DIFFERENTIAL**EQUATIONS**

Including Linear Algebra Topics
And Computer-Aided Problem-Solving
Using Maxima or SageMath

by Charles Bergeron, Jiří Lebl, and other open-source creators

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Cover photograph description: The shape of the Gateway Arch in Saint Louis is called a *catenary*; this shape is ideal to support its own weight. The formula used in the design is inscribed inside the arch: $y = -127.7\,\text{ft} \cdot \cosh(x/127.7\,\text{ft}) + 757.7\,\text{ft}$. This function involves a hyperbolic function. These functions are important tools to describe and analyze the solutions of several differential equations.
Cover photograph attribution: Taken by Bev Sykes on 14 April 2005, licensed under the Creative Commons (CC BY 2.0) United States License, https://www.flickr.com/photos/11399912@N00/9741415.

Attributions

This work is heavily based upon the open-source book *Notes on Diffy Qs: Differential Equations for Engineers* by Jiří Lebl. That book's home page is `http://www.jirka.org/diffyqs/`.

Most linear algebra content in chapter 4 was adapted from part of the open-source book *Linear Algebra* by Jim Hefferon. That book's home page is `http://joshua.smcvt.edu/linearalgebra/`.

Some content was adapted from the open-source book *Community Calculus: Multivariable, Early Transcendentals* by David Guichard et al. That book's home page is `http://communitycalculus.org/`.

Some content was adapted from the open-source book *Precalculus* by Carl Stitz and Jeff Zeager. That book's home page is `http://www.stitz-zeager.com/`.

Maxima is an open-source computer algebra system. That software's home page is `http://maxima.sourceforge.net/`.

SageMath is an open-source computer algebra system. That software's home page is `http://www.sagemath.org/`.

Further resources

[B] Gregory V. Bard, *Sage for Undergraduates*, American Mathematical Society, 2015. Free online version: `http://www.gregorybard.com/`

[BM] Paul W. Berg and James L. McGregor, *Elementary Partial Differential Equations*, Holden-Day, San Francisco, CA, 1966.

[BD] William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 9th edition, John Wiley & Sons Inc., New York, NY, 2008.

[EP] C.H. Edwards and D.E. Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling*, 4th edition, Prentice Hall, 2008.

[F] Stanley J. Farlow, *An Introduction to Differential Equations and Their Applications*, McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)

[G] David Guichard et al., *Community Calculus: Multivariable, Early Transcendentals*, Open-source publication. 2014. `http://communitycalculus.org`.

[H] Jim Hefferon, *Linear Algebra*, Open-source publication. 2015. `http://joshua.smcvt.edu/linearalgebra/`.

[I] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, Inc., New York, NY, 1956.

[JH] David Joyner and Marshall Hampton, *Introductory Differential Equations using Sage*, Open-source publication. 2009. `https://www.usna.edu/Users/math/wdj/teaching/differential-equations.php`.

[L] Jiří Lebl, *Notes on Diffy Qs: Differential Equations for Engineers*, Open-source publication. 2014. `http://www.jirka.org/diffyqs/`.

[SZ] Carl Stitz and Jeff Zeager, *Precalculus*, Open-source publication. 2013. `http://www.stitz-zeager.com`.

[T] William F. Trench, *Elementary Differential Equations with Boundary Value Problems*. Books and Monographs. Book 9. 2013. `http://digitalcommons.trinity.edu/mono/9`

Contents

Preface

Dr. Bergeron's notes about this book

This is a text for a first undergraduate Differential Equations course. It does not assume previous coverage of Linear Algebra. This book includes applications from many disciplines, and integrates the use of the open-source computer algebra system Maxima or SageMath throughout. This free online book should be usable as a stand-alone textbook or as a supplementary resource.

I used Dr. Lebl's book for my first offering of MAT 235 at Albany College of Pharmacy and Health Sciences in Fall 2014. I was immediately drawn to the book's simple yet powerful, informal yet precise, design. This book heavily adapts his.

In creating this book, I wanted to weaken the influence of applications from physics and engineering, and strengthen the presence of applications from include biology, chemistry, healthcare and pharmacy.

This book incorporates computer algebra system use. Up to 2015, I used Maxima. In 2016, with the advent of SageMathCloud, I switched to SageMath. This book presents both systems, but readers need only adopt one. Two sections specifically introduce Maxima or SageMath skills (one each in Chapter 1 and Chapter 4). Remaining computer algebra system skills are integrated into the appropriate sections, including Section 2.1, Section 3.1, Section 3.2, Section 4.6, Section 4.7, Section 4.8, and Section 6.4. In my course, students are encouraged to actively use Maxima or SageMath, being their primary tool for verifying their answers to the exercises.

Another intent behind this book's design is the fact that my course is taught using a *flipped* model. The course structure is simplest if each class meeting is one-to-one to a section of the book. That's why, for instance, I split Dr. Lebl's section on solving nonhomogeneous equations into Section 5.6 and Section 5.7. My students are expected to (and do) come to each class prepared, having read a section of the book, and attempted two written activities that are a mix of reading questions and introductory exercises. The incentive to complete this before-class work is that, at the beginning of class, I will answer their questions about it. The remainder of class is spent working through the section's exercises, in small groups, on whiteboards. After-class homework allow students to reinforce their learning, and practice their skills.

This book presents strong review content in critical areas:

- For many students, the need to solve L2ODEs with constant coefficient by factoring a characteristic equation over the complex numbers is their first real (pun intended) contact with complex numbers. Section 2.1 addresses this gap.

- Students are often too-briefly exposed to hyperbolic series in *Calculus Two*. This book provides a full exposition in Section 2.3.

- Most students learn about Taylor series in the last few weeks of *Calculus Two*—as a result, their retention of this topic is often weak. Section 2.5 revisits this topic. Then Section 7.1 provides a gentle introduction to DE series solutions, by verifying series solutions to differential equations. This practices series manipulation skills, and makes it easier for students to handle the methods for finding series solutions themselves in subsequent sections.

- My college does not (yet) offer a *Linear Algebra* course. In order to alleviate the distraction of learning enough linear algebra to solve a system of differential equations by the eigenmethod, I now integrate one week of solving systems of linear equations via Gauss-Jordan reduction (the first portion of Chapter 4) into my *Calculus Two* curriculum. That leaves more time, in my *Differential Equations* course, to comfortably discuss determinants, eigenvalues, and eigenvectors.

This book is available from `http://www.differentialequations.net`. Check there for any possible updates or errata. The LaTeX source is also available from the same site for possible modification and customization under the Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License

I am grateful to all authors listed on the Attributions page, and indeed all open-source creators, for making their materials available. Without this publishing model, I probably would not have been able to to realize this project. What a nice way for us to *collaborate*! I am also grateful to the authors listed on the Further Reading page, for influencing my thinking on this subject, and its presentation.

Dr. Lebl's notes about *Notes on Diffy Qs*

This book originated from my class notes for teaching Math 286 at the University of Illinois at Urbana-Champaign in Fall 2008 and Spring 2009. It is a first course on differential equations for engineers. I have also taught Math 285 at UIUC and Math 20D at UCSD using this book. The standard book at UIUC was Edwards and Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling* [EP], fourth edition. The standard book at UCSD is Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems* [BD]. As the syllabus at UIUC was based on [EP], the early chapters in the book have some resemblance to [EP] in choice of material and its sequence, and examples used. Among other books I have used as sources of information and inspiration are E.L. Ince's classic (and inexpensive) *Ordinary Differential Equations* [I], Stanley Farlow's *Differential Equations and Their Applications* [F], now available from Dover, Berg and McGregor's *Elementary Partial Differential Equations* [BM], and William Trench's free book *Elementary Differential Equations with Boundary Value Problems* [T]. See the Further Reading chapter at the end of the book.

I taught the UIUC courses using the IODE software (`http://www.math.uiuc.edu/iode/`). IODE is a free software package that works with Matlab (proprietary) or Octave (free software). Projects and labs from the IODE website are referenced throughout the book. They need not be used, but I recommend using them. This gives students some practice with Matlab, very useful for engineers. The graphs in the book were made with the Genius software (see `http://www.jirka.org/genius.html`). I have used Genius in class to show these (and other) graphs.

This book is available from `http://www.jirka.org/diffyqs/`. Check there for any possible updates or errata. The LaTeX source is also available from the same site for possible modification and customization.

Firstly, I would like to acknowledge Rick Laugesen. I used his handwritten class notes the first time I taught Math 286. My organization of this book through chapter 5, and the choice of material covered, is heavily influenced by his notes. Many examples and computations are taken from his notes. I am also heavily indebted to Rick for all the advice he has given me, not just on teaching Math 286. For spotting errors and other suggestions, I would also like to acknowledge (in no particular order): John P. D'Angelo, Sean Raleigh, Jessica Robinson, Michael Angelini, Leonardo Gomes, Jeff Winegar, Ian Simon, Thomas Wicklund, Eliot Brenner, Sean Robinson, Jannett Susberry, Dana Al-Quadi, Cesar Alvarez, Cem Bagdatlioglu, Nathan Wong, Alison Shive, Shawn White, Wing Yip Ho, Joanne Shin, Gladys Cruz, Jonathan Gomez, Janelle Louie, Navid Froutan, Grace Victorine, Paul Pearson, Jared Teague, Ziad Adwan, Martin Weilandt, Sönmez Şahutoğlu, Pete Peterson, Thomas Gresham, Prentiss Hyde, Jai Welch, Simon Tse, Andrew Browning, James Choi, Dusty Grundmeier, John Marriott, Jim Kruidenier, and probably others I have forgotten. Finally I would like to acknowledge NSF grants DMS-0900885 and DMS-1362337.

Chapter 1

Introduction

The laws of physics are generally written down as differential equations. Therefore, all of science uses differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science is concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes. In this chapter, we begin to learn the language of differential equations.

1.1 Learning about differential equations

Attribution: §0.2 in [L]
Further reading: §1.1 in [EP], chapter 1 in [BD], §1.1,4.1 in [T]

A *differential equation* is an equation that contains one or more derivatives of an unknown function. The *order* of a differential equation is the order of the highest derivative that it contains.

Example 1.1.1. The equations

$$\begin{aligned} \frac{dy}{dx} - x^2 &= 0 \\ y' + 2xy^2 &= -2 \\ y'' + 2y' + y = 2x &= 2x \\ xy''' &= \sin x \\ y^{(n)} + xy' + 3y &= x \end{aligned}$$

have order 1, 1, 2, 3, and n, respectively. ■

Another example of a first order differential equation is

$$\frac{dx}{dt} + x = 2\cos t, \tag{1.1}$$

since it involves only the first derivative of the dependent variable. Here x is the *dependent variable* and t is the *independent variable*. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

1.1.1 Solutions of differential equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we will call x, such that when we plug x, t, and $\frac{dx}{dt}$ into (1.1), the equation holds. It is the same idea as it would be for a normal (algebraic) equation of just x and t.

Example 1.1.2. We claim that $x = \cos t + \sin t$ is a solution. How do we check? We simply plug x into equation (1.1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left hand side of (1.1).

$$\frac{dx}{dt} + x = (-\sin t + \cos t) + (\cos t + \sin t) = 2\cos t.$$

Yay! We got precisely the right hand side. This solution is plotted in red in Figure 1.1 on the facing page.

But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

Again plugging into the left hand side of (1.1)

$$\frac{dx}{dt} + x = (-\sin t + \cos t - e^{-t}) + (\cos t + \sin t + e^{-t}) = 2\cos t.$$

And it works yet again! This solution is plotted in orange in Figure 1.1 on the next page.

So there can be many different solutions. In fact, for this equation all solutions can be written in the form $x = \cos t + \sin t + Ce^{-t}$ for some constant C. Figure 1.1 on the facing page plots a few more such solutions for us, corresponding to $C = 2$ in blue and $C = -1$ in green. We will see how we find these solutions soon. ■

Remark 1.1.1. *In the previous example, we found* explicit solutions *to the differential equation. This is, we were able to express the dependent variable x as a function of t. Indeed, the solutions that we found were combinations of sinusoids and a decaying exponential that are all functions. Often, the solution of a differential equation is not a function. For instance, if a circle solves a differential equation, we could write down the equation of that circle, and this would be a valid solution. This would be called an* implicit solutions.

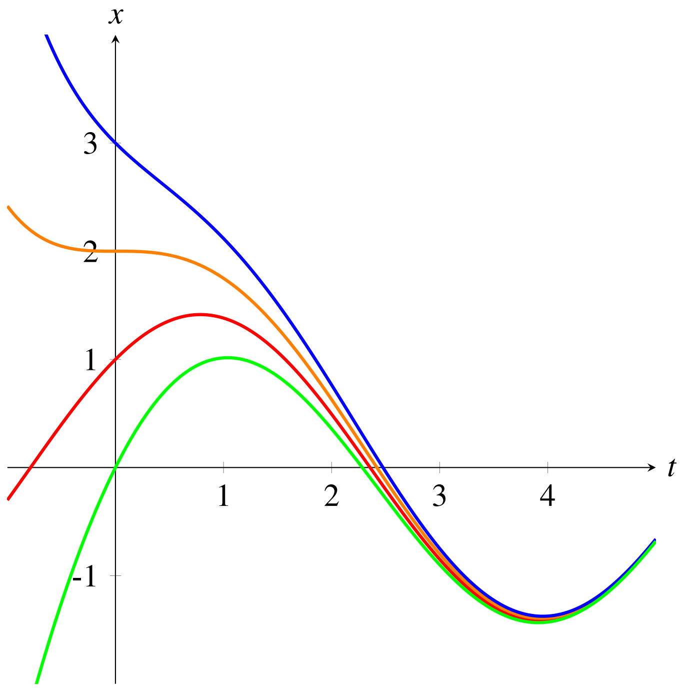


Figure 1.1: A few solutions of $\frac{dx}{dt} + x = 2\cos t$: $x = \cos t + \sin t$ (red), $x = \cos t + \sin t + e^{-t}$ (orange), $x = \cos t + \sin t + 2e^{-t}$ (blue), and $x = \cos t + \sin t - e^{-t}$ (green).

It turns out that solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions.

In this course we will look at *ordinary differential equations* or ODEs, by which we mean that there is only one independent variable and derivatives are only with respect to this one variable. If there are several independent variables, we will get *partial differential equations* or PDEs that you may study in an advanced Differential Equations course.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. It is important to know when it is easy to find solutions and how to do so. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may need to make certain assumptions and changes in your model to achieve this.

To be a successful scientist, you will be required to solve problems in your job that you have never seen before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

1.1.2 Fundamental differential equations

Let us get to what we will call the four fundamental equations. These equations appear very often and it is useful to just memorize what their solutions are. Their solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. There is no need to wonder if you have remembered the solution correctly.

First, we have the first order equation

$$\boxed{\frac{dy}{dx} = ky} \tag{1.2}$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. What function is proportional to its derivative? The zero function and an exponential function. The solution $y(x) = 0$ is not very useful in practice, and indeed is referred to as the *trivial solution*. The other solution

$$\boxed{y(x) = Ce^{kx}.} \tag{1.3}$$

is much more interesting, and is called a *nontrivial solution*. Because the arbitrary constant C can take on the value of any real number, our DE has infinitely many solutions. Also, the case $C = 0$ encapsulates the trivial solution, which is another reason why we don't care about trivial solutions very often. Later in this course, we will prove that, with (1.3), we have expressed all of the possible solutions to (1.2). Therefore, (1.3) is called the *general solution*.

Exercise* 1.1.1*: *Check that the y given is really a solution to the equation.*

Second, we have the first order equation

$$\boxed{\frac{dy}{dx} = -ky} \tag{1.4}$$

for some constant $k > 0$. The general solution for this equation is

$$\boxed{y(x) = Ce^{-kx}.} \tag{1.5}$$

Exercise* 1.1.2*: *Check that the y given is really a solution to the equation.*

Third, consider the second order differential equation

$$\boxed{\frac{d^2y}{dx^2} = -k^2y} \tag{1.6}$$

for some constant $k > 0$. The general solution for this equation is

$$\boxed{y(x) = C_1 \cos(kx) + C_2 \sin(kx).} \tag{1.7}$$

Note that because our differential equation is second order, we have two constants in our general solution.

Exercise* 1.1.3*: *Check that the y given is really a solution to the equation.*

Fourth, take the second order differential equation

$$\boxed{\boxed{\frac{d^2y}{dx^2} = k^2 y}} \tag{1.8}$$

for some constant $k > 0$. The general solution for this equation is either

$$\boxed{y(x) = C_1 e^{kx} + C_2 e^{-kx}} \tag{1.9}$$

or

$$\boxed{y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).} \tag{1.10}$$

If you are unfamiliar with hyperbolic cosine and sine functions (cosh and sinh), they are reviewed in Section 2.3 on page 49.

Exercise* 1.1.4*: *Check that both forms of y given are really solutions to the equation. Hint: Apply Definition 2.3.1 to express the hyperbolic functions in terms of exponential functions.*

These solutions stated above for the fundamental differential equations are called *general solutions*, as each one involves one or several arbitrary constants. For a specific application, additional information is needed to determine them, and is usually provided in the form of one or several *initial conditions* (IC). When a differential equation is specified with one or several ICs, it is called an *initial value problem* (IVP).

Example 1.1.3. Solve $y'' = 16y$ subject to the initial conditions $y(0) = 3$ and $y'(0) = 2$.

We notice that our differential equation is fundamental having the form (1.8) with $k = 4$.

First solution We immediately write down the first form (1.9) of this equation's general solution as

$$y(x) = C_1 e^{4x} + C_2 e^{-4x}. \tag{1.11}$$

Since one of our initial conditions involves $y'(x)$, we differentiate $y(x)$ to obtain

$$y'(x) = 4C_1 e^{4x} - 4C_2 e^{-4x}$$

and now we can enforce the conditions to solve for C_1 and C_2:

$$\begin{aligned} y(0) = 3 \quad &\Rightarrow \quad C_1 e^{4\cdot 0} + C_2 e^{-4\cdot 0} = 3 \quad \Rightarrow \quad C_1 + C_2 = 3 \\ y'(0) = 2 \quad &\Rightarrow \quad 4C_1 e^{4\cdot 0} - 4C_2 e^{-4\cdot 0} = 2 \quad \Rightarrow \quad 4C_1 - 4C_2 = 2. \end{aligned}$$

At this stage, we have a system of 2 linear equations in 2 unknowns. Perhaps easiest would isolate C_1 in the first equation to get $C_1 = 3 - C_2$ and then substitute it into the second equation to get

$$4(3 - C_2) - 4C_2 = 2 \quad \Rightarrow \quad 12 - 4C_2 - 4C_2 = 2 \quad \Rightarrow \quad -8C_2 = -10 \quad \Rightarrow \quad C_2 = 5/4.$$

Knowing C_2, it is easy to determine that $C_1 = 7/4$. Hence

$$y(x) = 7/4\, e^{4x} + 5/4\, e^{-4x}. \tag{1.12}$$

Second solution We immediately write down the second form (1.10) of this equation's general solution as

$$y(x) = D_1 \cosh(4x) + D_2 \sinh(4x). \tag{1.13}$$

Since one of our initial conditions involves $y'(x)$, we differentiate $y(x)$ to obtain

$$y'(x) = 4D_1 \sinh(4x) + 4D_2 \cosh(4x)$$

and now we can enforce the conditions to solve for D_1 and D_2:

$$\begin{aligned} y(0) = 3 \quad &\Rightarrow \quad D_1 \cosh(4\cdot 0) + D_2 \sinh(4\cdot 0) = 3 \quad \Rightarrow \quad D_1 = 3 \\ y'(0) = 2 \quad &\Rightarrow \quad 4D_1 \sinh(4\cdot 0) + 4D_2 \cosh(4\cdot 0) = 2 \quad \Rightarrow \quad 4D_2 = 2 \quad \Rightarrow \quad D_2 = 1/2. \end{aligned}$$

Easy! Our solution is

$$y(x) = 3\cosh(4x) + 1/2\, \sinh(4x). \tag{1.14}$$

■

Exercise* 1.1.5*: *Check that both solutions to the previous example are equal. Hint: Exploit Definition 2.3.1 on page 49.*

1.1.3 Differential equations in practice

So how do we use differential equations in science? First, we have some *real world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real world problem we started with. Figure 1.2 on the next page summarizes this process.

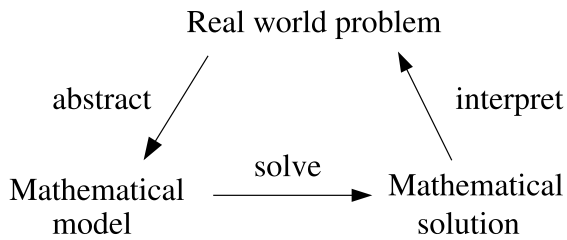


Figure 1.2: Using differential equations in science: an iterative loop.

Learning how to formulate the mathematical model and how to interpret the results is what your science classes do. In this course we will focus mostly on the mathematical analysis. Sometimes we will work with simple real world examples, so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process: the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive k. This equation, where a function is proportional to its derivative, is called a *Malthusian model*.

Example 1.1.4. Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0?

First we solve the differential equation. Since it is fundamental, we immediately write down that the general solution is

$$P(t) = Ce^{kt},$$

where C is a constant.

OK, so what now? We do not know C and we do not know k. But we know something. We know $P(0) = 100$, and we also know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$100 = P(0) = Ce^{k0} = C,$$
$$200 = P(10) = 100\, e^{k10}.$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So we know that

$$P(t) = 100\, e^{(\ln 2)t/10} \approx 100\, e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 1.3 on the following page for the graph of this solution.

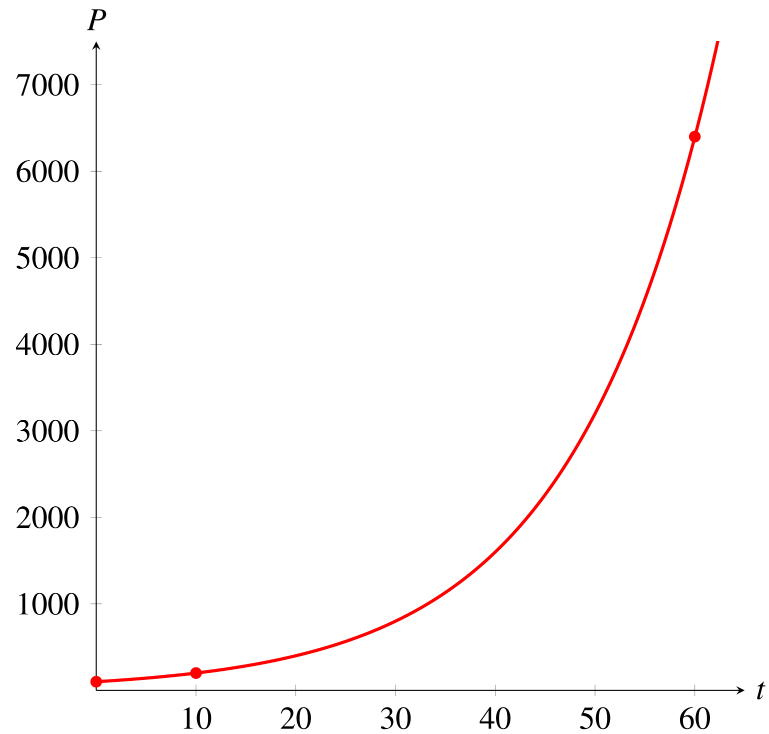


Figure 1.3: Bacteria growth.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60 seconds? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. Yet, our model has no problem saying that $P(61) \approx 6859.35$.

What happens as t grows without bound? As t increases, our exponential model keeps growing. Will the number of bacteria continue to increase indefinitely? No! Presumably, at some point in time, the bacteria will lack space, or food, or both, and the population will stabilize. At that time, our assumptions no longer hold. In Section 3.7, we will discuss a more sophisticated growth model that's called the *Verhulst model*. ■

Here's another example: the standard *exponential decay model*. Radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass Q of a radioactive material present at time t satisfies

$$\frac{dQ}{dt} = -kQ,$$

where k is a positive number that we'll call the *decay constant* of the material.

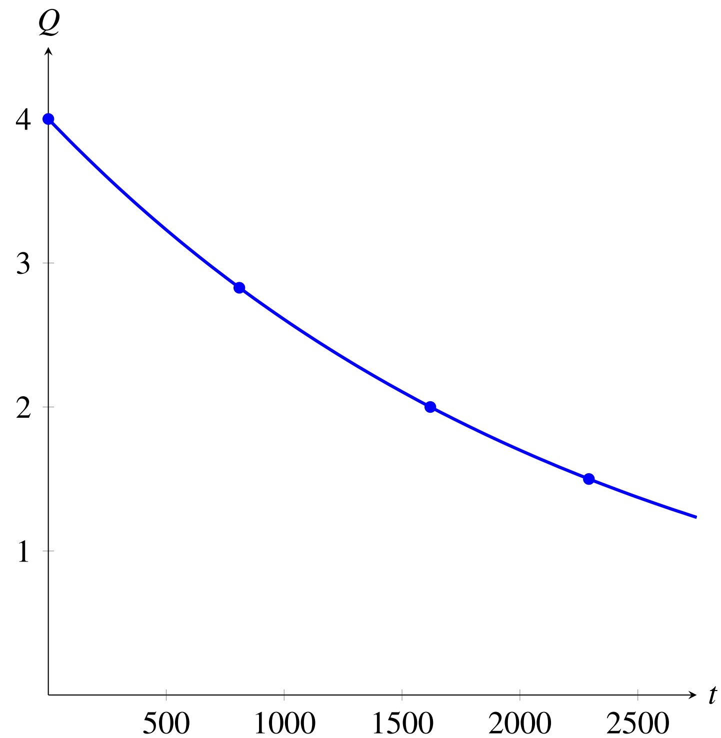


Figure 1.4: Radioactive decay. After 1620 years, half of the radioactive material has decayed.

Example 1.1.5. A radioactive substance has a half-life of 1620 years. If its mass is currently 4 grams, how much will be left 810 years from now? Then, find how long it will take for 1.5 gram of the substance remain.

Let us work with independent variable t in years and dependent variable Q that is the mass of the radioactive substance in grams. We easily solve the solve the differential equation because it is fundamental:

$$Q(t) = Ce^{-kt}.$$

Then, we enforce the initial condition $Q(0) = 4$ to find that $C = 4$, so our model is now

$$Q(t) = 4e^{-kt}.$$

To determine the value for k, we use the half-life. Indeed, the *half-life*, often denoted τ, of a radioactive material is defined to be the time required for half of its mass to decay. We started off with 4 grams, so τ is the time that it takes for 2 grams to remain. Put otherwise, $Q(\tau) = 2$ or $Q(1620) = 2$. We can now enforce this condition:

$$Q(1620) = 4e^{-1620k} = 2 \quad \Rightarrow \quad e^{-1620k} = \frac{1}{2} \quad \Rightarrow \quad -1620k = -\ln 2 \quad \Rightarrow \quad k = \frac{\ln 2}{1620}.$$

Our model is thus

$$Q(t) = 4e^{-\frac{\ln 2}{1620}t}.$$

See Figure 1.4 on the previous page for the graph of this solution.

In 810 years, there will remain

$$Q(810) = 4e^{-\frac{810\ln 2}{1620}} = 4e^{-\frac{\ln 2}{2}} = 4\left(e^{\ln 2}\right)^{-\frac{1}{2}} = 4\cdot 2^{-\frac{1}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.828\,\text{grams}.$$

To find how long it will take for there to remain 1.5 grams of radioactive material, we solve

$$Q(t) = 1.5 \quad\Rightarrow\quad 4e^{-\frac{\ln 2}{1620}t} = 1.5 \quad\Rightarrow\quad e^{-\frac{\ln 2}{1620}t} = \frac{3}{8} \quad\Rightarrow\quad -\frac{\ln 2}{1620}t = \ln\frac{3}{8} \quad\Rightarrow$$

$$\frac{\ln 2}{1620}t = \ln\frac{8}{3} \quad\Rightarrow\quad t = 1620\frac{\ln\frac{8}{3}}{\ln 2} \approx 2292.4\,\text{years}.$$

■

Exercises

Exercise* 1.1.6*: *Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.*

Exercise* 1.1.7*: *Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.*

Exercise* 1.1.8*: *Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.*

Exercise* 1.1.9*: *Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.*

Exercise* 1.1.10*: *Verify that $x = C_1 e^{-t} + C_2 e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.*

Exercise* 1.1.11*: *Solve $\frac{dA}{dt} = -10A,\ \ A(0) = 5$. Hint: This equation is fundamental.*

Exercise* 1.1.12*: *Solve $\frac{dH}{dx} = 3H,\ \ H(0) = 1$.*

Exercise* 1.1.13*: *Solve $\frac{d^2y}{dx^2} = 4y,\ \ y(0) = 0,\ \ y'(0) = 1$.*

Exercise* 1.1.14*: *Solve $\frac{d^2x}{dy^2} = -9x,\ \ x(0) = 1,\ \ x'(0) = 0$.*

Exercise* 1.1.15*: *Is there a solution to $y' = y$, such that $y(0) = y(1)$?*

Exercise* 1.1.16*: *Glucose is absorbed by the body at a rate proportional to the amount G of glucose present in the bloodstream, measured in milligrams per decilitre. Suppose there are G_0 milligrams per decilitre of glucose in the bloodstream when $t = 0$, and let $t > 0$ denote time in days. Let λ denote the (positive) constant of proportionality. Since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation $G' = -\lambda G$. Solve this fundamental differential equation, enforcing the initial condition.*

Exercise* 1.1.17*: *Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. The half-life of Iodine-131 is approximately 8 days. Suppose the decay of Iodine-131 follows the model $\frac{dm}{dt} = -km$ for a positive constant k. If 5 grams of Iodine-131 is present initially, find a function which gives the mass m in grams of Iodine-131, t days later.*

Exercise* 1.1.18*: *Chromium-51 is used to track red blood cells. Its half-life is 27.7 days. Following an injection of 75 milligrams, how much Chromium-51 is present in the bloodstream after one week? At what rate is the amount of Chromium-51 changing at that time?*

Exercise* 1.1.19*: *Verify that $x = t^3 + 5$ is a solution to the differential equation $x' = 3t^2$.*

Exercise* 1.1.20*: *Verify that, for any constant c, the function* $x(t) = 1 + ce^{-t}$ *solves* $x' + x = 1$*. Find the c for which this function solves the IVP* $x' + x = 1$ *with* $x(0) = 3$*.*

Exercise* 1.1.21*: *Verify that* $x = (C + t)\cos t$ *solves the differential equation* $x' + x\tan t - \cos t = 0$ *and find the value of C that satisfies the initial condition* $x(2\pi) = 0$*.*

Exercise* 1.1.22** (Hermite's equation):*** *The differential equation*

$$y'' - 2xy' + 2ny = 0$$

is called Hermite's equation* *of order n. This DE arises with modeling in physics (quantum harmonic oscillators), systems theory (in connection with nonlinear operations on Gaussian noise), and numerical analysis (Gaussian quadrature, a numerical integration technique). Some solutions to this DE are called* Hermite polynomials *of order n, denoted* $H_n(x)$*. These polynomials are:*

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \end{aligned}$$

Verify that scalar multiples of the zeroth, first, second, and third order Hermite polynomials solve the zeroth, first, second, and third order Hermite's equation, respectively.

Exercise* 1.1.23** (Chebyshev's equation):*** *The differential equation*

$$(1 - x^2)y'' - xy' + p^2y = 0$$

is called Chebyshev's equation† *of order p.*

Some solutions to this DE are called Chebyshev polynomials of the first kind *of order p, denoted* $T_p(x)$*. These polynomials are:*

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

Verify that scalar multiples of the zeroth, first, second, and third order Chebyshev polynomials solve the zeroth, first, second, and third order Chebyshev's equation, respectively.

*Named after the French mathematician Charles Hermite (1822–1901).

†Named after the Russian mathematician Pafnuty Chebyshev (1821–1894).

Note: Exercises with numbers 101 and higher have solutions in the back of the book.

Exercise* 1.1.101*: *Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.*

Exercise* 1.1.102*: *Is $y = x^2$ a solution to $x^2y'' - 2y = 0$? Justify.*

Exercise* 1.1.103*: *Verify that $x = C_1e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.*

Exercise* 1.1.104*: *Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.*

Exercise* 1.1.105*: *Let $xy'' - y' = 0$. Guess monomial solutions of the form $y = x^r$. By trial and error, you should quickly find two integer values of r that work. Write down the general solution.*

Exercise* 1.1.106*: *Let $x'' + x' - 6x = 0$. Guess exponential solutions of the form $x = e^{rt}$. By trial and error, you should quickly find two integer values of r that work. Write down the general solution.*

Exercise* 1.1.107*: *Let $y'' - 2y' - 8y = 0$. Guess exponential solutions of the form $y = e^{rt}$. By trial and error, you should quickly find two integer values of r that work. Write down the general solution.*

Exercise* 1.1.108*: *Guess a solution to the DE $y'' + y' + y = 5$, using your knowledge of calculus. Don't overthink it!*

Exercise* 1.1.109*: *Guess a solution to the DE $y'' + 5y = 10x + 5$, using your knowledge of calculus.*

Exercise* 1.1.110*: *Consider the differential equation $(x')^2 + x^2 = 4$. Guess a solution, using your knowledge of algebraic identities and derivative formulas. Once you find one solution, you might find it easy to guess more solutions.*

Exercise* 1.1.111*: *Carbon-14 has a half-life of approximately 5730 years. Prior to the nuclear tests of the 1950s, which raised the level of Carbon-14 in the atmosphere, the ratio of Carbon-14 to Carbon-12 in the air, plants, and animals was 10^{-15}. If this ratio is measured in an archeological sample of bone and found to be 10^{-17}, how old is the sample?*

1.2 Solving differential equations using Maxima

We can verify many of our answers to the exercises of Section 1.1 using the computer algebra system Maxima. All Maxima examples in this book were generated using Maxima version 5.31.2 and wxMaxima version 13.04.2 on a personal computer running Windows, but other configurations should generate similar results.

Example 1.2.1. Exercise 1.1.101 states: *Show that* $x = e^{-2t}$ *is a solution to* $x'' + 4x' + 4x = 0$. We will use the **ode_check** command. Type the syntax appearing in blue exactly as shown in Maxima program 1.2.1. Then, press Ctrl-R to execute your Maxima worksheet so that the result of our manipulations are displayed in black.

Maxima program 1.2.1 Verifying a solution to a differential equation

```
(%i1)  kill(all)$ reset()$ ratprint:false$ logabs:true$
(%i4)  load(contrib_ode)$
(%i5)  sol : x = exp(-2*t);
```

(%o5) $x = e^{-2\,t}$

```
(%i6)  de : 'diff(x,t,2) + 4*'diff(x,t,1) + 4*x = 0;
```

(%o6) $\frac{d^2}{d\,t^2}\,x + 4\left(\frac{d}{d\,t}\,x\right) + 4\,x = 0$

```
(%i7)  ode_check(de,sol);
```

(%o7) 0

Let us discuss each line in turn.

- The first input cell consists of configuration commands that are necessary, albeit tedious to include. The first two configuration commands (`kill(all)$ reset()$`) are necessary, telling Maxima that we're starting a new program. The third configuration command (`ratprint:false$`) is desirable because it prevents Maxima from generating excessive amounts of unnecessary output to the screen. Finally, the fourth configuration command (`logabs:true$`) forces Maxima to include the absolute value when antidifferentiating the function $\frac{1}{x}$ to get $\ln|x|$. This command is very helpful anytime we use Maxima to integrate (or manipulate differential equations).

- The second input cell loads the `contrib_ode` library for solving differential equations.

- In the third input cell, we define our proposed solution: $x = e^{-2t}$. To the left of the colon, we give this equation a name: *sol*.

- In the fourth input cell, we define our differential equation. The syntax `'diff(x,t,1)` represents the *the first derivative of x with respect to t* while the syntax `'diff(x,t,2)` represents the *the second derivative of x with respect to t*. To the left of the colon, we give this differential equation a name: *de*.
- Finally, in the fifth input cell, we exploit the `ode_check` command to ask the question: is *de* solved by *sol*? Maxima answers with the number zero. What Maxima means is that, if we plug *sol* and its derivatives into *de*, everything cancels out and we're left with zero.

Without much fanfare, Maxima has verified that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$. ■

Example 1.2.2. What happens if we feed a different candidate solution into `ode_check` that doesn't solve the DE? For instance, let's use Maxima to show that $x = e^{5t}$ doesn't solve $x'' + 4x' + 4x = 0$? Simply add the following syntax to the previous example's worksheet, as in Maxima program 1.2.2, and press Ctrl-R.

Maxima program 1.2.2 Verifying another solution to a differential equation

(continuing Maxima program 1.2.1)

```
(%i8)  sol2 : x = exp(5*t);
```

(%o8) $x = e^{5\,t}$

```
(%i9)  ode_check(de,sol2);
```

(%o9) $49\,e^{5\,t}$

In this program, we define our proposed solution, $x = e^{5t}$, and to the left of the colon, we named this equation *sol2*. The differential equation has already been defined; we don't need to type it in again. We therefore immediately call on `ode_check` to ask our question. Maxima answers as follows: $49e^{5t}$. By this, Maxima is telling us that if we plug *sol2* and its derivatives into *de*, we're stuck with an extra term that doesn't cancel out. Hence, we conclude that $x = e^{5t}$ does not solve $x'' + 4x' + 4x = 0$. ■

Exercise* 1.2.1*: *Use Maxima to confirm your answer to Exercise 1.1.102: Is $y = x^2$ a solution to $x^2y'' - 2y = 0$?*

Example 1.2.3. In Exercise 1.1.105, we guessed the general solution to the DE $xy'' - y' = 0$. Let us use Maxima's **`contrib_ode`** command to solve this DE. Examine Maxima program 1.2.3 on the following page.

In this worksheet, we input our usual configuration commands and load our library. We then define our DE, and aptly name it *de*. Finally, we exploit the `contrib_ode` command to ask the question: what is the general solution $y(x)$ of our differential equation *de*? To the left of the colon,

Maxima program 1.2.3 Solving a differential equation

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$
(%i4) load('contrib_ode)$
(%i5) de : x*'diff(y,x,2) - 'diff(y,x,1) = 0;
```

(%o5) $x\left(\frac{d^2}{d\,x^2}\,y\right)-\frac{d}{d\,x}\,y=0$

```
(%i6) gsoln : contrib_ode(de,y,x);
```

(%o6) $[y=\%k2\,x^2-\frac{\%k1}{2}]$

```
(%i7) method;
```

(%o7) *exact*

```
(%i8) ode_check(de,gsoln[1]);
```

(%o8) 0

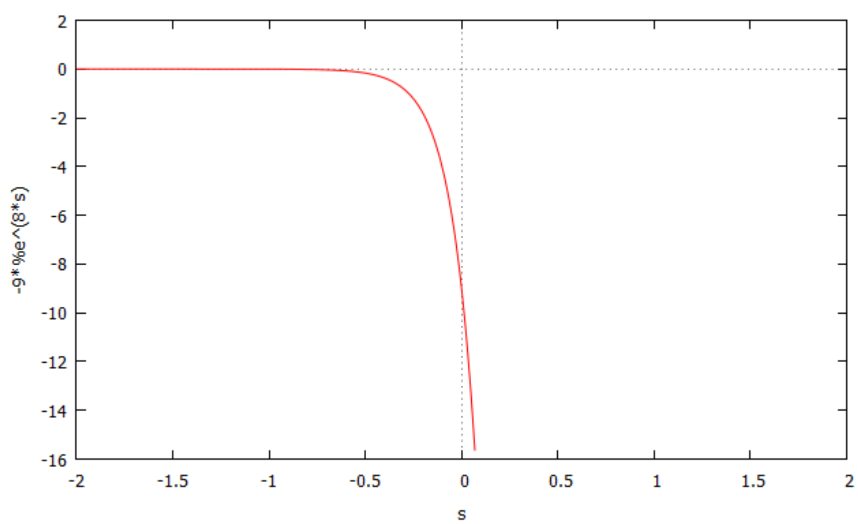


Figure 1.5: The solution to the fundamental differential equation $\frac{d\varphi}{ds} = 8\varphi$ subject to the initial condition $\varphi(0) = -9$ is the exponential function $\phi = -9e^{8s}$, generated using Maxima. We remind ourselves that 8 times this function equals its derivative, and we notice that the initial condition belongs to the curve.

Maxima program 1.2.4 Solving a differential equation with one initial condition

(%i1) `kill(all)$ reset()$ ratprint:false$ logabs:true$`

(%i4) `load('contrib_ode)$`

(%i5)
```
s0 : 0;
phi0 : -9;
```

(%o5) 0

(%o6) -9

(%i7) `de : 'diff(phi,s,1) = 8*phi;`

(%o7) $\frac{d}{d\,s}\phi = 8\,\phi$

(%i8) `gsoln : contrib_ode(de,phi,s);`

(%o8) $[\phi = \%c\,e^{8\,s}]$

(%i9) `soln : ic1(gsoln,s=s0,phi=phi0);`

(%o9) $[\phi = -9\,e^{8\,s}]$

(%i10) `define( phi(s),rhs(soln[1]) );`

(%o10) $\phi\,(s) := -9\,e^{8\,s}$

(%i11) `plot2d( phi(s),[s,-2,2],[y,-16,2],[color,red] );`

plot2d : somevalueswereclipped.

(%o11)

Maxima program 1.2.5 Solving a differential equation with two initial conditions

```
(%i1)  kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4)  load('contrib_ode)$

(%i5)  t0 : 0;
       x0 : 10;
       xp0 : 100;
```

(%o5) 0
(%o6) 10
(%o7) 100

```
(%i8)  de : 'diff(x,t,2) - 'diff(x,t,1) = 0;
```

(%o8) $\frac{d^2}{d\,t^2}\,x - \frac{d}{d\,t}\,x = 0$

```
(%i9)  gsoln : contrib_ode(de,x,t);
```

(%o9) $[x = \%k1\,e^t + \%k2]$

```
(%i10) soln : ic2(gsoln,t=t0,x=x0,'diff(x,t,1)=xp0);
```

(%o10) $[x = 100\,e^t - 90]$

```
(%i11) define( x(t),rhs(soln[1]) );
```

(%o11) $\mathrm{x}\,(t) := 100\,e^t - 90$

```
(%i12) plot2d( x(t),[t,-3,1],[y,-100,200],[color,green] );
```

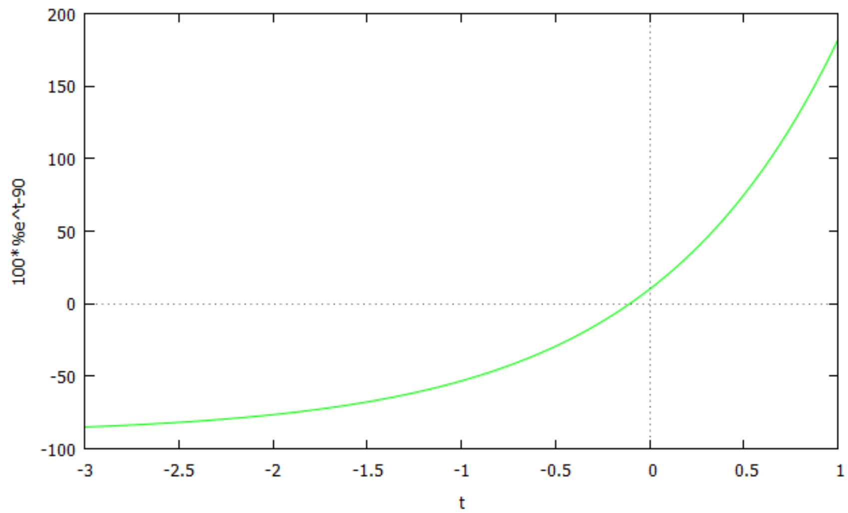


Figure 1.6: The solution to the differential equation $x'' - x' = 0$ subject to the initial conditions $x(0) = 10$ and $x'(0) = 100$ is the exponential function $x = 100e^t - 90$, generated using Maxima.

we give this general solution a name: *gsoln*. The general solution is a binomial consisting of a constant term and a quadratic term. Notice that Maxima sometimes uses k_1 and k_2 to denote the arbitrary constants when there are two of them! The remaining two commands are optional:

- `method;` asks Maxima how it solved the differential equation; sometimes the answer is insightful, sometimes not.
- The last input cell shows how we can use `ode_check` to verify Maxima's own answer. Unnecessary, but a nice way to practice our skills!

■

***Exercise* 1.2.2:** *Use Maxima to find the general solution to* $y'' + 2y' - 8y = 0$*. Compare with your answer to Exercise 1.1.107.*

Maxima can also apply an initial condition.

Example 1.2.4. In this example, we re-solve Exercise 1.1.104 that states: *Solve* $\dfrac{d\varphi}{ds} = 8\varphi$ *and* $\varphi(0) = -9$. To this end, we exploit the `contrib_ode` command to find the general solution followed by the **`ic1`** command to enforce an initial condition. See Maxima program 1.2.4 on page 27. The **`plot2d`** command plots the solution that appears as Figure 1.5 on page 26. ■

***Exercise* 1.2.3:** *Use Maxima to find the general solution to* $x' = -2x$*, and then enforce the initial condition* $x(0) = 100$*. Compare with your answer to Exercise 1.1.9. Graph this solution.*

Maxima can also handle two initial conditions.

Example 1.2.5. In this example, we re-solve Exercise 1.1.103 that states: *Verify that* $x = C_1 e^t + C_2$ *is a solution to* $x'' - x' = 0$*. Find* C_1 *and* C_2 *so that* x *satisfies* $x(0) = 10$ *and* $x'(0) = 100$. To this end, we use `contrib_ode` to find a general solution and then invoke the **`ic2`** command to apply two initial conditions. See Maxima program 1.2.5 on page 28. Once again, we plotted our solution; it appears as Figure 1.6 on the preceding page. ■

Exercise* 1.2.4*: *Use Maxima to find the general solution to* $x'' - x' - 2x = 0$*, and then enforce the initial conditions* $x(0) = 10$ *and* $x'(0) = 0$*. Compare with your answer to Exercise 1.1.10. Graph this solution.*

Exercises

Exercise* 1.2.5*: *Use Maxima to verify that* $x = e^{4t}$ *solves the* $x''' - 12x'' + 48x' - 64x = 0$ *but that* $x = e^t$ *does not, confirming your answers to Exercise 1.1.6 and Exercise 1.1.7.*

Exercise* 1.2.6*: *Use Maxima to solve* $\dfrac{dA}{dt} = -10A, \;\; A(0) = 5.$ *Compare with your answer to Exercise 1.1.11. Graph this solution.*

Exercise* 1.2.7*: *Use Maxima to solve* $\dfrac{dH}{dx} = 3H, \;\; H(0) = 1.$ *Compare with your answer to Exercise 1.1.12. Graph this solution.*

Exercise* 1.2.8*: *Use Maxima to solve* $\dfrac{d^2y}{dx^2} = 4y, \;\; y(0) = 0, \;\; y'(0) = 1.$ *Compare with your answer to Exercise 1.1.13. Graph this solution.*

Exercise* 1.2.9*: *Use Maxima to solve* $\dfrac{d^2x}{dy^2} = -9x, \;\; x(0) = 1, \;\; x'(0) = 0.$ *Compare with your answer to Exercise 1.1.14. Graph this solution.*

Exercise* 1.2.10*: *Use Maxima to find the general solution to* $x'' + x' - 6x = 0.$ *Compare with your answer to Exercise 1.1.106.*

1.3 Solving differential equations using SageMath

Further reading: §4.22 in [B], §1.1-1.2 in [JH]

We can verify many of our answers to the exercises of Section 1.1 using the computer algebra system SageMath. All SageMath examples in this book were generated using SageMathCloud in 2016, but other configurations should generate similar results.

Example 1.3.1. In Exercise 1.1.105, we guessed the general solution to the DE $xy'' - y' = 0$. Let us find this general solution using SageMath. Examine SageMath program 1.3.1.

SageMath program 1.3.1 Solving a differential equation

Input

```
# define symbolic variable x
x = var('x')
# define symbolic function y
y = function('y')(x)
# define a differential equation
de = x*diff(y,x,2) - diff(y,x) == 0
print "solve the differential equation"
gsoln = desolve(de,y)
print gsoln
```

Output

```
solve the differential equation
_K2*x^2 - 1/2*_K1
```

Let us discuss each line in turn.

- The first, third, and fifth lines begin with a hash, meaning that those lines are comments intended to help us document and understand the steps in our worksheet.
- The second line uses the **`var`** command to define t as a *symbolic variable* (or independent variable).
- The fourth line uses the **`function`** command to define x as a *symbolic function* (dependent variable) that depends on t.
- The sixth line defines our differential equation. We chose to call it `de`. That name is followed by the equality symbol (=). Then we specify our differential equation, replacing the equality

symbol that we usually write using paper and pencil to a double equality symbol (==). The syntax `diff(x,t,1)` represents the *the first derivative of x with respect to t* while the syntax `diff(x,t,2)` represents the *the second derivative of x with respect to t.*

- In the seventh line, we use the **`print`** command to send a phrase to the output, specified within double quotes ("). Such labeling—sometimes called *annotating the output*—is a good general practice.

- In the eighth line, we exploit the **`desolve`** command that takes two arguments: the name of the differential equation and the name of the symbolic function. We chose to call this general solution `gsoln`.

- In the ninth line, we use the **`print`** command to show us the general solution that SageMath found.

The general solution is a binomial consisting of a constant term and a quadratic term. Notice that SageMath generally denotes two arbitrary constants as `_K1` and `_K2`. (SageMath generally denotes one arbitrary constant as `_C`.) Also, notice that SageMath could have collapsed `1/2*_K1` into a single constant `_K1`, but didn't. ■

Exercise* 1.3.1*: *Use SageMath to find the general solution to* $y'' + 2y' - 8y = 0$. *Compare with your answer to Exercise 1.1.107.*

SageMath can also apply an initial condition.

Example 1.3.2. In this example, we re-solve Exercise 1.1.104 that states: *Solve* $\dfrac{d\varphi}{ds} = 8\varphi$ *and* $\varphi(0) = -9$. See SageMath program 1.3.2 on the next page.

We highlight differences and additions over our previous program:

- In the eighth line, we exploit the `desolve` command that now takes a third argument that specifies the IC. Since the solution is a function, we name our solution `f(s)`. Sadly, we cannot reuse `phi` without confusing SageMath.

- In the eleventh line, we evaluate our solution at $s = 0$ to convince ourselves that our solution satisfies the IC.

- In the twelfth line, we use the **`plot`** command to configure the graph that we wish to create.

- In the thirteenth line, we use the **`show`** command to view this graph.

Our graph appears as Figure 1.7 on page 36. ■

Exercise* 1.3.2*: *Use SageMath to find the general solution to* $x' = -2x$, *and then enforce the initial condition* $x(0) = 100$. *Compare with your answer to Exercise 1.1.9. Graph this solution.*

SageMath program 1.3.2 Solving a differential equation with one initial condition

Input

```
# define symbolic variable s
s = var('s')
# define symbolic function phi
phi = function('phi')(s)
# define the differential equation
de = diff(phi,s,1) == 8*phi
print "solve the DE with an IC"
f(s) = desolve( de, phi, ics=[0,-9] )
print f(s)
print "verify that the IC is satisfied"
f(0)
P = plot( f(s), (s,-2,2), ymin=-16, ymax=2, color="red" )
show(P)
```

Output

```
solve the DE with an IC
-9*e^(8*s)
verify that the IC is satisfied
-9
```

SageMath program 1.3.3 Solving a differential equation with two initial conditions

Input

```
# define symbolic variable t
t = var('t')
# define symbolic function x
x = function('x')(t)
# define the differential equation
de = diff(x,t,2) - diff(x,t,1) == 0
print "solve the DE with two ICs"
f(t) = desolve( de, x, ics=[0,10,100] )
print f(t)
print "differentiate it"
fp(t) = diff(f,t)
print fp(t)
print "verify that both ICs are satisfied"
f(0)
fp(0)
P = plot( f(t), (t,-3,1), ymin=-100, ymax=200, color="orange" )
show(P)
```

Output

```
solve the DE with two ICs
100*e^t - 90
differentiate it
100*e^t
verify that both ICs are satisfied
10
100
```

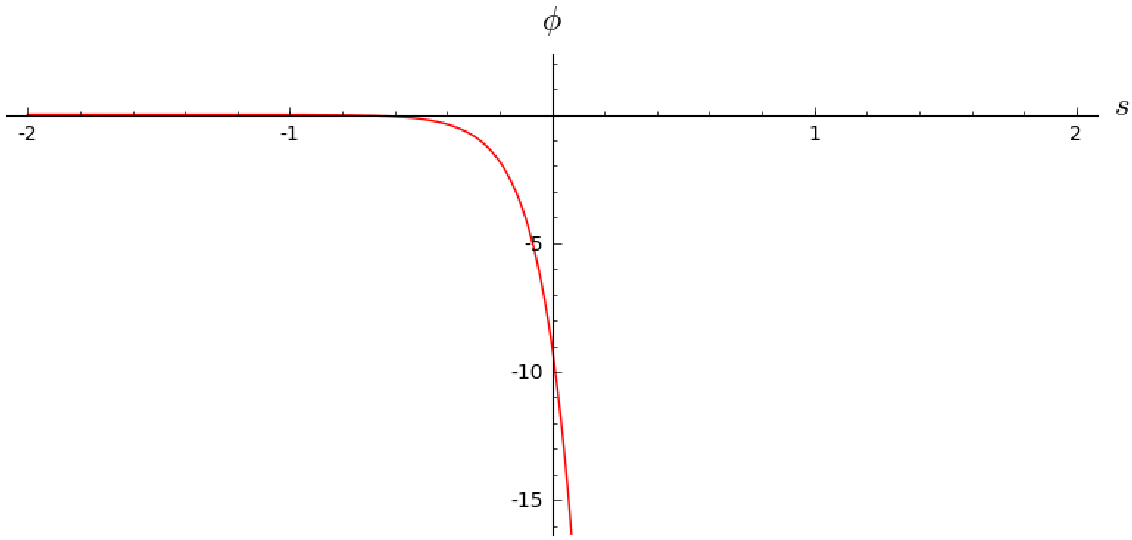


Figure 1.7: The solution to the fundamental differential equation $\frac{d\varphi}{ds} = 8\varphi$ subject to the initial condition $\varphi(0) = -9$ is the exponential function $\phi = -9e^{8s}$, generated using SageMath. We remind ourselves that 8 times this function equals its derivative, and we notice that the initial condition $\phi(0) = -10$ is satisfied.

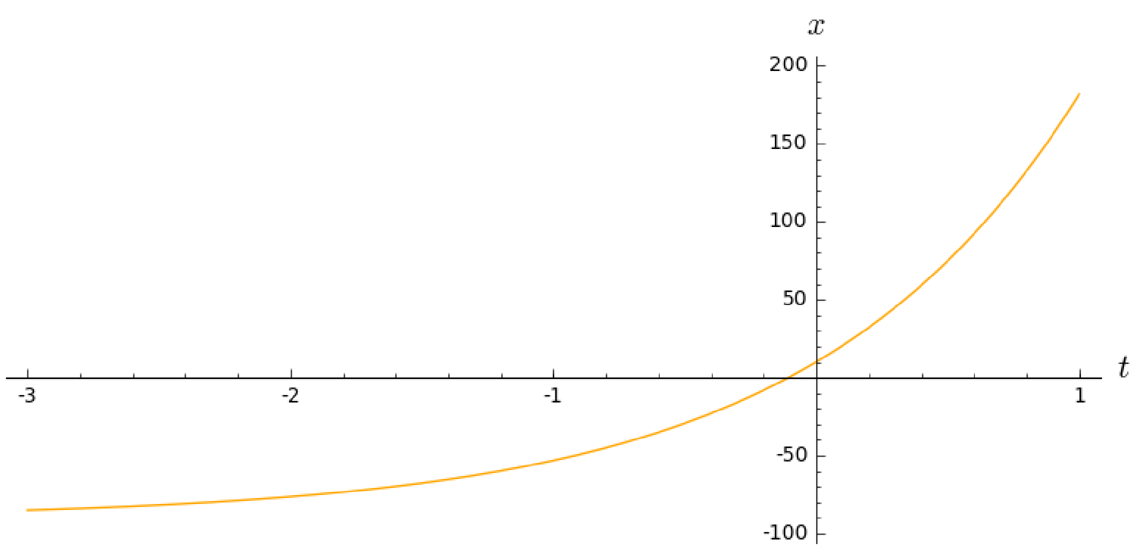


Figure 1.8: The solution to the differential equation $x'' - x' = 0$ subject to the initial conditions $x(0) = 10$ and $x'(0) = 100$ is the exponential function $x = 100e^{t} - 90$, generated using SageMath.

SageMath can also handle two initial conditions.

Example 1.3.3. In this example, we re-solve Exercise 1.1.103 that states: *Verify that* $x = C_1 e^t + C_2$ *is a solution to* $x'' - x' = 0$*. Find* C_1 *and* C_2 *so that* x *satisfies* $x(0) = 10$ *and* $x'(0) = 100$. See SageMath program 1.3.3 on page 35.

We highlight new syntax items:

- First, when using the `desolve` command to solve the DE while enforcing the ICs, the third argument is expanded from two entries to three.

- Second, prior to verifying that the DEs are enforced, the solution `f(t)` is differentiated using the **`diff`** command to produce `fp(t)` (with the 'p' standing for *prime* as SageMath doesn't permit the use of apostrophes in function names).

Once again, we plotted our solution; it appears as Figure 1.8 on the facing page. ■

Exercise* 1.3.3*: *Use SageMath to find the general solution to* $x'' - x' - 2x = 0$*, and then enforce the initial conditions* $x(0) = 10$ *and* $x'(0) = 0$*. Compare with your answer to Exercise 1.1.10. Graph this solution.*

Exercises

Exercise* 1.3.4*: *Use SageMath to solve* $\frac{dA}{dt} = -10A,\ \ A(0) = 5.$ *Compare with your answer to Exercise 1.1.11. Graph this solution.*

Exercise* 1.3.5*: *Use SageMath to solve* $\frac{dH}{dx} = 3H,\ \ H(0) = 1.$ *Compare with your answer to Exercise 1.1.12. Graph this solution.*

Exercise* 1.3.6*: *Use SageMath to solve* $\frac{d^2y}{dx^2} = 4y,\ \ y(0) = 0,\ \ y'(0) = 1.$ *Compare with your answer to Exercise 1.1.13. Graph this solution.*

Exercise* 1.3.7*: *Use SageMath to solve* $\frac{d^2x}{dy^2} = -9x,\ \ x(0) = 1,\ \ x'(0) = 0.$ *Compare with your answer to Exercise 1.1.14. Graph this solution.*

Exercise* 1.3.8*: *Use SageMath to verify that* $x = e^{-2t}$ *is a solution to* $x'' + 4x' + 4x = 0$*, confirming your answer to Exercise 1.1.101.*

Exercise* 1.3.9*: *Use SageMath to verify that* $y = x^2$ *is a solution to* $x^2y'' - 2y = 0$*, confirming your answer to Exercise 1.1.102.*

Exercise* 1.3.10*: *Use SageMath to find the general solution to* $x'' + x' - 6x = 0.$ *Compare with your answer to Exercise 1.1.106.*

Chapter 2

Review

2.1 Complex numbers

Attribution: §3.4 in [SZ]

Consider the polynomial $p(x) = x^2 + 1$. The zeros of p are the solutions to $x^2 + 1 = 0$, or $x^2 = -1$. This equation has no real solutions, but as an intellectual exercise we could extract the square roots of both sides to get $x = \pm\sqrt{-1}$. The quantity $\sqrt{-1}$ is usually re-labeled i, the so-called imaginary unit. Complex numbers may seem a strange concept, but it serves as a nice shortcut sometimes. Despite the terminology, there is nothing imaginary or really complicated about complex numbers.

The number i, while not a real number, plays along well with real numbers, and acts very much like any other radical expression. For instance,

$$3(2i) = 6i,$$
$$7i - 3i = 4i,$$
$$(2 - 7i) + (3 + 4i) = 5 - 3i,$$

and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition 2.1.1. *The imaginary unit i satisfies the following properties: $i^2 = -1$, and if c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$.*

Remark 2.1.1. *It is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get*

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. We want to enlarge the number system so we can solve things like $x^2 = -1$, but not at the cost of the established rules already set in place. For that reason, the general properties of radicals simply do not apply for even roots of negative quantities.

We are now in the position to define the complex numbers.

Definition 2.1.2. *A complex number is a number of the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and i is the imaginary unit.*

Complex numbers include things you'd normally expect, like $3 + 2i$ and $2/5 - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers, so 0 and $\pi - \sqrt{21}$ are both considered complex numbers.

For a complex number $a + ib$ we call a the *real part* and b the *imaginary part* of the number. Often the following notation is used,

$$\begin{aligned} \mathrm{Re}(a + ib) &= a, \\ \mathrm{Im}(a + ib) &= b. \end{aligned}$$

The arithmetic of complex numbers is as you would expect.

Example 2.1.1. Simplify $(1 - 2i) - (3 + 4i)$. As mentioned earlier, we treat expressions involving i as we would any other radical. We combine like terms to get

$$(1 - 2i) - (3 + 4i) = 1 - 2i - 3 - 4i = -2 - 6i.$$

■

Example 2.1.2. Simplify $(1 - 2i)(3 + 4i)$. Using the distributive property, we get

$$(1 - 2i)(3 + 4i) = (1)(3) + (1)(4i) - (2i)(3) - (2i)(4i) = 3 + 4i - 6i - 8i^2.$$

Since $i^2 = -1$, we get $3 + 4i - 6i - 8i^2 = 3 - 2i - (-8) = 11 - 2i$. ■

Example 2.1.3. Simplify $\frac{1 - 2i}{3 - 4i}$. We deal with the denominator $3 - 4i$ as we would any other denominator containing a radical, and multiply both numerator and denominator by $3 + 4i$ (the conjugate of $3 - 4i$). Doing so produces

$$\frac{1 - 2i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{(1 - 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{11 - 2i}{25} = \frac{11}{25} - \frac{2}{25}\, i.$$

■

Example 2.1.4. Simplify $\sqrt{-3}\,\sqrt{-12}$ and $\sqrt{(-3)(-12)}$.

$$\sqrt{-3}\,\sqrt{-12} = \left(i\sqrt{3}\right)\left(i\sqrt{12}\right) = i^2\sqrt{3 \cdot 12} = -\sqrt{36} = -6$$

$$\sqrt{(-3)(-12)} = \sqrt{36} = 6.$$

■

Example 2.1.5. Simplify $(x - [1 + 2i])(x - [1 - 2i])$. We can brute force multiply using the distributive property and see that

$$\begin{aligned}(x - [1 + 2i])(x - [1 - 2i]) &= x^2 - x[1 - 2i] - x[1 + 2i] + [1 - 2i][1 + 2i] \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &= x^2 - 2x + 5\end{aligned}$$

■

Some remarks about the last example are in order. First, the *complex conjugate* of a number $a + bi$ is the number $a - bi$. The notation commonly used for conjugation is a 'bar': $\overline{a + bi} = a - bi$. For example:

$$\begin{aligned}\overline{3 + 2i} &= 3 - 2i \\ \overline{3 - 2i} &= 3 + 2i \\ \overline{5} &= 5 \\ \overline{4i} &= -4i\end{aligned}$$

The real part of a complex number z can be computed as $\frac{z + \bar{z}}{2}$. The imaginary part of a complex number z can be computed as $\frac{z - \bar{z}}{2i}$. We state this mathematically as follows:

$$\begin{aligned}\operatorname{Re} z &= \frac{z + \bar{z}}{2} \\ \operatorname{Im} z &= \frac{z - \bar{z}}{2i}.\end{aligned}$$

The set of complex numbers is denoted $\mathbb{C}$.

We can also define the exponential e^{a+ib} of a complex number. Most properties that we know from real exponential functions extend to complex exponential functions, including $e^{x+y} = e^x e^y$. This means that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called *Euler's formula*.

Theorem 2.1.1 (Euler's formula)**.**

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

Surprisingly, this formula can be used to verify several trigonometric identities. Several of this section's exercises will develop this ability.

Exercises

Exercise* 2.1.1*: *Exploit properties of exponentials to show the following identities*

$$\begin{aligned} e^{-ix} &= \cos x - i \sin x \\ e^{ix}e^{iy} &= e^{i(x+y)} \\ \left(e^{ix}\right)^n &= e^{i(nx)} \\ \frac{e^{ix}}{e^{iy}} &= e^{i(x-y)} \end{aligned}$$

for all real numbers x and y and any natural number n.

Exercise* 2.1.2*: *Use Euler's formula to show that* $e^{i\pi} + 1 = 0$. *This famous equation relates five constants.*

Exercise* 2.1.3*: *Verify the identities*

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

Hint: Apply Euler's formula to the identity's right hand side and simplify as much as you can.

Exercise* 2.1.4*: *Starting with* $e^{i(2\theta)} = \left(e^{i\theta}\right)^2$*, apply Euler's formula to each side to deduce these double-angle identities*

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta. \end{aligned}$$

Use the given complex numbers z and w to find the following:

- $z + w$
- zw
- z^2
- $\frac{1}{z}$
- $\frac{z}{w}$
- $\frac{w}{z}$
- $\bar{z}$
- $z\bar{z}$
- $(\bar{z})^2$

Simplify your answers to the form $a + bi$.

Exercise* 2.1.101*: $z = 2 + 3i$ *and* $w = 4i$

Exercise* 2.1.102*: $z = 4i$ *and* $w = 2 - 2i$

Exercise* 2.1.103*: $z = -5 + i$ *and* $w = 4 + 2i$

Exercise* 2.1.104*: $z = 1 - i\sqrt{3}$ *and* $w = -1 - i\sqrt{3}$

Exercise* 2.1.105*: $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ *and* $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

2.2 Factoring

Attribution: §3.4 in [SZ]

We now return to the business of zeros.

Theorem 2.2.1. *Given a quadratic equation* $ax^2 + bx + c = 0$*, where* a*,* b *and* c *are real numbers and* a *is nonzero, let* $\Delta = b^2 - 4ac$ *be the discriminant.*

- *If* $\Delta > 0$*, there are two distinct real number solutions to the equation.*
- *If* $\Delta = 0$*, there is one (repeated) real number solution.*
- *If* $\Delta < 0$*, there are two non-real solutions which form a complex conjugate pair.*

These solutions are obtained from the quadratic formula $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.

Example 2.2.1. Suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic doesn't factor nicely, so we resort to the quadratic formula and obtain

$$\Delta = (-2)^2 - 4(1)(5) = -16$$

$$x = \frac{-(-2) \pm \sqrt{\Delta}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. Next, we note that in Example 2.1.5 we found $(x-[1+2i])(x-[1-2i]) = x^2-2x+5$. This demonstrates that the factor theorem holds even for non-real zeros, i.e, $x = 1 + 2i$ is a zero of f, and, sure enough, $(x - [1 + 2i])$ is a factor of $f(x)$. ■

But how do we know if a general polynomial has any complex zeros at all? A pair of theorems gives us the answer. Taken together, the next two theorems tell us that, if we have a polynomial function with real coefficients, we can always factor it down enough so that any nonreal zeros come from irreducible quadratics.

Theorem 2.2.2 (Conjugate pairs theorem)**.** *If* f *is a polynomial function with real number coefficients and* z *is a zero of* f*, then so is* $\overline{z}$*.*

So, if f is a polynomial function with real coefficients, Theorem 2.2.2 tells us that if $a + bi$ is a nonreal zero of f, then so is $a - bi$. In other words, nonreal zeros of f come in conjugate pairs. The next theorem kicks in to give us both $(x - [a + bi])$ and $(x - [a - bi])$ as factors of $f(x)$. When multiplied, we find that $(x - [a + bi])(x - [a - bi]) = x^2 + 2ax + \left(a^2 + b^2\right)$ is an irreducible quadratic factor of f.

Theorem 2.2.3 (Real factorization theorem). *Suppose f is a polynomial function with real coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.*

Example 2.2.2. Factor $x^2 + x + 1$ over $\mathbb{R}$ and $\mathbb{C}$. Seeing no obvious factoring, we attempt to solve $x^2 + x + 1 = 0$ using the quadratic formula. The discriminant is $\Delta = 1 - 4 = -3$, and so this equation possesses no real roots. Hence, this quadratic polynomial is irreducible over the reals. Continuing to use the quadratic formula towards factoring over the complex numbers,

$$x = \frac{-1 \pm \sqrt{\Delta}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

Therefore

$$x^2 + x + 1 = \left(x - \frac{-1 + \sqrt{3}i}{2}\right)\left(x - \frac{-1 - \sqrt{3}i}{2}\right) = \left(x + \frac{1 - \sqrt{3}i}{2}\right)\left(x + \frac{1 + \sqrt{3}i}{2}\right).$$

■

Example 2.2.3. Factor $x^2 - x - 1$ over $\mathbb{R}$ and $\mathbb{C}$. Seeing no obvious factoring, we attempt to solve $x^2 - x - 1 = 0$ using the quadratic formula. The discriminant is $\Delta = 1 + 4 = 5$, and

$$x = \frac{1 \pm \sqrt{\Delta}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore

$$x^2 - x - 1 = \left(x - \frac{1 + \sqrt{5}}{2}\right)\left(x - \frac{1 - \sqrt{5}}{2}\right).$$

This is the factoring over the real numbers, and also over the complex numbers. ■

Theorem 2.2.4 (Fundamental theorem of algebra). *Suppose f is a polynomial function of degree n with real coefficients. Then $f(x)$ can be factored into a product of n linear factors—including all real, complex, and repeated factors.*

Let us do an example which pulls together all of the major ideas of this section.

Example 2.2.4. Let $f(x) = x^4 + 64$. Knowing that $2 + 2i$ is a root of $f(x)$, factor $f(x)$ over the real numbers and over the complex numbers. If $2 + 2i$ is a root of $f(x)$, then so is its complex conjugate $2 + 2i$. Thus, $x - (2 + 2i)$ and $x - (2 - 2i)$ are both complex factors of $f(x)$. To obtain the remaining factors, we could long divide $x^4 + 64$ by $x - (2 + 2i)$ and then long divide that quotient by $x - (2 - 2i)$. Try it! Or, simpler, we could multiply both factors to obtain $(x - (2 + 2i))(x - (2 - 2i)) = x^2 - 4x + 8$, and then long divide $x^4 + 64$ by $x^2 - 4x + 8$ to obtain $x^2 + 4x + 8$. At this point, we know that

$$x^4 + 64 = \left(x^2 - 4x + 8\right)\left(x^2 + 4x + 8\right).$$

As it turns out, this is the full factorization over the reals, because the second factor's discriminant is negative: $\Delta = 4^2 - 4 \cdot 8 = 16 - 32 = -16$. Indeed, both roots are complex:

$$x = \frac{-4 \pm \sqrt{\Delta}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i,$$

and the complex factors are $x - (-2 + 2i)$ and $x - (-2 - 2i)$. The full factoring over the complex numbers is therefore

$$\begin{aligned} x^4 + 64 &= (x - [2 + 2i])\,(x - [2 - 2i])\,(x - [-2 + 2i])\,(x - [-2 - 2i]) \\ &= (x - 2 - 2i)\,(x - 2 + 2i)\,(x + 2 - 2i)\,(x + 2 + 2i)\,. \end{aligned}$$

■

Let us conclude this section by learning how to manipulate polynomials using technology. In Maxima, the imaginary number is denoted with the percent symbol followed by the lowercase letter i: `%i`.In SageMath, the imaginary number is denoted with the uppercase letter `I`. In Maxima program 2.2.1 and SageMath program 2.2.1, we learn how to manipulate polynomials: adding them, multiplying them, expanding them, and factoring them over the reals. Using SageMath, we can also see how to find all of a polynomial's roots over the complex numbers.

Maxima program 2.2.1 Manipulating polynomials

