Special Mathematics Lecture
Introduction to functional analysis

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Handwritten notes taken by L. Zhang

Introduction to Functional Analysis

Distribution theory

1) Test functions and distributions

Aim: to understand

$$\int f(x)\,\delta_0(x)\,dx = f(0)$$

Who is $\delta_0 \equiv \delta$?

Answer: δ is a linear and continuous functional on smooth functions with compact support.

Reminder: Consider $f : \mathbb{R}^n \longmapsto \mathbb{R}$ (or $\mathbb{C}$) and

$$[\partial_j f](x) \equiv [D_j f](x) \equiv [\nabla_j f](x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon E_j) - f(x)}{\varepsilon}$$

for $x \in \mathbb{R}^n$, $E_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ position j and $j \in \{1, \cdots, n\}$

For $m \in \mathbb{N}$, we set

$$\partial_j^m f := \underbrace{\partial_j \partial_j \cdots \partial_j}_{m \text{ times}} f$$

If $\alpha \in \mathbb{N}^n = \{\alpha = (\alpha_1, \overset{\mathbb{N}}{\overset{\cup}{\alpha_2}}, \cdots, \alpha_n)\}$ multi-index then

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$$

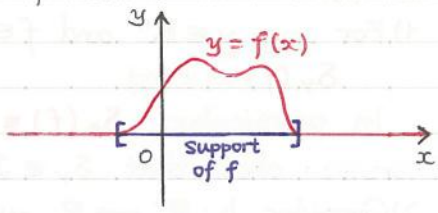
Def. A function $f : \mathbb{R}^n \longmapsto \mathbb{C}$ is smooth if $\partial^\alpha f$ exists and is a continuous function for any $\alpha \in \mathbb{N}^n$.

Remark: 1) For such function, $\partial_j \partial_k f = \partial_k \partial_j f$ for $j, k \in \{1, \cdots, n\}$

2) Any derivative of a smooth function is a smooth function.

Def. For any $f : \mathbb{R}^n \longmapsto \mathbb{C}$ the support of f is

$\overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ ← closure in $\mathbb{R}^n$



Example: $n = 1$, $f(x) = x \Rightarrow$ the support of f is $\mathbb{R}$.

We write $\mathrm{supp}(f)$ for the support.

Notation: We write $C^\infty(\mathbb{R}^n)$ either $C^\infty(\mathbb{R}^n, \mathbb{R})$ or $C^\infty(\mathbb{R}^n, \mathbb{C})$ for the set of all smooth functions on $\mathbb{R}^n$.

Def. $f \in C^\infty(\mathbb{R}^n)$ is a test function if its support is compact ($\equiv$ bounded).

We write $\mathcal{D}(\mathbb{R}^n)$ for the set of all test functions.

Example: For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we set $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{1 - \|x\|}\right) & \text{if } \|x\| \leq 1 \\ 0 & \text{if } \|x\| > 1 \end{cases}$$

This function has support in $\overline{B(\underset{\mathbb{R}^n}{\underset{\cap}{\mathbf{0}}}, \underset{\mathbb{R}}{\underset{\cap}{1}})}$ ball of radius 1 and centred at $\mathbf{0} \in \mathbb{R}^n$ $= \overline{\{y \in \mathbb{R}^n \mid \|y\| < 1\}}$

Remark: $\mathcal{D}(\mathbb{R}^n)$ is stable for addition $f, g \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \forall \lambda \in \mathbb{R}$ or $\mathbb{C}$: $f + \lambda g \in \mathcal{D}(\mathbb{R}^n)$ and for multiplication. $(fg \in \mathcal{D}(\mathbb{R}^n))$

This is an algebra.

No.

Date

Def. (Convergence in $\mathcal{D}(\mathbb{R}^n)$)

A sequence of function $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$ if

1) $\sup_{x\in\mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \xrightarrow{j\to\infty} 0$ for any $\alpha \in \mathbb{N}^n$

2) $\operatorname{supp}(f_j) \subset B(0, R)$ for some large $R \in \mathbb{R}$ and all $j \in \mathbb{N}$. (independent of j)

Remark: This is stronger (and including) than uniform convergence.

Def. A distribution T on $\mathbb{R}^n$ is a continuous linear function from $\mathcal{D}(\mathbb{R}^n)$ to $\mathbb{C}$ or $\mathbb{R}$.

