Date 3 . 10 . 19

Calculus I

I) Number, limits & functions

I.1) Reminder

$\mathbb{N} = \{0, 1, 2 \dots\}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q} = \{\frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0\}$

π, $0.999\ldots = 1$

$\mathbb{R}$ = all numbers, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

Def: for $a, b \in \mathbb{R}$,
$(a,b) =]a,b[$
open interval $= \{x \in \mathbb{R}$ with $a < x < b\}$
$[a,b]$
closed interval $= \{x \in \mathbb{R} \mid a \leq x \leq b\}$

- Powers:

Let $x \in \mathbb{R}$, $x \geq 0$

$x^n = x \cdot x \cdot x \dots x$ for $n \in \mathbb{N}$ with $x^0 := 1$

$x^{\frac{1}{n}} = y \geq 0$ with
$y^n = x$, $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$

$x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}} = (x^{\frac{1}{m}})^n$

$x^{-\frac{n}{m}} = 1/x^{\frac{n}{m}}$

$\boxed{\text{What is } \sqrt{2}^{\pi} \text{ ?}}$

I.2) Limit

Let X be a subset of $\mathbb{R}$

Def: $(a_n)_{n \in \mathbb{N}}$ is called a "sequence" in X if $a_i \in X$ for any i.

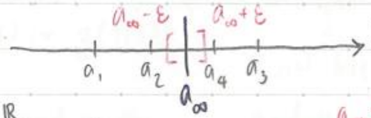
Question: What is the limit of $(a_n)_n \in \mathbb{N}$ when n goes to ∞ ?

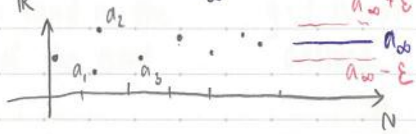
We write $\lim_{n \to \infty} a_n$ if it exists.

Ex. $a_n = \frac{1}{1+n}$, the limit

$\lim_{n \to \infty} a_n = 0$

Def: A sequence $(a_n)_{n \in \mathbb{N}} \subset X$ is convergent in X.
if $\exists a_\infty \in X$ such that for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $\underbrace{|a_n - a_\infty| \leq \varepsilon}_{a_\infty - \varepsilon \leq a_n \leq a_\infty + \varepsilon}$ for any $n \geq N$.





⚠ There are plenty of sequences which do not converge:

ex: • $a_n = (-1)^n$

• $a_n = n$

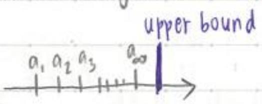
Q • $a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$ → $a_1 = 1$, $a_2 = 1 - \frac{1}{3}$, $a_3 = 1 - \frac{1}{3} + \frac{1}{5}$, $a_4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

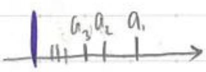
ex: $\lim_{n \to \infty} a_n = \frac{\pi}{4}$???

This sequence in $\mathbb{Q}$ does not converge in $\mathbb{Q}$, but it does converge in $\mathbb{R}$, and the limit is $\pi/4$.

Properties: Suppose $(a_n)_{n \in \mathbb{N}}$ converges to a_∞ and $(b_n)_{n \in \mathbb{N}}$ converges to b_∞. Then $(a_n + b_n)_{n \in \mathbb{N}}$ converges to $a_\infty + b_\infty$, and $(a_n b_n)_{n \in \mathbb{N}}$ converges to $a_\infty b_\infty$.

$\boxed{\text{Thm}}$ Any increasing sequence which is upper bounded is convergent, and any decreasing sequence which is lower bounded is convergent. in $\mathbb{R}$!!!





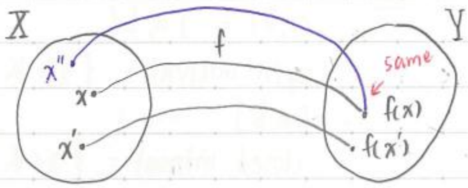
I.3) Function

Recall that a set X is a collection of object

$$X = \{x_1, x_2, \dots\}$$

Def A function (or a map) f from a set X to set Y is a rule which associates one element Y to every element of X.

It means, to any $x \in X$ $\exists!$ $y = f(x) \in Y$ — There exists unique



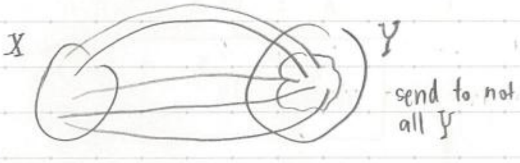
X is called the "domain" of f, denoted by $\mathrm{Dom}(f)$.

Y ——— "codomain" of f (or tangent space)

$f(X) = \mathrm{Ran}(f) = \mathrm{Im}(f)$ range image (don't use this one)

$= \{ y \in Y \mid \exists x \in X \text{ with } y = f(x) \}$

is called the range of f.



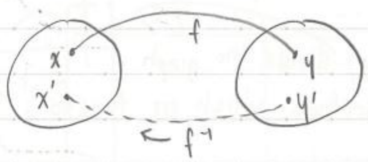
Notation: We write $X \ni x \longmapsto f(x) \in Y$

or $f : X \to Y$

$x \to f(x)$

⚠ $f(x)$ is not a function. It is an element of Y.
We can write f or $f(\cdot)$ for the function.

Def Let $f: X \to Y$
- f is injective if $\forall x_1, x_2 \in X$ but $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$
  all x maps to different y
- f is surjective if $Ran(f) = Y$
  $\Leftrightarrow \forall y \in Y, \exists x \in X$ with $f(x) = y$.
  reaches everybody
- f is bijective if f is both injective and surjective.

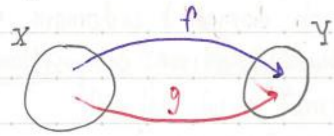


Remark Whenever $f: X \to Y$ is bijective, we can define $f^{-1}: Y \to X$ with
$f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

Ex./
1) $f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$
   ⇨ not inj ex. $x = +2, -2$
   ⇨ not surj ex. $y = -3$ doesn't reach
2) $f: \mathbb{R}_+ \ni x \mapsto x^2 \in \mathbb{R}$
   ⇨ inj, not surj.
3) $f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}^+$
   ⇨ surj, not inj
4) $f: \mathbb{R}^+ \ni x \mapsto x^2 \in \mathbb{R}^+$
   ⇨ bijective

Remark: A function is always surjective in its range.
$$Ran(f) = Ran(f).$$

I.4) Operation with function



define first

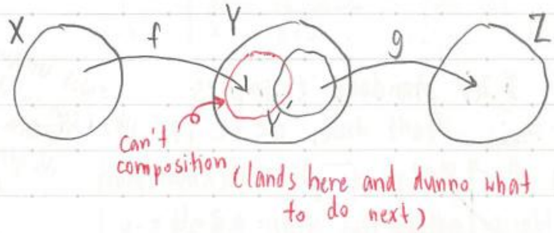
Question: can you define $f+g$? (no)

Addition: Let X be a set, but let $Y = \mathbb{R}$
If $f, g: X \to \mathbb{R}$, then we can define $f+g$ by $(f+g)(x) = \underbrace{f(x)}_{\in \mathbb{R}} + \underbrace{g(x)}_{\in \mathbb{R}}$

not always trivial

Multiplication: We can also define fg by
$(fg)(x) = f(x)\, g(x)$

Composition: If $f: X \mapsto Y$ and $g: Y' \mapsto Z$ with $Y' \subset Y$



Then we can define $g \circ f$ if $\underbrace{Ran(f) \subset Y'}$
by $(g \circ f)(x) := g(f(x))$

if not true, then $g \circ f$ is not well defined.

→ have to land in the domain of the next function.

No.

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Reminder :

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- limits : ... for any ε, find $N(\varepsilon)$
- functions : injective, surjective, bijective

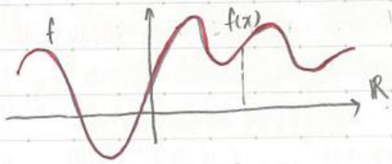
  △ depends on domain & codomain
- addition, multiplication, composition

I.) Graphs & Curves

I.1) Graphs

Let $f: \mathbb{R} \to \mathbb{R}$, and condiser the "picture".

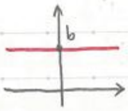
$\mathbb{R}^2$ The plane (2-dimensional)



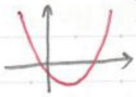
The graph of f in
$\{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y = f(x)\}$
For any x, there is only 1 y.

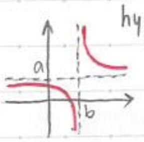
I.2) Standard examples.

1) 
graph of a function
$f(x) = ax + b$
↑ slope

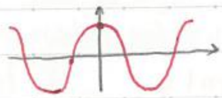
2) 

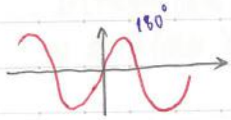
graph of $f(x) = b$.

3) 
parabola
graph of the function
$f(x) = ax^2 + bx + c$

4) 

hyperbola
graph of a function
$f(x) = a + \frac{c}{x-b}$

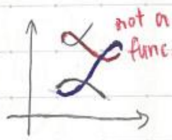
5) 
$f(x) = \cos(x)$
$f(-x) = f(x)$
even function

6) 

$f(x) = \sin(x)$
$f(-x) = -f(x)$
odd function

I.3) Curves

(1 x ⇒ many y)



to study, divide the graph to small sections, which are functions

Many objects like that are not the graph of a function.

Examples:

① $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

② $\{(x,y) \in \mathbb{R}^2 \mid (\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, with $a, b > 0\}$

③ $\{(x,y) \in \mathbb{R}^2 \mid x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}, t \in \mathbb{R}\}$

"Not unique" (can substitute t with 2t and the graph is same)
↑
has an expression in t

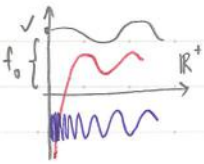
parametric curve with parameter t


" The folium of Descartes"

These are not the graph of any func. from $\mathbb{R}$ to $\mathbb{R}$, but they are curves or parametric curves.

△ A curve can be locally studied as a function.

III) The Derivative

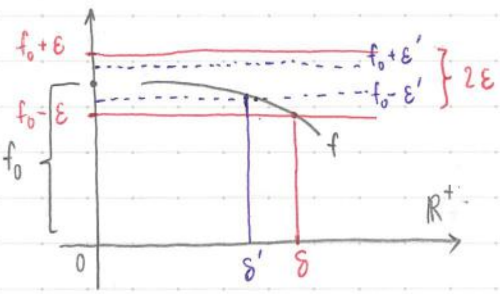


III.1) Limits and Continuity

<u>Def</u>: Let $f: (0,\infty) \to \mathbb{R}$

We say that f has a limit at 0 if there exists a value $f_0 \in \mathbb{R}$ such that for any $\varepsilon > 0$ (small but arbitrary), there exists $\delta > 0$ with

$$\underbrace{|f(x) - f_0| \leq \varepsilon}_{f_0 - \varepsilon \leq f(x) \leq f_0 + \varepsilon} \quad \text{for any } x \in (0, \delta).$$



If f has a limit at 0, we write

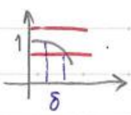
$$\lim_{x \searrow 0} f(x) = f_0$$
$$\|$$
$$\lim_{x \to 0^+} f(x) = f_0$$

[our job is to find δ now.]

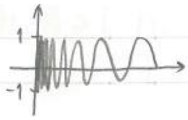
Examples

① $f: (0,\infty) \to \mathbb{R}$, $f(x) = x^n$ has a limit

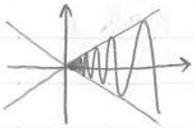
② $f: (0,\infty) \to \mathbb{R}$, $f(x) = \cos(x)$ has a limit.



③ $f(x) = x \sin\left(\frac{1}{x}\right)$



$f(x) = \sin\left(\frac{1}{x}\right)$ has no limit at 0.



$f(x) = x \sin\left(\frac{1}{x}\right)$ has the limit 0 at 0.