```
(%i1) kill(all)$ reset()$
(%i2) define( a(x), x^2 - 5*x + 6 );
```

(%o2) $a(x) := x^2 - 5\,x + 6$

```
(%i3) define( b(x), x^2 - 8*x + 13 );
```

(%o3) $b(x) := x^2 - 8\,x + 13$

```
(%i4) define( c(x), a(x)+b(x) );
```

(%o4) $c(x) := 2\,x^2 - 13\,x + 19$

```
(%i5) define( p(x), a(x)*b(x) );
```

(%o5) $p(x) := \left(x^2 - 8\,x + 13\right)\left(x^2 - 5\,x + 6\right)$

```
(%i6) expand(p(x));
```

(%o6) $x^4 - 13\,x^3 + 59\,x^2 - 113\,x + 78$

```
(%i7) factor(p(x));
```

(%o7) $(x-3)\,(x-2)\,\left(x^2 - 8\,x + 13\right)$

```
(%i8) define( q(x), a(x)/b(x) );
```

(%o8) $q(x) := \dfrac{x^2 - 5\,x + 6}{x^2 - 8\,x + 13}$

```
(%i9) expand(q(x));
```

(%o9) $\dfrac{x^2}{x^2 - 8\,x + 13} - \dfrac{5\,x}{x^2 - 8\,x + 13} + \dfrac{6}{x^2 - 8\,x + 13}$

```
(%i10) factor(q(x));
```

(%o10) $\dfrac{(x-3)\,(x-2)}{x^2 - 8\,x + 13}$

SageMath program 2.2.1 Manipulating polynomials

Input

```
a(x) = x^2 - 5*x + 6
b(x) = x^2 - 8*x + 13
print "adding a(x) and b(x)"
c(x) = a(x)+b(x)
print c(x)
print "multiplying a(x) and b(x)"
p(x) = a(x)*b(x)
print p(x)
print "expanding p(x)"
expand(p(x))
print "factoring p(x) over the real numbers"
factor(p(x))
print "finding the roots of p(x) over the complex numbers"
solve(p(x)==0,x)
print "dividing a(x) by b(x)"
q(x) = a(x)/b(x)
print q(x)
print "expanding q(x)"
expand(q(x))
print "factoring q(x) over the real numbers"
factor(q(x))
```

Output

```
adding a(x) and b(x)
2*x^2 - 13*x + 19
multiplying a(x) and b(x)
(x^2 - 5*x + 6)*(x^2 - 8*x + 13)
expanding p(x)
x^4 - 13*x^3 + 59*x^2 - 113*x + 78
factoring p(x) over the real numbers
(x^2 - 8*x + 13)*(x - 2)*(x - 3)
finding the roots of p(x) over the complex numbers
[x == 3, x == 2, x == -sqrt(3) + 4, x == sqrt(3) + 4]
dividing a(x) by b(x)
(x^2 - 5*x + 6)/(x^2 - 8*x + 13)
expanding q(x)
x^2/(x^2 - 8*x + 13) - 5*x/(x^2 - 8*x + 13) + 6/(x^2 - 8*x + 13)
factoring q(x) over the real numbers
(x - 2)*(x - 3)/(x^2 - 8*x + 13)
```

Exercises

Factor each polynomial over $\mathbb{R}$ and $\mathbb{C}$. Verify these factorizations by expanding them. You can also use Maxima or SageMath to verify the factoring over $\mathbb{R}$.

Exercise* 2.2.101*: $x^2 - 4x + 13$

Exercise* 2.2.102*: $x^2 - 2x + 5$

Exercise* 2.2.103*: $3x^2 + 2x + 10$

Exercise* 2.2.104*: $x^3 - 2x^2 + 9x - 18$

Exercise* 2.2.105*: $x^3 + 6x^2 + 6x + 5$. *Hint:* $x = -5$ *is a root.*

Exercise* 2.2.106*: $3x^3 - 13x^2 + 43x - 13$. *Hint:* $3x - 1$ *is a factor.*

Exercise* 2.2.107*: $x^3 + 3x^2 + 4x + 12$

Exercise* 2.2.108*: $x^4 + x^3 + 7x^2 + 9x - 18$

Exercise* 2.2.109*: $x^4 + 9x^2 + 20$

Exercise* 2.2.110*: $x^4 + 5x^2 - 24$

Exercise* 2.2.111*: $x^4 - 2x^3 + 27x^2 - 2x + 26$. *Hint:* $x = i$ *is a root.*

Exercise* 2.2.112*: $2x^4 + 5x^3 + 13x^2 + 7x + 5$. *Hint:* $x = -1 + 2i$ *is a root.*

2.3 Hyperbolic functions

Attribution: §4.11 in [G]

The hyperbolic functions appear with some frequency in applications. Surprisingly, while the hyperbolic functions are quite similar in many respects to the trigonometric functions, their definitions involve exponential functions.

Definition 2.3.1. *The hyperbolic sine is the function* $\sinh x = \dfrac{e^x - e^{-x}}{2}$, *and the hyperbolic cosine is the function* $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Graphs of $\sinh x$ and $\cosh x$ are shown in Figure 9.5. The graph of hyperbolic cosine is the exact shape a hanging chain or flexible cable will make if suspended between two points of equal height. This shape is called a *catenary*; contrary to popular belief this is not a parabola. If you invert the graph of cosh it is also the ideal arch for supporting its own weight. For example, the Gateway Arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall down. The formula used in the design is inscribed inside the arch:

$$y = -127.7\,\text{ft} \cdot \cosh(x/127.7\,\text{ft}) + 757.7\,\text{ft}.$$

Notice that hyperbolic cosine is even

$$\cosh(-x) = \cosh x$$

while hyperbolic sine is odd

$$\sinh(-x) = -\sinh x.$$

Theorem 2.3.1. *The domain of* $\cosh x$ *is* $\mathbb{R}$.

Proof. Let $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$. The domain of the growing exponential function e^x is $\mathbb{R}$. The domain of the decaying exponential function e^{-x} is $\mathbb{R}$. The domain of a sum of functions is the intersection of their domains, so $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Multiplying a function by scalar $\frac{1}{2}$ does not affect its domain. $\square$

Theorem 2.3.2. *The range of* $\cosh x$ *is* $[1, \infty)$.

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned}
y &= \frac{e^x + e^{-x}}{2} \\
2y &= e^x + e^{-x} \\
2ye^x &= e^{2x} + 1 \\
0 &= e^{2x} - 2ye^x + 1 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}
\end{aligned}$$

In the first equation, we clearly see that $y > 0$, since y is a weighted sum of exponential functions that are themselves positive. In this last equation, to ensure that the radicand is nonnegative, we see that $y^2 \geq 1$. Putting both conditions together, it follows that $y \geq 1$.

Now suppose $y \geq 1$, so $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh x$. □

In this course, we will mainly use the hyperbolic sine and cosine functions. In the remainder of this section, we mention the other hyperbolic functions, as well as learn some helpful properties of sinh and cosh.

Definition 2.3.2. *The other hyperbolic functions are*

$$\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} \\
\operatorname{sech} x &= \frac{1}{\cosh x} \\
\operatorname{csch} x &= \frac{1}{\sinh x}.
\end{aligned}$$

Graphs of these functions appear in Figure 9.6 and Figure 9.7.

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identities.

Theorem 2.3.3. *The following identities hold wherever they are defined:*

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 \\
1 - \tanh^2 x &= \operatorname{sech}^2 x \\
\coth^2 x - 1 &= \operatorname{csch}^2 x.
\end{aligned}$$

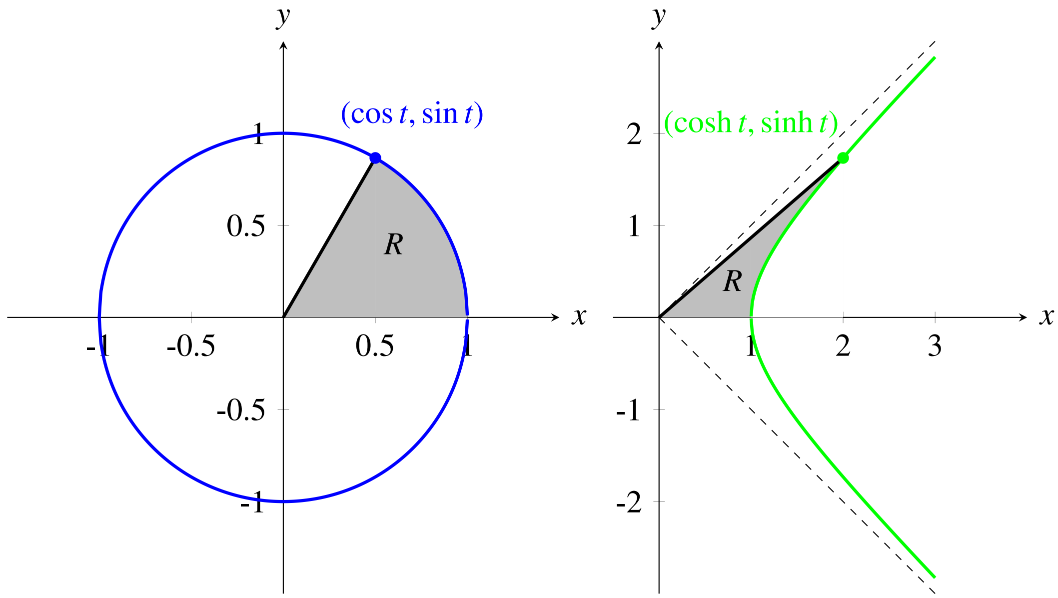


Figure 2.1: Geometric definitions of sin, cos, sinh, cosh: in both trigonometric and hyperbolic cases, t is twice the area of shaded region R.

This first identity also helps to provide a geometric motivation. Recall that the graph of $x^2-y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm\sqrt{x^2 - 1} = \pm\sqrt{\cosh^2 x - 1} = \pm \sinh t$. So for some suitable t, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that t is twice the area shown in the first graph of Figure 2.1. Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and t is twice the area shown in the second graph of Figure 2.1.

We will identify more identities of hyperbolic functions in the exercises. Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Again, we will see similarities with the trigonometric functions. Each derivative expression also gives us an anti-derivative expression.

Theorem 2.3.4 (Derivatives of hyperbolic functions)**.**

- $\dfrac{d}{dx}\sinh x = \cosh x$

- $\dfrac{d}{dx}\cosh x = \sinh x$

- $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$

- $\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$
- $\dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x$

Proof. We prove the first derivative formula using the exponential definition for hyperbolic sine:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x - -e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

We prove the third derivative formula using the quotient rule and an identity:

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Remaining proofs are left as exercises. □

Exercises

Exercise* 2.3.1*: *Use the definition of the hyperbolic functions in terms of exponentials to show the following properties:*

- $\sinh 0 = 0$,
- $\cosh 0 = 1$,
- $\sinh x$ *is odd,*
- $\cosh x$ *is even,*
- $\sinh x$ *intersects the x-axis at the origin,*
- $\cosh x$ *never intersects the x-axis.*

Exercise* 2.3.2*: *Use the definition of hyperbolic sine and cosine in terms of exponentials to verify the first identity of Theorem 2.3.3:* $\cosh^2 x - \sinh^2 x = 1$. *Then, manipulate this identity to prove the other two identities.*

Exercise* 2.3.3*: *Draw a sketch of the increasing exponential curve* $y = \frac{1}{2}e^x$ *and the decreasing exponential curve* $y = \frac{1}{2}e^{-x}$. *Add them to obtain a sketch of the curve* $y = \cosh x$. *Subtract them to obtain a sketch of the curve* $y = \sinh x$. *Compare with Figure 9.5.*

Exercise* 2.3.4*: *Draw a sketch of the curve* $y = \tanh x$. *Identify all asymptotes. Compare with Figure 9.6.*

Prove the identity.

Exercise* 2.3.5*: $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

Exercise* 2.3.6*: $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

Exercise* 2.3.7*: $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Exercise* 2.3.8*: $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

Exercise* 2.3.9*: $\sinh 2x = 2 \sinh x \cosh x$

Exercise* 2.3.10*: $\cosh 2x = \cosh^2 x + \sinh^2 x$

Exercise* 2.3.11*: *Use the definition of the hyperbolic functions in terms of exponentials to show that* $\dfrac{d}{dx}\cosh x = \sinh x$, *similarly to the proof to the first part of this section's theorem.*

Compute the derivatives.

Exercise* 2.3.12*: $\dfrac{d}{dx}\sinh x\cosh x$

Exercise* 2.3.13*: $\dfrac{d}{dx}\sinh(\cosh x)$

Exercise* 2.3.14*: $\dfrac{d}{dx}\cosh^2 x - \sinh^2 x$

Exercise* 2.3.15*: $\dfrac{d}{dx}\tanh 3x$

Exercise* 2.3.16*: $\dfrac{d}{dx}\cosh 4x$

Exercise* 2.3.17*: $\dfrac{d}{dx}\tanh e^x$

Exercise* 2.3.18*: *Find an expression for the second derivative of each of the six hyperbolic functions.*

Compute the integrals. Verify each answer by differentiating it to retrieve the integrand.

Exercise* 2.3.19*: $\displaystyle\int \sinh^3 x\cosh x\,dx$

Exercise* 2.3.20*: $\displaystyle\int \frac{\sinh x}{\cosh^3 x}\,dx$

Exercise* 2.3.21*: $\displaystyle\int \operatorname{sech}^2 x\tanh x\,dx$

Exercise* 2.3.22*: $\displaystyle\int_0^2 (\cosh^2 x + \sinh^2 x)\,dx$

2.4 Calculus

Differential equations involve functions and their derivatives. Therefore, recalling some integration formulas will be helpful. You should be able to differentiate and integrate simple functions involving power, exponential, logarithmic, trigonometric and hyperbolic functions. Some expressions are more challenging to integrate. The list of antiderivative formulas in Section 9.5 on page 321 should prove useful. The limit of a function is another topic that will also come up as we analyze Differential Equations.

Exercises

Exercise* 2.4.1*: *Evaluate* $\displaystyle\int \left(t^2 + t\right)\,dt$.

Exercise* 2.4.2*: *Evaluate* $\displaystyle\int \left(e^t + t\right)\,dt$.

Exercise* 2.4.3*: *Evaluate* $\displaystyle\int \frac{dx}{x}$.

Exercise* 2.4.4*: *Evaluate* $\displaystyle\int \frac{dx}{x^2}$.

Exercise* 2.4.5*: *Evaluate* $\displaystyle\int \frac{1 + 2x^2}{x}$.

Exercise* 2.4.6*: *Evaluate* $\displaystyle\int \sin 5y\,dy$.

Exercise* 2.4.7*: *Antidifferentiate* $f(t) = t^2 e^{t^3}$.

Exercise* 2.4.8*: *Differentiate* $f(x) = xe^x$ *using the product rule, and evaluate* $f'(0)$ *and* $f'(1)$.

Exercise* 2.4.9*: *Let* $f(x) = xe^x$*. Antidifferentiate* $f(x)$ *by parts. Verify your answer by comparing with the table of antiderivatives in this section. Then, evaluate* $\int_0^x f(s)\,ds$.

Exercise* 2.4.10*: *Differentiate* $g(x) = \dfrac{1}{a^2 + x^2}$ *for* $a > 0$ *using the quotient rule. Then, evaluate* $g'(0)$ *and* $g'(1)$.

Exercise* 2.4.11*: *Let* $g(x) = \dfrac{1}{a^2 + x^2}$ *for* $a > 0$*. Antidifferentiate* $g(x)$ *using trigonometric substitution. Verify your answer by comparing with the list of antiderivatives in this section. Then, evaluate* $\int_0^x g(s)\,ds$.

Exercise* 2.4.12*: *Let* $h(x) = \dfrac{1}{x^2 - a^2}$ *for* $a > 0$*. Antidifferentiate* $h(x)$ *using partial fraction decomposition. Verify your answer by comparing with the table of antiderivatives in this section. Then, evaluate* $\int_0^x h(s)\,ds$.

Exercise* 2.4.13*: *Antidifferentiate* $\sin^2\theta$ *by exploiting a trigonometric identity to verify the formula appearing in this section's table of antiderivatives.*

Exercise* 2.4.14*: *Antidifferentiate* $e^{aw}\sin bw$ *for real constants a and b by integrating by parts twice to verify the formula appearing in this section's table of antiderivatives.*

2.5 Power series

Attribution: §7.1 in [L]
Further reading: §8.1 in [EP], §5.1 in [BD]

A *power series* is an expression such as

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots \tag{2.1}$$

where $a_0, a_1, a_2, \ldots, a_k, \ldots$ and x_0 are constants. Many functions can be expressed as a power series. So can the solution to many differential equations. Let us review some results and concepts about power series.

Let

$$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

denote the so-called *partial sum*. If for some x, the limit

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x - x_0)^k$$

exists, then we say that the series (2.1) *converges* at x. Note that for $x = x_0$, the series always converges to a_0. When (2.1) converges at any other point $x \neq x_0$, we say that (2.1) is a *convergent power series*. In this case we write

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x - x_0)^k.$$

If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$.

We say that (2.1) *converges absolutely* at x whenever the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} |a_k|\, |x - x_0|^k$$

exists. That is, the series $\sum_{k=0}^{\infty} |a_k|\, |x - x_0|^k$ is convergent. If (2.1) converges absolutely at x, then it converges at x. However, the opposite implication is not true.

2.5.1 Radius of convergence

If a power series converges absolutely at some x_1, then for all x such that $|x - x_0| \leq |x_1 - x_0|$ (that is, x is closer than x_1 to x_0) we have $|a_k(x - x_0)^k| \leq |a_k(x_1 - x_0)^k|$ for all k. As the numbers $|a_k(x_1 - x_0)^k|$ sum to some finite limit, summing smaller positive numbers $|a_k(x - x_0)^k|$ must also have a finite limit. Therefore, the series must converge absolutely at x. We have the following result.

Theorem 2.5.1. *For a power series* (2.1)*, there exists a number ρ (we allow $\rho = \infty$) called the* radius of convergence *such that the series converges absolutely on the interval* $(x_0 - \rho, x_0 + \rho)$ *and diverges for $x < x_0 - \rho$ and $x > x_0 + \rho$. We write $\rho = \infty$ if the series converges for all x.*

See Figure 2.2.

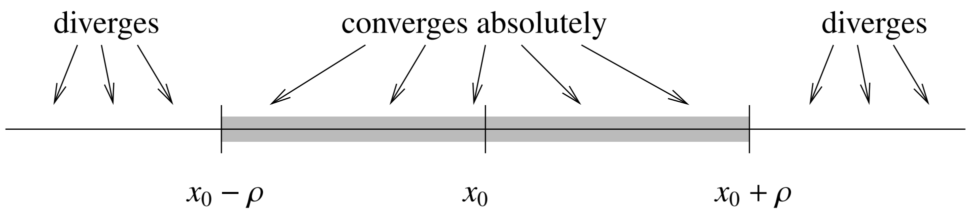


Figure 2.2: Convergence of a power series.

In Example 2.5.1 the radius of convergence is $\rho = \infty$ as the series converges everywhere. In Example 2.5.2 the radius of convergence is $\rho = 1$. We note that $\rho = 0$ is another way of saying that the series is divergent.

A useful test for convergence of a series is the *ratio test*.

Theorem 2.5.2 (Ratio test)**.** *Consider the series* $\sum_{k=0}^{\infty} c_k$. *Suppose that* $\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = L$. *Then:*

- *If $L < 1$ the series converges (absolutely).*
- *If $L > 1$ the series diverges.*
- *If $L = 1$ this test is inconclusive.*

Example 2.5.1. Let us determine the radius and interval of convergence of the power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. The k^{th} term of this series is $c_k = (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. The $(k+1)^{\text{th}}$ term of this series is $c_{k+1} = (-1)^{k+1} \frac{x^{2k+3}}{(2k+3)!}$.

We apply the ratio test:

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{x^{2k+3}}{(2k+3)!}}{(-1)^k\frac{x^{2k+1}}{(2k+1)!}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| = x^2\lim_{k\to\infty}\frac{1}{(2k+3)(2k+2)} = 0 < 1.$$

Since 0 is less than 1 for all x, the series converges regardless of what x is. So the radius of convergence is ∞ and the interval of convergence is $\mathbb{R}$. ■

Example 2.5.2. Suppose we have the series $\displaystyle\sum_{k=0}^{\infty}\frac{(x-1)^k}{2^k}$. We compute

$$\lim_{k\to\infty}\left|\frac{\frac{(x-1)^{k+1}}{2^{k+1}}}{\frac{(x-1)^k}{2^k}}\right| = \lim_{k\to\infty}\left|\frac{x-1}{2}\right| = \frac{|x-1|}{2} < 1.$$

What values of x satisfy this inequality?

$$\frac{|x-1|}{2} < 1 \quad\Rightarrow\quad |x-1| < 2 \quad\Rightarrow\quad -2 < x-1 < 2 \quad\Rightarrow\quad -1 < x < 3$$

Therefore the interval of convergence is $(-1,3)$ and the radius of convergence is 2. ■

The ratio test does not always apply. That is, the limit of $\left|\frac{a_{k+1}}{a_k}\right|$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter.

Exercise* 2.5.1*: *Show that the Taylor series for e^x which is $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ is absolutely convergent for any x.*

Exercise* 2.5.2*: *Show that the series $\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$ converges absolutely for all x in the interval $(-1,1)$.*

2.5.2 Analytic functions

Functions represented by power series are called *analytic functions*. Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function $f(x)$ is equal to its *Taylor series** near a point x_0. That is, for x near x_0 we have

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{2.2}$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0.

*Named after the English mathematician Sir Brook Taylor (1685 – 1731).

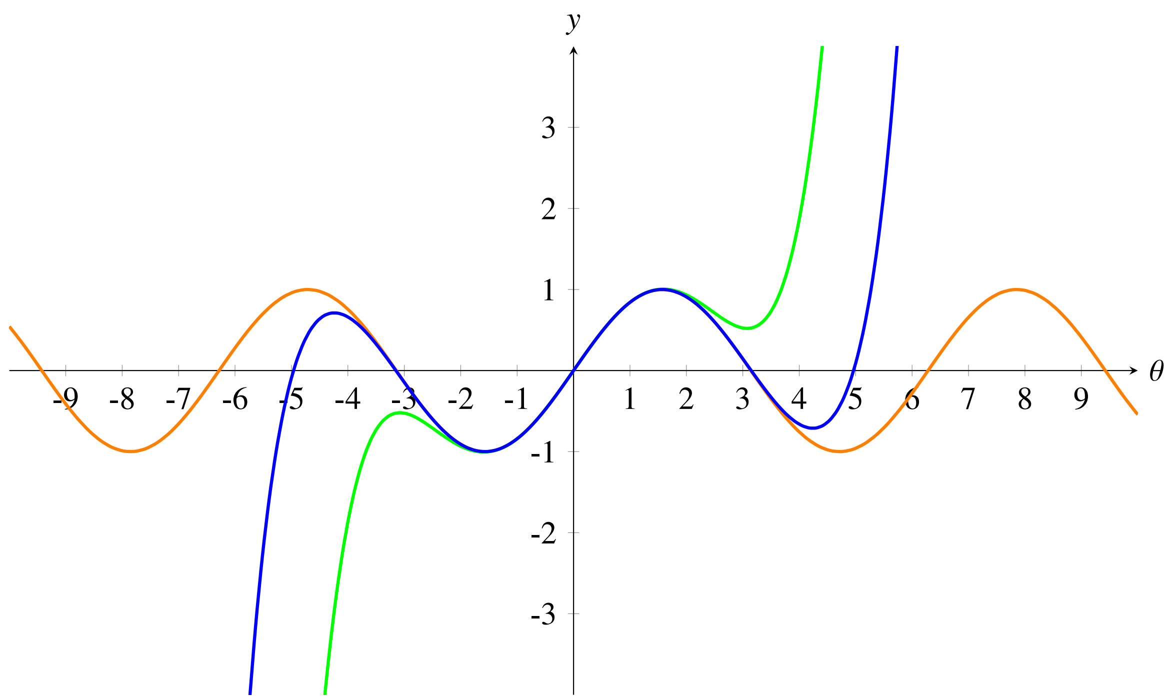


Figure 2.3: The sine function (orange) and its Taylor approximations around $x_0 = 0$ of 5^{th} degree (green) and 9^{th} degree (blue).

For example, sine is an analytic function and its Taylor series around $x_0 = 0$ is given by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

In Figure 2.3 we plot $\sin x$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for x further away from 0. This is what happens in general. To get a good approximation far away from x_0 you need to take more and more terms of the Taylor series.

2.5.3 Manipulating power series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is, suppose that $\sum a_k (x - x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k (x-x_0)^k\right] = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}.$$

Notice that the term corresponding to $k = 0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example 2.5.3. Let us show that the exponential $y = e^x$ solves $y' = y$. Put otherwise, let us show that the derivative of e^x is itself using its Taylor series. First write

$$y = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Now differentiate

$$y' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}.$$

We *reindex* the series by simply replacing k with $k + 1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k = 0$ again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k+1=1}^{\infty} \frac{1}{((k+1)-1)!} x^{(k+1)-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for e^x that we started with, so we showed that $\frac{d}{dx}[e^x] = e^x$. ■

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. First, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) + \left(\sum_{k=0}^{\infty} b_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} (a_k + b_k)(x - x_0)^k.$$

We can multiply by constants,

$$\alpha \left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} \alpha a_k (x - x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) \left(\sum_{k=0}^{\infty} b_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} c_k (x - x_0)^k,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

2.5.4 Power series for rational functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the a_k are zero for all k large enough. We can always expand a polynomial as a power series about any point

x_0 by writing the polynomial as a polynomial in $(x - x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x - 1) + 2(x - 1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$ as the polynomial is of degree 2. We write $a_0 + a_1(x - 1) + a_2(x - 1)^2$, we expand, and we solve for a_0, a_1, and a_2. We could have also differentiated at $x = 1$ and used the Taylor series formula (2.2).

Let us look at rational functions, that is, ratios of polynomials. An important fact is that a series for a function only defines the function on an interval even if the function is defined elsewhere. For example, for $-1 < x < 1$ we have

$$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0. Note that as for polynomials, we could equivalently use the Taylor series expansion (2.2).

Example 2.5.4. Expand $\dfrac{x}{1 + 2x + x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

First, write $1 + 2x + x^2 = (1 + x)^2 = (1 - (-x))^2$. Now we compute

$$\begin{aligned} \frac{x}{1 + 2x + x^2} &= x \left(\frac{1}{1 - (-x)} \right)^2 \\ &= x \left(\sum_{k=0}^{\infty} (-1)^k x^k \right)^2 \\ &= x \left(\sum_{k=0}^{\infty} c_k x^k \right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1}, \end{aligned}$$

where using the formula for the product of series we obtain, $c_0 = 1$, $c_1 = -1 - 1 = -2$, $c_2 = 1 + 1 + 1 = 3$, etc. . . . Therefore

$$\frac{x}{1 + 2x + x^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k}\right| = \lim_{k\to\infty}\frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1. ∎

When the rational function is more complicated, it is also possible to use method of partial fractions. For example, to find the Taylor series for $\frac{x^3+x}{x^2-1}$, we write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{k=0}^{\infty}(-1)^k x^k - \sum_{k=0}^{\infty} x^k = -x + \sum_{\substack{k=3\\ k \text{ odd}}}^{\infty}(-2)x^k.$$

Exercises

Is the power series convergent? If so, what is the interval of convergence?

Exercise* 2.5.3*: $\displaystyle\sum_{k=0}^{\infty} e^k x^k$

Exercise* 2.5.4*: $\displaystyle\sum_{k=0}^{\infty} k x^k$

Exercise* 2.5.5*: $\displaystyle\sum_{k=0}^{\infty} k! x^k$

Exercise* 2.5.6*: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-10)^k$

Exercise* 2.5.7*: *Determine the Taylor series for* $\sin x$ *around the point* $x_0 = \pi$*.*

Exercise* 2.5.8*: *Determine the Taylor series for* $\ln x$ *around the point* $x_0 = 1$*, and find the interval of convergence.*

Exercise* 2.5.9*: *Determine the Taylor series and its interval of convergence of* $\dfrac{1}{1+x}$ *around* $x_0 = 0$*.*

Exercise* 2.5.10*: *Determine the Taylor series and its interval of convergence of* $\dfrac{x}{4-x^2}$ *around* $x_0 = 0$*. Hint: You will not be able to use the ratio test.*

Exercise* 2.5.11*: *Expand* $x^5 + 5x + 1$ *as a power series around* $x_0 = 5$*.*

Exercise* 2.5.12*: *Suppose that the ratio test applies to a series* $\displaystyle\sum_{k=0}^{\infty} a_k x^k$*. Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.*

Exercise* 2.5.13*: *Suppose that* f *is an analytic function such that* $f^{(n)}(0) = n$*. Find* $f(1)$*.*

Exercise* 2.5.14*: *Show that the second derivative of* $\sin x$ *is negative itself using Taylor series, in a manner similar to Example 2.5.3.*

Is the power series convergent? If so, what is the interval of convergence?

Exercise* 2.5.101*: $\displaystyle\sum_{n=1}^{\infty} 0.1^n x^n$

Exercise* 2.5.102** (challenging):*** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Exercise* 2.5.103*: *Using the geometric series, expand* $\dfrac{1}{1-x}$ *around* $x_0 = 2$*. For what* x *does the series converge?*

Exercise* 2.5.104** (challenging):*** *Find the Taylor series for* $x^7 e^x$ *around* $x_0 = 0$*.*

Exercise* 2.5.105** (challenging):*** *Imagine* f *and* g *are analytic functions such that* $f^{(k)}(0) = g^{(k)}(0)$ *for all large enough* k*. What can you say about* $f(x) - g(x)$*?*

2.6 Linear independence of functions

Attribution: §2.3.1 in [L]

When we have a set of n functions $\{y_1, y_2, \ldots, y_n\}$, we say that these functions are linearly independent if none can be written as a linear combination of the others. We state this as follows.

Definition 2.6.1. *The set of functions* $\{y_1, y_2, \ldots, y_n\}$ *is* linearly independent *if*

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$*, where the equation must hold for all* x*.*

If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If a set of functions is not linearly independent, it is *linearly dependent*.

Example 2.6.1. Show that $\{e^x, e^{-x}, \cosh x\}$ is a linearly dependent set of functions. Let us write down

$$c_1 e^x + c_2 e^{-x} + c_3 \cosh x = 0.$$

We recall from the definition of the hyperbolic cosine that

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \Rightarrow \quad 2\cosh x - e^x - e^{-x} = 0 \quad \Rightarrow \quad -e^x - e^{-x} + 2\cosh x = 0.$$

Therefore $c_1 = -1$, $c_2 = -1$, and $c_3 = 2$. Since it is possible to write $c_1 e^x + c_2 e^{-x} + c_3 \cosh x = 0$ in such a way that the c_i's are not all zeroes, we conclude that our set is linearly dependent. ■

Example 2.6.2. Show that the set $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent. Let us give several ways to show this fact.* There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

First Solution Let us write down

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We use rules of exponentials and write $z = e^x$. Hence $z^2 = e^{2x}$ and $z^3 = e^{3x}$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0.$$

The left hand side is a third degree polynomial in z. It is either identically zero, or it has at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

*Many textbooks (including [EP] and [F]) introduce Wronskians, but it is difficult to see why they work and they are not really necessary here.

Second Solution As before we write

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0.$$

This equation has to hold for all x. We divide through by e^{3x} to get

$$c_1e^{-2x} + c_2e^{-x} + c_3 = 0.$$

As the equation is true for all x, let $x \to \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1e^x + c_2e^{2x} = 0.$$

Rinse, repeat! We divide through by e^{2x} to get

$$c_1e^{-x} + c_2 = 0.$$

As before, let $x \to \infty$. After taking the limit we see that $c_2 = 0$. Hence our equation becomes

$$c_1e^x = 0.$$

Rinse, repeat! We divide through by e^x and find that $c_1 = 0$. Since the only solution to

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$$

is $c_1 = c_2 = c_3 = 0$, we conclude that the set of functions are linearly independent.

Third Solution We again write

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0.$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2e^x + c_3e^{2x} = 0.$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2e^x + 2c_3e^{2x} = 0.$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

Fourth Solution We again write

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0.$$

Differentiate this equation twice:

$$\begin{aligned} c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} &= 0 \\ c_1e^x + 4c_2e^{2x} + 9c_3e^{3x} &= 0. \end{aligned}$$

We know have 3 equations. Evaluate all of them at $x = 0$ to obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \quad \clubsuit \\ c_1 + 2c_2 + 3c_3 &= 0 \quad \diamondsuit \\ c_1 + 4c_2 + 9c_3 &= 0. \quad \heartsuit \end{aligned}$$

Subtracting $\clubsuit$ from $\diamondsuit$ yields

$$c_2 + 2c_3 = 0 \quad \dagger$$

and subtracting $\clubsuit$ from $\heartsuit$ yields

$$c_2 + 8c_3 = 0. \quad \ddagger$$

Finally, subtracting $\dagger$ from $\ddagger$ gives

$$7c_3 = 0$$

which means $c_3 = 0$. Plugging this into $\dagger$ or $\ddagger$, we find $c_2 = 0$. Plugging both into $\clubsuit$, $\diamondsuit$ or $\heartsuit$, we find $c_1 = 0$. Since the only solution to

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$$

is $c_1 = c_2 = c_3 = 0$, we conclude that the set of functions are linearly independent. ∎

Example 2.6.3. The functions $\sin x$ and $\cos x$ are linearly independent. To show this, we write $c_1 \sin x + c_2 \cos x = 0$. Manipulating this expression a little, we get $\sin x = -\frac{c_2}{c_1} \cos x$, and setting $A = -\frac{c_2}{c_1}$, we get $\sin x = A \cos x$. So, when dealing with only two functions, we can ask ourselves whether one is a constant multiple of the other. It is not hard to see that sine and cosine are not constant multiples of each other. Letting $x = 0$ implies that $A = 0$. But then $\sin x = 0$ for all x, which is preposterous. So $\sin x$ and $\cos x$ are linearly independent. ∎

Example 2.6.4. The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent we could write $e^{4x} = Ce^{2x}$ for some constant C, implying that $e^{2x} = C$ for all x, which is clearly not possible. The easiest way to see this is by expanding the left-hand side $e^{2x}e^{2x} = Ce^{2x}$ meaning that $e^{2x} = C$ which is not true since an exponential function is never constant. ∎

Exercises

Exercise* 2.6.1*: *Show that* $y = e^x$ *and* $y = e^{2x}$ *are linearly independent.*

Exercise* 2.6.2*: *Are* e^{4x} *and* xe^{4x} *linearly independent?*

Exercise* 2.6.3*: *Are* $\sin x$ *and* $\tan x$ *linearly independent?*

Exercise* 2.6.4*: *Are* $\sin(x)$ *and* $\sin(2x)$ *linearly independent?*

Exercise* 2.6.5*: *Let* $s \in \mathbb{R}$ *and* $t \in \mathbb{R}$ *with* $s \neq t$. *Determine whether the set of functions* $\{x^s, x^t\}$ *is linearly dependent or independent.*

Exercise* 2.6.6*: *Let* $r \in \mathbb{R}$. *Determine whether the set of functions* $\{x^r, x^r \ln x\}$ *is linearly dependent or independent.*

Exercise* 2.6.7*: *Let* $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}$. *Are* $e^{\alpha x}\cos(\beta x)$ *and* $e^{\alpha x}\sin(\beta x)$ *linearly independent?*

Exercise* 2.6.8*: *Let* $f(x) = e^x - \cos x$, $g(x) = e^x + \cos x$, *and* $h(x) = \cos x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

Exercise* 2.6.9*: *Let* $f(x) = 1$, $g(x) = \sin x$, *and* $h(x) = \cos x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

Exercise* 2.6.10*: *Are* x, x^2, *and* x^4 *linearly independent? If so, show it, if not, find a linear combination that works.*

Exercise* 2.6.11*: *Are* e^{-x}, e^x, *and* e^{3x} *linearly independent? If so, show it, if not, find a linear combination that works.*

Exercise* 2.6.12*: *Are* e^x, xe^x, *and* x^2e^x *linearly independent? If so, show it, if not, find a linear combination that works.*

Exercise* 2.6.101*: *Are* $\sin x$ *and* e^x *linearly independent? Justify.*

Exercise* 2.6.102*: *Are* e^x *and* e^{x+2} *linearly independent? Justify.*

Exercise* 2.6.103*: *Are* e^x, e^{x+1}, e^{2x}, $\sin x$ *linearly independent? If so, show it, if not find a linear combination that works.*

Exercise* 2.6.104*: *Are* $\sin x$, x, $x\sin x$ *linearly independent? If so, show it, if not find a linear combination that works.*

Chapter 3

First order ODEs

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. Throughout this chapter we will look at special cases where solutions are not difficult to obtain.

3.1 Integrals as solutions

Attribution: §1.1 in [L]
Further reading: §1.2 in [EP],covered in §1.2,2.1 in [BD]

For now, let us assume that f is a function of independent variable x alone, that is, the equation is

$$\frac{dy}{dx} = f(x) \quad \text{or} \quad y' = f(x). \tag{3.1}$$

We could just indefinitely integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)\,dx = \int f(x)\,dx + C,$$

that is

$$y(x) = \int f(x)\,dx + C.$$

This $y(x)$ is actually the general solution. So to solve (3.1), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The indefinite integral is really the *antiderivative* (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)\,dx + C$ as

$$\int_{x_0}^{x} f(t)\,dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral.

Example 3.1.1. Find the general solution of $y' = 3x^2$.

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We have gotten *precisely* our equation back. ■

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(s)\,ds.} \tag{3.2}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = y_0 + \int_{x_0}^{x_0} f(x)\,dx = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (3.2) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 3.1.2. Solve $y' = e^{-x^2}$ with $y(0) = 1$.

By the preceding discussion, the solution must be

$$y(x) = 1 + \int_{0}^{x} e^{-s^2}\,ds.$$

You might spend a little time trying to evaluate this integral. Substitution? Parts? Another technique? No matter what you try, this integral doesn't possess a closed form solution, and there is absolutely nothing wrong with writing the solution as a definite integral. The integral $\int_0^x e^{-s^2}$ is in fact very important in statistics, so important that it's been given a name, even though it doesn't possess a closed form. It's called the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds.$$

Armed with this information, we could express our solution as

$$\boxed{y(x) = 1 + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x).} \tag{3.3}$$

This function has the shape of a sigmoid, and is graphed in Figure 3.1. We will encounter $\operatorname{erf}(x)$ again, in Exercise 3.1.10 and Exercise 3.3.12. ■

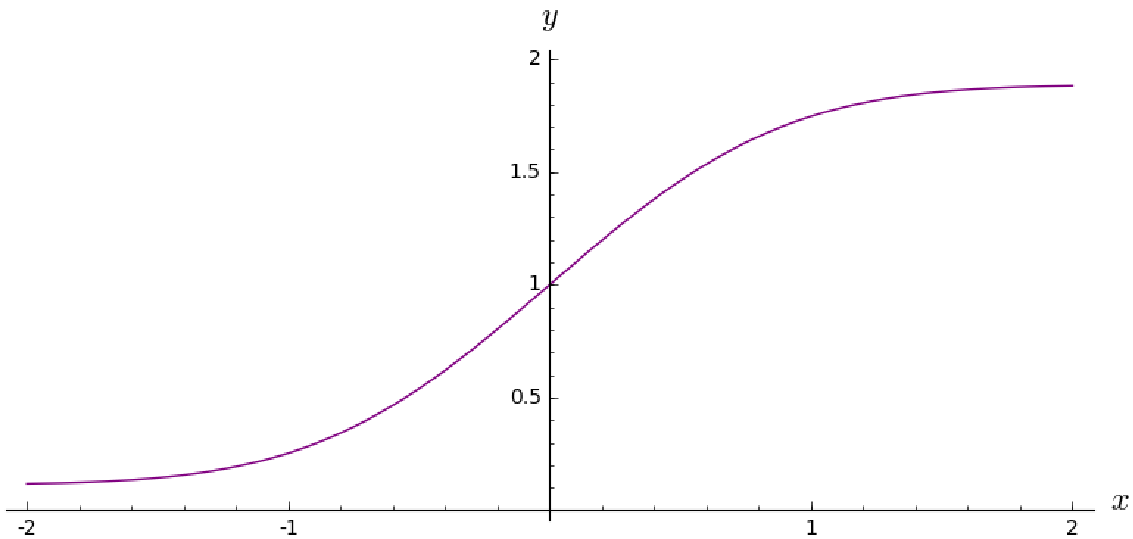


Figure 3.1: The solution to the differential equation $y' = e^{-x^2}$ subject to the initial condition $y(0) = 1$ is a function possessing a sigmoidal shape: $y(x) = 1 + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. This graph was generated using SageMath.

Maxima and SageMath both know about the error function. Examine Maxima program 3.1.1 or SageMath program 3.1.1, and notice that the answer matches the one above.

In the remainder of this section, we solve equations of the form

$$\frac{dy}{dx} = f(y) \quad \text{or} \quad y' = f(y). \tag{3.4}$$

Notice that, unlike in (3.1) where the right-hand side of the DE involves independent variable x, now in (3.4), the right-hand side of the DE involves dependent variable y. How do we handle this situation?

Maxima program 3.1.1 A solution containing the error function

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4) load('contrib_ode)$

(%i5) x0 : 0;
      y0 : 1;
```

(%o5) 0

(%o6) 1

```
(%i7) de : 'diff(y,x,1) = exp(-x^2);
```

(%o7) $\frac{d}{d\,x}\,y = e^{-x^2}$

```
(%i8) gsoln : contrib_ode(de,y,x);
```

(%o8) $[y = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x)}{2} + \%c]$

```
(%i9) soln : ic1(gsoln,x=x0,y=y0);
```

(%o9) $[y = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x) + 2}{2}]$

```
(%i10) define( y(x),rhs(soln[1]) );
```

(%o10) $\mathrm{y}\,(x) := \frac{\sqrt{\pi}\,\mathrm{erf}\,(x) + 2}{2}$

```
(%i11) plot2d( y(x),[x,-2,2],[color,purple] );
```

SageMath program 3.1.1 A solution containing the error function

Input

```
# define independent variable x
x = var('x')
# define dependent variable y
y = function('y')(x)
de = lambda y : diff(y,x,1) == exp(-x^2)
print "solve the DE with an IC"
f(x) = desolve( de(y), [y,x], ics=[0,1] )
print f(x)
# plot the solution
P = plot( f(x), (x,-2,2), color="purple" )
P.show()
```

Output

```
solve the DE with an IC
1/2*sqrt(pi)*erf(x) + 1
```

Let us use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)}\, dy + C.$$

Finally, we try to solve for y.

Example 3.1.3. Previously, we exploited our knowledge of calculus to guess that the fundamental DE $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln|y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD}e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$. ■

Example 3.1.4. Find the general solution of $y' = y^2$.

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C-x} \qquad \text{or} \qquad y = 0.$$

Note the singularities of the solution. If for example $C = 1$, then the solution "blows up" as we approach $x = 1$. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞). ■

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

Example 3.1.5. Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}.$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and
$$x(t) = 2e^{t/2} - 2.$$
Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain
$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \qquad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

■

Example 3.1.6. Suppose that the car accelerates at a rate of t^2 meters per squared second. At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 meters per second. Where is the car at time $t = 10$.

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is
$$x'' = t^2, \qquad x(0) = 1, \qquad x'(0) = 10.$$
What if we say $x' = v$. Then we have the problem
$$v' = t^2, \qquad v(0) = 10.$$
Once we solve for v, we can integrate and find x. ■

Exercise* 3.1.1*: *Finish the preceding example. Solve for v, and then solve for x. Finally, find $x(10)$.*

Exercises

Maxima or SageMath can be used to verify your answers.

Exercise* 3.1.2*: *Solve* $\frac{dy}{dx} = x^2 + x$ *for* $y(1) = 3$.

Exercise* 3.1.3*: *Solve* $\frac{dy}{dx} = \sin(5x)$ *for* $y(0) = 2$.

Exercise* 3.1.4*: *Solve* $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ *for* $y(0) = 0$.

Exercise* 3.1.5*: *Solve* $y' = y^3$ *for* $y(0) = 1$.

Exercise* 3.1.6** (little harder):*** *Solve* $y' = (y - 1)(y + 1)$ *for* $y(0) = 3$.

Exercise* 3.1.7*: *Solve* $\frac{dy}{dx} = \frac{1}{y + 1}$ *for* $y(0) = 0$.

Exercise* 3.1.8** (harder):*** *Solve* $y'' = \sin x$ *for* $y(0) = 0$, $y'(0) = 2$.

Exercise* 3.1.9*: *A spaceship is traveling at the speed* $2t^2 + 1$ *kilometers per second (t is time in seconds). It is pointing directly away from earth and at time* $t = 0$ *it is 1000 kilometers from earth. How far from earth is it at one minute from time* $t = 0$*?*

Exercise* 3.1.10*: *Solve* $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. *It is OK to leave your answer as a definite integral.*

The following three exercises model chemical reactions using differential equations. In all cases, the differential equations are of first order, as is the topic of this chapter. Yet, in each case, the order of the chemical reactions varies. Don't confuse the terms *order of a differential equations* versus *order of a chemical reaction*.

Exercise* 3.1.11*: *For zero-order chemical reactions, the reaction rate is independent of the concentration of a reactant, so that changing its concentration has no effect on the speed of the reaction. Suppose a reaction of the type* $A \to$ *Products where A is a compound. We denote* $[A]$ *as the concentration of A. Our model is rate* $= -\frac{d[A]}{dt} = k$ *with initial concentration* $[A](0) = A_0$. *Integrate to show that the solution takes the form* $[A](t) = -kt + A_0$. *Draw a sketch of* $[A]$ *against t.*

Exercise* 3.1.12*: *The rate of a first-order chemical reaction is proportional to the concentration. Suppose a reaction of the type* $A \to$ *Products where A is a compound. We denote* $[A]$ *as the concentration of A. Our model is rate* $= -\frac{d[A]}{dt} = k[A]$ *with initial concentration* $[A](0) = A_0$. *Integrate to show that the solution takes the form* $\ln [A](t) = -kt + \ln A_0$. *Draw a sketch of* $\ln [A]$ *against t.*

Exercise* 3.1.13:* *The rate of a second-order chemical reaction is proportional to the squared concentration. Suppose a reaction of the type A $\rightarrow$ Products where A is a compound. We denote* $[A]$ *as the concentration of* A*. Our model is rate* $= -\frac{d[A]}{dt} = k[A]^2$ *with initial concentration* $[A](0) = A_0$*. Integrate to show that the solution takes the form* $\frac{1}{[A](t)} = kt + \frac{1}{A_0}$*. Draw a sketch of* $\frac{1}{[A](t)}$ *against t.*

Exercise* 3.1.101:* *Solve the initial value problem* $\frac{dy}{dx} = e^x + x$ *and* $y(0) = 10$*.*

Exercise* 3.1.102:* *Solve the IVP* $x' = \frac{1}{x^2}$*,* $x(1) = 1$*.*

Exercise* 3.1.103:* *Solve the IVP* $x' = \frac{1}{\cos(x)}$*,* $x(0) = \frac{\pi}{2}$*.*

Exercise* 3.1.104:* *Sid is in a car traveling at speed* $10t + 70$ *miles per hour away from Las Vegas, where t is in hours. At t* $= 0$*, Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?*

Exercise* 3.1.105:* *Solve* $y' = y^n$*,* $y(0) = 1$*, where n is a positive integer. Hint: Consider different cases for n.*

3.2 Slope fields

Attribution: §1.2 in [L]
Further reading: §1.3 in [EP],§1.1 in [BD]

Most of this chapter's sections are devoted to specific strategies for finding solutions to first order ODEs having the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y).$$

So far, we have often found that this ODE possesses multiple general solutions involving an arbitrary constant. Then, when we impose a initial condition,

$$y(x_0) = y_0,$$

we have often found that this initial value problem possesses a single solution. In this section, we take a step back, and ask some foundational questions about these IVPs:

- Does a solution *exist*?
- If a solution exists, is it *unique*?
- If a solution exists, what insights about the solution(s) does the form of the DE reveal?

We begin by considering this last question.

3.2.1 Slope fields

So many first order ODEs are very difficult to solve explicitly, or cannot be solved explicitly. Yet, it would be nice if we could at least figure out the shape and behavior of the solutions, or if we could find approximate solutions. Fortunately, the form $\frac{dy}{dx} = f(x, y)$ is very interpretable, because it describes slope. Blatantly, the DE gives us a formula for the slope of the solution $y(x)$ at each point in the (x, y)-plane. Therefore, for each point (x, y), we can calculate the solution's slope $f(x, y)$, and then draw a little arrow or line segment at (x, y) having that slope. We call this picture the *slope field* of the equation.

Example 3.2.1. In this example, we consider the differential equation $y' = xy$.

We haven't solved this DE, but we suppose that there might exist some solutions $y(x)$. As a reminder, not only is this a DE, but it's also a formula for slope. The left hand side of this DE represents the slope $\dfrac{dy}{dx}$ of the solution. The right hand side of this DE is a formula for that slope: $f(x, y) = xy$. Let us try to find its slope field.

We start by drawing a blank (x, y) plane consisting of two labeled axes. At this stage, we could create a table of values with three columns: x, y, and $f(x, y)$, and compute an exhaustive number of rows to fill in our slope field, but that sounds like a lot of work! Let us try a different approach. Our thought process is broken down into the following steps.

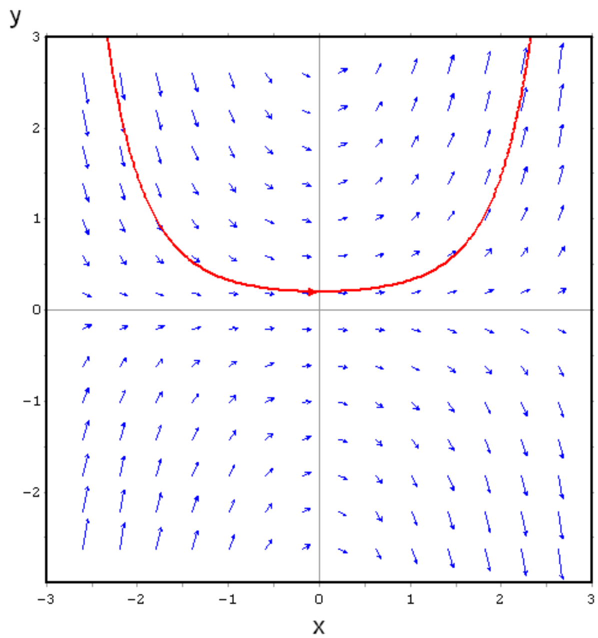
1. We notice that the domain of f is all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$. This means that we can find the slope corresponding to all pairs (x, y).
2. Second, we recognize that, if $x = 0$ or $y = 0$, then the slope is 0. We can immediately draw short horizontal line segments along both axes.
3. We take note of the fact that all positive slopes are located in the quadrants I and III, while all negative slopes are located in quadrants II and IV. This fully characterizes slope direction (increasing or decreasing).
4. We take note that, as we move away from either axis, slopes become steeper. So, in the top right and bottom left corners of our plane, we can draw short steep and increasing (almost vertical) line segments. And, in the other two corners, we can draw short steep and decreasing (almost vertical) line segments. From here, we can fill in the blanks of our slope field.
5. If we wanted to do a particularly nice job with our slope field (and we do), we could find all points (x, y) where the slope is 1. We therefore consider the equation $xy = 1$. Restricting $x \neq 0$, we can divide by x on both sides to obtain $y = \frac{1}{x}$. We might recall from precalculus that this curve is a hyperbola with one *branch* contained within quadrant I and the other branch contained within quadrant III. The axes serve as the hyperbola's asymptotes. Along this hyperbola, we draw short (increasing) line segments with a slope of 45 degrees from the horizontal axis.
6. We repeat this process to find all points (x, y) where the slope is -1. We find two more hyperbola branches, one each in quadrants II and IV. Along this hyperbola, we draw short (decreasing) line segments with a slope of -45 degrees from the horizontal axis. Having identified these points where the slope is 1 or -1, we can more accurately fill in the blanks of our slope field.

Compare your sketch to those of Figure 3.2 on the facing page. ■

But wait! There's more! By looking at the slope field we can get a lot of information about the behavior of solutions. How? If we are given a specific initial condition $y(x_0) = y_0$, we can find its location (x_0, y_0) on the slope field and sketch an approximate solution by following the slopes.

Example 3.2.1 (continuing from p. 79)**.** In this example, we revisit the slope field of $y' = xy$ that we drew and the previous example and also appears as Figure 3.2 on the next page.

If we impose the initial condition $y(0) = 0$, what does the solution look like? We begin by locating the point $(0, 0)$ on our slope field. The slope is horizontal there, so we take a small step in



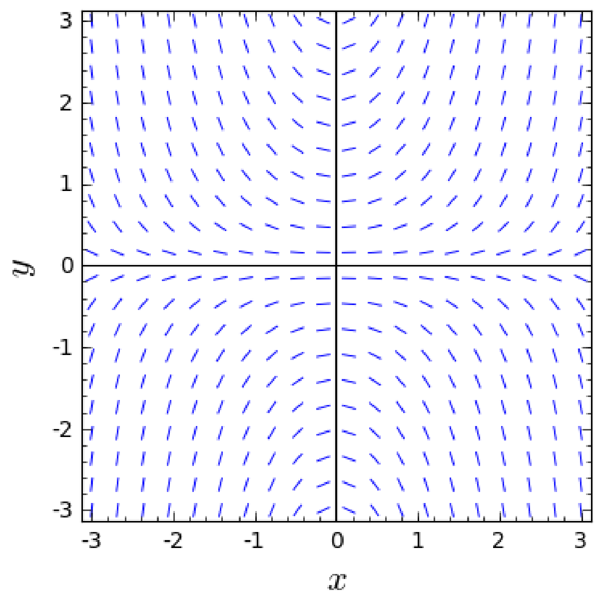


Figure 3.2: Slope field of $y' = xy$ (blue). The left panel was generated using Maxima, the right panel was generated using SageMath. By default, Maxima displays little arrows while SageMath displays short line sements. Maxima's slope field includes the solution satisfying initial condition $y(0) = 0.2$ (red curve).

that direction, which is due right. We're still on the x-axis, and the slope there is still horizontal, so we take another small step to the right. And so on. We quickly come to realize that our IVP's solution is the x-axis, or $y(x) = 0$. What happens to our solution as $x \to \infty$? Looking at our graph, it appears that $y \to 0$.

If we impose the initial condition $y(0) = 0.2$, what does the solution look like? We locate the point $(0, 0.2)$ on our slope field. The slope is horizontal. We take a small step in that direction, moving a little to the right. At this new point, the slope is positive and gentle. We take a step in this new direction, a little to the right and a little upwards. At this new point, we are further away from the axes, and so our slope becomes a little steeper. We take a new step in this direction, and so on. We get a U-shaped solution that appears in red on the left panel of Figure 3.2. When plotted on a slope field, this solution is often called a *trajectory*.

Convince yourself that, starting with any initial condition where $y(0) > 0$, we get similarly-shaped U curves. What happens to these trajectories as $x \to \infty$? Looking at our graph, it appears that $y \to \infty$.

If we impose the initial condition $y(0) = -0.2$, you will draw a solution curve resembling an upside-down U. Convince yourself that, starting with any initial condition where $y(0) < 0$, we get similarly-shaped upside-down U trajectories. For these trajectories, when $x \to \infty$, $y \to -\infty$.

We note that that a small change in the initial condition indices very different behavior. ■

In Maxima, you can generate Figure 3.2 by executing Maxima program 3.2.1. In SageMath, execute SageMath program 3.2.1.

Maxima program 3.2.1 Generating a slope field

```
(%i1)  kill(all)$ reset()$
(%i2)  f(x,y) := x*y;
```

(%o2) $f(x,y) := x\,y$

```
(%i3)  x0 : 0;
       y0 : 0.2;
```

(%o3) 0
(%o4) 0.2

```
(%i5)  plotdf( f(x,y), [x,y], [x,-3,3], [y,-3,3], [trajectory_at,x0,y0] )$
```

SageMath program 3.2.1 Generating a slope field

Input

```
x=var('x')
y=function('y')(x)
# define the slope function
f(x,y) = x*y
# generate the slope field
P = plot_slope_field( f(x,y), (x,-3,3), (y,-3,3), ...
        color="blue", aspect_ratio=1 )
show(P)
```

***Exercise* 3.2.1:** *Sketch the slope field for $y' = x - y$. Choose some initial conditions and sketch their trajectories.*

Example 3.2.2. Plotting the slope field of the differential equation $y' = -y$, and extrapolating several trajectories, we see that no matter what $y(0)$ is, all solutions y to zero as x grows without bound. See Figure 3.3. Looking at the shape of these trajectories, you might suspect the form of the general solution is. Confirm your suspicion by noticing that the DE is fundamental. ∎

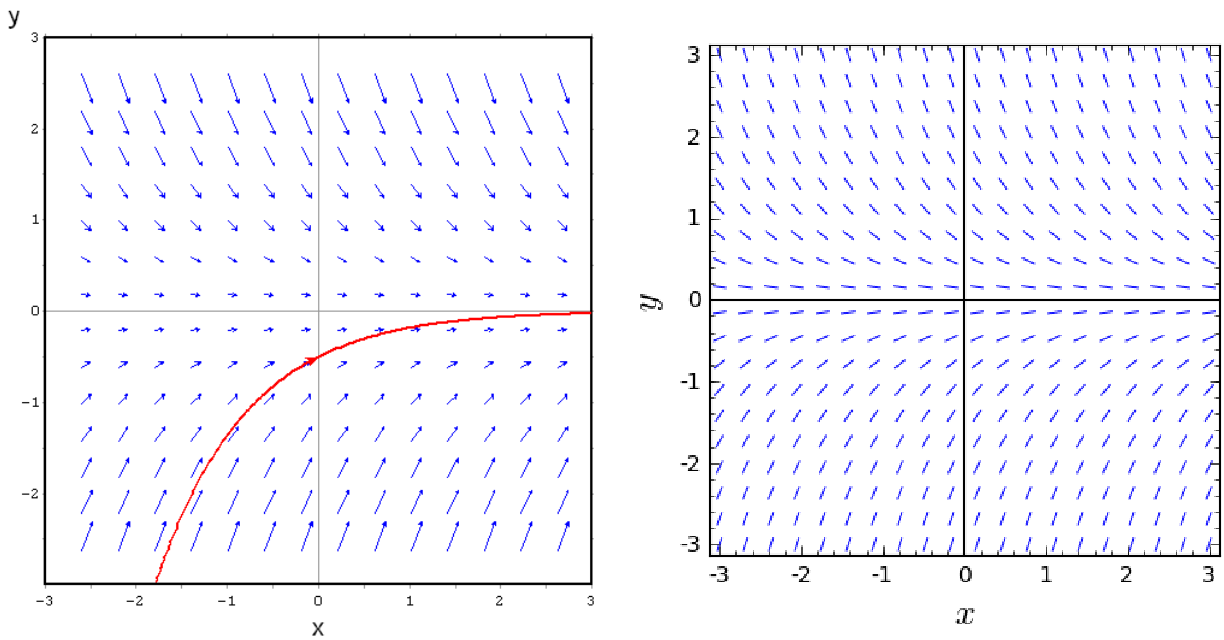


Figure 3.3: Slope field of $y' = -y$ (blue). The left panel was generated using Maxima, the right panel was generated using SageMath. Maxima's slope field includes a trajectory (red curve).

3.2.2 Existence and uniqueness

We wish to ask two fundamental questions about the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 3.2.3. Attempt to solve:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure 3.4. ■

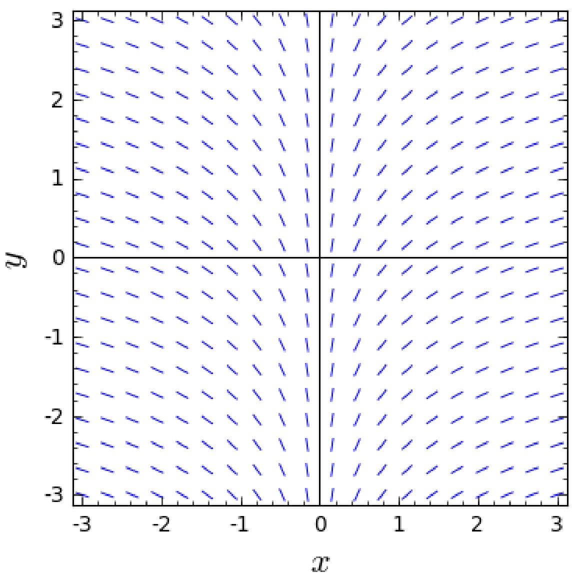


Figure 3.4: Slope field of $y' = \frac{1}{x}$. This graph was generated using SageMath.

