... the Sequel

Unabashedly excerpted from Davis & Hersh's
The Mathematical Experience

Some Questions

How would you answer these questions?

Is new mathematics created or discovered?

In what sense does a mathematical object exist?

π

What about the 2 trillionth digit of π? Does it already exist, even though we don't know what it is? Or is it created when Kanada (or his machine) eventually gets there?

Did the natural numbers have an order type before Cantor gave it a name?

Did quartic equations have solutions before Ferrari developed his formula?

... if you find yourself answering yes to these questions, then you are a Platonist!

Platonism

According to Platonism, mathematical objects are real. Their existence is an objective fact, quite independent of our knowledge of them. Infinite sets, uncountable infinite sets, space filling curves – all the members of the mathematical zoo are definite objects, with definite properties, some known, many unknown. These objects are, of course, not physical or material. They exist outside the space and time of physical existence. They are immutable – they were not created, and they will not change or disappear. Any meaningful question about a mathematical object has a definite answer, whether we are able to determine it or not. According to Platonism, mathematicians are empirical scientists like geologists; they cannot invent anything, because it is already there. All they can do is discover.

Formalism

But there is another view :-

According to Formalism, on the other hand, there are *no* mathematical objects! Mathematics just consists of axioms, definitions and theorems – in other words formulas. In an extreme view, there are rules by which one derives one formula from another, but the formulas are not *about* anything; they are just strings of symbols. Of course the formalist knows that mathematical formulas are sometimes applied to physical problems. When a formula is given a physical interpretation, it acquires a meaning, and may be true or false. But this truth or falsity has to do with the particular physical interpretation. As a purely mathematical formula, it has no meaning and no truth value.

The Continuum Hypothesis

The reactions to the proofs given by Gödel and Cohen that Cantor's Continuum Hypothesis can neither be proved nor disproved from the axioms of set theory (including the Axiom of Choice) serves to distinguish these two fundamental dogmas.

To a Platonist, this means that our axioms are incomplete as a description of the set of real numbers. They are not strong enough to tell us the whole truth. The continuum hypothesis is either true or false, but we don't understand the set of real numbers well enough to find the answer.

The Continuum Hypothesis

To a formalist, on the other hand, the Platonist's interpretation makes no sense, because there *is* no real number system, except as we choose to create it by laying down axioms to describe it. Of course we are free to change this axiom system if we desire to do so. Such a change can be for convenience or usefulness or some other criterion we choose to introduce; it cannot be a matter of a better correspondence with reality, because there is no reality there!

Constructivism

Formalists and Platonists are at opposite sides on the question of existence and reality; but they have no quarrel with each other on what principles of reasoning should be permissible in mathematical practice. Opposed to both of them are the constructivists. The constructivists regard as genuine mathematics only what can be obtained by a finite construction. Existence proofs are non-constructive and so are discarded. The logical law of the excluded middle is not to be used. The set of real numbers, or any other infinite set can not be obtained by a finite construction. Consequently, the constructivist regards Cantor's hypothesis as meaningless talk. Any answer at all would be a sheer waste of breath.

The Working Mathematician

According to J.D. Monk, the mathematical world is populated with 65% Platonists, 30% formalists and 5% constructivists. This is a bit misleading. Lets leave aside the constructivists, who are a rare breed, whose status in the mathematical world sometimes seems to be that of tolerated heretics surrounded by orthodox members of an established church. The rest of us could probably be best described by saying that we are Platonists on weekdays and formalists on Sunday. That is, when doing mathematics we are convinced that we are dealing with an objective reality whose properties we are attempting to determine. But then, when challenged to give a philosophical account of this reality, we find it easiest to pretend that we do not believe in it at all.

The Working Mathematician

On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say, "Mathematics is just a combination of meaningless symbols," and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but it is very convenient. That is Bourbaki's attitude toward foundations.

- J.A. Dieudonné

(a leading member of Bourbaki)

The Working Mathematician

To the average mathematician who merely wants to know his work is accurately based, the most appealing choice is to avoid difficulties by means of Hilbert's program. Here one regards mathematics as a formal game and one is only concerned with the question of consistency ... The Realist [i.e., Platonist] *position is probably the one which most mathematicians would prefer to take. It is not until he becomes aware of some of the difficulties in set theory that he would even begin to question it. If these difficulties particularly upset him, he will rush to the shelter of Formalism, while his normal position will be somewhere between the two, trying to enjoy the best of two worlds.*

\- *P.J. Cohen*

The Euclid Myth

The textbook picture of the philosophy of mathematics is a strangely fragmentary one. The reader gets the impression that the whole subject appeared for the first time in the late 19th century, in response to contradictions in Cantor's set theory. At that time there was talk of a "crisis in the foundations." To repair the foundations, three schools appeared on the scene, and spent some thirty or forty years quarreling with each other. It turned out that none of the three could really do much about the foundations, and the story ends in mid-air some sixty years ago, with Whitehead and Russell having abandoned logicism, Hilbert's formalism defeated by Gödel's theorem, and Brouwer left to preach constructivism in Amsterdam, disregarded by all the rest of the mathematical world.

