Solving Polynomial Equations

Introduction

We will spend the next few lectures looking at the history of the solutions of polynomial equations. We will organize this examination by the degree of the equations, looking at linear, quadratic, cubic, etc. equations in turn. This organization however is itself ahistorical. The concept of organizing polynomials by degree was not generally accepted until the beginning of the 17th Century. We also note that except in India, coefficients were always positive and so were the roots (solutions) that were sought. Our free-wheeling ways with negative coefficients and non-positive solutions is of relatively recent origin.

Linear Equations

Solutions of linear equations can be found in the Ahmes Papyrus. In one of the problems, the equation

$$x + \frac{1}{7}x = 19$$

is to be solved. Ahmes' solution goes like this:

Suppose the answer is 7. The LHS is 8, so this is not the correct answer. To get the correct answer, multiply 7 by 19/8.

This is an example of the "method of false position" which amounts to guessing an answer and then modifying the false answer to get the correct one. This method, or one of its many variants, was used for many centuries and can even be found as the recommended method as late as 1884.[1]

Method of Double False

We'll look at a variant of this method applied to the linear equation

$$ax + b = 0.$$

(But note that an equation would never be written this way and if a and b are positive, the root is negative so would have been ignored and the equation would be said to not have a solution.)

Let g_1 and g_2 be two guesses for x and f_1 and f_2 the false values computed from them. That is,

$$ag_1 + b = f_1 \qquad \text{and} \quad ag_2 + b = f_2.$$

From these equations we obtain $a(g_1 - g_2) = f_1 - f_2$ and $b(g_2 - g_1) = f_1g_2 - f_2g_1$, so we can get a rule for the solution :

$$x = -\frac{b}{a} = \frac{f_1 g_2 - f_2 g_1}{f_1 - f_2}.$$

Robert Recorde

- *Ground of Artes* (c. 1542)

Gesse at this woorke as happe doth leade.
By chaunce to truthe you may procede.
And firste woorke by the question,
Although no truthe therein be done.
Such falsehode is so good a grounde,
That truth by it will soone be founde.
From many bate to many mo,
From to fewe take to fewe also.
With to much ioyne to fewe againe,
To to fewe adde to manye plaine.
In crossewaies multiplye contrary kinde,
All truthe by falsehode for to fynde.[2]

Political Correctness[3]

"The name "Rule of False" was thought to demand an apology in a science whose function it is to find the truth, and various writers made an effort to give it. Thus Humphrey Baker (1568) says:

> The Rule of falsehoode is so named not for that it teacheth anye deceyte or falsehoode, but that by fayned numbers taken at all aduentures, it teacheth to finde out the true number that is demaunded, and this of all the vulgar Rules which are in practise) is ye most excellence.

Besides the "Rule of False" the method was also called the "Rule of Increase and Diminution," from the fact that the error is sometimes positive and sometimes negative. Indeed, in the 16th Century the symbols + and – were much more frequently used in this connection than as symbols of operation."

Quadratic Equations

In Smith's *History of Mathematics* (1925)[4] we find:
The first known solution of a quadratic equation appears in the ***Berlin Papyrus*** dated to the Middle Kingdom of Egypt (c. 2160 – 1700 B.C.). In modern terms, the problem there asks for the solution of the equations $x^2 + y^2 = 100$, and $y = \frac{3}{4} x$. The solution given is an application of the method of false position.

However, many more quadratic problems can be found in the Babylonian clay tablets which also date to this period (and some are even older). These tablets show a much higher level of algebraic manipulation and understanding than can be found in the Egyptian papyri. An appreciation of what was contained in these tablets did not occur until after 1935.

Babylonians

Since the first half of the 19th Century, archeologists working in Mesopotamia have unearthed some half-million inscribed clay tablets. Many peoples and civilizations, among them the Sumerians, Akkadians, Chaldeans, and Assyrians have inhabited the area at one time or another. It is common practice to use the term *Babylonian* to refer to all of them, even though the city of Babylon was only prominent for a short period.