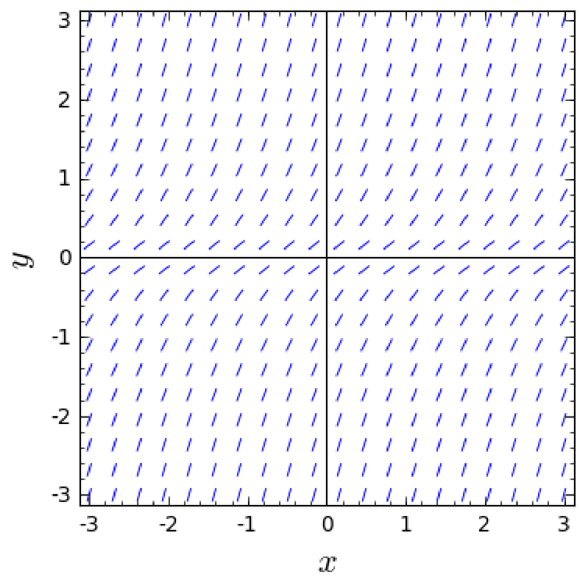


Figure 3.5: Slope field of $y' = 2\sqrt{|y|}$. Notice that all solutions satisfy $y(0) = 0$. This graph was generated using SageMath.

Example 3.2.4. Solve:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

See Figure 3.5. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

■

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem*.

*Named after the French mathematician Charles Émile Picard (1856 – 1941)

Theorem 3.2.1 (Picard's theorem on existence and uniqueness). *If $f(x,y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to*

$$y' = f(x,y), \quad y(x_0) = y_0,$$

exists (at least for some small interval of x's) and is unique.

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example 3.2.5. For some constant A, solve:

$$y' = y^2, \quad y(0) = A.$$

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = \frac{-1}{y} + C$, so $y = \frac{1}{C - x}$. If $y(0) = A$, then $C = 1/A$ so

$$y = \frac{1}{1/A - x}.$$

If $A = 0$, then $y = 0$ is a solution. For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice. ■

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

Exercises

You can verify your answers to many exercises by using Maxima or SageMath to plot the slope field.

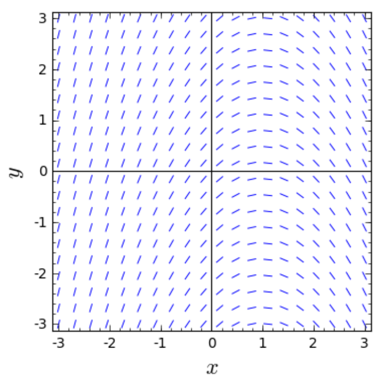
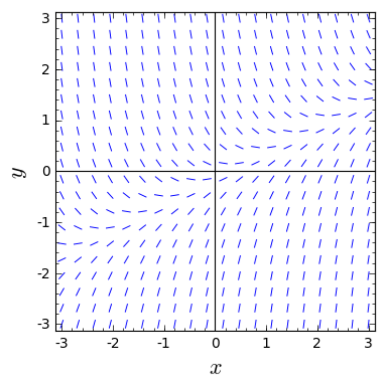
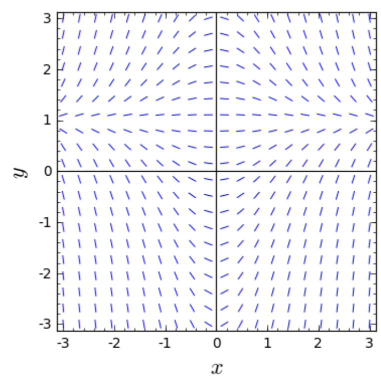
***Exercise* 3.2.2:** *Sketch the slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Choose some initial conditions and sketch their trajectories.*

***Exercise* 3.2.3:** *Sketch the slope field for $\frac{dy}{dx} = x^2$. Choose some initial conditions and sketch their trajectories.*

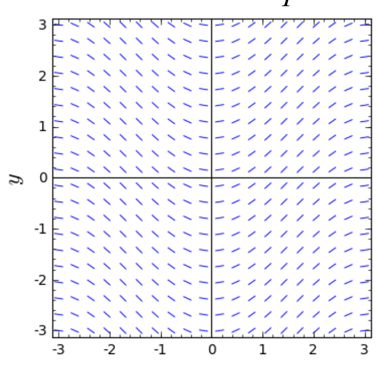
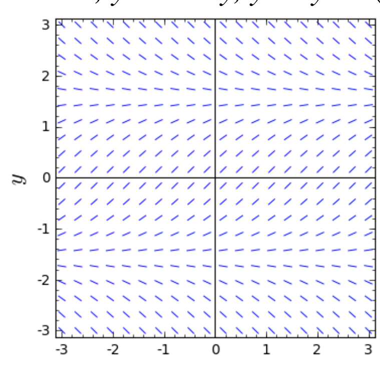
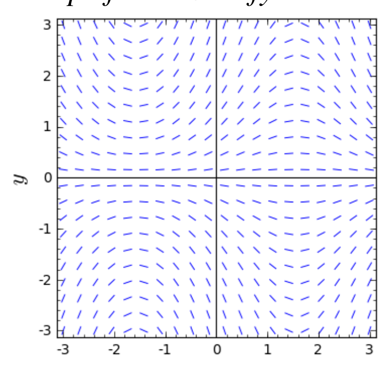
***Exercise* 3.2.4:** *In Example 3.2.5, we explored the existence and uniqueness of solutions to the differential equation $y' = y^2$. In this exercise, sketch its slope field. Choose some initial conditions and sketch their trajectories.*

***Exercise* 3.2.5:** *Sketch the slope field for $\frac{dy}{dx} = x^2 + y^2$. Sketch the trajectories associated with each of these initial conditions: $y(0) = -1$, $y(0) = 0$, $y(0) = 1$.*

***Exercise* 3.2.6:** *Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.*

a)  *b)*  *c)* 

***Exercise* 3.2.7:** *Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify.*

a)  *b)*  *c)* 

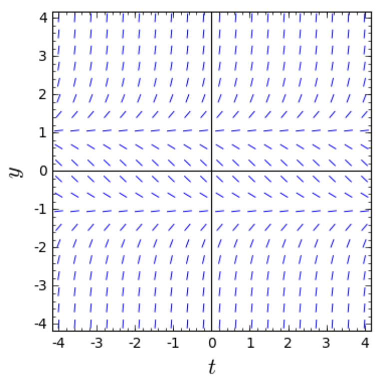
Exercise* 3.2.8*: *Is it possible to solve the equation* $y' = \frac{xy}{\cos x}$ *for* $y(0) = 1$*? Justify. Hint: Your goal is to successfully apply Picard's theorem to show existence and uniqueness, or else find two solutions that prevent uniqueness. Sketching a slope field can help you along either route, but is not sufficient.*

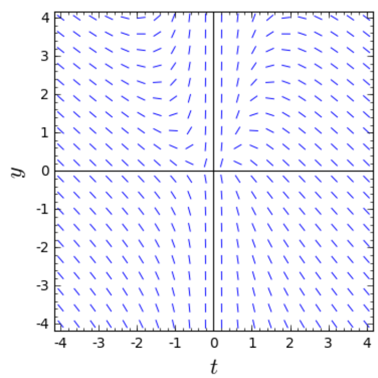
Exercise* 3.2.9*: *Is it possible to solve the equation* $y' = y\sqrt{|x|}$ *for* $y(0) = 0$*? Is the solution unique? Justify.*

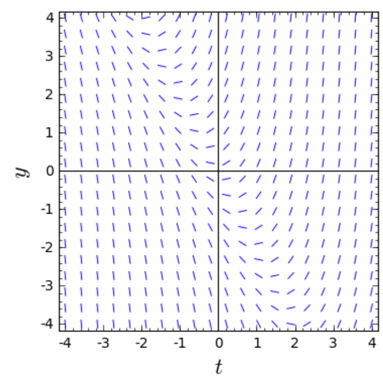
Exercise* 3.2.10** (challenging):*** *Take* $y' = f(x, y)$, $y(0) = 0$, *where* $f(x, y) > 1$ *for all x and y. If the solution exists for all x, can you say what happens to* $y(x)$ *as x goes to positive infinity? Explain.*

Exercise* 3.2.11** (challenging):*** *Take* $(y - x)y' = 0$, $y(0) = 0$*. Find two distinct solutions. Explain why this does not violate Picard's theorem.*

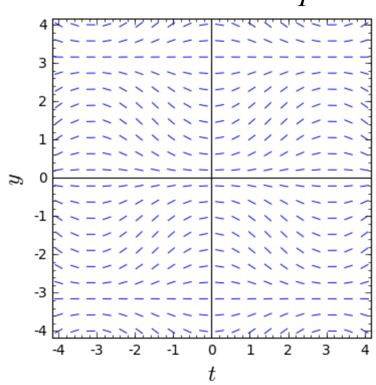
Exercise* 3.2.12*: *Match equations* $y' = y^2 - 1$, $y' = \frac{y}{t^2} - 1$, $y' = 2t + y$ *to slope fields. Justify.*

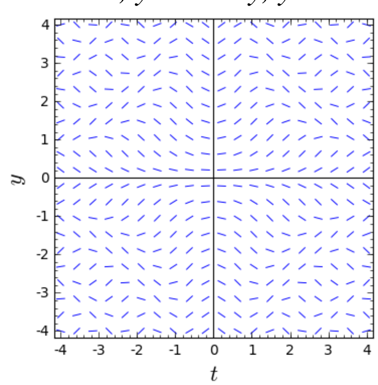
a)


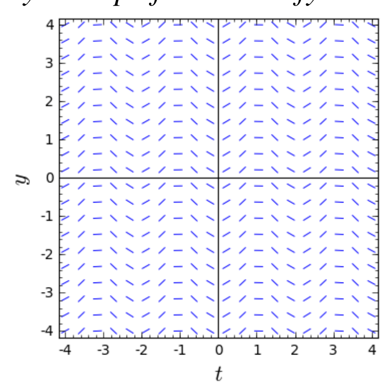
b)


c)


Exercise* 3.2.13*: *Match equations* $y' = \sin 3t$, $y' = \sin ty$, $y' = \sin t \sin y$ *to slope fields. Justify.*

a)


b)


c)


Exercise* 3.2.101*: *Sketch the slope field of* $y' = y^3$*. Can you visually find the solution that satisfies* $y(0) = 0$*?*

Exercise* 3.2.102*: *Is it possible to solve* $y' = xy$ *for* $y(0) = 0$*? Is the solution unique?*

Exercise* 3.2.103*: *Is it possible to solve* $y' = \frac{x}{x^2 - 1}$ *for* $y(1) = 0$*?*

3.3 Separable equations

Attribution: §1.3 in [L]
Further reading: §1.4 in [EP], §2.2 in [BD], chapter 17 in [G]

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)\,dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)\,dx + C.$$

Notice the dependence on y in the integral.

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Exercise* 3.3.1*: *Identify from the following list of equations the ones that are separable:* $y' = \sin(ty)$*,* $y' = e^t e^y$*,* $yy' = t$*,* $y' = (t^3 - t)\arcsin(y)$*, and* $y' = t^2 \ln y + 4t^3 \ln y$*.*

Example 3.3.1. Take the equation

$$y' = xy.$$

First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x\,dx + C.$$

We compute the antiderivatives to get

$$\ln |y| = \frac{x^2}{2} + C.$$

Or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy.$$

Yay! ■

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemed to be doing a different operation to each side. Let us work this method out more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y).$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

We integrate both sides with respect to x.

$$\int \frac{1}{g(y)} \frac{dy}{dx}\, dx = \int f(x)\, dx + C.$$

We can use the change of variables formula.

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx + C.$$

And we are done.

Implicit solutions It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2+1}.$$

We separate variables,

$$\frac{y^2+1}{y}\, dy = \left(y + \frac{1}{y}\right) dy = x\, dx.$$

We integrate to get

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2\ln|y| = x^2 + D.$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x to get

$$y'\left(2y + \frac{2}{y}\right) = 2x.$$

It is simple to see that the differential equation holds. If you want to compute values for y, you might have to be tricky. For example, you can graph x as a function of y, and then flip your paper. Computers are also good at some of these tricks.

We note that the above equation also has the solution $y = 0$. The general solution is

$$y^2 + 2\ln|y| = x^2 + C \quad \text{and} \quad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

Example 3.3.2. Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

First factor the right hand side to obtain

$$x^2y' = (1 - x^2)(1 + y^2).$$

Separate variables, integrate, and solve for y.

$$\begin{aligned}
\frac{dy}{1+y^2} &= \frac{1-x^2}{x^2}\,dx,\\
\frac{dy}{1+y^2} &= \left(\frac{1}{x^2} - 1\right)dx,\\
\arctan(y) &= \frac{-1}{x} - x + C,\\
y &= \tan\left(\frac{-1}{x} - x + C\right).
\end{aligned}$$

Now solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $2 + \pi$, etc...). The solution we are seeking is, therefore,

$$y = \tan\left(-\frac{1}{x} - x + 2\right).$$

■

Example 3.3.3. Bob made a cup of coffee, and Bob likes to drink coffee only once it will not burn him at 60 degrees. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Let T be the temperature of the coffee, and let A be the ambient (room) temperature. Newton's law of cooling states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned} \frac{1}{T-A}\frac{dT}{dt} &= -k, \\ \ln(T - A) &= -kt + C, \qquad \text{(note that } T - A > 0\text{)} \\ T - A &= D\,e^{-kt}, \\ T &= A + D\,e^{-kt}. \end{aligned}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln \frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. That is, we solve $60 = 22 + 67e^{-0.0616t}$ to get $t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take. ∎

Example 3.3.4. Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

First note that $y = 0$ is a solution (a singular solution). So assume that $y \neq 0$ and write

$$\begin{aligned} \frac{-3}{y^2} y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}. \end{aligned}$$

∎

Exercises

If you can express the solution explicitly, report that function as your answer. Otherwise, report the equation of your implicit solution.

Maxima or SageMath can be used to verify your answers.

***Exercise* 3.3.2:** *Solve* $y' = \frac{x}{y}$ *to show that the general solution in implicit form is* $y^2 - x^2 = C$. *Recognizing this equation as representing a family of curves, sketch the slope field. Hint: Consider the cases* $C > 0$, $C = 0$ *and* $C < 0$ *separately.*

***Exercise* 3.3.3:** *Solve* $y' = x^2 y$.

***Exercise* 3.3.4:** *Solve* $\frac{dx}{dt} = (x^2 - 1)\,t$, *for* $x(0) = 0$.

***Exercise* 3.3.5:** *Solve* $\frac{dx}{dt} = x \sin t$, *for* $x(0) = 1$.

***Exercise* 3.3.6:** *Solve* $\frac{dy}{dx} = xy + x + y + 1$. *Hint: Factor the right hand side.*

***Exercise* 3.3.7:** *Solve* $xy' = y + 2x^2 y$, *where* $y(1) = 1$.

***Exercise* 3.3.8:** *Solve* $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, *for* $y(0) = 1$.

***Exercise* 3.3.9:** *Solve* $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, *for* $y(0) = 1$.

***Exercise* 3.3.10:** *Solve* $y' = xe^{-y}$, *with* $y(0) = 1$.

***Exercise* 3.3.11:** *Solve* $xy' = e^{-y}$, *for* $y(1) = 1$.

***Exercise* 3.3.12:** *Solve* $y' = ye^{-x^2}$, $y(0) = 1$. *Hint: Your answer will involve the error function* $\operatorname{erf}(x)$ *that we previously encountered in Example 3.1.2.*

***Exercise* 3.3.13:** *Suppose a cup of coffee is at 100 degrees Celsius at time* $t = 0$, *it is at 70 degrees at* $t = 10$ *minutes, and it is at 50 degrees at* $t = 20$ *minutes. Compute the ambient temperature.*

***Exercise* 3.3.14:** *Consider a house as if it was a partly insulated box. The indoor temperature* U *is affected by the outdoor temperature* T. *By Newton's law of cooling,* $\frac{dU}{dt} = -k(U - T)$ *,where* t *represents time and the indoor air is assumed to be well-mixed. Assume that* T *is constant. Solve the DE. Evaluate a limit that will tell us what happens to* U *on the long run. Interpret this result. Choose* $T = 60$ *F and* $k = 1.5$/*day to plot and analyze the slope field using Maxima or SageMath. How do different initial conditions affect the indoor temperature?*

Exercise* 3.3.15*: *Show that* $\ln\left|\frac{y+1}{y-1}\right| = 2t + C$ *is an implicit general solution to the separable differential equation* $y' = y^2 - 1$*. Then, identify two singular solutions for this DE.*

Exercise* 3.3.16*: *Sherlock Holmes is awoken by a phone call from a policeman at 3:30 a.m. A body has been discovered and foul play is suspected. Sherlock tells the police to take the body's temperature and, when he arrives at the scene 45 minutes later, he takes the temperature again. The two readings that cold 60 degree Fahrenheit morning were 80 and 70 degrees. Write down a differential equation governing the body's temperature. Use 98.6 degrees for the normal body temperature. Then, solve this DE. Finally, determine the time of death.*

Exercise* 3.3.101*: *Solve* $y' = 2xy$*.*

Exercise* 3.3.102*: *Solve* $x' = 3xt^2 - 3t^2$*,* $x(0) = 2$*.*

Exercise* 3.3.103*: *Solve* $x' = \frac{1}{3x^2+1}$*,* $x(0) = 1$*.*

Exercise* 3.3.104*: *Solve* $xy' = y^2$*,* $y(1) = 1$*.*

Exercise* 3.3.105*: *Solve* $y' = \frac{\sin x}{\cos y}$*.*

Exercise* 3.3.106*: *Solve* $y' = \frac{1}{1+t^2}$*.*

Exercise* 3.3.107*: *Solve the initial value problem* $y' = t^n$ *with* $y(0) = 1$ *and* $n \geq 0$*.*

Exercise* 3.3.108*: *Solve* $y' = \ln t$*.*

Exercise* 3.3.109*: *Solve* $yy' = t$*.*

Exercise* 3.3.110*: *Solve* $y' = y^2 - 1$*.*

Exercise* 3.3.111*: *Solve* $y' = \frac{t}{y^3-5}$*.*

Exercise* 3.3.112*: *Find a non-constant solution of the initial value problem* $y' = y^{1/3}$*,* $y(0) = 0$*, using separation of variables. Note that the constant function* $y(t) = 0$ *also solves the initial value problem. This shows that an initial value problem can have more than one solution.*

Exercise* 3.3.113*: *Solve* $y' + \frac{y}{1+t^2} = 0$*.*

Exercise* 3.3.114*: *Solve the initial value problem* $y' + ye^t = 0$*,* $y(0) = e$*.*

Exercise* 3.3.115*: *Solve the IVP* $ty' - 2y = 0$*,* $y(1) = 4$*.*

Exercise* 3.3.116*: *Solve the IVP* $t^2y' + y = 0$*,* $y(1) = -2$*,* $t > 0$*.*

Exercise* 3.3.117*: *Solve the IVP* $t^3y' = 2y$*,* $y(1) = 1$*,* $t > 0$*.*

Exercise* 3.3.118*: *Solve the IVP* $t^3y' = 2y$*,* $y(1) = 0$*,* $t > 0$*.*

Exercise* 3.3.119*: *Solve the IVP* $t \ln t \, x' = -x + t(\ln t)^2$*,* $x(e) = 1$*.*

3.4 Linear equations and the integrating factor

Attribution: §1.4 in [L]
Further reading: §1.5 in [EP],§2.1 in [BD],§2.1,2.4 in [T],§17.3 in [G]

One of the most important types of equations we will learn how to solve are the so-called *linear equations*. In fact, the majority of the course is about linear equations. In this section we focus on the *first order linear equation*. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{3.5}$$

Here the word *linear* means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Example 3.4.1. The first order equations

$$\begin{aligned} x^2y' + 3y &= x^2 \\ xy' - 8x^2y &= \sin x \\ xy' + (\ln x)y &= 0 \\ y' &= x^2y - 2 \end{aligned}$$

are not in the form $y' + p(x)y = f(x)$, but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1 \\ y' - 8xy &= \frac{\sin x}{x} \\ y' + \frac{\ln x}{x}y &= 0 \\ y' - x^2y &= -2 \end{aligned}$$

■

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left hand side of (3.5) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big].$$

This is the left hand side of (3.5) multiplied by $r(x)$. So if we multiply (3.5) by $r(x)$, we obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the *integrating factor* and the method is called the *integrating factor method*.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)\,dx}.$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)\,dx} y' + e^{\int p(x)\,dx} p(x)y &= e^{\int p(x)\,dx} f(x), \\ \frac{d}{dx}\left[e^{\int p(x)\,dx} y\right] &= e^{\int p(x)\,dx} f(x), \\ e^{\int p(x)\,dx} y &= \int e^{\int p(x)\,dx} f(x)\,dx + C, \\ y &= e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx} f(x)\,dx + C\right). \end{aligned}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 3.4.2. Solve

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx}\left[e^{x^2} y\right] &= e^{x}. \end{aligned}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^{x} + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

■

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise* 3.4.1*: *Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.*

An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(y_0 + \int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt\right).} \tag{3.6}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise* 3.4.2*: *Check that $y(x_0) = y_0$ in formula* (3.6).

Remark 3.4.1. *Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula* (3.6) *for the solution. Second, it follows from the formula* (3.6) *that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 3.2.1.*

Exercises

In the exercises, leave your answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that. Maxima and SageMath can be used to verify your answers.

***Exercise* 3.4.3:** *Solve* $y' + xy = x$.

***Exercise* 3.4.4:** *Solve* $y' + 6y = e^x$.

***Exercise* 3.4.5:** *Solve* $y' + 3x^2y = \sin(x)\, e^{-x^3}$, *with* $y(0) = 1$.

***Exercise* 3.4.6:** *Solve* $y' + \cos(x)y = \cos(x)$.

***Exercise* 3.4.7:** *Solve* $\dfrac{1}{x^2+1}\, y' + xy = 3$, *with* $y(0) = 0$.

***Exercise* 3.4.8:** *Solve* $y' + y = e^{x^2-x}$ *with* $y(0) = 10$.

***Exercise* 3.4.101:** *Solve* $y' + 3x^2y = x^2$.

***Exercise* 3.4.102:** *Solve* $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\pi/2) = 3$.

***Exercise* 3.4.103:** *Solve* $y' + 4y = 8$.

***Exercise* 3.4.104:** *Solve* $y' + ty = 5t$.

***Exercise* 3.4.105:** *Solve* $y' + e^t y = -2e^t$.

***Exercise* 3.4.106:** *Solve* $y' - y = t^2$.

***Exercise* 3.4.107:** *Solve* $2y' + y = t$.

***Exercise* 3.4.108:** *Solve* $ty' - 2y = \dfrac{1}{t}$, $t > 0$.

***Exercise* 3.4.109:** *Solve* $ty' + y = \sqrt{t}$, $t > 0$.

***Exercise* 3.4.110:** *Solve* $y' \cos t + y \sin t = 1$, $-\pi/2 < t < \pi/2$.

***Exercise* 3.4.111:** *Solve* $y' + y \sec t = \tan t$, $-\pi/2 < t < \pi/2$.

3.5 Substitution

Attribution: §1.5 in [L]
Further reading: §1.6 in [EP]

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

Example 3.5.1. The equation

$$y' = (x - y + 1)^2$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try

$$v = x - y + 1.$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2.$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2}\, dv = dx.$$

So

$$\begin{aligned} \frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| &= x + C, \\ \left| \frac{v+1}{v-1} \right| &= e^{2x+2C}, \end{aligned}$$

or $\frac{v+1}{v-1} = De^{2x}$ for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to "unsubstitute" to obtain

$$\frac{x - y + 2}{x - y} = De^{2x},$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.

$$\begin{aligned} x - y + 2 &= (x - y)De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y\,(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$. ■

Another type of equations we can solve by substitution are the so-called *homogeneous equations*. This is a confusing name, because we will use this term again to describe a completely different kind of DE in Chapter 5.)

Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right).$$

Here we try the substitutions

$$v = \frac{y}{x} \qquad \text{and therefore} \qquad y' = v + xv'.$$

We note that the equation is transformed into

$$v + xv' = F(v) \qquad \text{or} \qquad xv' = F(v) - v \qquad \text{or} \qquad \frac{v'}{F(v) - v} = \frac{1}{x}.$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v}\, dv = \ln|x| + C.$$

Example 3.5.2. Solve $x^2 y' = y^2 + xy$ with $y(1) = 1$.

We put the equation into the form

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

We substitute $v = \frac{y}{x}$ to get the separable equation

$$xv' = v^2 + v - v = v^2,$$

which has a solution

$$\begin{aligned}
\int \frac{1}{v^2}\, dv &= \ln|x| + C, \\
\frac{-1}{v} &= \ln|x| + C, \\
v &= \frac{-1}{\ln|x| + C}.
\end{aligned}$$

Finally, we unsubstitute

$$\begin{aligned}
\frac{y}{x} &= -\frac{1}{\ln|x| + C}, \\
y &= \frac{-x}{\ln|x| + C}.
\end{aligned}$$

We want $y(1) = 1$, so

$$1 = y(1) = -\frac{1}{\ln|1| + C} = \frac{-1}{C}.$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{\ln|x| - 1}.$$

■

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general patterns to look for. We summarize a few of these in a table.

When you see	Try substituting
y/x	$v = y/x$
$y\,y'$	$v = y^2$
$y^2\,y'$	$v = y^3$
$(\cos y)\,y'$	$v = \sin y$
$(\sin y)\,y'$	$v = \cos y$
$e^y\,y'$	$v = e^y$

Usually you try to substitute in the "most complicated" part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

Exercises

Solve the DE or IVP by substitution.

Exercise* 3.5.1*: $2yy' + 1 = y^2 + x$ *with* $y(0) = 1$

Exercise* 3.5.2*: $yy' + x = \sqrt{x^2 + y^2}$

Exercise* 3.5.3*: $y' = (x + y - 1)^2$

Exercise* 3.5.4*: $y' = \dfrac{x^2 - y^2}{xy}$ *with* $y(1) = 2$

Exercise* 3.5.5*: $y' = \dfrac{2xy}{x^2 - y^2}$ *with* $y(0) = 2$

Exercise* 3.5.101*: $y^2y' = y^3 - 3x$ *with* $y(0) = 2$

Exercise* 3.5.102*: $2yy' = e^{y^2 - x^2} + 2x$

Exercise* 3.5.103*: $y' = (y + x + 3)^2$

Exercise* 3.5.104*: $xy' + y + x = 0$ *with* $y(1) = 1$

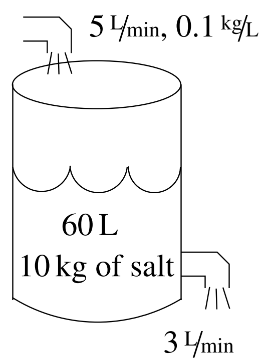
Exercise* 3.5.105*: $y' = \dfrac{y}{x} + 2$

3.6 Mixing problems and other applications

Attribution: §1.4 in [L]

Let us discuss a common simple application of linear equations. Indeed, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes). These are called *mixing problems*.

Example 3.6.1. A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilogram per liter is flowing in at the rate of 5 liters per minute. The solution in the tank is well stirred and flows out at a rate of 3 liters per minute. How much salt is in the tank when the tank is full?



Let x denote the mass of salt in kilograms in the tank, and let t denote the time in minutes. The model is

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned}$$

Therefore, our differential equation is

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60 + 2t}\right) \text{ kilograms per minute,}$$

or we can rewrite it in the form (3.5) so that it looks more like a first-order linear differential equation:

$$\boxed{\frac{dx}{dt} + \frac{3x}{60 + 2t} = 0.5 \text{ kilograms per minute.}}$$

An initial condition is associated with this DE: the starting mass of salt is 10 kilograms. Therefore

$$\boxed{x(0) = 10 \text{ kilograms}}.$$

As we set up our DE and IC, it is helpful to include the units of each.

We can now solve our model. We begin by determining the integrating factor:

$$r(t) = \exp\left(\int \frac{3}{60+2t}\,dt\right) = \exp\left(\frac{3}{2}\ln|60+2t|\right) = \exp\left(\ln|60+2t|^{\frac{3}{2}}\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation by $r(t)$ to get

$$\begin{aligned}
(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2},\\
\frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= \frac{(60+2t)^{3/2}}{2},\\
\int_0^t \frac{d}{ds}\left[(60+2s)^{3/2}x\right]ds &= \int_0^t \frac{(60+2s)^{3/2}}{2}\,ds,\\
(60+2s)^{3/2}x(s)\Big|_0^t &= \left.\frac{(60+2s)^{5/2}}{10}\right|_0^t,\\
(60+2t)^{3/2}x(t) - (60+2\cdot 0)^{3/2}x(0) &= \frac{(60+2t)^{5/2}}{10} - \frac{(60+2\cdot 0)^{5/2}}{10},\\
(60+2t)^{3/2}x(t) - 60^{3/2}10 &= \frac{(60+2t)^{5/2}}{10} - \frac{60^{5/2}}{10}.
\end{aligned}$$

Solving for $x(t)$, we get

$$\begin{aligned}
(60+2t)^{3/2}x(t) &= \frac{(60+2t)^{5/2}}{10} - \frac{60^{5/2}}{10} + 60^{3/2}10,\\
(60+2t)^{3/2}x(t) &= \frac{(60+2t)^{5/2}}{10} - \frac{60^{3/2}60}{10} + 60^{3/2}10,\\
(60+2t)^{3/2}x(t) &= \frac{(60+2t)^{5/2}}{10} - 60^{3/2}6 + 60^{3/2}10,\\
(60+2t)^{3/2}x(t) &= \frac{(60+2t)^{5/2}}{10} + 4\cdot 60^{3/2},\\
x(t) &= \frac{(60+2t)^{5/2}}{10(60+2t)^{3/2}} + \frac{4\cdot 60^{3/2}}{(60+2t)^{3/2}},\\
x(t) &= \frac{60+2t}{10} + \frac{4\cdot 60^{3/2}}{(60+2t)^{3/2}},\\
x(t) &= \frac{60+2t}{10} + 4\left(\frac{60}{60+2t}\right)^{3/2},\\
x(t) &= 6 + \frac{t}{5} + 4\left(\frac{60}{60+2t}\right)^{3/2}.
\end{aligned}$$

We are interested in x when the tank is full. So we note that the tank is full when $60 + 2t = 100$, or when $t = 20$. So

$$x(20) = 6 + \frac{20}{5} + 4\left(\frac{60}{60 + 2 \cdot 20}\right)^{3/2} = 10 + 4\left(\frac{3}{5}\right)^{3/2} \approx 11.86\,\text{kilograms}.$$

To summarize, we began with 10 kilograms of salt in the tank. After 20 seconds, the tank filled up, and had gained ≈ 1.86 kilograms of salt. Yet, the concentration of salt in the tank decreased from 0.167 kilogram per liter at $t = 0$ to 0.1186 kilogram per liter at $t = 20$ seconds. ■

Consider a falling body of mass m on which exactly three forces act:

- gravitation, F_{grav},
- air resistance, F_{res},
- an external force, $F_{ext} = f(t)$, where $f(t)$ is specified.

Let $x(t)$ denote the distance fallen from some fixed initial position. The velocity is denoted by $v = x'$ and the acceleration by $a = x''$. We choose an orientation so that downwards is positive. In this case, $F_{grav} = mg$, where $g > 0$ is the *gravitational constant*. On the surface of the Earth, the gravitational constant is $g = 9.8$ meters per second squared or newtons per kilogram. We assume that air resistance is proportional to velocity (a common assumption in physics), and write $F_{res} = -kv = -kx'$, where $k > 0$ is a *friction constant*. The total force, F_{total}, is by hypothesis,

$$F_{total} = F_{grav} + F_{res} + F_{ext}.$$

Newton's Second Law states that force equals mass times acceleration, so

$$F_{total} = ma = mx''.$$

Putting these together, we have

$$mx'' = mg - kx' + f(t),$$

or

$$mx'' + kx' = mg + f(t).$$

This is a second order differential equation in x. Substituting v for x' results in a first order differential equation in v:

$$mv' + kv = mg + f(t).$$

We divide both sides by m to fetch the usual form:

$$v' + \frac{k}{m}v = \frac{g + f(t)}{m}.$$

Example 3.6.2. You drop an object with mass 10 kilograms from a height of 2000 meters. Suppose the air resistance friction constant is $k = 10$ newton seconds per meter. Find the velocity after 10 seconds.

This example implies that the object begins at rest, so we have an initial condition: $v(0) = 0$. We also assume that no external force acts on the object, so $f(t) = 0$. The differential equation is therefore

$$\frac{dv}{dt} + \frac{k}{m}v = \frac{g}{m}.$$

The integrating factor is

$$r(t) = \exp\left(\int \frac{k}{m}\,dt\right) = \exp\left(\frac{k}{m}t\right).$$

We multiply both sides of the equation by $r(t)$, and then solve for $v(t)$:

$$\begin{aligned}
\exp\left(\frac{k}{m}t\right)\frac{dv}{dt} + \frac{k}{m}\exp\left(\frac{k}{m}t\right)v &= \frac{g}{m}\exp\left(\frac{k}{m}t\right) \\
\frac{d}{dt}\left[\exp\left(\frac{k}{m}t\right)v\right] &= \frac{g}{m}\exp\left(\frac{k}{m}t\right), \\
\int_0^t \frac{d}{ds}\left[\exp\left(\frac{k}{m}s\right)v\right]\,ds &= \frac{g}{m}\int_0^t \exp\left(\frac{k}{m}s\right)ds, \\
\exp\left(\frac{k}{m}s\right)v(s)\Big|_0^t &= \frac{g}{m}\frac{m}{k}\exp\left(\frac{k}{m}s\right)\Big|_0^t, \\
\exp\left(\frac{k}{m}s\right)v(s)\Big|_0^t &= \frac{g}{k}\exp\left(\frac{k}{m}s\right)\Big|_0^t, \\
\exp\left(\frac{k}{m}t\right)v(t) - \exp\left(\frac{k}{m}0\right)v(0) &= \frac{g}{k}\exp\left(\frac{k}{m}t\right) - \frac{g}{k}\exp\left(\frac{k}{m}0\right) \\
\exp\left(\frac{k}{m}t\right)v(t) &= \frac{g}{k}\exp\left(\frac{k}{m}t\right) - \frac{g}{k} \\
v(t) &= \frac{g}{k} - \frac{g}{k}\exp\left(-\frac{k}{m}t\right) \\
\exp\left(\frac{k}{m}t\right)v(t) &= \frac{g}{k}\exp\left(\frac{k}{m}t\right) - \frac{g}{k} \\
v(t) &= \frac{g}{k}\left(1 - \exp\left(-\frac{k}{m}t\right)\right).
\end{aligned}$$

Plugging in the parameters, we obtain

$$\begin{aligned}
v(t) &= \frac{9.8}{10}\left(1 - \exp\left(-\frac{10}{10}t\right)\right), \\
v(t) &= 0.98\left(1 - e^{-t}\right).
\end{aligned}$$

Ten second after being dropped, the velocity of our object is $v(10) = 0.98\left(1 - e^{-10}\right) \approx 0.98$ meters per second.

■

Exercises

For these problems, you can use Maxima or SageMath to verify your calculations, but it is perhaps better to check with a classmate that your modeling is correct before proceeding.

Exercise* 3.6.1*: *Suppose there is a lake located along a stream. Clean water flows into the lake, and lakewater flows out of it. The in- and out-flow from the lake is 500 liters per hour. The lake contains 100 thousand liters of water. A truck with 500 kilograms of toxic substance crashes into the lake. Assume that the water is being continually mixed perfectly by the stream. Find the concentration of toxic substance as a function of time in the lake. How long does it take for the concentration in the lake be below 0.001 kilogram per liter?*

Exercise* 3.6.2*: *This exercise revisits Exercise 3.3.14. Now, allow the outside temperature T to reflect daily variations according to the model $T = T_0 + A \sin(\omega t)$. Solve the DE. Evaluate a limit that will tell us what happens to U on the long run. Interpret this result. Choose $T_0 = 60$ F, $A = 15$ F, $\omega = 2\pi$ and $k = 1.5$/day to plot and analyze the slope field using Maxima or SageMath. What happens to the temperature of the house? On the long run, how does the initial condition affect the indoor temperature?*

Exercise* 3.6.3*: *Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters/minute. The tank is mixed well and is drained at 3 liters/minute. How long does the process continue until there are 20 grams of salt in the tank?*

Exercise* 3.6.4*: *Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter/minute. The water is mixed well and drained at the same rate so that the volume of water in the tank remains constant. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?*

Exercise* 3.6.5*: *In this exercise, we revisit the glucose absorption model that we examined in Exercise 1.1.16. Suppose that glucose is injected into a patient's bloodstream at a constant rate of r milligrams per decilitre per day. Then $G' = -\lambda G + r$, where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim\limits_{t\to\infty} G(t)$.*

Exercise* 3.6.6*: *A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t. Also, what is the mass of the radioactive substance on the long run?*

Exercise* 3.6.7*: *You drop an object with mass 100 kilograms from a height of 1000 meters. Suppose the air resistance friction constant is $k = 10$ newton seconds per meter. Find the time and speed of impact.*

Exercise* 3.6.8*: *A parachutist has mass 100 kilograms (including the chute). The chute is released 30 seconds after the jump from a height of 2000 m. The air resistance friction constant is $k = 15$ newton seconds per meter when the chute is closed, and $k = 100$ newton seconds per meter when the chute is open. Find the distance and velocity functions during the time interval during which the chute is closed. Then. find the distance and velocity functions during the time interval during which the chute is open. What is the parachutist's landing time and speed.*

Exercise* 3.6.9*: *A tank initially contains 100 liters of water and 50 kilograms of salt dissolved in the water. Suppose that pure water flows into the tank at 5 liters per minute, and well-mixed water flows out of it at the same rate. Find the mass of salt in the tank as a function of time.*

Exercise* 3.6.101*: *Suppose a water tank is being pumped out at 3 liters per minute. The water tank is filled to capacity with 10 liters of clean water. Water with toxic substance is flowing into the tank at 2 liters per minute, with concentration $20t$ grams per liter at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?*

Exercise* 3.6.102*: *Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.*

Exercise* 3.6.103*: *In the previous exercise, the DE is both linear and separable, and can be solved via the integrating factor or by separating the variables. Solve the previous exercise a second time by using a different technique.*

3.7 Autonomous equations

Attribution: §1.6 in [L]
Further reading: §2.2 in [EP],§2.5 in [BD]

Let us consider problems of the form

$$\frac{dx}{dt} = f(x),$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called *autonomous equations*. If we think of t as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (see Example 3.3.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x - A),$$

where x is the temperature, t is time, k is some constant, and A is the ambient temperature. See Figure 3.6 on the next page for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the *equilibrium solutions*. The points on the x axis where $f(x) = 0$ are called *critical points*. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \to \infty$ we get $x(t) \to A$. We call such critical points *stable*. In this simple example it turns out that all solutions in fact go to A as $t \to \infty$. If a critical point is not stable we would say it is *unstable*.

Let us consider the *logistic equation* that is also called the *Verhulst model*

$$\boxed{\frac{dx}{dt} = kx(M - x)} \tag{3.7}$$

for some positive k and M. This equation is commonly used to model population if we know the limiting population M, that is the maximum sustainable population. The logistic equation leads to less catastrophic population predictions than we studied than the *Malthusian model* $\frac{dx}{dt} = kx$ that we studied in Section 1.1 on page 11. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math.

See Figure 3.7 on the next page for an example. Note two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$ is stable. On the other hand the critical point at $x = 0$ is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the

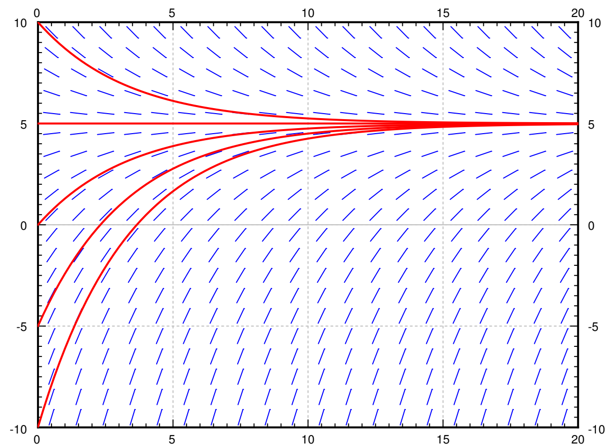


Figure 3.6: Slope field and some solutions of $x' = -0.3\,(x-5)$.

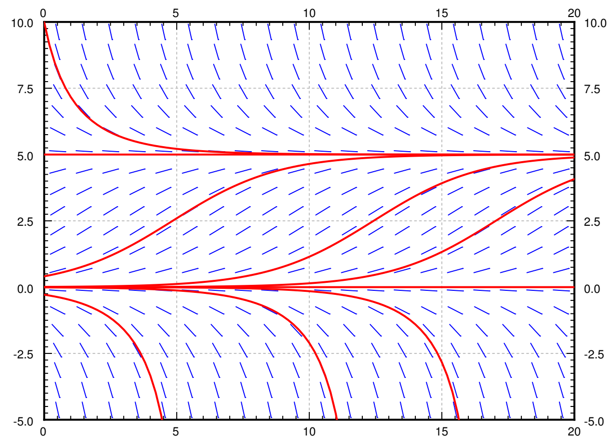


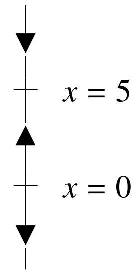
Figure 3.7: Slope field and some solutions of $x' = -0.1\,x\,(x-5)$.

solutions. For example, from the above we can easily see that

$$\lim_{t\to\infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ -\infty & \text{if } x(0) < 0. \end{cases}$$

From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞. Think of the equation $x' = x^2$; we have seen that solutions only exist for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

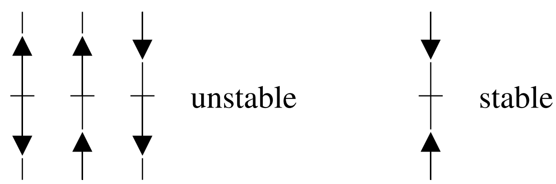
Often we are interested only in the long term behavior of the solution and we would be doing unnecessary work if we solved the equation exactly. It is easier to just look at the *phase diagram* or *phase portrait*, which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x. We draw the x axis, we mark all the critical points, and then we draw arrows in between. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow.



Armed with the phase diagram, it is easy to sketch the solutions approximately.

Exercise* 3.7.1*: *Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.*

Once we draw the phase diagram, we can easily classify critical points as stable or unstable.*



Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting.

Example 3.7.1. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose x is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done. The number $k > 0$ is a constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h.$$

and we expand the right hand side

$$\frac{dx}{dt} = -kx^2 + kMx - h.$$

Solving this quadratic polynomial for x can be done to find the critical points, let us call them A and B, using the quadratic formula to get

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k},$$
$$B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}.$$

*The unstable points that have one of the arrows pointing towards the critical point are sometimes called *semistable*.

Note that there are three possibilities: A and B are real and distinct, A and B are real and equal, and A and B are complex conjugate pairs.

A slightly easier strategy involves factoring $-k$ from the expanded right-hand side of the differential equation

$$\frac{dx}{dt} = -k\left(x^2 - Mx + \frac{h}{k}\right)$$

so that, if the roots are real, the quadratic can be factored as

$$-k\left(x^2 - Mx + \frac{h}{k}\right) = -k\,(x - A)\,(x - B)$$

Not surprisingly, the quadratic formula gives us similar expressions as before for the roots

$$A = \frac{M + \sqrt{M^2 - 4\,{}^h\!/\!_k}}{2},$$
$$B = \frac{M - \sqrt{M^2 - 4\,{}^h\!/\!_k}}{2}$$

but the factored form makes it easier to draw the phase diagram for different possibilities because $k > 0$. ■

Exercise* 3.7.2*: *Draw the phase diagram for different possibilities. Then, sketch a slope field for different possibilities. Finally, compare your answers against the interpretations below.*

Example 3.7.1 (continuing from p. 111)**.** When A and B are distinct and positive, we find that, as long as the population starts above B million, then the population will not die out. It will in fact tend towards A million. If ever some catastrophe happens and the population drops below B, humans will die out, and the fast food restaurant serving them will go out of business.

When $A = B$, there is only one critical point and it is unstable. When the population starts above A million it will tend towards A million. If it ever drops below A, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business. There is no room for error.

Finally if A and B are complex, then there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts. ■

Exercise* 3.7.3*: *Let $M = 8$ and $k = 0.1$. Use Maxima or SageMath to generate a slope field for the following cases: $h = 1$, $h = 1.6$, and $h = 2$. Compare them to your sketches from the previous exercise. For fixed values of M and k, what happens to the human race as the harvesting rate h increases?*

Exercises

You can verify your answers to many exercises by using Maxima or SageMath to plot the slope field.

Exercise* 3.7.4*: *Take $x' = x^2$. Draw the phase diagram, find the critical points, and mark them stable or unstable. Sketch typical solutions of the equation. Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = -1$.*

Exercise* 3.7.5*: *Take $x' = \sin x$. Draw the phase diagram for $-4\pi \leq x \leq 4\pi$. On this interval mark the critical points stable or unstable. Sketch typical solutions of the equation. Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 1$.*

Exercise* 3.7.6*: *Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x. Draw the phase diagram for $x' = f(x)$, find the critical points, and mark them stable or unstable. Sketch typical solutions of the equation. Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.*

Exercise* 3.7.7*: *Start with the logistic equation $\frac{dx}{dt} = kx(M - x)$. Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx). Construct the differential equation. Show that if kM > h, then the equation is still logistic. What happens when kM < h?*

Exercise* 3.7.101*: *Let $x' = (x - 1)(x - 2)x^2$. Sketch the phase diagram and find critical points. Classify the critical points. If $x(0) = 0.5$ then find $\lim\limits_{t\to\infty} x(t)$.*

Exercise* 3.7.102*: *Let $x' = e^{-x}$. Find and classify all critical points. Find $\lim\limits_{t\to\infty} x(t)$ given any initial condition.*

Exercise* 3.7.103*: *Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M - x)$. Now suppose that fish are continually added at A fish per unit of time. Find the differential equation for x. What is the new limiting population?*

3.8 Euler's method

Attribution: §1.7 in [L]
Further reading: §2.4 in [EP],§8.1 in [BD]

As we said before, unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is *Euler's method**. It works as follows: We take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. We follow the line for an interval of length h on the x axis. Hence if $y = y_0$ at x_0, then we will say that y_1 (the approximate value of y at $x_1 = x_0 + h$) will be $y_1 = y_0 + hk$. Rinse, repeat! That is, compute x_2 and y_2 using x_1 and y_1. For an example of the first two steps of the method see Figure 3.8.

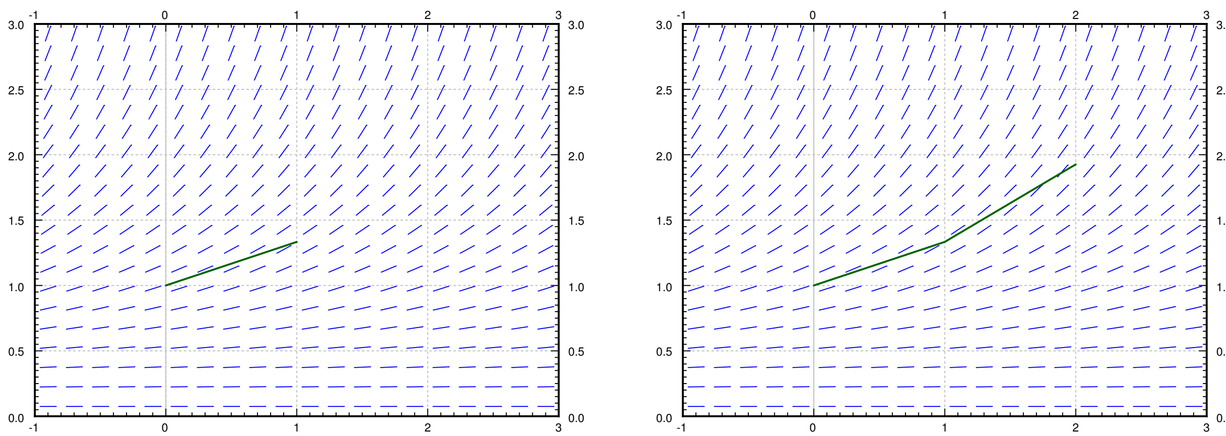


Figure 3.8: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

More abstractly, for any $i = 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\, f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 3.9 on the next page for the plot of the real solution and the approximation.

*Named after the Swiss mathematician Leonhard Paul Euler (1707 – 1783). Do note the correct pronunciation of the name sounds more like "oiler."

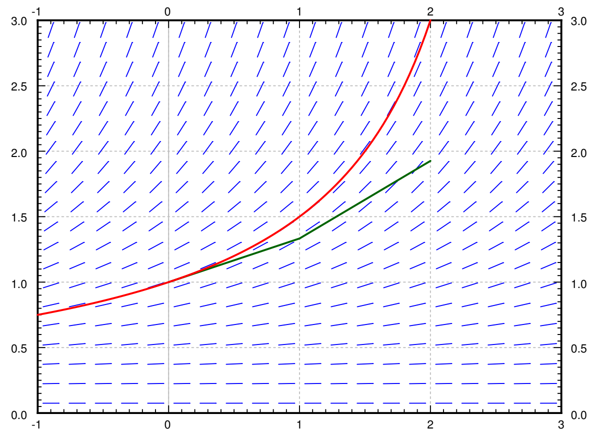


Figure 3.9: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

Let us see what happens with the equation $y' = \dfrac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 3.8 and 3.9 we have graphically approximated $y(2)$ with step size 1. With step size 1 we have $y(2) \approx 1.926$. The real answer is 3. So we are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 3.1 on the following page gives the values computed for various parameters.

Exercise* 3.8.1*: *Solve this equation exactly and show that $y(2) = 3$.*

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly . . . what is the point of doing the approximation?

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a *first order method*. In the IODE Project II you are asked to implement a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $1/4 = 1/2 \times 1/2$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we might need to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 3.1: Euler's method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

***Exercise* 3.8.2:** *In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?*

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 3.2 on the next page for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

Another case where things go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see exercises).

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table 3.2: Attempts to use Euler's to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $1/16 = 1/2 \times 1/2 \times 1/2 \times 1/2$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.

- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse. There is a certain optimum step size such that the precision increases as we approach it, but then starts getting worse as we make our step size smaller still. Toruble is: this optimum may be hard to find.

- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers seem to have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

Exercises

Exercise* 3.8.3*: *Consider* $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. *Use Euler's method with step size* $h = 0.5$ *to approximate* $x(1)$.

Exercise* 3.8.4*: *Consider* $\frac{dx}{dt} = t - x$, $x(0) = 1$. *a) Use Euler's method with step sizes* $h = 1, 1/2, 1/4, 1/8$ *to approximate* $x(1)$. *b) Solve the equation exactly. c) Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.*

Exercise* 3.8.5*: *Approximate the value of e by looking at the initial value problem* $y' = y$ *with* $y(0) = 1$ *and approximating* $y(1)$ *using Euler's method with a step size of* 0.2.

Exercise* 3.8.6*: *Example of numerical instability: Take* $y' = -5y$, $y(0) = 1$. *We know that the solution should decay to zero as x grows. Using Euler's method, start with* $h = 1$ *and compute* y_1, y_2, y_3, y_4 *to try to approximate* $y(4)$. *What happened? Now halve the interval. Keep halving the interval and approximating* $y(4)$ *until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.*

The simplest method used in practice is the *Runge-Kutta method*. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\big(x_i + h/2, y_i + k_1\, h/2\big), & x_{i+1} &= x_i + h, \\ k_3 &= f\big(x_i + h/2, y_i + k_2\, h/2\big), & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\, h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned}$$

Exercise* 3.8.7*: *Consider* $\frac{dy}{dx} = yx^2$, $y(0) = 1$. *a) Use Runge-Kutta (see above) with step sizes* $h = 1$ *and* $h = 1/2$ *to approximate* $y(1)$. *b) Use Euler's method with* $h = 1$ *and* $h = 1/2$. *c) Solve exactly, find the exact value of* $y(1)$*, and compare.*

Exercise* 3.8.101*: *Let* $x' = \sin(xt)$*, and* $x(0) = 1$. *Approximate* $x(1)$ *using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.*

Exercise* 3.8.102*: *Let* $x' = 2t$*, and* $x(0) = 0$. *Approximate* $x(4)$ *using Euler's method with step sizes 4, 2, and 1. Solve the IVP exactly, and compute the errors. Compute the factor by which the errors changed.*

Exercise* 3.8.103*: *Let* $x' = xe^{xt+1}$*, and* $x(0) = 0$. *Approximate* $x(4)$ *using Euler's method with step sizes 4, 2, and 1. Guess an exact solution, and compute the errors.*

Chapter 4

Linear algebra

Linear algebra is the branch of mathematics that is concerned with space and linear mappings therein. Most students are initially exposed to linear algebra from the process of solving a system of linear equations containing several unknowns. That will be our starting point. In doing so, we will begin to define mathematical concepts such as vectors and matrices that stem from this process.

4.1 Solving linear systems: Gauss's method

Attribution: §One.I in [H]

Systems of linear equations are common in science and mathematics.

Example 4.1.1. We can mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water (conditions should be very well controlled—trinitrotoluene is better known as TNT). In what proportion should we mix them? The number of atoms of each element present before the reaction

$$x\,C_7H_8 \;+\; y\,HNO_3 \quad\longrightarrow\quad z\,C_7H_5O_6N_3 \;+\; w\,H_2O$$

must equal the number present afterward. Applying that in turn to the elements C, H, N, and O gives this system.

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

In this linear system, the equations involve only the first power of each variable. This section shows how to solve any such system. ■

Definition 4.1.1. *A* linear combination *of $x_1, \ldots, x_n$ has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the combination's coefficients. *A* linear equation *in the variables $x_1, \ldots, x_n$ has the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$ where $d \in \mathbb{R}$ is the* constant.

An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a solution *of, or* satisfies*, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A* system of linear equations

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= d_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m \end{aligned}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations.

Example 4.1.2. The combination $3x_1 + 2x_2$ of x_1 and x_2 is linear. The combination $3x_1^2 + 2\sin x_2$ is not linear, nor is $3x_1^2 + 2x_2$. ■

Example 4.1.3. The ordered pair $(-1, 5)$ is a solution of this system.

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -x_1 + x_2 &= 6 \end{aligned}$$

In contrast, $(5, -1)$ is not a solution. ■

Finding the set of all solutions is *solving* the system. We don't need guesswork or good luck, there is an algorithm that always works. This algorithm is *Gauss's method* (or *Gaussian elimination* or *linear elimination*).

Example 4.1.4. To solve this system

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 &= 3 \end{aligned}$$

we transform it, step by step, until it is in a form that we can easily solve.

The first transformation rewrites the system by interchanging the first and third row.

$$\xrightarrow{\text{swap row 1 with row 3}} \quad \begin{aligned} \tfrac{1}{3}x_1 + 2x_2 &= 3 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The second transformation rescales the first row by a factor of 3.

$$\xrightarrow{\text{multiply row 1 by 3}} \quad \begin{aligned} x_1 + 6x_2 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The third transformation is the only nontrivial one in this example. We mentally multiply both sides of the first row by -1, mentally add that to the second row, and write the result in as the new second row.

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \quad \begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ & & -x_2 & - & 2x_3 & = & -7 \\ & & & & 3x_3 & = & 9 \end{array}$$

These steps have brought the system to a form where we can easily find the value of each variable. The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$. Thus the system has a unique solution; the solution set is $\{(3, 1, 3)\}$. ■

In the remainder of this section consists of examples of solving linear systems by Gauss's method, because it is fast and easy.

Theorem 4.1.1 (Gauss's method)**.** *If a linear system is changed to another by one of these operations*

1. *an equation is swapped with another (swapping)*
2. *an equation is multiplied by a nonzero constant (rescaling)*
3. *an equation is replaced by the sum of itself and a multiple of another (row combination)*

then the two systems have the same set of solutions. These three operations are called the elementary reduction operations*, or* elementary reduction operations*, or* elementary row operations*, or* Gaussian operations.

Remark 4.1.1. *Each of the three Gauss's method operations has a restriction. Multiplying a row by* 0 *is not allowed because obviously that can change the solution set. Similarly, adding a multiple of a row to itself is not allowed because adding* -1 *times the row to itself has the effect of multiplying the row by* 0*. We also disallow swapping a row with itself, because it's pointless.*

When writing out the calculations, we will abbreviate 'row i' by 'ρ_i'. For instance, we will denote a row combination operation by $k\rho_i + \rho_j$, with the row that changes written second. To save writing we will often combine addition steps when they use the same ρ_i as in the next example.

Example 4.1.5. Gauss's method systematically applies the row operations to solve a system. Here is a typical case.

$$\begin{array}{rcrcrcr} x & + & y & & & = & 0 \\ 2x & - & y & + & 3z & = & 3 \\ x & - & 2y & - & z & = & 3 \end{array}$$

We begin by using the first row to eliminate the $2x$ in the second row and the x in the third. To get rid of the $2x$ we mentally multiply the entire first row by -2, add that to the second row, and write the result in as the new second row. To eliminate the x in the third row we multiply the first row by -1, add that to the third row, and write the result in as the new third row.

$$\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x + \quad y \qquad &= 0 \\ -3y + 3z &= 3 \\ -3y - \ \ z &= 3 \end{aligned}$$

We finish by transforming the second system into a third, where the bottom equation involves only one unknown. We do that by using the second row to eliminate the y term from the third row.

$$\xrightarrow{-\rho_2+\rho_3} \begin{aligned} x + \quad y \qquad &= 0 \\ -3y + \ 3z &= 3 \\ -4z &= 0 \end{aligned}$$

Now finding the system's solution is easy. The third row gives $z = 0$. Substitute that back into the second row to get $y = -1$. Then substitute back into the first row to get $x = 1$. ■

Example 4.1.6. The reduction

$$\begin{aligned} x + \ y + \ z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x + \ y + \ z &= 9 \\ 2y - 5z &= -17 \\ 3y - 8z &= -27 \end{aligned}$$

$$\xrightarrow{-3/2\rho_2+\rho_3} \begin{aligned} x + \ y + \qquad z &= 9 \\ 2y - \quad 5z &= -17 \\ -1/2\, z &= -3/2 \end{aligned}$$

shows that $z = 3$, $y = -1$, and $x = 7$. ■

As illustrated above, the point of Gauss's method is to use the elementary reduction operations to set up back-substitution.

Definition 4.1.2 (Leading variable)**.** *In each row of a system, the first variable with a nonzero coefficient is the row's* leading variable.

Definition 4.1.3 (Echelon form)**.** *A system is in* echelon form *if each leading variable is to the right of the leading variable in the row above it, except for the leading variable in the first row, and any all-zero rows are at the bottom.*

Definition 4.1.4. *In each row of a system, the first variable with a nonzero coefficient is the row's* leading variable. *A system is in* echelon form *if each leading variable is to the right of the leading variable in the row above it, except for the leading variable in the first row, and any all-zero rows are at the bottom.*

Example 4.1.7. The prior three examples only used the operation of row combination. This linear system requires the swap operation to get it into echelon form because after the first combination

$$\begin{array}{rcrcrcrcl} x & - & y & & & & & = & 0 \\ 2x & - & 2y & + & z & + & 2w & = & 4 \\ & & y & & & + & w & = & 0 \\ & & & & 2z & + & w & = & 5 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcrcrcrcl} x & - & y & & & & & = & 0 \\ & & & & z & + & 2w & = & 4 \\ & & y & & & + & w & = & 0 \\ & & & & 2z & + & w & = & 5 \end{array}$$

the second equation has no leading y. We exchange it for a lower-down row that has a leading y.

$$\xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{array}{rcrcrcrcl} x & - & y & & & & & = & 0 \\ & & y & & & + & w & = & 0 \\ & & & & z & + & 2w & = & 4 \\ & & & & 2z & + & w & = & 5 \end{array}$$

(Had there been more than one suitable row below the second then we could have used any one.) With that, Gauss's method proceeds as before.

$$\xrightarrow{-2\rho_3+\rho_4} \begin{array}{rcrcrcrcr} x & - & y & & & & & = & 0 \\ & & y & & & + & w & = & 0 \\ & & & & z & + & 2w & = & 4 \\ & & & & & & -3w & = & -3 \end{array}$$

Back-substitution gives $w = 1$, $z = 2$, $y = -1$, and $x = -1$. ∎

Strictly speaking, to solve linear systems we don't need the row rescaling operation. We have introduced it here because it is convenient and because we will use it later in this chapter as part of a variation of Gauss's method, the Gauss-Jordan method.

All of the systems so far have the same number of equations as unknowns. All of them have a solution and for all of them there is only one solution. What else can happen?

Example 4.1.8. This system has more equations than variables.

$$\begin{array}{rcrcr} x & + & 3y & = & 1 \\ 2x & + & y & = & -3 \\ 2x & + & 2y & = & -2 \end{array}$$

Gauss's method helps us understand this system also, since this

$$\xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcrcr} x & + & 3y & = & 1 \\ & & -5y & = & -5 \\ & & -4y & = & -4 \end{array}$$

shows that one of the equations is redundant. Echelon form

$$\xrightarrow{-(4/5)\rho_2+\rho_3} \begin{array}{rcrcr} x & + & 3y & = & 1 \\ & & -5y & = & -5 \\ & & 0 & = & 0 \end{array}$$

gives that $y = 1$ and $x = -2$. The '$0 = 0$' reflects the redundancy. ∎

Gauss's method is also useful on systems with more variables than equations. Many more examples follow.

Another way that linear systems can differ from the examples shown above is that some linear systems do not have a unique solution. This can happen in two ways. The first is that a system can fail to have any solution at all.

Example 4.1.9. Contrast the system in the last example with this one.

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned} \quad \underset{-2\rho_1+\rho_3}{\xrightarrow{-2\rho_1+\rho_2}} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ -4y &= -2 \end{aligned}$$

Here the system is inconsistent: no pair of numbers (s_1, s_2) satisfies all three equations simultaneously. Echelon form makes the inconsistency obvious.

$$\xrightarrow{-(4/5)\rho_2+\rho_3} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ 0 &= 2 \end{aligned}$$

The solution set is empty, because the third equation is never satisfied, no matter the choice of x and y. ■

Definition 4.1.5 (Inconsistent system, contradictory equation)**.** *A system of linear equations is an* inconsistent system *if there is no set of values for the unknowns that satisfy all of the equations. If a system is inconsistent, then it is possible to manipulate the equations in such a way as to obtain a* contradictory equation*.*

Example 4.1.10. The prior system has more equations than unknowns but that is not what causes the inconsistency— Example 4.1.8 has more equations than unknowns and yet is consistent. Nor is having more equations than unknowns necessary for inconsistency, as we see with this inconsistent system that has the same number of equations as unknowns.

$$\begin{aligned} x + 2y &= 8 \\ 2x + 4y &= 8 \end{aligned} \quad \xrightarrow{-2\rho_1+\rho_2} \quad \begin{aligned} x + 2y &= 8 \\ 0 &= -8 \end{aligned}$$

Instead, inconsistency has to do with the interaction of the left and right sides; in the first system above the left side's second equation is twice the first but the right side's second constant is not twice the first. ■

The other way that a linear system can fail to have a unique solution, besides having no solutions, is to have many solutions.

Example 4.1.11. In this system

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 8 \end{aligned}$$

any pair of numbers satisfying the first equation also satisfies the second. The solution set $\{(x, y) \mid x + y = 4\}$ is infinite; some example member pairs are $(0, 4)$, $(-1, 5)$, and $(2.5, 1.5)$.

The result of applying Gauss's method here contrasts with the prior example because we do not get a contradictory equation.

$$\xrightarrow{-2\rho_1+\rho_2} \begin{array}{rl} x + y &= 4 \\ 0 &= 0 \end{array}$$

■

Don't be fooled by that example: a $0 = 0$ equation is not the signal that a system has many solutions.

Example 4.1.12. The absence of a $0 = 0$ equation does not keep a system from having many different solutions. This system is in echelon form, has no $0 = 0$, but has infinitely many solutions, including $(0, 1, -1)$, $(0, 1/2, -1/2)$, $(0, 0, 0)$, and $(0, -\pi, \pi)$ (any triple whose first component is 0 and whose second component is the negative of the third is a solution).

$$\begin{array}{rl} x + y + z &= 0 \\ y + z &= 0 \end{array}$$

Nor does the presence of $0 = 0$ mean that the system must have many solutions. Example 4.1.8 shows that. So does this system, which does not have any solutions at all despite that in echelon form it has a $0 = 0$ row.

$$\begin{array}{rl} 2x \qquad - 2z &= 6 \\ y + \ \ z &= 1 \\ 2x + \ y - \ \ z &= 7 \\ 3y + 3z &= 0 \end{array} \xrightarrow{-\rho_1+\rho_3} \begin{array}{rl} 2x \qquad - 2z &= 6 \\ y + \ \ z &= 1 \\ y + \ \ z &= 1 \\ 3y + 3z &= 0 \end{array}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{array}{rl} 2x \qquad - 2z &= \ \ 6 \\ y + \ \ z &= \ \ 1 \\ 0 &= \ \ 0 \\ 0 &= -3 \end{array}$$

■

In summary, Gauss's method uses the row operations to set a system up for back substitution. If any step shows a contradictory equation then we can stop with the conclusion that the system has no solutions. If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has a unique solution and we find it by back substitution. Finally, if we reach echelon form without a contradictory equation, and there is not a unique solution—that is, at least one variable is not a leading variable—then the system has infinitely many solutions.

Example 4.1.13. This system has many solutions because in echelon form

$$\begin{array}{rcrcr} 2x & & +z & = & 3 \\ x & -y & -z & = & 1 \\ 3x & -y & & = & 4 \end{array} \xrightarrow[-3/2\rho_1+\rho_3]{-1/2\rho_1+\rho_2} \begin{array}{rcrcr} 2x & & +z & = & 3 \\ & -y & -3/2\,z & = & -1/2 \\ & -y & -3/2\,z & = & -1/2 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{rcrcr} 2x & & +z & = & 3 \\ & -y & -3/2\,z & = & -1/2 \\ & & 0 & = & 0 \end{array}$$

not all of the variables are leading variables. Theorem 4.1.1 shows that an (x, y, z) satisfies the first system if and only if it satisfies the third. So we can describe the solution set

$$\{(x, y, z) \mid 2x + z = 3 \text{ and } x - y - z = 1 \text{ and } 3x - y = 4\}$$

in this way. This description

$$\{(x, y, z) \mid 2x + z = 3 \text{ and } -y - 3/2\,z = -1/2\} \tag{4.1}$$

is better because it has two equations instead of three but it is not optimal because it still has some hard to understand interactions among the variables.