The Euclid Myth

This impression that the philosophy of mathematics was an active field for only forty years, awakened by the contradictions in set theory, and after a while went back to sleep, is only illusionary. In reality there has always been a philosophical background, more or less explicit, to mathematical thinking. The "foundationist" period differed because it was one in which leading *mathematicians* were overtly concerned with philosophical issues, and engaged in public controversy about them. The crisis was a manifestation of a long-standing discrepancy between the traditional ideal of mathematics, which we will call the Euclid myth, and the actual practice of mathematical activity at any particular time.

The Euclid Myth

Bishop Berkeley recognized this discrepancy in 1734, in his book ***The Analyst***, which was subtitled

A Discourse Addressed to an Infidel Mathematician, Wherein it is Examined Whether the Object, Principles and Inferences of the Modern Analysis are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith. "First cast out the beam of thine own Eye; and then shall thou see clearly to cast the mote out of thy brother's Eye".

(The infidel was Edmund Halley.)

Berkeley exposed the obscurities and inconsistencies of differential calculus, as explained in his time by Newton, Leibnitz, and their followers. That is to say, he showed how far the calculus fell short of fitting the idea of mathematics according to the Euclid myth.

The Euclid Myth

What is the Euclid myth? It is the belief that the books of Euclid contain truths about the universe which are clear and indubitable. Starting from self-evident truths, and proceeding by rigorous proof, Euclid arrives at knowledge which is certain, objective, and eternal. Even now, it seems that most educated people believe in the Euclid myth. Up to the middle or late 19th century, the myth was unchallenged. Everyone believed it. It has been the major support for metaphysical philosophy, that is, for philosophy which sought to establish some a priori certainty about the nature of the universe.

For the Greeks, mathematics meant geometry, and the philosophy of mathematics in Plato and Aristotle is the philosophy of geometry!

Foundations, Found and Lost

In the 19th century, several disasters took place. One disaster was the discovery of non-Euclidean geometries, which showed that there was more than one thinkable geometry. A greater disaster was the development of analysis so that it overtook geometrical intuition, as in the discovery of space-filling curves and continuous nowhere-differentiable curves. These stunning surprises exposed the vulnerability of the one solid foundation – geometrical intuition – on which mathematics had been thought to rest. The loss of certainty in geometry was philosophically intolerable, because it implied the loss of all certainty in human knowledge. Geometry had served, from the time of Plato, as the supreme exemplar of the possibility of certainty in human knowledge.

Foundations, Found and Lost

Led by Dedekind and Weierstrass, the mathematicians of the 19^{th} century met the challenge by turning from geometry to arithmetic as the foundation for mathematics. To do this, it was necessary to give a construction of the real number system, to show how it could be built up from the integers. Three different methods of doing this were proposed by Dedekind, Cantor and Weierstrass. In all three methods one had to use some infinite set of rational numbers in order to define or construct a real number. Thus, in the effort to reduce analysis and geometry to arithmetic, one was led to introduce infinite sets into the foundations of mathematics.

Foundations, Found and Lost

The theory of sets was developed by Cantor as a new and fundamental branch of mathematics in its own right. It seemed that the idea of a set – an arbitrary collection of distinct objects – was so simple and fundamental that it could be the building block out of which all of mathematics could be constructed. Even arithmetic could be changed from a fundamental to a secondary structure, for Frege showed how the natural numbers could be constructed from nothing – i.e., from the empty set – by using operations of set theory.

Foundations, Found and Lost

Set theory at first seemed to be almost the same as logic. The set-theory relation of inclusion – A is a subset of B – can always be rewritten as the logical relation of implication – if A then B. So it seemed possible the set-theory-logic could serve as the foundation for all of mathematics. “Logic”, as understood in this context, refers to the fundamental laws of reason, the bedrock of the universe. The law of contradiction and rules of implication are regarded as objective and indubitable. To show that all of mathematics is just an elaboration of the laws of logic would have been to justify Platonism, by passing on to the rest of mathematics the indubitability of logic itself. This was the “logistic program” pursued by Russell and Whitehead in their ***Principia Mathematica.***

Foundations, Found and Lost

Since all mathematics can be reduced to set theory, all one need consider is the foundation of set theory. However, it was Russell himself who discovered that the seemingly transparent notion of set contained unexpected traps.

The controversies of the late nineteenth and early twentieth century came about because of the discovery of contradictions in set theory. A special word - “antinomies” - was used as a euphemism for contradictions of this type.

The paradoxes arise from the belief that any reasonable predicate – any verbal description that seemed to make sense – could be used to define a set, the set of things that shared the stated property.

Russell's Paradox

Definition: A set is said to be ***ordinary*** if it is not a member of itself.