More explicitly, $T: \mathcal{D}(\mathbb{R}^n) \longmapsto \mathbb{C}$ or $\mathbb{R}$ such that

1) $\underbrace{T(\underbrace{f_1 + \lambda f_2}_{\in\mathcal{D}(\mathbb{R}^n)})}_{\in\mathbb{C}} = \underbrace{Tf_1}_{\in\mathbb{C}} + \underbrace{\lambda \underbrace{Tf_2}_{\in\mathbb{C}}}_{\in\mathbb{C}}$.

2) $\underbrace{T(f_j)}_{\in\mathbb{C}} \xrightarrow{j\to\infty} \underbrace{T(f_\infty)}_{\in\mathbb{C}}$ if $\{f_j\}$ converges to f_∞ in $\mathcal{D}(\mathbb{R}^n)$

The family of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$ dual space of $\mathcal{D}(\mathbb{R}^n)$.

Again, 2 distributions can be added:

$$(T_1 + \lambda T_2) f := T_1 f + \lambda T_2 f$$

⚠ Multiplication is not well-defined.

We often write $Tf = T(f) = \langle T, f \rangle$ duality.

Examples:

1) For any $y \in \mathbb{R}^n$ and $f \in \mathcal{D}(\mathbb{R}^n)$, we set Dirac delta distribution at y

$$\delta_y(f) := f(y)$$

In particular $\delta_0(f) \equiv \delta(f) = f(0)$

Exercise: check that $\delta_y \in \mathcal{D}(\mathbb{R}^n)$

2) Consider $h: \mathbb{R}^n \longmapsto \mathbb{R}$ with

$$h \in L^1_{loc}(\mathbb{R}^n) := \{h: \mathbb{R}^n \longmapsto \mathbb{R} \mid \int_{B(y,1)} |h(x)|\, dx < \infty \ \forall y \in \mathbb{R}^n\}$$ very big set!

and set

$$T_h(f) := \int_{\mathbb{R}^n} f(x) h(x)\, dx$$

and observe that $T_h(f)$ is a distribution.

Exercise: check ⟶

Since for any $h \in L^1_{loc}(\mathbb{R}^n)$ we can define $T_h \in \mathcal{D}'(\mathbb{R}^n)$

we can identify $L^1_{loc}(\mathbb{R}^n)$ AS A SUBSET OF $\mathcal{D}'(\mathbb{R}^n)$.

"Any" function is a distribution by identifying h with T_h.

For that reason, we sometimes write $\int f(x)\delta_0(x)\,dx$ for $\delta_0(f)$.

Examples:

3) For any $\alpha \in \mathbb{N}^n$, we set

$$\delta_y^\alpha(f) := \delta_y(\partial^\alpha f) = [\partial^\alpha f](y)$$

Exercise: check that $\delta_y^\alpha \in \mathcal{D}'(\mathbb{R}^n)$

Proposition: (your friend)

$T \in \mathcal{D}'(\mathbb{R}^n)$ if and only if

1) $T : \mathcal{D}(\mathbb{R}^n) \longmapsto \mathbb{R}$ is linear and

2) $\forall y \in \mathbb{R}^n$ and $R > 0$: $\exists c > 0$ and $m \in \mathbb{N}$ with

$$|T(f)| \leq c \sum_{\substack{|\alpha| \leq m \\ \| \\ \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq m \,;\ \alpha \in \mathbb{N}^n}} \boxed{\|\partial^\alpha f\|_\infty} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^n) \text{ with } \operatorname{supp}(f) \subset B(y, R).$$

Remark: T is of finite order if we can choose one $\overset{\text{order}}{\overset{\|}{m}}$ for any y and R.

↳ indep of

Then, T_h and δ_y are of order 0

while δ_y^β is of order $|\beta| := \beta_1 + \cdots + \beta_n$.

The set of test functions is very limited so that it is difficult to find an example. So is the def of distribution $T : \mathcal{D}(\mathbb{R}^n) \longmapsto \mathbb{R}$ or $\mathbb{C}$ too limited? The distribution δ_y can act on almost every function which is continuous at y. Can other distributions have larger domains too?

