 9\pi$

(b) Pick a hat with the loop as a brim, pick the right N and use Stokes' theorem.

$\oint (x+z)\, dx + (x+z)\, dy + (x+y)\, dz$ on the brim

$$= \int \text{curl } F \cdot N\, dS \text{ on the hat}$$

curl F = (0,0,1)

FIRST HAT

I'll use the spherical hat (one-eighth of the whole sphere) in Fig A.
The righthanded N is outer.

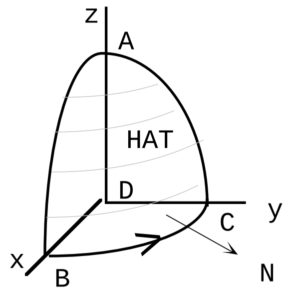


FIG A

method 1 for doing the surface integral

The hat has parametric equations

$x = 6 \sin\phi \cos\theta$
$y = 6 \sin\phi \sin\theta$
$z = 6 \cos\phi$
$0 \le \theta \le \pi/2, \ 0 \le \phi \le \pi/2$

outer $N = e_\rho = (\cdot, \ \cdot, \ \cos\phi)$

curl $F \cdot$ outer $N = \cos\phi$

$dS = h_\phi \, h_\theta \, d\phi \, d\theta = 36 \sin\phi \, d\phi \, d\theta$

$\int$ curl $F \cdot$ outer $N \, dS$ on the hat

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \cos\phi \; 36 \sin\phi \, d\phi \, d\theta = \frac{\pi}{2} \, 36 \cdot \frac{1}{2} \sin^2\phi \Big|_0^{\pi/2} = 9\pi$$

method 2 for doing the surface integral

Use x and y as parameters. The one-eighth sphere has parametric equations

$x = x$
$y = y$
$z = \sqrt{36 - x^2 - y^2}$

where the parameter world is the projection in the x,y plane, a quarter disk with radius 3.

The hat has equation $g(x,y,z) = 36$ where $g = x^2 + y^2 + z^2$.

$n = \dfrac{\nabla g}{\partial g/\partial z}$ = (who cares, who cares, 1) which happens to be outer.

$$\int \text{curl } F \cdot N \, dS = \int_{x,y \text{ projection}} \text{curl } F \cdot n \, dx \, dy$$

$$= \int_{x,y \text{ projection}} 1 \, dx \, dy$$

$$= \text{area of projection} = 9\pi$$

SECOND HAT

Use the three-cornered hat in Fig B consisting of the plane faces ABD, ACD, BCD. FIG C shows the right N's.

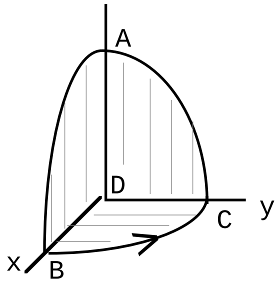


FIG B

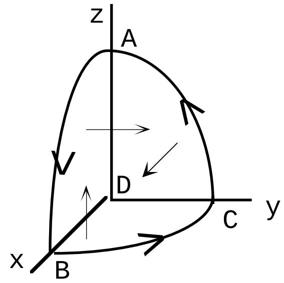


FIG C

Now find $\int$ curl $\mathbf{F}\cdot\mathbf{N}\,dS$ on the three-sided hat

On surface ABD (a quarter-disk in the x,z plane), $\mathbf{N} = \mathbf{j}$, curl $\mathbf{F}\cdot\mathbf{N} = 0$, surface integral is 0.

For surface ADC, $\mathbf{N} = \mathbf{i}$, curl $\mathbf{F}\cdot\mathbf{N} = 0$, surface integral is 0.

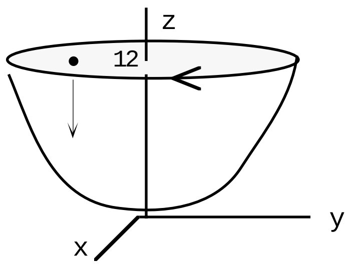
For surface DBC (quarter-disk in the x,y plane), $\mathbf{N} = \mathbf{k}$, curl $\mathbf{F}\cdot\mathbf{N} = 1$,
$\int$ curl $\mathbf{F}\cdot\mathbf{N}\,dS = \int 1\,dA$ = area of face DBC $= 9\pi$.

Final answer is $0 + 0 + 9\pi = 9\pi$

6. By Green's theorem,

$$\frac{1}{2}\oint_{ccl} -y\,dx + x\,dy = \frac{1}{2}\int_{\text{inside curve}} 2\,dA = \int_{\text{inside curve}} dA = \text{area inside curve.}$$

7. There are two hats in the diagram with the curve as its brim.
method 1 Use the flat hat (a lid).



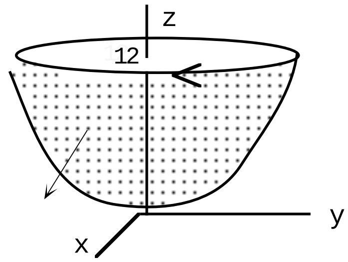
The N that is righthanded w.r.t. T is $-\mathbf{k}$.
By Stokes' theorem, $\oint \mathbf{F}\cdot\mathbf{T}\,ds$ on the rim $= \int$ curl $\mathbf{F}\cdot\mathbf{N}\,dS$ on the flat hat
Curl $\mathbf{F} = (0, y-1, -z)$
On the lid, $z = 12$, curl $\mathbf{F} = (0, y-1, -12)$, curl $\mathbf{F}\cdot\mathbf{N} = 12$, $dS = dA$
$\int$ curl $\mathbf{F}\cdot\mathbf{N}\,dS$ on the flat hat $= \int 12\,dA$ on the lid
The lid is bounded by ellipse $2x^2 + y^2 = 12$ If you know the area of an ellipse you can continue like this

$$\int 12\,dS \text{ on the lid} = 12 \times \text{area of lid} = 12\,\pi\,\sqrt{6}\,\sqrt{12} = 72\sqrt{2}\,\pi$$

Otherwise you can set up the integral like this.

$$\int 12\,dA \text{ on the lid} = \int_{x=-\sqrt{6}}^{\sqrt{6}} \int_{y=-\sqrt{12-2x^2}}^{\sqrt{12-2x^2}} 12\,dy\,dx \quad [\,= 72\sqrt{2}\,\pi\,]$$

method 2 Use the cup as the hat



The cup has parametric equations

$x = x$

$y = y$

$z = 2x^2 + y^2$

The parameter world is the projection of the cup in the x,y plane, the inside of ellipse $2x^2 + y^2 = 12$.

Let $g = 2x^2 + y^2 - z$

$$n = \frac{\nabla g}{\partial g/\partial z} = \frac{(4x,\ 2y,\ -1)}{-1} = (-4x,\ -2y,\ 1)$$

Or use $n = \text{vel}_x \times \text{vel}_y = (1,\ 0,\ 4x) \times (0,\ 1,\ 2y) = (-4x,\ -2y,\ 1)$

The N that is righthanded w.r.t. the T direction points downish so I'll use $-n$

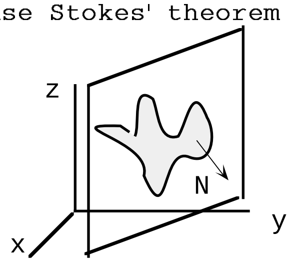
$\text{curl } F = (0,\ y - 1,\ -z)$

$\text{curl } F \cdot -n = 2y(y - 1) + z$

$\int$ curl F·N dS on the cup hat = $\int$ curl F·−n dy dx over x,y world

$$= \int_{x=-\sqrt{6}}^{\sqrt{6}} \int_{y=-\sqrt{12-2x^2}}^{\sqrt{12-2x^2}} [\ 2y(y-1) + 2x^2 + y^2\ ]\ dy\ dx$$

8. I'm going to use Stokes' theorem with a plane hat.



Let $F = zi + (x+y)j + xk$.

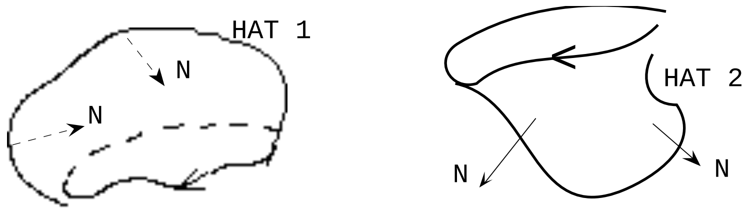
$\oint z\,dx + (x + y)dy + x\,dz = \int$ curl F·N dS on hat

Curl $F = (0,0,1)$

N has a zero third component (because the plane is perp to the x,y plane) so curl F·N = 0 on the hat and

$\oint z\,dx + (x + y)dy + x\,dz = \int 0\ dS = 0.$

9. There are two hats in the diagram with the given curve as brim.



So there are two possible answer. The diagram shows the correctly oriented N on each hat.

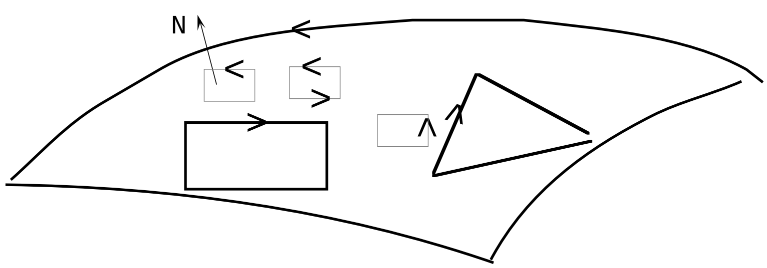
Answer 1 $\oint F\cdot T\ ds$ on the loop $= \int$ curl $F\cdot$ *inner* N dS on Hat 1

Answer 2 $\oint F\cdot T\ ds$ on the loop $= \int$ curl $F\cdot$ *outer* N dS on Hat 2

warning

1. The answer is *not* $\int$ curl $F\cdot N\ dS$ on the whole potato skin. *The whole potato skin is not a hat*.
2. And no answer is complete unless it clearly specifies in the diagram which hat and which N to use.