Ex: A = {1,2,3,A} is not an ordinary set.
{abstract ideas} is not an ordinary set.

Now consider the set P = {all ordinary sets}.

Is P ordinary?

If P were an ordinary set then $P \in P$, so P would not be ordinary.

$\rightarrow\leftarrow$

So, P is not ordinary ....

but then $P \notin P$, so P is an ordinary set. $\rightarrow\leftarrow$!!!!!!!

The only way out of this paradox ....

P is not a set!

Foundations, Found and Lost

The Russell paradox and the other antinomies showed that intuitive logic, far from being more secure than classical mathematics, was actually much riskier, for it could lead to contradictions in a way that never happens in arithmetic or geometry.

This was the "crisis in foundations," the central issue in the famous controversies of the first quarter of this century. Three principle remedies were proposed.

Logicism

The program of “logicism,” the school of Frege and Russell, was to find a reformulation of set theory which could avoid the Russell paradox and thereby save the Frege-Russell-Whitehead project of establishing mathematics upon logic as a foundation.

The work on this program played a major role in the development of logic. But it was a failure in terms of its original intention. By the time set theory had been patched up to exclude the paradoxes, it was a complicated structure which one could hardly identify with logic in the philosophical sense of “the rules for correct reasoning.” So it became untenable to argue that mathematics is nothing but logic – that mathematics is one vast tautology.

Constructivism

After the logicist, the next major school was the constructivist. This originated with the Dutch topologist L.E.J. Brouwer about 1908. Brower's position was that the natural numbers are given to us by a fundamental intuition, which is the starting point for all mathematics. He demanded that all mathematics should be based constructively on the natural numbers. That is to say, mathematical objects may not be considered meaningful, may no be said to exist, unless they are given by a construction, in finitely many steps, starting from the natural numbers. It is not sufficient to show that the assumption of nonexistence would lead to a contradiction.

For the constructivists, many of the standard proofs in classical mathematics are invalid. In some cases they are able to supply a constructive proof. But in other cases they show that a constructive proof is impossible: theorems which are considered to be well-established in classical mathematics are actually declared to be false for constructivist mathematics.

Formalism

Although many prominent mathematicians had expressed misgivings and disagreements with nonconstrutive methods and free use of infinite sets, Brouwer's call for restructuring analysis from the ground up seemed to most mathematicians unreasonable, indeed fanatical!

Hilbert was particularly alarmed. *"What Weyl and Brouwer do comes to the same thing as to follow in the footsteps of Kronecker! They seek to save mathematics by throwing overboard all that which is troublesome ... They would chop up and mangle the science. If we would follow such a reform as the one they suggest, we would run the risk of losing a great part of our most valuable treasure!"* (*Hilbert*, C. Reid, p.155)

Formalism

Hilbert undertook to defend mathematics from Brouwer's critique by giving a *mathematical proof* of the consistency of classical mathematics. Furthermore, he proposed to do so by arguments of a purely finitistic, combinatorial type – arguments that Brouwer himself could not reject.

This program involved three steps.

(1) Introduce a formal language and formal rules of inference, sufficient so that every "correct proof" of a classical theorem could be represented by a formal derivation, starting from axioms, with each step mechanically checkable. This had already been accomplished in large part by Frege, Russell and Whitehead.

(2) Develop a theory of the combinatorial properties of this formal language, regarded as a finite set of symbols subject to permutation and rearrangements as provided by the rules of inference, now regarded as rules for transforming formulas. This theory was called "meta-mathematics."

Formalism

(3) Prove by purely finite arguments that a contradiction, for example $1 = 0$, cannot be derived within this system.

In this way, mathematics would be given a secure foundation – in the sense of a guarantee of consistency.

The formalist foundation of Hilbert, like the logicist foundation, offered certainty and reliability at a price. As the logicist interpretation tried to make mathematics safe by turning it into a tautology, the formalist interpretation tried to make it safe by turning it into a meaningless game. The "proof-theoretic program" comes into action only after mathematics has been coded in a formal language and its proofs written in a way checkable by machine. As to the *meaning* of the symbols, that becomes something extra-mathematical.

Formalism

Hilbert's writings and conversation display full conviction that mathematical problems are questions about real objects, and having meaningful answers. If he was prepared to advocate a formalist interpretation of mathematics, this was the price he considered necessary for the sake of obtaining certainty.

The goal of my theory is to establish once and for all the certitude of mathematical methods ... The present state of affairs where we run up against the paradoxes is intolerable. Just think, the definitions and deductive methods which everyone learns, teaches and uses in mathematics, the paragon of truth and certitude, lead to absurdities! If mathematical thinking is defective, where are we to find truth and certitude?

- D. Hilbert

RIP Formalism

As it happened, certainty was not to be had, even at this price. In 1930 Gödel's incompleteness theorems showed that the Hilbert program was unattainable – that any consistent formal system strong enough to contain elementary arithmetic would be unable to prove its own consistency. The search for secure foundations has never recovered from this defeat.