2) Derivatives of distributions

Extension: support of a distribution

Recall that for $f, g : \mathbb{R} \mapsto \mathbb{R}$,

$$\int_a^b f'(x) g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f(x) g'(x)\, dx$$

If $a = -\infty$, $b = +\infty$, and $f, g \in \mathcal{D}(\mathbb{R})$, then

$$\int_a^b f'(x) g(x)\, dx = -\int_a^b f(x) g'(x)\, dx$$

Similarly, if $f, g \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\partial_j f)(x)\, g(x)\, dx = -\int_{\mathbb{R}^n} f(x)\, (\partial_j g)(x)\, dx$$

By induction, one gets

$$\int_{\mathbb{R}^n} (\partial^\alpha f)(x)\, g(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\, (\partial^\alpha g)(x)\, dx$$

Def. For $T \in \mathcal{D}'(\mathbb{R}^n)$ and for $f \in \mathcal{D}(\mathbb{R}^n)$ one sets

$$(\partial^\alpha T)(f) := (-1)^{|\alpha|}\, T(\partial^\alpha f)$$

Remark: We want to have

$$(\partial^\alpha T_h)(f) = (-1)^{|\alpha|}\, T_h(\partial^\alpha f) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} h(x) (\partial^\alpha f)(x)\, dx$$
$$= \int_{\mathbb{R}^n} (\partial^\alpha h)(x) f(x)\, dx = T_{\partial^\alpha h}(f)$$

if h is regular enough.

Lemma: $\partial^\alpha T \in \mathcal{D}'(\mathbb{R}^n)$

Exercise: proof ↗ (use the prop of last week)

Remark: In particular, T_h can be diffrenciated any times for any $h \in L^1_{loc}(\mathbb{R}^n)$

It means, any function in $L^1_{loc}(\mathbb{R}^n)$ is differenciable in the sense of distribution.

Examples:

1) Let $H : \mathbb{R} \mapsto \mathbb{R}$ be the Heaviside function:

$$H(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Clearly $H \in L^1_{loc}(\mathbb{R}) \Rightarrow T_H' \in \mathcal{D}'(\mathbb{R})$. What is ∂T_H?

$$(\partial T_H)(f) = -T_H(\partial f) = -T_H(f') = -\int_{\mathbb{R}} H(x) f'(x)\, dx = -\int_0^\infty f'(x)\, dx$$
$$= \underbrace{-f(\infty)}_{\substack{\shortparallel \\ 0}} + f(0) = f(0) = \delta_0(f)$$

$\Rightarrow \partial T_H = \delta_0$ (Horribly written as $\partial H = \delta_0$)

2) Recall that $P_V \frac{1}{x} \in \mathcal{D}'(\mathbb{R})$ (principal value) defined by

$$\left(P_V \tfrac{1}{x}\right)(f) := \lim_{\varepsilon \searrow 0} \left[\int_{-\infty}^{-\varepsilon} \underbrace{\tfrac{1}{x}}_{= \ln(|x|)' =} f(x)\, dx + \int_\varepsilon^\infty \tfrac{1}{x} f(x)\, dx \right]$$

Exercise: $\hookrightarrow \in \mathcal{D}'(\mathbb{R})$
1)
2)
3)
$\ln(x) \in L^1_{loc}(\mathbb{R})$

$$= \lim_{\varepsilon \searrow 0} \left[\int_{-\infty}^{-\varepsilon} \ln(|x|)' f(x)\, dx + \int_\varepsilon^\infty \ln(|x|)' f(x)\, dx \right]$$

$$= -\int_{\mathbb{R}} \ln(|x|) f'(x)\, dx = \partial T_{\ln(|x|)}(f)$$

$\Rightarrow \partial T_{\ln(|x|)} = P_V \frac{1}{x}$ (Also $\partial \ln(|\cdot|) = P_V \frac{1}{x}$)

Extension: For $n=3$ consider the function

$$h : \mathbb{R}^3 \setminus \{0\} \ni x \longmapsto \frac{1}{\|x\|} \in \mathbb{R}$$

and observe that $h \in L^1_{loc}(\mathbb{R}^3) \Rightarrow T_h \in \mathcal{D}'(\mathbb{R}^3)$

Recall that

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

Question: what is $\Delta T_h = -4\pi \delta_0$?

(in Physics $\Delta \frac{1}{\|x\|} = -4\pi \delta_0^3$) ← for dim

Ref: Section 3.5 of the reference book

3) Other operations with distribution

Def.

- **Multiplication by smooth function:**

  Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$. We set

  $$(gT)(f) := T(\underbrace{gf}_{\in \mathcal{D}(\mathbb{R}^n)}) \quad \text{for } f \in \mathcal{D}(\mathbb{R}^n)$$

Exercise: check that $gT \in \mathcal{D}'(\mathbb{R}^n)$.

