Squaring the Circle

A Case Study in the History of Mathematics

Part II

π

It is lost in the mists of pre-history who first realized that the ratio of the circumference of a circle to its diameter is a constant. All the ancient civilizations knew this fact. Today we call this ratio π and express this relationship by saying that for ***any*** circle, the circumference C and the diameter d satisfy: $C = \pi d$.

The use of the symbol "π" for this ratio is of relatively recent origin; the Greeks did not use the symbol.

"π was first used by the English mathematicians Oughtred (1647), Isaac Barrow (1664) and David Gregory (1697) to represent the circumference of a circle. The first use of "π" to represent the ratio of circumference to diameter was the English writer William Jones (1706). However, it did not come into common use until Euler adopted the symbol in 1737.[1]

Archimedes

Euclid XII.2 says that the ratio of the area of any circle to the square of its diameter is also a constant, but does not determine the value of this constant.

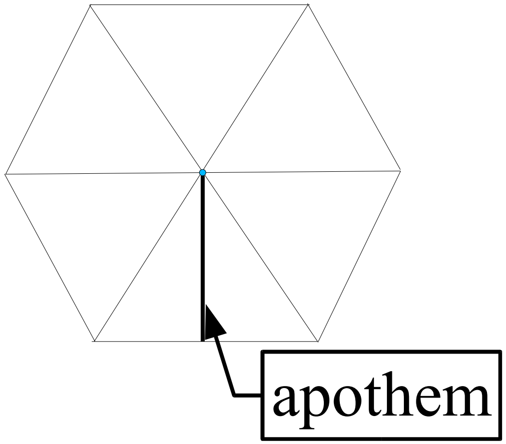
It was Archimedes (287 – 212 B.C.) who determined the constant in his remarkable treatise *Measurement of a Circle*. There are only three propositions in this short work (or at least, that is all of that work that has come down to us) and the second proposition is out of place – indicating that what we have is probably not the original version.[2]

We shall look at the first and third proposition.

Archimedes

A few preliminary ideas:

The area of a regular polygon is easily determined if you know that the area of a triangle = ½bh (b the length of a side, h the length of the altitude drawn to that side). In a regular polygon of n sides (all sides equal, all angles equal), draw the lines from the center to each of the vertices creating n congruent triangles.

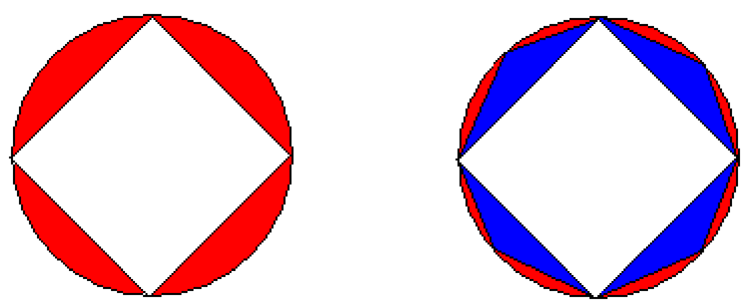


The area is thus n times the area of one of the triangles = n(½sa) where s is the length of a side and a the length of an ***apothem*** (line drawn from center perpendicular to a side). Since ns is the perimeter (Q) of the polygon we get :

$$A = \tfrac{1}{2}aQ.$$

Archimedes

Consider a regular polygon inscribed in a circle. Let K be the difference in the areas (area of circle – area of polygon). If you now double the number of sides of the polygon, the area you have added to the original polygon is more than ½K.



By repeating the procedure, you can make the area difference *as small as you like*, or in other words, for any positive number K, you can find a regular polygon (with enough sides) inscribed in a circle so that the area difference is less than K. This is Eudoxus' *method of exhaustion*. It also works for circumscribed polygons.

Archimedes

Proposition 1: *The area of any circle equals the area of a right triangle one of whose sides is a radius of the circle and the other has length equal to the circumference of the circle.*

Pf: Let C be the length of the circumference of our circle and r its radius. Denote by A the area of the circle. The area of the triangle is $T = \frac{1}{2}rC$.

Case I: Assume that $A > T$.