10. Fill the mask with little directed loops.
The diagram also shows the N that is righthanded w.r.t. the direction on the loops.



I'll describe directions as viewed from above.

Sum of little ccl circs
= ccl circ on outer loop
(1) + clockwise rectangle circ
+ clockwise triangle circ

Each little circ is a curl $F\cdot N\ dS$ (since F does not blow up inside any of the little loops). So

(2) sum of little ccl circs = sum of curl $F\cdot N\ dS$'s

Put (1) and (2) together to get this answer:

$\oint F\cdot T\ ds$ ccl on outer boundary loop

$+ \oint F\cdot T\ ds$ clockwise on rectangle

$+ \oint F\cdot T\ ds$ clockwise on triangle

$= \int$ curl $F\cdot$upper N dS on the mask

SOLUTIONS Section 4.4

1. (a) Yes (These are the opposites of the similarly oriented N's in Fig 1.)
(b) Yes (These are the opposites of the similarly oriented N's in Fig 2

2. The surface of an apple is a closed surface.
Div F = 2xy − 2xy = 0. Item (4) on the zero div list holds so item (1) also holds and $\oint$ F·N dS = 0.

3. F itself is irrelevant except that you must know that it doesn't blow up.
answer 1 This is $\oint$ G·N dS where G is a curl. Item (3) on the zero div list holds for G. So item (1) also holds and $\oint$ G·N dS, i.e., $\oint$ curl F·N dS, is always 0.

answer 2 Div curl F = 0 (famous identity). So item (4) on the zero div list holds for curl F. So item (1) holds also and $\oint$ curl F·N dS = 0.

4. (a) a,b,d,f,h,m,n,r are vector fields, the others are scalar fields
(b) a is a vector potential for b
c is a scalar potential for d
m is a vector potential for h
p is a scalar potential for n

5. F has an anticurl iff div F = 0

$$\text{iff } 3x^2y + \frac{\partial q}{\partial y} + 2xyz = 0$$

$$\text{iff } \frac{\partial q}{\partial y} = -3x^2y - 2xyz$$

$$\text{iff } q = -\frac{3}{2}x^2y^2 - xy^2z + \text{any } x,z \text{ stuff}$$

6. (a) $F = \dfrac{1}{x^2+y^2+z^2}(x\,i + yj + zk) = \dfrac{1}{\rho^2}\rho e_\rho = \dfrac{1}{\rho}e_\rho$
(b) *method 1* By inspection, $F = \nabla \ln\rho$ so the (scalar) potential is $\ln\rho$.

Mathematicians would point out that there is no potential at the origin ($\ln\rho$ is not defined when $\rho = 0$) where the original F blows up.

method 2 Compute curl F. When it comes out to be $\vec{0}$ you know that F has a (scalar) potential (although with this method you don't know what the potential actually is yet).

(c) $\text{div } F = \dfrac{1}{\rho^2 \sin\phi}\dfrac{\partial \rho \sin\phi}{\partial\rho} = \dfrac{1}{\rho^2\sin\phi}\sin\phi = \dfrac{1}{\rho^2}$.

Since div F is not 0, F doesn't have an anti-curl so there is no vector potential.

7. (a) Div F = 0 so the surface integral is 0.
(b) Div F = 2x + 2y + 2z, not zero everywhere.
Item (4) on the zero-div list fails. So item (1) fails.
So not *every* $\oint$ F·N dS is 0, i.e., at least one $\oint$ F·N dS is nonzero, but we don't know which one(s). So we have no conclusion about $\oint$ F·N dS here. Might be 0, might not.

8. (a)(b) Can't conclude anything because the umbrella is not a *closed* surface.

9. (a) Don't know anything about div F just from knowing that the surface integral on one particular closed surface is 0. (If you knew that the surface integral was 0 on *every* closed surface that would be a different story. Then div F = 0.)

(b) Item (1) on the zero div list fails. So item (4) fails also. So div F is not 0 everywhere, i.e., div F is nonzero somewhere.

(c) Doesn't matter what $\int \mathbf{F}\cdot\mathbf{N}\,dS$ is on the umbrella. That won't give you information about div F because an umbrella is not a closed surface.

10. Div F = 2x − 2x = 0.
The surfaces are hats with a common brim.
By the zero-div rule, $\int \mathbf{F}\cdot\mathbf{N}\,dS$ is the same on the three hats provided the N's are similarly oriented.
outer N_{cone} and outer $N_{\text{hemisphere}}$ and INNER N_{glove} have the same orientation.

By the zero div rule,

$\oint$ F·outer N dS on hemisphere = 7

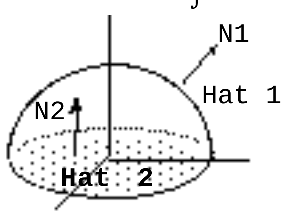
$\oint$ F·outer N dS on glove = −7

11. (a) Div $\vec{k} = 0$. So everything on the zero div list holds.
In particular the surface integral is independent of surface.
Call the top half of the ellipsoid Hat 1.
To make the computation easier, switch to Hat 2, the plane hat in the diagram (both have the same brim).
Hat 2 is a disk in the x,y plane, the inside of circle $2x^2 + 2y^2 = 6$ (radius $\sqrt{3}$).
The diagram shows similarly oriented N's.

$$\int \mathbf{k}\cdot\mathbf{N1}\,dS \text{ on hat 1} = \int \mathbf{k}\cdot\mathbf{N2}\,dS \text{ on hat 2}$$

$$= \int \mathbf{k}\cdot\mathbf{k}\,dS \text{ on hat 2}$$

$$= \int 1\,dS = \text{disk area} = 3\pi.$$



(b) The two halves are hats with a common brim. But the outer N on the bottom half and the outer N on the top half are oppositely oriented.
So $\int$ k·outer N dS on bottom half = MINUS $\int$ k·outer N dS on top half = −3π

(c) Div $\vec{k} = 0$. So everything on the zero div list holds.
In particular, $\oint \vec{k}\cdot\mathbf{N}\,dS$ is 0 on any closed surface. So the answer here is 0.

SOLUTIONS Section 4.5

1. (a) *on the lid* Outer N=k, z=6, $F \cdot N = y^2z^2 = 36y^2$, dS = dA

The circle around the lid has equations $x^2+y^2 = 6$, z=6.

$$\int F\cdot \text{outer } N\, dS \text{ on lid} = \int_{\text{lid}} 36y^2\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{6}} 36\, r^2 \sin^2\theta\; r\, dr\, d\theta = 324\pi$$

on the paraboloid The paraboloid has parametric equations

x=x
y=y
$z=x^2+y^2$

where the parameter world is the projection of the surface in the x,y plane, the inside of circle $x^2 + y^2 = 6$,

Let $g = z-x^2-y^2$. Then $n = \dfrac{\nabla g}{\partial g/\partial z} = (-2x,-2y,1)$.

Or use $n = (\partial x/\partial x,\ \partial y/\partial x,\ \partial z/\partial x) \times (\partial x/\partial y,\ \partial y/\partial y,\ \partial z/\partial y)$

The outer N points down so use $-n$.

$$\int F\cdot \text{outer } N\, dS = \int_{x,y \text{ projection}} F\cdot -n\; dx\, dy$$

$$= \int_{x,y \text{ projection}} [\, 2xy-y^2(x^2+y^2)^2 \,]\; dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{6}} (2\, r\cos\theta\; r\sin\theta - r^2 \sin^2\theta\;\; r^4\;\; r\, dr\, d\theta$$

$$= -162\pi$$

Final answer $= 324\pi - 162\pi = 162\pi$

(b) $\oint F\cdot \text{outer } N\, dS$ on paraboloid-plus-lid $= \int_{\text{inside}} \text{div } F\, dV = \int_{\text{region inside}} 2y^2 z\, dV$

I'll do the triple integral in cylindrical coordinates.

The lower boundary of the region is the paraboloid $z=x^2+y^2$ which is $z=r^2$ in cyl coords.

The upper boundary is the plane z = 6.

The projection of the solid in the x,y plane is a disk with radius $\sqrt{6}$.

$dV = r\, dz\, dr\, d\theta$

$$\int_{\text{region inside}} 2y^2 z\, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{6}} \int_{z=r^2}^{6} 2r^2 \sin^2\theta\; z\;\; r\, dz\, dr\, d\theta = 162\pi$$

warning The z limits are *not* $\int_{z=0}^{r^2}$. That would go with Fig A. But the limits should go with Fig B.

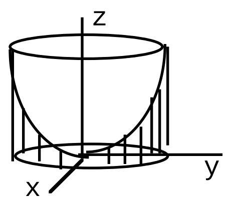


FIG A

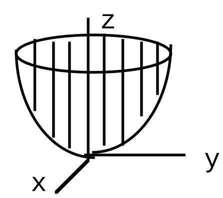


FIG B

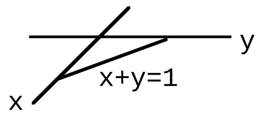
warning The z in the integrand stays z. It does not become r^2.

On the solid inside the cup z goes *from* r^2 *to* 6.

2. Flux out $= \oint_{\text{tet surface}} F\cdot\text{outer } N\, dS = \int \text{div } F\, dV$ over the solid tetrahedron

Here's one way to set up the triple integral.