Remark: If $T = T_h$ then $gT_h = T_{\underbrace{gh}_{\in L^1_{loc}(\mathbb{R}^n) \text{ if } h \in L^1_{loc}(\mathbb{R}^n)}}$

Examples:
- $g\delta_0 = g(0)\delta_0$
- $g\delta_0' = g(0)\delta_0' - g'(0)\delta_0$ (exercise)

- **Convergence in $\mathcal{D}'(\mathbb{R}^n)$**

  A sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to $T_\infty \in \mathcal{D}'(\mathbb{R}^n)$ if

  $$T_i(f) \longrightarrow T_\infty(f) \quad \forall f \in \mathcal{D}(\mathbb{R}^n)$$

Remark: If $\{T_i(f)\}$ converges in $\mathbb{C}$ or in $\mathbb{R}$, for any $f \in \mathcal{D}(\mathbb{R}^n)$ then $\exists T_\infty \in \mathcal{D}(\mathbb{R}^n) : T_i \xrightarrow{i\in\infty} T_\infty$ } completeness

Remark: If $T_i \xrightarrow{i\to\infty} T_\infty$ in distribution,

then $\partial^\alpha T_i \xrightarrow{i\to\infty} \partial^\alpha T_\infty$ in distribution.

Example:

Set $h_i : \mathbb{R} \longmapsto \mathbb{R}$, $h_i(x) = \frac{\sin(ix)}{x}$, $i \in \mathbb{N}$

Then $T_{h_i} \xrightarrow{i\to\infty} \pi\delta_0$ in distribution.

- Convolution of distributions

If $f, g : \mathbb{R} \mapsto \mathbb{R}$ we set

$$(f * g)(x) := \text{const} \int f(x-y)\, g(y)\, dy$$

for the convolution of f and g.

Extension: sometimes we can also define the convolution of 2 distributions.

Remark: Every distribution can be written locally as $\partial^\alpha T_h$ for some $h \in C(\mathbb{R}^n)$ and some $\alpha \in \mathbb{N}^n$.

$$\left(\begin{array}{l}\text{Take } R > 0 \text{ and } f \in \mathcal{D}(\mathbb{R}^n) \text{ with } \operatorname{supp}(f) \in B(0, R) \\ \text{Then } T(f) \equiv \partial^\alpha T_h(f) \ \exists \alpha \in \mathbb{N}^n \text{ and } h \in C(\mathbb{R}^n)\end{array}\right)$$

4) Fourier transform, Schwartz functions and tempered distributions.

Def. For any $f \in L^1(\mathbb{R}^n) := \{ f : \mathbb{R}^n \mapsto \mathbb{R} \mid \int_{\mathbb{R}^n} |f(x)|\, dx < \infty \}$ we set

scalar product in $\mathbb{R}^n$

$$[\mathcal{F} f](x) \equiv \hat{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)\, dy$$

We call it the Fourier transform of f.

Properties:

- $\mathcal{F}$ is linear
- $|\hat{f}(x)| \leq \int |f(y)|\, dy < \infty$
- $\hat{f} \in C_0(\mathbb{R}^n) := \{ g \in C(\mathbb{R}^n) \mid \lim_{\|x\| \to \infty} g(x) = 0 \}$
- $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}$ (const $= \frac{1}{(2\pi)^{n/2}}$)
- If $f, \partial_j f \in L^1(\mathbb{R}^n)$ then $\widehat{-i\partial_j f}(x) = x_j \hat{f}(x)$

with $i^2 = -1$

Prop. $\mathcal{F}$ extends to a bijective map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$

$(f \in L^2(\mathbb{R}^n)$ iff $\int |f(x)|^2 dx < \infty)$ and

$$[\mathcal{F}^{-1} f](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{+ix\cdot y} f(y)\, dy$$

and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F} f\|_{L^2(\mathbb{R}^n)} \quad \text{(Plancherel equality)}$$

$$\| \quad \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{\frac{1}{2}}$$

⚠ $\int e^{-ix\cdot y} f(y)\, dy$ for $f \in L^2(\mathbb{R}^n)$ is not well-defined. We have to consider

$$\lim_{R\to\infty} \int_{[-R,R]^n} e^{-ix\cdot y} f(y)\, dy$$

We would like to define $\mathcal{F}T$ (Fourier transform for a distribution) for $T \in \mathcal{D}'(\mathbb{R}^n)$, and the normal expression should be $[\mathcal{F}T](\varphi) = T(\mathcal{F}\varphi)$.