<u>Remark</u> We can also consider limit from the left for a function

$$f: (-\infty, 0) \to \mathbb{R}$$

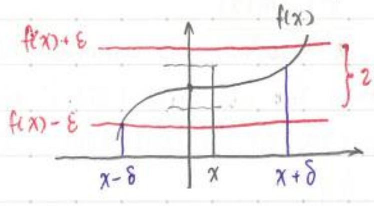
If it has a limit, one writes

$$\lim_{x \nearrow 0} f(x) = f_0 = \lim_{x \to 0^-} f(x)$$

<u>Def</u> $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

for any — there exists

$$\underbrace{|f(x+h) - f(x)| \leq \varepsilon}_{f(x) - \varepsilon \leq f(x+h) \leq f(x) + \varepsilon} \quad \text{for any } h \in [-\delta, \delta]$$



$\iff$ $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$|f(y) - f(x)| \leq \varepsilon$ for $y \in \mathbb{R}$ with
$|y - x| \leq \delta$ (y is just another variable)

if and only if

<u>Remark</u> f is continuous at x iff

$$\lim_{y \searrow x} f(y) = \lim_{y \nearrow x} f(x)$$

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f is continuous on a set $I \subset \mathbb{R}$ if f is continuous at any $x \in I$.

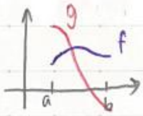
For fun: What means $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{Z}$?

(a,b), $[a,b]$, etc.

The set of all continuous function on I (interval) is denoted by $C(I)$.

These are all continuous.

Properties Consider $f, g \in C(I)$ and let $\lambda \in \mathbb{R}$



1) $\lambda f + g \in C(I)$
2) $fg \in C(I)$
3) f/g is continuous if $g(x) \neq 0 \quad \forall x \in I$

Remark How to know that

$$f(x) = \frac{x^3 + \sin(x)}{x^2 + \cos^2(x)}$$ is continuous at 0.

Show that both the numerator (aka. f) and the denominator (aka. g) are continuous. (from the properties: f/g)

④ If we consider $f(x) = \cos(x^2+3)$ we can not deduce the continuity from ① - ③.

More generally,

If $f \in C(I)$ and $g \in C(f(I))$, then, (Range of f defined on I)

the function $g \circ f$ defined by

$$I \ni x \longrightarrow g(f(x))$$

is continuous.

Reminder

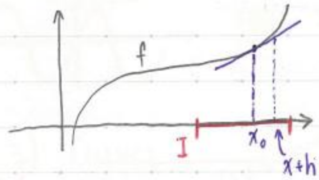
- graphs and curves
- limits at x_0

$\lim_{\substack{x \searrow x_0 \\ x \to x^+}} f(x)$, $\lim_{\substack{x \nearrow x_0 \\ x \to x^-}} f(x)$, if they are the same; $\lim_{x \to x_0} f(x)$

- continuity at one point

continuity on an interval $\rightsquigarrow C(I)$

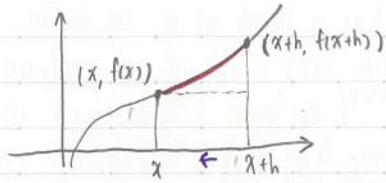
III. 2) Slope of a function at a point



Let $f: I \longrightarrow \mathbb{R}$ and let $x, x+h \in I$

an open interval

$\Rightarrow (x, f(x))$ and $(x+h, f(x+h))$ belong to the graph of f.



The slope of f at x is given by

$$\frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}$$ as

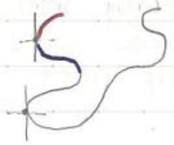
h goes to 0 if this limit exists.

Def: Let $f : I \to \mathbb{R}$ an $x \in I$. The slope at x is given $\lim_{h \to 0} = \frac{f(x+h) - f(x)}{h}$ if this limit exists.

Remarks.

- This slope has a geometric interpretation: it corresponds to the slope of the tangent of f at x

- We can define the slope at almost every point of a curve, by working locally.



(the slope is not defined when it is equal to "$\pm\infty$").

* critical point

III.3) The derivative of a function

Def: From $f:(a,b)\to\mathbb{R}$, and for $x\in(a,b)$ the limit $\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$ is called the "derivative of f at x", if it exists.

We write $f'(x)$ or $\frac{df}{dx}(x)$ for the derivative (which is a number).

If $f'(x)$ exists for all $x\in(a,b)$, we call "the derivative of f" the function.

$$f':(a,b)\to\mathbb{R}$$
$$x\mapsto f'(x)$$

If f' exists, we say the f is "differentiable".

Examples: consider

① $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$

One has $\frac{f(x+h)-f(x)}{h}$

$$=\frac{(x+h)^2-x^2}{h}=\frac{2xh+h^2}{h}=2x+h\xrightarrow{h\to 0}2x$$

Since $\frac{f(x+h)-f(x)}{h}$ has a limit for $h\to a$ and for any $x\in\mathbb{R}$, f is differentiable with derivative f' given by $f'(x)=2x$

② Consider $f:\mathbb{R}\to\mathbb{R}$, given $f(x)=|x|$.

One has $\frac{f(x+h)-f(x)}{h}=\frac{|x+h|-|x|}{h}$

$$=\begin{cases}\frac{x+h-x}{h} & \text{if } x>0 \text{ and } |h| \text{ small enough}\\ \frac{-(x+h)-(-x)}{h} & \text{if } x<0 \text{ and } |h| \text{ small enough}\end{cases}$$

$$=\begin{cases}1 & \dots\\ -1 & \dots\end{cases}$$

$\Rightarrow$ f is differentiable at x for any $x\neq 0$

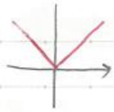
⚠ For $x=0$, one has

$$\frac{f(0+h)-f(0)}{h}=\frac{|h|-0}{h}$$

$$=\begin{cases}1 & \text{if } h>0\\ -1 & \text{if } h<0\end{cases}$$

$\Rightarrow$ Since $\lim_{x\to 0^-}\frac{f(0+h)-f(h)}{h}=1$

$\neq -1 = \lim_{h \to 0^-} \frac{f(0+h) - (0)}{h}$



then f has no derivative at 0.

Remark

$\lim_{h \to 0} \ldots$ means ε, δ !

More precisely, f is differentiable at x if $\exists f'(x) \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \varepsilon \qquad |h| \leq \delta$$

for any $h \in [-\delta, \delta]$

Def : If $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b) and f' is continuous, we write $f \in C^1((a,b))$

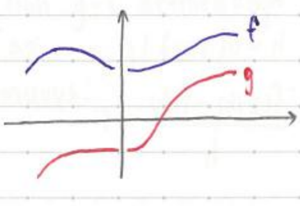
$\Leftrightarrow f, f'$ continuous

III.4) Back to limits.

Suppose $f, g : \mathbb{R}^* \to \mathbb{R}$ are continuous, and assume that

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} g(x) = g(0)$$



Then we know that

- $\lim_{x \to 0} (f+g)(x) = f(0) + g(0)$
- $\lim_{x \to 0} (fg)(x) = f(0)\, g(0)$
- $\lim_{x \to 0} \frac{f}{g}(x) = \frac{f(0)}{g(0)}$

What happen if $f(0) = 0 = g(0)$, what is $\frac{f(0)}{g(0)}$? Or more precisely, what $\lim_{x \to 0} \frac{f(x)}{g(x)} = ?$

Examples

① $f(x) = x^2$, $g(x) = x^2 + x$

then $\frac{f(x)}{g(x)} = \frac{x^2}{x^2 + x} = \frac{x}{x+1} \xrightarrow{x \to 0} 0$.

② $f(x) = x^2$, $g(x) = x^3$

$\frac{f(x)}{g(x)} = \frac{x^2}{x^3} = \frac{1}{x}$ has no limit as $x \to 0$.

③ $f(x) = x^2$, $g(x) = x^2 + x^3$,

then $\frac{f(x)}{g(x)} = \frac{x^2}{x^2 + x^3} = \frac{1}{1+x} \xrightarrow{x \to 0} 1$

L' Hospital's Rule

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable on (a,b).

Let $x_0 \in (a,b)$, and assume $f(x_0) = 0 = g(x_0)$

Assume also that $g'(x) \neq 0$ for any $x \in (a,b) \setminus \{x_0\}$.

Then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ if this limit exists.

Proof. (we assume that $g'(x_0) \neq 0$

skip this assumption later on)

One has $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)}$ we can subtract because both are 0.

$$= \lim_{x \to x_0} \frac{\frac{f(x) - f(x_0)}{x - x_0}}{\frac{g(x) - g(x_0)}{x - x_0}}$$

$= \frac{f(x_0+h) - f(x_0)}{h}$ if $h = x - x_0$

$$= \frac{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

$$= \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \#$$

III.5) Properties of differentiation

let $f, g : I \to \mathbb{R}$ be differentiable on I, and let $\lambda \in \mathbb{R}$. Then

1) $(\lambda f)'(x) = \lambda f'(x)$
2) $(f+g)'(x) = f'(x) + g'(x)$
3) $(fg)'(x) = f'(x) g(x) + f(x) g'(x)$
4) $\left(\frac{f}{g}\right)' = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}$ if $g(x) \neq 0$.

$\forall x \in I$

Proof.

1)
$$\frac{(\lambda f)(x+h) - (\lambda f)(x)}{h} = \frac{\lambda f(x+h) - \lambda f(x)}{h} = \lambda \frac{f(x+h) - f(x)}{h} \xrightarrow{h \to 0} \lambda f'(x)$$

2)
$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \xrightarrow{h \to 0} f'(x) + g'(x)$$

3)
$$\frac{(fg)(x+h) - (fg)(x)}{h} = \frac{f(x+h) + g(x+h) - f(x) g(x)}{h}$$

$$= \frac{f(x+h)\left(g(x+h) - g(x)\right) + f(x+h) g(x) - f(x) g(x)}{h}$$

$$= f(x+h) \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x)$$

$$\xrightarrow{h = 0} f(x) g'(x) + f'(x) g(x)$$

A differentiable func. is continuous.

No.

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4) $$\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \frac{1}{h} \frac{f(x+h)\,g(x) - g(x+h)\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{\frac{f(x+h)-f(x)}{h}\,g(x) - \frac{g(x+h)-g(x)}{h}\,f(x)}{g(x+h)\,g(x)}$$

$$\xrightarrow{h\to 0} \frac{f'(x)\,g(x) - g'(x)\,f(x)}{g(x)^2}$$

#

Something we are about to prove:

small

- Lemma
- Proposition
- Theorem
- Corollary : almost obvious

big

get from the smaller ones

Reminder:

- Derivatives
- L'Hospital's rule
- 4 properties for differentiation

Applications

1.) Consider $f: \mathbb{R} \to \mathbb{R}$ differentiable and $f(x) > 0$ $\forall x \in \mathbb{R}$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

2.) Consider $x \mapsto x^{-n} := \frac{1}{x^n}$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{n x^{n-1}}{x^{2n}} = -n \frac{1}{x^{n+1}}$$
$$= -n\, x^{-n-1}$$

$\boxed{\text{Chain Rule}}$:

Let $f, g: \mathbb{R} \to \mathbb{R}$, differentiable and consider $(f \circ g)(x) = f(g(x))$

$$(f \circ g)'(x) = f(g(x))' = f'(g(x))\, g'(x)$$

Lemma If $f, g: \mathbb{R} \to \mathbb{R}$ are differentiable, then $f(g(x))' = f'(g(x))\, g'(x)$

Proof: We consider

$$\frac{f(g(x+h)) - f(g(x))}{h}$$

set $u := g(x)$, $k := g(x+h) - g(x)$

$$= \frac{f(u+k) - f(u)}{k} \cdot \frac{k}{h}$$

$k = 0$? This can be 0 but the proof can be adapted.

$$= \underbrace{\frac{f(u+k) - f(u)}{k}}_{k \to 0 \Rightarrow f'(u)} \cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{h \to 0 \Rightarrow g'(x)}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= f'(u)\, g'(x)$$

$$= f'(g(x))\, g'(x) \quad \#$$

Observe that $k \to 0$ when $h \to 0$ since g is continuous.