The lower boundary of the solid is the plane $z = 0$.
The upper boundary is the plane $x + y + z = 1$.
The inner limits come from the projection of the solid in the x,y plane



$$\text{Flux out} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x + y + z)\, dz\, dy\, dx$$

3. $\oint F\cdot\text{INNER } N\, dS = \text{MINUS} \int_{\text{interior}} \text{div } F\, dV$

$$= -\int 3z\, dV$$

By inspection, this is 0 because, by symmetry, the positive z dV's from above the x,y plane cancel out the negative z dV's below so the sum is 0. If you don't believe me then set up the triple integral in spherical coords like this:

$$-\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} 3\rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \ldots = 0$$

4. $\oint F\cdot\text{outer } N\, dS$ on the cylinder plus top and bottom lids

$$= \int_{\text{inside}} \text{div } F\, dV = \int 3\, dV = 3 \times \text{volume of cylinder} = 3\pi R^2 H$$

On the top lid, $z = H$, outer $N = k$, $F\cdot N = z = H$,

$$\int_{\text{top lid}} F\cdot\text{outer } N\, dS = \int H\, dA = H\cdot\text{lid area} = H\cdot\pi R^2 = \pi R^2 H$$

On the bottom lid, $z=0$, outer $N = -k$, $F = xi + yj + 0k$, $F\cdot\text{outer } N = 0$,

$$\int_{\text{bottom lid}} F\cdot\text{outer } N\, dS = 0.$$

So $\int F\cdot\text{outer } N\, dS$ on the cyl surface itself $= 3\pi R^2 H - \pi R^2 H - 0 = 2\pi R^2 H$.

5. (a) (See (A) in Section 1.2.) It's the rate of change of g w.r.t. distance as you walk in the N direction (a directional derivative).
To compute it, take the component of ∇g in the direction of N, i.e., $\nabla g\cdot N$.
(b) $\nabla^2 g$ is the same as $\nabla\cdot\nabla g$ and it means $\text{div}(\nabla g)$ or, equivalently, Lapl g.
(c) $\oint f \dfrac{\partial g}{\partial \text{ outer } N}\, dS$

$$= \oint f(\nabla g\cdot\text{outer } N)\, dS$$

$$= \oint (f\nabla g)\cdot\text{outer } N\, dS \qquad \text{dot rule } k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot v$$

$$= \int_{\text{inside}} \operatorname{div}(f\nabla g)\, dV \qquad \text{divergence theorem}$$

$$= \int (f\ \nabla\cdot\nabla g + \nabla f\cdot\nabla g)\, dV \qquad \text{divergence product rule}$$

$$= \int (f\ \nabla^2 g + \nabla f\cdot\nabla g)\, dV \qquad \text{QED}$$

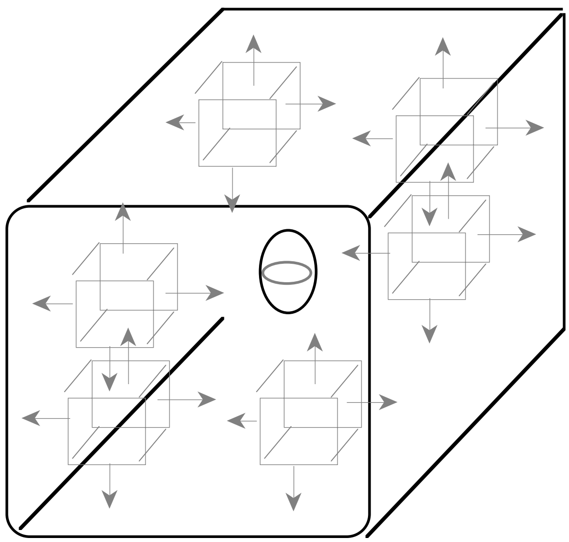
(d) Use Green's identity from part (c) with f and f, instead of with f and g, to get

$$\oint_{\text{surface}} f\ \frac{\partial f}{\partial\ \text{outer}\ N}\, dS = \int_{\text{inside}} (f\ \nabla^2 f + \nabla f\cdot\nabla f)\, dV$$

But $\nabla f\cdot\nabla f$ is $\|\nabla f\|^2$ and $\nabla^2 f$ = Lapl f = 0 because f is harmonic so the blank is

$$\int_{\text{inside}} \|\nabla f\|^2\, dV$$

6. Fill the space between the egg shell and the box with little boxes.



$\sum$ flux out of the little boxes = flux into the egg shell + flux out of the box

Also

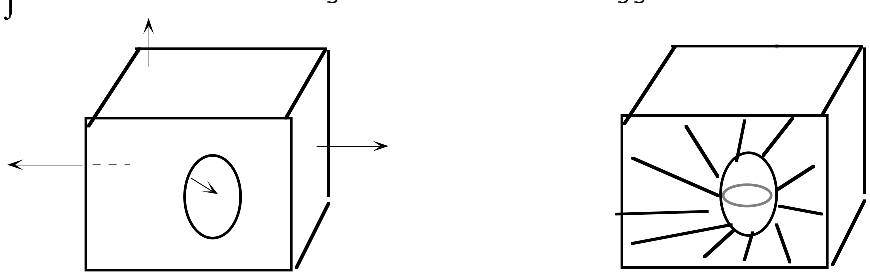
flux out of a little box = div F dV

$\sum$ flux out of the little boxes = $\sum$ div F dV

Put this together and translate into integrals:

$$\int F\cdot\ \textit{inner}\ N\, dS \text{ on the egg shell} + \int F\cdot\textit{outer}\ N\, dS \text{ on the box surface}$$

$$= \int \operatorname{div} F\, dV \text{ on the region between the eggshell and the box}$$



This is an extension of the div theorem.

SOLUTIONS review problems for Chapter 4

1. $F = \nabla \frac{1}{2} r^2$

$\int F\cdot T\, ds$ on the closed curve = 0

$\int F\cdot T\, ds$ on the B to A curve $= \left.\frac{1}{2} r^2\right|_{B}^{A}$

$$= \left.\frac{1}{2} r^2\right|_{r=\sqrt{2}}^{r=0} \qquad \text{(the cyl coord r is } \sqrt{x^2+y^2}\text{, dist to z-axis)}$$

$$= -1$$

2. (a) $\oint_{\text{loop}} = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{left}} + \int_{\text{right}}$

The top curve has parametric equations $x = y^2$, $y = y$, $z = 5$.
The bottom curve has parametric equations $x = y^2$, $y = y$, $z = 0$.
The left line has parametric equations $x = 4$, $y = -2$, $z = z$.
The right line has parametric equations $x = 4$, $y = 2$, $z = z$

$$\oint_{\text{loop}} = \int_{y=-2}^{2} (4y^2 - 5)\, dy + \int_{y=2}^{-2} 4y^2\, dy + \int_{z=0}^{5} 16\, dz + \int_{z=5}^{0} 16\, dz$$
$$= \int_{-2}^{2} -5\, dy = -20$$

(b) (Stokes' theorem) The loop is the brim of the hat in Fig A (a bent newspaper), lying in surface $x = y^2$.

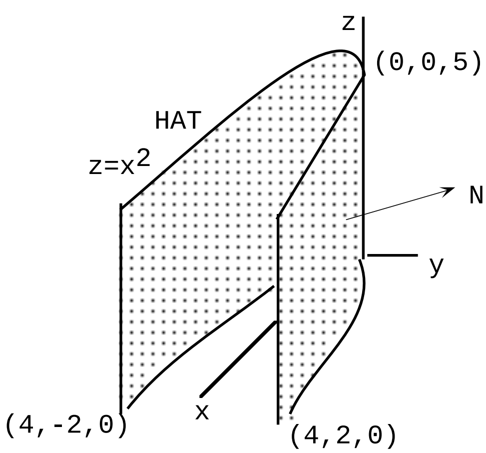


FIG A

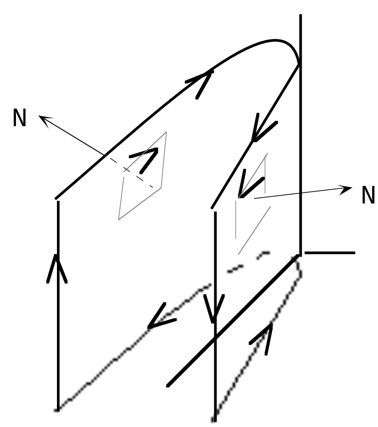


FIG B

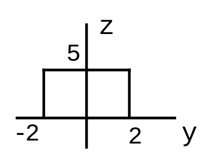


FIG C

Let $F = 2y\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$
The N that is oriented w.r.t. to the direction on the curve is an outer N.
I used the two little loops in Fig B to decide. In each case my thumb pointed out of the hat. So

$$\oint_{\text{loop}} F\cdot T\, ds = \int_{\text{hat}} \text{curl } F\cdot \text{outer } N\, dS$$

Now I have to do the surface integral.

The surface has parametric equations $x=y^2$, $y=y$, $z=z$ where the parameter world is the projection of the hat in the y,z plane *method 1 for getting n* Rewrite the equation as $x-y^2 = 0$. Let $g=x-y^2$. Then

$$n = \frac{\nabla g}{\partial g/\partial x} = (1, -2y, 0)$$

method 2 for getting n

$$n = \text{vel}_y \times \text{vel}_z = (2y,1,0) \times (0,0,1) = (1,-2y,0)$$

This n is a forward normal (because it has a positive x-component) which makes it an inner normal so $n_{\text{outer}} = -n$.