To improve it, use the variable that does not lead any equation, z, to describe the variables that do lead, x and y. The second equation gives $y = 1/2 - 3/2\,z$ and the first equation gives $x = 3/2 - 1/2\,z$. Thus we can describe the solution set in this way:

$$\{(x, y, z) = (3/2 - 1/2\,z, 1/2 - 3/2\,z, z) \mid z \in \mathbb{R}\}. \tag{4.2}$$

■

Compared with (4.1), the advantage of (4.2) is that z can be any real number. This makes the job of deciding which tuples are in the solution set much easier. For instance, taking $z = 2$ shows that $(1/2, -5/2, 2)$ is a solution.

Definition 4.1.6 (Free variable). *In an echelon form, the variables that are not leading are* free variables.

Example 4.1.14. Reduction of a linear system can end with more than one variable free. Gauss's method on this system

$$\begin{array}{rrrrcr} x+ & y+ & z- & w & = & 1 \\ & y- & z+ & w & = & -1 \\ 3x & & +6z- & 6w & = & 6 \\ & -y+ & z- & w & = & 1 \end{array} \xrightarrow{-3\rho_1+\rho_3} \begin{array}{rrrrcr} x+ & y+ & z- & w & = & 1 \\ & y- & z+ & w & = & -1 \\ & -3y+ & 3z- & 3w & = & 3 \\ & -y+ & z- & w & = & 1 \end{array}$$

$$\xrightarrow[\rho_2+\rho_4]{3\rho_2+\rho_3} \begin{array}{rcr} x+y+z-w & = & 1 \\ y-z+w & = & -1 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}$$

leaves x and y leading and both z and w free. To get the description that we prefer, we work from the bottom. We first express the leading variable y in terms of z and w, as $y = -1 + z - w$. Moving up to the top equation, substituting for y gives $x + (-1 + z - w) + z - w = 1$ and solving for x leaves $x = 2 - 2z + 2w$. The solution set

$$\{(2 - 2z + 2w, -1 + z - w, z, w) \mid z, w \in \mathbb{R}\}$$

has the leading variables in terms of the variables that are free. ■

Example 4.1.15. The list of leading variables may skip over some columns. After this reduction

$$\begin{array}{rcl} 2x - 2y \qquad\qquad &=& 0 \\ z + 3w &=& 2 \\ 3x - 3y \qquad\qquad &=& 0 \\ x - y + 2z + 6w &=& 4 \end{array} \xrightarrow[-1/2\rho_1+\rho_4]{-3/2\rho_1+\rho_3} \begin{array}{rcl} 2x - 2y \qquad\qquad &=& 0 \\ z + 3w &=& 2 \\ 0 &=& 0 \\ 2z + 6w &=& 4 \end{array}$$

$$\xrightarrow{-2\rho_2+\rho_4} \begin{array}{rcl} 2x - 2y \qquad\qquad &=& 0 \\ z + 3w &=& 2 \\ 0 &=& 0 \\ 0 &=& 0 \end{array}$$

x and z are the leading variables, not x and y. The free variables are y and w and so we can describe the solution set as $\{(y, y, 2 - 3w, w) \mid y, w \in \mathbb{R}\}$. For instance, $(1, 1, 2, 0)$ satisfies the system— take $y = 1$ and $w = 0$. The four-tuple $(1, 0, 5, 4)$ is not a solution since its first coordinate does not equal its second. ■

A variable that we use to describe a family of solutions is a *parameter*. We say that the solution set in the prior example is *parametrized* with y and w.

Example 4.1.16. This is another system with infinitely many solutions.

$$\begin{array}{rcl} x + 2y \qquad\qquad &=& 1 \\ 2x \qquad + z \qquad &=& 2 \\ 3x + 2y + z - w &=& 4 \end{array} \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x + 2y \qquad\qquad &=& 1 \\ -4y + z \qquad &=& 0 \\ -4y + z - w &=& 1 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{rcl} x + 2y \qquad\qquad &=& 1 \\ -4y + z \qquad &=& 0 \\ -w &=& 1 \end{array}$$

The leading variables are x, y, and w. The variable z is free. Notice that, although there are infinitely many solutions, the value of w doesn't vary but is constant at -1. To parametrize, write w in terms of z with $w = -1 + 0z$. Then $y = z/4$. Substitute for y in the first equation to get $x = 1 - 1/2\, z$. The solution set is $\{(1 - 1/2\, z, 1/4\, z, z, -1) \mid z \in \mathbb{R}\}$. ■

Parametrizing solution sets shows that systems with free variables have infinitely many solutions. For instance, above z takes on all of infinitely many real number values, each associated with a different solution.

Exercises

Exercise* 4.1.1*: *In this section we had to do a bit of work to show that there are only three types of solution sets—singleton, empty, and infinite. But this is easy to see geometrically in the case of systems with two equations and two unknowns. Draw each two-unknowns equation as a line in the plane, and then the two lines could have a unique intersection point, be parallel, or be the same line.*

Exercise* 4.1.2*: *If a system of equations has infinitely many solutions, does that imply that every possible solution is indeed a solution? Find an example in this section that supports your position.*

Apply Gauss's method to find the unique solution.

Exercise* 4.1.101*:

$$\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$$

Exercise* 4.1.102*:

$$\begin{aligned} x \phantom{{}+y} - z &= 0 \\ 3x + y \phantom{{}+z} &= 1 \\ -x + y + z &= 4 \end{aligned}$$

Apply Gauss's method to solve the system or conclude 'many solutions' or 'no solution'.

Exercise* 4.1.103*:

$$\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$$

Exercise* 4.1.104*:

$$\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$$

Exercise* 4.1.105*:

$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

Exercise* 4.1.106*:

$$\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$$

Exercise* 4.1.107*:

$$\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x \phantom{{}-2y} + z &= 5 \\ x + y - z &= 10 \end{aligned}$$

Exercise* 4.1.108*:

$$\begin{aligned}
2x \phantom{{}+y} + z + w &= 5 \\
y \phantom{{}+2z} - w &= -1 \\
3x \phantom{{}+y} - z - w &= 0 \\
4x + y + 2z + w &= 9
\end{aligned}$$

Exercise* 4.1.109*: *For which values of k are there no solutions, many solutions, or a unique solution to this system?*

$$\begin{aligned}
x - y &= 1 \\
3x - 3y &= k
\end{aligned}$$

4.2 Transition to matrix notation

Attribution: §One.I in [H]

In this section, we develop a streamlined notation for linear systems and their solution sets.

Definition 4.2.1. *An $m \times n$* matrix *is a rectangular array of numbers with m* rows *and n* columns. *Each number in the matrix is an* entry.

We usually denote a matrix with an upper case roman letters. For instance,

$$A = \begin{pmatrix} 1 & 2.2 & 5 \\ 3 & 4 & -7 \end{pmatrix}$$

has 2 rows and 3 columns and so is a 2×3 matrix. Read that aloud as "two-by-three"; the number of rows is always given first. (The matrix has parentheses on either side so that when two matrices are adjacent we can tell where one ends and the other begins.) We name matrix entries with the corresponding lower-case letter so that $a_{2,1} = 3$ is the entry in the second row and first column of the above array. Note that the order of the subscripts matters: $a_{1,2} \neq a_{2,1}$ since $a_{1,2} = 2.2$.

We use matrices to do Gauss's method in essentially the same way that we did it for systems of equations: we perform row operations to arrive at *matrix echelon form*, where the leading entry in lower rows are to the right of those in the rows above. We switch to this notation because it lightens the clerical load of Gauss's method— the copying of variables and the writing of +'s and ='s.

Example 4.2.1. We can abbreviate this linear system

$$\begin{aligned} x + 2y \phantom{{}-2z} &= 4 \\ y - z &= 0 \\ x \phantom{{}+2y} + 2z &= 4 \end{aligned}$$

with this matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right).$$

The vertical bar reminds a reader of the difference between the coefficients on the system's left hand side and the constants on the right. With a bar, this is an *augmented* matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row stands for $y - z = 0$ and the first row stands for $x + 2y = 4$ so the solution set is $\{(4 - 2z, z, z) \mid z \in \mathbb{R}\}$. ■

Matrix notation also clarifies the descriptions of solution sets. Example 4.1.14's

$$\{(2-2z+2w,-1+z-w,z,w) \mid z,w\in\mathbb{R}\}$$

is hard to read. We will rewrite it to group all of the constants together, all of the coefficients of z together, and all of the coefficients of w together. We write them vertically, in one-column matrices:

$$\left\{\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}+\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}z+\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\right\}.$$

For instance, the top line says that $x=2-2z+2w$ and the second line says that $y=-1+z-w$.

Definition 4.2.2. *A* vector *(or* column vector*) is a matrix with a single column. A matrix with a single row is a* row vector. *The entries of a vector are its* components. *A column or row vector whose components are all zeros is a* zero vector.

Vectors are an exception to the convention of representing matrices with capital roman letters. We use lower-case roman or greek letters overlined with an arrow: $\vec{a},\vec{b},\ldots$ or $\vec{\alpha},\vec{\beta},\ldots$ (boldface is also common: $\boldsymbol{a}$ or $\boldsymbol{\alpha}$). For instance, this is a column vector with a third component of 7.

$$\vec{v}=\begin{pmatrix}1\\3\\7\end{pmatrix}$$

We are set to use matrix and vector notation to both solve systems and express the solution.

Example 4.2.2. This system

$$\begin{array}{rcl} 2x+y-w &=& 4\\ y+w+u &=& 4\\ x-z+2w &=& 0 \end{array}$$

reduces in this way:

$$\left(\begin{array}{ccccc|c}2&1&0&-1&0&4\\0&1&0&1&1&4\\1&0&-1&2&0&0\end{array}\right)\xrightarrow{-(1/2)\rho_1+\rho_3}\left(\begin{array}{ccccc|c}2&1&0&-1&0&4\\0&1&0&1&1&4\\0&-1/2&-1&5/2&0&-2\end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3}\left(\begin{array}{ccccc|c}2&1&0&-1&0&4\\0&1&0&1&1&4\\0&0&-1&3&1/2&0\end{array}\right).$$

The solution set is $\{(w+1/2\,u,4-w-u,3w+1/2\,u,w,u) \mid w,u\in\mathbb{R}\}$. We write that in vector form:

$$\left\{\begin{pmatrix}x\\y\\z\\w\\u\end{pmatrix}=\begin{pmatrix}0\\4\\0\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-1\\3\\1\\0\end{pmatrix}w+\begin{pmatrix}1/2\\-1\\1/2\\0\\1\end{pmatrix}u \mid w,u\in\mathbb{R}\right\}.$$

Note how well vector notation sets off the coefficients of each parameter. For instance, the third row of the vector form shows plainly that if u is fixed then z increases three times as fast as w. ■

Example 4.2.3. In the same way, the system

$$\begin{aligned} x - y + z &= 1 \\ 3x \phantom{{}-2y} + z &= 3 \\ 5x - 2y + 3z &= 5 \end{aligned}$$

reduces

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & -2 & 3 & 5 \end{array}\right) \xrightarrow[-5\rho_1+\rho_3]{-3\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

to give the one-parameter solution set $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R}\right\}$. ■

Example 4.2.4. The Chemistry problem of Example 4.1.1 has many solutions.

$$\left(\begin{array}{cccc|c} 7 & 0 & -7 & 0 & 0 \\ 8 & 1 & -5 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 3 & -6 & -1 & 0 \end{array}\right) \xrightarrow{(-8/7)\rho_1+\rho_2} \left(\begin{array}{cccc|c} 7 & 0 & -7 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 3 & -6 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 7 & 0 & -7 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & -6 & 2 & 0 \\ 0 & 0 & -15 & 5 & 0 \end{array}\right) \xrightarrow{-(5/2)\rho_3+\rho_4} \left(\begin{array}{cccc|c} 7 & 0 & -7 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & -6 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set $\left\{\begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R}\right\}$ has many vectors besides the zero vector (if we interpret w as a number of molecules then solutions make sense only when w is a nonnegative multiple of 3). ■

Remark 4.2.1. *In the previous example, we note than no row operation modifies the vector of zeros appearing in our augmented matrix, and so we could have saved ourselves some writing by omitting it—while remembering that it's there:*

$$\begin{pmatrix} 7 & 0 & -7 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix} \xrightarrow{(-8/7)\rho_1+\rho_2} \begin{pmatrix} 7 & 0 & -7 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{pmatrix} 7 & 0 & -7 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -15 & 5 \end{pmatrix} \xrightarrow{-(5/2)\rho_3+\rho_4} \begin{pmatrix} 7 & 0 & -7 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercises

Exercise* 4.2.1*: *Calcium hydroxide* $Ca(OH)_2$ *has numerous uses, including the treatment of human sewage. Calcium hydroxide treats phosphoric acid* H_3PO_4 *and converts it to water and tricalcium phosphate* $Ca_3(PO_4)_2$*. Suppose that the coefficients of this chemical reaction are unknown:*

$$x\,Ca(OH)_2 \;+\; y\,H_3PO_4 \quad\longrightarrow\quad z\,Ca_3(PO_4)_2 \;+\; w\,H_2O.$$

Restate this problem as an augmented matrix, solve this system, and express the solution set using vectors. What value makes sense for the free variable?

Exercise* 4.2.2*: *Choose a chemical equation of interest to you, consisting of at least 4 molecules involving at least 4 elements. Restate this problem as an augmented matrix, solve this system, and express the solution set using vectors. Is your answer consistent with the true coefficients?*

Apply Gauss' method using matrix notation, and express the solution set using vectors.

Exercise* 4.2.101*:

$$\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$$

Exercise* 4.2.102*:

$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$$

Exercise* 4.2.103*:

$$\begin{aligned} x_1 \phantom{{}- x_2} + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$$

Exercise* 4.2.104*:

$$\begin{aligned} x + 2y - z \phantom{{}+w} &= 3 \\ 2x + y \phantom{{}+z} + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$$

Apply Gauss' method using matrix notation, and conclude that it possesses a unique solution. Express this solution as a vector.

Exercise* 4.2.105*:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned}$$

Exercise* 4.2.106*:

$$\begin{aligned} 2a + b - c &= 2 \\ 2a \phantom{{}+b} + c &= 3 \\ a - b \phantom{{}+c} &= 0 \end{aligned}$$

Apply Gauss' method using matrix notation, and conclude that it possesses no solution.

Exercise* 4.2.107*:

$$\begin{aligned} x \phantom{{}+y} + z + w &= 4 \\ 2x + y \phantom{{}+z} - w &= 2 \\ 3x + y + z \phantom{{}+w} &= 7 \end{aligned}$$

Exercise* 4.2.108*:

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

The following systems possess infinitely many solutions. Apply Gauss' method using matrix notation, and express the solution set using vectors.

Exercise* 4.2.109*:

$$\begin{aligned} 2x + y - z &= 1 \\ 4x - y \phantom{{}-z} &= 3 \end{aligned}$$

Exercise* 4.2.110*:

$$\begin{aligned} x \phantom{{}+2y} - z \phantom{{}-w} &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

Exercise* 4.2.111*:

$$\begin{aligned} x - y + z \phantom{{}+w} &= 0 \\ y \phantom{{}+3z} + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \phantom{{}+3z} - w &= 0 \end{aligned}$$

Exercise* 4.2.112*:

$$\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - b + c + d + e &= 3 \end{aligned}$$

Exercise* 4.2.113*:

$$\begin{aligned} x_1 + 2x_2 + + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

Apply Gauss' method using matrix notation, and express the solution set using vectors.

Exercise* 4.2.114*:

$$\begin{aligned} 3x + 2y + z &= 1 \\ x - y + z &= 2 \\ 5x + 5y + z &= 0 \end{aligned}$$

Exercise* 4.2.115*:

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 0 \\ x & - & y & & & = & -3 \\ 3x & - & y & - & 2z & = & -6 \\ & & 2y & - & 2z & = & 3 \end{array}$$

Exercise* 4.2.116*:

$$\begin{array}{rcrcrcrcr} 2x & - & y & - & z & + & w & = & 4 \\ x & + & y & + & z & & & = & -1 \end{array}$$

Exercise* 4.2.117*:

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 0 \\ x & - & y & & & = & -3 \\ 3x & - & y & - & 2z & = & 0 \end{array}$$

4.3 Gauss-Jordan reduction

Attribution: §One.III in [H]

After developing the mechanics of Gauss's method, we observed that it can be done in more than one way. For example, from this matrix

$$\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$$

we could derive any of these three echelon form matrices:

$$\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first results from $-2\rho_1 + \rho_2$. The second comes from doing $(1/2)\rho_1$ and then $-4\rho_1 + \rho_2$. The third comes from $-2\rho_1 + \rho_2$ followed by $2\rho_2 + \rho_1$ (after the first row combination the matrix is already in echelon form so the second one is extra work but it is nonetheless a legal row operation).

The fact that echelon form is not unique raises questions. Will any two echelon form versions of a linear system have the same number of free variables? If yes, will the two have exactly the same free variables? In this section we will give a way to decide if one linear system can be derived from another by row operations. The answers to both questions, both 'yes', will follow from this.

Here is an extension of Gauss's method that has some advantages. It's called the *Gauss-Jordan method* or *Gauss-Jordan reduction*.

Example 4.3.1. To solve

$$\begin{aligned} x_1 + x_2 - 2x_3 &= -2 \\ x_2 + 3x_3 &= 7 \\ x_1 \qquad - x_3 &= -1, \end{aligned}$$

we think of this as the product of a matrix of coefficients with a vector of unknowns

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix}$$

or even more succinctly as

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & 1 \end{pmatrix}\vec{x} = \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix}.$$

We generate our augmented matrix and reduce it to echelon form:

$$\xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right).$$

We can keep going to a second stage by making the leading entries into 1's

$$\xrightarrow{(1/4)\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

and then to a third stage that uses the leading entries to eliminate all of the other entries in each column by combining upwards:

$$\xrightarrow[2\rho_3+\rho_1]{-3\rho_3+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

The answer is $x_1 = 1$, $x_2 = 1$, and $x_3 = 2$. ∎

Using one entry to clear out the rest of a column is *pivoting* on that entry. Note that the row combination operations in the first stage move left to right, from column one to column three, while the combination operations in the third stage move right to left.

Example 4.3.2. The middle stage operations that turn the leading entries into 1's don't interact so we can combine multiple ones into a single step.

$$\left(\begin{array}{cc|c} 2 & 1 & 7 \\ 4 & -2 & 6 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & -4 & -8 \end{array}\right) \xrightarrow[(-1/4)\rho_2]{(1/2)\rho_1} \left(\begin{array}{cc|c} 1 & 1/2 & 7/2 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{-(1/2)\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & 2 \end{array}\right)$$

The answer is $x = 5/2$ and $y = 2$. ∎

Definition 4.3.1 (Reduced echelon form). *A matrix or linear system is in* reduced echelon form *if, in addition to being in echelon form, each leading entry is a* 1 *and is the only nonzero entry in its column.*

The cost of using Gauss-Jordan reduction to solve a system is the additional arithmetic. The benefit is that we can just read off the solution set description.

Example 4.3.3. Another example of Gauss-Jordan reduction:

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 3 & 5 \end{array}\right) \xrightarrow{-(1/2)\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 5/2 & 5 \end{array}\right) \xrightarrow[(2/5)\rho_2]{(1/2)\rho_1} \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{-(1/2)\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right).$$

∎

In any echelon form system, reduced or not, we can read off when the system has an empty solution set because there is a contradictory equation. We can read off when the system has a one-element solution set because there is no contradiction and every variable is the leading variable in some row. And, we can read off when the system has an infinite solution set because there is no contradiction and at least one variable is free.

However, in reduced echelon form we can read off not just the size of the solution set but also its description. We have no trouble describing the solution set when it is empty, of course. Example 4.3.1 and Example 4.3.2 show how in a single element solution set case the single element is in the column of constants.

Example 4.3.4. This example shows how to read the parametrization of an infinite solution set.

$$\left(\begin{array}{cccc|c} 2 & 6 & 1 & 2 & 5 \\ 0 & 3 & 1 & 4 & 1 \\ 0 & 3 & 1 & 2 & 5 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 2 & 6 & 1 & 2 & 5 \\ 0 & 3 & 1 & 4 & 1 \\ 0 & 0 & 0 & -2 & 4 \end{array}\right) \underset{(1/3)\rho_2-(1/2)\rho_3}{\xrightarrow{(1/2)\rho_1}} \cdots \underset{-\rho_3+\rho_1}{\xrightarrow{-(4/3)\rho_3+\rho_2}} \cdots$$

$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 0 & -9/2 \\ 0 & 1 & 1/3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

As a linear system this is

$$\begin{array}{rcrcrcr} x_1 & & -\ 1/2\,x_3 & & & = & -9/2 \\ & x_2 & +\ 1/3\,x_3 & & & = & 3 \\ & & & & x_4 & = & -2 \end{array}$$

so a solution set description is

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -9/2 \\ 3 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/3 \\ 1 \\ 0 \end{pmatrix} x_3 \mid x_3 \in \mathbb{R} \right\}.$$

∎

Exercises

Exercise* 4.3.1*: *Fill in the missing steps in Example 4.3.4.*

Apply Gauss-Jordan reduction to solve the system and state the solution set.

Exercise* 4.3.2*: $\begin{pmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Exercise* 4.3.3*: $\begin{pmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

Exercise* 4.3.4*: $\begin{pmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$

Exercise* 4.3.5*: $\begin{pmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} -33 \\ 24 \\ 5 \end{pmatrix}$

Exercise* 4.3.6*: $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 8 \\ 5 \end{pmatrix}$

Exercise* 4.3.7*:

$$\begin{aligned} 2x + y - z &= 1 \\ 4x - y \phantom{{}-z} &= 3 \end{aligned}$$

Compare with your answer to Exercise 4.2.109.

Exercise* 4.3.8*:

$$\begin{aligned} x \phantom{{}+2y} - z \phantom{{}-w} &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

Compare with your answer to Exercise 4.2.110.

Exercise* 4.3.9*:

$$\begin{aligned} x - y + z \phantom{{}+w} &= 0 \\ y \phantom{{}+3z} + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \phantom{{}+3z} - w &= 0 \end{aligned}$$

Compare with your answer to Exercise 4.2.111.

Exercise* 4.3.10*:

$$\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - b + c + d + e &= 3 \end{aligned}$$

Compare with your answer to Exercise 4.2.112.

Exercise* 4.3.11*:

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Compare with your answer to Exercise 4.2.108.

Exercise* 4.3.12*:

$$\begin{aligned} x_1 + 2x_2 + + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

Compare with your answer to Exercise 4.2.113.

Exercise* 4.3.101*: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \vec{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$

Exercise* 4.3.102*:

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

Exercise* 4.3.103*:

$$\begin{aligned} x \phantom{{}+2y} - z &= 4 \\ 2x + 2y \phantom{{}-z} &= 1 \end{aligned}$$

Exercise* 4.3.104*:

$$\begin{aligned} 3x - 2y &= 1 \\ 6x + y &= 1/2 \end{aligned}$$

Exercise* 4.3.105*:

$$\begin{aligned} 2x - y \phantom{{}-2z} &= -1 \\ x + 3y - z &= 5 \\ y + 2z &= 5 \end{aligned}$$

Exercise* 4.3.106*:

$$\begin{aligned} x + y - z &= 3 \\ 2x - y - z &= 1 \\ 3x + y + 2z &= 0 \end{aligned}$$

Exercise* 4.3.107*:

$$\begin{aligned} x + y + 2z &= 0 \\ 2x - y + z &= 1 \\ 4x + y + 5z &= 1 \end{aligned}$$

4.4 Solving systems of linear equations using Maxima

Previously in this chapter, we discovered the nature of the solution to the system of linear equations represented by $A\vec{x} = \vec{b}$ by obtaining the reduced echelon form of the augmented matrix $[A|b]$ and determining whether there are no, one or many solutions. We now learn how to compute a matrix's reduced echelon form using the computer algebra system Maxima, through three examples.

Example 4.4.1. In this example, we use Maxima to find the unique solution of the system of linear equations

$$\begin{aligned} 2x - y \phantom{{}-z} &= -1 \\ x + 3y - z &= 5 \\ y + 2z &= 5 \end{aligned}$$

that we studied in Exercise 4.3.105. Sadly, Maxima doesn't provide a command to obtain the reduced echelon form, and so we must define this *macro* ourselves. Simply copy and paste this tedious syntax from Maxima program 4.4.1 into an otherwise blank Maxima worksheet and use that as your source each time you need it!

What does this syntax do? First, we define the vector $\vec{b}$ and the matrix A using Maxima's **`matrix`** command. Second, we use the **`addcol`** command to generate an augmented matrix that we call M. Third, we apply the **`ref`** macro to obtain the reduced echelon form of M. Since its leftmost 3-by-3 portion is the identity matrix, we conclude that its rightmost column is the unique solution $\vec{x}$. ■

Exercise* 4.4.1*: *Use both approaches to solve the system of linear equations*

$$\begin{aligned} x + y - z &= 3. \\ 2x - y - z &= 1 \\ 3x + y + 2z &= 0 \end{aligned}$$

from Exercise 4.3.106.

Example 4.4.2. In this example, we use Maxima to characterize the many solutions of the system of linear equations

$$\begin{aligned} x + y + 2z &= 0 \\ 2x - y + z &= 1 \\ 4x + y + 5z &= 1 \end{aligned}$$

that we studied in Exercise 4.3.107. Our syntax appears in Maxima program 4.4.2. Looking at the augmented matrix in reduced echelon form, we recognize the presence of free variables suggesting that this system possesses infinitely many solutions. We know how to express the solution set from there.

■

Maxima program 4.4.1 Solving a system of linear equations, unique solution

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4) ref(a):=block([p,q,k],[p,q]:matrix_size(a),a:echelon(a),k:min(p,q),
        for i thru k do (if a[i,i]=0 then (k:i-1,return())),
        for i:k thru 2 step -1 do
        (for j from i-1 thru 1 step -1 do a:rowop(a,j,i,a[j,i])),
        a)$

(%i5) b : matrix( [-1],[5],[5] );
```

(%o5) $\begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix}$

```
(%i6) A : matrix( [2,-1,0],[1,3,-1],[0,1,2] );
```

(%o6) $\begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

```
(%i7) M : addcol(A,b);
```

(%o7) $\begin{pmatrix} 2 & -1 & 0 & -1 \\ 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 5 \end{pmatrix}$

```
(%i8) ref(M);
```

(%o8) $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{3}{2} \end{pmatrix}$

Maxima program 4.4.2 Solving a system of linear equations, infinitely many solutions

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4) rref(a):=block([p,q,k],[p,q]:matrix_size(a),a:echelon(a),k:min(p,q),
        for i thru k do (if a[i,i]=0 then (k:i-1,return())),
        for i:k thru 2 step -1 do
        (for j from i-1 thru 1 step -1 do a:rowop(a,j,i,a[j,i])),
        a)$

(%i5) b : matrix( [4],[2],[7] );
```

(%o5) $\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$

```
(%i6) A : matrix( [1,0,1,1],[2,1,0,-1],[3,1,1,0] );
```

(%o6) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{pmatrix}$

```
(%i7) M : addcol(A,b);
```

(%o7) $\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 7 \end{pmatrix}$

```
(%i8) rref(M);
```

(%o8) $\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise* 4.4.2*: *Attempt both approaches to solve the system of linear equations*

$$\begin{aligned} x \quad\ \ - z \quad\ \ &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

from Exercise 4.3.8.

Example 4.4.3. In this example, we use Maxima to recognize that the system of linear equations

$$\begin{aligned} x \quad\ + z + w &= 4 \\ 2x + y \quad\ - w &= 2 \\ 3x + y + z \quad\ &= 7 \end{aligned}$$

that we studied in Exercise 4.2.107 possesses no solution. Examine Maxima program 4.4.3. Looking at the augmented matrix in reduced echelon form, we notice the contradictory row meaning that this system is inconsistent. Hence, there is no solution, and we're done!

■

Exercise* 4.4.3*: *Attempt both approaches to solve the system of linear equations*

$$\begin{aligned} x + y - 2z &= 0 \\ x - y \quad\ &= -3 \\ 3x - y - 2z &= 0 \end{aligned}$$

from Exercise 4.2.117.

Maxima program 4.4.3 Solving a system of linear equations, no solution

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4) ref(a):=block([p,q,k],[p,q]:matrix_size(a),a:echelon(a),k:min(p,q),
          for i thru k do (if a[i,i]=0 then (k:i-1,return())),
          for i:k thru 2 step -1 do
          (for j from i-1 thru 1 step -1 do a:rowop(a,j,i,a[j,i])),
          a)$

(%i5) b : matrix( [4],[2],[7] );
```

(%o5) $\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$

```
(%i6) A : matrix( [1,0,1,1],[2,1,0,-1],[3,1,1,0] );
```

(%o6) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{pmatrix}$

```
(%i7) invA : invert(A);
```

$determinant : matrixmustbesquare; found2rows, 3columns. - -anerror.Todebugthistry : debugmode(true);$

```
(%i8) M : addcol(A,b);
```

(%o8) $\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 7 \end{pmatrix}$

```
(%i9) ref(M);
```

(%o9) $\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Exercises

***Exercise* 4.4.4:** *Use Maxima to solve the system*

$$\begin{aligned} 2x + y - z &= 1 \\ 4x - y \quad\;\; &= 3 \end{aligned}$$

and compare with your solution to Exercise 4.3.7.

***Exercise* 4.4.5:** *Use Maxima to solve the system*

$$\begin{aligned} x - \;\; y + \;\; z \qquad &= 0 \\ y \qquad + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \qquad - w &= 0 \end{aligned}$$

and compare with your solution to Exercise 4.3.9.

***Exercise* 4.4.6:** *Use Maxima to solve the system*

$$\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - \;\; b + \;\; c + d + e &= 3 \end{aligned}$$

and compare with your solution to Exercise 4.3.10.

***Exercise* 4.4.7:** *Verify your answers to as many exercises from Section 4.3 as you wish to feel comfortable using Maxima and solving systems of linear equations.*

4.5 Solving systems of linear equations using SageMath

Further reading: §1.5,4.4 in [B]

Previously in this chapter, we discovered the nature of the solution to the system of linear equations represented by $A\vec{x} = \vec{b}$ by obtaining the reduced echelon form of the augmented matrix $[A|b]$ and determining whether there are no, one or infinitely many solutions. We now learn how to compute a matrix's reduced echelon form using the computer algebra system SageMath, through three examples.

Example 4.5.1. In this example, we use SageMath to find the unique solution of the system of linear equations

$$\begin{aligned} 2x - \ \ y \qquad &= -1 \\ x + 3y - \ \ z &= \ \ 5 \\ y + 2z &= \ \ 5 \end{aligned}$$

that we studied in Exercise 4.3.105. Copy and paste the syntax from SageMath program 4.5.1 into a blank SageMath execution block and click *Run*!

What does this syntax do? First, we define the matrix A and the vector $\vec{b}$ using SageMath's **`matrix`** command. Second, we use the **`augment`** command to generate an augmented matrix that we call M. Third, we apply the **`rref`** command to obtain the reduced echelon form of M. Since its leftmost 3-by-3 portion is the identity matrix, we conclude that its rightmost column is the unique solution $\vec{x}$. ■

Remark 4.5.1. *Why is the SageMath command that generates the* reduced echelon form *called* `rref` *and not simply* `ref`*? Simply put, because some textbooks call this the* reduced-row echelon form*. Hence, the extra 'r'.*

Exercise* 4.5.1*: *Use both approaches to solve the system of linear equations*

$$\begin{aligned} x + y - \ \ z &= 3. \\ 2x - y - \ \ z &= 1 \\ 3x + y + 2z &= 0 \end{aligned}$$

from Exercise 4.3.106.

Example 4.5.2. In this example, we use SageMath to characterize the many solutions of the system of linear equations

$$\begin{aligned} x + y + 2z &= 0 \\ 2x - y + \ \ z &= 1 \\ 4x + y + 5z &= 1 \end{aligned}$$

that we studied in Exercise 4.3.107. Our syntax appears in SageMath program 4.5.2. Looking at the augmented matrix in reduced echelon form, we recognize the presence of free variables suggesting

SageMath program 4.5.1 Solving a system of linear equations, unique solution

Input

```
print "define a matrix A"
A = matrix( 3,3,[2,-1,0,1,3,-1,0,1,2] );
print A
print "define a vector b"
b = matrix( 3,1,[-1,5,5] );
print b
print "generate the augmented matrix M"
M = A.augment(b)
print M
print "generate its reduced echelon form"
M.rref()
```

Output

```
define a matrix A
[ 2 -1  0]
[ 1  3 -1]
[ 0  1  2]
define a vector b
[-1]
[ 5]
[ 5]
generate the augmented matrix M
[ 2 -1  0 -1]
[ 1  3 -1  5]
[ 0  1  2  5]
generate its reduced echelon form
[  1   0   0 1/2]
[  0   1   0   2]
[  0   0   1 3/2]
```

SageMath program 4.5.2 Solving a system of linear equations, infinitely many solutions

Input

```
print "define a matrix A"
A = matrix( 3,3,[1,1,2,2,-1,1,4,1,5] );
print A
print "define a vector b"
b = matrix( 3,1,[0,1,1] );
print b
print "generate the augmented matrix M"
M = A.augment(b)
print M
print "generate its reduced echelon form"
M.rref()
```

Output

```
define a matrix A
[ 1  1  2]
[ 2 -1  1]
[ 4  1  5]
define a vector b
[0]
[1]
[1]
generate the augmented matrix M
[ 1  1  2  0]
[ 2 -1  1  1]
[ 4  1  5  1]
generate its reduced echelon form
[   1    0    1  1/3]
[   0    1    1 -1/3]
[   0    0    0    0]
```

that this system possesses infinitely many solutions. We know how to express the solution set from there.

■

***Exercise* 4.5.2:** *Attempt both approaches to solve the system of linear equations*

$$\begin{aligned} x \quad\quad - z \quad\quad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

from Exercise 4.3.8.

Example 4.5.3. In this example, we use SageMath to recognize that the system of linear equations

$$\begin{aligned} x \quad + z + w &= 4 \\ 2x + y \quad - w &= 2 \\ 3x + y + z \quad &= 7 \end{aligned}$$

that we studied in Exercise 4.2.107 possesses no solution. Examine SageMath program 4.5.3. Looking at the augmented matrix in reduced echelon form, we notice the contradictory row meaning that this system is inconsistent. Hence, there is no solution, and we're done!

■

***Exercise* 4.5.3:** *Attempt both approaches to solve the system of linear equations*

$$\begin{aligned} x + y - 2z &= 0 \\ x - y \quad &= -3 \\ 3x - y - 2z &= 0 \end{aligned}$$

from Exercise 4.2.117.

SageMath program 4.5.3 Solving a system of linear equations, no solution

Input

```
print "define a matrix A"
A = matrix( 3,4,[1,0,1,1,2,1,0,-1,3,1,1,0] );
print A
print "define a vector b"
b = matrix( 3,1,[4,2,7] );
print b
print "generate the augmented matrix M"
M = A.augment(b)
print M
print "generate its reduced echelon form"
M.rref()
```

Output

```
define a matrix A
[ 1  0  1  1]
[ 2  1  0 -1]
[ 3  1  1  0]
define a vector b
[4]
[2]
[7]
generate the augmented matrix M
[ 1  0  1  1  4]
[ 2  1  0 -1  2]
[ 3  1  1  0  7]
generate its reduced echelon form
[ 1  0  1  1  0]
[ 0  1 -2 -3  0]
[ 0  0  0  0  1]
```

Exercises

Exercise* 4.5.4*: *Use SageMath to solve the system*

$$\begin{aligned} 2x + y - z &= 1 \\ 4x - y \phantom{{}-z} &= 3 \end{aligned}$$

and compare with your solution to Exercise 4.3.7.

Exercise* 4.5.5*: *Use SageMath to solve the system*

$$\begin{aligned} x - y + z \phantom{{}+w} &= 0 \\ y \phantom{{}+3z} + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \phantom{{}+3z} - w &= 0 \end{aligned}$$

and compare with your solution to Exercise 4.3.9.

Exercise* 4.5.6*: *Use SageMath to solve the system*

$$\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - b + c + d + e &= 3 \end{aligned}$$

and compare with your solution to Exercise 4.3.10.

Exercise* 4.5.7*: *Verify your answers to as many exercises from Section 4.3 as you wish to feel comfortable using SageMath and solving systems of linear equations.*

4.6 Matrix operations

Attribution: §Three.IV in [H]
Further reading: §3.2 in [L]

Previously, we introduced matrix notation to simplify the notation associated with solving a system of linear equations. We now explore how matrices operate.

A *matrix* is an $m \times n$ array of numbers (m rows and n columns). For example, we denote a 3×5 matrix as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}.$$

By a *vector* we will usually mean a *column vector*, that is an $m \times 1$ matrix. If we mean a *row vector* we will explicitly say so (a row vector is a $1 \times n$ matrix). We will usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as $\vec{x}$ or $\vec{b}$. By $\vec{0}$ we will mean the vector of all zeros.

4.6.1 Addition and scalar multiplication

It is easy to define some operations on matrices. Note that we will want 1×1 matrices to really act like numbers, so our operations must be compatible with this viewpoint.

First, we can multiply by a *scalar* (a number). This means just multiplying each entry by the same number. For instance,

$$2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}.$$

Scalar multiplication is also called *scaling*. Matrix addition is also easy. We add matrices element by element. For instance,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{pmatrix}.$$

If the sizes do not match, then addition is not defined.

If we denote by 0 the matrix of with all zero entries, by c, d scalars, and by A, B, C matrices, we have the following familiar rules.

$$\begin{aligned} A + 0 = A &= 0 + A, \\ A + B &= B + A, \\ (A + B) + C &= A + (B + C), \\ c(A + B) &= cA + cB, \\ (c + d)A &= cA + dA. \end{aligned}$$

Example 4.6.1. To compute $4\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}+5\begin{pmatrix}-1 & 4\\ -2 & 1\end{pmatrix}$, perform these steps:

$$4\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}+5\begin{pmatrix}-1 & 4\\ -2 & 1\end{pmatrix}=\begin{pmatrix}4 & 8\\ 12 & -4\end{pmatrix}+\begin{pmatrix}-5 & 20\\ -10 & 5\end{pmatrix}=\begin{pmatrix}-1 & 28\\ 2 & 1\end{pmatrix}.$$

■

Let us now work through this computation using technology. With Maxima, the syntax appears in Maxima program 4.6.1. With SageMath, the syntax appears in SageMath program 4.6.1. Using either software, the asterisk serves as the scalar multiplication symbol.

Maxima program 4.6.1 Scaling and adding matrices

```
(%i1)  kill(all)$ reset()$ ratprint:false$ logabs:true$
(%i4)  A : matrix( [1,2],[3,-1] );
```

(%o4) $\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$

```
(%i5)  B : matrix( [-1,4],[-2,1] );
```

(%o5) $\begin{pmatrix}-1 & 4\\ -2 & 1\end{pmatrix}$

```
(%i6)  C : 4*A + 5*B;
```

(%o6) $\begin{pmatrix}-1 & 28\\ 2 & 1\end{pmatrix}$

***Exercise* 4.6.1:** *Compute* $6\begin{pmatrix}2 & -1 & -1\\ 1 & 2 & 3\end{pmatrix}$. *Verify your answer using Maxima or SageMath.*

SageMath program 4.6.1 Scaling and adding matrices

Input

```
print "define the 2x2 matrix A"
A = matrix( 2,2,[1,2,3,-1] )
print A
print "define the 2x2 matrix B"
B = matrix( 2,2,[-1,4,-2,1] )
print B
print "compute a linear combination of these matrices"
C = 4*A + 5*B
print C
```

Output

```
define the 2x2 matrix A
[ 1  2]
[ 3 -1]
define the 2x2 matrix B
[-1  4]
[-2  1]
compute a linear combination of these matrices
[-1 28]
[ 2  1]
```

4.6.2 Matrix multiplication

After representing addition and scalar multiplication of linear maps in the prior subsection, the natural next operation to consider is matrix multiplication.

First we define the so-called *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

And similarly for larger (or smaller) vectors.

Armed with the dot product we define the *product of matrices*: the $(i, j)^{\text{th}}$ entry of AB is the dot product of the i^{th} row of A with the j^{th} column of B. This is formalized in a definition.

Definition 4.6.1. *The* matrix-multiplicative product *of the $m{\times}r$ matrix A and the $r{\times}n$ matrix B is the $m{\times}n$ matrix C, where*

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,r}b_{r,j}$$

so that the i, j-th entry of the product is the dot product of the i-th row of the first matrix with the j-th column of the second:

$$C = AB = \begin{pmatrix} & \vdots & & \\ a_{i,1} & a_{i,2} & \cdots & a_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & b_{1,j} & \\ \cdots & b_{2,j} & \cdots \\ & \vdots & \\ & b_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & c_{i,j} & \cdots \\ & \vdots & \end{pmatrix}.$$

Example 4.6.2.

$$\begin{pmatrix} 2 & 0 \\ 4 & 6 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 0 \cdot 5 & 2 \cdot 3 + 0 \cdot 7 \\ 4 \cdot 1 + 6 \cdot 5 & 4 \cdot 3 + 6 \cdot 7 \\ 8 \cdot 1 + 2 \cdot 5 & 8 \cdot 3 + 2 \cdot 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 34 & 54 \\ 18 & 38 \end{pmatrix}$$

■

Example 4.6.3. Some products are not defined, such as the product of a $2{\times}3$ matrix with a $2{\times}2$, because the number of columns in the first matrix must equal the number of rows in the second. But the product of two $n{\times}n$ matrices is always defined. Here are two $2{\times}2$'s.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 2 & 1 \cdot 0 + 2 \cdot (-2) \\ 3 \cdot (-1) + 4 \cdot 2 & 3 \cdot 0 + 4 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -8 \end{pmatrix}$$

■

Example 4.6.4. Let $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix}$. Compute EF.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 + 1\cdot 5 & 1\cdot 6 + 1\cdot 7 & 1\cdot 8 + 1\cdot 9 & 1\cdot 2 + 1\cdot 3 \\ 0\cdot 4 + 1\cdot 5 & 0\cdot 6 + 1\cdot 7 & 0\cdot 8 + 1\cdot 9 & 0\cdot 2 + 1\cdot 3 \\ 1\cdot 4 + 0\cdot 5 & 1\cdot 6 + 0\cdot 7 & 1\cdot 8 + 0\cdot 9 & 1\cdot 2 + 0\cdot 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & 9 & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

■

How do we implement matrix multiplication using technology? With Maxima, the syntax appears in Maxima program 4.6.2; notice that a period serves as the matrix multiplication symbol. With SageMath, the syntax appears in SageMath program 4.6.2; the asterisk serves as the matrix multiplication symbol.

Maxima program 4.6.2 Multiplying matrices

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$
(%i4) E : matrix( [1,1],[1,0],[0,1] );
```

(%o4) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
(%i5) F : matrix( [4,6,8,2],[5,7,9,3] );
```

(%o5) $\begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix}$

```
(%i6) G : E . F;
```

(%o6) $\begin{pmatrix} 9 & 13 & 17 & 5 \\ 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix}$

Exercise 4.6.2: *Compute* $\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 0 & 0.5 \end{pmatrix}$. *Verify your answer using Maxima or SageMath.*

Example 4.6.5. Matrix multiplication is not commutative: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$ is not the same as $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$. ■

SageMath program 4.6.2 Multiplying matrices

Input

```
print "define a 2x3 matrix E"
E = matrix( 3,2,[1,1,0,1,1,0] )
print E
print "define a 2x4 matrix F"
F = matrix( 2,4,[4,6,8,2,5,7,9,3] )
print F
print "multiply these matrices"
G = E * F
print G
```

Output

```
define a 2x3 matrix E
[1 1]
[0 1]
[1 0]
define a 2x4 matrix F
[4 6 8 2]
[5 7 9 3]
multiply these matrices
[ 9 13 17  5]
[ 5  7  9  3]
[ 4  6  8  2]
```

Example 4.6.6. Commutativity can fail more dramatically: $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 23 & 34 & 0 \\ 31 & 46 & 0 \end{pmatrix}$, while $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ isn't even defined. ■

Remark 4.6.1. *The fact that matrix multiplication is not commutative can seem odd at first, perhaps because most mathematical operations in prior courses are commutative. But matrix multiplication represents function composition and function composition is not commutative: if* $f(x) = 2x$ *and* $g(x) = x + 1$ *then* $g \circ f(x) = 2x + 1$ *while* $f \circ g(x) = 2(x + 1) = 2x + 2$.

Except for the lack of commutativity, matrix multiplication is algebraically well-behaved. The next result gives some nice properties.

Theorem 4.6.1. *If* F, G, *and* H *are matrices, and the matrix products are defined, then the product is associative*

$$(FG)H = F(GH)$$

and distributes over matrix addition

$$F(G + H) = FG + FH$$

and

$$(G + H)F = GF + HF.$$

Definition 4.6.2. *A* square matrix *possesses the same number of rows as columns.*

Definition 4.6.3. *An* identity matrix *is square and every entry is* 0 *except for* 1*'s in the main diagonal:*

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 1. \end{pmatrix}$$

Example 4.6.7. Here is the 2×2 identity matrix leaving its multiplicand unchanged when it acts from the right:

$$\begin{pmatrix} 1 & -2 \\ 0 & -2 \\ 1 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -2 \\ 1 & -1 \\ 4 & 3 \end{pmatrix}.$$

■

Example 4.6.8. Here the 3×3 identity leaves its multiplicand unchanged both from the left

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 6 \\ 1 & 3 & 8 \\ -7 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 3 & 8 \\ -7 & 1 & 0 \end{pmatrix}$$

and from the right

$$\begin{pmatrix} 2 & 3 & 6 \\ 1 & 3 & 8 \\ -7 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 3 & 8 \\ -7 & 1 & 0 \end{pmatrix}.$$

■

In short, an identity matrix is the identity element of the set of $n \times n$ matrices with respect to the operation of matrix multiplication.

We can generalize the identity matrix by relaxing the ones to arbitrary reals. The resulting matrix rescales whole rows or columns.

Definition 4.6.4. *A* diagonal matrix *is square and has* 0*'s off the main diagonal*

$$\begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & a_{n,n}. \end{pmatrix}$$

Definition 4.6.5. *The* transpose *of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.*

4.6.3 Inverse

A few warnings are in order.

(i) $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.

(ii) $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$. For example take $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For the last two items to hold we would need to "divide" by a matrix. This is where the *matrix inverse* comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA.$$

Then we call B the inverse of A and we denote B by A^{-1}. If the inverse of A exists, then we call A *invertible*. If A is not invertible we sometimes say A is *singular*.

If A is invertible, then $AB = AC$ does imply that $B = C$ (in particular the inverse of A is unique). We just multiply both sides by A^{-1} to get $A^{-1}AB = A^{-1}AC$ or $IB = IC$ or $B = C$. It is also not hard to see that $(A^{-1})^{-1} = A$.

Example 4.6.9. To find the inverse of $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, do Gauss-Jordan reduction, meanwhile performing the same operations on the identity. For clerical convenience we write the matrix and the identity side-by-side and do the reduction steps together.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right)$$
$$\xrightarrow{-1/3\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$
$$\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$

This calculation has found the inverse matrix:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

To ensure that we haven't made a calculation error along the way, we check our answer by multiplying the matrix by its inverse:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1/3 + 1 \cdot 2/3 & 1 \cdot 1/3 + 1 \cdot (-1/3) \\ 2 \cdot 1/3 + (-1) \cdot 2/3 & 2 \cdot 1/3 + (-1) \cdot (-1/3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain the identity matrix, as we should. Or, just as effective, we can multiply them in reverse order:

$$\begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1/3 \cdot 1 + 1/3 \cdot 2 & 1/3 \cdot 1 + 1/3 \cdot (-1) \\ 2/3 \cdot 1 + (-1/3) \cdot 2 & 2/3 \cdot 1 + (-1/3) \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

■

Example 4.6.10. Perform Gauss-Jordan reduction to invert $D = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

Form the augmented matrix $[D|I]$. Apply elementary row operations, starting with a row swap, to obtain reduced echelon form.

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right)$$
$$\vdots$$
$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right)$$

Hence, the inverse matrix is

$$A^{-1} = \begin{pmatrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{pmatrix}.$$

We can check our answer by multiplying D by D^{-1} to obtain the identity I:

$$DD^{-1} = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Success! Equivalently, we can check our answer by multiplying D^{-1} by D to obtain the identity I:

$$D^{-1}D = \begin{pmatrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{pmatrix}\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

■

***Exercise* 4.6.3:** *Fill in the missing steps in the example above.*

Let us now invert matrices using technology. Maxima possesses an **`invert`** command; SageMath possesses an **`inverse`** command. Refer to Maxima program 4.6.3 or SageMath program 4.6.3.

Maxima program 4.6.3 Inverting a matrix

```
(%i1) kill(all)$ reset()$ ratprint:false$ logabs:true$
(%i4) D : matrix( [0,3,-1],[1,0,1],[1,-1,0] );
```

(%o4) $\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

```
(%i5) invD : invert(D);
```

(%o5) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} \end{pmatrix}$

Example 4.6.11. This algorithm detects a non-invertible matrix when the left half doesn't reduce to the identity.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

■

SageMath program 4.6.3 Inverting a matrix

Input

```
print "define a 3x3 matrix D"
D = matrix( 3,3,[0,3,-1,1,0,1,1,-1,0] )
print D
print "invert this matrix"
invD = D.inverse()
print invD
```

Output

```
define a 3x3 matrix D
[ 0  3 -1]
[ 1  0  1]
[ 1 -1  0]
invert this matrix
[ 1/4  1/4  3/4]
[ 1/4  1/4 -1/4]
[-1/4  3/4 -3/4]
```

With this procedure we can give a formula for the inverse of a general 2×2 matrix, which is worth memorizing.

Theorem 4.6.2. *The inverse for a 2×2 matrix exists and equals*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

Exercises

You can use Maxima or SageMath to verify your answers to many of these exercises.

***Exercise* 4.6.4:** *Find 3 nonzero 2×2 matrices A, B, and C such that $AB = AC$ but $B \neq C$.*

***Exercise* 4.6.5:** *Perform Gauss-Jordan reduction on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to prove Theorem 4.6.2.*

***Exercise* 4.6.6:** *Use Theorem 4.6.2 to decide which of the following matrices are invertible: $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 4 \\ 1 & -3 \end{pmatrix}$, and $\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$. For those that are, use Theorem 4.6.2 to find their inverses.*

Compute the inverse of matrix A. Verify your answer by computing AA^{-1}, or $A^{-1}A$, or both.

***Exercise* 4.6.7:** $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

***Exercise* 4.6.8:** $A = \begin{pmatrix} 5 & 3 & 6 \\ 4 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix}$.

***Exercise* 4.6.9:** *Solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ by using the matrix inverse.*

***Exercise* 4.6.10:** *For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, compute the product $A^2 = AA$.*

***Exercise* 4.6.11:** *Another useful operation for matrices is the so-called* transpose*. This operation just swaps rows and columns of a matrix. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. The transpose of A is denoted by A^T. Hence, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. Compute AA^T and A^TA.*

Compute the matrix expression.

***Exercise* 4.6.101:** $\begin{pmatrix} 5 & -1 & 2 \\ 6 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$

Exercise* 4.6.102*: $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

Exercise* 4.6.103*: $3\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$

Exercise* 4.6.104*: $\begin{pmatrix} 1 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Exercise* 4.6.105*: $\begin{pmatrix} 2 & -7 \\ 7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ -1 & 1 & 1 \\ 3 & 8 & 4 \end{pmatrix}$

Exercise* 4.6.106*: $\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$

Find the inverse of matrix A, if it exists, by using Gauss-Jordan reduction. Verify your answer by using Theorem 4.6.2.

Exercise* 4.6.107*: $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

Exercise* 4.6.108*: $A = \begin{pmatrix} 2 & 1/2 \\ 3 & 1 \end{pmatrix}$

Exercise* 4.6.109*: $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Find the inverse of matrix A, if it exists, by using Gauss-Jordan reduction. If it exists, verify your answer by computing AA^{-1}, or $A^{-1}A$, or both.

Exercise* 4.6.110*: $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

Exercise* 4.6.111*: $A = \begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$

Exercise* 4.6.112*: $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$

Exercise* 4.6.113*: *Suppose a, b, c are nonzero numbers. Let $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $N = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$. Compute M^{-1} and N^{-1}.*

4.7 Determinants

Attribution: §Four.I in [H]
Further reading: §3.2 in [L]

Previously in this chapter, we noticed the special case of linear systems with the same number of equations as unknowns, those of the form $T\vec{x} = \vec{b}$ where T is a square matrix. We noted that there are only two kinds of T's:

- If T is associated with a unique solution for any $\vec{b}$, then we call such a matrix nonsingular.
- The other kind of T, where every linear system for which it is the matrix of coefficients has either no solution or infinitely many solutions, we call singular.

We now know that an $n\times n$ matrix T is nonsingular if and only if each of these holds:

- any system $T\vec{x} = \vec{b}$ has a solution and that solution is unique;
- Gauss-Jordan reduction of T yields an identity matrix;
- an inverse matrix T^{-1} exists.

So when we look at a square matrix, one of the first things that we ask is whether it is nonsingular. This section develops a formula that determines whether T is nonsingular. More precisely, we will develop a formula for 1×1 matrices, one for 2×2 matrices, etc. These are naturally related; that is, we will develop a family of formulas, a scheme that describes the formula for each size. Since we will restrict the discussion to square matrices, in this section we will often simply say 'matrix' in place of 'square matrix'.

Determining nonsingularity is trivial for 1×1 matrices:

$$\begin{pmatrix} a \end{pmatrix} \quad \text{is nonsingular iff} \quad a \neq 0.$$

Theorem 4.6.2 gives the 2×2 formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is nonsingular iff} \quad ad - bc \neq 0.$$

We can produce the 3×3 formula as we did the prior one, although the computation is intricate:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{is nonsingular iff} \quad aei + bfg + cdh - hfa - idb - gec \neq 0.$$

With these cases in mind, we posit a family of formulas: a, $ad - bc$, etc. For each n the formula defines a *determinant* function such that an $n\times n$ matrix T is nonsingular if and only if $\det(T) \neq 0$.

Before digging any deeper, let us first note the meaning of the determinant. Consider an $n \times n$ matrix as a mapping of the n dimensional euclidean space $\mathbb{R}^n$ to itself, where $\vec{x}$ gets sent to $A\vec{x}$. In particular, a 2×2 matrix A is a mapping of the plane to itself. The determinant of A is the factor by which the area of objects gets changed. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes changing of orientation (negative if the axes get flipped). For example, let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $\det(A) = 1 + 1 = 2$. Let us see where the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ gets sent. Clearly $(0, 0)$ gets sent to $(0, 0)$.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The image of the square is another square with vertices $(0, 0)$, $(1, -1)$, $(1, 1)$, and $(2, 0)$. The image square has a side of length $\sqrt{2}$ and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0, 0)$, (a, c), (b, d) and $(a + b, c + d)$. And it is precisely

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|.$$

The vertical lines above mean absolute value. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ carries the unit square to the given parallelogram.

Now we define the determinant for larger matrices. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. To compute the determinant of a matrix, pick one row, say the i^{th} row and compute:

$$\boxed{\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).}$$

For the first row we get

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \cdots \begin{cases} +a_{1n} \det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n} \det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

We alternately add and subtract the determinants of the submatrices A_{ij} for a fixed i and all j. For a 3×3 matrix, picking the first row, we get $\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$.

The numbers $(-1)^{i+j} \det(A_{ij})$ are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above).

Example 4.7.1. Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute $\det B$.

First Solution Perform the cofactor expansion along the first row:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \\ &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= 1(-3) - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Second Solution Perform the cofactor expansion along the second row:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= -4 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 5 \cdot \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 6 \cdot \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \\ &= -4(2 \cdot 9 - 3 \cdot 8) + 5(1 \cdot 9 - 3 \cdot 7) - 6(1 \cdot 8 - 2 \cdot 7) \\ &= -4(18 - 24) + 5(9 - 21) - 6(8 - 14) \\ &= -4(-6) + 5(-12) - 6(-6) \\ &= 24 - 60 + 36 \\ &= 0. \end{aligned}$$

Third Solution Perform the cofactor expansion along the first column:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 7 \cdot \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 4(2 \cdot 9 - 3 \cdot 8) + 7(2 \cdot 6 - 3 \cdot 5) \\ &= 1(45 - 48) - 4(18 - 24) + 7(12 - 15) \\ &= 1(-3) - 4(-6) + 7(-3) \\ &= -3 + 24 - 21 \\ &= 0. \end{aligned}$$

■

We can verify our answer to the example above using Maxima's **determinant** command. See Maxima program 4.7.1. Or, we can use SageMath's **det** command. See SageMath program 4.7.1. Using either software, the syntax is quite straightforward.

Maxima program 4.7.1 Computing the determinant of a matrix

```
(%i1)  kill(all)$ reset()$ ratprint:false$ logabs:true$

(%i4)  B : matrix( [1,2,3],[4,5,6],[7,8,9] );
```

(%o4) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

```
(%i5)  determinant(B);
```

(%o5) 0

SageMath program 4.7.1 Computing the determinant of a matrix

Input

```
print "define a matrix B"
B = matrix( 3,3,[1,2,3,4,5,6,7,8,9] )
print B
print "compute its determinant"
det(B)
```

Output

```
define a matrix B
[1 2 3]
[4 5 6]
[7 8 9]
compute its determinant
0
```

Exercises

You can use Maxima or SageMath to verify your answers to most of these exercises.

Exercise* 4.7.1*: *Use a cofactor expansion to show that*

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

Evaluate the determinant of the matrix.

Exercise* 4.7.2*: $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

Exercise* 4.7.3*: $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

Exercise* 4.7.4*: $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

Exercise* 4.7.5*: $\begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Exercise* 4.7.6*: $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{pmatrix}$

Exercise* 4.7.7*: $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

Exercise* 4.7.8*: $\begin{pmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{pmatrix}$

Exercise* 4.7.9*: *Compute the determinant of* $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{pmatrix}$ *choosing two different cofactor expansions. Show that they agree. Hint: Choose expansions that are simpler to compute.*

Exercise* 4.7.10*: *For which h is* $\begin{pmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{pmatrix}$ *not invertible? Find all such h.*

Exercise* 4.7.11*: *For which h is* $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{pmatrix}$ *not invertible? Is there only one such h? Are there several? Infinitely many?*

Exercise* 4.7.12*: *In Exercise 4.2.108, you solved a system of linear equations* $A\vec{x} = \vec{b}$ *and found that this system possesses no solution. Based on this fact alone, what can be said about* $\det A$ *?*

Exercise* 4.7.13*: *In Exercise 4.2.113, you solved a system of linear equations* $A\vec{x} = \vec{b}$ *and found that this system possesses infinitely many solutions. Based on this fact alone, what can be said about* $\det A$ *?*

Exercise* 4.7.14*: *In Exercise 4.2.105, you solved a system of linear equations* $A\vec{x} = \vec{b}$ *and found that this system possesses a single solution. Based on this fact alone, what can be said about* $\det A$ *?*

Exercise* 4.7.101*: *Find t such that* $\begin{pmatrix} 1 & t \\ -1 & 2 \end{pmatrix}$ *is not invertible.*

Exercise* 4.7.102*: *Compute the determinant of* $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{pmatrix}$.

4.8 Eigenvalues and eigenvectors

Attribution: §Five.II.3 in [H]
Further reading: §3.4.1 in [L]

Let A be a constant square matrix. Suppose there is a scalar λ and a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

We then call λ an *eigenvalue* of A and $\vec{v}$ is said to be a corresponding *eigenvector*.

Example 4.8.1. The matrix $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ has an eigenvalue of $\lambda = 2$ with a corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

■

Let us see how to compute the eigenvalues for any matrix. We rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

We notice that this equation has a nonzero solution $\vec{v}$ only if $A - \lambda I$ is not invertible. Were it invertible, we could write $(A - \lambda I)^{-1}(A - \lambda I)\vec{v} = (A - \lambda I)^{-1}\vec{0}$, which implies $\vec{v} = \vec{0}$. Therefore, A has the eigenvalue λ if and only if λ solves the equation

$$\det(A - \lambda I) = 0.$$

Consequently, we will be able to find an eigenvalue of A without finding a corresponding eigenvector. An eigenvector will be found later, once λ is known.

***Exercise* 4.8.1** (easy)*: Let A be a 3×3 matrix with an eigenvalue of 3 and a corresponding eigenvector $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. Find $A\vec{v}$.*

Definition 4.8.1. *The* characteristic polynomial of a square matrix A *is the determinant* $\det(A - \lambda I)$ *where λ is a variable. The* characteristic equation *is* $\det(A - \lambda I) = 0$.

Note that for an $n \times n$ matrix, the characteristic polynomial will be of degree n, and hence we will in general have n eigenvalues. Some may be repeated, some may be complex.

Example 4.8.2. To the find the eigenvalues of matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we write

$$\begin{aligned} \det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) &= \det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)((2-\lambda)^2 - 1) \\ &= -(\lambda-1)(\lambda-2)(\lambda-3). \end{aligned}$$

Notice that the cofactor expansion was performed along the third row, since that appeared the be the simplest approach. By factoring the characteristic polynomial, we see that the roots are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. ■

To find an eigenvector corresponding to an eigenvalue λ, we write

$$(A - \lambda I)\vec{v} = \vec{0},$$

and solve for a nonzero vector $\vec{v}$. If λ is an eigenvalue, this will always be possible.

Example 4.8.2 (continuing from p. 175)**.** Now, let us find the eigenvector corresponding to the eigenvalue $\lambda = 3$. We write down the system of linear equations that we must solve:

$$(A - \lambda I)\vec{v} = \left(\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}.$$

We perform row operations to obtain the reduced echelon form. We save ourselves some writing by omitting the vector of zeros from our augmented matrix, as discussed in Remark 4.2.1.

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\rho_1 + \rho_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\rho_2 + \rho_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that v_1 and v_3 are *leading*, and that v_2 is *free*. It's easy to read off the solution to obtain $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (or a scalar multiple thereof). Let us verify that $\vec{v}$ really is an eigenvector corresponding to $\lambda = 3$:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Yay! It worked. ■

***Exercise* 4.8.2:** *Finish the previous example by finding both remaining eigenvectors. Then, verify each one.*

Sometimes, a matrix possesses a repeated eigenvalue.

Example 4.8.3. If

$$T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

then to find the scalars λ such that $T\vec{v} = \lambda\vec{v}$ for nonzero eigenvectors $\vec{v}$, bring everything to the left-hand side

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \lambda\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}$$

and factor $(T - \lambda I)\vec{v} = \vec{0}$. This linear system

$$\begin{pmatrix} 1-\lambda & 2 & 1 \\ 2 & 0-\lambda & -2 \\ -1 & 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution $\vec{z}$ if and only if the matrix is singular. We can determine when that happens:

$$\begin{aligned} 0 &= \det(T - \lambda I) \\ &= \det\begin{pmatrix} 1-\lambda & 2 & 1 \\ 2 & 0-\lambda & -2 \\ -1 & 2 & 3-\lambda \end{pmatrix} \\ &= \lambda^3 - 4\lambda^2 + 4\lambda \\ &= \lambda(\lambda-2)^2. \end{aligned}$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. To find the associated eigenvectors plug in each eigenvalue. Plugging in $\lambda_1 = 0$ gives

$$\begin{pmatrix} 1-0 & 2 & 1 \\ 2 & 0-0 & -2 \\ -1 & 2 & 3-0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} a$$

for $a \neq 0$ (a must be nonzero because eigenvectors are defined to be nonzero). We can choose $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ to be our eigenvector associated with eigenvalue 0, or any nonzero scalar multiple thereof. Plugging in $\lambda_2 = 2$ gives

$$\begin{pmatrix} 1-2 & 2 & 1 \\ 2 & 0-2 & -2 \\ -1 & 2 & 3-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} b$$

with $b \neq 0$. We can choose $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ to be our eigenvector associated with eigenvalue 2, or any nonzero scalar multiple thereof. ∎

In the next example, we apply our procedure for finding eigenvalues and eigenvectors on a smaller, 2-by-2, matrix.

Example 4.8.4. If

$$S = \begin{pmatrix} \pi & 1 \\ 0 & 3 \end{pmatrix}$$

then

$$\det\begin{pmatrix} \pi-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (\lambda-\pi)(\lambda-3)$$

so S has eigenvalues of $\lambda_1 = \pi$ and $\lambda_2 = 3$. To find associated eigenvectors, first plug in λ_1 for λ

$$\begin{pmatrix} \pi-\pi & 1 \\ 0 & 3-\pi \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 0 & 1 \\ 0 & 3-\pi \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} a$$

for a scalar $a \neq 0$. Then plug in λ_2

$$\begin{pmatrix} \pi-3 & 1 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \pi-3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1/(\pi-3) \\ 1 \end{pmatrix} b$$

where $b \neq 0$. ∎

We can use technology solve this previous Example 4.8.4.

In Maxima program 4.8.1 on the following page, the **`eigenvalues`** command returns two pairs of values. The first pair lists the eigenvalues themselves. Indeed, we previously found π and 3. The second pair tells us how many times each eigenvalue appears. Since both eigenvalues appear only once, both entries are 1. The output to **`eigenvectors`** requires a little more effort to discern. It begins by repeating the output from the `eigenvalues` command. Then, it lists an eigenvector associated with eigenvalue π: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. That's the vector we previously found. Then, it lists an eigenvector associated with eigenvalue 3: $\begin{pmatrix} 1 \\ 3-\pi \end{pmatrix}$. We can do a little manipulation to show that it's a constant multiple of the vector we previously found.

Maxima program 4.8.1 Computing the eigenvalues and eigenvectors of a matrix

```
(%i1) kill(all)$ reset()$

(%i2) S : matrix( [%pi,1],[0,3] );
```

(%o2) $\begin{pmatrix} \pi & 1 \\ 0 & 3 \end{pmatrix}$

```
(%i3) eigenvalues(S);
```

(%o3) $[[\pi, 3], [1, 1]]$

```
(%i4) eigenvectors(S);
```

(%o4) $[[[\pi, 3], [1, 1]], [[[1, 0]], [[1, 3 - \pi]]]]$

In SageMath program 4.8.1 on the next page, the **eigenvalues** command simply displays the two eigenvalues that we previously found: π and 3. That was easy! The output to **eigenvectors_right** requires a little more effort to discern. Notice that it's split into two portions. The first portion declares the eigenvalue that is *pi*, followed by its associated eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. That's the vector we previously found. The first portion declares the other eigenvalue that is 3, followed by its associated eigenvector $\begin{pmatrix} 1 \\ -\pi + 3 \end{pmatrix}$. We can do a little manipulation to show that it's a constant multiple of the vector we previously found.

Sometimes, technology computes eigenvalues and eigenvectors in a less helpful way. Consider SageMath program 4.8.2 on page 180 that computes the eigenvalues and eigenvectors of matrix $T = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$. The answers that come out are all in decimal notation, when your pencil-and-paper solution probably involves fractions. (The question mark indicates that SageMath is unsure about the accuracy of the last displayed decimal.) C'est la vie!

A matrix might very well have complex eigenvalues even if all the entries are real.

Example 4.8.5. Suppose that we have the matrix

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Let us compute the eigenvalues of P.

$$\det(P - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

SageMath program 4.8.1 Computing the eigenvalues and eigenvectors of a matrix

Input