Remark One often see

$$\frac{d\, f(g(x))}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

∇ Prof. doesn't like this form. Use the other one please. :)

Example: Consider

$$x \mapsto g(x)^3$$

f is given by $x \mapsto x^3$

Then $(g(x)^3)' = 3(g(x))^2\, g'(x)$

III.6) Higher derivatives

Let $f : I \to \mathbb{R}$ be differentiable with $f' : I \to \mathbb{R}$

If f' is differentiable, then we can compute $(f')' \equiv f'' \equiv f^{(2)}$

Notations

$f^{(j)}$ is the j^{th} derivative of f.

If $f^{(j)}$ is continuous, we write $f \in C^j(I)$

Example: let $P_n : \mathbb{R} \to \mathbb{R}$ polynomial degree n

$$P_n(x) = x^n$$

Then $P_n' = n P_{n-1}$ equality of functions

$$P_n^{(2)} = n(n-1) P_{n-2}$$

$$P_n^{(3)} = n(n-1)(n-2) P_{n-3}$$

$$\vdots$$

$$P_n^{(n)} = n(n-1)(n-2) \dots 1 =: n!$$

$$P_n^{(n+1)} = 0$$

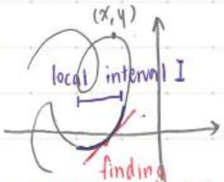
function equal to 0 for any x

It may look like a zero, but it's a function. function = function

$$\left[(x^n)' = n x^{n-1} = n P_{n-1}(x) \right]$$

equality of numbers

III.7) Implicit differentiation



Often, curves can be described by an equation, it means a relation between x and y.

Example

we can't solve this, but it described the curve.

$$x^7 y^6 + x^2 y^3 + x^{25} y + x y^{23} = 0$$

We know some points like $(0,0)$.

In general, we write $F(x,y) = 0$

with $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\begin{matrix} \mathbb{R} & \times & \mathbb{R} & \to & \mathbb{R} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ x & & y & & F(x,y) \end{matrix}$$

a vertical "∈"

Example Consider the curve given by

$$F(x,y) = 3x^3y - y^4 + 5x^2 + 5 = 0$$

Suppose that we can solve this equation, and write $y = f(x)$ (at least locally). Then we can consider $F(x, f(x)) \equiv F(x, y(x))$ and take its derivative.

One has $\frac{d}{dx} F(x, f(x)) \equiv \frac{d}{dx} F(x, y(x))$

$$= 9x^2 y(x) + 3x^3 \underline{y'(x)} - 4y^3(x)\, \underline{y'(x)} + 10x = 0$$

We need this to find the slope

because $F(x, y(x)) = 0$ $\forall x \in I$

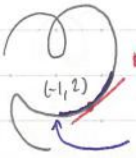
We get $(3x^3 - 4y(x)^3)\, y'(x) = -9x^2 y(x) - 10x$

$$\Rightarrow \boxed{y'(x) = \frac{9x^2 y(x) + 10x}{4y(x)^3 - 3x^3}}$$

Observe that $(1,2)$ satisfies $F(1,2) = 0$, which means that $(1,2)$ belongs to the curve.

$\Rightarrow$ $y(1) = 2$ and we get

$$y'(1) = \frac{18+10}{32-3} = \frac{28}{29}$$



slope of this tangent is $y'(1) = \frac{28}{29}$

So we can get the slope without knowing the (local) function YAY!

IV.) Some Basic Functions

IV.1) Sine and cosine functions

$\sin, \cos : \mathbb{R} \to [-1, 1]$, 2π periodic and satisfy

- $\cos^2(x) + \sin^2(x) = 1$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\cos(x+y) = \ldots$
- $\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
- $\cos(x) - \cos(y) = \ldots$

Lemma: $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$

Corollary: $\sin(x)' \equiv \sin'(x) = \cos(x)$

Proof: $\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\xrightarrow{h\to 0} 1} \underbrace{\cos\left(x+\frac{h}{2}\right)}_{\xrightarrow{h\to 0} \cos(x)}$$

from lemma

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right)$$

$$= 1 \cdot \cos(x)$$

$$= \cos(x)$$

#

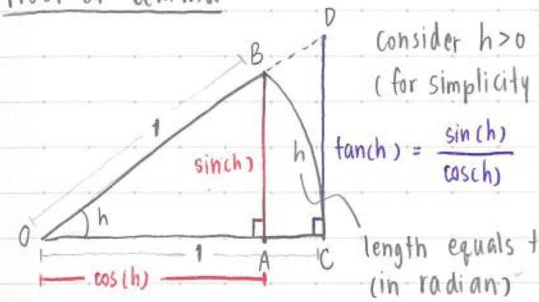
Corallary (Homework)

1.) $\cos'(x) = -\sin(x)$

2.) $\lim\limits_{x \to 0} \dfrac{\cos(x)-1}{x} = 0$

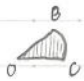
3.) $\lim\limits_{x \to 0} \dfrac{\cos(x)-1}{x^2} = -\dfrac{1}{2}$

Proof of lemma



consider $h>0$ (for simplicity)

length equals to h (in radian)

One has $\Delta OAB \leq$  $\leq \Delta OCD$

$\Leftrightarrow \frac{1}{2}\cos(h)\sin(h) \leq \frac{h}{2\pi}\pi \cdot 1^2 \leq \frac{1}{2}\tan(h) = \frac{1}{2}\frac{\sin(h)}{\cos(h)}$

fraction of the disk

area of the disk

$\Leftrightarrow \cos(h)\sin(h) \leq h \leq \dfrac{\sin(h)}{\cos(h)}$

$\Leftrightarrow \cos(h) \leq \dfrac{h}{\sin(h)} \leq \dfrac{1}{\cos(h)}$

$\Leftrightarrow \dfrac{1}{\cos(h)} \leq \dfrac{\sin(h)}{h} \leq \cos(h)$

Since $\lim\limits_{h \to 0} \cos(h) = 1$

We infer that $\lim\limits_{h \to 0} \dfrac{\sin(h)}{h} = 1$ #.

(squeeze theorem)

IV.2) Exponential functions

In Homework 4 we define

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \sum\limits_{j=0}^{\infty} \dfrac{1}{j!} x^j$

and we obtain that

$f'(x) = f(x)$

name of the function defined by

We call f the exponential, and write

$f(x) = e^x$

Lemma Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $g'(x) = g(x)$.

Then $g(x) = ce^x$ for some $c \in \mathbb{R}$

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Reminder

- chain rule = differentiable for composition
- higher order derivatives, $C^n(\mathbb{R})$
- implicit differentiation
- $\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \rightsquigarrow \sin'(x) = \cos(x)$
- $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{x\prime} = e^x$

Lemma If $f: \mathbb{R} \to \mathbb{R}$ differentiable satisfies $f' = f$, then $f(x) = ce^x$ for some $c \in \mathbb{R}$

Proof Let us show that $\frac{f(x)}{e^x}$ is a constant independent of x

Indeed, one has

$$\left(\frac{f(x)}{e^x}\right)' = \frac{f'(x)e^x - \overbrace{(e^x)'}^{=e^x} f(x)}{(e^x)^2}$$

$f(x)$ assumption

$$= \frac{f(x)e^x - f(x)e^x}{(e^x)^2} = 0$$

Since $\left(\frac{f(x)}{e^x}\right)' = 0$, one infers that $\frac{f(x)}{e^x} = c$ for some $c \in \mathbb{R}$

and then $f(x) = ce^x \quad \forall x \in \mathbb{R}$ #

Remark We will show that $e^x \neq 0$ for any $x \in \mathbb{R}$

Note

$$\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1-a} \qquad (a \neq 1)$$

$$(1-a)\sum_{n=0}^{N} a^n = (1-a)(a^0 + a^1 + a^2 + \dots + a^N)$$

$$= a^0 + \cancel{a^1} + \cancel{a^2} + \dots + \cancel{a^N}$$
$$- \cancel{a^1} - \cancel{a^2} - \cancel{a^3} - \dots - \cancel{a^N} - a^{N+1}$$

$$= a^0 - a^{N+1}$$
$$= 1 - a^{N+1}$$

$$\therefore \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

If $|a| < 1$; $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

V.) Mean Value Theorem

V.1) Local minimum and maximum

<u>Def</u>: Let $f: I \to \mathbb{R}$. A point $x_0 \in I$ in a local maximum for f if $\exists \delta > 0$ such that $f(x_0) \geq f(x_0 + h)$ for and $|h| < \delta$
$\iff f(x_0) \geq f(y)$ for any $y \in [x_0 - \delta, x_0 + \delta]$

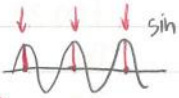
x_0 is a local minimum for f. if $\exists \delta > 0$ such that $f(x_0) \leq f(x_0 + h)$ for any $|h| < \delta$,

x_0 is a local maximum of f if
$f(x_0) \geq f(y) \quad \forall y \in I$ (with $x_0 + h \in I$)

x_0 is a global minimum of f if
$f(x_0) \leq f(y) \quad \forall y \in I$

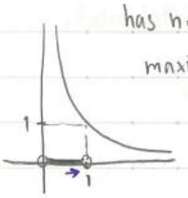
<u>Question</u>: Do they alway exist? and are they unique?

<u>Ans</u>: No. For uniqueness:



local $\neq$ global maximum

$\frac{1}{x}$ on $(0,1)$ ← open interval
has no local or global maximum (and also minimum)



But on $(0,1]$, the function $x \mapsto \frac{1}{x}$ has a a local (and global) minimum at $x = 1$.

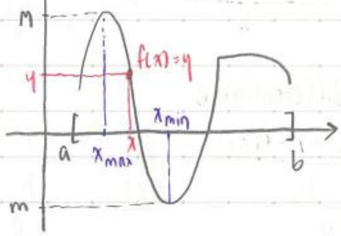
$\boxed{\text{Thm}}$ Extreme Value thm.

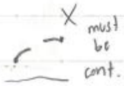
If $f: [a,b] \to \mathbb{R}$ is continuous, then there exists $x_{min} \in [a,b]$ and $x_{max} \in [a,b]$ with x_{min} a global minimum, and x_{max} a global maximum.

⚠ They might not be unique

$\boxed{\text{Thm}}$ Intermidiate value thm.

Let $f: [a,b] \to \mathbb{R}$ be continuous, and set $m := f(x_{min})$ and $M := f(x_{max})$. Then for any $y \in [m, M]$ there exist $x \in [a,b]$ with $f(x) = y$. x might not be unique





<u>Def</u> Let $f: I \to \mathbb{R}$ be differentiable
A point $x_0 \in I$ is a critical point for f if $f'(x_0) = 0$

examples.
1) $f(x) = \sin(x)$, $f'(x) = \cos(x)$, $f: \mathbb{R} \to \mathbb{R}$
and $\cos(x) = 0 \iff x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$
Critical point $= \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$

2) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

$f'(x) = 3x^2$

$3x^2 = 0 \iff x = 0$

critical point $= \{0\}$

Thm Let $f: I \to \mathbb{R}$ be differentiable, and let $x_0 \in I$ with x_0 a local maximum (minimum). Then $f'(x_0) = 0$ (which means x_0 is a critical point for f)

(a,b) open

Proof: Consider $h \in \mathbb{R}$, small with $x_0 + h \in I$

For $h > 0$, $\dfrac{f(x_0+h) - f(x_0)}{h} \leq 0$ for h small enough

For $h < 0$, $\dfrac{f(x_0+h) - f(x_0)}{h} \geq 0$ for h small enough

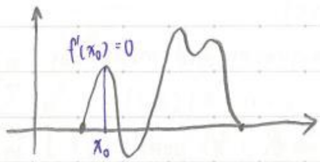
Since f is differentiable,

$$\underbrace{\lim_{h \searrow 0} \frac{f(x_0+h) - f(x_0)}{h}}_{\leq 0} = \underbrace{\lim_{h \nearrow 0} \frac{f(x_0+h) - f(x_0)}{h}}_{\geq 0}$$

which implies that $\underbrace{\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}}_{f'(x)} = 0$

Do the same for local minimum. #

V.2) The mean value theorem



Thm (Rolle's Thm.)