Problem: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\mathcal{F}\varphi \in \underbrace{C_0(\mathbb{R}^n)}_{:= \{f \in C(\mathbb{R}^n) \mid \lim_{|x|\to\infty} f(x) = 0\}}$ but $\mathcal{F}\varphi \notin \mathcal{D}(\mathbb{R}^n)$ in general.

We need to replace $\mathcal{D}(\mathbb{R}^n)$ by a set invariant under Fourier transform.

Def. Let $\mathcal{S}(\mathbb{R}^n)$ be the subset of $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty \quad \forall f \in \mathcal{S}(\mathbb{R}^n),\ \alpha, \beta \in \mathbb{N}^n$$

$$\forall x \in \mathbb{R}^n,\ \beta \in \mathbb{N}^n:\ x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

$\mathcal{S}(\mathbb{R}^n)$ is called the Schwartz space.

Clearly

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

$$h \longleftrightarrow T_h$$

Properties:

- $\mathcal{S}(\mathbb{R}^n)$ is a vector space.
- If $f \in \mathcal{S}(\mathbb{R}^n)$, then $x^\beta \partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$.
- $\mathcal{F}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ easy

Def. (Convergence in $\mathcal{S}(\mathbb{R}^n)$)

A sequence $(f_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{S}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha (f_j - f_\infty)(x)| \xrightarrow{j\to\infty} 0 \quad \forall \alpha, \beta \in \mathbb{N}^n$$

⚠ Not the same notation as in $\mathcal{D}(\mathbb{R}^n)$!

Def. A tempered distribution is a distribution which is linear and continuous on $\mathcal{S}(\mathbb{R}^n)$.

It means if $f_j \xrightarrow{j\to\infty} f_\infty$ in $\mathcal{S}(\mathbb{R}^n)$ then $T(f_j) \xrightarrow{j\to\infty} T(f_\infty)$.

The set of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Clearly $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and observe that $\mathcal{D}'(\mathbb{R}^n) \ni T_{x\mapsto e^{x^2}} \notin \mathcal{S}'(\mathbb{R}^n)$.

Indeed $T_{x\mapsto e^{x^2}}(\varphi) = \int e^{x^2} \varphi(x)\, dx$ not well defined for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$L^1_{loc}(\mathbb{R}^n) \not\subset S'(\mathbb{R}^n)$ since $[x \mapsto e^{x^2}] \in L^1_{loc}(\mathbb{R}^n)$ but $T_{x \mapsto e^{x^2}} \notin S'(\mathbb{R}^n)$.

Observe that if $T \in S'(\mathbb{R}^n)$ then $\partial^\alpha T \in S'(\mathbb{R}^n)$

since $\partial^\alpha \varphi \in S(\mathbb{R}^n)$ if $\varphi \in S(\mathbb{R}^n)$; $(\partial^\alpha T)(\varphi) = (-1)^\alpha T(\partial^\alpha \varphi)$

and one can take care of the continuity.

Prop. The distribution T is tempered iff :

$\exists c > 0, m \in \mathbb{N}$:

$$|\langle T, \varphi \rangle| \equiv |T(\varphi)| \leq c \sum_{|\alpha|, |\beta| \leq m} \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \varphi(x)| \quad \forall \varphi \in S(\mathbb{R}^n).$$

No localization compared to the creterion for $T \in \mathcal{D}'(\mathbb{R}^n)$ (p. 3).

Def. For any $T \in S'(\mathbb{R}^n)$ we set

$$[\mathcal{F}T](\varphi) := T(\underbrace{\mathcal{F}\varphi}_{\in S(\mathbb{R}^n)}) \quad \forall \varphi \in S(\mathbb{R}^n)$$

⚠ The Fourier transform is not defined for all distributions but on the tempered distributions.