```
print "define a matrix"
S = matrix( 2,2,[pi,1,0,3] )
print S
print "compute the eigenvalues"
S.eigenvalues()
print "compute the eigenvalues and eigenvectors"
S.eigenvectors_right()
```

Output

```
define a matrix
[pi  1]
[ 0  3]
compute the eigenvalues
[pi, 3]
compute the eigenvalues and eigenvectors
[(pi, [(1, 0)], 1), (3, [(1, -pi + 3)], 1)]
```

Thus $\lambda = 1 \pm i$. The corresponding eigenvectors are also complex. First take $\lambda = 1 - i$,

$$(P - (1 - i)I)\vec{v} = \vec{0},$$
$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\vec{v} = \vec{0}.$$

The equations $iv_1 + v_2 = 0$ and $-v_1 + iv_2 = 0$ are multiples of each other. So we only need to consider one of them. After picking $v_2 = 1$, for example, we have an eigenvector $\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$. In similar fashion we find that $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $1 + i$. ■

As it turns out, in the last example, we don't need to do much work to find the second eigenvalue and eigenvector. Here's why.

Theorem 4.8.1. *Let A be a square matrix with real entries. If $\lambda = \alpha + i\beta$ is an eigenvalue of A and $\vec{v}$ is its associated eigenvector, then its complex conjugate $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of A and $\overline{\vec{v}}$ is its associated eigenvector.*

Put otherwise, complex eigenvalues of a real matrix come in complex-conjugate pairs. If $\vec{v}$ is an eigenvector corresponding to the eigenvalue $\lambda = a + ib$, then $\overline{\vec{v}}$ is an eigenvector corresponding to

SageMath program 4.8.2 Computing the eigenvalues and eigenvectors of another matrix

Input

```
print "define a matrix"
T = matrix( 2,2,[4,1,1,3] )
print T
print "compute the eigenvalues"
T.eigenvalues()
print "compute the eigenvalues and eigenvectors"
T.eigenvectors_right()
```

Output

```
define a matrix
[4 1]
[1 3]
compute the eigenvalues
[2.381966011250106?, 4.618033988749895?]
compute the eigenvalues and eigenvectors
[(2.381966011250106?, [(1, -1.618033988749895?)], 1), ...
 (4.618033988749895?, [(1, 0.618033988749895?)], 1)]
```

the eigenvalue $\overline{\lambda} = a - ib$.

To conclude this section, let us use technology to solve Exercise 4.8.5, so that we can see how complex numbers appear. Indeed, the notation is consisdent with what we saw in Section 2.1: Maxima denotes the imaginary number with the percent symbol followed by the lowercase letter i: `%i`, and SageMath denotes the imaginary number with the uppercase letter `I`. See Maxima program 4.8.2 or SageMath program 4.8.3.

Maxima program 4.8.2 Computing eigenvalues and eigenvectors, with complex numbers

```
(%i1) kill(all)$ reset()$
(%i2) P : matrix( [1,1],[-1,1] );
```

(%o2) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

```
(%i3) eigenvalues(P);
```

(%o3) $[[1 - i, i + 1], [1, 1]]$

```
(%i4) eigenvectors(P);
```

(%o4) $[[[1 - i, i + 1], [1, 1]], [[[1, -i]], [[1, i]]]]$

SageMath program 4.8.3 Computing eigenvalues and eigenvectors, with complex numbers

Input

```
print "define a matrix"
P = matrix( 2,2,[1,1,-1,1] )
print P
print "compute the eigenvalues"
values = P.eigenvalues()
print values
print "compute the eigenvalues and eigenvectors"
P.eigenvectors_right()
```

Output

```
define a matrix
[ 1  1]
[-1  1]
compute the eigenvalues
[1 - 1*I, 1 + 1*I]
compute the eigenvalues and eigenvectors
[(1 - 1*I, [(1, -1*I)], 1), (1 + 1*I, [(1, 1*I)], 1)]
```

Exercises

You can use Maxima or SageMath to verify your answers to most of these exercises.

Exercise* 4.8.3*: *Let a, b, c, d, e, f be real numbers. Find the eigenvalues of* $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

Find the characteristic equation, the eigenvalues, and associated eigenvectors of the matrix.

Exercise* 4.8.4*: $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Exercise* 4.8.5*: $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

Exercise* 4.8.6*: $\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

Exercise* 4.8.7*: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise* 4.8.8*: $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Exercise* 4.8.9*: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

Exercise* 4.8.10*: *Compute the eigenvalues and eigenvectors of* $A = \begin{pmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{pmatrix}$.

Exercise* 4.8.11*: *Consider the matrix* $A = \begin{pmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{pmatrix}$. *Show that the characteristic equation is* $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$ *and that the eigenvalues are* $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. *Find the eigenvectors.*

Exercise* 4.8.101*: *Compute eigenvalues and eigenvectors of* $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Exercise* 4.8.102*: *Compute the eigenvalues and eigenvectors of* $A = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$.

Chapter 5

Higher order ODEs

5.1 Second order linear ODEs

Attribution: §2.1 in [L]
Further reading: first part of §3.1 in [EP], parts of §3.1 and §3.2 in [BD]

Let us consider the general *second order linear differential equation* (L2ODE)

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x), \tag{5.1}$$

where $p(x) = \dfrac{B(x)}{A(x)}$, $q(x) = \dfrac{C(x)}{A(x)}$, and $f(x) = \dfrac{F(x)}{A(x)}$. The word *linear* means that the equation contains no powers nor functions of y, y', and y''.

In the special case when $f(x) = 0$, we have a *homogeneous* equation

$$y'' + p(x)y' + q(x)y = 0. \tag{5.2}$$

In the more general case when $f(x) \neq 0$, we have a *nonhomogeneous linear equation*.

In Section 1.1, we have already seen some homogeneous L2ODEs that are fundamental:

$$y'' + k^2 y = 0,$$
$$y'' - k^2 y = 0.$$

If we know two solutions of a linear homogeneous equation, we know a lot more of them. This section formalizes this, beginning with a theorem.

Theorem 5.1.1 (Superposition)**.** *Suppose y_1 and y_2 are two solutions of the homogeneous equation* (5.2)*. Then*

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

also solves (5.2) *for arbitrary constants C_1 and C_2.*

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2.

Proof. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work. Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1 (y_1'' + p y_1' + q y_1) + C_2 (y_2'' + p y_2' + q y_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \end{aligned}$$

□

Remark 5.1.1. *Two different solutions to the second equation $y'' - k^2 y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Reminding ourselves that $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, these are solutions by superposition as they are linear combinations of the two exponential solutions.*

Exercise* 5.1.1: Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $y'' + py' + qy = f$ and y_2 is a solution to $y'' + py' + qy = g$. Show that $y = y_1 + y_2$ solves $y'' + py' + qy = f + g$.*

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem 5.1.2 (Existence and uniqueness)**.** *Suppose p, q, f are continuous functions on some interval I, a is a number in I, and a, b_0, b_1 are constants. The equation*

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Using cosh and sinh in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Question: Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (5.2). Can every solution be written (using superposition) in the form $y = C_1y_1 + C_2y_2$?

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We will say y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem 5.1.3. *Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation* (5.2)*. Then every other solution is of the form*

$$y = C_1y_1 + C_2y_2.$$

That is, $y = C_1y_1 + C_2y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. Since y_1 and y_2 are linearly independent,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

We will study the solution of nonhomogeneous equations in Section 5.6. We will first focus on finding general solutions to homogeneous equations.

Exercises

Exercise* 5.1.2*: *Equations of the form* $ax^2y'' + bxy' + cy = 0$ *are called* Euler's equations *or* Cauchy-Euler equations. *They are solved by trying* $y = x^r$ *and solving for* r*. Assume that* $x \geq 0$ *for simplicity. Suppose that* $(b - a)^2 - 4ac > 0$*. Find two solutions of* $ax^2y'' + bxy' + cy = 0$ *by assuming a solution of the form* $y = x^r$*. Recall from Exercise 2.6.5 that both solutions are linearly independent. Write down the general solution, basing yourself on Theorem 5.1.3.*

Exercise* 5.1.3*: *Solve the Cauchy-Euler equation* $ax^2y'' + bxy' + cy = 0$ *for the case where* $(b - a)^2 - 4ac = 0$*. Similarly to Exercise 5.1.2, try a solution of the form* $y = x^r$*. Show that you only obtain one solution that way. Then, assume that the second solution has the form* $y = x^t \ln x$ *works. In doing so, show that* $r = t$*. Recall from Exercise 2.6.6 that both solutions are linearly independent. Write down the general solution, basing yourself on Theorem 5.1.3.*

Exercise* 5.1.4*: *Solve the second-order Cauchy-Euler DE* $x^2y'' + 10xy' - 10y = 0$*.*

Exercise* 5.1.5*: *Solve the second-order Cauchy-Euler DE* $x^2y'' - 7xy' + 16y = 0$*.*

Exercise* 5.1.6*: *For the equation* $x^2y'' - xy' = 0$*, find two solutions, show that they are linearly independent and find the general solution.*

Exercise* 5.1.101*: *Find the general solution to* $xy'' + y' = 0$*. Hint: Notice that it is a first order ODE in* y'*.*

5.2 Constant coefficient L2ODEs

Attribution: §2.2 in [L]
Further reading: second part of §3.1 in [EP], §3.1 in [BD], §17.5 in [G]

Suppose we have the problem

$$y'' - 6y' + 8y = 0, \qquad y(0) = -2, \qquad y'(0) = 6.$$

This is a homogeneous L2ODE with constant coefficients. *Constant coefficients* means that the functions in front of y'', y', and y are constants, not depending on x.

To guess a solution, think of a function that you know stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero.

Let us try a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0 \\ r^2e^{rx} - 6re^{rx} + 8e^{rx} &= 0 \\ e^{rx}\left(r^2 - 6r + 8\right) &= 0 \\ e^{rx}(r-2)(r-4) &= 0. \end{aligned}$$

Hence, if $e^{rx} = 0$, $r - 2 = 0$, or $r - 4 = 0$, then e^{rx} is a solution. We ignore the first equation because exponential functions of the form e^{rx} are asymptotic to the x-axis. The remaining two equations are easy to solve: we find that $r = 2$ or $r = 4$. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Exercise* 5.2.1*: *Check that y_1 and y_2 are solutions.*

The functions e^{2x} and e^{4x} are linearly independent as seen in Example 2.6.4 on page 68. Hence, we can write the general solution as

$$y = C_1e^{2x} + C_2e^{4x}.$$

We need to solve for C_1 and C_2. To apply the initial conditions we first find $y' = 2C_1e^{2x} + 4C_2e^{4x}$. We plug in $x = 0$ and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2, \\ 6 &= y'(0) = 2C_1 + 4C_2. \end{aligned}$$

We have a system of two linear equations in two unknowns to solve. Applying the Gauss-Jordan reduction to an augmented matrix as we learned in Section 4.3 will give us the solution.* Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}.$$

---

* Or, just solve these equations by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$.

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{5.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$\begin{aligned} ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0, \\ ar^2 + br + c &= 0. \end{aligned}$$

The equation $ar^2 + br + c = 0$ is called the *characteristic equation* of the ODE. Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore, we have e^{r_1x} and e^{r_2x} as solutions.

Now suppose that the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}.$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}.$$

However, the exponential is now complex valued. We would need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$\begin{aligned} y_1 &= e^{(\alpha+i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x), \\ y_2 &= e^{(\alpha-i\beta)x} = e^{\alpha x}\cos(-\beta x) + ie^{\alpha x}\sin(-\beta x). \end{aligned}$$

We can simplify y_2 a little bit by recalling that cosine is an even function and sine is an odd function:

$$\begin{aligned} y_1 &= e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x), \\ y_2 &= e^{\alpha x}\cos(\beta x) - ie^{\alpha x}\sin(\beta x). \end{aligned}$$

At this stage, both y_1 and y_2 are still complex-valued. Fortunately, we can take linear combinations of these solutions to obtain different solutions that are real-valued:

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{\alpha x}\cos(\beta x) \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{\alpha x}\sin(\beta x). \end{aligned}$$

In Exercise 2.6.7, we showed that y_3 and y_4 are linearly independent. Further, we notice that y_3 is the real part of y_1 and y_4 is the imaginary part of y_1. Our two solutions are therefore

$$y_3 = \operatorname{Re} y_1 = \operatorname{Re} e^{(\alpha+i\beta)x} = e^{\alpha x}\cos(\beta x)$$
$$y_4 = \operatorname{Im} y_1 = \operatorname{Im} e^{(\alpha+i\beta)x} = e^{\alpha x}\sin(\beta x).$$

The following theorem summarizes this.

Theorem 5.2.1. *Consider the differential equation*

$$ay'' + by' + cy = 0.$$

Suppose that r_1 and r_2 are the roots of its characteristic equation.

(i) If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then the general solution is

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

(ii) If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then the general solution is

$$y = (C_1 + C_2 x)\, e^{r_1 x}.$$

(iii) If the roots are a complex conjugate pair $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x).$$

Proof. Let us give a short proof for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case. □

Exercise 5.2.2: *Verify that the functions stated in the above theorem are solutions to $ay'' + by' + cy = 0$.*

Remark 5.2.1. *We should note that in practice, doubled root rarely happens. If coefficients are picked truly randomly we are very unlikely to get a doubled root.*

Remember this summary of Theorem 5.2.1:

When you get this root	Write down a solution in this form
real root r	e^{rx}
complex roots $r = \alpha \pm \beta i$	$e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$
repeated roots	repeated solutions, each boosted by x

Example 5.2.1. Find the general solution of

$$y'' - 8y' + 16y = 0.$$

The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = e^{4x}(C_1 + C_2 x) = C_1 e^{4x} + C_2 x e^{4x}.$$

In Exercise 2.6.2, you showed that e^{4x} and xe^{4x} are linearly independent. That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0.$$

■

Example 5.2.2. Solve $y'' - 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 10$. The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r-3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By the theorem we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x).$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1.$$

Hence $C_1 = 0$ and $y = C_2 e^{3x} \sin(2x)$. We differentiate

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x).$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. Hence our solution is

$$y = 5e^{3x} \sin(2x).$$

■

Example 5.2.3. Use the method of this section to verify the general solution to the fundamental differential equation $y'' - k^2 y = 0$, $k > 0$. The characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. By the theorem, we therefore confirm that e^{-kx} and e^{kx} are the two linearly independent solutions. ■

Example 5.2.4. Use the method of this section to verify the general solution to the fundamental differential equation $y'' + k^2 y = 0$, $k > 0$. The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$ and by the theorem we have $\cos(kx)$ and $\sin(kx)$ as the two linearly independent solutions. ■

Exercises

Exercise* 5.2.3*: *Find the general solution of* $2y'' + 2y' - 4y = 0$.

Exercise* 5.2.4*: *Find the general solution of* $y'' + 9y' - 10y = 0$.

Exercise* 5.2.5*: *Solve* $y'' - 8y' + 16y = 0$ *for* $y(0) = 2$*,* $y'(0) = 0$.

Exercise* 5.2.6*: *Solve* $y'' + 9y' = 0$ *for* $y(0) = 1$*,* $y'(0) = 1$.

Exercise* 5.2.7*: *Find the general solution of* $2y'' + 50y = 0$.

Exercise* 5.2.8*: *Find the general solution of* $y'' + 6y' + 13y = 0$.

Exercise* 5.2.9*: *Find the general solution of* $y'' = 0$ *using the methods of this section.*

Exercise* 5.2.10*: *The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation* $2y' + 3y = 0$ *using the method of this section.*

Exercise* 5.2.11*: *Let us revisit the Cauchy-Euler equations of Exercise 5.1.2 on page 187. Suppose now that* $(b - a)^2 - 4ac < 0$*. Find a formula for the general solution of* $ax^2y'' + bxy' + cy = 0$*. Hint: Note that* $x^r = e^{r \ln x}$.

Exercise* 5.2.101*: *Find the general solution to* $y'' + 4y' + 2y = 0$.

Exercise* 5.2.102*: *Find the general solution to* $y'' - 6y' + 9y = 0$.

Exercise* 5.2.103*: *Find the solution to* $2y'' + y' + y = 0$*,* $y(0) = 1$*,* $y'(0) = -2$.

Exercise* 5.2.104*: *Find the solution to* $2y'' + y' - 3y = 0$*,* $y(0) = a$*,* $y'(0) = b$.

Exercise* 5.2.105*: *Find the solution to* $z''(t) = -2z'(t) - 2z(t)$*,* $z(0) = 2$*,* $z'(0) = -2$.

Exercise* 5.2.106*: *Solve the initial value problem* $y'' + 6y' + 5y = 0$*,* $y(0) = 1$*,* $y'(0) = 0$.

Exercise* 5.2.107*: *Solve the initial value problem* $y'' - y' - 12y = 0$*,* $y(0) = 0$*,* $y'(0) = 14$.

Exercise* 5.2.108*: *Solve the initial value problem* $y'' + 12y' + 36y = 0$*,* $y(0) = 5$*,* $y'(0) = -10$.

Exercise* 5.2.109*: *Solve the initial value problem* $y'' + 12y' + 37y = 0$*,* $y(0) = 4$*,* $y'(0) = 0$.

Exercise* 5.2.110*: *Solve the initial value problem* $y'' + 6y' + 18y = 0$*,* $y(0) = 0$*,* $y'(0) = 6$.

5.3 Reduction of order

If you have found a solution to a homogeneous L2ODE

$$y'' + p(x)y' + q(x)y = 0, \tag{5.4}$$

then you can find another one using the *reduction of order* formula. Let us state this formula as a theorem.

Theorem 5.3.1. *Suppose that y_1 is a solution to* (5.4). *Then*

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx \tag{5.5}$$

is another solution.

The proof is left as an exercise:

Exercise* 5.3.1*: *Prove the reduction of order formula* (5.5) *by verifying that it solves* (5.4).

Exercises

Exercise* 5.3.2*: *Come up with the reduction of order formula* (5.5) *yourself. Start by trying* $y_2(x) = y_1(x)v(x)$*. Then plug* y_2 *into the equation, use the fact that* y_1 *is a solution, substitute* $w = v'$*, and you have a first order linear equation in* w*. Solve for* w *and then for* v*. When solving for* w*, make sure to include a constant of integration.*

Exercise* 5.3.3*: *In Exercise 5.2.4, we found that the solutions to the second-order constant-coefficient DE* $y'' + 9y' - 10y = 0$ *are linear combinations of* $y_1 = e^x$ *and* $y_2 = e^{-10x}$ *. Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.4*: *In Exercise 5.2.5, we found that the solutions to the second-order constant-coefficient DE* $y'' - 8y' + 16y = 0$ *are linear combinations of* $y_1 = e^{4x}$ *and* $y_2 = xe^{4x}$ *. Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.5*: *In Exercise 5.1.4, we found that the solutions to the second-order Cauchy-Euler DE* $x^2y'' + 10xy' - 10y = 0$ *are linear combinations of* $y_1 = x$ *and* $y_2 = x^{-10}$ *. Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.6*: *In Exercise 5.1.5, we found that the solutions to the second-order Cauchy-Euler DE* $x^2y'' - 7xy' + 16y = 0$ *are linear combinations of* $y_1 = x^4$ *and* $y_2 = x^4 \ln x$ *. Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.7*: *In Exercise 5.2.7, we found that the solutions to the second-order constant-coefficient DE* $2y'' + 50y = 0$ *are linear combinations of* $y_1 = \sin 5x$ *and* $y_2 = \cos 5x$ *(we can also obtain these solutions by noticing that the DE is fundamental). Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.8*: *If we change a sign from the previous exercise, the DE becomes* $2y'' - 50y = 0$*, whose solutions are linear combinations of* $y_1 = \sinh 5x$ *and* $y_2 = \cosh 5x$ *. Now suppose that you had found* y_1 *. Use the reduction of order formula to verify* y_2 *.*

Exercise* 5.3.9*: *Given that* x *solves* $x^2y'' + 5xy' - 5y = 0$ *, use reduction of order to find another linearly independent solution to this DE. State its general solution.*

Exercise* 5.3.10*: *Repeat any exercise from above by switching the* y_1 *and the* y_2 *. That is, suppose that you had found* y_2 *, and use the reduction of order formula to verify* y_1*.*

Exercise* 5.3.11** (Chebyshev's equation of order 1):*** *In Exercise 1.1.23, we verified that* $y = x$ *solves the first-order Chebyshev equation* $(1 - x^2)y'' - xy' + y = 0$*. Use reduction of order to find a second linearly independent solution. Then, write down the general solution.*

Exercise* 5.3.12** (Hermite's equation of order 2):*** *In Exercise 1.1.22, we verified that* $y = 4x^2 - 2$ *solves the second-order Hermite equation* $y'' - 2xy' + 4y = 0$*. Use reduction of order to find a second linearly independent solution. Then, write down the general solution.*

5.4 Higher order linear ODEs

Attribution: §2.3 in [L]
Further reading: §3.2 and §3.3 in [EP], §4.1 and §4.2 in [BD]

We briefly study higher order equations. Equations appearing in applications tend to be second order. Higher order equations do appear from time to time, but generally the world around us is "second order."

The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. For higher order constant coefficient ODEs, the methods developed are also somewhat harder to apply, but we will not dwell on these complications. It is also possible to use the methods for systems of linear equations from Chapter 6 to solve higher order constant coefficient equations.

Let us start with a general homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{5.6}$$

Theorem 5.4.1 (Superposition). *Suppose $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation* (5.6). *Then*

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

also solves (5.6) *for arbitrary constants $C_1, C_2, \ldots, C_n$.*

In other words, a *linear combination* of solutions to (5.6) is also a solution to (5.6). We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 5.4.2 (Existence and uniqueness). *Suppose p_0 through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order we need to find n linearly independent solutions. It is best seen by example.

Example 5.4.1. Find the general solution to

$$y''' - 3y'' - y' + 3y = 0. \tag{5.7}$$

Try: $y = e^{rx}$. We plug in and get

$$r^3e^{rx} - 3r^2e^{rx} - re^{rx} + 3e^{rx} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0.$$

The trick now is to find the roots. A good strategy is to plug a few easy guesses for r into the polynomial: $r = 0, 1, -1, 2, -2$, etc. Our polynomial happens to have two such roots, $r_1 = -1$ and $r_2 = 1$. There should be 3 roots and the last root is reasonably easy to find. The constant term in a monic polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3.$$

You should check that $r_3 = 3$ really is a root. Hence we know that e^{-x}, e^x and e^{3x} are solutions to (5.7). They are linearly independent as can easily be checked by revisiting Exercise 2.6.11, and there are 3 of them, which happens to be exactly the number we need. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}.$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 + C_3, \\ 2 &= y'(0) = -C_1 + C_2 + 3C_3, \\ 3 &= y''(0) = C_1 + C_2 + 9C_3. \end{aligned}$$

The sensible way to solve a system of equations such as this is to use matrix algebra to obtain $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$. The specific solution to the ODE is

$$y = -1/4\, e^{-x} + e^x + 1/4\, e^{3x}.$$

■

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, \quad xe^{rx}, \quad x^2 e^{rx}, \quad \ldots, \quad x^{k-1} e^{rx}.$$

We take a linear combination of these solutions to find the general solution.

Example 5.4.2. Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0.$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r - 1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2 x + C_3 x^2)\, e^x}_{\text{terms coming from } r = 1} + \underbrace{C_4}_{\text{from } r = 0}.$$

■

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1 x + \cdots + C_{k-1} x^{k-1})\, e^{\alpha x} \cos(\beta x) + (D_0 + D_1 x + \cdots + D_{k-1} x^{k-1})\, e^{\alpha x} \sin(\beta x).$$

where $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}$ are arbitrary constants.

Example 5.4.3. Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0.$$

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0, \\ (r^2 - 2r + 2)^2 &= 0, \\ ((r-1)^2 + 1)^2 &= 0. \end{aligned}$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2 x)\, e^x \cos x + (C_3 + C_4 x)\, e^x \sin x.$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots. ■

Exercises

Maxima's `contrib_ode` and SageMath's `desolve` commands cannot handle differential equations of third or higher order. However, if you're a Maxima user, you can use the `ode_check` command to verify your answers to all exercises in this section.

Exercise* 5.4.1*: *Find the general solution for* $y''' - y'' + y' - y = 0$.

Exercise* 5.4.2*: *Find the general solution for* $y^{(4)} - 5y''' + 6y'' = 0$.

Exercise* 5.4.3*: *Find the general solution for* $y''' + 2y'' + 2y' = 0$.

Exercise* 5.4.4*: *Suppose the characteristic equation for a differential equation is*

$$(r-1)^2(r-2)^2 = 0.$$

Find such a differential equation. Find its general solution.

Exercise* 5.4.5*: *Suppose that a fourth order equation satisfying four specified initial conditions has the solution* $y = 2e^{4x} x \cos x$*. Write down what the general solution must have been. Then, find what the DE must have been. Finally, determine what the initial conditions were.*

Exercise* 5.4.101*: *Find the general solution of* $y^{(5)} - y^{(4)} = 0$

Exercise* 5.4.102*: *Suppose that the characteristic equation of a third order differential equation has roots* $3, \pm 2i$*. What is the characteristic equation? Find the corresponding differential equation. Find the general solution.*

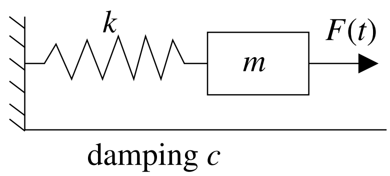
Exercise* 5.4.103*: *Solve* $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

5.5 Mechanical vibrations

Attribution: §2.4 in [L]
Further reading: §3.4 in [EP], §3.7 in [BD]

Let us look at an application of a linear second order constant coefficient equation.

Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps there is a damper connected).



Let x be the displacement of the mass ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

$$mx'' + cx' + kx = F(t).$$

This is a linear second order constant coefficient ODE. We set up some terminology about this equation. We say the motion is

(i) *forced*, if $F \not\equiv 0$ (if F is not identically zero),

(ii) *unforced* or *free*, if $F \equiv 0$ (if F is identically zero),

(iii) *damped*, if $c > 0$, and

(iv) *undamped*, if $c = 0$.

Many real world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right?

In this section we will only consider free or unforced motion, as we cannot yet solve nonhomogeneous equations.

5.5.1 Free undamped motion

Let us start with undamped motion where $c = 0$. We have the equation

$$mx'' + kx = 0.$$

If we divide by m and let $\omega_0 = \sqrt{k/m}$, then we can write the equation as

$$x'' + \omega_0^2 x = 0.$$

The general solution to this equation is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

By a trigonometric identity, we have that for two different constants C and γ, we have

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \gamma).$$

It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = \dfrac{B}{A}$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C\cos(\omega_0 t - \gamma)$.

Exercise* 5.5.1: Justify the above identity and verify the equations for C and γ. Hint: Start with $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and multiply by C. Then think what should α and β be.*

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have very nice interpretation. We look at the form of the solution

$$x(t) = C\cos(\omega_0 t - \gamma).$$

We can see that the *amplitude* is C, ω_0 is the (angular) frequency, and γ is the so-called *phase shift*. The phase shift just shifts the graph left or right. We call ω_0 the *natural (angular) frequency*. This entire setup is usually called *simple harmonic motion*.

Let us pause to explain the word *angular* before the word *frequency*. The units of ω_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because one cycle is 2π radians, the usual frequency is given by $\dfrac{\omega_0}{2\pi}$. It is simply a matter of where we put the constant 2π, and that is a matter of taste.

The *period* of the motion is one over the frequency (in cycles per unit time) and hence $\dfrac{2\pi}{\omega_0}$. That is the amount of time it takes to complete one full cycle.

Example 5.5.1. Suppose that $m = 2$ kilograms and $k = 8$ newtons per meter. The whole mass and spring setup is sitting on a truck that was traveling at 1 meter per second. The truck crashes and hence stops. The mass was held in place 0.5 meter forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at 1 meter per second, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation and what is the amplitude?

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at 1 meter per second. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \qquad x(0) = 0.5, \qquad x'(0) = 1.$$

We can directly compute $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $2/2\pi = 1/\pi \approx 0.318$.

The general solution is

$$x(t) = A\cos(2t) + B\sin(2t).$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5)\sin(2t) + 2B\cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5\cos(2t) + 0.5\sin(2t).$$

A plot of $x(t)$ is shown in Figure 5.1. ■

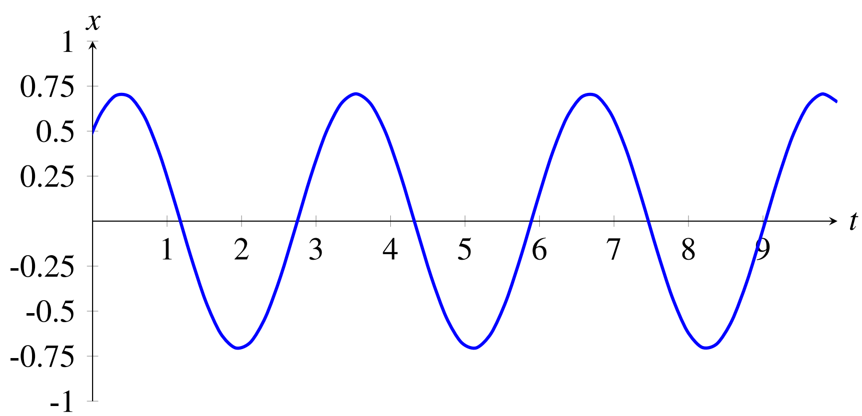


Figure 5.1: Simple undamped oscillation.

In general, for free undamped motion, a solution of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = \omega_0 B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B. In the example, we have already found the amplitude C. Let us compute the phase shift.

We know that $\tan\gamma = {}^{B}\!/_{A} = 1$. We take the arctangent of 1 and get approximately 0.785. We still need to check if this γ is in the correct quadrant (and add π to γ if it is not). Since both A and B are positive, then γ should be in the first quadrant, and 0.785 radians really is in the first quadrant.

Remark 5.5.1. *Many calculators and computer software have, in addition to an* `atan` *function for arctangent, what is usually called* `atan2`*. This function takes two arguments, B and A, and returns a γ in the correct quadrant for you. This holds true for Maxima and SageMath.*

5.5.2 Free damped motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + cx' + kx = 0,$$

as

$$x'' + 2px' + \omega_0^2 x = 0,$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad p = \frac{c}{2m}.$$

The characteristic equation is

$$r^2 + 2pr + \omega_0^2 = 0.$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - \omega_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

The sign of $p^2 - \omega_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

Overdamping

When $c^2 - 4km > 0$, we say the system is *overdamped*. In this case, there are two distinct real roots r_1 and r_2. Both roots are negative: As $\sqrt{p^2 - \omega_0^2}$ is always less than p, then $-p \pm \sqrt{p^2 - \omega_0^2}$ is negative in either case.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

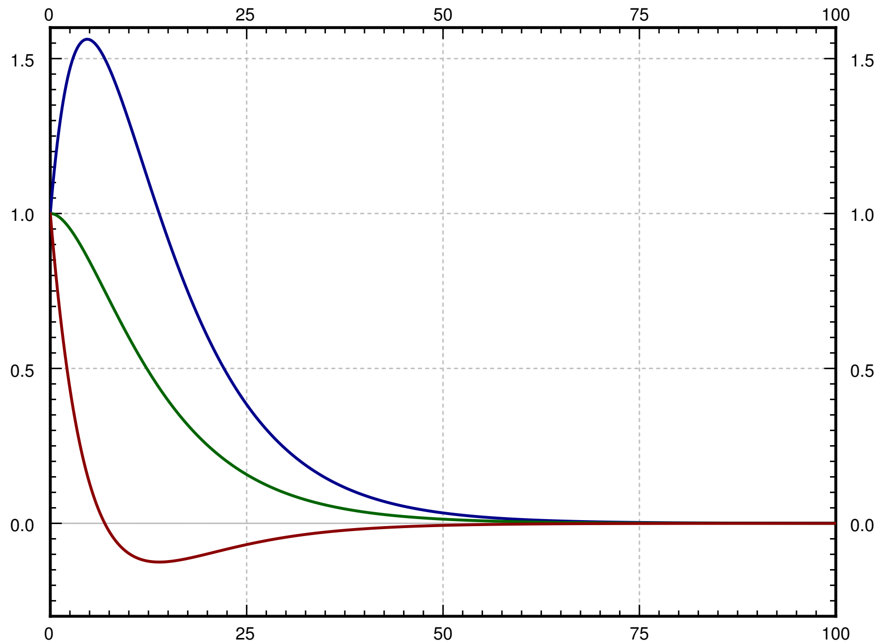


Figure 5.2: Overdamped motion for several different initial conditions.

Since r_1, r_2 are negative, $x(t) \to 0$ as $t \to \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions, see Figure 5.2.

Do note that no oscillation happens. In fact, the graph will cross the x axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$-\frac{C_1}{C_2} = e^{(r_2 - r_1)t}.$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph will never cross the x axis, as is evident from the sample graphs.

Example 5.5.2. Suppose the mass is released from rest. That is $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2}\left(r_1 e^{r_2 t} - r_2 e^{r_1 t}\right).$$

It is not hard to see that this satisfies the initial conditions. ■

Critical damping

When $c^2 - 4km = 0$, we say the system is *critically damped*. In this case, there is one root of multiplicity 2 and this root is $-p$. Our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}.$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

Underdamping

When $c^2 - 4km < 0$, we say the system is *underdamped*. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - \omega_0^2} \\ &= -p \pm \sqrt{-1}\sqrt{\omega_0^2 - p^2} \\ &= -p \pm i\omega_1, \end{aligned}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt}(A\cos(\omega_1 t) + B\sin(\omega_1 t)),$$

or

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \gamma).$$

An example plot is given in Figure 5.3 on the next page. Note that we still have that $x(t) \to 0$ as $t \to \infty$.

In the figure we also show the *envelope curves* Ce^{-pt} and $-Ce^{-pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example if you are bungee jumping, you are really interested in computing the envelope curve so that you do not hit the concrete with your head.

The phase shift γ just shifts the graph left or right but within the envelope curves (the envelope curves do not change if γ changes).

Finally note that the angular *pseudo-frequency* (we do not call it a frequency since the solution is not really a periodic function) ω_1 becomes smaller when the damping c (and hence p) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we want ω_1 to approach 0.

On the other hand when c becomes smaller, ω_1 approaches ω_0 (ω_1 is always smaller than ω_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence p) goes to 0.

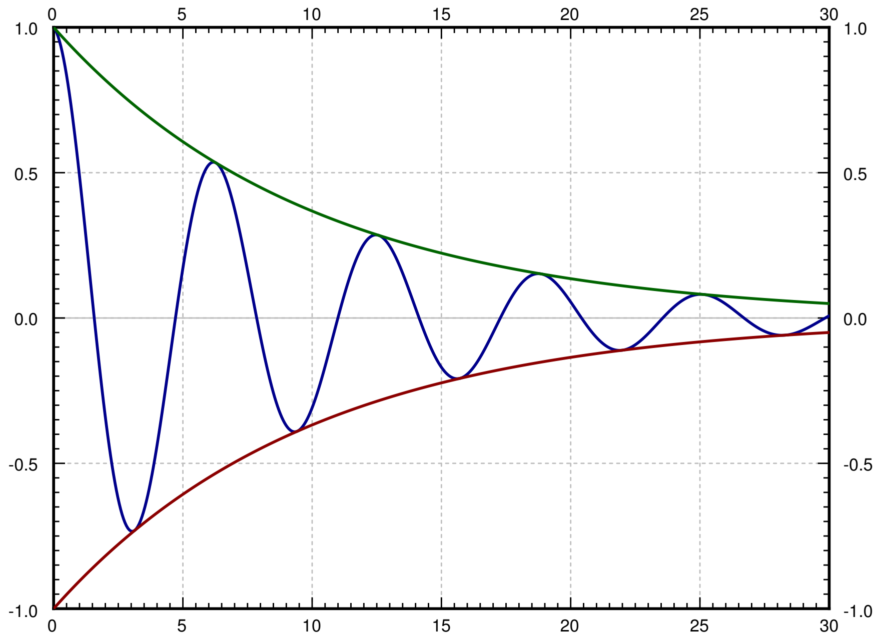


Figure 5.3: Underdamped motion with the envelope curves shown.

Exercises

***Exercise* 5.5.2:** *Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$. a) Set up and find the general solution of the system. b) Is the system underdamped, overdamped or critically damped? c) If the system is not critically damped, find a c that makes the system critically damped.*

***Exercise* 5.5.3:** *Do Exercise 5.5.2 for $m = 3$, $k = 12$, and $c = 12$.*

***Exercise* 5.5.4:** *Suppose you have a spring with spring constant 4 newtons per meter. You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion. a) You count and find that the frequency is 0.8 hertz (cycles per second). What is the mass? b) Find a formula for the mass m given the frequency ω in hertz.*

***Exercise* 5.5.5:** *Suppose we add possible friction to Exercise 5.5.4. Further, suppose you do not know the spring constant, but you have two reference weights 1 kilogram and 2 kilograms to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kilogram weight you measured 1.1 hertz, for the 2 kilogram weight you measured 0.8 hertz. a) Find k (spring constant) and c (damping constant). b) Find a formula for the mass in terms of the frequency (in hertz).* Note that there may be more than one possible mass for a given frequency. *c) For an unknown object you measured 0.2 hertz, what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.*

Exercise* 5.5.6*: *Suppose you wish to measure the friction a mass of 0.1 kilogram experiences as it slides along a floor (you wish to find c). You have a spring with spring constant $k = 5$ newtons per meter. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency 1 hertz. What is the friction?*

Exercise* 5.5.101*: *A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time $t = 0$ the mass is kicked and starts traveling at 2 meters per second. How large must k be to so that the mass does not go further than 3 meters from the rest position?*

Exercise* 5.5.102*: *A 5000 kilogram railcar hits a bumper (a spring) at 1 meter per second, and the spring compresses by 0.1 meter. Assume no damping. a) Find k. b) Find out how far does the spring compress when a 10000 kilogram railcar hits the spring at the same speed. c) If the spring would break if it compresses further than 0.3 meter, what is the maximum mass of a railcar that can hit it at 1 meter per second? d) What is the maximum mass of a railcar that can hit the spring without breaking at 2 meters per second?*

5.6 Nonhomogeneous L2ODEs: Undetermined coefficients

Attribution: §2.5 in [L], §17.6 in [G]
Further reading: §3.5 in [EP], §3.5 and §3.6 in [BD]

We have solved homogeneous L2ODEs. What about nonhomogeneous equations? That is, suppose we have an equation such as

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \tag{5.8}$$

perhaps accompanied with some initial conditions. We solve (5.8) in the following manner:

- First, we find the general solution y_c to the *associated homogeneous equation*

  $$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

  We call y_c the *complementary solution*.

- Second, we find a *particular solution* y_p to (5.8). The *method of undetermined coefficients* is a framework for finding y_p. In this section, we will work through several examples to understand this framework.

- Third, we write down the general solution to (5.8) as

  $$y = y_c + y_p.$$

- Fourth, we enforce the initial conditions, should there be any.

Example 5.6.1. Apply the 4-step process above to solve the nonhomogenous differential equation

$$y'' + 5y' + 6y = 2x + 1 \tag{5.9}$$

with $y(0) = 0$ and $y'(0) = 1/3$.

First, we solve the associated homogeneous equation

$$y'' + 5y' + 6y = 0. \tag{5.10}$$

This equation is constant coefficient second order, and so solving it is easy to accomplish using Section 5.2 methods. The complementary solution is therefore

$$y_c = C_1e^{-2x} + C_2e^{-3x}.$$

Second, we find a particular solution y_p. The trick to doing this is to somehow, in a smart way, guess one particular solution to (5.9). Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B.$$

We plug in to obtain

$$y_p'' + 5y_p' + 6y_p = (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) = 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B).$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore, $A = 1/3$ and $B = -1/9$. That means

$$y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x - 1}{9}.$$

Third, we write down the general solution to (5.9) that is the sum of the complementary and particular solutions that we found:

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x - 1}{9}.$$

Fourth, we enforce the initial conditions. Find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + 1/3$. Then

$$0 = y(0) = C_1 + C_2 - 1/9, \qquad 1/3 = y'(0) = -2C_1 - 3C_2 + 1/3.$$

We solve to get $C_1 = 1/3$ and $C_2 = -2/9$. Our solution is therefore

$$y = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x - 1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

■

Exercise* 5.6.1*: *Check that y_c solves* (5.10). *Then, check that y_p solves* (5.9). *Finally, check that the final answer y solves* (5.9) *and both initial conditions.*

Remark 5.6.1. *A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will* not *work. You need to first compute $y = y_c + y_p$ and* only then *solve for the constants using the initial conditions.*

Example 5.6.2. Apply our 4-step process to solve the differential equation

$$y'' - y' - 6y = 18t^2 + 5.$$

First, we easily find that the general solution of the associated homogeneous equation

$$y'' - y' - 6y = 0$$

is $Ae^{3t} + Be^{-2t}$. This is our y_c.

Second, we guess that a solution to the nonhomogeneous equation might look like $f(t)$ itself, namely, a quadratic $y = Ct^2 + Dt + E$. Substituting this guess into the differential equation we get

$$y'' - y' - 6y = 2C - (2Ct + D) - 6(Ct^2 + Dt + E) = -6Ct^2 + (-2C - 6D)t + (2C - D - 6E).$$

We want this to equal $18t^2 + 5$, so we need

$$\begin{aligned} -6C &= 18 \\ -2C - 6D &= 0 \\ 2C - D - 6E &= 5. \end{aligned}$$

This is a system of three equations in three unknowns and is not hard to solve: $C = -3$, $D = 1$, $E = -2$. Our y_p is therefore $-3t^2 + t - 2$.

Third, we write down the general solution as being $Ae^{3t} + Be^{-2t} - 3t^2 + t - 2$.

The fourth step doesn't apply as no initial conditions were supplied. ■

Until now, we've looked at nonhomogenous equations whose right hand side were polynomials. If the right hand side contains exponentials we try exponentials for our y_p. Let us now see how that plays out.

Example 5.6.3. Find the general solution to $y'' + 7y' + 10y = e^{3t}$. The characteristic equation is $r^2 + 7r + 10 = (r + 5)(r + 2)$, so the solution to the homogeneous equation is $Ae^{-5t} + Be^{-2t}$. For a particular solution to the inhomogeneous equation we guess Ce^{3t}. Substituting we get

$$9Ce^{3t} + 21Ce^{3t} + 10Ce^{3t} = e^{3t}40C.$$

When $C = 1/40$ this is equal to $f(t) = e^{3t}$, so the solution is $Ae^{-5t} + Be^{-2t} + 1/40\, e^{3t}$. ■

***Exercise* 5.6.2:** *Solve*

$$y'' + 5y' + 6y = e^{3x},$$

by noticing that we previously found that the complementary solution is $y_c = C_1e^{-2x} + C_2e^{-3x}$ *and then guessing* $y_p = Ae^{3x}$ *for the particular solution.*

A right hand side consisting of sines and cosines can be handled similarly.

Example 5.6.4. Solve

$$y'' + 2y' + 2y = \cos(2x).$$

Let us find some y_p. We start by guessing the solution includes some multiple of $\cos(2x)$. We also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x).$$

We plug y_p into the equation and we get

$$-4A\cos(2x) - 4B\sin(2x) - 4A\sin(2x) + 4B\cos(2x) + 2A\cos(2x) + 2B\sin(2x) = \cos(2x).$$

The left hand side must equal to right hand side. We group terms and we get that $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = -1/10$ and $B = 1/5$. So

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}.$$

■

When the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess.

Example 5.6.5. For the differential equation

$$y'' + 5y' + 6y = (1 + 3x^2)\, e^{-x} \cos(\pi x),$$

we would guess

$$y_p = (A + Bx + Cx^2)\, e^{-x} \cos(\pi x) + (D + Ex + Fx^2)\, e^{-x} \sin(\pi x).$$

We will plug in and then hopefully get equations that we can solve for A, B, C, D, E, and F. As you can see this can make for a very long and tedious calculation very quickly. C'est la vie! ■

Exercise 5.6.3: *Complete the missing steps to solve the equation of the previous example.*

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation.

Example 5.6.6. Suppose we have

$$y'' - 9y = e^{3x}.$$

We would love to guess $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way we can choose A to make the left hand side be e^{3x}. The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is first we compute y_c (solution to $y'' + 5y' + 6y = 0$)

$$y_c = C_1 e^{-3x} + C_2 e^{3x}$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y = Axe^{3x}$ and notice there is no duplication anymore. Let us try. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = 1/6$. We can now write the general solution as

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{3x} + 1/6\, xe^{3x}.$$

■

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Axe^{3x}$ would not get us anywhere because xe^{3x} already appears in the complementary solution. In this case we want to guess $y_p = Ax^2 e^{3x}$. Basically, we want to multiply our guess by x until all duplication is gone. *But no more!* Multiplying too many times will not work.

Here is another example.

Example 5.6.7. Find the general solution to $y'' + 16y = -\sin(4t)$. The roots of the characteristic equation are $\pm 4i$, so the solution to the homogeneous equation is $A\cos(4t) + B\sin(4t)$. Since both $\cos(4t)$ and $\sin(4t)$ are solutions to the homogeneous equation, $C\cos(4t) + D\sin(4t)$ is also, so it cannot be a solution to the nonhomogeneous equation. Instead, we guess $Ct\cos(4t) + Dt\sin(4t)$. Then substituting:

$$\begin{aligned}(-16Ct\cos(4t) - 16D\sin(4t) + 8D\cos(4t) - 8C\sin(4t))) + 16(Ct\cos(4t) + Dt\sin(4t))\\ = 8D\cos(4t) - 8C\sin(4t).\end{aligned}$$

Thus $C = 1/8$, $D = 0$, and the solution is $C\cos(4t) + D\sin(4t) + 1/8\, t\cos(4t)$. ■

The method of undetermined coefficients will work for many basic problems that crop up. But it does not work all the time. It only works when the right hand side of the equation $ay'' + by' + cy = f(x)$ has only finitely many linearly independent derivatives, so that we can write a guess that consists of them all.

Example 5.6.8. Let us try to solve the equation

$$y'' + y = \tan x.$$

The homogenous equation

$$y'' + y = 0$$

is fundamental that is best seen by rewriting it as:

$$y'' = -y.$$

We therefore immediately write down its solution as

$$y = A\cos x + B\sin x.$$

This is the complementary solution to the homogenous equation. In choosing a form for the particular solution, we inspire ourselves from the right hand side of the DE as well as its derivatives. A problem arises. The derivative of $\tan x$ is $\sec^2 x$. Its derivative is $2\sec^2 x\tan x$, whose derivative is $4\sec^2 x\tan^2 x + 2\sec^4 x$. The next derivative is $8\sec^2 x\tan^3 x + 16\sec^4 x\tan x$. The more we differentiate, the more terms we pick up. At this stage, our form for y_p is

$$y = C\tan x + D\sec^2 x + E\sec^2 x\tan x + F\sec^2 x\tan^2 x + G\sec^4 x + H\sec^2 x\tan^3 x + I\sec^4 x\tan x + \ldots$$

with no end in sight. Since we cannot handle a y_p consisting of infinitely many terms, we give up! ■

This example calls for a different method. Or, rather, we are motivated to learn a second method for finding a particular solution. That method is called variation of parameters, and is the topic of the next section.

Exercises

Exercise* 5.6.4*: *Find a particular solution of* $y'' - y' - 6y = e^{2x}$ *by undetermined coefficients. What form do you assume for the particular solution?*

Exercise* 5.6.5*: *Find a particular solution of* $y'' - 4y' + 4y = e^{2x}$ *by undetermined coefficients. What form do you assume for the particular solution? Hint: If you picked* $y_p = Ee^{2x}$ *for some constant* E *, show that this choice won't work, and try another form.*

Exercise* 5.6.6*: *Solve the initial value problem* $y'' + 9y = \cos 3x + \sin 3x$ *for* $y(0) = 2$*,* $y'(0) = 1$ *by undetermined coefficients. Hint: The homogeneous equation is fundamental.*

Exercise* 5.6.7*: *Consider the ODE* $y^{(4)} - 2y''' + y'' = e^x$*. Its homogeneous equation's general solution is* $y = A + Bx + Ce^x + Dxe^x$*. Write down the form of a particular solution. Do not solve for the coefficients.*

Exercise* 5.6.8*: *Consider the ODE* $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$*. Its homogeneous equation's general solution is* $y = A + Bx + Ce^x + Dxe^x$*. Write down the form of a particular solution. Do not solve for the coefficients.*

Exercise* 5.6.9*: *Find a particular solution of* $y'' - 2y' + y = e^x$ *by undetermined coefficients.*

Exercise* 5.6.10*: *Explain why we cannot solve* $y'' - 2y' + y = \sin x^2$ *by undetermined coefficients.*

Exercise* 5.6.11*: *Solve* $y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}$ *by undetermined coefficients.*

Exercise* 5.6.12*: *Solve* $y'' - 4y' + 3y = e^{3x}\left(6 + 8x + 12x^2\right)$ *by undetermined coefficients.*

Exercise* 5.6.13*: *Explain why we cannot solve* $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$ *by undetermined coefficients.*

Exercise* 5.6.14*: *For an arbitrary constant* c *find a particular solution to* $y'' - y = e^{cx}$*. Hint: Make sure to handle every possible real* c*.*

Exercise* 5.6.15*: *Consider the ODE* $x'' + 9x = 10t^2e^t + 2t\cos 3t$*. Write down the form of a particular solution. Do not solve for the coefficients.*