Let $f: [a,b] \to \mathbb{R}$, continuous, and differentiable on (a,b). Suppose $f(a) = f(b) = 0$. Then there exists $x_0 \in (a,b)$ with $f'(x_0) = 0$.

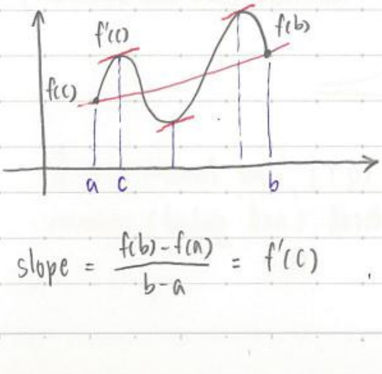
$f: [0,1)$ continuous at 0, but not differentiable at 0.

example: $|x|$, $\sqrt{x}$

proof
- If $f(x) = 0 \ \forall x \in [a,b]$, then $f' = 0$, the statement is satisfied for any $x \in (a,b)$
- Suppose $\exists x \in (a,b)$ with $f(x) > 0$ (or < 0)

Then, by the extreme value thm, $\exists x_{max} \in (a,b)$ which is a global maximum.

By the previous thm, $f'(x_{max}) = 0$, which proves the statement. #



$$\text{slope} = \frac{f(b) - f(a)}{b - a} = f'(c)$$

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Thm (mean value thm.)

cont.

diff. let $f: [a,b]$ be continuous, and differentiable on (a,b). Then $\exists c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof Equation of the line:

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

consider $h: [a,b] \to \mathbb{R}$,

$$h(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right)$$

continue next class.

(continue)

One $h(a) = 0$, $h(b) = 0$

h is cont. on $[a,b]$, diff on (a,b)

By Roffe's thm, $\exists c \in (a,b)$ with $h'(c) = 0$

But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Reminder

- local / global max / min
- extreme value thm.
- intermediate value thm.
- critical point
- Roffe's thm.
- mean value thm.

Corollary Let $f: I \to \mathbb{R}$ be diff. and such that $f'(x) = 0 \ \forall x \in I$.

Then f = constant. const.

interval

Proof: Choose $a, b \in I$ with $a < b$

Then $f: [a,b] \to \mathbb{R}$ is continuous and diff. on (a,b)

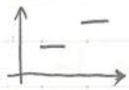
By the mean value thm, $\exists c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Since $f'(c) = 0$, then $f(b) = f(a)$

$\Rightarrow$ f takes the same value.

On any pair of points in I



$\Rightarrow$ f = const. #

New proof of L'Hospital's Rule:

Step 1: Cauchy mean value thm.

Lemma: Let $f, g: [a,b] \to \mathbb{R}$ be cont. and diff. on (a,b). Then $\exists c \in (a,b)$ with

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$$

(*)

Proof 1) if $g(b) = g(a)$, then we can apply Rolle's thm. to $x \mapsto g(x) - g(x)$, and get $g'(c) = 0$ for some $c \in (a,b)$

2) if $g(b) \neq g(a)$, set $h: [a,b] \to \mathbb{R}$

$$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x) - \frac{f(a)f(b) - f(b)f(a)}{g(b)-g(a)}$$

Then h is cont. on $[a,b]$ and diff. on (a,b). One has $h(a) = 0 = h(b)$. By Rolle's thm, $\exists c \in (a,b)$ with $h'(c) = 0$

$$\Leftrightarrow f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0 \Rightarrow (*)$$

#

Step 2: proof of L'Hospital's thm.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}$$

by step 1, when $c \in (x_0, x)$

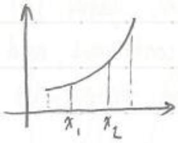
$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c)}{g'(c)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}$$

if the limit exists. #

Def A function $f: I \to \mathbb{R}$ is increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.

And f is decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$.

We say that it is strictly increasing / decreasing if $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$ for $x_1 < x_2$.



Lemma Let $f: [a,b] \to \mathbb{R}$ be cont. and diff. on (a,b).

1.) If $f'(x) \geq 0 \quad \forall x \in (a,b)$, then f is increasing on $[a,b]$.

2) If $f'(x) \leq 0 \quad \forall x \in (a,b)$, the f is decreasing on $[a,b]$

Proof of 1)

Choose $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Then by the mean value thm. $\exists c \in (x_1, x_2)$ with

$0 \geq$

$$0 \leq f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) \overset{\leq}{\geq} 0$$

$$\Leftrightarrow f(x_2) \overset{\leq}{\geq} f(x_1)$$

Similar proof for 2.)

VI.) Sketching Curves.

look at the website.

VI.) Inverse Function

VI.1) Definition of inverse function

Let $f: I \to \mathbb{R}$ and set $y = f(x)$

Question: Can we solve $x = g(y)$ for some g?

More precisely, can we find $g: \ldots \to I$ such that $g(f(x)) = x \quad \forall x \in I$

$f(g(y)) = y \quad \forall y \in f(I)$ or $Ran(f)$

Examples:

① $f: \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^2$

If we set $g: \mathbb{R}_+ \to \mathbb{R}_+$

$x \mapsto \sqrt{x} = g(x)$

then $g(f(x)) = x \quad \forall x \in \mathbb{R}_+$

$f(g(y)) = y \quad \forall y \in \mathbb{R}_+$

② $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto x^2$

ex. $x = 3$ and -3 have the same y

We cannot find an inverse because f is not injective.

Remark

Since f is always surjective on its image (on $f(I)$ or $Ran(f)$) then $f: I \to f(I)$ is bijective if and only if f is injective.

In such a case, we can define an inverse.

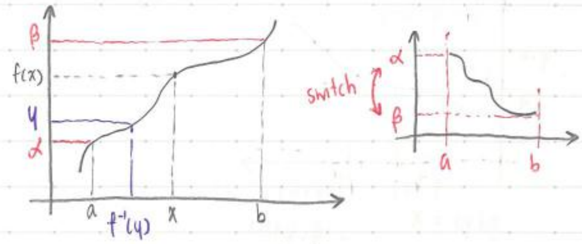
$\boxed{\text{Thm}}$: Let $f: [a,b] \to \mathbb{R}$ be strictly increasing and cont. Set $\alpha := f(a)$ and $\beta := f(b)$. Then there exists

notation $f^{-1}: [\alpha, \beta] \to [a,b]$ with

$f^{-1}(f(x)) = x \quad \forall x \in [a,b]$ and

$f(f^{-1}(y)) = y \quad \forall y \in [\alpha, \beta]$

Proof as an exercise.



Similar statement if f is strictly decreasing.

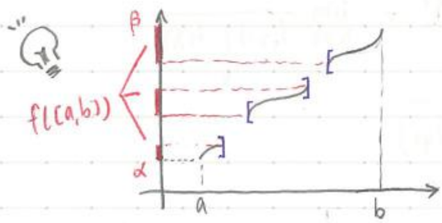


image is not $[\alpha, \beta]$

VI.2) Derivative of the inverse function

$\boxed{\text{Thm.}}$ Let $f: [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $f'(x) > 0 \quad \forall x \in (a,b)$.

Set $\alpha := f(a)$, $\beta := f(b)$

Then $f^{-1}: [\alpha, \beta] \to [a,b]$ is diff. and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

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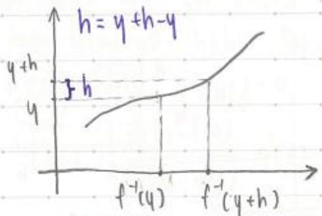
Proof Set $g(y) := f^{-1}(y)$. One has

$$\frac{g(y+h)-g(y)}{h} = \frac{x+k-x}{f(x+k)-f(x)} = \frac{1}{\frac{f(x+k)-f(x)}{k}}$$

when $h \to 0$, then $k \to 0$.



$g(y) := x$ $\quad = g(y+h) = x+k$

$$\lim_{h\to 0} \frac{g(y+h)-g(y)}{h} = \lim_{h\to 0} \frac{1}{\frac{f(x+k)-f(x)}{k}} = \frac{1}{f'(x)} = \frac{1}{f(f^{-1}(y))}$$

Thus, $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$ #

Reminder

- mean value thm.
- proof of L'Hospital's thm
- increasing / decreasing function
- If f differentiable, $f' \geq 0 \Rightarrow f$ increasing. $f' \leq 0 \Rightarrow f$ decreasing.
- Inverse function
- $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$

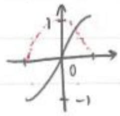
Examples

1.) $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \ni x \mapsto \sin(x) \in [-1, 1]$

$\sin'(x) = \cos(x) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$

since sin is invertible, we can consider its inverse, $\sin^{-1} =: \arcsin$ and

$$\arcsin'(y) = \frac{1}{\cos(\arcsin(y))} = \frac{1}{\sqrt{1-\sin(\arcsin(y))^2}}$$



$$= \frac{1}{\sqrt{1-y^2}} \quad \text{for any } y \in (-1,1)$$

2.) $\cos : [0,\pi] \ni x \mapsto \cos(x) \in [-1,1]$

$\cos'(x) = -\sin(x) < 0$ for $x \in (0,\pi)$

since cos is invertible, we set $\cos^{-1} = \arccos : (-1,1) \mapsto (0,\pi)$ with

$$\arccos'(y) = \frac{1}{-\sin(\arccos(y))} = -\frac{1}{\sqrt{1-y^2}}$$

3.) $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \ni x \mapsto \tan(x) \in \mathbb{R}$

$\tan^{-1}(x) = 1 + \tan^2(x) \geq 1 > 0$

tan is invertible, and $\tan^{-1} = \arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\arctan'(y) = \frac{1}{1+\tan(\arctan(y))^2} = \frac{1}{1+y^2}$$

VI.3) Exponential and Logarithmic functions

<u>Reminder</u> $e^x := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, and $e^{x'} = e^x$, $e^0 = 1$

<u>Remark</u> : $e^{-x} := \frac{1}{e^x} \overset{?}{\neq} e^{-x}$

<u>Lemma</u> : let us set $f(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

1.) $f(x) f(-1) = 1 \quad \forall x \in \mathbb{R}$
2.) $f(x) > 0 \quad \forall x \in \mathbb{R}$
3.) f is increasing
4.) $f(x+y) = f(x) f(y) \quad \forall x, y \in \mathbb{R}$

<u>Proof</u>

1.) $(f(x)f(-x))' = f'(x)f(-x) + f(x)(f(-x))'$

$= f(x)f(-x) + f(x)\underbrace{f'(-x)}_{= f(-x)}(-1) = 0$

$\Rightarrow x \mapsto f(x)f(-x)$ is a constant function since $f(0) = 1 = f(-0)$

$\Rightarrow f(x)f(-1) = 1 \quad \forall x \in \mathbb{R}$

2.) It follows from 1.) that $f(x) \neq 0$ for any x (otherwise $f(x)f(-x) \neq 1$)

By the intermediate value thm, if there exists $x \in \mathbb{R}$ such that $f(x) < 0$, the $[f(x), 1]$ is in the range of f

$\Rightarrow 0 \in \mathrm{Ran}(f)$, which is not possible.

3.) $f'(x) = f(x) > 0 \Rightarrow f$ is increasing

4.) For fixed x, consider the function

$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(y) := \frac{f(x+y)}{f(y)}$

($f(x+y)$ is a composition of two functions. $f(x+y) = f(g(x))$ with $g(x) = x+y$.)

One has $\phi'(y) = \frac{f'(x+y)f(y) - f(x+y)f'(y)}{f(y)^2} = 0$

because $f' = f \Rightarrow y \mapsto \phi(y)$ is a constant function.