Curl $F = i - 2xj - 2k$

Curl $F\cdot -n = -1-4y^3$

$$\int_{\text{hat}} \text{curl } F\cdot \text{outer } N \, dS = \int_{y,z \text{ proj}} \text{curl } F\cdot n_{\text{outer}} \, dy\, dz$$

$$= \int_{z=0}^{5} \int_{y=-2}^{2} (-1-4y^3)\, dy\, dz \quad [\, = -20 \,]$$

3.(a) On the top face $F\cdot N = F\cdot k = 0$.
On the bottom face $F\cdot N = F\cdot -k = 0$.
On the left face $F\cdot N = F\cdot -j = -2y = 0$ (since $y = 0$ on left face).
On the right face $F\cdot N = F\cdot j = 2y = 4$.

$$\int_{\text{right face}} F\cdot N \, dS = \int 4 \, dA = 4 \times \text{area} = 80$$

On the forward face $F\cdot N = F\cdot i = x = 4$.

$$\int_{\text{forward face}} F\cdot N \, dS = \int 4 \, dA = 4 \times \text{area} = 40$$

On the rear face $F\cdot N = F\cdot -i = -x = 0$.

So $\oint_{\text{box}} F\cdot \text{outer } N \, dS = 80 + 40 = 120$

(b) $\oint F\cdot \text{outer } N \, dS = \int_{\text{inside}} \text{div } F \, dV = \int 3 \, dV = 3\cdot \text{vol} = 3\cdot 40 = 120$

4. $\text{div } F = \dfrac{1}{\rho^2 \sin\phi} \dfrac{\partial}{\partial \rho}\left(\rho^2 \sin\phi \; \dfrac{1}{\rho^2}\right) = 0$ but F blows up at the origin.

By the deformation of surface principle (§4.4), $\oint F\cdot \text{outer } N \, dS$ is the same on any surface enclosing the origin.

Switch from the tin can to a unit sphere around the origin, a convenient surface because on the sphere, $F = e_\rho$ and $F\cdot \text{outer } N = e_\rho\cdot e_\rho = 1$.

$$\oint_{\text{tin can}} F\cdot \text{outer } N \, dS = \oint_{\text{sphere}} F\cdot \text{outer } N \, dS = \oint_{\text{sphere}} dS = \text{surface area of sphere} = 4\pi$$

5. (a) Don't see any applicable theory.

(b) $\oint xy\, dx + x^2\, dy$ CLOCKWISE

$$= \text{MINUS} \int_{\text{inside}} (2x-x)\, dA \text{ (Green's theorem) (minus because of clockwise)}$$

$$= \text{MINUS} \int_{y=-1}^{0} \int_{x=y^2}^{1} x\, dx\, dy \quad [\, = -\tfrac{2}{5} \,]$$

(c) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} = 2x$ so F has an antigradient. In particular, $F = \nabla x^2 y$ so

$$\int_{C \text{ to } A} 2xy\,dx + x^2\,dy = x^2 y \Big|_C^A = 0 - -1 = 1$$

(d) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ so the line integral is 0 by the zero-curl rule.

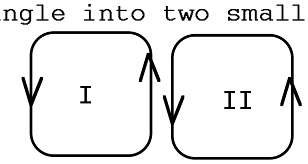
6. (a) $\oint_{\text{ccl circle}} = \oint_{\text{ccl triangle}} = 4$

(b) $\oint_{\text{ccl pentagon}} = 0$ since there is no blowup inside the pentagon

(c) $\oint_{\text{figure eight}} = \oint_{\text{ccl left half}} + \oint_{\text{clock right half}}$

$= \oint_{\text{ccl triangle}} + \oint_{\text{clock square}} = 4 + -6 = -2$

(d) Split the rectangle into two smaller rectangles.



$$\oint_{\text{ccl rectangle}} = \oint_{\text{ccl on I}} + \oint_{\text{ccl on II}}$$

$$= \oint_{\text{ccl triangle}} + \oint_{\text{ccl square}} = 4 + 6 = 10$$

Question

Beside the fact that the triangle = 4 and the square = 6, do you remember what else the hypothesis was (that made this game playable in the first place)

Answer The hypothesis was curl $F = \vec{0}$ (or in this 2-dim case, $\partial q/\partial x = \partial p/\partial y$) but F blows up at the two asterisks.

7. (a) No because curl F is not $\vec{0}$ at every point (it's $2z\vec{j} - \vec{k}$ which is $\vec{0}$ at some points but not all).

(b) Yes because div F = 2x − 2x = 0.

(c) Makes no sense. There is no way to get div of something to be F because divergence is a *scalar* field and F is a *vector* field.

8. (a) $\frac{\partial q}{\partial x} \neq \frac{\partial p}{\partial y}$ so all the items on the zero-curl list fail.

$\int F \cdot T\,ds$ is *not* independent of path: There exist two points A and B and two A-to-B paths on which $\int F \cdot T\,ds$ has different values. But we do't know if it is paths I and II for which this happens. Can't draw any conclusion.

(b) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ so everything on the zero-curl list holds, including independence of path. So $\int_I = \int_{II}$ for sure.

9. (a) It means that if two curves have the same endpoints and the same direction then $\int F \cdot T\,ds$ has the same value on the two curves (the zero-curl rule tells you *when* this happens).

(b) It means that if two hats have the same brim and the N's are similarly oriented then $\int F \cdot N\,dS$ has the same value on the two hats (the zero-div rule tells you *when* this happens).

10. (a) Don't know anything about curl F just from knowing that the line integral on one particular loop is 0. (If you knew that the line integral was 0 on *every* closed curve, that would be a different story. Then curl F = $\vec{0}$.)

(b) Curl F can't be $\vec{0}$. Statement (1) on the zero-curl rule fails so statement (4) fails also. Curl F can be $\vec{0}$ at some points but not everywhere.

(c) Can't tell anything. There is no connection between curl F and $\int$ F·T ds on one particular *non*-closed curve.

(d) Same answer as (c).

11. (a) The surfaces are hats with the same brim, namely a circle in the x,y plane with center at the origin and radius 3.

Div F = 0 so by the zero-div rule, $\int$ F·N dS is independent of surface so $\int$F·N dS is the same on the hats provided the N's are similarly oriented.

So all I have to do is find the surface integral on one of the hats.
I'll use the first one.

On the plane hat in (i), indicated N = k, dS = dA, F·N = x^2,

$\int$F·N dS on surface (i)

$= \int x^2$ dA on the circular region with center at the origin and radius 3

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^2 \cos^2 \theta \;\; r \, dr \, d\theta \;=\; \frac{81\pi}{4}$$

The N's on hats (i) and (ii) are similarly oriented so $\int$F·N dS on hat (ii) = $81\pi/4$.

The N's on (i) and (iii) have opposite orientation so $\int$F·N dS on hat (iii) = $-81\pi/4$

(b) An anticurl must exist because div F = 0.

The anticurl is a vector field $p\vec{i} + q\vec{j} + r\vec{k}$ so that

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (p,q,r) = (0,0,x^2).$$

There are lots of anticurls. One possibility is p=0, $q = \frac{1}{3}x^3$, r = 0. An anticurl is $\frac{1}{3}x^3 \, \vec{j}$ (another is $-x^2y \, \vec{i}$).

Now read Stokes' theorem backwards: If N and T are oriented righthandedly then

$$\int \text{F·N dS on a hat} = \oint \text{anticurl F·T ds on the brim}$$

The three hats in this problem all have the same brim.
View directions on the brim from above.
For surfaces (i) and (ii) the T direction on the brim that is righthanded w.r.t. the given N is ccl.
The brim is a circle with parametric equations

$x = 3 \cos t$
$y = 3 \sin t$
$z = 0$
$0 \le t \le 2\pi$.

$\int$F·N dS on hat (i) = $\oint$ anticurl F·T ds on the ccl brim

$$= \oint 0\,dx + \tfrac{1}{3}x^3\,dy + 0\,dz \text{ on ccl circle}$$

$$= \int_{t=0}^{2\pi} \frac{1}{3}(3\cos t)^3 \cdot 3\cos t \, dt = \frac{81\pi}{4}$$

Same for hat (ii).

For hat (iii) the T direction on the brim that is righthanded w.r.t. the given N is clockwise.

$$\int F\cdot N\ dS \text{ on hat (iii)} = \oint \text{anticurl } F\cdot T\ ds \text{ on the } \textit{clockwise} \text{ brim} = -\frac{81\pi}{4}$$

12. Pick one of the two T directions. Then for that T,

$$\oint \nabla(fg)\cdot T\ ds = \oint (f\nabla g + g\nabla f)\cdot T\ ds \quad \text{(gradient product rule)}$$

$$= \oint f\nabla g\cdot T\ ds + \oint g\nabla f\cdot T\ ds$$

The left side is 0 because $\int F\cdot T\ ds$ on a loop is 0 if F is a gradient. So

$$0 = \oint f\nabla g\cdot T\ ds + \oint g\nabla f\cdot T\ ds$$

$$\oint f\nabla g\cdot T\ ds = -\oint g\nabla f\cdot T\ ds$$

$$= \oint g\nabla f\cdot \text{opposite } T\ ds \quad \text{QED}$$

SOLUTIONS Section 5.1

1. *method 1* $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u & v \\ -v & u \end{vmatrix} = u^2 + v^2$

method 2 (works because the parabolic coord system is orthogonal and righthanded)

$h_u = h_v = \sqrt{u^2 + v^2}$, $\dfrac{\partial(x,y)}{\partial(u,v)} = h_u h_v = u^2 + v^2$

2. (a) Consider the v-curve $u = u_0$. It has parametric equations

$x = u_0 + v$

$y = u_0 - v$

It has the plain equation $x + y = 2u_0$ (I added the two parametric equations to eliminate the parameter v.

The v-curves are lines (see the diagram below).

Consider the u-curve $v = v_0$. It has parametric equations

$x = u + v_0$

$y = u - v_0$

It has the plain equation $x - y = 2v_0$ (I subtracted to eliminate the parameter u).

The u-curves are also lines.