Observation : $\mathcal{F}S'(\mathbb{R}^n) = S'(\mathbb{R}^n)$

(exercise) follows from $\mathcal{F}S(\mathbb{R}^n) = S(\mathbb{R}^n)$

Examples :

1) $\mathcal{F}\delta_0 = T_{(\frac{1}{2\pi})^{n/2} \cdot 1}$ } constant function

Proof :

$$\begin{aligned}
[\mathcal{F}\delta_0](\varphi) &= \delta_0(\mathcal{F}\varphi) = \delta_0\left(x \mapsto \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot y} \varphi(y)\, dy\right) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(y)\, dy \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \cdot 1\, \varphi(y)\, dy \\
&= T_{(\frac{1}{2\pi})^{n/2} \cdot 1}(\varphi) \qquad \forall \varphi \in S(\mathbb{R}^n)
\end{aligned}$$

(Often written as $\mathcal{F}\delta_0 = \frac{1}{(2\pi)^{n/2}}$ but a little misleading)

2) $\mathcal{F}T_1 = (2\pi)^{n/2} \delta_0$

3) $\mathcal{F}T_h = T_{\hat{h}}$ for $h \in L^2(\mathbb{R}^n)$

$\hat{h} := \mathcal{F}(h)$

5) Continuous extensions

Def. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is summable if

$\exists\, m \geq 0$ and $c > 0$:

$$|T(\varphi)| \leq c \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_\infty \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

No localization (m, c are global)

The smallest m satisfying this is called the summability order of T

(denoted sum-order $(T) = m$)

Remark: The space of summable distributions with sum-order 0 coincides with the set of bounded measures on $\mathbb{R}^n$

(denoted by $M_b(\mathbb{R}^n)$)

Remarks:

sum-order $(\delta_0) = 0$;

sum-order $(\delta_0^\alpha) = |\alpha|$

For any $m \in \mathbb{N}$, we define

$$C_b^m(\mathbb{R}^n) := \{ f \in C^m(\mathbb{R}^n) \mid \partial^\alpha f \text{ is bounded for } \|\alpha\| \leq m \}$$

$C_b^m(\mathbb{R}^n)$ is endowed with a norm

$$\|f\|_{C_b^m(\mathbb{R}^n)} = \sum_{\|\alpha\| \leq m} \|\partial^\alpha f\|_\infty$$

and $C_b^m(\mathbb{R}^n)$ is complete with this norm.

Remark: $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C_b^m(\mathbb{R}^n)$

Lemma: If T is summable with sum-order m

then T can be (continuously) extended

to a continuous linear functional on $C_b^m(\mathbb{R}^n)$

$$T : C_b^m(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

In summary: even if, for the general theory, one starts with $\mathcal{D}(\mathbb{R}^n)$ a very small set of function

we can always extend the validity of a given distribution

to a much larger set.

Lebesgue Theory

1) Reminder on Riemann integration

Let $\mathcal{P}$ be a partition of $[a,b]$. It means

$$\mathcal{P} = \{x_0, x_1, \cdots, x_n\} \text{ with } x_0 = 0,\ x_n = b,\ x_j < x_{j+1}.$$

Let $f \in \mathcal{L}^\infty([a,b])$, it means

$$f : [a,b] \longmapsto \mathbb{R} \quad \text{with } \sup_{x\in[a,b]} |f(x)| \equiv \|f\|_\infty < \infty$$

One sets

$$\underset{\substack{\uparrow \\ \text{lower}}}{L}(f,\mathcal{P}) = \sum_j \inf_{x\in[x_j,x_{j}]} f(x)\,(x_j - x_{j-1}) \geq (b-a) \underbrace{\inf_{x\in[a,b]} f(x)}_{:= m}$$

$$U(f,\mathcal{P}) = \sum_j \sup_{x\in[x_j,x_{j}]} f(x)\,(x_j - x_{j-1}) \leq (b-a) \underbrace{\sup_{x\in[a,b]} f(x)}_{:= M}$$

We define the lower integral by $\sup_{\mathcal{P}} L(f,\mathcal{P})$, $\leq$ Exercise: check
and the uper integral by $\inf_{\mathcal{P}} U(f,\mathcal{P})$.

Def. f is Riemann integrable if $\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$,
and we write $\int_a^b f(x)\,dx :=$

Remark: Consider the function $f : [0,1] \longmapsto \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Observe that $L(f,\mathcal{P}) = 0$ and $U(f,\mathcal{P}) = 1$ $\forall \mathcal{P}$.

$\Rightarrow f$ is not Riemann integrable.

Theorem: If $f \in C([a,b],\mathbb{R})$ then f is Riemann integrable.

Main ingrediant: use that f is uniformly continuous on $[a,b]$.
$\uparrow \forall \varepsilon > 0\ \exists \delta$ (indep of x) ...

We would like to have a more general theory of integration.