By considering $\phi(0) = \frac{f(x+0)}{f(0)} = f(x)$

we get $\phi(y) = f(x)$

$\Leftrightarrow f(x+y) = f(x)f(y)$ #.

<u>Remark</u> : If we use the notation $f(x) = e^x$ we have obtained :

① $e^x e^{-x} = 1 \quad (\Rightarrow e^{-x} = \frac{1}{e^x})$
② $e^x > 0$
③ $x \mapsto e^x$ is increasing
④ $e^{x+y} = e^x e^y$

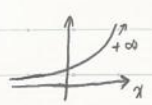
We call this function the "exponential function".

Since $(e^x)' = e^x > 0$

$\Rightarrow e^x$ is strictly increasing, and then it is invertible. Its inverse in denoted by

$\ln : \mathbb{R}^+ \to \mathbb{R}$ with $\ln(e^x) = x \quad \forall x \in \mathbb{R}$

and $e^{\ln(y)} = y \quad \forall y \in \mathbb{R}^+$



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<u>Lemma</u>: (HW 7)

1.) $\ln(y)' = \frac{1}{y}$
2.) $\ln(xy) = \ln(x) + \ln(y)$ $\Big\}\ \forall x, y \in \mathbb{R}^+$
3.) $\ln(y^x) = x \ln(y) \quad \forall y \in \mathbb{R}^+,\ x \in \mathbb{Q}$

<u>Remark</u>

Since for any $x \in \mathbb{R}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{Q}$ and $x_n \xrightarrow{n \to \infty} x$, we would like to have:

$$\ln y^x \cong \ln y^{\lim_{n\to\infty} x_n} \cong \lim_{n\to\infty} \ln y^{x_n}$$

$$= \lim_{n\to\infty} x_n \ln y = x \ln(y)$$

because $x_n \in \mathbb{Q}$

<u>Def</u>: For any $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$ one set $y^x = e^{\ln(y^x)} := e^{x \ln y}$

$\Rightarrow \ln(y^x) = \ln(e^{x \ln y})$
$= x \ln y$.

midterm is everything until this point

Good luck !!

VIII.) Integration

1.) We would like to find an inverse operation for the differentiation
2.) Compute some area

VIII.1) Indefinite integral

Prof. doesn't like this word

= antiderivative

<u>Def</u>: Let $f: I \to \mathbb{R}$ be a function. An <u>indefinite integral</u> for f is a function $F: I \to \mathbb{R}$, differentiable with $F' = f$

<u>Examples</u>

1. $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^n \quad n \in \mathbb{N}$

   then $F(x) = \frac{1}{n+1} x^{n+1} + k$ with $k \in \mathbb{R}$

2. $f: \mathbb{R} \to \mathbb{R},\ f(x) = \cos(x),\ F(x) = \sin(x)$

<u>Remarks</u>:

1.) If F is an indefinite integral for f, then $F + k$, $k \in \mathbb{R}$, is also an indefinite integral
2.) We set $\int f = \int f(x)dx$ for an independent integral.

⚠ If we look for an indefinite integral of f on a domain made of more than 1 piece, one has to add several constants.

exclude zero

<u>Example</u>: consider $f: \mathbb{R}^* \to \mathbb{R}$
$$x \mapsto \frac{1}{x}$$

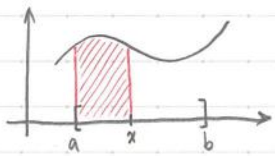
indefinite integral

What is the most general indef. int. for f?

$$F(x) := \begin{cases} \ln(x) + k_1 & \text{for } x > 0 \\ \ln(-x) + k_2 & \text{for } x < 0 \end{cases}$$

VIII.2) Some areas

Consider $f: [a,b] \to \mathbb{R}^+$, continuous
For any $x \in [a,b]$, set $F(x) :=$ area under the graph of f between a and x.



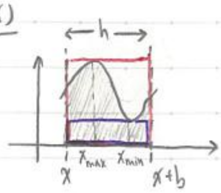
Observe that $F(a) = 0$

$\boxed{\text{Thm.}}$: The function $F: [a,b] \to \mathbb{R}$ is differentiable, and $F(x) = f(x)$, $\forall x \in (a,b)$

<u>Remark</u>: The statement cannot be very precise because we have not defined what is the area under the curve.

<u>Proof</u>. Consider $x \in (a,b)$ and $h \in \mathbb{R}$ with h small enough such that $x+h \in (a,b)$

Consider $\dfrac{F(x+h) - F(x)}{h}$



and consider
$$x_{max} \in [x, x+h]$$
$$x_{min} \in [x, x+h]$$

$$h f(x_{min}) \leq F(x+h) - F(x) \leq h f(x_{max})$$
$$\Longleftrightarrow f(x_{min}) \leq \frac{F(x+h) - F(x)}{h} \leq f(x_{max})$$

When $h \to 0$, x_{min} and x_{max} converge to x and then
$$f(x) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq f(x) \quad \#$$

No. Final

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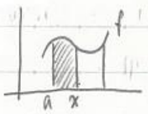
Reminder

- Indefinite integral (not unique)
- $f: [a,b] \to \mathbb{R}^+$, continuous.

and $F(x) =$ 

Then $F: [a,b] \to \mathbb{R}$, differentiable, and $F' = f$.

Remark

The function F is an indefinite integral for f on (a,b).

The area under the curve between a and b is given by $F(b) = F(b) - F(a)$

Corollary Let $f: [a,b] \to \mathbb{R}^+$ be continuous, and let $G: [a,b] \to \mathbb{R}$ be an indefinite integral for f for (a,b). Then the area under the curve between a and b is given by $G(b) - G(a)$.

Proof Let f be the function defined by the previous thm,

Then $G = F +$ const. (see previous lemma)

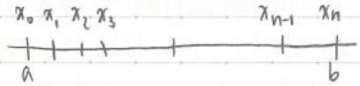
$\Rightarrow G(b) - G(a) = F(b) + \cancel{\text{const.}} - (\underbrace{F(a)}_{0} + \cancel{\text{const.}})$

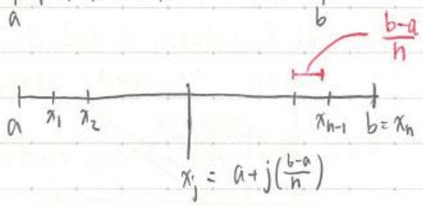
$= F(b)$

VIII.3) Riemann's Sum

Question: How can one compute the surface below the curve more precisely.

Def For any interval $[a,b]$, we call a n-partition $\mathcal{P}$ of $[a,b]$ a collection $\{x_0, x_1, \dots, x_n\} = \{x_j\}_{j=0}^{n}$ with $x_0 = a$, $x_n = b$ and $x_j < x_{j+1}$

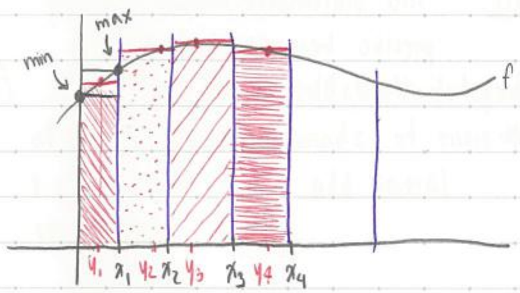


Example 

Def For any $f: [a,b] \to \mathbb{R}$, bounded for any n-partition $\mathcal{P}$, and for any family $\{y_j\}_{j=1}^{n}$ with $y_j \in [x_{j-1}, x_j]$

We set $\mathcal{R}_{\mathcal{P}, \{y\}_{j=1}^{n}}(f) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(y_j)$

We call this a "Riemann sum" for f.



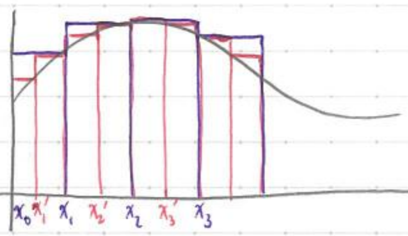
Remark

In each subinterval, we can choose y_j such that $f(y_j)$ takes the maximum value in the interval $[x_{j-1}, x_j]$, or choose y_j such that $f(y_j)$ takes the minimum value in $[x_{j-1}, x_j]$

The first Riemann sum is called $\mathcal{R}_P^{max}(f)$, and the second one is $\mathcal{R}_P^{min}(f)$.

We call them the upper/lower Riemann sum. One has

$$\mathcal{R}_P^{min}(f) \le \mathcal{R}_{P,\{y_j\}_{j=1}^n}(f) \le \mathcal{R}_P^{max}(f)$$



Idea: Consider lower & upper partition of $[a,b]$

· Consider a partition $P = \{x_j\}_{j=0}^n$, and a finer partition $P' = \{x_0, x_1', x_1, x_2', x_2, \dots, x_n\}$

Then $\mathcal{R}_P^{max}(f) \geq \mathcal{R}_{P'}^{max}(f)$ thinner → decreases

$\geq$ bound by each other

$\mathcal{R}_P^{min}(f) \le \mathcal{R}_{P'}^{min}(f)$ thinner → increases

By taking finer partitions, $\mathcal{R}^{max}(f)$ is decreasing, while $\mathcal{R}^{min}(f)$ is increasing.

But $\mathcal{R}^{max}(f)$ is lower bounded (by $\mathcal{R}_P^{min}(f)$) and $\mathcal{R}^{min}(f)$ is upper bounded by $\mathcal{R}^{max}(f)$.

This implies that, by taking linear partition, the upper Riemann sums are converging, and the lower Riemann sums are also converging.

Def A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if

supremum "maximum" infimum "minimum"

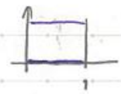
$$\sup_{P \text{ any partition of } [a,b]} \mathcal{R}_P^{min}(f) = \inf_{P \text{ partion of } [a,b]} \mathcal{R}_P^{max}(f)$$

In this case, we write

$\int_a^b f(x)\,dx$ for thin value.

just a notation for thin number.

Example



Consider $[a,b] = [0,1]$

and $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \end{cases}$

Take any partition P of $[0,1]$

$\mathcal{R}_P^{min}(f) = 0$ because there are $y \in \mathbb{R} \setminus \mathbb{Q}$ in any subinterval.

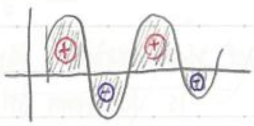
$(x_1 - \cancel{x_0}^{0}) + (x_2 - x_1) + \dots + (x_n - x_{n-1})$

$$\mathcal{R}_P^{max}(f) = \sum_{j=1}^n (x_j - x_{j-1})\, 1 = 1$$

$\Rightarrow$ f is not Riemann integrable.

$\boxed{\text{Thm}}$ Let $f: [a,b] \to \mathbb{R}$ be continuous, then f is Riemann integrable on $[a,b]$. (It means $\int_a^b f(x)dx$ is well defined).

Remark: If f is positive and continuous, the construction with Riemann sums give a precise meaning to the surface below the curve. But it is more general since f has not to be positive.



Properties of this integral

If f is Riemann integrable on $[a,b]$ and if $c \in (a,b)$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx .$$

already defined

If $a<b$, we set

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

Then we can write

$$\int_x^z f(s)\,ds = \int_x^y f(s)\,ds + \int_y^z f(s)\,ds$$

for any x, y, z.

$\boxed{\text{Thm}}$ (Extremion of previous thm.)

