Module MAU23203: Analysis in Several Real Variables
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Section 4: Open and Closed Sets in Euclidean Spaces

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4 Open and Closed Sets in Euclidean Spaces

4.1 Open Sets in Euclidean Spaces

Definition Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a positive real number η, the *open ball* $B(\mathbf{p}, \eta)$ in $\mathbb{R}^n$ of *radius* η centred on the point $\mathbf{p}$ consists of all points of $\mathbb{R}^n$ whose Euclidean distance from the point $\mathbf{p}$ is less than η.

We see therefore that

$$B(\mathbf{p}, \eta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \eta\}.$$

for all points $\mathbf{p}$ of $\mathbb{R}^n$ and positive real numbers η.

The *open ball* $B(\mathbf{p}, \eta)$ of radius η centred on a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius η centred on $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = \eta\}.$$

Definition A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point of V, there exists an open ball of positive radius, centred on that point, which is wholly contained within the set V.

By convention the empty set $\emptyset$ is also considered to be an open set (on the grounds that there does not exist any point of the empty set that is not the centre of some open ball contained in the empty set).

Thus a subset V of $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$ if and only if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset V of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ centred on the origin. This open ball is an open set. Indeed let $\mathbf{q}$ be a point of V. Then $|\mathbf{q}| < 3$. Let $\delta = 3 - |\mathbf{q}|$. Then $\delta > 0$. Moreover if $\mathbf{x}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{x} - \mathbf{q}| < \delta$ then

$$|\mathbf{x}| = |\mathbf{q} + (\mathbf{x} - \mathbf{q})| \leq |\mathbf{q}| + |\mathbf{x} - \mathbf{q}| < |\mathbf{q}| + \delta = 3,$$

and therefore $\mathbf{x} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius centred on any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 4.1).

4.2 Open Sets in Subsets of Euclidean Spaces

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a positive real number η, the *open ball* $B_X(\mathbf{p}, \eta)$ in X of *radius* η centred on the point $\mathbf{p}$ consists of all points of the set X whose Euclidean distance from the point $\mathbf{p}$ is less than η.

We see therefore that

$$B_X(\mathbf{p}, \eta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \eta\}.$$

for all points $\mathbf{p}$ of X and positive real numbers η.

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A subset V of X is said to be *open in* X if, given any point of V, there exists an open ball in X of positive radius, centred on that point, which is wholly contained within the set V.

By convention the empty set $\emptyset$ is also considered to be open in the given set X (on the grounds that there does not exist any point of the empty set that is not the centre of some open ball contained in the empty set).

Thus given any subset X of $\mathbb{R}^n$, and given any subset V of X, the set V is said to be open in X if and only if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where

$$B_X(\mathbf{p}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example Let V be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $V \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any positive real number δ , the open ball (in $\mathbb{R}^3$) of radius δ centred on $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ centred on the point $(0, 0, 1)$ is not a subset of N.

Lemma 4.1 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number η, the open ball $B_X(\mathbf{p}, \eta)$ in X of radius η centred on $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{q}$ be an element of $B_X(\mathbf{p}, \eta)$. We must show that there exists some positive real number δ such that $B_X(\mathbf{q}, \delta) \subset B_X(\mathbf{p}, \eta)$. Let $\delta = \eta - |\mathbf{q} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{q} - \mathbf{p}| < \eta$. Moreover if $\mathbf{x} \in B_X(\mathbf{q}, \delta)$ then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{q}| + |\mathbf{q} - \mathbf{p}| < \delta + |\mathbf{q} - \mathbf{p}| = \eta,$$

by the Triangle Inequality, and hence $\mathbf{x} \in B_X(\mathbf{p}, \eta)$. Thus $B_X(\mathbf{q}, \delta) \subset B_X(\mathbf{p}, \eta)$. This shows that $B_X(\mathbf{p}, \eta)$ is an open set, as required. ∎

Lemma 4.2 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number η, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > \eta\}$ is an open set in X.*

Proof Let $\mathbf{q}$ be a point of X satisfying $|\mathbf{q} - \mathbf{p}| > \eta$, and let $\mathbf{x}$ be any point of X satisfying $|\mathbf{x} - \mathbf{q}| < \delta$, where $\delta = |\mathbf{q} - \mathbf{p}| - \eta$. Then

$$|\mathbf{q} - \mathbf{p}| \leq |\mathbf{q} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{x} - \mathbf{p}| \geq |\mathbf{q} - \mathbf{p}| - |\mathbf{x} - \mathbf{q}| > |\mathbf{q} - \mathbf{p}| - \delta = \eta.$$

Thus $B_X(\mathbf{q}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 4.3 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{C}$ be any collection of open sets in X, and let W denote the union of all the open sets belonging to $\mathcal{C}$. We must show that W is itself open in X. Let $\mathbf{p} \in W$. Then $\mathbf{p} \in V$ for some set V belonging to the collection $\mathcal{C}$. It follows that there exists some positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$. But $V \subset W$, and thus $B_X(\mathbf{p}, \delta) \subset W$. This shows that W is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{p} \in V$. Now $\mathbf{p} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{p}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{p}, \delta) \subset B_X(\mathbf{p}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{p}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 centred on the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 centred on the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ centred on the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ centred on the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 4.4 *Let X be a subset of $\mathbb{R}^n$, and let W be a subset of X. Then W is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $W = V \cap X$.*

Proof First suppose that $W = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{p} \in W$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset W.$$

This shows that W is open in X.