(b) $e_u = (\frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j)_{\text{unit}} = (i + j)_{\text{unit}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

$e_v = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

(c) Yes because $e_u \cdot e_v = 0$

(d) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

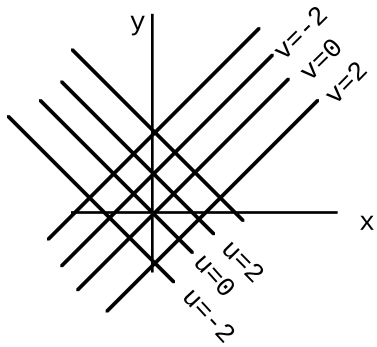
Also $\dfrac{\partial(x,y)}{\partial(u,v)} = -h_u h_v$ since the coord system is orthogonal and lefthanded.

$$h_u = \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2} = \sqrt{2},\quad h_v = \sqrt{2},\quad \frac{\partial(x,y)}{\partial(u,v)} = -\sqrt{2}\sqrt{2} = -2$$

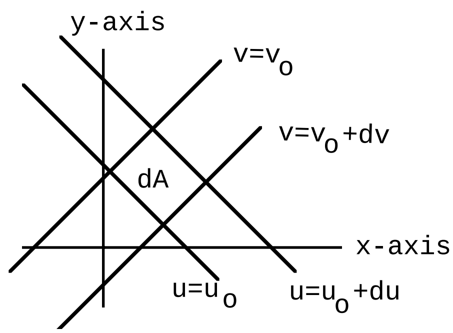
(d) $dA = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = |-2|\, du\,dv = 2\, du\,dv$.

Also $dA = h_u h_v\, du\,dv = 2\, du\,dv$ because the coord system is orthogonal. Would not work otherwise.

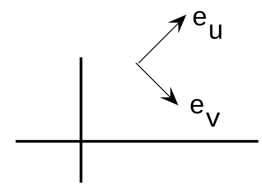
(f) Lefthanded. The Jacobian is negative at every point and besides, you can just plot the vectors e_u and e_v from part (b) and see that e_u is always ahead of e_v.



(a)



(d)



(e)

3. (a) $e_u = (6u\vec{i})_{unit}$, $e_v = (7\vec{i} + 2\vec{j})_{unit}$. Since $e_u \cdot e_v$ is not always 0, the system is not orthogonal.

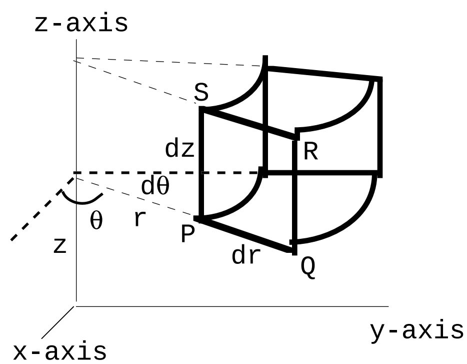
(b) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 6u & 0 \\ 7 & 2 \end{vmatrix} = 12u$$

$dA = |12u|\,du\,dv$ (don't leave out the absolute value signs)

The coordinate system is not orthogonal so there's no other way to get dA.

(c) Neither. At points where $u > 0$, the basis vectors are righthanded but at points where $u < 0$, they are lefthanded.

4. (a) The diagram shows a cylindrical coordinate volume element.



method 1 for finding dV

The cylindrical coord system is orthogonal so

$$dV = h_r h_\theta h_z\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz$$

method 2 for finding dV

$x = r\cos\theta$, $y = r\sin\theta$, $z=z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$dV = |r|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$ (since $r \ge 0$ in cylindrical coords).

(b) The face PQRS is traced out, a *surface area element* with area dS in the plane hinged along the z-axis and swung around at angle θ.
You can see from the picture that $dS = dr\,dz$.
Also, since cyl coords are orthog, $dS = h_r h_z\,dr\,dz = dr\,dz$.

5. $$\begin{vmatrix} 0 & 2 & vw \\ 3v^2 w & 3 & uw \\ v^3 & 4 & uv \end{vmatrix}$$

$$= -3v^2w \begin{vmatrix} 2 & vw \\ 4 & uv \end{vmatrix} + v^3 \begin{vmatrix} 2 & vw \\ 3 & uw \end{vmatrix}$$

$$= -4uv^3w + 12v^3w^2 - 3v^4w$$

6. $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,z)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \\ \partial x/\partial z & \partial y/\partial z & \partial z/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$ expand across row 3 or down col 3

So the two Jacobians are equal.

footnote

Here's the more careful version of how the two Jacs are related:
The blue Jac is a scalar field in *two-space*. It's a function of u and v.
The blue *cylindrical* Jac has the same formula but it's a scalar field in *three-space*; it's a function of u, v, z but does not depend on z.

SOLUTIONS Section 5.2

1. (a) $\frac{\partial(u,v)}{\partial(x,y)}) = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -11$

$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{11}$ by (5)

(b) $x = \frac{1}{11}(4u + v)$, $y = \frac{1}{11}(3u - 2v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4/11 & 3/11 \\ 1/11 & -2/11 \end{vmatrix} = -\frac{1}{11}$$

2. (a) $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 8xy$

(b) $x = \sqrt{\frac{1}{2}(u+v)}$, $y = \sqrt{\frac{1}{2}(v-u)}$

(I took the *positive* square roots because the problem said $x \ge 0$, $y \ge 0$.)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4\sqrt{\frac{1}{2}(u+v)}} & \frac{1}{4\sqrt{\frac{1}{2}(u+v)}} \\ -\frac{1}{4\sqrt{\frac{1}{2}(v-u)}} & -\frac{1}{4\sqrt{\frac{1}{2}(v-u)}} \end{vmatrix} = \frac{1}{8\sqrt{\frac{1}{2}(u+v)}\ \sqrt{\frac{1}{2}(v-u)}}$$

This *is* the reciprocal of the answer in (a) since $x = \sqrt{\frac{1}{2}(u+v)}$ and $y = \sqrt{\frac{1}{2}(v-u)}$

3. (a) $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -6w^2 & 0 & 0 \end{vmatrix} = -6w^2$

(b) If $x=61$, $y=12$, $z=5$ then $61 = u + v - 2w^3$, $12 = u+2v$, $5 = v$ so $v=5$, $u=2$, $w=-3$.

$$\left.\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|_{\text{at } u=1, v=2, w=-3} = -6w^2 = -54$$

$$\left.\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|_{\text{at } x=61, y=12, z=5} = -\frac{1}{54}$$

4. (a) ABC is a right triangle (because line $y=-x$ is perp to line $y=x$).

Area $= \frac{1}{2}$ BC $\times$ AC $= \frac{1}{2} \cdot \frac{2}{3}\sqrt{2} \cdot 2\sqrt{2} = \frac{4}{3}$

(b) *method 1*

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = -3$$

The equations map from an x,y plane to a u,v plane.
The area mag factor is 3 and it's constant so any region in the x,y plane should map to a region in the u,v plane with triple the area.
So the image area in the u,v plane should be $3 \times 4/3 = 4$

method 2

Imagine the mapping equations solved for x and y (but I don't actually have to do the solving). Call them $x = x(u,v)$, $y = y(u,v)$.
They map from a u,v plane to an x,y plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = 1/\frac{\partial(u,v)}{\partial(x,y)} = -1/3$$

The area mag factor is 1/3.

Any region in the u,v plane maps to a region with 1/3 the area in the x,y plane, i.e.,

u,v area × 1/3 = x,y area

u,v area = 3 × x,y area = 3 × 4/3 = 4

(c) *Image of side AC*

On AC, $y = x$ so $u = 2x-x = x$, $v = x-2x = -x$ so $v = -u$.

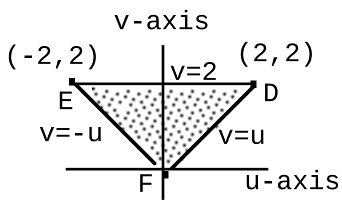
The image is line EF in the diagram below.

Image of side CB

On CB, $y = -x$ so $u = 3x$, $v = 3x$, $v = u$. The image is line DF.

Image of side AB

On AB, $x - 2y = 2$ so $v = 2$. The image is line ED.



DEF is a right triangle (FE is perp to FD).

DEF area = $\frac{1}{2}$ EF × FD = $\frac{1}{2}\sqrt{8}\,\sqrt{8} = 4$ It checks out.

5. Let $u = x+y+z$, $v = 2y+5z$, $w=3z$. Then the image in the u,v,w world of the given surface is the sphere $u^2 + v^2 + w^2 = R^2$ with volume $\frac{4}{3}\pi R^3$. I'll think of the x,y,z world as old and the u,v,w world as new (vice versa is OK too as long as you are consistent). Then

$$\frac{\partial(\text{new1},\text{new2},\text{new3})}{\partial(\text{old1},\text{old2},\text{old3})} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 5 & 3 \end{vmatrix} = 6$$

Volume from the old world gets multiplied by 6 when it's mapped to the new world. The given region is in the old world so its volume is 1/6 the new volume. So the old volume is $\frac{1}{6}\,\frac{4}{3}\,\pi R^3 = \frac{2}{9}\,\pi R^3$

6. These are mappings from a u,v plane to an x,y plane.

(a) The mapping translates points (see the lefthand diagram below). To get an image point, shift the original point right 2 and down 1. The mapping doesn't change areas and doesn't change orientation. Expect the Jacobian to be +1. And

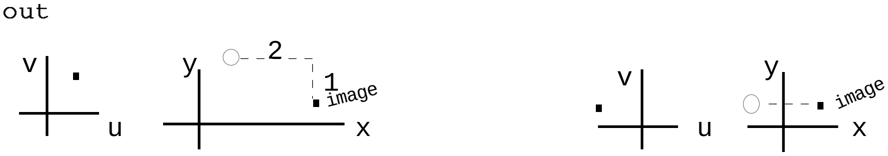
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

which checks out

(b) The mapping reflects points across the vertical axis. The mapping preserves area and reverses orientation. The Jacobian should be −1.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

which checks out



(a) (b)

7. It's a function of r and s, i.e., the letters r and s will be in the answer. Just like the derivative dw/du is a function of u.