Let $f: [a,b] \to \mathbb{R}$ be continuous (⇒ Riemann integrable) and for $x \in [a,b]$ we set $F(x) = \int_a^x f(y)\,dy$ limit of Riemann sum

Then $F: (a,b) \to \mathbb{R}$ is differentiable, and $F' = f$

The proof for $h<0$ is similar

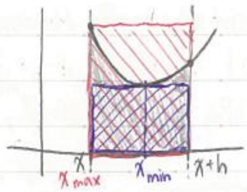
Proof: Consider $x \in (a,b)$ and $h>0$ such that $x+h \in (a,b)$

Then $$\frac{F(x+h)-F(x)}{h} = \frac{\int_a^{x+h} f(y)\,dy - \int_a^x f(y)\,dy}{h}$$

$$= \frac{\int_h^{x+h} f(y)\,dy}{h}$$

continue next class

(continue)



Let $x_{min} \in [x, x+h]$ be the global minimum of f on $[x, x+h]$, and x_{max} its global maximum. Then $f(x_{min}) \le f(y) \le f(x_{max})$ $\forall y \in [x, x+h]$

$$\Rightarrow f(x_{min})h \le \int_x^{x+h} f(y)\,dy \le h f(x_{max})$$

$$\Rightarrow f(x_{min}) \le \frac{\int_x^{x+h} f(y)\,dy}{h} \le f(x_{max})$$

Since $\lim_{h\to 0} x_{max} = x = \lim_{h\to 0} x_{min}$

One gets that

$$f(x) \le \lim_{h\to 0} \frac{\int_x^{x+h} f(y)\,dy}{h} \le f(x)$$

$\Rightarrow$ F is differentiable, and $F' = f$ #

The equality

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x)$$

called the Fundamental Theorem of Calculus.

Date 12 . 12 . 19

Reminder

- Indefinite integral for f
- Riemann sums
- $f: [a,b] \to \mathbb{R}$ Riemann integrable if
  $\lim_{\mathcal{P}} R_{min}(f, \mathcal{P}) = \lim_{\mathcal{P}} R_{max}(f, \mathcal{P})$

VIII.4) Properties of the Integral

Let $f: [a,b] \to \mathbb{R}$ be continuous, let $x \in [a,b]$ and set $F(x) = \int_a^x f(y)\,dy$.
is an indefinite integral for f

Let $G: [a,b] \to \mathbb{R}$ be another indefinite integral for $f \Rightarrow G = F + c$, with $c = \mathbb{R}$.

Since $F(a) = 0 \Rightarrow G(a) = F(a) + c = c$

Then $\int_a^b f(y)\,dy = F(b) - F(a)$
$= G(b) - G(a) = G(x)\Big|_a^b$

<u>Lemma</u>: If $f, g: [a,b] \to \mathbb{R}$, continuous, then

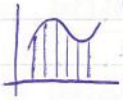
1.) $\int_a^b (f(y) + \lambda g(y))\,dy$
$= \int_a^b f(y)\,dy + \lambda \int_a^b g(y)\,dy$

2.) ⚠ $\int_a^b f(y) g(y)\,dy$
$\neq \int_a^b f(y)\,dy \cdot \int_a^b g(y)\,dy$

3.) If $f(x) \leq g(x) \Rightarrow \int_a^b f(y)\,dy \leq \int_a^b g(y)\,dy$

<u>Proof for 3.</u>

$\Rightarrow g(x) - f(x) \geq 0 \Rightarrow \int_a^b (g(x) - f(x))\,dx$



≥ 0 integral of positive function

$\Rightarrow \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0$

<u>Proposition</u> Let $f: [a,b] \to \mathbb{R}$ be continuous

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

<u>Proof</u> Observe that $|f|$ is also continuous, which means that RHS is well-defined

Since $|f(x)| \geq f(x)$

$\overset{③}{\Rightarrow} \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

Observe also that $-|f(x)| \leq f(x)$

$\Rightarrow \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx$

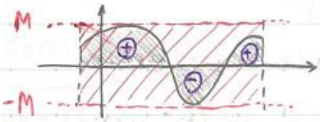
$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx$

$\Rightarrow$ statement. #

<u>Corollary</u>

If $f: [a,b] \to \mathbb{R}$ is continuous and $|f(x)| \leq M < \infty \quad \forall x \in [a,b]$

then $\left| \int_a^b f(x)\,dx \right| \leq M(b-a)$



If $f(x) \leq M \Rightarrow \int_a^b f(x)\,dx \leq (b-a)M$

If $|f(x)| \leq M \Rightarrow \left| \int_a^b f(x)\,dx \right| \leq (b-a)M$

IX.) Techniques of Integration

A few tricks.

IX.1) Substitution

Look for $\int f(x)\,dx = \int g(u(x))\,u'(x)\,dx$
$= G(u(x))$
with G an indefinite integral for g.
(Fund. Thm. of calculus)

Indeed $\frac{d}{dx}\int_a^x g(u(y))\,u'(y)\,dy = g(u(x))\,u'(x)$

and $\frac{d}{dx} G(u(x)) = g(u(x))\,u'(x)$

Examples

1) $\int (\overbrace{x^3+x}^{u(x)})^9 (\overbrace{3x^2+1}^{u'(x)})\,dx$ $\quad g(x) = x^9$

$= \frac{1}{10}(x^3+x)^{10} + \text{const.}$

2.) $\frac{1}{2}\int \overbrace{2x}^{u'(x)} \sin(\overbrace{x^2}^{u(x)})\,dx$

$= -\cos(x^2) + \text{const.}$

Convenient trick:

$$\int_a^b g(u(x))\,u'(x)\,dx = \int_a^b g(u(x))\,\underline{\frac{du}{dx}(x)\,dx}$$

"⇒" $u'(x)\,dx = du$

$$= G(u(x))\Big|_{x=a}^{x=b}$$

$$= G(u(b)) - G(u(a)) = G(y)\Big|_{u(a)}^{u(b)}$$

$$= \int_{u(a)}^{u(b)} g(u)\,du$$

Example

$$\int x^5\sqrt{1-x^2}\,dx$$
$$= -\frac{1}{2}\int x^4\sqrt{1-x^2}\,(-2x)\,dx$$
$$= -\frac{1}{2}\int (1-u)^2 u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{2}\int \left(u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$$
$$= -\frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{3/2} - \frac{2}{5/2}u^{\frac{5}{2}} + \frac{1}{7/2}u^{\frac{7}{2}}\right) + \text{const.}$$
$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{2}{3}(1-x^2)^{\frac{5}{2}} - \frac{1}{7}(1-x^2)^{\frac{7}{2}} + \text{const.}$$

IX.2) Integration by Parts.

Recall that $(fg)' = f'g + fg'$

$$\Rightarrow \boxed{\int fg' = \int (fg)' - \int f'g = fg - \int f'g}$$

indefinite integral

Examples

1.) $\int \underbrace{x}_{f(x)} \overbrace{e^x}^{g'(x)}\,dx = xe^x - \int e^x\,dx$
$= xe^x - e^x = e^x(x-1)$

2.) $\int \underbrace{1}_{g'(x)} \overbrace{\ln(x)}^{f(x)}\,dx = x\ln(x) - \int x\,\frac{1}{x}\,dx$
$= x\ln(x) - x$

IX.3) Trigonometric Integrals

Remember

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

$\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

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Examples

1.) $\int \sin^2(x)\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx$

$= \frac{1}{2}x - \frac{1}{4}\sin(2x)$

2.) $\int \cos^3(x)\,dx = \int \cos^2(x)\cos(x)\,dx$

$= \int (1-\sin^2(x))\cos(x)\,dx$

$= \sin(x) - \frac{1}{3}\sin^3(x)$

IX.4) Partial fraction

Partial Fraction

Examples

1.) $\int \frac{1}{x-a}\,dx = \ln(x-a)$

2.) $\int \frac{1}{(x-a)^n}\,dx = \int (x-a)^{-n}\,dx = \frac{1}{-n+1}(x-a)^{-n+1}$ $(n \neq 1)$

3.) $\int \frac{1}{x^2+1}\,dx = \arctan(x)$

More generally, consider $\frac{f(x)}{g(x)}$ with f, g polynomials with degree $(f) <$ degree (g).

In such a case, one has to factor the denominator into terms of the form

$(x-\alpha_j)^n$ or $((x-\beta_j)^2+\gamma_j^2)^m$

as for example

$$\frac{2x+5}{(x^2+1)^2(x-3)} = \frac{C_1+C_2x}{(x^2+1)} + \frac{C_3+C_4x}{(x^2+1)^2} + \frac{C_5}{x-3}$$

with $C_1, \dots, C_5 \in \mathbb{R}$

$$= \underbrace{\frac{C_1}{x^2+1}}_{\text{arctan}} + \frac{C_2x}{x^2+1} + \underbrace{\frac{C_3}{(x^2+1)^2}}_{???} + \frac{C_4x}{(x^2+1)^2} + \frac{C_5}{x-3}$$

integral

$$= \frac{C_1(x^2+1)(x-3) + C_2x(x^2+1)(x-3) + C_3(x-3) + C_4x(x-3) + C_5(x^2+1)^2}{(x^2+1)^2(x-3)}$$

$$= \frac{1}{(x^2+1)^2(x-3)}\left(\overset{=0}{(C_2+C_5)}x^4 + \overset{=0}{(C_1-3C_2)}x^3 + \overset{=0}{(-3C_1+C_2+C_4+2C_5)}x^2 + \underbrace{(C_1-3C_2+C_3-3C_4)}_{=2}x + \underbrace{(-3C_1-3C_3+C_5)}_{=5} \right)$$

We find

$$\begin{cases} C_1 = -\frac{33}{100} & C_3 = -\frac{130}{100} \quad C_5 = \frac{11}{100} \\ C_2 = -\frac{11}{100} & C_4 = -\frac{110}{100} \end{cases}$$

The integrals of four terms are simple, but what about $\int \frac{1}{(x^2+1)^2}\,dx$?

Consider $\int \underbrace{1}_{g(x)} \cdot \underbrace{\frac{1}{(x^2+1)}}_{f(x)}\,dx = \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{(x^2+1)^2}\,dx$ $\quad = \arctan(x)$

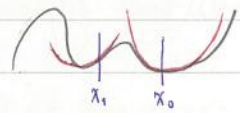
$$= \frac{x}{x^2+1} + \underbrace{2\int \frac{1}{x^2+1}\,dx}_{2\arctan(x)} - 2\int \frac{1}{(x^2+1)^2}\,dx$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan(x)$$

Remark: Similar tricks exist for $\int \frac{1}{(x^2+1)^m}\,dx$ for $m \in \mathbb{N}$.

X) Taylor's Formula

Idea: approximate a function locally by a polynomial



Recall that for $f: [a,b] \to \mathbb{R}$, sufficiently differentiable, we write $f^{(0)} = f$

$f^{(1)} = f'$

$f^{(2)} = f''$

$f^{(n)} = f^{\overbrace{''''\dots'}^{n \text{ times}}} = f^{(n-1)'}$

Def Consider $f: [a,b] \to \mathbb{R}$, such differentiable, and let $x_0 \in (a,b)$. Let $n \in \mathbb{N}$

For any $x \in [a,b]$ we define $P_n(x_0, x)$

$$:= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$= \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(x_0)(x-x_0)^j$$

$P_n(x_0, \cdot)$ is a polynomial of degree n, called the Taylor.

Question

$$\left.\begin{aligned} f(x_0) - P_n(x_0, x_0) &= 0 \\ f'(x_0) - P_n'(x_0, x_0) &= 0 \\ f''(x_0) - P_n''(x_0, x_0) &= 0 \\ &\vdots \\ f^{(n)}(x_0) - P_n^{(n)}(x_0, x_0) &= 0 \end{aligned}\right\}$$

$f^{(j)}$ and $P_n^{(j)}(x_0, \cdot)$ take the same value at x_0 for any $j \in \{0, 1, 2, \dots, n\}$

What about $f(x) - P_n(x_0 - x)$ for $x \neq x_0$?

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Example

1.) $f(x) = \sin(x)$, $x_0 = 0$, $n=5$

$P_5(0,x) = x - \frac{1}{3}x^5 + \frac{1}{5}x^5$

What about $\sin(x) - x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5$ for $x = 0$?