Exercise* 5.6.16*: *Consider the ODE* $x' - 2x = 10t^10e^{2t}$*. Write down the form of a particular solution. Do not solve for the coefficients.*

Exercise* 5.6.101*: *Find a particular solution to* $y'' - y' + y = 2\sin 3x$ *by undetermined coefficients.*

Solve the DE or IVP by undetermined coefficients.

Exercise* 5.6.102*: $y'' + 2y = e^x + x^3$*. Hint: The homogeneous equation is fundamental.*

Exercise* 5.6.103*: $y'' + 2y' + y = x^2$, $y(0) = 1$, $y'(0) = 2$

Exercise* 5.6.104*: $y'' + 12y' + 36y = 6e^{-6t}$. *Hint: In Exercise 5.2.108, you found that the general solution to* $y'' + 12y' + 36y = 0$ *is* $y = Ae^{-6t} + Bte^{-6t}$.

Exercise* 5.6.105*: $y'' - 8y' + 16y = -2e^{4x}$. *Hint: In Example 5.2.1, we found that the general solution to* $y'' - 8y' + 16y = 0$ *is* $y = e^{4x}(C_1 + C_2x)$.

Exercise* 5.6.106*: $y'' + 6y' + 5y = 4$. *Hint: In Exercise 5.2.106, you found that the general solution to* $y'' + 6y' + 5y = 0$ *is* $y = Ae^{-5t} + Be^{-t}$.

Exercise* 5.6.107*: $y'' - y' - 12y = t$. *Hint: In Exercise 5.2.107, you found that the general solution to* $y'' - y' - 12y = 0$ *is* $y = Ae^{-3t} + Be^{4t}$.

Exercise* 5.6.108*: $y'' + 12y' + 37y = 10e^{-4t}$, $y(0) = 4$, $y'(0) = 0$. *Hint: In Exercise 5.2.109, you found that the general solution to* $y'' + 12y' + 37y = 0$ *is* $y = e^{-6t}(A\cos t + B\sin t)$.

Exercise* 5.6.109*: $y'' + 6y' + 18y = \cos t - \sin t$, $y(0) = 0$, $y'(0) = 2$. *Hint: In Exercise 5.2.110, you found that the general solution to* $y'' + 6y' + 18y = 0$ *is* $y = e^{-3t}(A\cos(3t) + B\sin(3t))$.

Exercise* 5.6.110*: *For an arbitrary constant c find the general solution to* $y'' - 2y = \sin(x + c)$.

5.7 Nonhomogeneous L2ODEs: Variation of parameters

Attribution: §2.5 in [L], §17.7 in [G]
Further reading: §3.5 in [EP], §3.5-3.6 in [BD]

We present the method of *variation of parameters* that will handle any equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

provided we can solve certain integrals. As in the previous section, we assume that we can find the complementary solution

$$y_c(x) = Ay_1(x) + By_2(x).$$

This method assumes that a particular solution takes the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

where u_1 and u_2 are *functions* and not constants. From this point on, to streamline the notation, we drop the independent variable from out notation and remember that y_p, u_1, u_2, p, q, and f are all functions of x. So, we simply write

$$y'' + py' + qy = f \tag{5.11}$$

and

$$y_p = u_1y_1 + u_2y_2. \tag{5.12}$$

We are trying to satisfy (5.11); that gives us one condition on the functions u_1 and u_2. Applying the product rule, we can differentiate y_p to get

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

We can still impose one more condition at our discretion to simplify computations—we have two unknown functions, so we should be allowed two conditions. We choose to require $u_1'y_1 + u_2'y_2 = 0$, so that our expression for y_p' is simpler and thus so is the second derivative:

$$y_p' = u_1y_1' + u_2y_2', \tag{5.13}$$
$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \tag{5.14}$$

Since y_1 and y_2 solve $y'' + py' + qy = 0$, we know that

$$y_1'' = -py_1' - qy_1, \tag{5.15}$$
$$y_2'' = -py_2' - qy_2. \tag{5.16}$$

So plugging (5.15) and (5.16) into (5.14) gives

$$\begin{aligned}
y_p'' &= u_1'y_1' + u_1\left(-py_1' - qy_1\right) + u_2'y_2' + u_2\left(-py_2' - qy_2\right) \\
&= u_1'y_1' - pu_1y_1' - qu_1y_1 + u_2'y_2' - pu_2y_2' - qu_2y_2 \\
&= u_1'y_1' + u_2'y_2' - p\left(u_1y_1' + u_2y_2'\right) - q\left(u_1y_1 + u_2y_2\right) \\
&= u_1'y_1' + u_2'y_2' - py_p' - qy_p
\end{aligned}$$

Rearranging gives us

$$y_p'' + py_p' + qy_p = u_1'y_1' + u_2'y_2',$$

and then we replace the left side using (5.11) to obtain

$$f = u_1'y_1' + u_2'y_2'.$$

What this shows is that, in order to find u_1 and u_2, we need to solve these two equations for u_1' and u_2' and then antidifferentiate both:

$$\boxed{\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= f. \end{aligned}}$$

These equations are sometimes called the *simultaneous equations*. Now that we're convinced that these simultaneous equations didn't come out of thin air, perhaps it is best to understand how the method of variation of parameters works by looking at some examples.

Example 5.7.1. In Example 5.6.8, we tried to solve the equation

$$y'' + y = \tan x.$$

Previously, we found that the complementary solution was $y_c = Ay_1 + By_2$ with

$$\begin{aligned} y_1 &= \cos x, \\ y_2 &= \sin x. \end{aligned}$$

Then we got stuck applying the method of undetermined coefficients to find a particular solution y_p.

We can now solve for u_1' and u_2' in terms of f, y_1 and y_2. In our case the two equations become

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0, \quad \clubsuit \\ -u_1' \sin x + u_2' \cos x &= \tan x. \quad \spadesuit \end{aligned}$$

The easiest way to solve this system of two equations in two unknown functions u_1' and u_2' is to multiply $\clubsuit$ by $\sin x$ and also multiply $\spadesuit$ by $\cos x$ to get the following system of equations:

$$\begin{aligned} u_1' \cos x \sin x + u_2' \sin^2(x) &= 0, \quad \diamondsuit \\ -u_1' \sin x \cos x + u_2' \cos^2(x) &= \sin x. \quad \heartsuit \end{aligned}$$

On the surface, it looks like we made the equations harder to solve, not easier, but things are about to unfold. Simply add $\diamondsuit$ and $\heartsuit$ together

$$u_2' \sin^2(x) + u_2' \cos^2(x) = \sin x \tag{5.17}$$

and apply a trigonometric identity to simplify

$$u_2' = \sin x. \tag{5.18}$$

Now that we have u_2', we can plug it into either $\clubsuit$ or $\spadesuit$ and fetch u_1'. Let's choose $\clubsuit$:

$$\begin{aligned}
u_1' \cos x + \sin^2 x &= 0 \\
u_1' \cos x &= -\sin^2 x \\
u_1' &= -\frac{\sin^2 x}{\cos x} \\
u_1' &= -\tan x \sin x.
\end{aligned}$$

Now we need to integrate u_1' and u_2' to get u_1 and u_2.

$$\begin{aligned}
u_1 &= \int u_1' \, dx = \int -\tan x \sin x \, dx = \frac{1}{2} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| + \sin x, \\
u_2 &= \int u_2' \, dx = \int \sin x \, dx = -\cos x.
\end{aligned}$$

So our particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{2} \cos x \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| + \cos x \sin x - \cos x \sin x = \frac{1}{2} \cos x \ln \left| \frac{\sin x - 1}{\sin x + 1} \right|.$$

The general solution to $y'' + y = \tan x$ is

$$y = A \cos x + B \sin x + \frac{1}{2} \cos x \ln \left| \frac{\sin x - 1}{\sin x + 1} \right|.$$

■

Example 5.7.2. The nonhomogenous differential equation $y'' - 2y' + y = \dfrac{e^t}{t^2}$ is not amenable to the method of undetermined coefficients, because the set of derivatives of the equations right side is not finite. The homogeneous equation $y'' - 2y' + y = 0$ is second-order constant-coefficient whose solution is easily found to be $Ae^t + Bte^t$. Setting

$$\begin{aligned}
y_1 &= e^t, \\
y_2 &= te^t,
\end{aligned}$$

we write down the simultaneous equations as

$$\begin{aligned} u_1' e^t + u_2' t e^t &= 0 \\ u_1' e^t + u_2' t e^t + u_2' e^t &= \frac{e^t}{t^2}. \end{aligned}$$

Subtracting the first equation from the second gives

$$u_2' e^t = \frac{e^t}{t^2}.$$

Solving for u_2' and antidifferentiating gives

$$\begin{aligned} u_2' &= \frac{1}{t^2} \\ u_2 &= -\frac{1}{t}. \end{aligned}$$

Substituting u_2' into the first simultaneous equation yields

$$u_1' e^t = -u_2' t e^t = -\frac{1}{t^2} t e^t.$$

Solving for u_1' and antidifferentiating gives

$$\begin{aligned} u_1' &= -\frac{1}{t} \\ u_1 &= -\ln t. \end{aligned}$$

The general solution is therefore $y = Ae^t + Bte^t - e^t \ln t - e^t$. ■

Exercises

Exercise* 5.7.1*: *In Exercise 5.6.6, you solved the initial value problem* $y'' + 9y = \cos(3x) + \sin(3x)$ *for* $y(0) = 2$*,* $y'(0) = 1$ *using undetermined coefficients. Now, solve this IVP again using variation of parameters.*

Exercise* 5.7.2*: *In Exercise 5.6.9, you found a particular solution to the differential equation* $y'' - 2y' + y = e^x$ *using undetermined coefficients. Now, find a particular solution using variation of parameters.*

Exercise* 5.7.3*: *In Exercise 5.6.10, you decided that* $y'' - 2y' + y = \sin(x^2)$ *cannot be solved by undetermined coefficients. Now, find a particular solution using variation of parameters. It is OK to leave the answer as a definite integral.*

Exercise* 5.7.4*: *In Exercise 5.6.13, you decided that* $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$ *cannot be solved by undetermined coefficients. Now, find a particular solution using variation of parameters. Then, write down the general solution.*

Exercise* 5.7.5*: *Solve* $\dfrac{d^2y}{dx^2} + 4y = \dfrac{2}{\cos 2x}$ *by variation of parameters.*

Exercise* 5.7.6*: *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = \dfrac{e^x}{x}$ *by variation of parameters.*

Find the DE's general solution by variation of parameters.

Exercise* 5.7.101*: $y'' + y = e^{2t}$ *Hint: The homogeneous equation is fundamental.*

Exercise* 5.7.102*: $y'' + 4y = \sec x$

Exercise* 5.7.103*: $y'' + y' - 6y = t^2 e^{2t}$

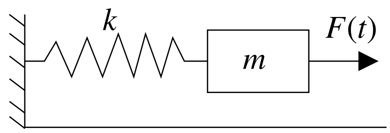
Exercise* 5.7.104*: $y'' - 2y' + 2y = e^t \tan t$

Exercise* 5.7.105*: *Use variation of parameters to find a particular solution of* $y'' - y = \dfrac{1}{e^x + e^{-x}}$*. Hint: The homogenous equation is fundamental whose solution involves hyperbolic functions, and the right hand side of the nonhomogenous equation can also be expressed using hyperbolics. If you choose this approach, the identity* $\cosh^2 x - \sinh^2 x = 1$ *might be useful.*

5.8 Forced oscillations and resonance

Attribution: §2.6 in [L]
Further reading: §3.6 in [EP], §3.8 in [BD]

Let us return back to the example of a mass on a spring.



We now examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t),$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant and $F(t)$ is an external force acting on the mass.

We are interested in periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

5.8.1 Undamped forced motion and resonance

First let us consider undamped ($c = 0$) motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t).$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ is the *natural frequency* (angular). It is the frequency at which the system "wants to oscillate" without external interference.

Let us suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A \cos(\omega t)$ and solve for A. Note that we do not need a sine in our trial solution as after plugging in we only have cosines. If you include a sine, it is fine; you will find that its coefficient is zero (I could not find a second rhyme).

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

We leave it as an exercise to do the algebra required.

The general solution is

$$\boxed{x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).}$$

Written another way

$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

The solution is a superposition of two cosine waves at different frequencies.

Example 5.8.1. Take

$$0.5x'' + 8x = 10 \cos(\pi t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

Let us compute. First we read off the parameters: $\omega = \pi$, $\omega_0 = \sqrt{8/0.5} = 4$, $F_0 = 10$, $m = 0.5$. The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t).$$

Solve for C_1 and C_2 using the initial conditions. It is easy to see that $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16 - \pi^2} \bigl(\cos(\pi t) - \cos(4t)\bigr).$$

Notice the “beating” behavior in Figure 5.4 on the following page. First use the trigonometric identity

$$2 \sin\left(\frac{A - B}{2}\right) \sin\left(\frac{A + B}{2}\right) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16 - \pi^2} \left(2 \sin\left(\frac{4 - \pi}{2} t\right) \sin\left(\frac{4 + \pi}{2} t\right)\right).$$

Notice that x is a high frequency wave modulated by a low frequency wave. ■

Now suppose that $\omega_0 = \omega$. Obviously, we cannot try the solution $A \cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we need to try $x_p = At \cos(\omega t) + Bt \sin(\omega t)$. This time we do need the sine term since the second derivative of $t \cos(\omega t)$ does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m} \cos(\omega t).$$

Plugging x_p into the left hand side we get

$$2B\omega \cos(\omega t) - 2A\omega \sin(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

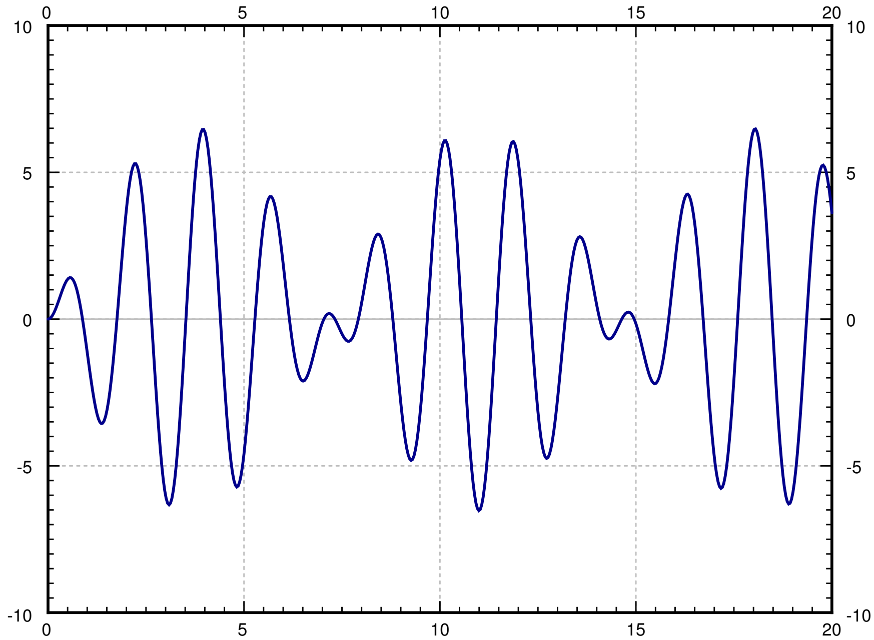


Figure 5.4: Graph of $\dfrac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$.

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}\, t \sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega}\, t \sin(\omega t).$$

The important term is the last one (the particular solution we found). This term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure 5.5 on the next page we see the graph with $C_1 = C_2 = 0$, $F_0 = 2$, $m = 1$, $\omega = \pi$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called *resonance* or perhaps *pure resonance*. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the "correct frequency"? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge

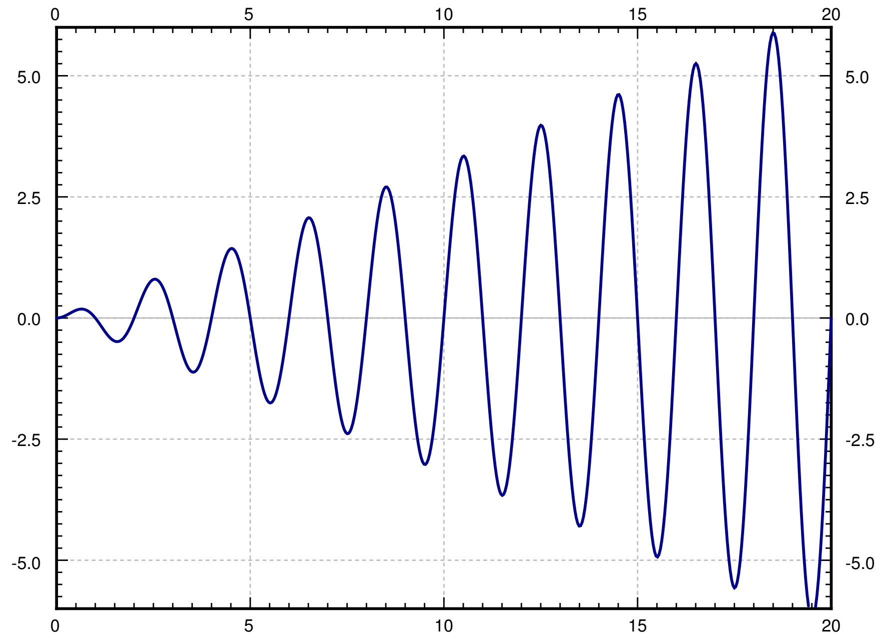


Figure 5.5: Graph of $\frac{1}{\pi} t \sin(\pi t)$.

failure. It turns out there was a different phenomenon at play*.

*K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, 59(2), 1991, 118–124, `http://www.ketchum.org/billah/Billah-Scanlan.pdf`

5.8.2 Damped forced motion and practical resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{5.19}$$

for some $c > 0$. We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

We replace equation (5.19) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t).$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as we have seen before. That is,

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } c^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}\big(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\big) & \text{if } c^2 < 4km\ , \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we see that $x_c(t) \to 0$ as $t \to \infty$. Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A \cos(\omega t) + B \sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$\big((\omega_0^2 - \omega^2)B - 2\omega p A\big) \sin(\omega t) + \big((\omega_0^2 - \omega^2)A + 2\omega p B\big) \cos(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

We solve for A and B:

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2},$$

$$B = \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t).$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega \neq \omega_0$)

$$\tan \gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}.$$

Hence we have

$$\boxed{x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}} \cos(\omega t - \gamma).}$$

If $\omega = \omega_0$ we see that $A = 0$, $B = C = \dfrac{F_0}{2m\omega p}$, and $\gamma = \frac{\pi}{2}$.

The exact formula is not as important as the idea. Do not memorize the above formula, you should instead remember the ideas involved. For a different forcing function F, you will get a different formula for x_p. So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we call x_c the *transient solution* and denote it by x_{tr}. We call the x_p we found above the *steady periodic solution* and denote it by x_{sp}. The general solution to our problem is

$$x = x_c + x_p = x_{tr} + x_{sp}.$$

We note that $x_c = x_{tr}$ goes to zero as $t \to \infty$, as all the terms involve an exponential with a negative exponent. So for large t, the effect of x_{tr} is negligible and we see essentially only x_{sp}. Hence the name *transient*. Notice that x_{sp} involves no arbitrary constants, and the initial conditions only affect x_{tr}. This means that the effect of the initial conditions is negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. See Figure 5.6 on the following page for a graph given several different initial conditions.

The speed at which x_{tr} goes to zero depends on p (and hence c). The bigger p is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This agrees with the observation that when $c = 0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. We look at the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed) we can find its maximum. We call the ω that achieves this maximum the *practical resonance frequency*. We call the maximal

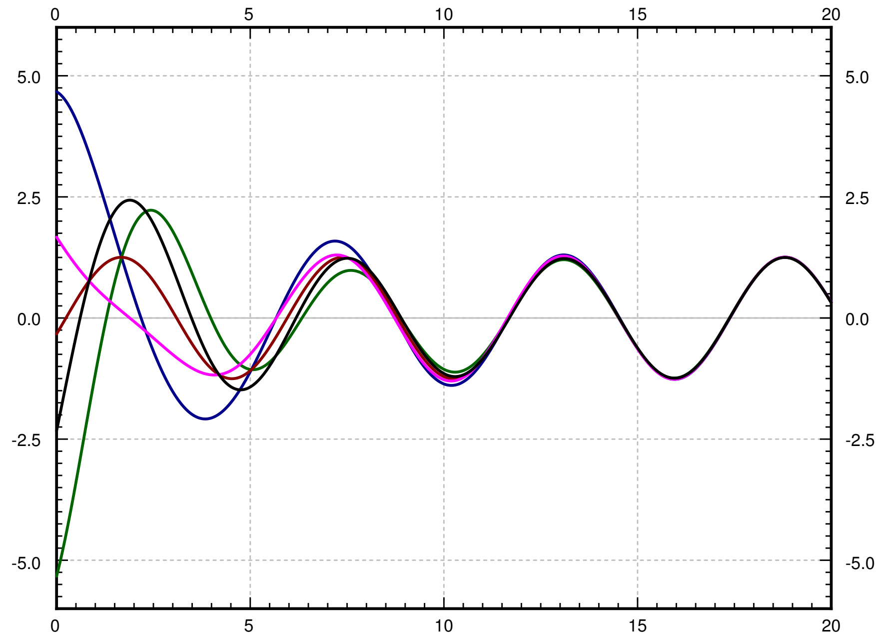


Figure 5.6: Solutions with different initial conditions for parameters $k = 1$, $m = 1$, $F_0 = 1$, $c = 0.7$, and $\omega = 1.1$.

amplitude $C(\omega)$ the *practical resonance amplitude*. Thus when damping is present we talk of *practical resonance* rather than pure resonance. A sample plot for three different values of c is given in Figure 5.7 on the next page. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-2\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m\big((2\omega p)^2 + (\omega_0^2 - \omega^2)^2\big)^{3/2}}.$$

This is zero either when $\omega = 0$ or when $2p^2 + \omega^2 - \omega_0^2 = 0$. In other words, $C'(\omega) = 0$ when

$$\boxed{\omega = \sqrt{\omega_0^2 - 2p^2} \quad \text{or} \quad \omega = 0.}$$

It can be shown that if $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal, note that in this case $C'(\omega) > 0$ for small ω). If $\omega = 0$ is the maximum, then essentially there is no practical resonance since we assume that $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence p) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0

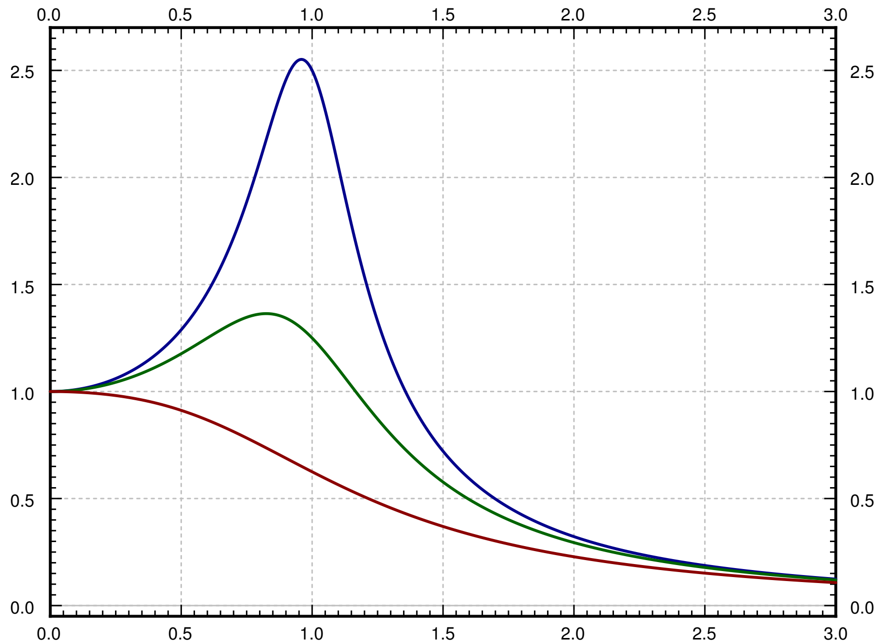


Figure 5.7: Graph of $C(\omega)$ showing practical resonance with parameters $k = 1$, $m = 1$, $F_0 = 1$. The top line is with $c = 0.4$, the middle line with $c = 0.8$, and the bottom line with $c = 1.6$.

is a good estimate of the resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. It will be good to come back to this section once we have learned about the Fourier series.

Exercises

Exercise* 5.8.1*: *Derive a formula for* x_{sp} *if the equation is* $mx'' + cx' + kx = F_0 \sin(\omega t)$*. Assume* $c > 0$*.*

Exercise* 5.8.2*: *Derive a formula for* x_{sp} *if the equation is* $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$*. Assume* $c > 0$*.*

Exercise* 5.8.3*: *Take* $mx'' + cx' + kx = F_0 \cos(\omega t)$*. Fix* $m > 0$*,* $k > 0$*, and* $F_0 > 0$*. Consider the function* $C(\omega)$*. For what values of* c *(solve in terms of* m*,* k*, and* F_0*) will there be no practical resonance (that is, for what values of* c *is there no maximum of* $C(\omega)$ *for* $\omega > 0$*)?*

Exercise* 5.8.4*: *Take* $mx'' + cx' + kx = F_0 \cos(\omega t)$*. Fix* $c > 0$*,* $k > 0$*, and* $F_0 > 0$*. Consider the function* $C(\omega)$*. For what values of* m *(solve in terms of* c*,* k*, and* F_0*) will there be no practical resonance (that is, for what values of* m *is there no maximum of* $C(\omega)$ *for* $\omega > 0$*)?*

Exercise* 5.8.5*: *Suppose a water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as a mass and the support acts as the spring, where the induced vibrations are horizontal. Suppose that the container with water has a mass of* $m = 10,000\,\mathrm{kg}$*. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving).*

Suppose that an earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$*.*

a) What is the natural frequency of the water tower?

b) If ω *is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.*

c) Suppose $A = 1$ *and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter from the rest position, the tower collapses. Will the tower collapse?*

Exercise* 5.8.101*: *A mass of 4 kg on a spring with* $k = 4$ *and a damping constant* $c = 1$*. Suppose that* $F_0 = 2$*. Using forcing function* $F_0 \cos(\omega t)$*, find the* ω *that causes practical resonance and find the amplitude.*

Exercise* 5.8.102*: *Derive a formula for* x_{sp} *for* $mx'' + cx' + kx = F_0 \cos(\omega t) + A$*, where* A *is some constant. Assume* $c > 0$*.*

Exercise* 5.8.103*: *Suppose there is no damping in a mass and spring system with* $m = 5$*,* $k = 20$*, and* $F_0 = 5$*. Suppose that* ω *is chosen to be precisely the resonance frequency. Find* ω*. Find the amplitude of the oscillations at time* $t = 100$*.*

Chapter 6

Systems of ODEs

6.1 Introduction to systems of ODEs

Attribution: §3.1 in [L]
Further reading: §4.1 in [EP], §7.1 in [BD]

Often we do not have just one dependent variable and one equation. And as we will see, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose $y_1, y_2, \ldots, y_n$, then we can have a differential equation involving all of them and their derivatives. For example, $y_1'' = f(y_1', y_2', y_1, y_2, x)$. Usually, when we have two dependent variables we have two equations such as

$$\begin{aligned} y_1'' &= f_1(y_1', y_2', y_1, y_2, x), \\ y_2'' &= f_2(y_1', y_2', y_1, y_2, x), \end{aligned}$$

for some functions f_1 and f_2. We call the above a *system of differential equations*. More precisely, the above is a second order system of ODEs.

Example 6.1.1. Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= y_1 - y_2, \end{aligned}$$

with initial conditions of the form $y_1(0) = 1$, $y_2(0) = 2$.

We note that $y_1 = C_1 e^x$ is the general solution of the first equation. We then plug this y_1 into the second equation and get the equation $y_2' = C_1 e^x - y_2$, which is a linear first order equation that is easily solved for y_2. By the method of integrating factor we get

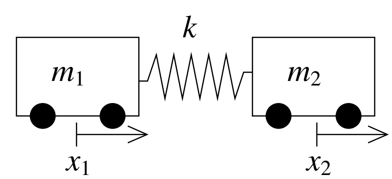
$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2,$$

or $y_2 = \frac{C_1}{2}e^x + C_2e^{-x}$. The general solution to the system is, therefore,

$$y_1 = C_1e^x,$$
$$y_2 = \frac{C_1}{2}e^x + C_2e^{-x}.$$

We solve for C_1 and C_2 given the initial conditions. We substitute $x = 0$ and find that $C_1 = 1$ and $C_2 = 3/2$. Thus the solution is $y_1 = e^x$, and $y_2 = \frac{1}{2}e^x + \frac{3}{2}e^{-x}$. ■

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will need to solve for all variables at once.



As an example application, let us think of mass and spring systems again. Suppose we have one spring with constant k, but two masses m_1 and m_2. We can think of the masses as carts, and we will suppose that they ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign. Newton's second law states that force equals mass times acceleration. So the system of equations governing the setup is

$$m_1x_1'' = k(x_2 - x_1),$$
$$m_2x_2'' = -k(x_2 - x_1).$$

In this system we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends exactly on where the second cart goes and vice-versa.

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an n^{th} order differential equation

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x).$$

We define new variables $u_1, u_2, \dots, u_n$ and write the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ &\vdots \\ u_{n-1}' &= u_n, \\ u_n' &= F(u_n, u_{n-1}, \dots, u_2, u_1, x). \end{aligned}$$

We solve this system for $u_1, u_2, \dots, u_n$. Once we have solved for the u's, we can discard u_2 through u_n and let $y = u_1$. We note that this y solves the original equation.

A similar process can be followed for a system of higher order differential equations. For example, a system of k differential equations in k unknowns, all of order n, can be transformed into a first order system of $n \times k$ equations and $n \times k$ unknowns.

Example 6.1.2. We can use this idea in reverse as well. Let us consider the system

$$x' = 2y - x, \quad y' = x,$$

where the independent variable is t. We wish to solve for the initial conditions $x(0) = 1$, $y(0) = 0$.

If we differentiate the second equation we get $y'' = x'$. We know what x' is in terms of x and y, and we know that $x = y'$. So,

$$y'' = x' = 2y - x = 2y - y'.$$

We now have the equation $y'' + y' - 2y = 0$. We know how to solve this equation and we find that $y = C_1 e^{-2t} + C_2 e^t$. Once we have y we use the equation $y' = x$ to get x.

$$x = y' = -2C_1 e^{-2t} + C_2 e^t.$$

We solve for the initial conditions $1 = x(0) = -2C_1 + C_2$ and $0 = y(0) = C_1 + C_2$. Hence, $C_1 = -C_2$ and $1 = 3C_2$. So $C_1 = {}^{-1}\!/\!_3$ and $C_2 = {}^1\!/\!_3$. Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3}.$$

■

***Exercise* 6.1.1:** *Plug in and check that this really is the solution.*

It is useful to go back and forth between systems and higher order equations for other reasons. For example, ODE approximation methods are generally only given as solutions for first order systems. The above example is what we call a *linear first order system*, as none of the dependent variables appear in any functions or with any higher powers than one. It is also *autonomous* as the equations do not depend on the independent variable t.

Exercises

Exercise* 6.1.2*: *Find the general solution of* $x_1' = x_2 - x_1 + t$, $x_2' = x_2$.

Exercise* 6.1.3*: *Find the general solution of* $x_1' = 3x_1 - x_2 + e^t$, $x_2' = x_1$.

Exercise* 6.1.4*: *Write* $x'' + 3x' + 5x = t$ *as a first order system of ODEs.*

Exercise* 6.1.5*: *Write* $ay'' + by' + cy = f(x)$ *as a first order system of ODEs.*

Exercise* 6.1.6*: *Write* $x'' + y^2y' - x^3 = \sin(t)$, $y'' + (x' + y')^2 - x = 0$ *as a first order system of ODEs.*

Exercise* 6.1.7*: *Find the general solution of* $x_1' = 2x_1$, $x_2' = 3x_2$.

Exercise* 6.1.8*: *This exercise revisits Exercise 3.6.1. Suppose that there is a second lake downstream, so that clean water flows into our original lake, then the water from that lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kilograms of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream. Previously, you found the concentration of toxic substance as a function of time in the first lake. Now find the concentration of toxic substance as a function of time in the second lake. When will the concentration in the second lake be maximal? What is this maximal concentration?*

Exercise* 6.1.9*: *This exercise revisits Exercise 3.6.9. Suppose that a second tank is linked in cascade with our first tank. That is, our original tank empties into the second. Initially, the second tank contains 300 liters of water and 50 kilograms of salt dissolved in it. Well-mixed water flows from the first tank to the second at 5 liters per minute, and well-mixed water also flows out of the second tank at the same rate. Previously, you found the mass of salt as a function of time in the first tank. Now find the mass of salt as a function of time in the second lake. When will the salt mass in the second lake be maximal? What is this maximal mass?*

Exercise* 6.1.101*: *Find the general solution to* $y_1' = 3y_1$, $y_2' = y_1 + y_2$, $y_3' = y_1 + y_3$.

Exercise* 6.1.102*: *Solve* $y' = 2x$, $x' = x + y$, $x(0) = 1$, $y(0) = 3$.

Exercise* 6.1.103*: *Write* $x''' = x + t$ *as a first order system.*

Exercise* 6.1.104*: *Write* $y_1'' + y_1 + y_2 = t$, $y_2'' + y_1 - y_2 = t^2$ *as a first order system.*

6.2 Linear systems of ODEs

Attribution: §3.3 in [L]
Further reading: second part of §5.1 in [EP], §7.4 in [BD]

First let us talk about matrix or vector valued functions. Such a function is just a matrix whose entries depend on some variable. If t is the independent variable, we write a *vector valued function* $\vec{x}(t)$ as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Similarly a *matrix valued function* $A(t)$ is

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

We can talk about the derivative $A'(t)$ or $\frac{dA}{dt}$. This is just the matrix valued function whose ij^{th} entry is $a'_{ij}(t)$.

Rules of differentiation of matrix valued functions are similar to rules for normal functions. Let $A(t)$ and $B(t)$ be matrix valued functions. Let c a scalar and let C be a constant matrix. Then

$$\begin{aligned} \bigl(A(t) + B(t)\bigr)' &= A'(t) + B'(t), \\ \bigl(A(t)B(t)\bigr)' &= A'(t)B(t) + A(t)B'(t), \\ \bigl(cA(t)\bigr)' &= cA'(t), \\ \bigl(CA(t)\bigr)' &= CA'(t), \\ \bigl(A(t)\,C\bigr)' &= A'(t)\,C. \end{aligned}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\vec{x}\,'(t) = P(t)\vec{x}(t) + \vec{f}(t),$$

where $P(t)$ is a matrix valued function, and $\vec{x}(t)$ and $\vec{f}(t)$ are vector valued functions. We will often suppress the dependence on t and only write $\vec{x}\,' = P\vec{x} + \vec{f}$. A solution of the system is a vector valued function $\vec{x}$ satisfying the vector equation.

For example, the equations

$$x_1' = 2tx_1 + e^t x_2 + t^2,$$
$$x_2' = \frac{x_1}{t} - x_2 + e^t,$$

can be written as

$$\vec{x}\,' = \begin{pmatrix} 2t & e^t \\ 1/t & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} t^2 \\ e^t \end{pmatrix}.$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix P will be constant; it will not depend on t.

Linear independence for vector functions is the same idea as for normal functions. The vector valued functions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$$

has only the solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all t.

Example 6.2.1. $\vec{x}_1 = \begin{pmatrix} t^2 \\ t \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 1+t \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} -t^2 \\ 1 \end{pmatrix}$ are linearly dependent because $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$, and this holds for all t. So $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$ above will work. ■

Example 6.2.2. If we change the previous example just slightly to $\vec{x}_1 = \begin{pmatrix} t^2 \\ t \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ t \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} -t^2 \\ 1 \end{pmatrix}$, then the functions are linearly independent. First write $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$ and note that it has to hold for all t. We get that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{pmatrix} c_1t^2 - c_3t^2 \\ c_1t + c_2t + c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In other words $c_1t^2 - c_3t^2 = 0$ and $c_1t + c_2t + c_3 = 0$. If we set $t = 0$, then the second equation becomes $c_3 = 0$. But then the first equation becomes $c_1t^2 = 0$ for all t and so $c_1 = 0$. Thus the second equation is just $c_2t = 0$, which means $c_2 = 0$. So $c_1 = c_2 = c_3 = 0$ is the only solution and $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are linearly independent. ■

When $\vec{f} = \vec{0}$ (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

Theorem 6.2.1 (Superposition)**.** *Let $\vec{x}\,' = P\vec{x}$ be a linear homogeneous system of ODEs. Suppose that $\vec{x}_1, \ldots, \vec{x}_n$ are n solutions of the equation, then*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n, \tag{6.1}$$

is also a solution. Furthermore, if this is a system of n equations (P is $n \times n$), and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent, then every solution $\vec{x}$ can be written as (6.1).

The linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ could always be written as

$$X(t)\,\vec{c},$$

where $X(t)$ is the matrix with columns $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$, and $\vec{c}$ is the column vector with entries $c_1, c_2, \dots, c_n$. The matrix valued function $X(t)$ is called the *fundamental matrix*, or the *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

Theorem 6.2.2. *Let $\vec{x}\,' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is one particular solution. Then every solution can be written as*

$$\vec{x} = \vec{x}_c + \vec{x}_p,$$

where $\vec{x}_c$ is a solution to the associated homogeneous equation ($\vec{x}\,' = P\vec{x}$).

So the procedure for systems is the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution of $\vec{x}\,' = P\vec{x} + \vec{f}$. Next suppose you are given an initial condition of the form $\vec{x}(t_0) = \vec{b}$ for some constant vector $\vec{b}$. Let $X(t)$ be the fundamental matrix solution of the associated homogeneous equation (i.e. columns of $X(t)$ are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\,\vec{c} + \vec{x}_p(t).$$

We are seeking a vector $\vec{c}$ such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\,\vec{c} + \vec{x}_p(t_0).$$

In other words, we are solving for $\vec{c}$ the nonhomogeneous system of linear equations

$$X(t_0)\,\vec{c} = \vec{b} - \vec{x}_p(t_0).$$

Example 6.2.3. In Section 6.1 we solved the system

$$x_1' = x_1,$$
$$x_2' = x_1 - x_2,$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$. Let us consider this problem in the language of this section.

The system is homogeneous, so $\vec{f}(t) = \vec{0}$. We write the system and the initial conditions as

$$\vec{x}\,' = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We found the general solution was $x_1 = c_1 e^t$ and $x_2 = \frac{c_1}{2} e^t + c_2 e^{-t}$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain the solution $\begin{pmatrix} e^t \\ \frac{1}{2} e^t \end{pmatrix}$. Letting $c_1 = 0$ and $c_2 = 1$, we obtain $\begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$. These two solutions are linearly independent, as can be seen by setting $t = 0$, and noting that the resulting constant vectors are linearly independent. In matrix notation, the fundamental matrix solution is, therefore,

$$X(t) = \begin{pmatrix} e^t & 0 \\ \frac{1}{2} e^t & e^{-t} \end{pmatrix}.$$

To solve the initial value problem we solve for $\vec{c}$ the equation

$$X(0)\,\vec{c} = \vec{b},$$

or in other words,

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

A single elementary row operation shows $\vec{c} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$. Our solution is

$$\vec{x}(t) = X(t)\,\vec{c} = \begin{pmatrix} e^t & 0 \\ \frac{1}{2} e^t & e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} e^t \\ \frac{1}{2} e^t + \frac{3}{2} e^{-t} \end{pmatrix}.$$

This new solution agrees with our previous solution from Section 6.1. ■

Exercises

Exercise* 6.2.1*: *Write the system*

$$x_1' = 2x_1 - 3tx_2 + \sin t,$$
$$x_2' = e^t x_1 + 3x_2 + \cos t$$

in the form $\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$.

Exercise* 6.2.2*: *Verify that the system* $\vec{x}\,' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{x}$ *has the two solutions* $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ *and* $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$. *Write down the general solution. Write down the general solution in the form* $x_1 = ?$, $x_2 = ?$ *(i.e. write down a formula for each element of the solution).*

Exercise* 6.2.3*: *Verify that* $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$ *and* $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$ *are linearly independent. Hint: Just plug in* $t = 0$.

Exercise* 6.2.4*: *Verify that* $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t}$ *and* $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{t}$ *and* $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t}$ *are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.*

Exercise* 6.2.5*: *Verify that* $\begin{pmatrix} t \\ t^2 \end{pmatrix}$ *and* $\begin{pmatrix} t^3 \\ t^4 \end{pmatrix}$ *are linearly independent.*

Exercise* 6.2.101*: *Write*

$$x' = 3x - y + e^t,$$
$$y' = tx$$

in matrix notation.

Exercise* 6.2.102*: *Write*

$$x_1' = 2tx_2,$$
$$x_2' = 2tx_2$$

in matrix notation. Solve this system. Write the solution in matrix notation.

Exercise* 6.2.103*: *Are* $\begin{pmatrix} e^{2t} \\ e^t \end{pmatrix}$ *and* $\begin{pmatrix} e^{t} \\ e^{2t} \end{pmatrix}$ *linearly independent? Justify.*

Exercise* 6.2.104*: *Are* $\begin{pmatrix} \cosh(t) \\ 1 \end{pmatrix}$, $\begin{pmatrix} e^{t} \\ 1 \end{pmatrix}$, *and* $\begin{pmatrix} e^{-t} \\ 1 \end{pmatrix}$ *linearly independent? Justify.*

6.3 Eigenmethod

Attribution: §3.4 in [L]
Further reading: §5.2 in [EP], part of §7.3, §7.5, and §7.6 in [BD]

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have such a system

$$\vec{x}\,' = P\vec{x},$$

where P is a constant square matrix. We wish to adapt the method for the single constant coefficient equation by trying the function $e^{\lambda t}$. However, $\vec{x}$ is a vector. So we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an arbitrary constant vector. We plug this $\vec{x}$ into the equation to get

$$\lambda\vec{v}e^{\lambda t} = P\vec{v}e^{\lambda t}.$$

We divide by $e^{\lambda t}$ and notice that we are looking for a scalar λ and a vector $\vec{v}$ that satisfy the equation

$$\lambda\vec{v} = P\vec{v}.$$

We find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix P, and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$. Now we notice that the functions $\vec{v}_1 e^{\lambda_1 t}$, $\vec{v}_2 e^{\lambda_2 t}$, $\ldots$, $\vec{v}_n e^{\lambda_n t}$ are solutions of the system of equations and hence $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$ is a solution.

Theorem 6.3.1. *Take $\vec{x}\,' = P\vec{x}$. If P is an $n\times n$ constant matrix that has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then there exist n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and the general solution to $\vec{x}\,' = P\vec{x}$ can be written as*

$$\boxed{\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}.}$$

The corresponding fundamental matrix solution is $X(t) = \begin{pmatrix} \vec{v}_1 e^{\lambda_1 t} & \vec{v}_2 e^{\lambda_2 t} & \cdots & \vec{v}_n e^{\lambda_n t} \end{pmatrix}$. That is, $X(t)$ is the matrix whose j^{th} column is $\vec{v}_j e^{\lambda_j t}$.

Example 6.3.1. Consider the system

$$\vec{x}\,' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\vec{x}.$$

Find the general solution.

In Example 4.8.2 and Exercise 4.8.2 we found the eigenvalues to be $1, 2, 3$ and the eigenvectors to be $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, respectively. Hence our general solution is

$$\vec{x} = c_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{2t} + c_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{3t} = \begin{pmatrix} c_1 e^{t} + c_3 e^{3t} \\ -c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{pmatrix} = \begin{pmatrix} e^{t} & 0 & e^{3t} \\ -e^{t} & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

■

Exercise* 6.3.1*: *Check that this $\vec{x}$ really solves the system.*

Note: If we write a homogeneous linear constant coefficient n^{th} order equation as a first order system (as we did in Section 6.1), then the eigenvalue equation

$$\det(P - \lambda I) = 0$$

is essentially the same as the characteristic equation we got in Section 5.2 and Section 5.4.

Example 6.3.2. Suppose that we have the system

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{x}.$$

In Exercise 4.8.5, we found that the eigenvalues are $1 - i$, $1 + i$ and the eigenvectors are $\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$, $\begin{pmatrix} -i \\ 1 \end{pmatrix}$, respectively. We *could* write the solution as

$$\vec{x} = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1-i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} c_1 i e^{(1-i)t} - c_2 i e^{(1+i)t} \\ c_1 e^{(1-i)t} + c_2 e^{(1+i)t} \end{pmatrix},$$

but we wonder whether the solution requires complex functions. We also fear the prospect of having to find complex values c_1 and c_2 to solve any initial conditions that may be imposed. ■

It is perhaps not completely clear that we can get a real solution. We could do the whole song and dance we did before, but we will not. We will do something a bit smarter first.

Suppose that $a + ib$ is a complex eigenvalue of P, and $\vec{v}$ is a corresponding eigenvector. Then

$$\vec{x}_1 = \vec{v} e^{(a+ib)t}$$

is a solution (complex valued) of $\vec{x}\,' = P\vec{x}$. Euler's formula shows that $\overline{e^{a+ib}} = e^{a-ib}$, and so

$$\vec{x}_2 = \overline{\vec{x}_1} = \overline{\vec{v}} e^{(a-ib)t}$$

is also a solution. The function

$$\vec{x}_3 = \operatorname{Re} \vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t} = \frac{\vec{x}_1 + \overline{\vec{x}_1}}{2} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

is also a solution. And $\vec{x}_3$ is real-valued! Similarly, we find that

$$\vec{x}_4 = \operatorname{Im} \vec{x}_1 = \frac{\vec{x}_1 - \overline{\vec{x}_1}}{2i} = \frac{\vec{x}_1 - \vec{x}_2}{2i}.$$

is also a real-valued solution. It turns out that $\vec{x}_3$ and $\vec{x}_4$ are linearly independent. We will use Euler's formula to separate out the real and imaginary part.

Example 6.3.2 (continuing from p. 239)**.** We can now convert our previously-found complex-valued solution and convert it to a real-valued solution. The major step invokes Euler's formula to convert the complex exponential into trigonometric functions:

$$
\begin{aligned}
\vec{x}_1 &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1-i)t} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^t e^{-it} \\
&= \begin{pmatrix} i \\ 1 \end{pmatrix} e^t \left(\cos(-t) + i \sin(-t)\right) \\
&= \begin{pmatrix} i \\ 1 \end{pmatrix} e^t \left(\cos t - i \sin t\right) \\
&= \begin{pmatrix} ie^t \cos t + e^t \sin t \\ e^t \cos t - ie^t \sin t \end{pmatrix} \\
&= \begin{pmatrix} e^t \sin t + ie^t \cos t \\ e^t \cos t - ie^t \sin t \end{pmatrix} \\
&= \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} + i \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}.
\end{aligned}
$$

(Notice that we exploited that the sine function is odd and the cosine function is even to simply a bit.) Then

$$
\begin{aligned}
\operatorname{Re} \vec{x}_1 &= \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} \\
\operatorname{Im} \vec{x}_1 &= \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}
\end{aligned}
$$

are the two real-valued linearly independent solutions we seek. The general solution is

$$
\vec{x} = c_1 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} + c_2 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} = \begin{pmatrix} c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t - c_2 e^t \sin t \end{pmatrix}.
$$

This solution is real-valued for real c_1 and c_2. Now we can solve for any initial conditions that we may have. ■

***Exercise* 6.3.2:** *Check that these really are solutions.*

Let us summarize this solution form as a theorem.

Theorem 6.3.2. *Let $\vec{x}\,' = P\vec{x}$ be a system of differential equations where P is a real-valued constant matrix. If P possesses a pair of complex-conjugate eigenvalues $a + ib$ and $a - ib$ and corresponding eigenvectors $\vec{v}$ and $\overline{\vec{v}}$, respectively, then the system possesses two linearly independent real-valued solutions*

$$
\vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t} \qquad \textit{and} \qquad \vec{x}_2 = \operatorname{Im} \vec{v} e^{(a+ib)t}.
$$

For each pair of complex eigenvalues $a+ib$ and $a-ib$, we get two real-valued linearly independent solutions. We then go on to the next eigenvalue, which is either a real eigenvalue or another complex eigenvalue pair. If we have n distinct eigenvalues (real or complex), then we end up with n linearly independent solutions.

We can now find a real-valued general solution to any homogeneous system where the matrix has distinct eigenvalues. When we have repeated eigenvalues, matters get a bit more complicated.

Exercises

In these exercises, do not express your solution using complex exponentials.

Exercise* 6.3.3*: *Write the system $x_1' = 2x_1$, $x_2' = 3x_2$ in the form $\vec{x}\,' = A\vec{x}$. Find the general solution using the eigenvalue method. Compare with your answer to Exercise 6.1.7.*

Exercise* 6.3.4*: *Find the general solution of $x_1' = 3x_1 + x_2$, $x_2' = 2x_1 + 4x_2$ using the eigenvalue method.*

Exercise* 6.3.5*: *Find the general solution of $x_1' = x_1 - 2x_2$, $x_2' = 2x_1 + x_2$ using the eigenvalue method.*

Exercise* 6.3.6*: *Use the eigenvalues and eigenvectors previously computed under Exercise 4.8.6 to find the general solution of $\vec{x}\,' = A\vec{x}$.*

Exercise* 6.3.7*: *Use the eigenvalues and eigenvectors previously computed under Exercise 4.8.11 to find the general solution of $\vec{x}\,' = A\vec{x}$.*

Exercise* 6.3.8*: *Consider the system $x' = 2y - x$, $y' = x$, where the independent variable is t. Use the eigenvalue method to solve this system. Then, enforce the initial conditions $x(0) = 1$, $y(0) = 0$. Compare with the solution that we found in Example 6.1.2.*

Exercise* 6.3.101*: *Use the eigenvalues and eigenvectors previously computed under Exercise 4.8.102 to find the general solution of $\vec{x}\,' = A\vec{x}$.*

Exercise* 6.3.102*: *Use the eigenvalues and eigenvectors previously computed under Exercise 4.8.101 to find the general solution of $\vec{x}\,' = A\vec{x}$.*

Exercise* 6.3.103*: *Solve $x_1' = x_2$, $x_2' = x_1$ using the eigenvalue method.*

Exercise* 6.3.104*: *Solve $x_1' = x_2$, $x_2' = -x_1$ using the eigenvalue method.*

6.4 Two dimensional systems and their vector fields

Attribution: §3.5 in [L]
Further reading: part of §6.2 in [EP], parts of §7.5 and §7.6 in [BD]

Let us take a moment to talk about constant coefficient linear homogeneous systems in the plane. Much intuition can be obtained by studying this simple case. Suppose we have a 2×2 matrix P and the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = P \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6.2}$$

The system is autonomous (compare this section to Section 3.7), and so we can draw the so-called *direction field* or *vector field*. That is, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude).

Example 6.1.2 (continuing from p. 231)**.** Let us revisit the system $x' = 2y - x$, $y' = x$ where the independent variable is t and the initial conditions are $x(0) = 1$, $y(0) = 0$. At the point (x, y), the direction in which we should travel to satisfy the equations should be the direction of the vector $(2y - x, x)$ with the speed equal to the magnitude of this vector. So we draw the vector $(2y - x, x)$ based at the point (x, y) and we do this for many points on the xy-plane. We may want to scale down the size of our vectors to fit many of them on the same direction field. See Figure 6.1 on page 245.

We can now draw a path of the solution in the plane. That is, suppose the solution is given by $x = f(t)$, $y = g(t)$, then we pick an interval of t (say $0 \leq t \leq 2$ for our example) and plot all the points $(f(t), g(t))$ for t in the selected range. The resulting picture is called the *phase portrait* (or phase plane portrait). The particular curve obtained is called the *trajectory* or *solution curve*. An example plot is given in Figure 6.2 on page 245. In this figure the line starts at $(1, 0)$ and travels along the vector field for a distance of 2 units of t. Since we previously solved this system, finding Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3},$$

we can easily compute that $x(2) \approx 2.475$ and $y(2) \approx 2.457$. This point corresponds to the top right end of the plotted solution curve in the figure. ■

Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But now note how much more complicated things became when we allowed just one more dimension.

In Maxima, you can generate a vector field similar to Figure 6.1 by executing Maxima program 6.4.1. In SageMath, execute SageMath program 6.4.1.

Now, we push our analysis a little further so as to describe what the vector field looks like and how the solutions behave, based solely on the eigenvalues and eigenvectors of the matrix P. For this section, we assume that P has two eigenvalues and two corresponding eigenvectors.

Maxima program 6.4.1 Generating a vector field

(%i1) `kill(all)$ reset()$`

(%i2) `f(x,y) := 2*y-x;`

(%o2) $\mathrm{f}(x,y) := 2\,y - x$

(%i3) `g(x,y) := x;`

(%o3) $\mathrm{g}(x,y) := x$

(%i4)
```
x0 : 1;
y0 : 0;
```

(%o4) 1
(%o5) 0

(%i6)
```
plotdf( [f(x,y),g(x,y)], [x,y], [x,-3,3], [y,-3,3], ...
            [trajectory_at,x0,y0] )$
```

SageMath program 6.4.1 Generating a vector field

Input