2) Lebesgue measure

Reminder: A subset $V \subset \mathbb{R}^n$ is open if

$$\forall x \in V\ \exists \varepsilon > 0 : B(x,\varepsilon) \subset V$$

$\hookrightarrow$ open ball centred at x with radius ε

V is closed in $\mathbb{R}^n$ if $\mathbb{R}^n \setminus V$ is open.

V is compact in $\mathbb{R}^n$ if V is closed and bounded

$\hookrightarrow \exists R > 0 : V \subset B(0,R)$

Def. A closed box in $\mathbb{R}^n$ is a set I of the form

$$I := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \ \forall i \in \{1, \cdots, n\} \text{ for some fixed } a_i, b_i \in \mathbb{R}\}$$

For $n=1$ [——] ; For $n=2$ [ ]

The volume of I is given by $\mathrm{vol}(I) := \prod_{j=1}^{n} (b_j - a_j)$.

For more general sets in $\mathbb{R}^n$, we use approximation procedure.

Def. For any set $V \subset \mathbb{R}^n$, the set

$S := \{I_j\}_j$ closed box for any j is a covering of V if $V \subset \bigcup_j I_j$.

S can be finite or countably infinite. We set

$$\sigma(S) := \sum_j \mathrm{vol}(I_j) \in [0, \infty]$$

Def. For any $V \subset \mathbb{R}^n$ the Lebesgue outer measure of V is

$$m^*(V) := \inf \{\sigma(S) \mid S \text{ a covering of } V\}$$

Exercise: $m^*(I) = \mathrm{vol}(I)$ ← closed box

Prop.

1) If $V \subset W \subset \mathbb{R}^n$, then $m^*(V) \leq m^*(W)$
2) $m^*(V \cup W) \leq m^*(V) + m^*(W)$
3) $m^*(\bigcup_j V_j) \leq \sum_j m^*(V_j)$

⚠ Even if $V_1 \cap V_2 = \phi$ one can have $m^*(V_1 \cup V_2) < m^*(V_1) + m^*(V_2)$ not desirable

Exercise: find examples.

Thm. Let $V \subset \mathbb{R}^n$ with $m^*(V) < \infty$.

Then for any $\varepsilon > 0$, $\exists W$ open with $V \subset W$ and $m^*(W) \leq m^*(V) + \varepsilon$. W is only slightly bigger

If $m^*(V) = \infty$, the statement is true for $W = \mathbb{R}^n$.

Note that from the Thm and the Property 2,

$$m^*(W) \leq m^*(V) + m^*(W \setminus V) \quad \text{for } W = V \sqcup (W \setminus V)$$

$$\Rightarrow m^*(W) - m^*(V) \leq m^*(W \setminus V) \quad \text{lower bound}$$

But can we get an upper estimate for $m^*(W \setminus V)$?

Def. A set $V \subset \mathbb{R}^n$ is Lebesgue measurable if

if for any $\varepsilon > 0$, $\exists W$ open with $V \subset W \subset \mathbb{R}^n$ such that $m^*(W \setminus V) \leq \varepsilon$.

For Lebesgue measurable sets, we set the Lebesgue measure as

$$m(V) := m^*(V)$$

⚠ It means not all sets are Lebesgue measurable.

Prop. (proof as exercise)

- If V is open, then V is Lebesgue measurable.
- If $m^*(V) = 0$, then V is Lebesgue measurable.
- If $V = \bigcup_j V_j$ with V_j L.m., then V is L.m. with $m(V) \leq \sum_j m(V_j)$
- If $V = \bigcap_j V_j$ with V_j L.m., then V is L.m.

- If V is closed, then V is L.m.

  In particular, any closed box is L.m., and $m(I) = \mathrm{vol}(I)$
  ↑ closed box

Thm.

If $\{V_j\}$ is a countable family of L.m. sets with pairwise empty intersecti
then $m(\bigcup_j V_j) = \sum_j m(V_j)$.

This was not true for m^*.

Remark: There exist sets which are not L.m.

Example: On $\mathbb{R}$ we consider the equivalence relation

$x \sim y$ if $x - y \in \mathbb{Q}$

We denote by $[x]$ the equivalence class containing x. E.g.

$[\sqrt{2}] \neq [\sqrt{3}] \neq [0] = \mathbb{Q}$

Consider $V \subset \mathbb{R}$ made of 1 representative of each equivalence class. Axiom of choice

Then V is NOT L. measurable.