SOLUTIONS Section 5.3

1. The graph of xy=16 is a hyperbola but I only drew the branch in quadrant III. Similarly for xy=4. Let u = y−x, v = xy. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ y & x \end{vmatrix} = -(x+y)$$

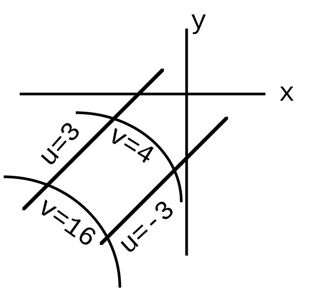
$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{x+y} \right| = -\frac{1}{x+y}$$

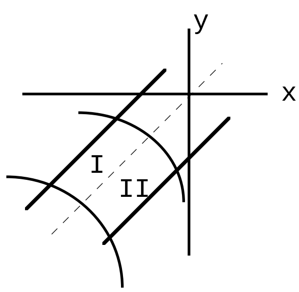
(*Keep* the minus sign to make the abs value *positive* since x+y itself is *negative* in the region.)

$$\int_{\text{region}} (x^2 - y^2)\, dA = \int_{u=-3}^{3} \int_{v=4}^{16} (x^2 - y^2) \cdot -\frac{1}{x+y}\, dv\, du$$

$$= \int_{u=-3}^{3} \int_{v=4}^{16} (y-x)\, dv\, du = \int_{u=-3}^{3} \int_{v=4}^{16} u\, dv\, du = 0$$

footnote I can see by inspection that the integral is 0. The integral adds (x^2-y^2)dA's. By symmetry, the positive (x^2-y^2)dA's in region I balance the negative (x^2-y^2)dA's in region II and the total sum is 0.

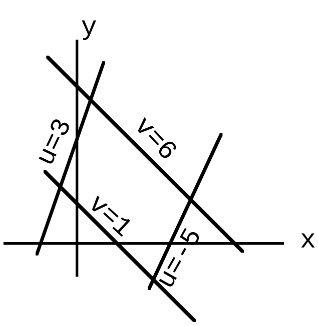




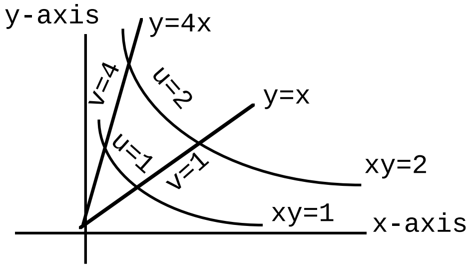
2. Let u = y − 2x, v = y + x.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3 \;, \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{3} \;, \quad x = \frac{1}{3}(v-u)$$

$$\int_{\text{region}} x^2\, dA = \int_{u=-5}^{3} \int_{v=1}^{6} \frac{1}{9}(v-u)^2 \, \frac{1}{3}\, dv\, du \quad \left(= \frac{3320}{81}\right)$$



3. Let $u = xy$, $v = y/x$.



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x}$$

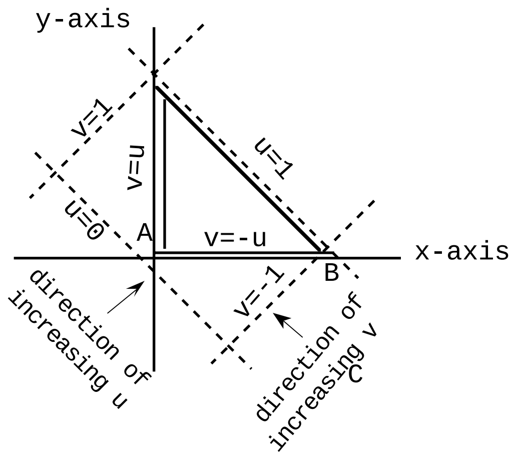
$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{2y}{x}$$ since y/x is positive in the region

$$\int_{\text{region}} \sin xy \, dA = \int_{u=1}^{2} \int_{v=1}^{4} \sin u \; \frac{1}{2v} \, dv \, du \qquad \left[= \frac{1}{2} \ln 4 \; (\cos 1 - \cos 2) \right]$$

4. Switch to the u,v system defined by $u = y + x$, $v = y - x$ to make the integrand simpler.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 , \qquad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2}$$

The region of integration is not a u,v rectangle. The entering u boundary (lefthand diagram) is the line AB. In the x,y system it's $y = 0$, so $u=x$, $v=-x$ and its equation in the u,v system is $v=-u$.
The exiting u boundary is the line AC. In the x,y system it's $x=0$, so $u=y$, $v=y$ and its equation in the u,v system is $v=u$.
The smallest u in the region is $u=0$ and the largest u in the region is $u=1$.
Another way to see the limits is to think mapping and get the image of the x,y region in the u,v plane (righthand diagram).



u,v coord paper

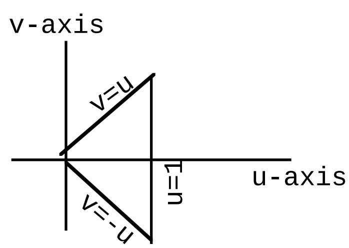


image of the x,y region

Either way,

$$\int_{\text{region}} e^{(y-x)/(y+x)} \, dA = \int_{u=0}^{1} \int_{v=-u}^{u} e^{v/u} \; \frac{1}{2} \, dv \, du$$

footnote I computed it to see if I agree with Mathematica

$$\text{inner integral} = \frac{1}{2}\, u e^{v/u}\Big|_{v=-u}^{u} = \frac{1}{2}\,(ue - ue^{-1}) = \frac{1}{2}\,(e-e^{-1})u$$

$$\text{outer} = \frac{1}{2}\,(e-e^{-1})\,\frac{u^2}{2}\Big|_{u=0}^{1} = \frac{1}{4}\,\left(e - \frac{1}{e}\right) \qquad \text{Yes, I did}$$

5. (a) Note that in this coord system, θ is not the polar coord angle θ and r is not the polar coord r.

The ellipse $\frac{x^2}{4} + \frac{y^2}{25} = r_0^2$ has parametric equations

$x = 2r_0 \cos\theta$, $y = 5r_0 \sin\theta$

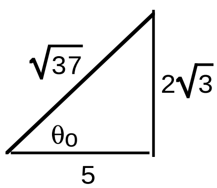
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 2\cos\theta & 5\sin\theta \\ -2r\sin\theta & 5r\cos\theta \end{vmatrix} = 10r, \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |10r| = 10r.$$

The bounding ellipses in the problem are $r = 1$ and $r = \sqrt{2}$.
The positive x-axis is where $y = 0$, $5r_0 \sin\theta = 0$, $\theta = 0$.

The ray at angle $\pi/3$ is *not* $\theta = \pi/3$. Here's how to find its r,θ equation.

The ray has the angle of inclination $\pi/3$, slope $\tan \pi/3 = \sqrt{3}$, equation $y = \sqrt{3}\,x$, $x \ge 0$. This equation converts to $5r\sin\theta = \sqrt{3}\; 2r\cos\theta$, $\tan\theta = 2\sqrt{3}/5$ (see the diagram).

You don't have to actually find the value of θ. Just call it θ_o for now.

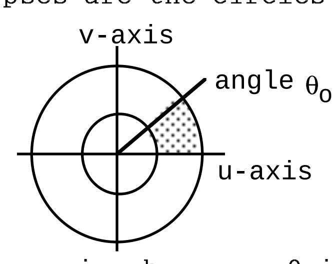


$$\int_{\text{region}} xy\, dA = \int_{\theta=0}^{\theta_o} \int_{r=1}^{\sqrt{2}} (2r\cos\theta)(5r\sin\theta)\; 10r\, dr\, d\theta$$

$$= \frac{75}{2}\sin^2\theta_o$$

$$= \frac{75}{2}\left[\frac{2\sqrt{3}}{\sqrt{37}}\right]^2 \qquad \text{(read } \sin\theta_o \text{ from the right triangle)}$$

(b) I'll think of the equations as mapping from an x,y plane to a u,v plane. The images of the ellipses are the circles $u^2 + v^2 = 1$ and $u^2 + v^2 = 2$.



The image of the x-axis where $y = 0$ is the u-axis where $v=0$.

The image of the line $y = \sqrt{3}\,x$ (the $\pi/3$ ray) is the line $5v = \sqrt{3}\; 2u$, $v = \frac{2}{5}\sqrt{3}\; u$. The line is at angle θ_0 where $\tan\theta_0 = \frac{2}{5}\sqrt{3}$ as in part (a).

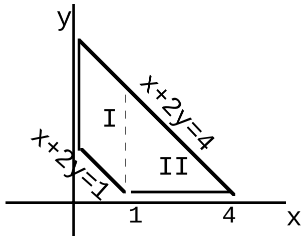
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10$$

$$\int_{\text{region}} xy\,dA = \int_{\text{image region}} 2u\cdot 5v\ 10\ du\,dv$$

I'll do the integration in polar coordinates in the u,v plane.

$$\int_{\text{region}} xy\,dA = \int_{\theta=0}^{\theta_0} \int_{r=1}^{\sqrt{2}} 2r\cos\theta \cdot 5r\sin\theta\ 10\ r\,dr\,d\theta \quad \text{etc. as in part (a)}$$

6.(a) Must divide up the region. The diagram shows one possibility.