$K = A - T$ is a positive number. By the method of exhaustion, one can find an inscribed regular polygon so that

$$A - (\text{area of the inscribed polygon}) < A - T.$$

From this we see that $T <$ area of the inscribed polygon.

Since the polygon is inscribed, its perimeter Q is less than C and its apothem a is less than r, so

$$\text{area of the inscribed polygon} = \tfrac{1}{2}aQ < \tfrac{1}{2}rC = T,$$

which gives a contradiction.

Archimedes

Proposition 1: *The area of any circle equals the area of a right triangle one of whose sides is a radius of the circle and the other has length equal to the circumference of the circle.*

Pf (cont.):

Case II : A < T

Now T – A is a positive number and we can find a regular polygon circumscribed about the circle so that

(area of circumscribed polygon) – A < T – A.

or, more simply

area of circumscribed polygon < T.

Since the polygon is circumscribed, the length of its apothem **is** r and its perimeter is greater than C so,

area of circumscribed polygon = ½rQ > ½rC = T.

Another contradiction!

Archimedes

Proposition 1: *The area of any circle equals the area of a right triangle one of whose sides is a radius of the circle and the other has length equal to the circumference of the circle.*

Pf (cont.):

Since A > T and A < T have both led to contradictions, the only other possibility, A = T, must be true. *QED*

The proof method is called a double *reductio ad absurdum,* exactly the same method used in Euclid XII.2. The method of exhaustion is used in both proofs to obtain the contradictions. The only difference in method is that Euclid (or Eudoxus) reduces the second case to the first case, so uses inscribed polygons in both parts of his proof, while Archimedes switches to circumscribed polygons for the second case. [Dunham is a bit too harsh on Euclid's result while he dumps praise on Archimedes!][3]

Archimedes

Archimedes has proved that for any circle, $A = \frac{1}{2}rC$, and since we know that $C = \pi d$, we get $A = \frac{1}{2}r(\pi d) = \frac{1}{2}r(\pi 2r) = \pi r^2$ our familiar high school formula.

Even though Archimedes showed the equivalence of a circle with a rectilinear figure, easily converted to a square, this is **not** a solution of the quadrature problem. The proof is indirect, it does not give a means for constructing the triangle with straightedge and compass.

Before looking at Proposition 3, let's consider one of several methods known to the Greeks of using curves to perform a quadrature of the circle – however, the curves used in this way can not themselves be constructed with straightedge and compass!

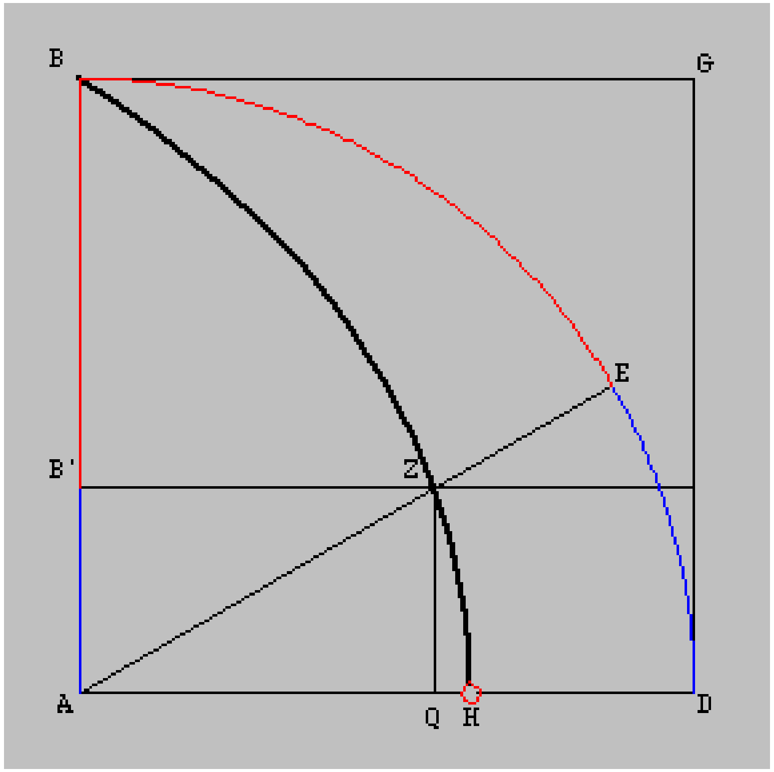
Quadratrix of Hippias

Hippias of Elis (ca. 425 B.C.) is credited with inventing a curve which he used to trisect angles. Dinostratus (ca. 350 B.C.) is usually given credit for using Hippias' curve to find a quadrature of the circle. He and his followers referred to the curve as the Quadratrix ("squarer") and this name stuck.