Thm (Taylor's expansion thm)

Let $f : [a,b] \to \mathbb{R}$. Let $n \in \mathbb{N}$ and suppose that f is $(n+1)$ times differentiable, with $f^{(n+1)}$ continuous ($\Leftrightarrow f \in C^{n+1}([a,b])$)

Let $x_0 \in (a,b)$ and $x \in [a,b]$

Then $f(x) = P_n(x_0, x) + \underbrace{R_{n+1}(x_0 + x)}$ Remainder term

with $R_{n+1}(x, x_0) = \int_{x_0}^{x} \frac{(x+t)^n}{n!} f^{(n+1)}(t)\,dt$

In addition,

$R_{n+1}(x, x_0) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}$

for some $c \in (x_0, x)$

Proof (by induction)

For $n=0$, one has

$$f(x) \overset{?}{=} f(x_0) + \underbrace{R_1(x_0, x)}_{= \int_{x_0}^{x} f'(t)\,dt}$$

$\Leftrightarrow f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt$ Property of any indefinite integral

$\Rightarrow$ the statement is true for $n=0$

Assume now that the statement is true for a certain n, and let us prove it for $n+1$.

$\hookrightarrow$ $f(x) = P_n(x_0, x) + \int_{x_0}^{x} \underbrace{\frac{(x-t)^n}{n!}}_{g'(t)} \underbrace{f^{(n+1)}(t)}_{h(t)}\,dt$

(int. by parts)

$$\int_{x_0}^{x} g'(t)h(t)\,dt = g(t)h(t)\Big|_{t=x_0}^{t=x} - \int_{x_0}^{x} g(t)h'(t)\,dt$$

$$= -\frac{(x-t)^{n+1}}{(n+1)\,n!} f^{(n+1)}(t)\Big|_{t=x_0}^{t=x} + \int_{x_0}^{x} \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\,dt$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(x_0) + R_{n+2}(x, x_0)$$

$$\Rightarrow f(x) = \underbrace{P_n(x_0, x) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0)} + R_{n+2}(x, x_0)$$

$$= P_{n+1}(x_0, x) + R_{n+2}(x, x_0)$$

$\Rightarrow$ We have proved the result for $n+1$. Since n is arbitrary, the first statement is proved.

For the second statement, since $f^{(n+1)}$ is continuous, there exist x_{max} and x_{min} in $[x_0, x]$ with $f^{(n+1)}(x_{min}) \leq f^{(n+1)}(t) \leq f^{(n+1)}(x_{max})$ $\forall t \in [x_0, x]$.

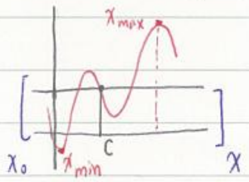
Suppose that $x_0 \leq x$, then

Then
$$\underbrace{\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(x_{min})\,dt}_{\frac{(x-x_0)^{n+1}}{(x+1)!} f^{(n+1)}(x_{min})} \leq \underbrace{\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}_{R_{n+1}(x, x_0)} \leq \underbrace{\int_{x_0}^{x} \frac{(n-t)^n}{n!} f^{(n+1)}(x_{max})\,dt}_{\frac{(x-x_0)^{n+1}}{(x-1)!} f^{(n+1)}(x_{max})}$$

$\Rightarrow \exists c \in [x_0, x]$ with $f^{(x+1)}(x_{min}) \leq f^{(x+1)}(c) \leq f^{(x+1)}(x_{max})$

with $f^{(x+1)}(c) \cdot \frac{(x-x_0)^{(n+1)}}{(n+1)!} = R_{n+1}(x, x_0)$

#



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Reminder

- For $f: \mathbb{R} \to \mathbb{R}$ differentiable many times, for fixed $x_0 \in \mathbb{R}$, and $n \in \mathbb{N}$, set

$$P_n(x_0, x) := \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(x_0)(x-x_0)^j$$

↑ polynomial of degree n

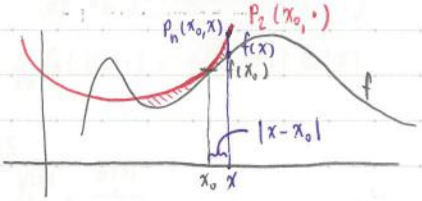
- Taylor: $f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$

with $R_{n+1}(x_0, x)$

$$= \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}$$

for some $c \in (x_0, x)$

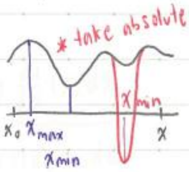


$$|R_{n+1}(x_0, x)| = \left|\frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}\right|$$

$$= \frac{1}{(n+1)!} |x-x_0|^{n+1} |f^{(n+1)}(c)|$$

$$\leq \frac{1}{(n+1)!} |x-x_0|^{n+1} \sup_{c \in [x, x_0]} |f^{(n+1)}(c)|$$

$$= \frac{1}{(n+1)!} |x-x_0|^{n+1} \max\left\{ |f^{(n+1)}(x_{max})|, |f^{(n+1)}(x_{min})| \right\}$$



Examples

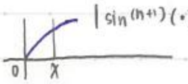
1.) $f(x) = \sin(x)$

$x_0 = 0$, n fixed

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + R_{n+1}(0, x)$$

with $|R_{n+1}(0, x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \cdot 1$

decreases as n increases



2) $f(x) = e^x$

$x_0 = 0$, $x > 0$

fixed $n \in \mathbb{N}$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + R_{n+1}(0+x)$$

with $|R_{n+1}(0, x)| \leq \frac{1}{(n+1)!} |x|^{n+1} e^x$ $\leq e$ if $x \leq 1$

Lemma For any $x \in \mathbb{R}$ one has

$$\lim_{n \to \infty} \frac{|x|^n}{n!} = 0$$

This implies that the remainder term is "usually" decreasing with n

$$\boxed{R_{n+1}(0, x) = \frac{1}{(n+1)!} |x|^{n+1} f^{(n+1)}(c)}$$

Proof Consider $x > 0$ (or add absolute value) and let $n_0 \in \mathbb{N}$ such that

$$x < \frac{n_0}{2} \iff \frac{x}{n_0} < \frac{1}{2}$$

Then $$\frac{x^n}{n!} = \underbrace{\frac{x}{1}\frac{x}{2}\frac{x}{3}\cdots}_{n_0 - 1 \text{ term}} \underbrace{\frac{x}{n_0}\frac{x}{(n_0+1)}\cdots\frac{x}{\cdots}\frac{x}{n}}_{n - n_0 + 1 \text{ term}}$$ for $n > n_0$

$$\leq \underbrace{\frac{x^{n_0-1}}{(n_0-1)!}}_{\substack{\text{indep. of } n \\ \text{(decreasing)}}} \left(\frac{1}{2}\right)^{n-n_0+1}$$ This goes to 0 as $n \to \infty$

$\left(\frac{1}{2}\right)^{n-n_0+1} \xrightarrow{n \to \infty} 0$

#

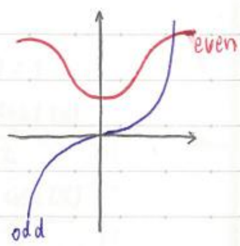
Odd vs. Even functions

Remember:

f is even if $f(-x) = f(x)$

f is odd if $f(-x) = -f(x)$

$\iff -f(-x) = f(x)$



Taylor expansion for odd/even functions.

- Recall that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable,
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Question : If f is even, what about f' ?

Lemma Let f be differentiable.
If f is even, f' is odd, and
if f is odd, f' is even.

Proof Suppose f odd. One has

f is odd
$$\frac{f(x+h) - f(x)}{h} \xrightarrow{h \to 0} f'(x)$$
$$= \frac{-f(-(x+h)) - (-f(-x))}{h}$$ f' is even
$$= \frac{f(-x-h) - f(-x)}{h} \xrightarrow{(-)h \to 0} f'(-x)$$

If f is even,

f is even
$$\frac{f(x+h) - f(x)}{h} \xrightarrow{h \to 0} f'(x)$$ f' is odd
$$- \frac{f(-x-h) - f(-x)}{-h} \xrightarrow{h \to 0} -f'(-x)$$

Remark
If f is odd, $f(0) = 0$
(since $f(-x) = -f(x)$, take $x = 0$)

$\boxed{\text{Thm}}$ Let f be sufficiently differentiable, and consider $P_n(0, x)$ its Talor polynomial at 0
$$\parallel$$
$$\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) (x)^j$$
If f is even, then $f^{(j)}(0) = 0$ whenever j is odd.
If f is odd, then $f^{(j)}(0) = 0$ whenever j is even.

Proof
If f is even, f' is odd, f'' is even, f''' is odd
... $f^{(j)}$ is odd whenever j is odd.
$\Rightarrow$ $f^{(j)}(0) = 0$ (by the remark)
If f is odd, $f^{(j)}$ is odd whenever j is even.
$\Rightarrow$ $f^{(j)} = 0$ #

$$f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$$
Question : Can we take $n \to \infty$?

motivation for next chapter !

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XI.) Series

XI.1) Convergent series.

indefinite sums

Consider $a_1, a_2, a_3, a_4, \dots$ be a sequence of real numbers.

When is $\sum_{j=1}^{\infty} a_j$ finite? or convergent?

For example, consider $a_j = \begin{cases} 1 \text{ if } j \text{ is even} \\ -1 \text{ if } j \text{ is odd} \end{cases}$

What is $-1+1-1+1-1+1-1+1\cdots$

$\underbrace{-1+1}_{0}\underbrace{-1+1}_{0}\underbrace{-1+1}_{0}\dots = 0$

$-1\underbrace{+1-1}_{0}\underbrace{+1-1}_{0}\underbrace{+1-1}_{0}\dots = -1$ etc.

For any sequence $a_1, a_2, a_3, \dots =: (a_j)_{j=1}^{\infty}$

we set $S_n = \sum_{j=1}^{n} a_j = a_1+a_2+a_3+\dots+a_n$

partial sum

Def The series $a_1+a_2+a_3+\dots$ is convergent if $\lim_{n\to\infty} S_n$ exists.

If the limit of S_n does not exist when $n\to\infty$, we say that the series is not convergent, or that it is divergent.

If the sequence S_n is convergent, then we write $\lim_{n\to\infty} S_n = \sum_{j=1}^{\infty} a_j$.

Examples

1.) Consider $a_j = \frac{1}{2^{j-1}}$

The series is $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$

(called the geometric series)

In this case, $S_n = a_1+a_2+a_3+\dots+a_n$

$= \sum_{j=1}^{n} \frac{1}{2^{j-1}} = \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{j-1} = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k$

$= \frac{1-\left(\frac{1}{2}\right)^n}{1/2} = 2\left(1-\left(\frac{1}{2}\right)^n\right) \xrightarrow{n\to\infty} 2$

It means that the partial sum S_n converge to 2, which means that the series is convergent. $\Rightarrow \sum_{j=1}^{\infty} a_j = 2$

2) $a_j = \begin{cases} 1 \text{ if } j \text{ is even} \\ -1 \text{ if } j \text{ is odd} \end{cases}$

Then $S_n = \begin{cases} -1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$

The partial sums are not converging, so the series does not converge.

3) $a_j = \frac{1}{j}$

$S_n = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>2\cdot\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>4\cdot\frac{1}{8}=\frac{1}{2}}+\underbrace{\dots+\frac{1}{16}}_{>8\cdot\frac{1}{16}=\frac{1}{2}}+\dots+\frac{1}{n}$

$1+\frac{1}{2}$ ... $\sim \ln(n)$

$\xrightarrow{n\to\infty} \infty$

The series does not converge

$\sum_{j=0}^{n} a^j = \frac{1-a^{n+1}}{1-a}$

$\sum a_j x^j$

Reminder:

- Taylor's formula for even / odd functions
- Sequence of number $\overset{\text{ordered}}{(a_j)_{j=1}^{\infty}} \neq \overset{\text{not ordered}}{\{a_j\}_{j=1}^{\infty}}$
  $\neq$ series of number $a_1 + a_2 + a_3 + \dots$
- Partial sums and convergent series

Lemma Let $(a_j)_{j=1}^{\infty}$, $(b_j)_{j=1}^{\infty}$ be sequence of real numbers with convergent associated series.