Conversely suppose that the subset W of X is open in X. For each point $\mathbf{p}$ of W there exists some positive real number $\delta_{\mathbf{p}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_{\mathbf{p}}\} \subset W.$$

For each $\mathbf{p} \in W$, let $B(\mathbf{p}, \delta_{\mathbf{p}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{p}}$ centred on the point $\mathbf{p}$, so that

$$B(\mathbf{p}, \delta_{\mathbf{p}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta_{\mathbf{p}}\}$$

for all $\mathbf{p} \in W$, and let V be the union of all the open balls $B(\mathbf{p}, \delta_{\mathbf{p}})$ as $\mathbf{p}$ ranges over all the points of W. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 4.1), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 4.3). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{p}, \delta_{\mathbf{p}}) \cap X \subset W$. for all $\mathbf{p} \in W$. Also every point of V belongs to $B(\mathbf{p}, \delta_{\mathbf{p}})$ for at least one point $\mathbf{p}$ of W. It follows that $V \cap X \subset W$. But $\mathbf{p} \in B(\mathbf{p}, \delta_{\mathbf{p}})$ and $B(\mathbf{p}, \delta_{\mathbf{p}}) \subset V$ for all $\mathbf{p} \in W$, and therefore $W \subset V$, and thus $W \subset V \cap X$. It follows that $W = V \cap X$, as required. ∎

4.3 Convergence of Sequences and Open Sets

Lemma 4.5 *An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set V which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in V$ for all positive integers j satisfying $j \geq N$.*

Proof Suppose that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ has the property that, given any open set V which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in V$ whenever $j \geq N$. Let some positive real number ε be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε centred on the point $\mathbf{p}$ is an open set by Lemma 4.1. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the infinite sequence converges to the point $\mathbf{p}$.

Conversely, suppose that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of $\mathbb{R}^n$ converges to the point $\mathbf{p}$. Let V be an open set to which that point $\mathbf{p}$ belongs. Then there exists some positive real number ε such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε centred on $\mathbf{p}$ is a subset of V. All points $\mathbf{x}$ of $\mathbb{R}^n$ that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. then belong to the open set V. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in V$ whenever $j \geq N$, as required. ∎

4.4 Closed Sets in Euclidean Spaces

Definition Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X.

(Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| \leq \eta\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| \geq \eta\}$ are closed in X for each non-negative real numbers η. In particular, the set $\{\mathbf{p}\}$ consisting of the single point $\mathbf{p}$ is a closed set in X. (These results follow immediately using Lemma 4.1 and Lemma 4.2 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned}
\mathbf{x} \in X \setminus \bigcup_{S\in\mathcal{A}} S &\iff \mathbf{x} \notin \bigcup_{S\in\mathcal{A}} S \\
&\iff \text{for all } S \in \mathcal{A}, \mathbf{x} \notin S \\
&\iff \text{for all } S \in \mathcal{A}, \mathbf{x} \in X \setminus S \\
&\iff \mathbf{x} \in \bigcap_{S\in\mathcal{A}} (X \setminus S),
\end{aligned}$$

and therefore

$$X \setminus \bigcup_{S\in\mathcal{A}} S = \bigcap_{S\in\mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned}
\mathbf{x} \in X \setminus \bigcap_{S\in\mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S\in\mathcal{A}} S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\
&\iff \mathbf{x} \in \bigcup_{S\in\mathcal{A}} (X \setminus S),
\end{aligned}$$

and therefore

$$X \setminus \bigcap_{S\in\mathcal{A}} S = \bigcup_{S\in\mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 4.3.

Proposition 4.6 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Proof The empty set $\emptyset$ is the complement in X of the whole set X. The set X is open in itself. It follows that the empty set $\emptyset$ is closed in X.

The whole set X is the complement in X of the empty set. The empty set is open in X. It follows that the whole set X is closed in itself.