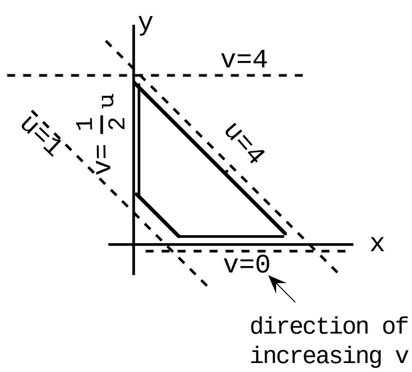


$$\int \frac{1}{x+2y}\,dA = \int_{x=0}^{1} \int_{\frac{1}{2}(1-x)}^{\frac{1}{2}(4-x)} \frac{1}{x+2y}\,dy\,dx + \int_{x=1}^{4} \int_{y=0}^{\frac{1}{2}(4-x)} \frac{1}{x+2y}\,dy\,dx$$

(b) Let $u = x + 2y$, $v = y$ [could also use $v=x$].

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = 1$$

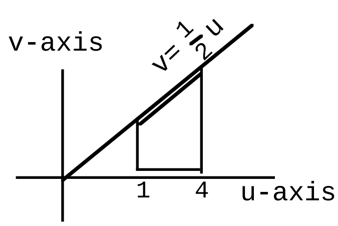
The entering v boundary is $v = 0$
The exiting v boundary is the y-axis where $x = 0$; so $u = 2y$, $v = y$ and its equation in the u,v system is $v = \frac{1}{2}u$.
The extreme u values are 1 and 4.



So

$$\int \frac{1}{x+2y}\,dA = \int_{u=1}^{4} \int_{v=0}^{u/2} \frac{1}{u}\,dv\,du.$$

Another way to see the u,v limits is to think mapping and find the image of the region of integration in the u,v plane.



7. The left plane ABGH has normal vector $v_1 \times v_2 = (-16,2,-4)$ and goes through the origin. Use the simpler normal $(-8,1,-2)$ to get equation $-8x + y - 2z = 0$
The right plane DEFC has the same normal and goes through $D = (-2,1,3)$ so its equation is

$$-8(x+2) + (y-1) - 2(z-3) = 0$$
$$-8x + y - 2z = 11$$

Similarly

front plane ABCD has equation $4x + 5y + z = 0$
rear plane HGFE has equation $4x + 5y + z = 22$
bottom plane ADEH has equation $14x + y + 9z = 0$
top plane BCFG has equation $14x + y + 9z = 22$

Use the coordinate system where

$$u = -8x + y - 2z$$
$$v = 4x + 5y + z$$
$$w = 14x + y + 9z$$

Then

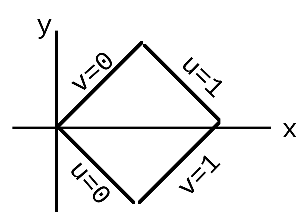
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -8 & 4 & 14 \\ 1 & 5 & 1 \\ -2 & 1 & 9 \end{vmatrix} = -242$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{242}$$

$$y = \frac{1}{11} u + \frac{2}{11} v \quad \text{(solve the equations for y)}$$

$$\int y^2 \, dV = \int_{u=0}^{11} \int_{v=0}^{22} \int_{w=0}^{22} \left(\frac{1}{11} u + \frac{2}{11} v\right)^2 \cdot \frac{1}{242} \, dw \, dv \, du \quad \left(= \frac{506}{3} \right)$$

8. Let $u=x+y$, $v=x-y$. Then



$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \quad x^2 - y^2 = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2 = uv$$

You can also get x^2-y^2 without solving for x and y:

$$x^2-y^2 = (x+y)(x-y) = uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

You can also get this Jac without solving for x and y:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \quad \text{flip to get } \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\int (x^2 - y^2)^N \, dA = \int_{v=0}^{1} \int_{u=0}^{1} (uv)^N \, \frac{1}{2} \, du \, dv$$

$$= \frac{1}{2} \left. \frac{u^{N+1}}{N+1} \right|_{u=0}^{1} \cdot \left. \frac{v^{N+1}}{N+1} \right|_{v=0}^{1} = \frac{1}{2(N+1)^2}$$

SOLUTIONS review problems for Chapter 5

1. Let $u = x^2 + y^2$, $v = x^2 - y^2$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2x \\ 2y & 2y \end{vmatrix} = -8xy$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{8xy} \right| = -\frac{1}{8xy} \quad \text{since } xy \text{ is negative in quadrant IV}$$

$$\int x\, dA = \int_{v=1}^{2} \int_{u=10}^{11} x \cdot -\frac{1}{8xy}\, du\, dv = -\frac{1}{8} \int_{v=1}^{2} \int_{u=10}^{11} \frac{1}{y}\, du\, dv$$

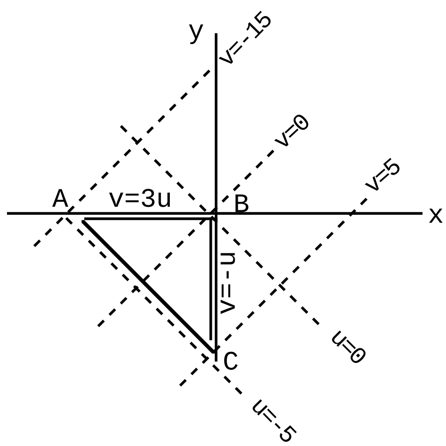
Solve for y to get $y = -\sqrt{\frac{1}{2}(u-v)}$ (choose the negative square root since y is negative in quadrant IV).

$$\int x\, dA = \frac{\sqrt{2}}{8} \int_{v=1}^{2} \int_{u=10}^{11} \frac{1}{\sqrt{u-v}}\, du\, dv$$

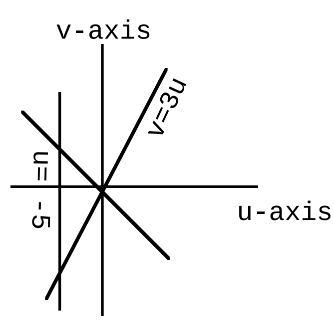
2. Let $u = x + y$, $v = 3x - y$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4, \qquad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{4}$$

The entering v-boundary is AB where $y = 0$, $u = x$, $v = 3x$. Its equation in the u,v system is $v = 3u$.
The exiting v-boundary is BC where $x = 0$, $u = y$, $v = -y$. Its equation in the u,v system is $v = -u$.
The extreme u values are $u = -5$ and $u = 0$.



Another way to get the limits is to think mapping and find the region in the u,v plane corresponding to triangle ABC in the x,y plane.



All in all, $\displaystyle\int (x + y)^n\,(3x - y)^m\, dA = \frac{1}{4} \int_{u=-5}^{0} \int_{v=3u}^{-u} u^n v^m\, dv\, du.$

Mathematica could do this u,v integral.

```
In[10]:=
- 1/8Integrate[  1/Sqrt[u - v], {v, 1, 2}, {u, 10, 11}] // Together

Out[10]=
27 - 8 Sqrt[2] - 5 Sqrt[10]
---------------------------
             3
```

3. Let $u = x+y$, $v = x+6y$. These equations map from the x,y plane to a u,v plane. The image of $(x+y)^2 + (x+6y)^2 = 100$ is the circle $u^2 + v^2 = 100$ with area 100π.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} = 5$$

Think of the x,y plane as old and the u,v plane as new. Areas from the old plane are magnified by 5 when they are mapped to the new plane. The region whose area we want is in the old plane. Its image in the new plane has area 100π. So the old area was $\frac{1}{5}\cdot 100\pi = 20\pi$.

4. (a) The parabolas are $\frac{y}{x^2} = 5$ and $\frac{y}{x^2} = 1$. The circles are $(x-2)^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ which multiply out to $x^2 + y^2 = 4x$ and $x^2 + y^2 = 2x$. Let

(*) $$u = \frac{y}{x^2}, \quad v = \frac{x^2+y^2}{x}$$

In the u,v coordinate system the region is the rectangle $1 \le u \le 5$, $2 \le v \le 4$.

(b) The problem is to choose ? so that $\int_R ?\, dA$ over the region is easy to do.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-2y}{x^3} & \frac{x^2-y^2}{x^2} \\ \frac{1}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{-3y^2-x^2}{x^4}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{x^4}{3y^2+x^2}$$

So $\int_R ?\, dA = \int_{u=1}^{5}\int_{v=2}^{4} ?\, \frac{x^4}{3y^2+x^2}\, dv\, du$.

To do the integral you have to be able to convert $?\,\frac{x^4}{3y^2+x^2}$ to u's and v's. I can't just replace the x's and y's because I can't solve the equations in (*) for x and y. But if I choose $? = \frac{3y^2+x^2}{x^4}$ then I get $\int \frac{3y^2+x^2}{x^4}\, dA = \int_{u=1}^{5}\int_{v=2}^{4} dv\, du = 4\cdot 2 = 8$.

So I would use $\int_R \frac{3y^2+x^2}{x^4}\, dA$ as the integral for the class to do.

(c) There are many possibilities. Here are a few.

$$\int \frac{3y^2+x^2}{x^4}\, e^{y/x^2}\, dA \text{ becomes } \int_{u=1}^{5}\int_{v=2}^{4} e^u\, dv\, du$$

$$\int \frac{3y^2+x^2}{x^4}\, \sin\frac{x^2+y^2}{x}\, dA \text{ becomes } \int_{u=1}^{5}\int_{v=2}^{4} \sin v\, dv\, du$$

$$\int \frac{3y^2+x^2}{x^4}\left(\frac{x^2+y^2}{x}\right)^6\left(\frac{x^2}{y}\right)^9 dA \text{ becomes } \int_{u=1}^{5}\int_{v=2}^{4} v^6 u^{-9}\, dv\, du$$

5. ($\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & v \\ -1 & u \end{vmatrix} = 2u^2 + v.$

At point u=0, v=−10, $\frac{\partial(x,y)}{\partial(u,v)} = -10.$

Area swept out = $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ du dv = 10 du dv.