Then

1.) $\sum_{j=1}^{\infty} (\lambda a_j) = \lambda \sum_j a_j \Rightarrow$ This series is convergent

2.) $\sum_j (a_j + b_j) = \sum_j a_j + \sum_j b_j$

3.) $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} b_k\right)$ this is partial sum converging to $\left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{k=1}^{\infty} b_k\right)$

4.) $\sum_{j=1}^{n} a_j b_j \neq \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right)$ in general

XI.2) Series with positive terms only

Remark When all $a_j \geq 0$, the sequence of partial sum $S_n = \sum_{j=1}^{n} a_j$ is an increasing sequence.

$\boxed{\text{Thm}}$ (monotone convergence theorem)

An increasing sequence $(S_n)_{n=1}^{\infty}$ of real numbers which is upper bounded is convergent

Corollary A series of positive numbers is convergent if the sequence of partial sums is upper bounded.

Examples

$$a_j = \frac{1}{j^2}$$

We consider the series

$$1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{\leq 2 \cdot \frac{1}{2^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}_{\leq 4 \cdot \frac{1}{4^2}} + \underbrace{\frac{1}{8^2} + \frac{1}{9^2} + \dots}_{\leq 8 \cdot \frac{1}{8^2}}$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{j=0}^{\infty} \frac{1}{2^j} = 2$$

$\Rightarrow \sum_j \frac{1}{j^2} < 2$ which implies that the series is convergent, in fact $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$ Basel Problem

Corollary (Comparison lemma)

Let $(a_j)_j$, $(b_j)_j$ be sequences of positive numbers, and assume that the series $\sum_{j=1}^{\infty} b_j$ is convergent. If there exists $d > 0$ such that $a_j \leq d b_j$ $\forall j$, then the series $\sum a_j$ is convergent, and $\sum_{j=1}^{\infty} a_j \leq d \sum_{j=1}^{\infty} b_j$

XI.3) Absolute Convergence

In this section, we do not assume $a_j \geq 0$.

Def: A series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent if the series $\sum |a_j|$ is convergent.

$\boxed{\text{Thm}}$ If a series is absolutely convergent, then it is convergent.

⚠ The converse is not true in general

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Counter example

The series $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$ is converging, but

$$\sum_{j=1}^{\infty} \left| (-1)^j \frac{1}{j} \right| = \sum_{j=1}^{\infty} \frac{1}{j} = \infty$$

Proof of the thm

Consider the sequence $(a_j)_{j=1}^{\infty}$, and suppose that $\sum_{j=1}^{\infty} |a_j|$ is convergent. Set $b_j = a_j$ if $a_j \geq 0$ and $b_j = 0$ if $a_j < 0$. Set $c_j = -a_j$ if $a_j < 0$, and $c_j = 0$ if $a_j \geq 0$.

Observe that $a_j = b_j - c_j$ and $|a_j| = b_j + c_j$

Since $\sum_{j=1}^{\infty} |a_j| < \infty$, then $\sum_{j=1}^{\infty} (b_j + c_j) < \infty \Leftrightarrow \sum_j b_j < \infty$ and $\sum_j c_j < \infty$

Now, consider $S_n = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} c_j$

Since $\lim_{n\to\infty} \sum_{j=1}^{n} b_j < \infty$ and $\lim_{n\to\infty} \sum_{j=1}^{n} c_j < \infty$

then $\lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} b_j - \lim_{n\to\infty} \sum_{j=1}^{n} c_j$

$= \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} c_j$ which exists.

#

⚠ When a series is not absolutely convergent, one has to manipulate it carefully.

$\boxed{\text{Thm}}$ (criterion for alternating series)

Let $(a_j)_{n=1}^{\infty}$ be a sequence of numbers such that

1.) $\lim_{j\to\infty} a_j = 0$

2.) $a_j a_{j+1} \leq 0 \quad \forall j$

3.) $|a_{j+1}| \leq |a_j| \quad \forall j$

Then the series $\sum_{j=1}^{\infty} a_j$ is converging.

Remark This thm implies that the series $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \dots$ is converging.

XI.4) Power Series

Def A power series is an expression of the form $\sum_{j=0}^{\infty} a_j x^j$ with $a_j \in \mathbb{R}$ and x a variable.

Example

The power series $\sum_{j=0}^{\infty} \frac{1}{j!} x^j$ corresponds to the exponential function.

$\boxed{\text{Thm}}$ Assume that there exists $r \geq 0$ such that $\sum_{j=0}^{\infty} |a_j| r^j$ is convergent. Then for any $x \in \mathbb{R}$ with $|x| \leq r$, the series $\sum_{j=0}^{\infty} a_j x^j$ is absolutely convergent.

Proof Since $|a_j x^j| = |a_j| |x|^j \leq |a_j| r^j$ and since the series $\sum_{j=0}^{\infty} |a_j| r^j$ is convergent, we got by the comparison lemma that $\sum_{j=0}^{\infty} |a_j x^j|$ is convergent, which means that $\sum_{j=1}^{\infty} a_j x^j$ is absolutely convergent.

#

Def: The least upper bound on $r \geq 0$ such that $\sum_{j=0}^{\infty} |a_j| r^j < \infty$ is called

"the radius of convergence"

of the power series $\sum a_j x^j$

Reminder

- Power series $f(x) := \sum_{j=0}^{\infty} a_j x^j$
  example $a_j = \frac{1}{j!} \rightsquigarrow f(x) = e^x$
- $\boxed{\text{Thm}}$:
  If $\sum_{j=0}^{\infty} |a_j| r^j < \infty$ for some $r \geq 0$, then $\sum_{j=0}^{\infty} a_j x^j$ converges for all $|x| < r$
- Def : The least upper bound on $r \geq 0$ such that $\sum_j |a_j| r^j < \infty$ is called the "radius of convergence"
- * can be 0, finite or ∞

Example

1.) $a_j = 1$, $f(x) = \sum_{j=0}^{\infty} x^j$
For $|x| < 1$ one has $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$
but for $x = 1$, one has $\sum_{j=0}^{\infty} 1^j = \infty$
$\Rightarrow$ The radius of convergence is 1

2.) $a_j = \frac{1}{j!}$, we consider $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$
Let us prove that the radius of convergence is ∞.
We consider $\sum_{j=0}^{\infty} \frac{1}{j!} r^j$ and use the ratio test to show that this series is convergent.
Ratio test:

$$\frac{\frac{1}{(j+1)!} r^{j+1}}{\frac{1}{j!} r^j} = \frac{j!}{(j+1)!} \frac{r^{j+1}}{r^j} = \frac{1}{j+1} r \leq C < 1 \quad \forall j \geq N$$

For a fixed $r \geq 0$. If we choose $N \geq r$, then $\frac{1}{j+1} r \leq \frac{1}{N+1} r \overset{\text{fix } c}{<} 1$

Conclusion : For any fixed r, we can find $N \geq r$ and $C := \frac{r}{r+1} < 1$ such that $\frac{1}{j+1} r \leq C < 1$ $\forall j > N$
$\Rightarrow$ The series $\sum_{j=0}^{\infty} \frac{1}{j!} r^j$ is convergent
Since r is arbitrary, the radius of convergence is ∞.

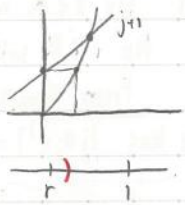
Question : Can we have $a_{j+1} > a_j$ and still a non-zero radius of convergence?

3) Consider $a_j = \ln(j)$, it means $f(x) = \sum_{j=1}^{\infty} \ln(j) x^j$
Let us consider $0 \leq r < 1$, and show that the series $\sum_{j=1}^{\infty} \ln(j) r^j < \infty$
By the ratio test we have

$$\frac{\ln(j+1)\, r^{j+1}}{\ln(j)\, r^j} = \underbrace{\frac{\ln(j+1)}{\ln(j)}}_{>1} r \overset{?}{\leq} C < 1 \quad \text{for } j \text{ large enough.}$$

Observe that $\lim_{j\to\infty} \frac{\ln(j+1)}{\ln(j)} = 1$

We show that for any $\alpha > 1$ (1 ← α)
One has $\frac{\ln(j+1)}{\ln(j)} \leq \alpha$ for j large enough.
Indeed $\ln(j+1) \leq \alpha \ln(j) = \ln(j^\alpha)$
$\Leftrightarrow e^{\ln(j+1)} \leq e^{\ln(j^\alpha)}$
$\Leftrightarrow j+1 \leq j^\alpha$ which is true for j large enough.
Conclusion: $\lim_{j\to\infty} \frac{\ln(j+1)}{\ln(j)} = 1$
$\Rightarrow$ for any $r < 1$, we can find N such that
$\underbrace{\frac{\ln(j+1)}{\ln(j)}}_{\text{very close to 1}} r \leq C < 1$ for $j \geq N$



$\Rightarrow \sum_{j=1}^{\infty} \ln(j) r^j$ is convergent for any $r < 1$
$\Rightarrow$ The radius of convergence in 1

Remark

The function $x \mapsto e^x$
$x \mapsto \sin(x)$
$x \mapsto \cos(x)$
can be expressed as power series with radius of convergence $= \infty$.

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<u>Question</u>: Suppose that the function
$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$
has a radius of convergence equals to $R>0$
Can we differentiate f, and is $f' = \sum_{j=1}^{\infty} a_j j x^{j-1}$?

XI.5) Differentiation and Integration of power series.

<u>Proposition</u>: Consider $\sum_{j=0}^{\infty} a_j x^j$ and suppose that its radius of convergence is $R>0$. let $0<r<R$

(Which means that $\sum_{j=0}^{\infty} a_j x^j$ is absolutely convergent for any $|x| \leq r$)

Then, the series $\sum_{j=1}^{\infty} j a_j x^{j-1}$ is also absolutely convergent for $|x|<r$.

<u>Proof</u> Fix $x \in \mathbb{R}$ with $x \neq 0$ and $|x|<r$ and let us fix $c \in \mathbb{R}$ with $|x|<c<r$.

From Homework 8, Ex. 3, one has $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

One has $|j a_j x^j| = |a_j (j^{1/j} x)^j|$
$$= |a_j| |j^{\frac{1}{j}} x|^j \leq |a_j| c^j$$
for f large enough.

$\Rightarrow |j a_j x^{j-1}| = \frac{1}{|x|} |a_j (j^{\frac{1}{j}} x)^j| \leq \frac{1}{|x|} |a_j| c^j$

Since the series $\sum_{j=0}^{\infty} a_j c^j$ is convergent, we get from the comparison lemma that $\sum_{j=1}^{\infty} j a_j x^{j-1}$ is absolutely convergent.

#.

$\boxed{\text{Theorem}}$ let $\sum_{j=0}^{\infty} a_j x^j$ be a power series with a radius of convergence $R>0$. Then the function $f: (-R,R) \ni x \mapsto \sum_{j=0}^{\infty} a_j x^j$ is differentiable, and $f'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}$.

Also, $\int f(x) dx = \sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^{j+1} + \text{const}.$

<u>Remark</u> By applying the thm to f' (which is a power series with radius of convergence $R>0$), we get that f' is differentiable and is given by $\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$. This implies that f is C^{∞}.

<u>Part of the Proof</u>

We don't prove the first, but deduce the second part from the first statement.

Indeed, the series $\sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^j$ is absolutely convergent for $|x|<R$ because $\frac{1}{j+1}|a_j| \leq |a_j|$ and the use of the comparison lemma, and thus the function
$$F(x) := \sum_{j=0}^{\infty} \frac{1}{j+1} a_j x^{j+1}$$
is differentiable (first part of the statement) and $F'(x) = \sum_{j=0}^{\infty} \frac{1}{j+1} a_j (j+1) x^j = f(x)$

#.

<u>Remark</u>: We have proved the initial manipulation on $\sum_{j=0}^{\infty} \frac{1}{j!} x^j$

<u>Remark</u>: Power series are very important for complex analysis.

The End ◡̈.