One of several types of quadratic problems that can be found in the tablets are those of the form:

Given $x + y = a$ and $xy = c$ find x and y.

Geometrically, such problems are relating the perimeters of rectangles to their areas, and this gives us a hint as to how they might have obtained their solutions.

Babylonians

The problems in the clay tablets are, as with the Egyptian papyri, always expressed with specific numbers, and the solutions are obtained by following certain algorithms. Examining the algorithms gives a reasonable explanation of a geometrical approach to their solutions.

Thus, for the problems of the form $x + y = b$ and $xy = c$, the algorithm is:

1. Take the square of ½ b.
2. Subtract c from this value.
3. Take the square root of this value.
4. Add it to ½ b to get x and subtract it from ½ b to get y.

In modern terms, this algorithm is equivalent to what we call the quadratic formula as taught in high school. We derive the formula algebraically, but how did the Babylonians come up with it?

Babylonians

No one really knows how the Babylonians arrived at their algorithm, and several suggestions have been made. Here is mine.

Problem: Find a rectangle whose area is c where the sum of the two sides is b.

Solution: By the method of false position.

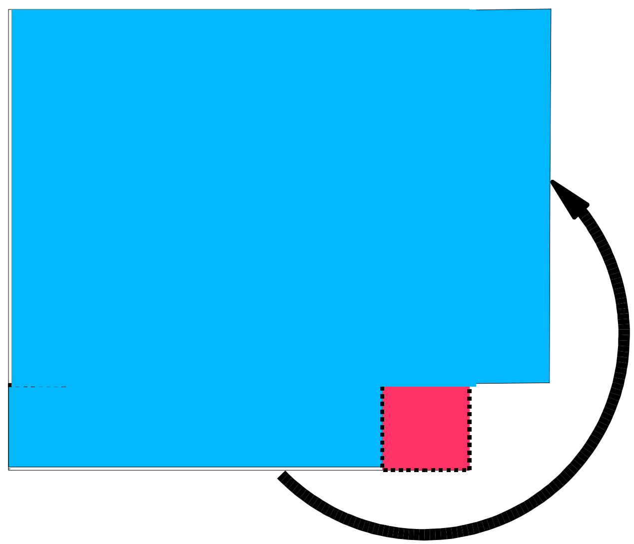
Let's guess that the answer is a square. The square would have side ½b and area the square of this. Step 1: Square ½b.

Since this was a guess, the error made in terms of areas is the difference between this value and c. Step 2: Subtract c.

A geometric square with area equal to this error would have a side equal to the square root of the area. Step 3: Take square root.

We now have to adjust the sides of the square to get a rectangle that has the right area. The procedure for this correction can be seen from examining the following diagram.

Babylonians



Thus, the correct dimensions are obtained by adding the side of the error square to b/2 and subtracting it from b/2. Step 4.

Greek Solutions

The Greek geometers would easily find solutions of quadratic equations geometrically. To solve $x^2 + bx + c = 0$, they would in principle find the intersection of the circle $y = x^2$ with the line $y = bx + c$ by straightedge and compass constructions.

In Euclid's Elements, one finds a considerable amount of discussion on whether or not the solutions are rational. This is interesting because the issue is of no consequence in the geometric constructions. It implies that the Greeks were concerned with numerical solutions, since if the solutions were irrational they had no way to represent them other than as lengths of line segments.

Hindu Solutions

There is evidence that the Hindu's may have been solving quadratic equations as early as 200 B.C., but we have no record of the methods used.

Brahmagupta (c. 628) gives the equation $x^2 – 10x = -9$ and its solution, showing that algorithms for the solutions of quadratics were known by this time.

Śrīdhara (c. 1025) was the first, so far as known, to give the so-called Hindu Rule for quadratics. He is quoted by Bhāskara (c. 1150) as saying:

Multiply both sides of the equation by a number equal to four times the [coefficient of the] square, and add to them a number equal to the square of the original [coefficient of the] unknown quantity. [Then extract the root.]