```
x=var('x')
y=var('y')
# define the slope functions
f(x,y) = 2*y-x
g(x,y) = x
# generate the slope field
P = plot_vector_field( (f(x,y),g(x,y)), (x,-3,3), (y,-3,3), ...
        color="blue", aspect_ratio=1 )
view(P)
```

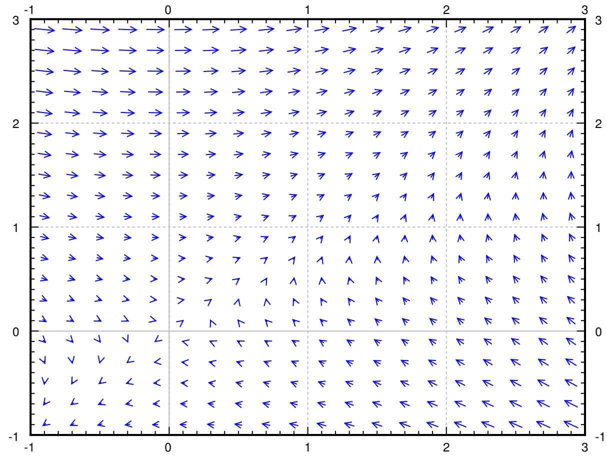


Figure 6.1: The vector field for $x' = 2y - x$, $y' = x$.

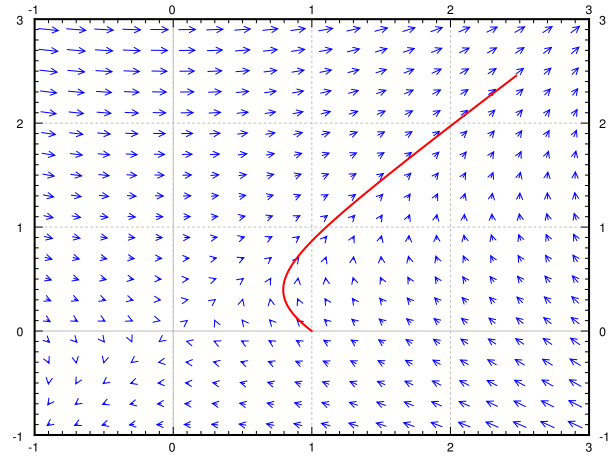


Figure 6.2: The vector field for $x' = 2y - x$, $y' = x$ with the trajectory of the solution starting at $(1, 0)$ for $0 \leq t \leq 2$.

Case I Suppose that the eigenvalues of P are real and positive. We find two corresponding eigenvectors and plot them in the plane. For example, take the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. The eigenvalues are 1 and 2 and corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now suppose that x and y are on the line determined by an eigenvector $\vec{v}$ for an eigenvalue λ. That is, $\begin{pmatrix} x \\ y \end{pmatrix} = a\vec{v}$ for some scalar a. Then

$$\begin{pmatrix} x \\ y \end{pmatrix}' = P \begin{pmatrix} x \\ y \end{pmatrix} = P(a\vec{v}) = a(P\vec{v}) = a\lambda\vec{v}.$$

The derivative is a multiple of $\vec{v}$ and hence points along the line determined by $\vec{v}$. As $\lambda > 0$, the derivative points in the direction of $\vec{v}$ when a is positive and in the opposite direction when a is negative. Let us draw the lines determined by the eigenvectors, and let us draw arrows on the lines to indicate the directions. See Figure 6.3 on the following page.

We fill in the rest of the arrows for the vector field and we also draw a few solutions. See Figure 6.4 on the next page. The picture looks like a source with arrows coming out from the origin. Hence we call this type of picture a *source* or sometimes an *unstable node*.

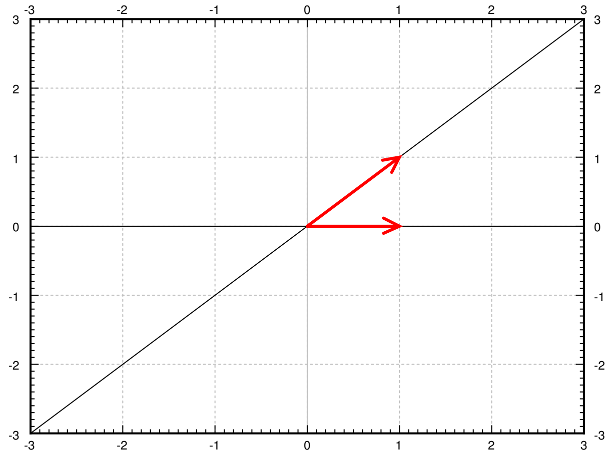


Figure 6.3: Eigenvectors of P with directions.

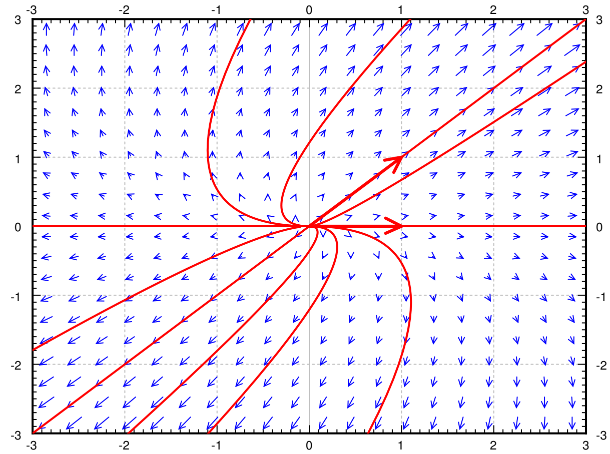


Figure 6.4: Example source vector field with eigenvectors and solutions.

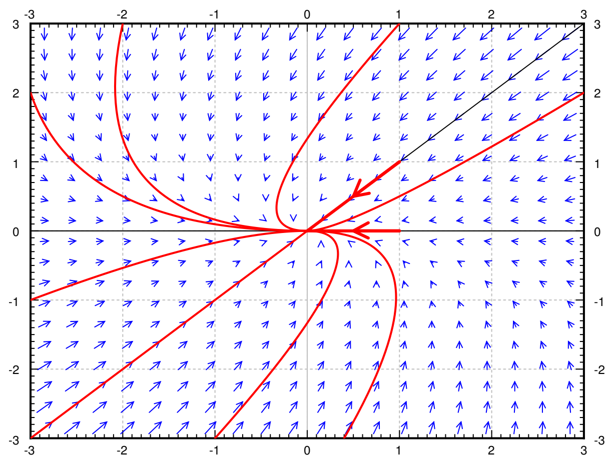


Figure 6.5: Example sink vector field with eigenvectors and solutions.

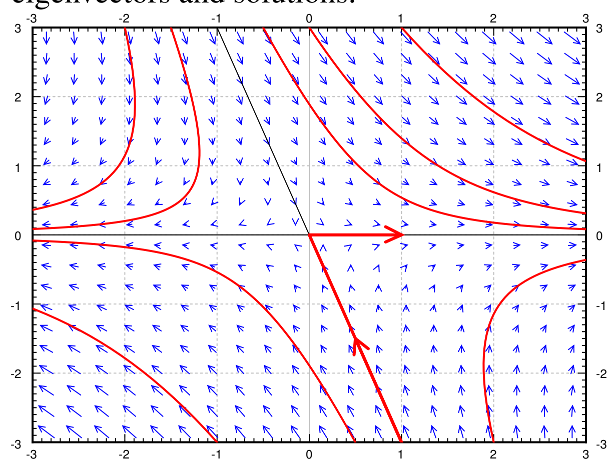


Figure 6.6: Example saddle vector field with eigenvectors and solutions.

Case II Suppose both eigenvalues are negative. For example, take the negation of the matrix in case I, $\begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$. The eigenvalues are -1 and -2 and corresponding eigenvectors are the same, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The calculation and the picture are almost the same. The only difference is that the eigenvalues are negative and hence all arrows are reversed. We get the picture in Figure 6.5. We call this kind of picture a *sink* or sometimes a *stable node*.

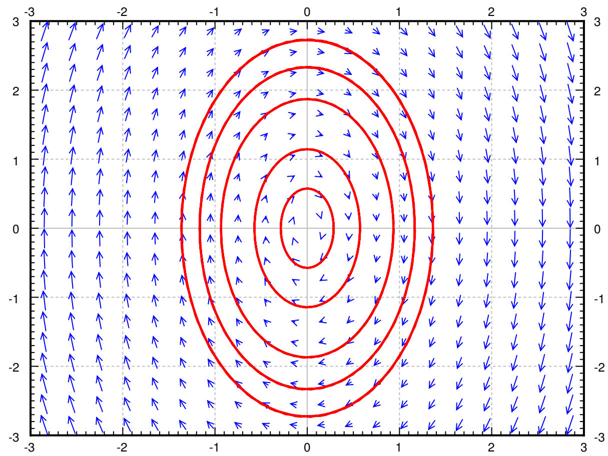


Figure 6.7: Example center vector field.

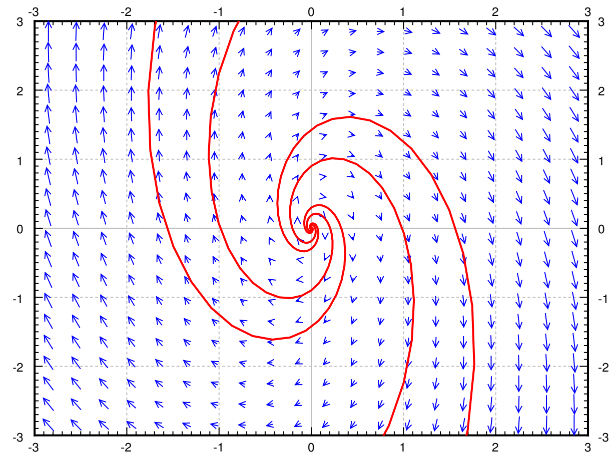


Figure 6.8: Example spiral source vector field.

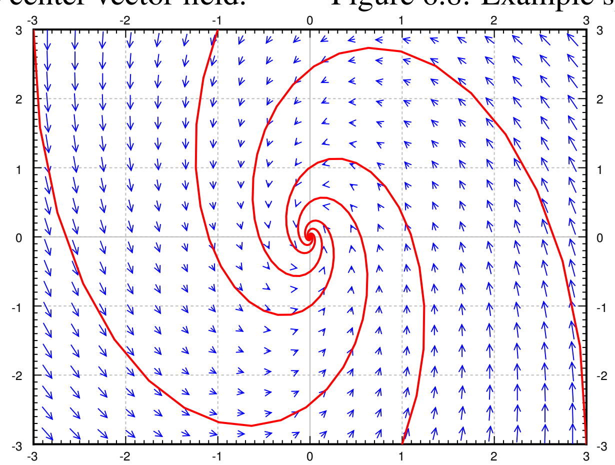


Figure 6.9: Example spiral sink vector field.

Case III Suppose one eigenvalue is positive and one is negative. For example the matrix $\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$. The eigenvalues are 1 and -2 and corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$. We reverse the arrows on one line (corresponding to the negative eigenvalue) and we obtain the picture in Figure 6.6 on the preceding page. We call this picture a *saddle point*.

For the next three cases we will assume the eigenvalues are complex. In this case the eigenvectors are also complex and we cannot just plot them in the plane.

Case IV Suppose the eigenvalues are purely imaginary. That is, suppose the eigenvalues are $\pm ib$. For example, let $P = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$. The eigenvalues turn out to be $\pm 2i$ and eigenvectors are $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$. Consider the eigenvalue $2i$ and its eigenvector $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$. The real and imaginary parts of $\vec{v}e^{i2t}$ are

$$\begin{aligned} \operatorname{Re} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{i2t} &= \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix}, \\ \operatorname{Im} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{i2t} &= \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}. \end{aligned}$$

We can take any linear combination of them to get other solutions, which one we take depends on the initial conditions. Now note that the real part is a parametric equation for an ellipse. Same with the imaginary part and in fact any linear combination of the two. This is what happens in general when the eigenvalues are purely imaginary. So when the eigenvalues are purely imaginary, we get *ellipses* for the solutions. This type of picture is sometimes called a *center*. See Figure 6.7 on the preceding page.

Case V Now suppose the complex eigenvalues have a positive real part. That is, suppose the eigenvalues are $a \pm ib$ with a being positive. For example, let $P = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. The eigenvalues turn out to be $1 \pm 2i$ and eigenvectors are $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$. We take $1 + 2i$ and its eigenvector $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and find the real and imaginary parts of $\vec{v}e^{(1+2i)t}$ are

$$\begin{aligned} \operatorname{Re} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(1+2i)t} &= e^t \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix}, \\ \operatorname{Im} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(1+2i)t} &= e^t \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}. \end{aligned}$$

Note the e^t in front of the solutions. This means that the solutions grow in magnitude while spinning around the origin. Hence we get a *spiral source*. See Figure 6.8 on the previous page.

Case VI Finally suppose the complex eigenvalues have a negative real part. That is, suppose the eigenvalues are $a \pm ib$ with a being negative. For example, let $P = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix}$. The eigenvalues turn out to be $-1 \pm 2i$ and eigenvectors are $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$. We take $-1 - 2i$ and its eigenvector $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and find the real and imaginary parts of $\vec{v}e^{(-1-2i)t}$ are

$$\begin{aligned}
\operatorname{Re} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(-1-2i)t} &= e^{-t} \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}, \\
\operatorname{Im} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(-1-2i)t} &= e^{-t} \begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix}.
\end{aligned}$$

Note the e^{-t} in front of the solutions. This means that the solutions shrink in magnitude while spinning around the origin. Hence we get a *spiral sink*. See Figure 6.9 on page 247.

We summarize the behavior of linear homogeneous two dimensional systems in Table 6.1.

Eigenvalues	Behavior
real and both positive	source / unstable node
real and both negative	sink / stable node
real and opposite signs	saddle
purely imaginary	center point / ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

Table 6.1: Summary of behavior of linear homogeneous two dimensional systems.

Exercises

Exercise* 6.4.1*: *Take the equation* $mx'' + cx' + kx = 0$*, with* $m > 0$*,* $c \geq 0$*,* $k > 0$ *for the mass-spring system. a) Convert this to a system of first order equations. b) Classify for what* m, c, k *do you get which behavior. c) Can you explain from physical intuition why you do not get all the different kinds of behavior here?*

Exercise* 6.4.2*: *Describe the case when* $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$*. In this case the eigenvalue is repeated and there is only one eigenvector. What does the picture look like?*

Exercise* 6.4.3*: *Describe the case when* $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$*. Does this look like any of the pictures we have drawn?*

Exercise* 6.4.101*: *Describe the behavior of the following systems without solving:*
a) $x' = x + y, \;\; y' = x - y.$
b) $x_1' = x_1 + x_2, \;\; x_2' = 2x_2.$
c) $x_1' = -2x_2, \;\; x_2' = 2x_1.$
d) $x' = x + 3y, \;\; y' = -2x - 4y.$
e) $x' = x - 4y, \;\; y' = -4x + y.$

Exercise* 6.4.102*: *Suppose that* $\vec{x}\,' = A\vec{x}$ *where A is a 2 by 2 matrix with eigenvalues* $2 \pm i$*. Describe the behavior.*

Exercise* 6.4.103*: *Take* $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$*. Draw the vector field and describe the behavior. Is it one of the behaviors that we have seen before?*

6.5 Second order systems and applications

Attribution: §3.6 in [L]
Further reading: §5.3 in [EP], not in [BD]

6.5.1 Undamped mass-spring systems

While we did say that we will usually only look at first order systems, it is sometimes more convenient to study the system in the way it arises naturally. For example, suppose we have 3 masses connected by springs between two walls. We could pick any higher number, and the math would be essentially the same, but for simplicity we pick 3 right now. Let us also assume no friction, that is, the system is undamped. The masses are m_1, m_2, and m_3 and the spring constants are k_1, k_2, k_3, and k_4. Let x_1 be the displacement from rest position of the first mass, and x_2 and x_3 the displacement of the second and third mass. We will make, as usual, positive values go right (as x_1 grows, the first mass is moving right). See Figure 6.10.

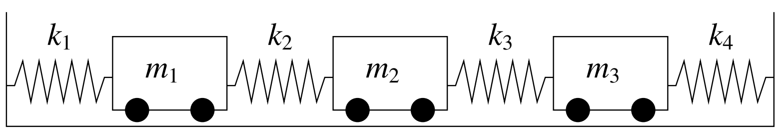


Figure 6.10: System of masses and springs.

This simple system turns up in unexpected places. For example, our world really consists of many small particles of matter interacting together. When we try the above system with many more masses, we obtain a good approximation to how an elastic material behaves. By somehow taking a limit of the number of masses going to infinity, we obtain the continuous one dimensional wave equation. But we digress.

Let us set up the equations for the three mass system. By Hooke's law we have that the force acting on the mass equals the spring compression times the spring constant. By Newton's second law we have that force is mass times acceleration. So if we sum the forces acting on each mass and put the right sign in front of each term, depending on the direction in which it is acting, we end up with the desired system of equations.

$$\begin{aligned}
m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) && = -(k_1 + k_2)x_1 + k_2 x_2, \\
m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) && = k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3, \\
m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 && = k_3 x_2 - (k_3 + k_4)x_3.
\end{aligned}$$

We define the matrices

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{pmatrix}.$$

We write the equation simply as

$$M\vec{x}'' = K\vec{x}.$$

At this point we could introduce 3 new variables and write out a system of 6 first order equations. We claim this simple setup is easier to handle as a second order system. We call $\vec{x}$ the *displacement vector*, M the *mass matrix*, and K the *stiffness matrix*.

Exercise* 6.5.1*: *Repeat this setup for 4 masses (find the matrices M and K). Do it for 5 masses. Can you find a prescription to do it for n masses?*

As with a single equation we want to “divide by M.” This means computing the inverse of M. The masses are all nonzero and M is a diagonal matrix, so comping the inverse is easy:

$$M^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{pmatrix}.$$

This fact follows readily by how we multiply diagonal matrices. As an exercise, you should verify that $MM^{-1} = M^{-1}M = I$.

Let $A = M^{-1}K$. We look at the system $\vec{x}'' = M^{-1}K\vec{x}$, or

$$\vec{x}'' = A\vec{x}.$$

Many real world systems can be modeled by this equation. For simplicity, we will only talk about the given masses-and-springs problem. We try a solution of the form

$$\vec{x} = \vec{v}e^{\alpha t}.$$

We compute that for this guess, $\vec{x}'' = \alpha^2\vec{v}e^{\alpha t}$. We plug our guess into the equation and get

$$\alpha^2\vec{v}e^{\alpha t} = A\vec{v}e^{\alpha t}.$$

We divide by $e^{\alpha t}$ to arrive at $\alpha^2\vec{v} = A\vec{v}$. Hence if α^2 is an eigenvalue of A and $\vec{v}$ is a corresponding eigenvector, we have found a solution.

In our example, and in other common applications, A has only real negative eigenvalues (and possibly a zero eigenvalue). So we study only this case. When an eigenvalue λ is negative, it means that $\alpha^2 = \lambda$ is negative. Hence there is some real number ω such that $-\omega^2 = \lambda$. Then $\alpha = \pm i\omega$. The solution we guessed was

$$\vec{x} = \vec{v}\big(\cos(\omega t) + i\sin(\omega t)\big).$$

By taking the real and imaginary parts (note that $\vec{v}$ is real), we find that $\vec{v}\cos(\omega t)$ and $\vec{v}\sin(\omega t)$ are linearly independent solutions.

If an eigenvalue is zero, it turns out that both $\vec{v}$ and $\vec{v}t$ are solutions, where $\vec{v}$ is an eigenvector corresponding to the eigenvalue 0.

***Exercise* 6.5.2:** *Show that if A has a zero eigenvalue and $\vec{v}$ is a corresponding eigenvector, then $\vec{x} = \vec{v}(a + bt)$ is a solution of $\vec{x}'' = A\vec{x}$ for arbitrary constants a and b.*

Theorem 6.5.1. *Let A be an $n \times n$ matrix with n distinct real negative eigenvalues we denote by $-\omega_1^2 > -\omega_2^2 > \cdots > -\omega_n^2$, and corresponding eigenvectors by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. If A is invertible (that is, if $\omega_1 > 0$), then*

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} \vec{v}_i \big(a_i \cos(\omega_i t) + b_i \sin(\omega_i t)\big),}$$

is the general solution of

$$\vec{x}'' = A\vec{x},$$

for some arbitrary constants a_i and b_i. If A has a zero eigenvalue, that is $\omega_1 = 0$, and all other eigenvalues are distinct and negative, then the general solution can be written as

$$\boxed{\vec{x}(t) = \vec{v}_1(a_1 + b_1 t) + \sum_{i=2}^{n} \vec{v}_i \big(a_i \cos(\omega_i t) + b_i \sin(\omega_i t)\big).}$$

We use this solution and the setup from the introduction of this section even when some of the masses and springs are missing. For example, when there are only 2 masses and only 2 springs, simply take only the equations for the two masses and set all the spring constants for the springs that are missing to zero.

6.5.2 Examples

Example 6.5.1. Suppose we have the system in Figure 6.11, with $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, and $k_2 = 2$.

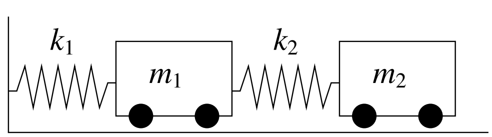


Figure 6.11: System of masses and springs.

The equations we write down are

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -(4+2) & 2 \\ 2 & -2 \end{pmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \vec{x}.$$

We find the eigenvalues of A to be $\lambda = -1, -4$ (exercise). We find corresponding eigenvectors to be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively (exercise).

We check the theorem and note that $\omega_1 = 1$ and $\omega_2 = 2$. Hence the general solution is

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (a_2 \cos(2t) + b_2 \sin(2t)).$$

The two terms in the solution represent the two so-called *natural* or *normal modes of oscillation*. And the two (angular) frequencies are the *natural frequencies*. The two modes are plotted in Figure 6.12.

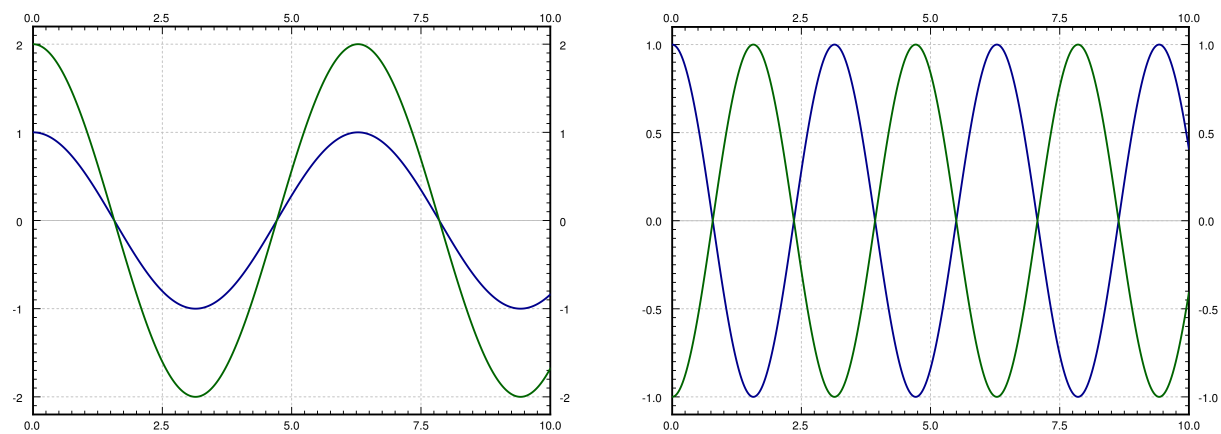


Figure 6.12: The two modes of the mass-spring system. In the left plot the masses are moving in unison and in the right plot are masses moving in the opposite direction.

Let us write the solution as

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_1 \cos(t - \alpha_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} c_2 \cos(2t - \alpha_2).$$

The first term,

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} c_1 \cos(t - \alpha_1) = \begin{pmatrix} c_1 \cos(t - \alpha_1) \\ 2c_1 \cos(t - \alpha_1) \end{pmatrix},$$

corresponds to the mode where the masses move synchronously in the same direction.

The second term,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} c_2 \cos(2t - \alpha_2) = \begin{pmatrix} c_2 \cos(2t - \alpha_2) \\ -c_2 \cos(2t - \alpha_2) \end{pmatrix},$$

corresponds to the mode where the masses move synchronously but in opposite directions.

The general solution is a combination of the two modes. That is, the initial conditions determine the amplitude and phase shift of each mode. ■

Example 6.5.2. We have two toy rail cars. Car 1 of mass 2 kilograms is traveling at 3 meters per second towards the second rail car of mass 1 kg. There is a bumper on the second rail car that engages at the moment the cars hit (it connects to two cars) and does not let go. The bumper acts like a spring of spring constant $k = 2$ newton meters. The second car is 10 meters from a wall. See Figure 6.13.

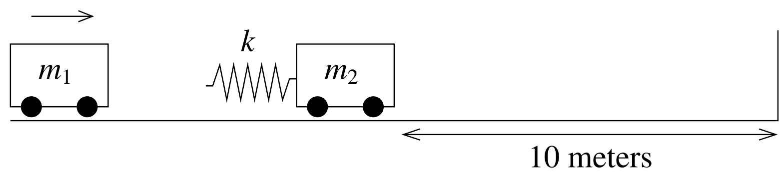


Figure 6.13: The crash of two rail cars.

We want to ask several questions. At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

OK, let us first set the system up. Let $t = 0$ be the time when the two cars link up. Let x_1 be the displacement of the first car from the position at $t = 0$, and let x_2 be the displacement of the second car from its original location. Then the time when $x_2(t) = 10$ is exactly the time when impact with wall occurs. For this t, $x_2'(t)$ is the speed at impact. This system acts just like the system of the previous example but without k_1. Hence the equation is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \vec{x}.$$

or

$$\vec{x}'' = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \vec{x}.$$

We compute the eigenvalues of A. It is not hard to see that the eigenvalues are 0 and -3 (exercise). Furthermore, eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ respectively (exercise). Then $\omega_2 = \sqrt{3}$ and by the second part of the theorem we find our general solution to be

$$\begin{aligned} \vec{x} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (a_1 + b_1 t) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \big(a_2 \cos(\sqrt{3}\, t) + b_2 \sin(\sqrt{3}\, t) \big) \\ &= \begin{pmatrix} a_1 + b_1 t + a_2 \cos(\sqrt{3}\, t) + b_2 \sin(\sqrt{3}\, t) \\ a_1 + b_1 t - 2a_2 \cos(\sqrt{3}\, t) - 2b_2 \sin(\sqrt{3}\, t). \end{pmatrix} \end{aligned}$$

We now apply the initial conditions. First the cars start at position 0 so $x_1(0) = 0$ and $x_2(0) = 0$. The first car is traveling at 3 meters per second, so $x_1'(0) = 3$ and the second car starts at rest, so $x_2'(0) = 0$. The first conditions says

$$\vec{0} = \vec{x}(0) = \begin{pmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{pmatrix}.$$

It is not hard to see that $a_1 = a_2 = 0$. We set $a_1 = 0$ and $a_2 = 0$ in $\vec{x}(t)$ and differentiate to get

$$\vec{x}\,'(t) = \begin{pmatrix} b_1 + \sqrt{3}\, b_2 \cos(\sqrt{3}\, t) \\ b_1 - 2\sqrt{3}\, b_2 \cos(\sqrt{3}\, t) \end{pmatrix}.$$

So

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \vec{x}\,'(0) = \begin{pmatrix} b_1 + \sqrt{3}\, b_2 \\ b_1 - 2\sqrt{3}\, b_2 \end{pmatrix}.$$

Solving these two equations we find $b_1 = 2$ and $b_2 = \frac{1}{\sqrt{3}}$. Hence the position of our cars is (until the impact with the wall)

$$\vec{x} = \begin{pmatrix} 2t + \frac{1}{\sqrt{3}} \sin(\sqrt{3}\, t) \\ 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t) \end{pmatrix}.$$

Note how the presence of the zero eigenvalue resulted in a term containing t. This means that the carts will be traveling in the positive direction as time grows, which is what we expect.

What we are really interested in is the second expression, the one for x_2. We have $x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t)$. See Figure 6.14 on the next page for the plot of x_2 versus time.

Just from the graph we can see that time of impact will be a little more than 5 seconds from time zero. For this we solve the equation $10 = x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t)$. Using a computer (or even a graphing calculator) we find that $t_{\text{impact}} \approx 5.22$ seconds.

As for the speed we note that $x_2' = 2 - 2\cos(\sqrt{3}\, t)$. At time of impact (5.22 seconds from $t = 0$) we get that $x_2'(t_{\text{impact}}) \approx 3.85$.

The maximum speed is the maximum of $2 - 2\cos(\sqrt{3}\, t)$, which is 4. We are traveling at almost the maximum speed when we hit the wall.

Suppose that Bob is a tiny person sitting on car 2. Bob has a Martini in his hand and would like not to spill it. Let us suppose Bob would not spill his Martini when the first car links up with car 2, but if car 2 hits the wall at any speed greater than zero, Bob will spill his drink. Suppose Bob can move car 2 a few meters towards or away from the wall (he cannot go all the way to the wall, nor can he get out of the way of the first car). Is there a “safe” distance for him to be at? A distance such that the impact with the wall is at zero speed?

The answer is yes. Looking at Figure 6.14 on the facing page, we note the “plateau” between $t = 3$ and $t = 4$. There is a point where the speed is zero. To find it we need to solve $x_2'(t) = 0$. This is when $\cos(\sqrt{3}\, t) = 1$ or in other words when $t = \frac{2\pi}{\sqrt{3}}, \frac{4\pi}{\sqrt{3}}, \ldots$ and so on. We plug in the first value to obtain $x_2\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \approx 7.26$. So a “safe” distance is about 7 and a quarter meters from the wall.

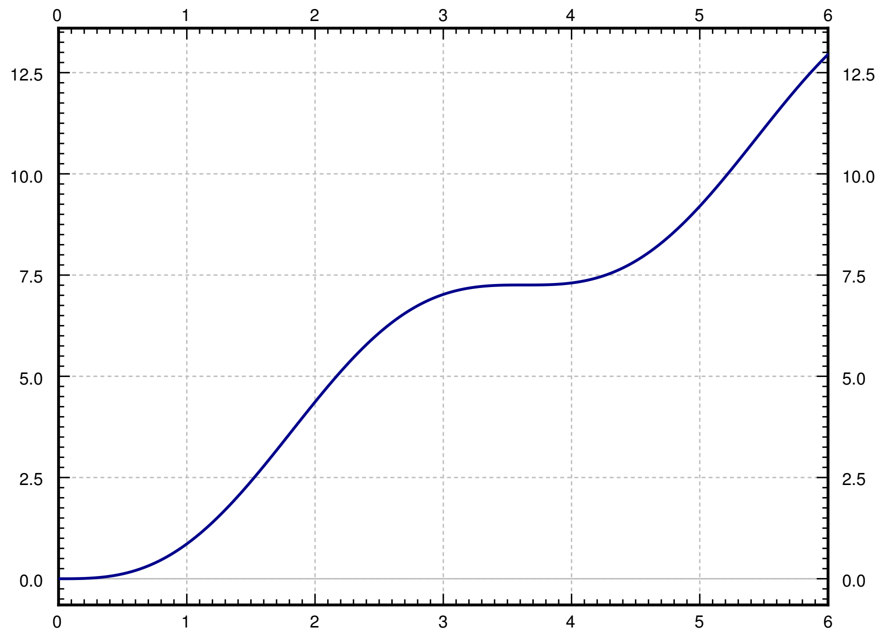


Figure 6.14: Position of the second car in time (ignoring the wall).

Alternatively Bob could move away from the wall towards the incoming car 2 where another safe distance is $\frac{8\pi}{\sqrt{3}} \approx 14.51$ and so on, using all the different t such that $x_2'(t) = 0$. Of course $t = 0$ is always a solution here, corresponding to $x_2 = 0$, but that means standing right at the wall. ■

6.5.3 Forced oscillations

Finally we move to forced oscillations. Suppose that now our system is

$$\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t). \tag{6.3}$$

That is, we are adding periodic forcing to the system in the direction of the vector $\vec{F}$.

As before, this system just requires us to find one particular solution $\vec{x}_p$, add it to the general solution of the associated homogeneous system $\vec{x}_c$, and we will have the general solution to (6.3). Let us suppose that ω is not one of the natural frequencies of $\vec{x}'' = A\vec{x}$, then we can guess

$$\vec{x}_p = \vec{c}\cos(\omega t),$$

where $\vec{c}$ is an unknown constant vector. Note that we do not need to use sine since there are only second derivatives. We solve for $\vec{c}$ to find $\vec{x}_p$. This is really just the method of *undetermined coefficients* for systems. Let us differentiate $\vec{x}_p$ twice to get

$$\vec{x}_p'' = -\omega^2\vec{c}\cos(\omega t).$$

Plug $\vec{x}_p$ and $\vec{x}_p''$ into the equation:

$$-\omega^2 \vec{c} \cos(\omega t) = A\vec{c} \cos(\omega t) + \vec{F} \cos(\omega t).$$

We cancel out the cosine and rearrange the equation to obtain

$$(A + \omega^2 I)\vec{c} = -\vec{F}.$$

So

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}).$$

Of course this is possible only if $(A + \omega^2 I) = (A - (-\omega^2)I)$ is invertible. That matrix is invertible if and only if $-\omega^2$ is not an eigenvalue of A. That is true if and only if ω is not a natural frequency of the system.

Example 6.5.3. Let us take the example in Figure 6.11 on page 253 with the same parameters as before: $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, and $k_2 = 2$. Now suppose that there is a force $2\cos(3t)$ acting on the second cart.

The equation is

$$\vec{x}'' = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(3t).$$

We solved the associated homogeneous equation before and found the complementary solution to be

$$\vec{x}_c = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \big(a_1 \cos(t) + b_1 \sin(t)\big) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \big(a_2 \cos(2t) + b_2 \sin(2t)\big).$$

The natural frequencies are 1 and 2. Hence as 3 is not a natural frequency, we can try $\vec{c}\cos(3t)$. We invert $(A + 3^2 I)$:

$$\left(\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} + 3^2 I\right)^{-1} = \begin{pmatrix} 6 & 1 \\ 2 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{pmatrix}.$$

Hence,

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}) = \begin{pmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{pmatrix}.$$

Combining with what we know the general solution of the associated homogeneous problem to be, we get that the general solution to $\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \big(a_1 \cos(t) + b_1 \sin(t)\big) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \big(a_2 \cos(2t) + b_2 \sin(2t)\big) + \begin{pmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{pmatrix} \cos(3t).$$

The constants a_1, a_2, b_1, and b_2 must then be solved for given any initial conditions. ■

If ω is a natural frequency of the system *resonance* occurs because we will have to try a particular solution of the form

$$\vec{x}_p = \vec{c}\, t \sin(\omega t) + \vec{d} \cos(\omega t).$$

That is assuming that the eigenvalues of the coefficient matrix are distinct. Next, note that the amplitude of this solution grows without bound as t grows.

Exercises

Exercise* 6.5.3*: *Find a particular solution to*

$$\vec{x}'' = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t).$$

Exercise* 6.5.4** (challenging):*** *Let us take the example in Figure 6.11 on page 253 with the same parameters as before: $m_1 = 2$, $k_1 = 4$, and $k_2 = 2$, except for m_2, which is unknown. Suppose that there is a force* $\cos(5t)$ *acting on the first mass. Find an m_2 such that there exists a particular solution where the first mass does not move.*

Note: This idea is called dynamic damping. *In practice there will be a small amount of damping and so any transient solution will disappear and after long enough time, the first mass will always come to a stop.*

Exercise* 6.5.5*: *Let us take the Example 6.5.2 on page 255, but that at time of impact, cart 2 is moving to the left at the speed of 3 meters per second. a) Find the behavior of the system after linkup. b) Will the second car hit the wall, or will it be moving away from the wall as time goes on? c) At what speed must the first car be traveling at for the system to essentially stay in place after linkup?*

Exercise* 6.5.6*: *Let us take the example in Figure 6.11 on page 253 with parameters* $m_1 = m_2 = 1$, $k_1 = k_2 = 1$. *Does there exist a set of initial conditions for which the first cart moves but the second cart does not? If so, find those conditions. If not, argue why not.*

Exercise* 6.5.101*: *Find the general solution to* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 6 & -3 \end{pmatrix} \vec{x} + \begin{pmatrix} \cos(2t) \\ 0 \\ 0 \end{pmatrix}$.

Exercise* 6.5.102*: *Suppose there are three carts of equal mass m and connected by two springs of constant k (and no connections to walls). Set up the system and find its general solution.*

Exercise* 6.5.103*: *Suppose a cart of mass 2 kilograms is attached by a spring of constant* $k = 1$ *to a cart of mass 3 kg, which is attached to the wall by a spring also of constant* $k = 1$. *Suppose that the initial position of the first cart is 1 meter in the positive direction from the rest position, and the second mass starts at the rest position. The masses are not moving and are let go. Find the position of the second mass as a function of time.*

Chapter 7

Power series methods

Recall that Taylor series allows us to express many functions as a polynomial possessing infinitely many terms, and gave us a framework for finding the coefficients of that polynomial. In this chapter, we assume that it is possible to express the solution to a differential equation as a polynomial possessing infinitely many terms, and we develop a framework for finding the coefficients.

7.1 Preliminaries

Further reading: §7.2-7.3 in [L]

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

and solve for the a_k to try to obtain a solution defined in some interval around x_0. The point x_0 is called an *ordinary point* if $p(x_0) \neq 0$. That is, the functions

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are defined for x near x_0. If $p(x_0) = 0$, then we say x_0 is a *singular point*.

If, furthermore, the limits

$$\lim_{x \to x_0} (x - x_0)\frac{q(x)}{p(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{r(x)}{p(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

Example 7.1.1. Consider $y'' - y = 0$. The point $x_0 = 0$ is an ordinary point. For this DE, every point is an ordinary point in fact, as the equation is constant coefficient. ■

Example 7.1.2. Consider

$$x^2y'' + x(1+x)y' + (\pi + x^2)y = 0.$$

For this DE, $p(x) = x^2$ and $q(x) = x(1+x)$. Let $x_0 = 0$. Since $p(x_0) = 0$, x_0 is not an ordinary point. Write

$$\begin{aligned}
\lim_{x\to 0} x\frac{q(x)}{p(x)} &= \lim_{x\to 0} x\frac{x(1+x)}{x^2} = \lim_{x\to 0} (1+x) = 1,\\
\lim_{x\to 0} x^2\frac{r(x)}{p(x)} &= \lim_{x\to 0} x^2\frac{(\pi + x^2)}{x^2} = \lim_{x\to 0} (\pi + x^2) = \pi.
\end{aligned}$$

So x_0 is a regular singular point. On the other hand if we make the slight change

$$x^2y'' + (1+x)y' + (\pi + x^2)y = 0,$$

then

$$\lim_{x\to 0} x\frac{q(x)}{p(x)} = \lim_{x\to 0} x\frac{(1+x)}{x^2} = \lim_{x\to 0} \frac{1+x}{x}$$

does not exist. The point x_0 is a singular point, but not a regular singular point. ■

With these definitions out of the way, let us practice our skills in manipulating series to verify series solutions of differential equations.

Example 7.1.3. Consider the differential equation

$$y'' - y = 0.$$

Verify that

$$y = C\sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} + D\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}$$

is a solution. We immediately notice that the proposed solution includes two power series, each scaled by an arbitrary constant C or D. This makes sense, as we're dealing with a homogeneous L2ODE.

$$\begin{aligned}
y &= C\sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} + D\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}\\
&= C\left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots\right) + D\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots\right)
\end{aligned}$$

We should differentiate y with respect to x:

$$\begin{aligned} y' &= C\left(x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\dots\right)+D\left(1+\frac{1}{2}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\dots\right) \\ &= C\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,x^{2n+1}+D\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,x^{2n} \end{aligned}$$

And again:

$$\begin{aligned} y' &= C\left(1+\frac{1}{2}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\dots\right)+D\left(x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\dots\right) \\ &= C\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,x^{2n}+D\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,x^{2n+1} \end{aligned}$$

At this stage, we notice that y and y'' are the same, so clearly $y''-y=0$. ■

Remark 7.1.1. *Looking back, the equation $y''-y=0$ is fundamental. We know that its solution is $y=C\cosh x+D\sinh x$. Indeed, we chose the Taylor series of $\cosh x$ and $\sinh x$ in the solution that we verified. No wonder it worked!*

Example 7.1.4. Consider *Airy's equation**:

$$y''-xy=0.$$

In physics, this DE models Schrödinger's equation for a particle in a one-dimensional constant force field. Verify that $y=Cy_1+Dy_2$ with

$$\begin{aligned} y_1(x) &= 1+\frac{1}{(2)(3)}x^3+\frac{1}{(2)(3)(5)(6)}x^6+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} \\ y_2(x) &= x+\frac{1}{(3)(4)}x^4+\frac{1}{(3)(4)(6)(7)}x^7+\cdots=\sum_{n=0}^{\infty}\frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} \end{aligned}$$

is a solution.

Let's be more slick than in the last example. Instead of plugging $y=Cy_1+Dy_2$ into our DE, let us start with something slightly easier: plugging y_1 Also, let's rearrange the DE:

$$y''=xy.$$

We compute xy_1:

$$\begin{aligned} xy_1(x) &= x\left(1+\frac{1}{(2)(3)}x^3+\frac{1}{(2)(3)(5)(6)}x^6+\cdots\right)=x\sum_{n=0}^{\infty}\frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} \\ &= x+\frac{1}{(2)(3)}x^4+\frac{1}{(2)(3)(5)(6)}x^7+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n+1} \end{aligned}$$

*Named after the English mathematician Sir George Biddell Airy (1801 – 1892).

Let us differentiate y_1 (twice):

$$y_1'(x) = \frac{3}{(2)(3)}x^2 + \frac{6}{(2)(3)(5)(6)}x^5 + \frac{9}{(2)(3)(5)(6)(8)(9)}x^8 + \cdots = \sum_{n=1}^{\infty} \frac{3n}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n-1}$$

$$= \frac{1}{2}x^2 + \frac{1}{(2)(3)(5)}x^5 + \frac{1}{(2)(3)(5)(6)(8)}x^8 + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2)(3)(5)(6)\cdots(3n-1)}x^{3n-1}$$

$$y_1''(x) = x + \frac{5}{(2)(3)(5)}x^4 + \frac{8}{(2)(3)(5)(6)(8)}x^7 + \cdots = \sum_{n=1}^{\infty} \frac{3n-1}{(2)(3)(5)(6)\cdots(3n-1)}x^{3n-2}$$

$$= x + \frac{1}{(2)(3)}x^4 + \frac{1}{(2)(3)(5)(6)}x^7 + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2)(3)(5)(6)\cdots(3n-4)(3n-3)}x^{3n-2}$$

We should re-index our summation notation for $y_1''(x)$, so that counting begins at zero. Set $m = n - 1$. Then:

$$y_1''(x) = x + \frac{1}{(2)(3)}x^4 + \frac{1}{(2)(3)(5)(6)}x^7 + \cdots = \sum_{m=0}^{\infty} \frac{1}{(2)(3)(5)(6)\cdots(3m-1)(3m)}x^{3m+1}$$

It works! The expressions for xy_1 and y_1'' are equal, so y_1 solves the DE.

Using very similar steps, y_2 is also verified as a solution.

Now, invoke Theorem 5.1.1. We claim that, since Airy's equation is an L2ODE, the superposition principle implies that $y = Cy_1 + Dy_2$ is also a solution for arbitrary constants C and D. And we're done! ■

Exercise* 7.1.1*: Complete the missing steps in the above example and verify that* y_2 *solves the differential equation.

Exercises

In each equation, classify the point $x = 0$ as *ordinary*, *regular singular*, or *singular but not regular singular*.

Exercise* 7.1.2*: $x^2(1 + x^2)y'' + xy = 0$

Exercise* 7.1.3*: $x^2y'' + y' + y = 0$

Exercise* 7.1.4*: $xy'' + x^3y' + y = 0$

Exercise* 7.1.5*: $xy'' + xy' - e^x y = 0$

Exercise* 7.1.6*: $x^2y'' + x^2y' + x^2y = 0$

Exercise* 7.1.7*: *Consider the differential equation $y'' + y = 0$ about $x_0 = 0$. Write down the general solution to this fundamental DE. Use Section 9.6 to obtain the Taylor series for this solution. Apply the ratio test to determine its interval of convergence. Verify that the Taylor series also solves the DE.*

Exercise* 7.1.8*: *Consider the differential equation $y' - y = 5$ about $x_0 = 0$. Verify that $y = Ce^x - 5$ solves the DE. Use Section 9.6 to obtain the Taylor series for this solution. Apply the ratio test to determine its interval of convergence. Verify that the Taylor series also solves the DE. Finally, enforce the initial condition $y(0) = -4$.*

Exercise* 7.1.9*: *Consider the differential equation $y' - xy = 0$ about $x_0 = 0$. Verify that $y = Ce^{x^2}$ solves the DE. Use Section 9.6 to obtain the Taylor series for this solution. Apply the ratio test to determine its interval of convergence. Verify that the Taylor series also solves the DE. Finally, enforce the initial condition $y(0) = 2$.*

Exercise* 7.1.10*: Bessel's equation *of zeroth order is*

$$x^2y'' + xy' + x^2y = 0.$$

The Bessel function of the first kind *of zeroth order*

$$J_o(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} \left(\frac{x}{2}\right)^{2k}$$

solves this DE. That is, verify that $y = CJ_o(x)$ is a solution. What is this solution's interval of convergence?

Exercise* 7.1.11** (Hermite equation of order 0):*** *In Exercise 1.1.22, we verified that $y_1 = 1$ solves the zeroth-order Hermite equation:*

$$y'' - 2xy' = 0.$$

A second solution is the odd infinite polynomial

$$y_2 = \sum_{k=0}^{\infty} 2^k \frac{(1)(3)(5)\cdots(2k-1))}{(2k+1)!} x^{2k+1}.$$

Verify this solution. Then, argue that $Cy_1 + Dy_2$ must be the general solution to the DE.

In each equation, classify the point $x = 0$ as *ordinary*, *regular singular*, or *singular but not regular singular*.

Exercise* 7.1.101*: $y'' + y = 0$

Exercise* 7.1.102*: $x^3y'' + (1 + x)y = 0$

Exercise* 7.1.103*: $xy'' + x^5y' + y = 0$

Exercise* 7.1.104*: $\sin(x)y'' - y = 0$

Exercise* 7.1.105*: $\cos(x)y'' - \sin(x)y = 0$

7.2 Solutions about ordinary points

Attribution: §7.2 in [L]
Further reading: §8.2 in [EP], §5.2-5.3 in [BD]

Handling singular points is harder than ordinary points and so we now focus only on ordinary points.

Example 7.2.1. Let us revisit Example 7.1.3 involving the DE

$$y'' - y = 0.$$

Suppose that we didn't know the series solution to this DE. Let us hypothesize that a power series solution near $x_0 = 0$ can be found. We already noted in the previous section that this is an ordinary point.

We assume a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^k$$

where the values of the coefficients a_k are not yet known. If we differentiate, the $k = 0$ term is a constant and hence disappears. We therefore get

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We reindex the series (replace k with $k + 2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2} x^k.$$

Now we plug y and y'' into the differential equation

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2} x^k \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)\, a_{k+2} x^k - a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)\, a_{k+2} - a_k \right) x^k. \end{aligned}$$

As $y'' - y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)\,(k+1)\,a_{k+2} - a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The above equation is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was. They can be arbitrary. But once we pick a_0 and a_1, then all other coefficients are determined by the recurrence relation.

Let us see what the coefficients must be. First, a_0 and a_1 are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \ldots$$

So we note that for even k, that is $k = 2n$ we get

$$a_k = a_{2n} = \frac{a_0}{(2n)!},$$

and for odd k, that is $k = 2n+1$ we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}.$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \left(\frac{a_0}{(2n)!}\, x^{2n} + \frac{a_1}{(2n+1)!}\, x^{2n+1} \right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}.$$

■

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 7.2.2. Let us do a more complicated example, by revisiting Airy's equation

$$y'' - xy = 0$$

that we previously examined in Example 7.1.4 near the point $x_0 = 0$. Suppose that we don't know its series solution. Note that $x_0 = 0$ is an ordinary point. We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}.$$

We plug y into the equation

$$\begin{aligned} 0 = y'' - xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2} \right) - x \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2} \right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1} \right). \end{aligned}$$

We reindex to make things easier to sum

$$\begin{aligned} 0 = y'' - xy &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k \right) - \left(\sum_{k=1}^{\infty} a_{k-1} x^k \right). \\ &= 2a_2 + \sum_{k=1}^{\infty} \Big((k+2)\,(k+1)\,a_{k+2} - a_{k-1} \Big)\, x^k. \end{aligned}$$

Again $y'' - xy$ is supposed to be 0 so first we notice that $a_2 = 0$ and also

$$(k+2)\,(k+1)\,a_{k+2} - a_{k-1} = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}.$$

Now we jump in steps of three. First we notice that since $a_2 = 0$ we must have that, $a_5 = 0$, $a_8 = 0$, $a_{11} = 0$, etc.... In general $a_{3n+2} = 0$.

The constants a_0 and a_1 are arbitrary and we obtain

$$a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}, \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}, \quad \ldots$$

For a_k where k is a multiple of 3, that is $k = 3n$ we notice that

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}.$$

For a_k where $k = 3n+1$, we notice

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}.$$

In other words, if we write down the series for y we notice that it has two parts

$$\begin{aligned} y &= \left(a_0 + \frac{a_0}{(2)(3)} x^3 + \frac{a_0}{(2)(3)(5)(6)} x^6 + \cdots + \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)} x^{3n} + \cdots \right) \\ &\quad + \left(a_1 x + \frac{a_1}{(3)(4)} x^4 + \frac{a_1}{(3)(4)(6)(7)} x^7 + \cdots + \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)} x^{3n+1} + \cdots \right) \\ &= a_0 \left(1 + \frac{1}{(2)(3)} x^3 + \frac{1}{(2)(3)(5)(6)} x^6 + \cdots + \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)} x^{3n} + \cdots \right) \\ &\quad + a_1 \left(x + \frac{1}{(3)(4)} x^4 + \frac{1}{(3)(4)(6)(7)} x^7 + \cdots + \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)} x^{3n+1} + \cdots \right). \end{aligned}$$

We define

$$y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{(2)(3)(5)(6)}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n}$$
$$y_2(x) = x + \frac{1}{12}x^4 + \frac{1}{(3)(4)(6)(7)}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1}$$

and write the general solution to the equation as $y(x) = a_0 y_1(x) + a_1 y_2(x)$. Notice from the power series that $y_1(0) = 1$ and $y_2(0) = 0$. Also, $y_1'(0) = 0$ and $y_2'(0) = 1$. Therefore $y(x)$ is a solution that satisfies the initial conditions $y(0) = a_0$ and $y'(0) = a_1$. ■

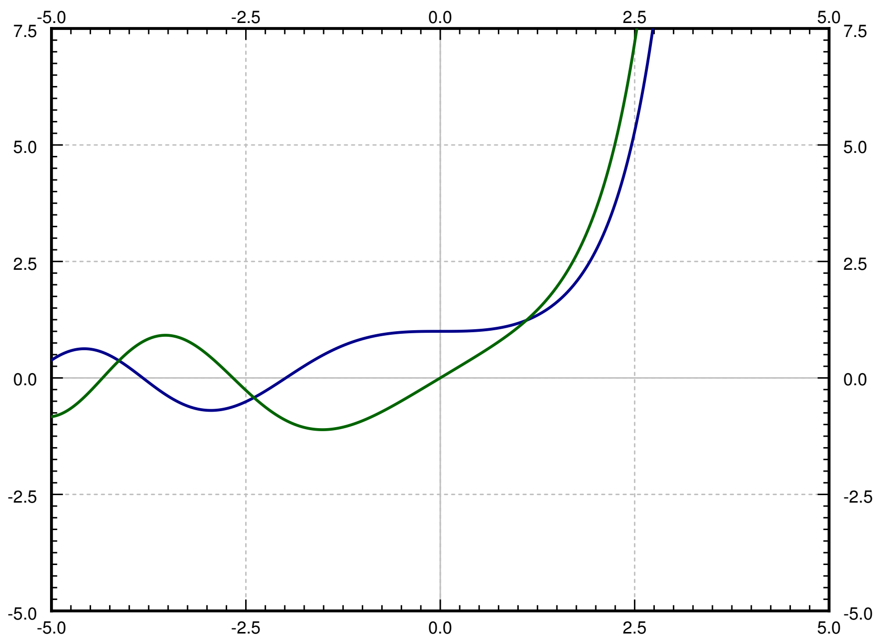


Figure 7.1: The two solutions y_1 and y_2 to Airy's equation.

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See Figure 7.1 for the plot of the solutions y_1 and y_2. These functions have many interesting properties. For example, they are oscillatory for negative x (like solutions to $y'' + y = 0$) and for positive x they grow without bound (like solutions to $y'' - y = 0$).

Sometimes a solution may turn out to be a polynomial.

Example 7.2.3. We previously encountered Hermite's equation of degree n

$$y'' - 2xy' + 2ny = 0$$

of order n in Exercise 1.1.22, Exercise 5.3.12, and Exercise 7.1.11. Let us apply our series solution method to solve Hermite's equations around the point $x_0 = 0$ that is an ordinary point. We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$\begin{aligned} y' &= \sum_{k=1}^{\infty} k a_k x^{k-1}, \\ y'' &= \sum_{k=2}^{\infty} k\,(k-1)\, a_k x^{k-2}. \end{aligned}$$

Now we plug into the equation

$$\begin{aligned} 0 = y'' - 2xy' + 2ny &= \left(\sum_{k=2}^{\infty} k\,(k-1)\, a_k x^{k-2}\right) - 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) + 2n\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\, a_k x^{k-2}\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k\right) \\ &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)\,(k+1)\, a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(2na_0 + \sum_{k=1}^{\infty} 2n a_k x^k\right) \\ &= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} \big((k+2)\,(k+1)\, a_{k+2} - 2k a_k + 2n a_k\big) x^k. \end{aligned}$$

As $y'' - 2xy' + 2ny = 0$ we have

$$(k+2)\,(k+1)\, a_{k+2} + (-2k+2n) a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{(2k-2n)}{(k+2)(k+1)} a_k.$$

This recurrence relation actually includes $a_2 = -na_0$ (which comes about from $2a_2 + 2na_0 = 0$). Again a_0 and a_1 are arbitrary.

$$\begin{aligned} a_2 &= \frac{-2n}{(2)(1)} a_0, \\ a_3 &= \frac{2(1-n)}{(3)(2)} a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)} a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)} a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)} a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)} a_1, \quad \ldots \end{aligned}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \cdots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \cdots. \end{aligned}$$

We then write

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

We also notice that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond a certain degree are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example, if $n = 4$, then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4.$$

■

Exercises

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

Exercise* 7.2.1*: *Use a power series around* $t_0 = 0$ *to find a recurrence relation for the coefficients that satisfy* $y'' + ty' + y = 0$.

Exercise* 7.2.2*: *Show that the coefficients of the power series to the initial value problem*

$$y'' - y' - y = 0,\ y(0) = 0,\ y'(0) = 1$$

have the form $a_n = \dfrac{F_n}{n!}$ *where* F_n *is the nth term of the* Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ *(each term is the sum of the previous two terms).*

Use power series methods to solve the differential equation using the provided point. Apply the ratio test to determine the interval of convergence.

Exercise* 7.2.3*: $y'' + y = 0$ *about* $x_0 = 0$
Compare with your answer to Exercise 7.1.7.

Exercise* 7.2.4*: $y'' + 4xy = 0$ *about* $x_0 = 0$

Exercise* 7.2.5*: $y'' - xy = 0$ *about* $x_0 = 1$
Your answer should resemble that of Example 7.2.2.

Exercise* 7.2.6*: $y'' + x^2 y = 0$ *about* $x_0 = 0$

Exercise* 7.2.7*: $y' - y = 5$ *with* $y(0) = 4$ *about* $x_0 = 0$
Compare with your answer to Exercise 7.1.8.

Exercise* 7.2.8*: $y' - xy = 0$ *with* $y(0) = 2$ *about* $x_0 = 0$
Compare with your answer to Exercise 7.1.9.

Exercise* 7.2.9** (Chebyshev's equation of order p):*** *We previously examined Chebyshev's equation* $(1 - x^2)y'' - xy' + p^2 y = 0$ *of order p in Exercise 1.1.23 and Exercise 5.3.11. Solve this equation using power series methods at* $x_0 = 0$*. For what p is there a polynomial solution? Determine the interval of convergence.*

Exercise* 7.2.10*: *Find a polynomial solution to* $(x^2 + 1)y'' - 2xy' + 2y = 0$ *using power series methods.*

Exercise* 7.2.11*: *Use power series methods to solve* $(1-x)y'' + y = 0$ *at the point* $x_0 = 0$*. Use this solution to find a solution for* $xy'' + y = 0$ *around the point* $x_0 = 1$*.*

Exercise* 7.2.101*: *Use power series methods to solve* $y'' + 2x^3 y = 0$*, at the point* $x_0 = 0$*.*

Exercise* 7.2.102** (challenging):*** *We can also use power series methods in nonhomogeneous equations. Use power series methods to solve* $y'' - xy = \frac{1}{1-x}$ *at the point* $x_0 = 0$*. Hint: Recall the geometric series. Now solve for the initial condition* $y(0) = 0$*,* $y'(0) = 0$*.*

Exercise* 7.2.103*: *Attempt to solve* $x^2 y'' - y = 0$ *at* $x_0 = 0$ *using the power series method of this section (*x_0 *is a singular point). Can you find at least one solution? Can you find more than one solution?*

7.3 Solutions about singular points and the method of Frobenius

Attribution: §7.3 in [L]
Further reading: §8.4,8.5 in [EP], §5.4-5.7 in [BD]

While behavior of ODEs at singular points is more complicated, certain singular points are not especially difficult to solve. Let us look at some examples before giving a general method. We may be lucky and obtain a power series solution using the method of the previous section, but in general we will require other strategies.

Example 7.3.1. Let us first look at a simple first order equation

$$2xy' - y = 0.$$

Note that $x = 0$ is a singular point. If we only try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) - \left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= a_0 + \sum_{k=1}^{\infty} (2ka_k - a_k)\, x^k. \end{aligned}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\, a_k$ for $k = 1, 2, 3, \dots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

Let us try $y = x^r$ for some real number r. Consequently our solution—if we can find one—may only make sense for positive x. Then $y' = rx^{r-1}$. So

$$0 = 2xy' - y = 2xrx^{r-1} - x^r = (2r-1)x^r.$$

Therefore $r = 1/2$, or in other words $y = x^{1/2}$. Multiplying by a constant, the general solution for positive x is

$$y = Cx^{1/2}.$$

If $C \neq 0$ then the derivative of the solution "blows up" at $x = 0$ (the singular point). There is only one solution that is differentiable at $x = 0$ and that's the trivial solution $y = 0$. ■

Not every problem with a singular point has a solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

Example 7.3.2. Suppose that we have the equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0,$$

and note that $x_0 = 0$ is a regular singular point. Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$\begin{aligned} y' &= \sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1}, \\ y'' &= \sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2}. \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} 0 &= 4x^2y'' - 4x^2y' + (1-2x)y \\ &= 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2} \right) - 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1} \right) + (1-2x) \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 2a_k x^{k+r+1} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 4(k+r-1)\, a_{k-1} x^{k+r} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} \right) \\ &= r\,(r-1)\, a_0 x^r + \left(\sum_{k=1}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 4(k+r-1)\, a_{k-1} x^{k+r} \right) \\ &\quad + a_0 x^r + \left(\sum_{k=1}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} \right) \\ &= 4r(r-1)\, a_0 x^r + a_0 x^r + \sum_{k=1}^{\infty} \Big(4(k+r)\,(k+r-1)\, a_k - 4(k+r-1)\, a_{k-1} + a_k - 2a_{k-1} \Big)\, x^{k+r} \\ &= \big(4r(r-1)+1\big)\, a_0 x^r + \sum_{k=1}^{\infty} \Big(\big(4(k+r)\,(k+r-1)+1\big)\, a_k - \big(4(k+r-1)+2\big)\, a_{k-1} \Big)\, x^{k+r}. \end{aligned}$$

To have a solution we must first have $(4r(r-1)+1)\,a_0 = 0$. Supposing that $a_0 \neq 0$ we obtain

$$4r(r-1)+1 = 0.$$

This equation is called the *indicial equation*. This particular indicial equation has a double root at $r = 1/2$.

OK, so we know what r has to be. That knowledge we obtained simply by looking at the coefficient of x^r. All other coefficients of x^{k+r} also should be zero so

$$\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} = 0.$$

If we plug in $r = 1/2$ and solve for a_k we get

$$a_k = \frac{4(k+1/2-1)+2}{4(k+1/2)\,(k+1/2-1)+1}\,a_{k-1} = \frac{1}{k}\,a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1}a_0 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}, \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}, \qquad \ldots$$

Extrapolating, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions.

We have one solution, let us call it $y_1 = x^{1/2}e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1, and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We now differentiate this equation, substitute into the differential equation and solve for b_k. A long computation ensues and we obtain some recursion relation for b_k. The reader can (and should) try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \qquad b_2 = \frac{2b_1 - 1}{4}, \qquad b_3 = \frac{6b_2 - 1}{18}, \qquad \ldots$$

We then fix b_0 and obtain a solution y_2. Then we write the general solution as $y = Ay_1 + By_2$. ■

Let us now discuss the general *Method of Frobenius**. This method applies when we have a regular singular point. Let us only consider the method at the point $x = 0$ for simplicity. The main idea is the following theorem.

Theorem 7.3.1 (Method of Frobenius). *Suppose that*

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{7.1}$$

has a regular singular point at $x = 0$, then there exists at least one solution of the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k.$$

A solution of this form is called a Frobenius-type solution.

The method usually breaks down like this.

(i) We seek a Frobenius-type solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation (7.1). We collect terms and write everything as a single series.

(ii) The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.

(iii) If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we have two linearly independent Frobenius-type solutions. Using the first root, we plug in

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and we solve for all a_k to obtain the first solution. Then using the second root, we plug in

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k,$$

and solve for all b_k to obtain the second solution.

*Named after the German mathematician Ferdinand Georg Frobenius (1849 – 1917).

(iv) If the indicial equation has a doubled root r, then there we find one solution

$$y_1 = x^r \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^r \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into equation (7.1) and solving for the constants b_k.

(v) If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C (\ln x) y_1,$$

where we plug y_2 into (7.1) and solve for the constants b_k and C.

(vi) Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

results in a complex-valued function—all the a_k are complex numbers. We obtain our two linearly independent solutions* by taking the real and imaginary parts of y.

The main idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or even a different method such as reduction of order (Section 5.3) to obtain a second solution.

*See Joseph L. Neuringera, *The Frobenius method for complex roots of the indicial equation*, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

Exercises

Apply the method of Frobenius to the differential equation about $x_0 = 0$.

Exercise* 7.3.1*: $x^2y'' + xy' + (1 + x)y = 0$

Exercise* 7.3.2*: $xy'' - y = 0$

Exercise* 7.3.3*: $y'' + \frac{1}{x}y' - xy = 0$

Find the general solution to the differential equation about $x_0 = 0$.

Exercise* 7.3.4*: $2xy'' + y' - x^2y = 0$

Exercise* 7.3.5*: $x^2y'' - xy' - y = 0$

Exercise* 7.3.101*: $x^2y'' - y = 0$

Exercise* 7.3.102*: $x^2y'' + (x - 3/4)y = 0$

Exercise* 7.3.103** (Tricky):*** $x^2y'' - xy' + y = 0$

7.4 Bessel functions

Attribution: §7.3 in [L]

An important class of functions that arises commonly in physics are the *Bessel functions**. These functions appear when solving the wave equation in two and three dimensions. We previously encountered the zeroth-order Bessel equation in Exercise 7.1.10. *Bessel's equation* of order p is

$$x^2y'' + xy' + \left(x^2 - p^2\right)y = 0$$

where $p \in \mathbb{R}$, although integers and multiples of $1/2$ are most important in applications.

When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

Therefore we obtain two roots $r_1 = p$ and $r_2 = -p$. If p is not an integer following the method of Frobenius and setting $a_0 = 1$, we obtain linearly independent solutions of the form

$$\begin{aligned} y_1 &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k+p)(k-1+p)\cdots(2+p)(1+p)}, \\ y_2 &= x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k-p)(k-1-p)\cdots(2-p)(1-p)}. \end{aligned}$$

Exercise* 7.4.1*: *Verify that the indicial equation of Bessel's equation of order p is* $(r-p)(r+p) = 0$*. Suppose that p is not an integer. Carry out the computation to obtain the solutions* y_1 *and* y_2 *above.*

Bessel functions will be convenient constant multiples of y_1 and y_2. First we must define the *gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt.$$

Notice that $\Gamma(1) = 1$. The gamma function also has a wonderful property

$$\Gamma(x+1) = x\Gamma(x).$$

From this property, one can show that $\Gamma(n) = (n-1)!$ when n is an integer, so the gamma function is a continuous version of the factorial. We compute:

$$\begin{aligned} \Gamma(k+p+1) &= (k+p)(k-1+p)\cdots(2+p)(1+p)\Gamma(1+p), \\ \Gamma(k-p+1) &= (k-p)(k-1-p)\cdots(2-p)(1-p)\Gamma(1-p). \end{aligned}$$

*Named after the German astronomer and mathematician Friedrich Wilhelm Bessel (1784 – 1846).

***Exercise* 7.4.2:** *Verify the above identities using* $\Gamma(x+1) = x\Gamma(x)$.

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$J_p(x) = \frac{1}{2^p\Gamma(1+p)} y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$
$$J_{-p}(x) = \frac{1}{2^{-p}\Gamma(1-p)} y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}.$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

In this case it turns out that

$$J_n(x) = (-1)^n J_{-n}(x),$$

and so we do not obtain a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$ as p approaches n in the following way:

$$Y_n(x) = \lim_{p \to n} \frac{\cos(p\pi) J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

As each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p, then as we take the limit as p goes to n, $Y_n(x)$ is a solution to Bessel's equation of order n. It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n

$$y = AJ_n(x) + BY_n(x),$$

for arbitrary constants A and B. Note that $Y_n(x)$ goes to negative infinity at $x = 0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, they have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore, these functions oscillate, although they are not periodic. See Figure 7.2 on the next page for graphs of Bessel functions.

Example 7.4.1. Other equations can sometimes be solved in terms of the Bessel functions. For example, given a positive constant λ,

$$xy'' + y' + \lambda^2 xy = 0,$$

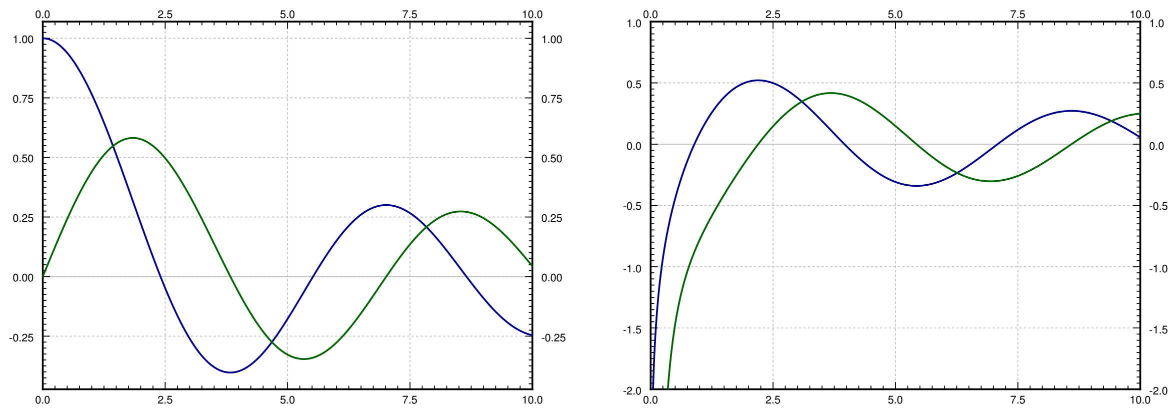


Figure 7.2: Plot of the $J_0(x)$ and $J_1(x)$ in the first graph and $Y_0(x)$ and $Y_1(x)$ in the second graph.

can be changed to $x^2y'' + xy' + \lambda^2x^2y = 0$. Then changing variables $t = \lambda x$ we obtain via chain rule the equation in y and t:

$$t^2y'' + ty' + t^2y = 0,$$

which can be recognized as Bessel's equation of order 0. Therefore the general solution is $y(t) = AJ_0(t) + BY_0(t)$, or in terms of x:

$$y = AJ_0(\lambda x) + BY_0(\lambda x).$$

This equation comes up for example when finding fundamental modes of vibration of a circular drum, but we digress. ■

Chapter 8

Laplace transform method

In this chapter we will discuss the Laplace transform*. The Laplace transform turns out to be a very efficient method to solve certain ODE problems. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere.

8.1 Preliminaries

Attribution: §6.1 in [L]
Further reading: , §10.1 in [EP], §6.1 and parts of §6.2 in [BD]

8.1.1 The transform

The Laplace transform gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

We can think of t as time and $f(t)$ as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable s is the frequency.

We can think of the *Laplace transform* as a black box. It eats functions and spits out functions in a new variable. We write $\mathcal{L}\{f(t)\} = F(s)$ for the Laplace transform of $f(t)$. It is common to write lower case letters for functions in the time domain and upper case letters for functions in the

*Just like the Laplace equation and the Laplacian, the Laplace transform is also named after Pierre-Simon, marquis de Laplace (1749 – 1827).

frequency domain. We use the same letter to denote that one function is the Laplace transform of the other. For example $F(s)$ is the Laplace transform of $f(t)$. Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t)\, dt.$$

We note that we are only considering $t \geq 0$ in the transform. Of course, if we think of t as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past). Let us compute some simple transforms.

Example 8.1.1. Suppose $f(t) = 1$, then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \lim_{h\to\infty} \left[\frac{e^{-st}}{-s}\right]_{t=0}^{h} = \lim_{h\to\infty} \left(\frac{e^{-sh}}{-s} - \frac{1}{-s}\right) = \frac{1}{s}.$$

The limit (the improper integral) only exists if $s > 0$. So $\mathcal{L}\{1\}$ is only defined for $s > 0$. ■

Example 8.1.2. Suppose $f(t) = e^{-at}$, then

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^{\infty} = \frac{1}{s+a}.$$

The limit only exists if $s + a > 0$. So $\mathcal{L}\{e^{-at}\}$ is only defined for $s + a > 0$. ■

Example 8.1.3. Suppose $f(t) = t$, then using integration by parts

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st} t\, dt \\
&= \left[\frac{-te^{-st}}{s}\right]_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}\, dt \\
&= 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} \\
&= \frac{1}{s^2}.
\end{aligned}$$

Again, the limit only exists if $s > 0$. ■

Example 8.1.4. A common function is the *unit step function*, which is sometimes called the *Heaviside function**. This function is generally given as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

*The function is named after the English mathematician, engineer, and physicist Oliver Heaviside (1850–1925). Only by coincidence is the function “heavy” on “one side.”

Let us find the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^\infty = \frac{e^{-as}}{s},$$

where of course $s > 0$ (and $a \geq 0$ as we said before). ■

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in Table 8.1.

$f(t)$	$\mathcal{L}\{f(t)\}$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^3	$\frac{6}{s^4}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$

Table 8.1: Some Laplace transforms (C, ω, and a are constants).

Exercise* 8.1.1*: *Verify Table 8.1.*

Since the transform is defined by an integral. We can use the linearity properties of the integral.

For example, suppose C is a constant, then

$$\mathcal{L}\{Cf(t)\} = \int_0^\infty e^{-st} Cf(t)\, dt = C \int_0^\infty e^{-st} f(t)\, dt = C\mathcal{L}\{f(t)\}.$$

So we can "pull out" a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

Theorem 8.1.1 (Linearity of the Laplace transform)**.** *Suppose that A, B, and C are constants, then*

$$\boxed{\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\},}$$

and in particular

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

Exercise* 8.1.2*: *Verify the theorem. That is, show that* $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$.

These rules together with Table 8.1 on the preceding page make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function $\frac{1}{t}$ does not have a Laplace transform as the integral diverges for all s. Similarly, $\tan t$ or e^{t^2} do not have Laplace transforms.

8.1.2 Existence and uniqueness

Let us consider when does the Laplace transform exist in more detail. First let us consider functions of exponential order. The function $f(t)$ is of *exponential order* as t goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants M and c, for sufficiently large t (say for all $t > t_0$ for some t_0). The simplest way to check this condition is to try and compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

Exercise* 8.1.3*: *Use L'Hopital's rule from calculus to show that a polynomial is of exponential order. Hint: Note that a sum of two exponential order functions is also of exponential order. Then show that t^n is of exponential order for any n.*

For an exponential order function we have existence and uniqueness of the Laplace transform.

Theorem 8.1.2 (Existence)**.** *Let $f(t)$ be continuous and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.*

The existence is not difficult to see. Let $f(t)$ be of exponential order, that is $|f(t)| \leq Me^{ct}$ for all $t > 0$ (for simplicity $t_0 = 0$). Let $s > c$, or in other words $(c - s) < 0$. By the comparison theorem from calculus, the improper integral defining $\mathcal{L}\{f(t)\}$ exists if the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\,dt = M\int_0^\infty e^{(c-s)t}\,dt = M\left[\frac{e^{(c-s)t}}{c-s}\right]_{t=0}^\infty = \frac{M}{c-s}.$$

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions we obtain that their Laplace transform decays at infinity:

$$\lim_{s\to\infty} F(s) = 0.$$

Theorem 8.1.3 (Uniqueness)**.** *Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.*

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function. Uniqueness, however, does not “see” values at the discontinuities. So we can only conclude that $f(t) = g(t)$ outside of discontinuities. For example, the unit step function is sometimes defined using $u(0) = 1/2$. This new step function, however, has the exact same Laplace transform as the one we defined earlier where $u(0) = 1$.

8.1.3 The inverse transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. If we have a function $F(s)$, to be able to find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$, we need to first know if such a function is unique. It turns out we are in luck by Theorem 8.1.3. So we can without fear make the following definition.

If $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \overset{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself—it requires complex numbers and path integrals. For us it will suffice to compute the inverse using Table 8.1 on page 285.

Example 8.1.5. Take $F(s) = \frac{1}{s+1}$. Find the inverse Laplace transform.

We look at the table to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

■

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, we also have $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$. Let us demonstrate how linearity can be used.

Example 8.1.6. Take $F(s) = \frac{s^2+s+1}{s^3+s}$. Find the inverse Laplace transform.

First we use the *method of partial fractions* to write F in a form where we can use Table 8.1 on page 285. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1) + s(Bs+C) = s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$, and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity of the inverse Laplace transform we get

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 1 + \sin t.$$

■

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\boxed{\mathcal{L}\{e^{-at}f(t)\} = F(s+a),}$$

where $F(s)$ is the Laplace transform of $f(t)$.

Exercise* 8.1.4*: *Derive the first shifting property from the definition of the Laplace transform.*

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We complete the square and write such quadratics as $(s+a)^2 + b$ and then use the shifting property.

Example 8.1.7. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\} = \frac{1}{2}\,e^{-2t}\sin(2t).$$

■

In general, we want to be able to apply the Laplace transform to rational functions, that is functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials. Since normally, for the functions that we are considering, the Laplace transform goes to zero as $s \to \infty$, it is not hard to see that the degree of $F(s)$ must be smaller than that of $G(s)$. Such rational functions are called *proper rational functions* and we can always apply the method of partial fractions. Of course this means we need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator.

Exercises

***Exercise* 8.1.5:** *Find the Laplace transform of* $3 + t^5 + \sin(\pi t)$.

***Exercise* 8.1.6:** *Find the Laplace transform of* $a + bt + ct^2$ *for some constants a, b, and c.*

***Exercise* 8.1.7:** *Find the Laplace transform of* $A\cos(\omega t) + B\sin(\omega t)$.

***Exercise* 8.1.8:** *Find the Laplace transform of* $\cos^2(\omega t)$.

***Exercise* 8.1.9:** *Find the inverse Laplace transform of* $\dfrac{4}{s^2-9}$.

***Exercise* 8.1.10:** *Find the inverse Laplace transform of* $\dfrac{2s}{s^2-1}$.

***Exercise* 8.1.11:** *Find the inverse Laplace transform of* $\dfrac{1}{(s-1)^2(s+1)}$.

***Exercise* 8.1.12:** *Find the Laplace transform of* $f(t) = \begin{cases} t & \text{if } t \geq 1, \\ 0 & \text{if } t < 1. \end{cases}$

***Exercise* 8.1.13:** *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+s+2)(s+4)}$.

***Exercise* 8.1.14:** *Find the Laplace transform of* $\sin(\omega(t-a))$.

***Exercise* 8.1.15:** *Find the Laplace transform of* $t\sin(\omega t)$. *Hint: Several integrations by parts.*

***Exercise* 8.1.101:** *Find the Laplace transform of* $4(t+1)^2$.

***Exercise* 8.1.102:** *Find the inverse Laplace transform of* $\dfrac{8}{s^3(s+2)}$.

***Exercise* 8.1.103:** *Find the Laplace transform of* te^{-t} *(Hint: integrate by parts).*

***Exercise* 8.1.104:** *Find the Laplace transform of* $\sin(t)\,e^{-t}$ *(Hint: integrate by parts).*

8.2 Transforms of derivatives and ODEs