6. $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$h_\rho\ h_\phi\ h_\theta = \rho^2 \sin\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho^2\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

I'll expand across row 3 (to take advantage of the zero entry).

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = -\rho\sin\phi\sin\theta\ \underbrace{\left[-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right]}_{-\rho\sin\theta}$$

$$= \rho^2\sin\phi\sin^2\theta + \rho^2\sin\phi\cos^2\theta$$

$$= \rho^2\sin\phi \quad \text{QED}$$

7.(a) *area* traced out (in the x, y plane) is $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ du dv

Question Why isn't the answer $dA = h_u\ h_v$ du dv.

Answer dA is $h_u\ h_v$ du dv if the u,v coordinate system is orthogonal but not in general.

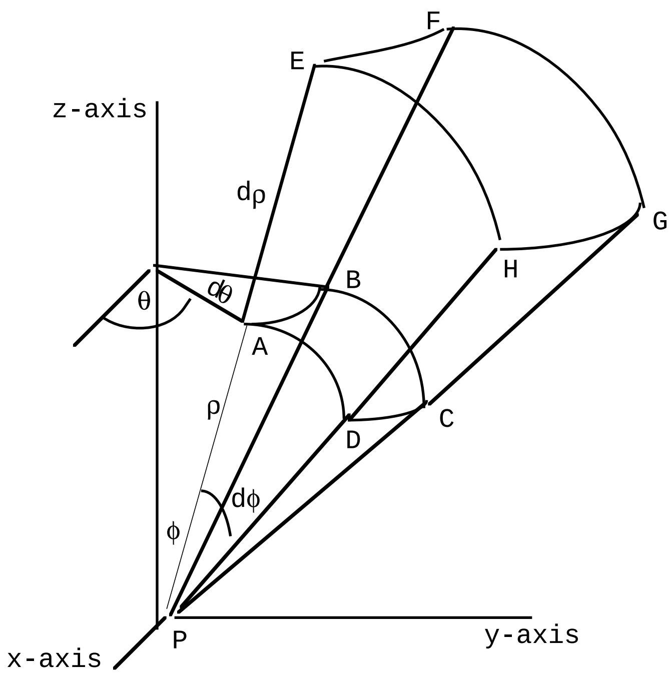
(b) *volume* traced out is $dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ du dv dw

SOLUTIONS summary of mag factors

1. The equations parametrize a surface in space. When u changes by du while v is fixed then an arclength element (little curve) is traced out on the surface.
$ds = \|vel_u\|\, du$

2. The equations determine a u,v coordinate system. (Or you can say they parametrize all of the x,y plane.)
If u changes by du while v is fixed then an arclength element is traced out (on a u-curve).
$ds = h_u\, du = \|vel_u\|\, du$
(doesn't matter here whether or not the coord system is orthog)

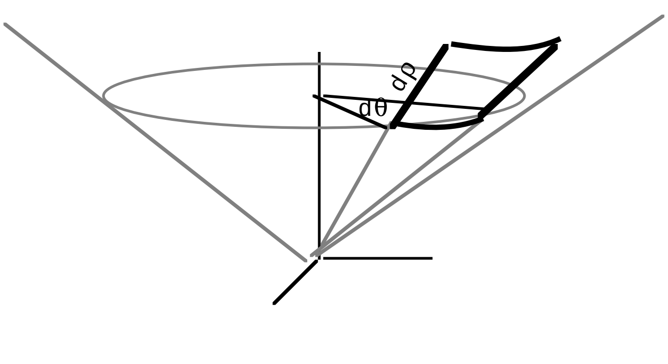
3. A volume element is traced out in 3-space.
$dV = h_\rho\, h_\phi\, h_\theta\, d\rho\, d\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$



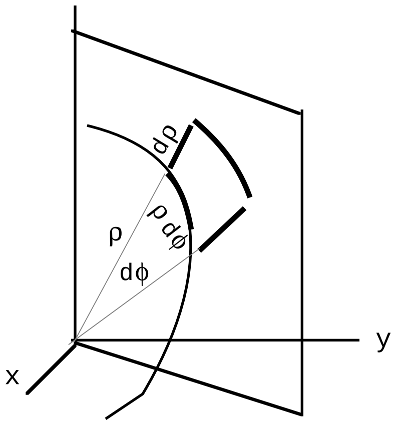
4. The equations define the usual spherical coordinate system.

A surface area element is traced out on a cone (where ϕ is constant)

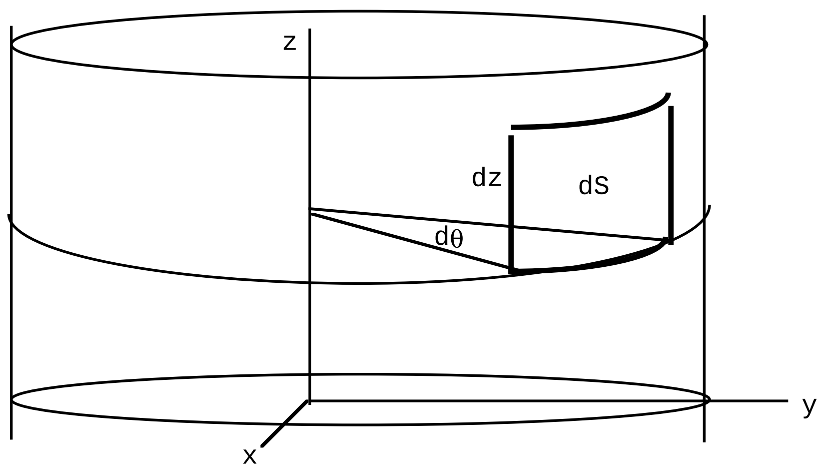
$dS = h_\rho\, h_\theta\, d\rho\, d\theta = \rho \sin\phi\, d\rho\, d\theta$



5. A surface area element is traced out on a half-plane (where θ is constant)
$dS = h_\rho\ h_\phi\ d\rho\ d\phi = \rho\ d\rho\ d\phi$



6. A surface area element is traced out on a cylinder (where r = constant).
$dS = h_\theta\ h_z\ d\theta\ dz = r\ d\theta\ dz$



REFERENCE PAGE TO BE GIVEN OUT WITH EXAMS

volume and surface area of a sphere with radius R

volume = $\frac{4}{3}\pi R^3$

surface area = $4\pi R^2$

integral tables

$$\int_{x=0}^{2\pi} \sin^2 x \, dx = \pi$$

$$\int_{x=0}^{2\pi} \cos^2 x \, dx = \pi$$

$$\int_{x=0}^{2\pi} \sin^4 x \, dx = \frac{3}{4}\pi$$

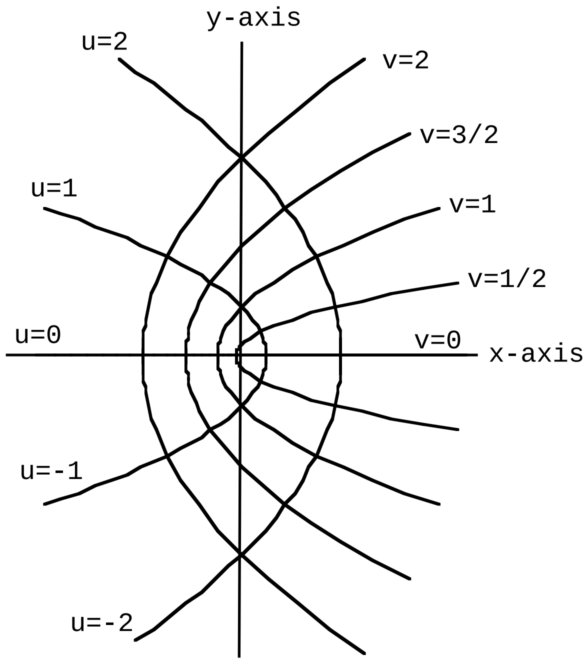
$$\int_{x=0}^{2\pi} \cos^4 x \, dx = \frac{3}{4}\pi$$

$$\int \sin^2 u \, du = \frac{1}{2}(u - \sin u \cos u) = \frac{1}{2}u - \frac{1}{4}\sin 2u$$

$$\int \cos^2 u \, du = \frac{1}{2}(u + \sin u \cos u) = \frac{1}{2}u + \frac{1}{4}\sin 2u$$

parabolic coord system

$$x = \frac{1}{2}(u^2 - v^2), \; y = uv, \quad v \ge 0$$



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REFERENCE PAGE CONTINUED

cylindrical coords

$x = r\cos\theta,\ y = r\sin\theta,\ z=z$

$e_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$e_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$h_r = 1$

$h_\theta = r$

$h_z = 1$

spherical coords

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$e_\rho = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$

$e_\phi = \cos\phi\cos\theta\,\vec{i} + \cos\phi\sin\theta\,\vec{j} - \sin\phi\,\vec{k}$

$e_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$h_\rho = 1$

$h_\phi = \rho$

$h_\theta = \rho\sin\phi$

vector derivatives in u,v,w coords

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}e_u + \frac{1}{h_v}\frac{\partial f}{\partial v}e_v + \frac{1}{h_w}\frac{\partial f}{\partial w}e_w$$

If $F(u,v,w) = p(u,v,w)e_u + q(u,v,w)\ e_v + r(u,v,w)\ e_w$ then

$$\operatorname{div} F = \frac{1}{h_u h_v h_w}\left(\frac{\partial(h_v h_w p)}{\partial u} + \frac{\partial(h_u h_w q)}{\partial v} + \frac{\partial(h_u h_v r)}{\partial w}\right)$$

$$\operatorname{curl} F = \begin{vmatrix} \frac{1}{h_v h_w}e_u & \frac{1}{h_u h_w}e_v & \frac{1}{h_u h_v}e_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ p h_u & q h_v & r h_w \end{vmatrix}$$