Quadratrix I[4]

Quadratrix II[5]

The defining property implies that :
arc BED = $(AB)^2/AH$
which expresses the arc in terms of straight line segments, thus permitting a quadrature of the circle.



Archimedes

In proposition 3 Archimedes turns his attention to the circumference of a circle. Again using inscribed and circumscribed regular polygons, their perimeters provide upper and lower bounds for the circumference of the circle. This gives him a means of calculating bounds for the number π.

Proposition 3: *The ratio of the circumference of any circle to its diameter is less than 3 1/7 (22/7) but greater than 3 10/71 (223/71).*

$$(3.140845... < \pi < 3.142857...)$$

What is remarkable about this result is not the underlying idea, but rather the skill of Archimedes in carrying out the computations. He started with inscribed and circumscribed hexagons, then doubled the size, and again, and again and yet again, ending with 96-sided polygons. At each step he calculates the perimeters. This involves approximating radicals which is where he shows his genius.

The Computation of π Early Phase

There has always been an interest in the precise value of π. As we have seen, the ancient Egyptians used $\pi = 3.1604938...$. Other ancient civilizations were not as precise, generally using $\pi = 3$. This can be seen in Babylonian clay tablets and in the Bible (I Kings: 7:23)

> *Then He made the molten sea (*circular*), ten cubits from brim to brim, while a line of 30 cubits measured it around.*

After Archimedes improvements were made by taking larger and larger polygons (except for the Romans – not very concerned with precision, they dropped back to the value 3 1/8)[6].

The Computation of π Early Phase[7]

ca. 150 A.D. - The first improvement over Archimedes values was given by Claudius Ptolemy of Alexandria in the *Almagest*, the most famous Greek work on astronomy. Ptolemy gives a value of $\pi = 377/120 = 3.1416$ ....

ca. 480 – The early Chinese worker in mechanics, Tsu Ch'ung-chih, gave the approximation $\pi = 355/113 = 3.1415929$ ... correct to 6 decimal places.

ca. 530 – The early Hindu mathematician Āryabhata gave $\pi = 62{,}832/20{,}000 = 3.1416$ ... as an approximation. It is not known how this was obtained, but it could have been calculated as the perimeter of a regular inscribed polygon of 384 sides.

The Computation of π Early Phase

ca. 1150 – The later Hindu mathematician Bhāskara gave several approximations. He gave 3927/1250 as an accurate value, 22/7 as an inaccurate value, and √10 for ordinary work.

1429 – Al-Kashi, astronomer royal to Ulugh Beg of Samarkand, computed π to sixteen decimal places using perimeters.

1579 – The eminent French mathematician François Viète found π correct to nine decimal places using polygons having 393,216 sides.

1593 – Adriaen van Roomen, more commonly known as Adrianus Romanus, of the Netherlands, found π correct to 15 places using polygons having 2^{30} sides.

The Computation of π   Early Phase

1610 – Ludolph van Ceulen of the Netherlands computed π to 35 decimal places using polygons having 2^{62} sides. He spent a large part of his life on this task, and his achievement was considered so extraordinary that his widow had the number engraved on his tombstone (now lost). To this day, the number is sometimes referred to as "the Ludophine number."

1621 – The Dutch physicist Willebrord Snell, best known for his discovery of the law of refraction, devised a trigonometric improvement of the classical method so that from each pair of bounds given by the classical method he was able to obtain considerably closer bounds. By his method, he was able to get van Ceulen's 35 places using a polygon with only 2^{30} sides.

The Computation of π Early Phase

1630 – Grienberger, using Snell's refinement, computed π to 39 decimal places. This was the last major attempt to compute π by the method of Archimedes.