Next let $\mathcal{C}$ be a collection of subsets of X that are closed in X, and let G be the intersection of all the sets that are members of the collection $\mathcal{C}$. Now the complement in X of the set G, being the complement of the intersection of all the members of the collection $\mathcal{C}$ is the union of the complements of the members of this collection $\mathcal{C}$. Now the complement of each member of the collection $\mathcal{C}$ is open in X. Consequently the union of the complements of the members of the collection must also be open in X. Thus the complement of the set G is open in X, and therefore the set G itself is closed in X.

Now suppose that the collection $\mathcal{C}$ is a finite collection of subsets of X that are closed in X, and let H be the union of all the sets that are members of the finite collection $\mathcal{C}$. Now the complement in X of the set H, being the complement of the union of all the members of the finite collection $\mathcal{C}$ is the intersection of the complements of the members of this finite collection $\mathcal{C}$. Now the complement of each member of the finite collection $\mathcal{C}$ is open in X. Consequently the intersection of the complements of the members of the finite collection must also be open in X. Thus the complement of the set H is open in X, and therefore the set H itself is closed in X. This completes the proof. ∎

Lemma 4.7 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points of F which converges to some point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 4.5 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

4.5 Limit Points of Subsets of Euclidean Spaces

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^n$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any positive real number δ, there exists some point $\mathbf{x}$ of X for which $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Lemma 4.8 *Let X be a subset of n-dimensional Euclidean space $\mathbf{R}^n$. A point $\mathbf{p}$ of $\mathbb{R}^n$ is a limit point of the set X if and only if, given any positive* *real number δ, the set*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set.

Proof Suppose that, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set. Then, for each positive real number δ, the set thus determined by δ must consist of more than just the single point $\mathbf{p}$, and therefore there exists $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus $\mathbf{p}$ is a limit point of the set X.

Now let $\mathbf{p}$ be an arbitrary point of $\mathbb{R}^n$. Suppose that there exists some positive real number η for which the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \eta\}$$

is finite. If this set does not contain any points of X distinct from the point $\mathbf{p}$ then $\mathbf{p}$ is not a limit point of the set X. Otherwise let δ be the minimum value of $|\mathbf{x} - \mathbf{p}|$ as $\mathbf{x}$ ranges over all points of the finite set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \eta\}$$

that are distinct from $\mathbf{p}$. Then $\delta > 0$, and $|\mathbf{x} - \mathbf{p}| \geq \delta$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Thus the point $\mathbf{p}$ is not a limit point of the set X. The result follows. ∎

Lemma 4.9 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$ and let $\mathbf{p} \in \mathbb{R}^n$. Then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.*

Proof Suppose that $\mathbf{p}$ is a limit point of X. Then, for each positive integer j, there exists a point $\mathbf{x}_j$ of X for which $0 < |\mathbf{x}_j - \mathbf{p}| < 1/j$. The points $\mathbf{x}_j$ satisfying this condition then constitute an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, that converge to the point $\mathbf{p}$.

Conversely suppose that $\mathbf{p}$ is some point of $\mathbb{R}^n$ that is the limit of some infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X that are all distinct from the point $\mathbf{p}$. Let some positive number δ be given. The definition of convergence ensures that there exists a positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. Moreover $|\mathbf{x}_j - \mathbf{p}| > 0$ for all positive integers j. Thus $0 < |\mathbf{x}_j - \mathbf{p}| < \delta$ when the positive integer j is sufficiently large. Thus the point $\mathbf{p}$ is a limit point of the set X, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{p}$ of X is said to be an *isolated point* of X if it is not a limit point of X.

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in X$. It follows immediately from the definition of isolated points that the point $\mathbf{p}$ is an isolated point of the set X if and only if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} = \{\mathbf{p}\}.$$

Lemma 4.10 *A subset F of n-dimensional Euclidean space $\mathbb{R}^n$ is closed in $\mathbb{R}^n$ if and only if it contains its limit points.*

Proof Let F be a closed set in $\mathbb{R}^n$ and let $\mathbf{p}$ be a limit point of F. It follows from Lemma 4.9 that there exists an infinite sequence of points of F that converges to the point $\mathbf{p}$. It then follows from Lemma 4.7 that $\mathbf{p} \in F$. Thus if the set F is closed then it contains its limit points.

Conversely let F be a subset of $\mathbb{R}^n$ that contains its limit points. Let $\mathbf{p} \in \mathbb{R}^n \setminus F$. Then $\mathbf{p}$ is not a limit point of F. It follows from the definition of limit points that there exists some positive real number δ for which

$$\{\mathbf{x} \in F : 0 < |\mathbf{x} - \mathbf{p}| < \delta\} = \emptyset.$$

It then follows from this that the open ball in $\mathbb{R}^n$ of radius δ centred on the point $\mathbf{p}$ is contained in the complement of F. We conclude therefore that the complement of F in $\mathbb{R}^n$ is open in $\mathbb{R}^n$, and thus F is closed in $\mathbb{R}^n$, as required. ∎