Hindu Solutions

Thus, for the equation $ax^2 + bx = c$ we would get:

$$ax^2 + bx = c$$
$$4a^2x^2 + 4abx = 4ac$$
$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac$$
$$(2ax + b)^2 = b^2 + 4ac$$
$$2ax + b = \sqrt{b^2 + 4ac}.$$

These Indian mathematicians worked with both positive and negative numbers and had a symbol for zero. They worked effectively with all types of numbers, but usually ignored any negative roots of equations.

Al-Khwārizmī

One of the earliest Islamic algebra texts, written about 825 by Muhammad ibn Mūsā al-Khwārizmī (c. 780-850) was titled

Al-kitāb al-muhtaşar fī hisāb al-jabr wa-l-muqābala

(The condensed book on the calculation of al-Jabr and al-Muqabala)

The term *al-jabr* can be translated as "restoring" and refers to transposing a term from one side to the other of an equation. The word *al-muqabala* is translated as "comparing" and refers to subtracting equal quantities from both sides of an equation. Thus

$3x + 2 = 4 - 2x \Rightarrow 5x + 2 = 4$ is *al-jabr* while

$5x + 2 = 4 \Rightarrow 5x = 2$ is *al-muqabala*.

Our term "algebra" is a corrupted form of the Arabic *al-jabr*.

Al-Khwārizmī

Al-Khwārizmī was interested in writing a practical guide to solving equations, but due to the influence of Greek texts which were being translated into arabic, he includes geometric justifications of his manipulations. The geometry however appears to come from the Babylonians rather than the Greeks.

He classifies the equations he will deal with into six types:

$ax^2 = bx$	$ax^2 = c$	$bx = c$
$ax^2 + bx = c$	$ax^2 + c = bx$	$bx + c = ax^2$

The reason for the six-fold classification is that Islamic mathematicians, unlike the Hindus, did not deal with negative numbers. In their system, coefficients, as well as the roots, had to be positive. The types listed are the only ones with positive solutions. [$ax^2 + bx + c = 0$ would make no sense]

Viète

François Viète (1540 – 1603) was a French lawyer who worked for kings Henri III and Henri IV as a cryptanalyst (a breaker of secret codes). He was so successful at this that his critics denounced him for being in league with the Devil.

Starting in 1591 he found the time to write several treatises that are collectively known as *The Analytic Art*, in which he effectively reformulated the study of algebra by replacing the search for solutions with a detailed study of the structure of equations.

Modern symbolism can be traced back to Viete for he was the first to use letters to represent numbers and manipulate them in accordance with the rules for manipulating numbers. Without this idea, his predecessors were forced to consider problems only with specific numerical coefficients and write out their algorithms in words.

Viète

Viète used vowels to represent unknowns and consonants to represent given constants. His symbolism was not complete, he still used words to indicate powers – A^2 is A *quadratum*, B^3 would be B *cubus*, and C^4 is C *quadrato-quadratum*. He did at times use abbreviations such as A *quad* or C *quad-quad*. His rules for combining powers had to be given verbally.

The equation $x^3 + cx = d$, as we write it today, would have appeared as

A cubus + C plano in A aequetus D solido.

Note that he used the symbol "+" for addition – he also uses "-" for subtraction, but the word "in" for multiplication. There is no symbol for equals (aequetus). The modifiers *plano* and *solido* are there to preserve the law of homogeneity, a Greek concept.

Viète

Viete does not yet have the concept of the general quadratic equation for he gives separate solutions for:

Si A quad. + B2 in A, aequatur Z plano. ($x^2 + 2bx = c$)
Si A quad. - B in A2, aequatur Z plano. ($x^2 – 2bx = c$) and
Si D2 in A – A quad., aequator Z plano. ($2dx – x^2 = c$)

His solutions are, however, algebraic and produce special cases of the familiar quadratic formula. Interestingly, in his solutions he replaces the variable by a sum to obtain the result (today, we call this a *change of variable method*) a purely algebraic operation.