Attribution: §6.2 in [L]
Further reading: §7.2 –7.3 in [EP], §6.2 and §6.3 in [BD]

8.2.1 Transforms of derivatives

Let us see how the Laplace transform is used for differential equations. First let us try to find the Laplace transform of a function that is a derivative. Suppose $g(t)$ is a differentiable function of exponential order, that is, $|g(t)| \leq Me^{ct}$ for some M and c. So $\mathcal{L}\{g(t)\}$ exists, and what is more, $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \left[e^{-st}g(t)\right]_{t=0}^\infty - \int_0^\infty (-s)\,e^{-st}g(t)\,dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in Table 8.2. The procedure also works for piecewise smooth functions, that is functions that are piecewise continuous with a piecewise continuous derivative.

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Table 8.2: Laplace transforms of derivatives ($G(s) = \mathcal{L}\{g(t)\}$ as usual).

Exercise* 8.2.1*: *Verify Table 8.2.*

8.2.2 Solving ODEs with the Laplace transform

Notice that the Laplace transform turns differentiation into multiplication by s. Let us see how to apply this fact to differential equations.

Example 8.2.1. Take the equation

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

We will take the Laplace transform of both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\},$$
$$s^2X(s) - sx(0) - x'(0) + X(s) = \frac{s}{s^2+4}.$$

We plug in the initial conditions now—this makes the computations more streamlined—to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}.$$

We solve for $X(s)$,

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

■

The procedure for linear constant coefficient equations is as follows. We take an ordinary differential equation in the time variable t. We apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the $x(t)$, $x'(t)$, $x''(t)$, and so on, will be converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. We solve the equation for $X(s)$. Then taking the inverse transform, if possible, we find $x(t)$.

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

8.2.3 Using the Heaviside function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure 8.1 on the next page for the graph.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as $u(t-a)$ for some constant a. This just shifts the graph to the right by a. That is, it is a

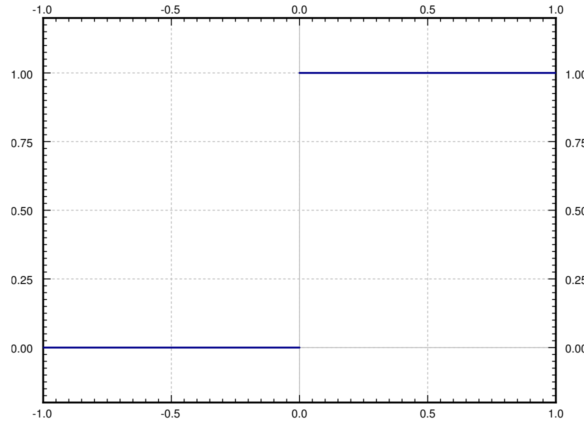


Figure 8.1: Plot of the Heaviside (unit step) function $u(t)$.

function that is 0 when $t < a$ and 1 when $t \geq a$. Suppose for example that $f(t)$ is a "signal" and you started receiving the signal $\sin t$ at time $t = \pi$. The function $f(t)$ should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, $f(t)$ can be written as

$$f(t) = u(t - \pi)\, \sin t.$$

Similarly the step function that is 1 on the interval $[1, 2)$ and zero everywhere else can be written as

$$u(t - 1) - u(t - 2).$$

The Heaviside function is useful to define functions defined piecewise. If you want to define $f(tj$ such that $f(t) = t$ when t is in $[0, 1]$, $f(t) = -t + 2$ when t is in $[1, 2]$ and $f(t) = 0$ otherwise, you can use the expression

$$f(t) = t\big(u(t) - u(t - 1)\big) + (-t + 2)\big(u(t - 1) - u(t - 2)\big).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s}.$$

This can be generalized into a *shifting property* or *second shifting property*.

$$\boxed{\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as}\mathcal{L}\{f(t)\}.} \tag{8.1}$$

Example 8.2.2. Suppose that the forcing function is not periodic. For example, suppose that we had a mass-spring system

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at $t = 1$. Or perhaps an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$, and then held steady again starting at $t = 5$.

We can write $f(t) = u(t-1) - u(t-5)$. We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

In other words $\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$. So using (8.1) we find

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-1)\big)\,u(t-1).$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-5)\big)\,u(t-5).$$

Hence, the solution is

$$x(t) = \big(1-\cos(t-1)\big)\,u(t-1) - \big(1-\cos(t-5)\big)\,u(t-5).$$

The plot of this solution is given in Figure 8.2 on the facing page.

■

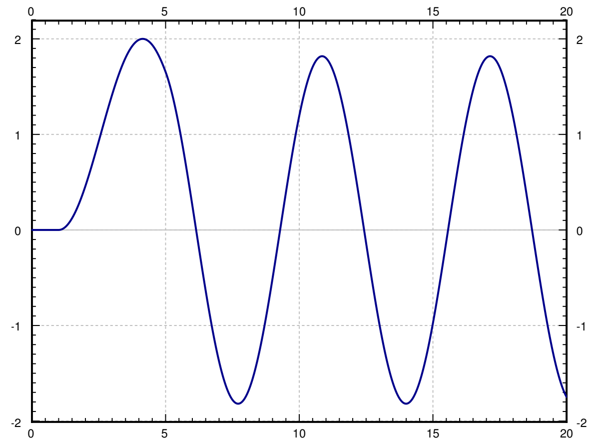


Figure 8.2: Plot of $x(t)$.

8.2.4 Transfer functions

Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Suppose we have an equation of the form

$$Lx = f(t),$$

where L is a linear constant coefficient differential operator. Then $f(t)$ is usually thought of as input of the system and $x(t)$ is thought of as the output of the system. For example, for a mass-spring system the input is the forcing function and output is the behavior of the mass. We would like to have an convenient way to study the behavior of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

Solving for the ratio $X(s)/F(s)$ we obtain the so-called *transfer function* $H(s) = 1/A(s)$.

$$H(s) = \frac{X(s)}{F(s)}.$$

In other words, $X(s) = H(s)F(s)$. We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function.

Example 8.2.3. Given $x'' + \omega_0^2 x = f(t)$, let us find the transfer function (assuming the initial conditions are zero).

First, we take the Laplace transform of the equation.

$$s^2X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function $X(s)/F(s)$.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input $f(t) = 1$. Hence $F(s) = 1/s$, and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2}\frac{1}{s}.$$

Taking the inverse Laplace transform of $X(s)$ we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

■

8.2.5 Transforms of integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s).}$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Example 8.2.4. To compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ we could proceed by applying this integration rule.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2+1}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\,d\tau = \int_0^t \sin\tau\,d\tau = 1 - \cos t.$$

■

Example 8.2.5. An equation containing an integral of the unknown function is called an *integral equation*. For example, take

$$t^2 = \int_0^t e^\tau x(\tau)\,d\tau,$$

where we wish to solve for $x(t)$. We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}\{e^t x(t)\} = \frac{1}{s}X(s-1),$$

where $X(s) = \mathcal{L}\{x(t)\}$. Thus

$$X(s-1) = \frac{2}{s^2} \qquad \text{or} \qquad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again

$$x(t) = 2e^{-t}t.$$

$\blacksquare$

Exercises

Exercise* 8.2.2*: *Using the Heaviside function write down the piecewise function that is 0 for* $t < 0$*,* t^2 *for t in* $[0, 1]$ *and t for* $t > 1$*.*

Exercise* 8.2.3*: *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$*,* $c > 0$*,* $k > 0$*, and* $c^2 - 4km > 0$ *(system is overdamped).*

Exercise* 8.2.4*: *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$*,* $c > 0$*,* $k > 0$*, and* $c^2 - 4km < 0$ *(system is underdamped).*

Exercise* 8.2.5*: *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$*,* $c > 0$*,* $k > 0$*, and* $c^2 = 4km$ *(system is critically damped).*

Exercise* 8.2.6*: *Solve* $x'' + x = u(t - 1)$ *for initial conditions* $x(0) = 0$ *and* $x'(0) = 0$*.*

Exercise* 8.2.7*: *Show the differentiation of the transform property. Suppose* $\mathcal{L}\{f(t)\} = F(s)$*, then show*

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Hint: Differentiate under the integral sign.

Exercise* 8.2.8*: *Solve* $x''' + x = t^3 u(t - 1)$ *for initial conditions* $x(0) = 1$ *and* $x'(0) = 0$*,* $x''(0) = 0$*.*

Exercise* 8.2.9*: *Show the second shifting property:* $\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as}\mathcal{L}\{f(t)\}$*.*

Exercise* 8.2.10*: *Let us think of the mass-spring system with a rocket from Example 8.2.2. We noticed that the solution kept oscillating after the rocket stopped running. The amplitude of the oscillation depends on the time that the rocket was fired (for 4 seconds in the example). a) Find a formula for the amplitude of the resulting oscillation in terms of the amount of time the rocket is fired. b) Is there a nonzero time (if so what is it?) for which the rocket fires and the resulting oscillation has amplitude 0 (the mass is not moving)?*

Exercise* 8.2.11*: *Define*

$$f(t) = \begin{cases} (t-1)^2 & \text{if } 1 \le t < 2, \\ 3 - t & \text{if } 2 \le t < 3, \\ 0 & \text{otherwise}. \end{cases}$$

a) Sketch the graph of $f(t)$*. b) Write down* $f(t)$ *using the Heaviside function. c) Solve* $x'' + x = f(t)$*,* $x(0) = 0$*,* $x'(0) = 0$ *using Laplace transform.*

Exercise* 8.2.12*: *Find the transfer function for* $mx'' + cx' + kx = f(t)$ *(assuming the initial conditions are zero).*

Exercise* 8.2.101*: *Using the Heaviside function* $u(t)$*, write down the function*

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ t - 1 & \text{if } 1 \leq t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}.$$

Exercise* 8.2.102*: *Solve* $x'' - x = (t^2 - 1)u(t - 1)$ *for initial conditions* $x(0) = 1$, $x'(0) = 2$ *using the Laplace transform.*

Exercise* 8.2.103*: *Find the transfer function for* $x' + x = f(t)$ *(assuming the initial conditions are zero).*

8.3 Convolution

Attribution: §6.3 in [L]
Further reading: , §7.2 in [EP], §6.6 in [BD]

8.3.1 The convolution

We said that the Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a "product." Take two functions $f(t)$ and $g(t)$ defined for $t \geq 0$, and define the *convolution** of $f(t)$ and $g(t)$ as

$$\boxed{(f * g)(t) \overset{\text{def}}{=} \int_0^t f(\tau) g(t-\tau)\, d\tau.} \tag{8.2}$$

As you can see, the convolution of two functions of t is another function of t.

Example 8.3.1. Take $f(t) = e^t$ and $g(t) = t$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t e^\tau (t-\tau)\, d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts. ■

Example 8.3.2. Take $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t \sin(\omega \tau) \cos(\omega (t-\tau))\, d\tau.$$

We apply the identity

$$\cos(\theta) \sin(\psi) = \frac{1}{2} \big(\sin(\theta + \psi) - \sin(\theta - \psi)\big).$$

Hence,

$$\begin{aligned} (f * g)(t) &= \int_0^t \frac{1}{2} \big(\sin(\omega t) - \sin(\omega t - 2\omega\tau)\big)\, d\tau \\ &= \left[\frac{1}{2} \tau \sin(\omega t) + \frac{1}{4\omega} \cos(2\omega\tau - \omega t) \right]_{\tau=0}^t \\ &= \frac{1}{2} t \sin(\omega t). \end{aligned}$$

*For those that have seen convolution defined before, you may have seen it defined as $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$. This definition agrees with (8.2) if you define $f(t)$ and $g(t)$ to be zero for $t < 0$. When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t. ■

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions then

$$\begin{aligned} &f * g = g * f, \\ &(cf) * g = f * (cg) = c(f * g), \\ &(f * g) * h = f * (g * h). \end{aligned}$$

The most interesting property for us, and the main result of this section is the following theorem.

Theorem 8.3.1. *Let $f(t)$ and $g(t)$ be of exponential type, then*

$$\boxed{\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.}$$

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

Example 8.3.3. Suppose we have the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\,\frac{1}{s^2}.$$

We recognize the two entries of Table 8.2. That is

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\,\frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)}\,d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts. ■

8.3.2 Solving ODEs

The next example demonstrates the full power of the convolution and the Laplace transform. We can give the solution to the forced oscillation problem for any forcing function as a definite integral.

Example 8.3.4. Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual.

$$s^2X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2 + \omega_0^2}.$$

We know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t) = \int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}\, d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} f(t-\tau)\, d\tau.$$

■

Let us notice one more feature of this example. We can now see how Laplace transform handles resonance. Suppose that $f(t) = \cos(\omega_0 t)$. Then

$$x(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} \cos(\omega_0(t-\tau))\, d\tau = \frac{1}{\omega_0}\int_0^t \sin(\omega_0 \tau)\cos(\omega_0(t-\tau))\, d\tau.$$

We have computed the convolution of sine and cosine in Example 8.3.2. Hence

$$x(t) = \left(\frac{1}{\omega_0}\right)\left(\frac{1}{2}\, t\, \sin(\omega_0 t)\right) = \frac{1}{2\omega_0}\, t\, \sin(\omega_0 t).$$

Note the t in front of the sine. The solution, therefore, grows without bound as t gets large, meaning we get resonance.

Similarly, we can solve any constant coefficient equation with an arbitrary forcing function $f(t)$ as a definite integral using convolution. A definite integral, rather than a closed form solution, is usually enough for most practical purposes. It is not hard to numerically evaluate a definite integral.

8.3.3 Volterra integral equation

A common integral equation is the *Volterra integral equation**

$$x(t) = f(t) + \int_0^t g(t-\tau)x(\tau)\, d\tau,$$

*Named for the Italian mathematician Vito Volterra (1860 – 1940).

where $f(t)$ and $g(t)$ are known functions and $x(t)$ is an unknown we wish to solve for. To find $x(t)$, we apply the Laplace transform to the equation to obtain

$$X(s) = F(s) + G(s)X(s),$$

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of $x(t)$, $f(t)$, and $g(t)$ respectively. We find

$$X(s) = \frac{F(s)}{1 - G(s)}.$$

To find $x(t)$ we now need to find the inverse Laplace transform of $X(s)$.

Example 8.3.5. Solve

$$x(t) = e^{-t} + \int_0^t \sinh(t-\tau)x(\tau)\,d\tau.$$

We apply Laplace transform to obtain

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2-1}X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not hard to apply Table 8.1 on page 285 to find

$$x(t) = \cosh\left(\sqrt{2}\,t\right) - \frac{1}{\sqrt{2}}\sinh\left(\sqrt{2}\,t\right).$$

■

Exercises

Exercise* 8.3.1*: *Let $f(t) = t^2$ for $t \geq 0$, and $g(t) = u(t-1)$. Compute $f * g$.*

Exercise* 8.3.2*: *Let $f(t) = t$ for $t \geq 0$, and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.*

Exercise* 8.3.3*: *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km > 0$ (system is overdamped). Write the solution as a definite integral.

Exercise* 8.3.4*: *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km < 0$ (system is underdamped). Write the solution as a definite integral.

Exercise* 8.3.5*: *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 = 4km$ (system is critically damped). Write the solution as a definite integral.

Exercise* 8.3.6*: *Solve*

$$x(t) = e^{-t} + \int_0^t \cos(t-\tau)x(\tau)\, d\tau.$$

Exercise* 8.3.7*: *Solve*

$$x(t) = \cos t + \int_0^t \cos(t-\tau)x(\tau)\, d\tau.$$

Exercise* 8.3.8*: *Compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$ using convolution.*

Exercise* 8.3.9*: *Write down the solution to $x'' - 2x = e^{-t^2}$, $x(0) = 0$, $x'(0) = 0$ as a definite integral. Hint: Do not try to compute the Laplace transform of e^{-t^2}.*

Exercise* 8.3.101*: *Let $f(t) = \cos t$ for $t \geq 0$, and $g(t) = e^{-t}$. Compute $f * g$.*

Exercise* 8.3.102*: *Compute $\mathcal{L}^{-1}\left\{\frac{5}{s^4+s^2}\right\}$ using convolution.*

Exercise* 8.3.103*: *Solve $x'' + x = \sin t$, $x(0) = 0$, $x'(0) = 0$ using convolution.*

Exercise* 8.3.104*: *Solve $x''' + x' = f(t)$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 0$ using convolution. Write the result as a definite integral.*

8.4 Dirac delta and impulse response

Attribution: §6.4 in [L]
Further reading: §7.6 in [EP], §6.5 in [BD]

8.4.1 Rectangular pulse

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behavior is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t < a, \\ M & \text{if } a \leq t < b, \\ 0 & \text{if } b \leq t. \end{cases}$$

See Figure 8.3 for a graph. Notice that

$$\varphi(t) = M\big(u(t-a) - u(t-b)\big),$$

where $u(t)$ is the unit step function.

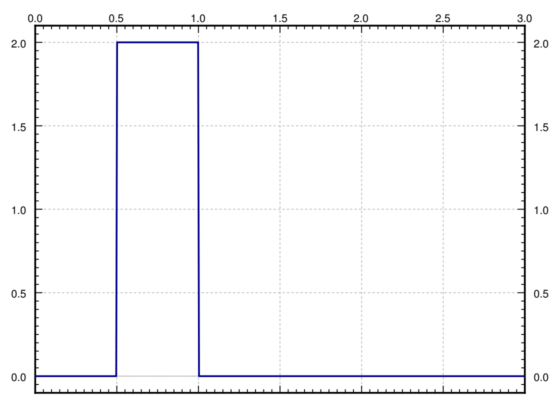


Figure 8.3: Sample square pulse with $a = 0.5$, $b = 1$ and $M = 2$.

Let us take the Laplace transform of a square pulse,

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\big\{M\big(u(t-a) - u(t-b)\big)\big\} = M\frac{e^{-as} - e^{-bs}}{s}.$$

For simplicity we let $a = 0$, and it is convenient to set $M = 1/b$ to have

$$\int_0^\infty \varphi(t)\, dt = 1.$$

That is, to have the pulse have "unit mass." For such a pulse we compute

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\left\{\frac{u(t) - u(t-b)}{b}\right\} = \frac{1 - e^{-bs}}{bs}.$$

We generally want b to be very small. That is, we wish to have the pulse be very short and very tall. By letting b go to zero we arrive at the concept of the Dirac delta function.

8.4.2 The delta function

The *Dirac delta function*[*] is not exactly a function; it is sometimes called a *generalized function*. We avoid unnecessary details and simply say that it is an object that does not really make sense unless we integrate it. The motivation is that we would like a "function" $\delta(t)$ such that for any continuous function $f(t)$ we have

$$\boxed{\int_{-\infty}^\infty \delta(t) f(t)\, dt = f(0).}$$

The formula should hold if we integrate over any interval that contains 0, not just $(-\infty, \infty)$. So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, for any interval $[c, d]$

$$\int_c^d \delta(t)\, dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } 0, \text{ i.e. } c \leq 0 \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

A good way to think about $\delta(t)$ is as a limit of short pulses whose integral is 1. For example, suppose that we have a square pulse $\varphi(t)$ as above with $a = 0$, $M = 1/b$, that is $\varphi(t) = \frac{u(t)-u(t-b)}{b}$. Compute

$$\int_{-\infty}^\infty \varphi(t) f(t)\, dt = \int_{-\infty}^\infty \frac{u(t) - u(t-b)}{b} f(t)\, dt = \frac{1}{b} \int_0^b f(t)\, dt.$$

If $f(t)$ is continuous at $t = 0$, then for very small b, the function $f(t)$ is approximately equal to $f(0)$ on the interval $[0, b]$. We approximate the integral

$$\frac{1}{b} \int_0^b f(t)\, dt \approx \frac{1}{b} \int_0^b f(0)\, dt = f(0).$$

[*]Named after the English physicist and mathematician Paul Adrien Maurice Dirac (1902–1984).

Therefore,

$$\lim_{b\to 0} \int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \lim_{b\to 0} \frac{1}{b} \int_0^b f(t)\, dt = f(0).$$

Let us therefore accept $\delta(t)$ as an object that is possible to integrate. We often want to shift δ to another point, for example $\delta(t-a)$. In that case we have

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a).$$

Note that $\delta(a-t)$ is the same object as $\delta(t-a)$. In other words, the convolution of $\delta(t)$ with $f(t)$ is again $f(t)$,

$$(f * \delta)(t) = \int_0^t \delta(t-s) f(s)\, ds = f(t).$$

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$\boxed{\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st} \delta(t-a)\, dt = e^{-as}.}$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Remark 8.4.1. *Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice*

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{“} \quad \frac{d}{dt}\big[u(t-a)\big] = \delta(t-a) \quad \text{”}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

Example 8.4.1. Let us compute $\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\}$. So far we have always looked at proper rational functions in the s variable. That is, the numerator was always of lower degree than the denominator. Not so with $\frac{s+1}{s}$. We write,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\} = \mathcal{L}^{-1}\left\{1 + \frac{1}{s}\right\} = \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \delta(t) + 1.$$

The resulting object is a generalized function and only makes sense when put underneath an integral. ■

8.4.3 Impulse response

As we said before, in the differential equation $Lx = f(t)$, we think of $f(t)$ as input, and $x(t)$ as the output. Often it is important to find the response to an impulse, and then we use the delta function in place of $f(t)$. The solution to

$$Lx = \delta(t)$$

is called the *impulse response*.

Example 8.4.2. Solve (find the impulse response)

$$x'' + \omega_0^2 x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \tag{8.3}$$

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$.

$$s^2 X(s) + \omega_0^2 X(s) = 1, \qquad \text{and so} \qquad X(s) = \frac{1}{s^2 + \omega_0^2}.$$

Taking the inverse Laplace transform we obtain

$$x(t) = \frac{\sin(\omega_0 t)}{\omega_0}.$$

■

Let us notice something about the above example. We have proved before that when the input was $f(t)$, then the solution to $Lx = f(t)$ was given by

$$x(t) = \int_0^t f(\tau) \frac{\sin(\omega_0 (t-\tau))}{\omega_0} \, d\tau.$$

Notice that the solution for an arbitrary input is given as convolution with the impulse response. Let us see why. The key is to notice that for functions $x(t)$ and $f(t)$,

$$(x * f)''(t) = \frac{d^2}{dt^2}\left[\int_0^t f(\tau) x(t-\tau) \, d\tau\right] = \int_0^t f(\tau) x''(t-\tau) \, d\tau = (x'' * f)(t).$$

We simply differentiate twice under the integral*, the details are left as an exercise. And so if we convolve the entire equation (8.3), the left hand side becomes

$$(x'' + \omega_0^2 x) * f = (x'' * f) + \omega_0^2 (x * f) = (x * f)'' + \omega_0^2 (x * f).$$

The right hand side becomes

$$(\delta * f)(t) = f(t).$$

*You should really think of the integral going over $(-\infty, \infty)$ rather than over $[0, t]$ and simply assume that $f(t)$ and $x(t)$ are continuous and zero for negative t.

Therefore $y(t) = (x * f)(t)$ is the solution to

$$y'' + \omega_0^2 y = f(t).$$

This procedure works in general for other linear equations $Lx = f(t)$. If you determine the impulse response, you also know how to obtain the output $x(t)$ for any input $f(t)$ by simply convolving the impulse response and the input $f(t)$.

8.4.4 Three-point beam bending

Let us give another quite different example where delta functions turn up. In this case representing point loads on a steel beam. Suppose we have a beam of length L, resting on two simple supports at the ends. Let x denote the position on the beam, and let $y(x)$ denote the deflection of the beam in the vertical direction. The deflection $y(x)$ satisfies the *Euler-Bernoulli equation**,

$$EI \frac{d^4 y}{dx^4} = F(x),$$

where E and I are constants† and $F(x)$ is the force applied per unit length at position x. The situation we are interested in is when the force is applied at a single point as in Figure 8.4.

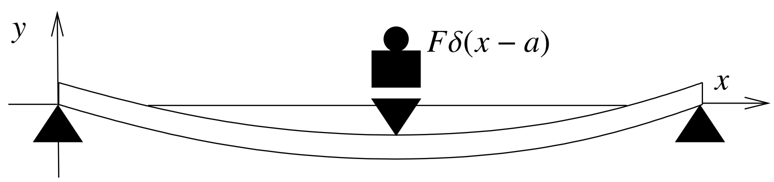


Figure 8.4: Three-point bending.

In this case the equation becomes

$$EI \frac{d^4 y}{dx^4} = -F\delta(x-a),$$

where $x = a$ is the point where the mass is applied. F is the force applied and the minus sign indicates that the force is downward, that is, in the negative y direction. The end points of the beam satisfy the conditions,

$$\begin{aligned} y(0) &= 0, & y''(0) &= 0, \\ y(L) &= 0, & y''(L) &= 0. \end{aligned}$$

*Named for the Swiss mathematicians Jacob Bernoulli (1654 – 1705), Daniel Bernoulli —nephew of Jacob— (1700 – 1782), and Leonhard Paul Euler (1707 – 1783).

†E is the elastic modulus and I is the second moment of area. Let us not worry about the details and simply think of these as some given constants.

Example 8.4.3. Suppose that length of the beam is 2, and suppose that $EI = 1$ for simplicity. Further suppose that the force $F = 1$ is applied at $x = 1$. That is, we have the equation

$$\frac{d^4y}{dx^4} = -\delta(x-1),$$

and the endpoint conditions are

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

We could integrate, but using the Laplace transform is even easier. We apply the transform in the x variable rather than the t variable. Let us again denote the transform of $y(x)$ as $Y(s)$.

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property (8.1) to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) + C_1x + \frac{C_2}{6}x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6}2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3}C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6}3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = -\frac{1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6}u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

■

Exercises

Exercise* 8.4.1*: *Solve (find the impulse response)* $x'' + x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 8.4.2*: *Solve (find the impulse response)* $x'' + 2x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 8.4.3*: *A pulse can come later and can be bigger. Solve* $x'' + 4x = 4\delta(t - 1)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 8.4.4*: *Suppose that* $f(t)$ *and* $g(t)$ *are differentiable functions and suppose that*

$$f(t) = g(t) = 0$$

for all $t \leq 0$. *Show that*

$$(f * g)'(t) = (f' * g)(t) = (f * g')(t).$$

Exercise* 8.4.5*: *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x = e^{-t}$ *for* $t > 0$. *Find the solution to* $Lx = t^2$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

Exercise* 8.4.6*: *Compute* $\mathcal{L}^{-1}\left\{\dfrac{s^2 + s + 1}{s^2}\right\}$.

Exercise* 8.4.7** (challenging):*** *Solve Example 8.4.3 via integrating 4 times in the x variable.*

Exercise* 8.4.8*: *Suppose we have a beam of length* 1 *simply supported at the ends and suppose that force* $F = 1$ *is applied at* $x = \frac{3}{4}$ *in the downward direction. Suppose that* $EI = 1$ *for simplicity. Find the beam deflection* $y(x)$.

Exercise* 8.4.101*: *Solve (find the impulse response)* $x'' = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 8.4.102*: *Solve (find the impulse response)* $x' + ax = \delta(t)$, $x(0) = 0$.

Exercise* 8.4.103*: *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x(t) = \cos(t)$ *for* $t > 0$. *Find (in closed form) the solution to* $Lx = \sin(t)$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

Exercise* 8.4.104*: *Compute* $\mathcal{L}^{-1}\left\{\dfrac{s^2}{s^2 + 1}\right\}$.

Exercise* 8.4.105*: *Compute* $\mathcal{L}^{-1}\left\{\dfrac{3s^2 e^{-s} + 2}{s^2}\right\}$.

Chapter 9

Useful formulas

9.1 Number sets

The set of integers is denoted $\mathbb{Z}$. The set of rational numbers is denoted $\mathbb{Q}$. The set of real numbers is denoted $\mathbb{R}$. The set of complex numbers is denoted $\mathbb{C}$.

9.2 Exponentials and logarithms

The irrational number e has value ≈ 2.718. An exponential function has the form a^x, where a is the base and x is the exponent. Common bases are integers and e. No matter what the base a is, $a^0 = 1$. Some exponential functions are easy to evaluate, e.g.

$$\begin{aligned} 10^5 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \\ 2^7 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 128 \\ 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 = 81 \\ 3.47^0 &= 1 \end{aligned}$$

Others are best left in symbolic form, e.g. e^4 or π^2 or 1.7^6. Properties of exponential functions include

$$\begin{aligned} a^b a^c &= a^{b+c} \\ \frac{a^b}{a^c} &= a^{b-c} \\ a^{b^c} &= a^{bc} \\ a^{\frac{1}{b}} &= \sqrt[b]{a} \end{aligned}$$

Provided that base a is positive, we can define the logarithmic function as $\log_a x$ as an inverse function to the exponential function a^x. We can use this insight to evaluate logarithms, e.g.

$$\log_{10} 10000 = 5 \text{ because } 10^5 = 10000$$

$$\log_2 128 = 7 \text{ because } 2^7 = 128$$
$$\log_3 81 = 4 \text{ because } 3^4 = 81$$
$$\log_{3.47} 1 = 0 \text{ because } a^0 = 1 \text{ for any base } a$$
$$\log_{22} 22 = 1 \text{ because } a^1 = a \text{ for any base } a$$

The natural logarithm has base e and is often abbreviated $\log_e x = \ln x$. The common logarithm has base 10 and is often abbreviated $\log_{10} x = \log x$. (Confusingly, engineers and computer scientists usually use the abbreviation $\log_e x = \log x$.) Logarithmic identities include

- the logarithm of a product is the sum of the logarithms: $\log_a xy = \log_a x + \log_a y$,
- the logarithm of a quotient is the difference of the logarithms: $\log_a \frac{x}{y} = \log_a x - \log_a y$,
- the logarithm of a power is that power times the logarithm: $\log_a x^c = c \log_a x$.

The change of base formula is sometimes useful when working with an expression containing several bases:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Sketching the graphs of exponential and logarithmic functions reveals many of their properties, such as:

- the location of x- and y-intercepts (e.g. the function e^x passes through point $(0, 1)$),
- the location of asymptotes (e.g. e^x possesses the horizontal asymptote $y = 0$),
- the domain and range of these functions (e.g. the domain of $\ln x$ is $(0, \infty)$ and its range is $\mathbb{R}$),
- e^x and $\ln(x)$ are inverse functions, reflecting each other across the line $y = x$, and
- other features (e.g. e^x passes through point $(1, e)$).

Functions of the form a^x model exponential growth and are increasing, while functions of the form a^{-x} model exponential decay and are decreasing.

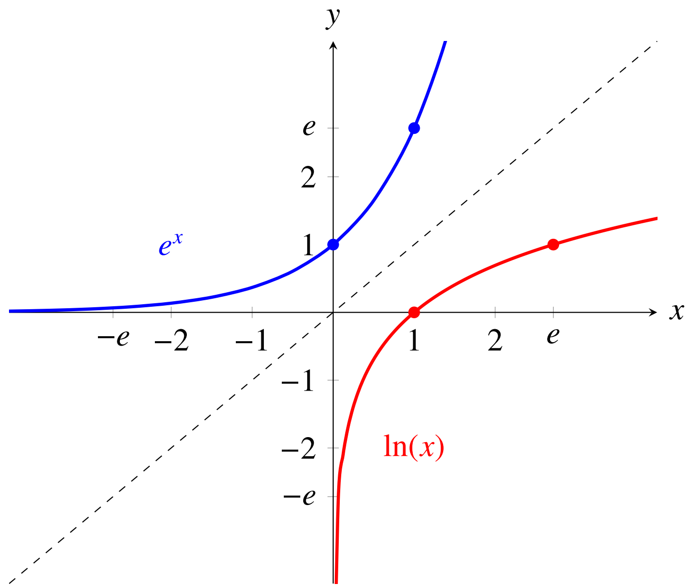


Figure 9.1: Graphs of e^x and $\ln(x)$. Being inverse functions, they are reflections of each other over the line $y = x$.

9.3 Trigonometry

The irrational number π has value ≈ 3.141. All trigonometric functions are defined in terms of sinusoids (sine and cosine):

$$\begin{aligned}
\sec\theta &= \frac{1}{\cos\theta} \\
\csc\theta &= \frac{1}{\sin\theta} \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
\cot\theta &= \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

Sketching the graphs of all trigonometric functions remind us of many features of these functions, such as:

- the location of the horizontal and vertical intercepts (e.g. the sine function passes through the origin),
- the location of asymptotes (e.g. $\tan\theta$ possesses infinitely many vertical asymptotes at $\theta = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$),

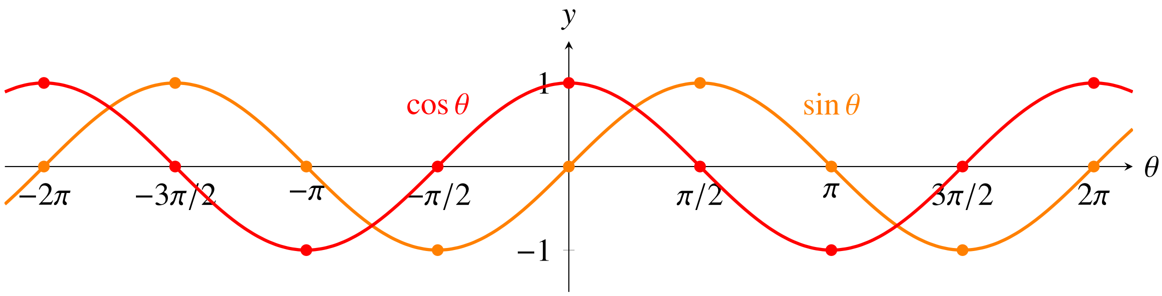


Figure 9.2: Graphs of the sinusoid functions, whose domains are the set of real numbers.

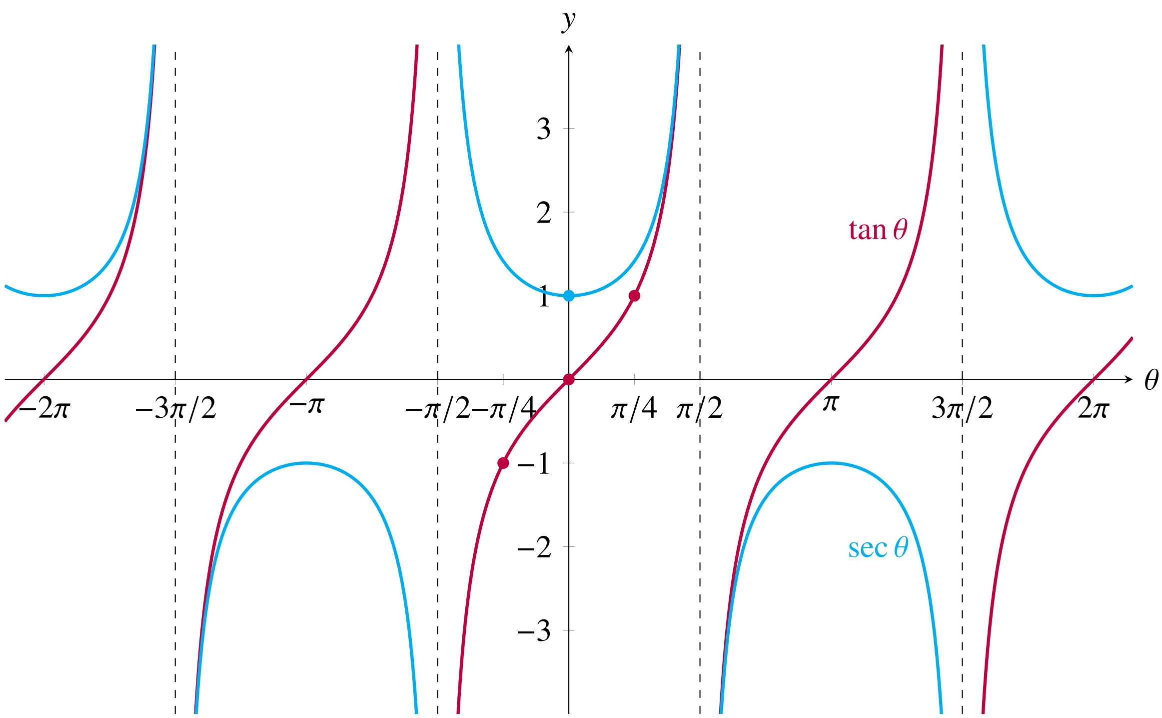


Figure 9.3: Graphs of the secant and tangent functions, with vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

- the domain and range of these functions (e.g. the domain of $\sin\theta$ is $\mathbb{R}$ and its range is $[-1, 1]$).

The sine and tangent functions are odd while the cosine function is even:

$$\begin{aligned}
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\tan(-\theta) &= -\tan\theta
\end{aligned}$$

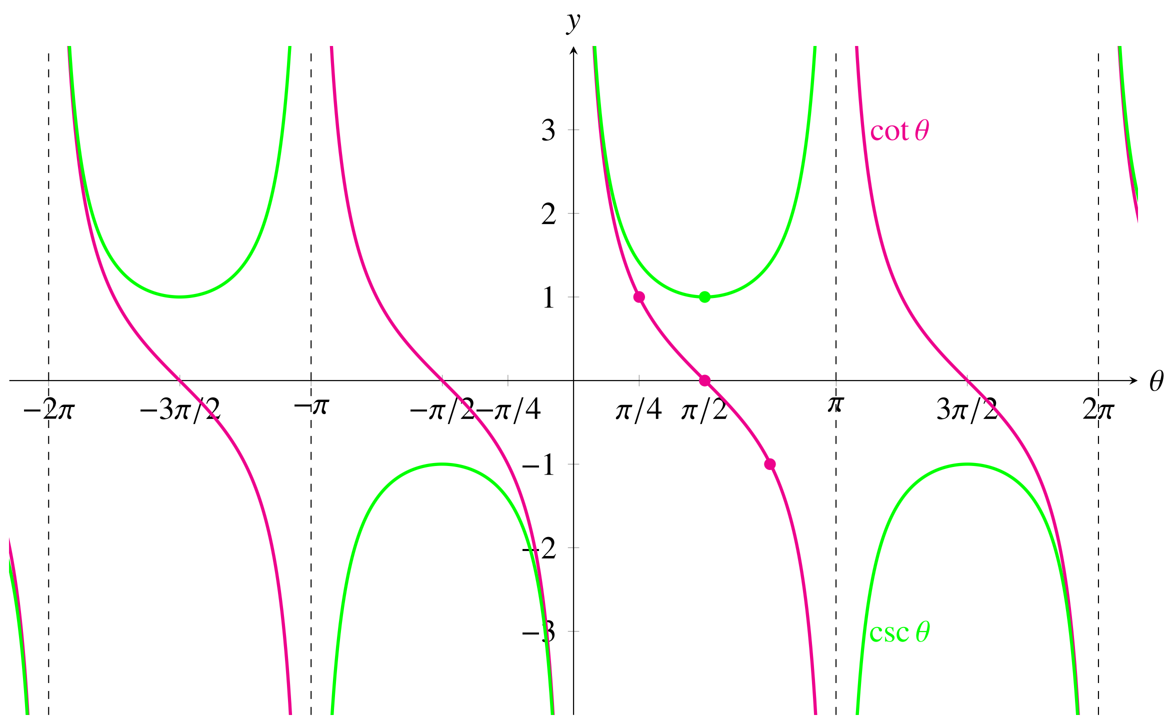


Figure 9.4: Graphs of the cosecant and cotangent functions, with vertical asymptotes at integer multiples of π.

Trig functions are periodic, e.g.

$$\begin{aligned}\sin(\theta + 2\pi) &= \sin\theta \\ \cos(\theta + 2\pi) &= \cos\theta \\ \tan(\theta + \pi) &= \tan\theta\end{aligned}$$

The Pythagorean trigonometric identity provide nice shortcuts:

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing all terms of this identity by $\cos^2\theta$ yields a second identity

$$\tan^2\theta + 1 = \sec^2\theta,$$

while dividing all terms of the Pythagorean identity by $\sin^2\theta$ yields a third identity

$$1 + \cot^2\theta = \csc^2\theta.$$

as do the double-angle identities

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.\end{aligned}$$

Rearranging this last identity yields

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2x)}{2} \\
\cos^2\theta &= \frac{1+\cos(2x)}{2}.
\end{aligned}$$

Finally, there are addition and subtraction identities:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.
\end{aligned}$$

You should know the sines and cosines at the following angles in the first quadrant of the unit circle:

$$\begin{aligned}
\sin 0 &= \frac{\sqrt{0}}{2} = 0 = \cos\frac{\pi}{2} \\
\sin\frac{\pi}{6} &= \frac{\sqrt{1}}{2} = \frac{1}{2} = \cos\frac{\pi}{3} \\
\sin\frac{\pi}{4} &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \\
\sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \\
\sin\frac{\pi}{2} &= \frac{\sqrt{4}}{2} = 1 = \cos 0.
\end{aligned}$$

Do you notice a pattern? The angles are increasing in the leftmost column and decreasing in the rightmost column. Values in the second column are all in the form $\frac{\sqrt{n}}{2}$ with $n = 0, 1, 2, 3, 4$.

Armed with a knowledge of the graph of each function, you can translate these to evaluate trig functions in other quadrants, e.g.

$$\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2} \quad \text{or} \quad \cos\frac{-\pi}{6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

You can also evaluate other trig functions at these angles, e.g.

$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 \quad \text{or} \quad \csc\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{2}{\sqrt{3}}.$$

9.4 Hyperbolics

All hyperbolic functions are defined in terms of exponential functions:

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} \\
\operatorname{sech} x &= \frac{1}{\cosh x} \\
\operatorname{csch} x &= \frac{1}{\sinh x}.
\end{aligned}$$

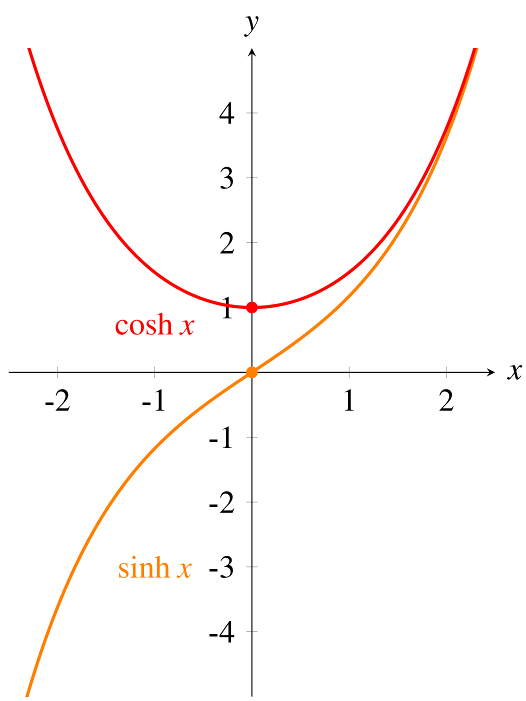


Figure 9.5: The hyperbolic sine function $\sinh x$ is an increasing function that passes through the origin. The hyperbolic cosine function $\cosh x$ intersects the vertical axis at point $(0, 1)$.

Sketching the graphs of hyperbolic functions remind us of many features of these functions, such as $y = \sinh x$ passes through the origin, while the $y = \cosh x$ intersects the y-axis at $y = 1$ and never intersects the x-axis.

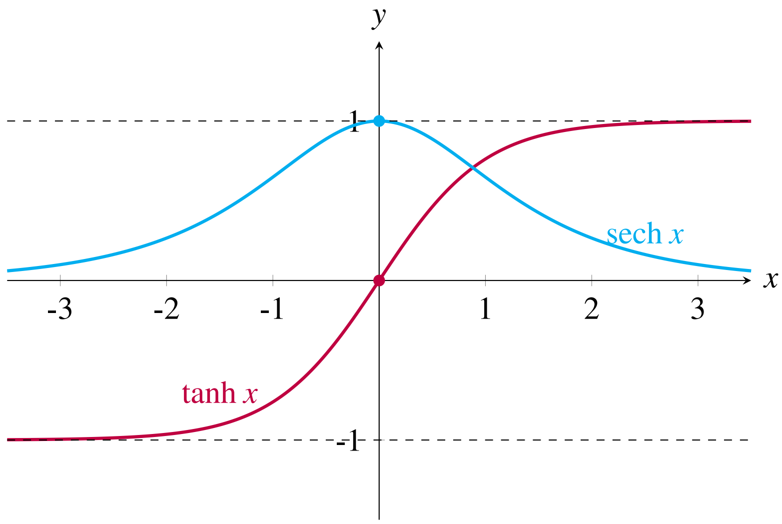


Figure 9.6: The hyperbolic tangent function $\tanh x$ is an increasing function bound between the horizontal asymptotes $y = \pm 1$. The hyperbolic secant function $\operatorname{sech} x$ is lower-bound by the x-axis that is a horizontal asymptote.

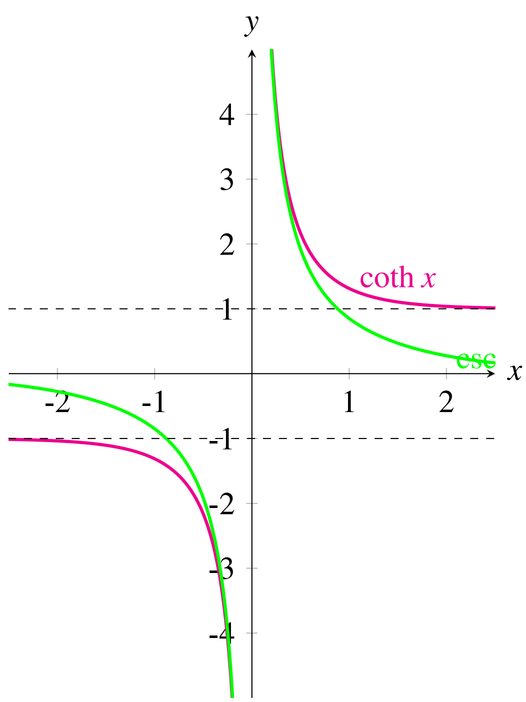


Figure 9.7: The hyperbolic cotangent function $\coth x$ and hyperbolic cosecant function $\operatorname{csch} x$ are not defined at $x = 0$ where a vertical asymptote appears.

Hyperbolic cosine is even

$$\cosh(-x) = \cosh x$$

while hyperbolic sine and tangent are odd

$$\begin{aligned}\sinh(-x) &= -\sinh x\\ \tanh(-x) &= -\tanh x.\end{aligned}$$

The Pythagorean hyperbolic identities are

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1\\ 1 - \tanh^2 x &= \operatorname{sech}^2 x\\ \coth^2 x - 1 &= \operatorname{csch}^2 x\end{aligned}$$

while the analogues of the double-angle identities are

$$\begin{aligned}\sinh(2x) &= 2\sinh x\cosh x\\ \cosh(2x) &= \cosh^2 x + \sinh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1.\end{aligned}$$

Rearranging this last identity yields

$$\begin{aligned}\sinh^2 x &= \frac{\cosh(2x) - 1}{2}\\ \cosh^2 x &= \frac{\cosh(2x) + 1}{2}.\end{aligned}$$

The addition/subtraction identities are

$$\begin{aligned}\sinh(x+y) &= \sinh x\cosh y + \cosh x\sinh y\\ \sinh(x-y) &= \sinh x\cosh y - \cosh x\sinh y\\ \cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y\\ \cosh(x-y) &= \cosh x\cosh y - \sinh x\sinh y.\end{aligned}$$

9.5 Antiderivatives

Let k, a and b be real constants. The following antiderivative formulas apply, where C denotes the arbitrary constant of integration.

$$\int kf\,dw = k\int f\,dw$$

$$\int f+g\,dw = \int f\,dw + \int g\,dw$$

$$\int u\,dv = uv - \int v\,du$$

$$\int k\,dw = kw + C$$

$$\int w^n\,dw = \frac{w^{n+1}}{n+1} + C,\ \ n \neq -1$$

$$\int \frac{dw}{w} = \ln|w| + C$$

$$\int e^w\,dw = e^w + C$$

$$\int we^w\,dw = (w-1)e^w + C$$

$$\int \frac{dw}{w^2+a^2} = \frac{1}{a}\arctan\left(\frac{w}{a}\right) + C$$

$$\int \frac{dw}{w^2-a^2} = \frac{1}{2a}\ln\left|\frac{w-a}{w+a}\right| + C$$

$$\int \frac{dw}{a^2-w^2} = \frac{1}{2a}\ln\left|\frac{w+a}{w-a}\right| + C$$

$$\int \frac{w\,dw}{(w+a)^3} = -\frac{2w+a}{2w^2+4aw+2a^2} + C$$

$$\int \frac{w\,dw}{(w-a)^3} = -\frac{2w-a}{2w^2-4aw+2a^2} + C$$

$$\int \frac{w\,dw}{(a-w)^3} = \frac{2w-a}{2w^2-4aw+2a^2} + C$$

$$\int \sin w\,dw = -\cos w + C$$

$$\int \cos w\,dw = \sin w + C$$

$$\int \tan w\,dw = \ln|\sec w| + C$$

$$\int \cot w\,dw = \ln|\sin w| + C$$

$$\int \sin^2 w\,dw = \frac{w}{2} - \frac{\sin(2w)}{4} + C$$

$$\int \cos^2 w\,dw = \frac{w}{2} + \frac{\sin(2w)}{4} + C$$

$$\int \tan^2 w\,dw = \tan w - w + C$$

$$\int \cot^2 w\,dw = -\cot w - w + C$$

$$\int \sec^2 w\,dw = \tan w + C$$

$$\int \csc^2 w\,dw = -\cot w + C$$

$$\int \sinh w\,dw = \cosh w + C$$

$$\int \cosh w\,dw = \sinh w + C$$

$$\int \tanh w\,dw = \ln(\cosh w) + C$$

$$\int \coth w\,dw = \ln|\sinh w| + C$$

$$\int \operatorname{sech}^2 w\,dw = \tanh w + C$$

$$\int \operatorname{csch}^2 w\,dw = -\coth w + C$$

$$
\begin{aligned}
\int \sin aw \tan aw \, dw &= -\frac{\sin aw}{a} - \frac{1}{2a} \ln \left| \frac{\sin aw - 1}{\sin aw + 1} \right| + C \\
\int \cos aw \cot aw \, dw &= \frac{\cos aw}{a} + \frac{1}{2a} \ln \left| \frac{\cos aw - 1}{\cos aw + 1} \right| + C \\
\int e^{aw} \sin bw \, dw &= \frac{e^{aw}}{a^2 + b^2} \left[a \sin bw - b \cos bw \right] + C \\
\int e^{aw} \cos bw \, dw &= \frac{e^{aw}}{a^2 + b^2} \left[a \cos bw + b \sin bw \right] + C
\end{aligned}
$$

9.6 Taylor series

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots, \ |x| < 1 \\
e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots, \ x \in \mathbb{R} \\
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \ldots, \ x \in \mathbb{R} \\
\cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots, \ x \in \mathbb{R} \\
\sinh x &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \ldots, \ x \in \mathbb{R} \\
\cosh x &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} = 1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \ldots, \ x \in \mathbb{R}
\end{aligned}
$$

Solutions to selected exercises

1.1.101: Compute $x' = -2e^{-2t}$ and $x'' = 4e^{-2t}$. Then $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$.
1.1.102: Yes.
1.1.103: $C_1 = 100$, $C_2 = -90$
1.1.104: $\varphi = -9e^{8s}$
1.1.105: $y = Cx^0 + Dx^2 = C + Dx^2$
1.1.106: $y = Ce^{2x} + De^{-3x}$
1.1.107: $y = Ce^{4x} + De^{-2x}$
1.1.108: $y = 5$
1.1.109: The easiest solution to identify is $y = 2x + 1$. Other solutions also exist, but can be tricky to find without learning techniques for solving DEs.
1.1.110: One group of solutions is $x = C \sin t$. Try to guess the remaining solutions.
1.1.111: The bone sample is approximately 38000 years old.
2.1.101: For $z = 2 + 3i$ and $w = 4i$, $z + w = 2 + 7i$, $zw = -12 + 8i$, $z^2 = -5 + 12i$, $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}\, i$, $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}\, i$, $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}\, i$, $\bar{z} = 2 - 3i$, $z\bar{z} = 13$, $(\bar{z})^2 = -5 - 12i$.
2.1.102: For $z = 4i$ and $w = 2 - 2i$, $z + w = 2 + 2i$, $zw = 8 + 8i$, $z^2 = -16$ $\frac{1}{z} = -\frac{1}{4}\, i$, $\frac{z}{w} = -1 + i$, $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}\, i$, $\bar{z} = -4i$, $z\bar{z} = 16$, $(\bar{z})^2 = -16$.
2.1.103: For $z = -5 + i$ and $w = 4 + 2i$, $z + w = -1 + 3i$, $zw = -22 - 6i$, $z^2 = 24 - 10i$, $\frac{1}{z} = -\frac{5}{26} - \frac{1}{26}\, i$, $\frac{z}{w} = -\frac{9}{10} + \frac{7}{10}\, i$, $\frac{w}{z} = -\frac{9}{13} - \frac{7}{13}\, i$, $\bar{z} = -5 - i$, $z\bar{z} = 26$, $(\bar{z})^2 = 24 + 10i$.
2.1.104: For $z = 1 - i\sqrt{3}$ and $w = -1 - i\sqrt{3}$, $z + w = -2i\sqrt{3}$, $zw = -4$, $z^2 = -2 - 2i\sqrt{3}$, $\frac{1}{z} = \frac{1}{4} + \frac{\sqrt{3}}{4}\, i$, $\frac{z}{w} = \frac{1}{2} + \frac{\sqrt{3}}{2}\, i$, $\frac{w}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\, i$, $\bar{z} = 1 + i\sqrt{3}$, $z\bar{z} = 4$, $(\bar{z})^2 = -2 + 2i\sqrt{3}$.
2.1.105: For $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\, i$ and $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\, i$, $-\sqrt{2}$, $zw = 1$, $z^2 = -i$, $\frac{1}{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\, i$, $\frac{z}{w} = -i$, $\frac{w}{z} = i$, $\bar{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\, i$, $z\bar{z} = 1$, $(\bar{z})^2 = i$.
2.5.101: Yes. Radius of convergence is 10.
2.5.102: Yes. Radius of convergence is e.
2.5.103: $\dfrac{1}{1-x} = -\dfrac{1}{1-(2-x)}$ so $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n$, which converges for $1 < x < 3$.

2.5.104: $\displaystyle\sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$

2.5.105: $f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

2.6.101: Yes. To justify try to find a constant A such that $\sin(x) = Ae^x$ for all x.

2.6.102: No. $e^{x+2} = e^2 e^x$.

2.6.103: No. $e^1 e^x - e^{x+1} = 0$.

2.6.104: Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x\sin(x)$ cannot be multiples of each other.)

3.1.101: $y = e^x + \frac{x^2}{2} + 9$

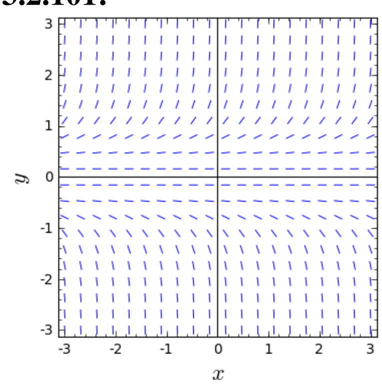
3.1.102: $x = (3t-2)^{1/3}$

3.1.103: $x = \sin^{-1}(t+1)$

3.1.104: 170 miles per hour

3.1.105: If $n \neq 1$, then $y = ((1-n)x+1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

3.2.101:



$y = 0$ is a solution such that $y(0) = 0$.

3.2.102: Yes a solution exists. $y' = f(x,y)$ where $f(x,y) = xy$. The function $f(x,y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0,0)$. So a solution exists and is unique. (In fact $y = 0$ is the solution).

3.2.103: No, the equation is not defined at $(x,y) = (1,0)$.

3.3.101: $y = Ce^{x^2}$

3.3.102: $x = e^{t^3} + 1$

3.3.103: $x^3 + x = t + 2$

3.3.104: $y = \dfrac{1}{1 - \ln x}$

3.3.105: $\sin(y) = -\cos(x) + C$

3.3.106: $y = \arctan t + C$

3.3.107: $y = \dfrac{t^{n+1}}{n+1} + 1$

3.3.108: $y = t \ln t - t + C$

3.3.109: $y = \pm\sqrt{t^2 + C}$

3.3.110: $y = \pm 1,\ y = \dfrac{1 + Ae^{2t}}{1 - Ae^{2t}}$

3.3.111: $\dfrac{y^4}{4} - 5y = \dfrac{t^2}{2} + C$

3.3.112: $y = \left(\dfrac{2t}{3}\right)^{\frac{3}{2}}$

3.3.113: $y = Ae^{-\arctan t}$

3.3.114: $y = e^2 e^{-e^t}$

3.3.115: $y = 4t^2$

3.3.116: $y = -2e^{(1/t)-1}$

3.3.117: $y = e^{1-t^{-2}}$

3.3.118: $y = 0$

3.3.119: $x(t) = t \ln|t| - 2t + \dfrac{2t - e + 1}{\ln|t|}$

3.4.101: $y = Ce^{-x^3} + {}^1\!/_3$

3.4.102: $y = 2e^{\cos(2x)+1} + 1$

3.4.103: $y = Ae^{-4t} + 2$

3.4.104: $y = Ae^{-(1/2)t^2} + 5$

3.4.105: $y = Ae^{-e^t} - 2$

3.4.106: $y = Ae^t - t^2 - 2t - 2$

3.4.107: $y = Ae^{-t/2} + t - 2$

3.4.108: $y = At^2 - \dfrac{1}{3t}$

3.4.109: $y = \dfrac{c}{t} + \dfrac{2}{3}\sqrt{t}$

3.4.110: $y = A\cos t + \sin t$

3.4.111: $y = \dfrac{A}{\sec t + \tan t} + 1 - \dfrac{t}{\sec t + \tan t}$

3.5.101: $y = \sqrt[3]{7e^{3x} + 3x + 1}$

3.5.102: $y = \sqrt{x^2 - \ln(C - x)}$

3.5.103: $y = \tan(x + C) - x - 3$

3.5.104: $y = \frac{3-x^2}{2x}$

3.5.105: $y = 2x \ln x + Cx$

3.6.101: 250 grams

3.6.102: $P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

3.7.101: The critical points are 0, 1, and 2. The point $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable. The limit is 1.

3.7.102: There are no critical points. The solution grows without bound: ∞.

3.7.103: The DE is $\dfrac{dx}{dt} = kx(M - x) + A$. The new limiting population is $\dfrac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$.

3.8.101: Approximately: 1.0000, 1.2397, 1.3829

3.8.102: The approximations for $x(4)$ are 0, 8, and 12, respectively, with step sizes 4, 2, and 1. The exact value is $x(4) = 16$, so the errors are 16, 8, and 4. All three factors are 0.5.

3.8.103: a) 0, 0, 0 b) $x = 0$ is a solution so errors are: 0, 0, 0.

4.1.101: The solution is $y = 3$ and $x = 2$.

4.1.102: Gauss's method here

$$\xrightarrow[\rho_1+\rho_3]{-3\rho_1+\rho_2} \begin{array}{rcrcl} x & & -z & = & 0 \\ & y & +3z & = & 1 \\ & y & & = & 4 \end{array} \xrightarrow{-\rho_2+\rho_3} \begin{array}{rcrcl} x & & -z & = & 0 \\ & y & +3z & = & 1 \\ & & -3z & = & 3 \end{array}$$

gives $x = -1$, $y = 4$, and $z = -1$.

4.1.103: Gaussian reduction shows that $y = 1/2$ and $x = 2$ is the unique solution.

4.1.104: Gauss's method gives $y = 3/2$ and $x = 1/2$ as the only solution.

4.1.105: Row reduction

$$\xrightarrow{-\rho_1+\rho_2} \begin{array}{rcrcrcr} x & - & 3y & + & z & = & 1 \\ & & 4y & + & z & = & 13 \end{array}$$

shows, because the variable z is not a leading variable in any row, that there are many solutions.

4.1.106: Row reduction shows that there is no solution.

4.1.107: Gauss's method

$$\xrightarrow{\rho_1\leftrightarrow\rho_4} \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ 2x & - & 2y & + & z & = & 0 \\ x & & & + & z & = & 5 \\ & & 4y & + & z & = & 20 \end{array} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ & & -4y & + & 3z & = & -20 \\ & & -y & + & 2z & = & -5 \\ & & 4y & + & z & = & 20 \end{array}$$

$$\xrightarrow[\rho_2+\rho_4]{-(1/4)\rho_2+\rho_3} \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ & & -4y & + & 3z & = & -20 \\ & & & & 5/4z & = & 0 \\ & & & & 4z & = & 0 \end{array}$$

gives the unique solution $(x, y, z) = (5, 5, 0)$.

4.1.108: Here Gauss's method gives

$$\xrightarrow[-2\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3} \begin{array}{rrrrrrrl} 2x & & + & z & + & w & = & 5 \\ & y & & & - & w & = & -1 \\ & & - & 5/2z & - & 5/2w & = & -15/2 \\ & y & & & - & w & = & -1 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_4} \begin{array}{rrrrrrrl} 2x & & + & z & + & w & = & 5 \\ & y & & & - & w & = & -1 \\ & & - & 5/2z & - & 5/2w & = & -15/2 \\ & & & & & 0 & = & 0 \end{array}$$

which shows that there are many solutions.

4.1.109: Do the reduction

$$\xrightarrow{-3\rho_1+\rho_2} \begin{array}{rcl} x - y & = & 1 \\ 0 & = & -3 + k \end{array}$$

to conclude this system has no solutions if $k \neq 3$ and if $k = 3$ then it has infinitely many solutions. It never has a unique solution.

4.2.101: $\left\{ \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} y \mid y \in \mathbb{R} \right\}$

4.2.102: $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

4.2.103: Gauss's method

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) \xrightarrow[-4\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

leaves x_1 and x_2 leading with x_3 free. The solution set is this.

$$\left\{ \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} x_3 \mid x_3 \in \mathbb{R} \right\}$$

4.2.104: This reduction is easy

$$\left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ 2 & 1 & 0 & 1 & 4 \\ 1 & -1 & 1 & 1 & 1 \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & 1 & -2 \\ 0 & -3 & 2 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and ends with x and y leading while z and w are free. Solving for y gives $y = (2 + 2z + w)/3$ and substitution shows that $x + 2(2 + 2z + w)/3 - z = 3$ so $x = (5/3) - (1/3)z - (2/3)w$, making this the solution set.

$$\left\{ \begin{pmatrix} 5/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \right\}$$

4.2.105: $\left\{ \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}$

4.2.106: This reduction

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow[-(1/2)\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & -3/2 & 1/2 & -1 \end{array}\right)$$
$$\xrightarrow{(-3/2)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -5/2 & -5/2 \end{array}\right)$$

shows that the solution set is a singleton set: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

4.2.107: The reduction

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 0 & 7 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 1 & -2 & -3 & -5 \end{array}\right)$$
$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

shows that there is no solution— the solution set is empty.

4.2.108: The solution set is empty.

4.2.109: $\left\{ \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/6 \\ 2/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$

4.2.110: This application of Gauss's method

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 3 & -1 & 7 \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & -1 & 6 \end{array}\right) \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

leaves x, y, and w leading. The solution set is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$.

4.2.111: This row reduction

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 3 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{-3\rho_1+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[\rho_2+\rho_4]{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

ends with z and w free. We have this solution set: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \right\}$.

4.2.112: Gauss's method done in this way

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 3 & -1 & 1 & 1 & 1 & 3 \end{array}\right) \xrightarrow{-3\rho_1+\rho_2} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 0 & -7 & -8 & -2 & 4 & 0 \end{array}\right)$$

ends with c, d, and e free. Solving for b shows that $b = (8c + 2d - 4e)/(-7)$ and then substitution $a + 2(8c + 2d - 4e)/(-7) + 3c + 1d - 1e = 1$ shows that $a = 1 - (5/7)c - (3/7)d - (1/7)e$ and we have the solution set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -5/7 \\ -8/7 \\ 1 \\ 0 \\ 0 \end{pmatrix} c + \begin{pmatrix} -3/7 \\ -2/7 \\ 0 \\ 1 \\ 0 \end{pmatrix} d + \begin{pmatrix} -1/7 \\ 4/7 \\ 0 \\ 0 \\ 1 \end{pmatrix} e \mid c, d, e \in \mathbb{R} \right\}$.

4.2.113: $\left\{\begin{pmatrix}-1\\4\\0\\0\end{pmatrix}+\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}a+\begin{pmatrix}3\\-2\\0\\1\end{pmatrix}b \mid a,b\in\mathbb{R}\right\}$.

4.2.114: This reduction

$$\left(\begin{array}{ccc|c}3&2&1&1\\1&-1&1&2\\5&5&1&0\end{array}\right)\xrightarrow[-(5/3)\rho_1+\rho_3]{-(1/3)\rho_1+\rho_2}\left(\begin{array}{ccc|c}3&2&1&1\\0&-5/3&2/3&5/3\\0&5/3&-2/3&-5/3\end{array}\right)$$
$$\xrightarrow{\rho_2+\rho_3}\left(\begin{array}{ccc|c}3&2&1&1\\0&-5/3&2/3&5/3\\0&0&0&0\end{array}\right)$$

gives this solution set

$$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+\begin{pmatrix}-3/5\\2/5\\1\end{pmatrix}z \mid z\in\mathbb{R}\right\}.$$

4.2.115: This is the reduction.

$$\left(\begin{array}{ccc|c}1&1&-2&0\\1&-1&0&3\\3&-1&-2&-6\\0&2&-2&3\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&1&-2&0\\0&-2&2&-3\\0&-4&4&-6\\0&2&-2&3\end{array}\right)$$
$$\xrightarrow[\rho_2+\rho_4]{-2\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&1&-2&0\\0&-2&2&-3\\0&0&0&0\\0&0&0&0\end{array}\right)$$

The solution set is $\left\{\begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}z \mid z\in\mathbb{R}\right\}$.

4.2.116: Gauss's method

$$\left(\begin{array}{cccc|c}2&-1&-1&1&4\\1&1&1&0&-1\end{array}\right)\xrightarrow{-(1/2)\rho_1+\rho_2}\left(\begin{array}{cccc|c}2&-1&-1&1&4\\0&3/2&3/2&-1/2&-3\end{array}\right)$$

gives the solution set

$$\left\{\begin{pmatrix}1\\-2\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}z\begin{pmatrix}-1/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\right\}.$$

4.2.117: Here is the reduction.

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & -3 \\ 3 & -1 & -2 & 0 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -3 \\ 0 & -4 & 4 & 0 \end{array}\right) \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

The solution set is empty (denoted $\varnothing$).

4.3.101: $\vec{x} = \begin{pmatrix} 15 \\ -5 \end{pmatrix}$

4.3.102: The solution set contains only a single element.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right) \xrightarrow{-(1/2)\rho_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

4.3.103: The solution set has one parameter.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 2 & 2 & -7 \end{array}\right) \xrightarrow{(1/2)\rho_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -7/2 \end{array}\right)$$

4.3.104: There is a unique solution.

$$\left(\begin{array}{cc|c} 3 & -2 & 1 \\ 6 & 1 & 1/2 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & 5 & -3/2 \end{array}\right) \xrightarrow[(1/5)\rho_2]{(1/3)\rho_1} \left(\begin{array}{cc|c} 1 & -2/3 & 1/3 \\ 0 & 1 & -3/10 \end{array}\right) \xrightarrow{(2/3)\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & 2/15 \\ 0 & 1 & -3/10 \end{array}\right)$$

4.3.105: A row swap in the second step makes the arithmetic easier:

$$\begin{aligned}
\left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right) &\xrightarrow{-(1/2)\rho_1+\rho_2} \left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & 7/2 & -1 & 11/2 \\ 0 & 1 & 2 & 5 \end{array}\right) \\
&\xrightarrow{\rho_2\leftrightarrow\rho_3} \left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 7/2 & -1 & 11/2 \end{array}\right) \xrightarrow{-(7/2)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -8 & -12 \end{array}\right) \\
&\xrightarrow[-(1/8)\rho_2]{(1/2)\rho_1} \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3/2 \end{array}\right) \xrightarrow{-2\rho_3+\rho_2} \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3/2 \end{array}\right) \\
&\xrightarrow{(1/2)\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3/2 \end{array}\right).
\end{aligned}$$

4.3.106:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -1 & -1 & 1 \\ 3 & 1 & 2 & 0 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & -1 & 1 \\ 0 & -2 & 5 & -9 \end{array}\right)$$

$$\xrightarrow{-(2/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 13/3 & -17/3 \end{array}\right) \xrightarrow[(3/13)\rho_3]{-(1/3)\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & -17/13 \end{array}\right)$$

$$\xrightarrow[(1/3)\rho_3+\rho_2]{\rho_3+\rho_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 22/13 \\ 0 & 1 & 0 & 16/13 \\ 0 & 0 & 1 & -17/13 \end{array}\right) \xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6/13 \\ 0 & 1 & 0 & 16/13 \\ 0 & 0 & 1 & -17/13 \end{array}\right)$$

4.3.107:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 1 \\ 4 & 1 & 5 & 1 \end{array}\right) \xrightarrow[-4\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & -3 & -3 & 1 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-(1/3)\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1/3 \\ 0 & 1 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

4.6.101: $\begin{pmatrix} 7 & 0 & 6 \\ 9 & 1 & 6 \end{pmatrix}$

4.6.102: $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}$

4.6.103: Not defined.

4.6.104: $\begin{pmatrix} 2 & -1 & -1 \\ 17 & -1 & -1 \end{pmatrix}$

4.6.105: Not defined.

4.6.106: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4.6.107: The reduction is routine.

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow[(1/2)\rho_2]{(1/3)\rho_1} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right)$$

$$\xrightarrow{-(1/3)\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & -1/6 \\ 0 & 1 & 0 & 1/2 \end{array}\right)$$

This answer agrees with the answer from the check.

$$\begin{pmatrix}3&1\\0&2\end{pmatrix}^{-1}=\frac{1}{3\cdot 2-0\cdot 1}\cdot\begin{pmatrix}2&-1\\0&3\end{pmatrix}=\frac{1}{6}\cdot\begin{pmatrix}2&-1\\0&3\end{pmatrix}$$

4.6.108: This reduction is easy.

$$\left(\begin{array}{cc|cc}2&1/2&1&0\\3&1&0&1\end{array}\right)\xrightarrow{-(3/2)\rho_1+\rho_2}\left(\begin{array}{cc|cc}2&1/2&1&0\\0&1/4&-3/2&1\end{array}\right)$$
$$\xrightarrow[4\rho_2]{(1/2)\rho_1}\left(\begin{array}{cc|cc}1&1/4&1/2&0\\0&1&-6&4\end{array}\right)\xrightarrow{-(1/4)\rho_2+\rho_1}\left(\begin{array}{cc|cc}1&0&2&-1\\0&1&-6&4\end{array}\right)$$

The check agrees.

$$\frac{1}{2\cdot 1-3\cdot(1/2)}\cdot\begin{pmatrix}1&-1/2\\-3&2\end{pmatrix}=2\cdot\begin{pmatrix}1&-1/2\\-3&2\end{pmatrix}$$

4.6.109: Trying the Gauss-Jordan reduction

$$\left(\begin{array}{cc|cc}2&-4&1&0\\-1&2&0&1\end{array}\right)\xrightarrow{(1/2)\rho_1+\rho_2}\left(\begin{array}{cc|cc}2&-4&1&0\\0&0&1/2&1\end{array}\right)$$

shows that the left side doesn't reduce to the identity, so no inverse exists. The check $ad-bc=2\cdot 2-(-4)\cdot(-1)=0$ agrees.

4.6.110: This produces an inverse:

$$\left(\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&2&4&0&1&0\\-1&1&0&0&0&1\end{array}\right)\xrightarrow{\rho_1+\rho_3}\left(\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&2&4&0&1&0\\0&2&3&1&0&1\end{array}\right)$$
$$\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&2&4&0&1&0\\0&0&-1&1&-1&1\end{array}\right)\xrightarrow[-\rho_3]{(1/2)\rho_2}\left(\begin{array}{ccc|ccc}1&1&3&1&0&0\\0&1&2&0&1/2&0\\0&0&1&-1&1&-1\end{array}\right)$$
$$\xrightarrow[-3\rho_3+\rho_1]{-2\rho_3+\rho_2}\left(\begin{array}{ccc|ccc}1&1&0&4&-3&3\\0&1&0&2&-3/2&2\\0&0&1&-1&1&-1\end{array}\right)$$
$$\xrightarrow{-\rho_2+\rho_1}\left(\begin{array}{ccc|ccc}1&0&0&2&-3/2&1\\0&1&0&2&-3/2&2\\0&0&1&-1&1&-1\end{array}\right).$$

4.6.111: This is one way to do the reduction.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\rho_3 \leftrightarrow \rho_1} \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 7 & 1 & 1/2 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-(1/2)\rho_2 \\ (1/7)\rho_3}]{(1/2)\rho_1} \left(\begin{array}{ccc|ccc} 1 & 3/2 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & -2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right)$$

$$\xrightarrow[\rho_3+\rho_1]{2\rho_3+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/7 & 1/14 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right)$$

$$\xrightarrow{-(3/2)\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/7 & 17/28 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right)$$

4.6.112: There is no inverse.

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 4 & -2 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -9/2 & -1/2 & 1 & 0 \\ 0 & -6 & -9 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -9/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

As a check, note that the third column of the starting matrix is $3/2$ times the second, and so it is indeed singular and therefore has no inverse.

4.6.113: $M^{-1} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}$ and $N^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$

4.7.101: -2

4.7.102: -15

4.8.101: eigenvalues: $\dfrac{1+\sqrt{3}i}{2}, \dfrac{1-\sqrt{3}i}{2}$, eigenvectors: $\begin{pmatrix} -2 \\ 1-\sqrt{3}i \end{pmatrix}, \begin{pmatrix} -2 \\ 1+\sqrt{3}i \end{pmatrix}$.

4.8.102: eigenvalues: $4, 0, -1$, eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$.

5.1.101: $y = C_1 \ln(x) + C_2$

5.2.101: $y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$

5.2.102: $y = C_1 e^{3x} + C_2 x e^{3x}$

5.2.103: $y = e^{-x/4} \cos\left(\frac{\sqrt{7}}{4}x\right) - \sqrt{7} e^{-x/4} \sin\left(\frac{\sqrt{7}}{4}x\right)$

5.2.104: $y(x) = \frac{2(a-b)}{5}\, e^{-3x/2} + \frac{3a+2b}{5}\, e^{x}$

5.2.105: $z(t) = 2e^{-t}\cos t$

5.2.106: $y = -{}^{1}\!/_{4}\, e^{-5t} + {}^{5}\!/_{4}\, e^{-t}$

5.2.107: $y = -2e^{-3t} + 2e^{4t}$

5.2.108: $y = 5e^{-6t} + 20te^{-6t}$

5.2.109: $y = e^{-6t}\,(4\cos t + 24\sin t)$

5.2.110: $y = 2e^{-3t}\sin(3t)$

5.4.101: $y = C_1 e^{x} + C_2 x^3 + C_3 x^2 + C_4 x + C_5$

5.4.102: The characteristic equation is $r^3 - 3r^2 + 4r - 12 = 0$, the differential equation is $y''' - 3y'' + 4y' - 12y = 0$, and the general solution is $y = C_1 e^{3x} + C_2 \sin(2x) + C_3 \cos(2x)$.

5.4.103: $y = 0$

5.5.101: $k = {}^{8}\!/_{9}$ (and larger)

5.5.102: a) $k = 500000$, b) $\frac{1}{5\sqrt{2}} \approx 0.141$, c) 45000 kilograms, d) 11250 kilograms

5.6.101: $y = \frac{-16\sin 3x + 6\cos 3x}{73}$

5.6.102: $y = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \frac{2e^{x} + 3x^3 - 9x}{6}$

5.6.103: $y(x) = x^2 - 4x + 6 + e^{-x}(x-5)$

5.6.104: $Ae^{-6t} + Bte^{-6t} + 3t^2 e^{-6t}$

5.6.105: $C_1 e^{4x} + C_2 x e^{4x} - t^2 e^{4x}$

5.6.106: $Ae^{-t} + Be^{-5t} + (4/5)$

5.6.107: $Ae^{4t} + Be^{-3t} + (1/144) - (t/12)$

5.6.108: $e^{-6t}(2\cos t + 20\sin t) + 2e^{-4t}$

5.6.109: $\left(-\frac{23}{325}\cos(3t) + \frac{592}{975}\sin(3t)\right) + \frac{23}{325}\cos t - \frac{11}{325}\sin t$

5.6.110: $y = \frac{-\sin(x+c)}{3} + C_1 e^{\sqrt{2}\,x} + C_2 e^{-\sqrt{2}\,x}$

5.7.101: $A\sin(t) + B\cos(t) + \frac{1}{5}e^{2t}$

5.7.102: $A\sin(2t) + B\cos(2t) + \cos t - \sin t \cos t \ln|\sec t + \tan t|$

5.7.103: $Ae^{2t} + Be^{-3t} + \frac{t^3}{15}e^{2t} - \left(\frac{t^2}{5} - \frac{2t}{25} + \frac{2}{125}\right)\frac{e^{2t}}{5}$

5.7.104: $Ae^t \sin t + Be^t \cos t - e^t \cos t \ln|\sec t + \tan t|$

5.7.105: $y = \frac{2xe^x - (e^x + e^{-x}) \log(e^{2x}+1)}{4}$

5.8.101: $\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984 \quad C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$

5.8.102: $x_{sp} = \dfrac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\cos(\omega t) + \dfrac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\sin(\omega t) + \dfrac{A}{k}$, where $p = \dfrac{c}{2m}$ and $\omega_0 = \sqrt{\dfrac{k}{m}}$.

5.8.103: $\omega = 2$ and the amplitude is 25

6.1.101: $y_1 = C_1 e^{3x}$, $y_2 = C_2 e^x + \frac{C_1}{2}e^{3x}$, $y_3 = C_3 e^x + \frac{C_1}{2}e^{3x}$

6.1.102: $x = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$, $y = \frac{5}{3}e^{2t} + \frac{4}{3}e^{-t}$

6.1.103: $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_1 + t$

6.1.104: $y_3' + y_1 + y_2 = t$, $y_4' + y_1 - y_2 = t^2$, $y_1' = y_3$, $y_2' = y_4$

6.2.101: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ t & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

6.2.102: $\vec{x}' = \begin{pmatrix} 0 & 2t \\ 0 & 2t \end{pmatrix}\vec{x}$, $\vec{x} = \begin{pmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{pmatrix}$

6.2.103: Yes.

6.2.104: No. $2\begin{pmatrix} \cosh(t) \\ 1 \end{pmatrix} - \begin{pmatrix} e^t \\ 1 \end{pmatrix} - \begin{pmatrix} e^{-t} \\ 1 \end{pmatrix} = \vec{0}$

6.3.101: $\vec{x} = C_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{4t} + C_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}e^{-t}$

6.3.102: $\vec{x} = C_1 e^{t/2}\begin{pmatrix} -2\cos(\frac{\sqrt{3}t}{2}) \\ \cos(\frac{\sqrt{3}t}{2}) + \sqrt{3}\sin(\frac{\sqrt{3}t}{2}) \end{pmatrix} + C_2 e^{t/2}\begin{pmatrix} -2\sin(\frac{\sqrt{3}t}{2}) \\ \sin(\frac{\sqrt{3}t}{2}) - \sqrt{3}\cos(\frac{\sqrt{3}t}{2}) \end{pmatrix}$

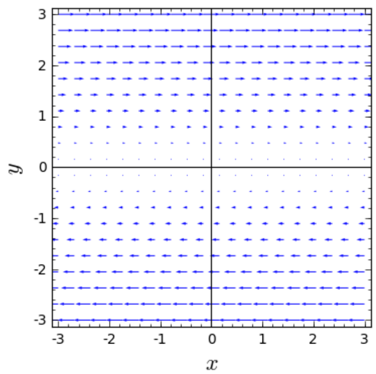
6.3.103: $\vec{x} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$

6.3.104: $\vec{x} = C_1\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + C_2\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

6.4.101: a) Two eigenvalues: $\pm\sqrt{2}$ so the behavior is a saddle. b) Two eigenvalues: 1 and 2, so the behavior is a source. c) Two eigenvalues: $\pm 2i$, so the behavior is a center (ellipses). d) Two eigenvalues: -1 and -2, so the behavior is a sink. e) Two eigenvalues: 5 and -3, so the behavior is a saddle.

6.4.102: Spiral source.

6.4.103:



The solution will not move anywhere if $y = 0$. When y is positive, then the solution moves (with constant speed) in the positive x direction. When y is negative, then the solution moves (with constant speed) in the negative x direction. It is not one of the behaviors we have seen.
Note that the matrix has a double eigenvalue 0 and the general solution is $x = C_1 t + C_2$ and $y = C_1$, which agrees with the above description.

6.5.101: $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \bigl(a_1 \cos(\sqrt{3}\,t) + b_1 \sin(\sqrt{3}\,t)\bigr) + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \bigl(a_2 \cos(\sqrt{2}\,t) + b_2 \sin(\sqrt{2}\,t)\bigr) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bigl(a_3 \cos(t) + b_3 \sin(t)\bigr) + \begin{pmatrix} -1 \\ 1/2 \\ 2/3 \end{pmatrix} \cos(2t)$

6.5.102: $\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \vec{x}'' = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \vec{x}$. Solution: $\vec{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bigl(a_1 \cos(\sqrt{3k/m}\,t) + b_1 \sin(\sqrt{3k/m}\,t)\bigr) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \bigl(a_2 \cos(\sqrt{k/m}\,t) + b_2 \sin(\sqrt{k/m}\,t)\bigr) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (a_3 t + b_3)$.

6.5.103: $x_2 = (2/5)\cos(\sqrt{1/6}\,t) - (2/5)\cos(t)$

7.1.101: ordinary

7.1.102: singular but not regular singular

7.1.103: regular singular

7.1.104: regular singular

7.1.105: ordinary

7.2.101:

$$\begin{aligned}
0 = y'' + 2x^3 y &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) + 2x^3\left(\sum_{k=0}^{\infty} a_k x^k\right) \\
&= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} 2a_k x^{k+3}\right). \\
&= \left(\sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) + \left(\sum_{k=3}^{\infty} 2a_{k-3} x^k\right). \\
&= 2a_2 + 6a_3 x + 12a_4 x^2 + \left(\sum_{k=3}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) + \left(\sum_{k=3}^{\infty} 2a_{k-3} x^k\right).
\end{aligned}$$

$a_2 = 0$, $a_3 = 0$, $a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = -2a_{k-5}$, so:
$y(x) = a_0 + a_1 x - 2a_0 x^5 - 2a_1 x^6 + 4a_0 x^{10} + 4a_1 x^{11} - 8a_0 x^{15} - 8a_1 x^{16} + \cdots$

7.2.102:

$$\begin{aligned}
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = y'' - xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - x\left(\sum_{k=0}^{\infty} a_k x^k\right) \\
&= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right). \\
&= \left(\sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} a_{k-1} x^k\right). \\
&= 2a_2 + \left(\sum_{k=1}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} a_{k-1} x^k\right).
\end{aligned}$$

a) $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = a_{k-3} + 1$, so
$y(x) = a_0 + a_1 x + \frac{1}{2}x^2 + (a_0+1)x^3 + (a_1+1)x^4 + \frac{3}{2}x^5 + (a_0+2)x^6 + (a_1+2)x^7 + \frac{5}{2}x^8 + (a_0+3)x^9 + (a_1+3)x^{10} + \cdots$
b) $y(x) = \frac{1}{2}x^2 + x^3 + x^4 + \frac{3}{2}x^5 + 2x^6 + 2x^7 + \frac{5}{2}x^8 + 3x^9 + 3x^{10} + \cdots$

7.2.103:

$$\begin{aligned}
0 = x^2 y'' - y &= x^2\left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=0}^{\infty} a_k x^k\right) \\
&= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^k\right) - a_0 - a_1 x - \left(\sum_{k=2}^{\infty} a_k x^k\right).
\end{aligned}$$

so $a_0 = 0$, $a_1 = 0$, $k(k-1)a_k = a_k$ Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

7.3.101: $y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$

7.3.102: $y = x^{3/2} \sum_{k=0}^{\infty} \frac{(-1)^{-1}}{k!(k+2)!} x^k$ (Note that for convenience we did not pick $a_0 = 1$.)

7.3.103: $y = Ax + Bx \ln(x)$

8.1.101: $\frac{8}{s^3} + \frac{8}{s^2} + \frac{4}{s}$

8.1.102: $2t^2 - 2t + 1 - e^{-2t}$

8.1.103: $\frac{1}{(s+1)^2}$

8.1.104: $\frac{1}{s^2+2s+2}$

8.2.101: $f(t) = (t-1)\big(u(t-1) - u(t-2)\big) + u(t-2)$

8.2.102: $x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{e^{-t}}{2} + \frac{3e^t}{2}$

8.2.103: $H(s) = \frac{1}{s+1}$

8.3.101: $\frac{1}{2}(\cos t + \sin t - e^{-t})$

8.3.102: $5t - 5 \sin t$

8.3.103: $\frac{1}{2}(\sin t - t \cos t)$

8.3.104: $\int_0^t f(\tau)\big(1 - \cos(t-\tau)\big)\, d\tau$

8.4.101: $x(t) = t$

8.4.102: $x(t) = e^{-at}$

8.4.103: $x(t) = (\cos * \sin)(t) = \frac{1}{2} t \sin(t)$

8.4.104: $\delta(t) - \sin(t)$

8.4.105: $3\delta(t-1) + 2t$

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