Report about this, or about G_δ sets. (see the book) §1.3

3) Lebesgue integration (on $\mathbb{R}$)

Aim: to define an integration more general than the Riemannian integration, which gives the same value for Riemann integrable functions.

Def. Let $f : [a,b] \longmapsto \mathbb{R}$. f is Lebesgue measurable on $[a,b]$ if

$\forall s \in \mathbb{R} : \{x \in [a,b] \mid f(x) > s\}$ is Lebesgue measurable.

$= f^{-1}((s,\infty))$

Lemma: The set of Lebesgue measurable functions on $[a,b]$ is a vector space (they can be added) and is an algebra (they can be multiplied) and $\frac{f}{g}$ is also L. measurable if f, g are L. measurable and $g(x) \neq 0$.

Proof as a (not so easy) exercise.

Def. Let $f, g : [a,b] \longmapsto \mathbb{R}$.

1) $f = g$ a.e. (= almost everywhere) if

$\{x \in [a,b] \mid f(x) \neq g(x)\}$ is Lebesgue measurable and has measure 0.

2) $f \leq g$ a.e. if

$\{x \in [a,b] \mid f(x) > g(x)\}$ is Lebesgue measurable and has measure 0.

Proposition

Let $f, g : [a,b] \longmapsto \mathbb{R}$ with $f = g$ a.e.

Then if f is L. measurable then g is also L. measurable.

Additional important notions

Let $\{f_n\}_{n\in\mathbb{N}}$ be a pointwise bounded sequence of functions defined on $[a,b]$, it means $\forall x \in [a,b] : \{f_n(x)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.

Then for fixed $x \in [a,b]$, we set

$$f^*(x) = \limsup_{n\to\infty} f_n(x) := \lim_{n\to\infty} \underbrace{\sup_{m\geq n} \{f_m(x)\}}_{\text{decreasing with } n \text{ and bounded below}}$$

always well-defined

and

$$f_*(x) = \liminf_{n\to\infty} f_n(x) := \lim_{n\to\infty} \inf_{m\geq n} \{f_m(x)\}$$

⚠ These 2 notations are always well-defined! ⇝ useful

Observe that $f_*(x) \leq f^*(x)$ and that

$f_*(x) = f^*(x)$ iff $\lim_{n\to\infty} f_n(x)$ exists.

Thm. If each f_n is Lebesgue measurable on $[a,b]$ and $\{f_n\}_n$ is pointwise bounded then $f^* : [a,b] \longmapsto \mathbb{R}$ and $f_* : [a,b] \longmapsto \mathbb{R}$ are Lebesgue measurable.

Corollary: If $\{f_n\}$ a sequence of L.m. functions on $[a,b]$, and if $\exists f : [a,b] \longmapsto \mathbb{R} : \lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Then f is L. measurable. useful for the Lebesgue dominated convergence thm

This notion is useful for defining the Lebesgue integral, of f, by a family of approximation (f_n). In particular, we can consider

$f_n = \sum_{k=1}^{n} c_k \chi_{E_k}$ with $c_k \in \mathbb{R}$ and E_k a measurable subset of $[a,b]$.

check that f_n is L. measurable.

$$\chi_{E_k}(x) := \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \end{cases}$$

Def. A measurable partition ~~of~~ $\mathcal{P}$ of $[a,b]$ consists in a finite collection $\{E_j\}_{j=1}^{N} \subseteq [a,b]$ s.t.

- E_j is L. m.
- $\bigcup_j E_j = [a,b]$
- $m(E_j \cap E_k) = 0 \ \forall j \neq k$

⚠ These partitions are much more general than the ones seen in Calculus I.

For $f \in L^\infty([a,b])$ and for P a measurable partition of $[a,b]$ we set

$$U(f,\mathcal{P}) := \sum_{j=1}^{N} \left(\sup_{x \in E_j} f(x) \right) m(E_j)$$

$$L(f,\mathcal{P}) := \sum_{j=1}^{N} \left(\inf_{x \in E_j} f(x) \right) m(E_j)$$

Def. f is Lebesgue integrable if

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

and if so, we write

$$\int_a^b f(x)\,dx := \sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

Remark:

If $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$

then f is L. integrable and $\int_0^1 f(x)\,dx = 0$. (Exercise)

⚠ This function was not Riemannian integrable.

Thm. If $f \in L^\infty([a,b])$ is Riemannian integrable then f is Lebesgue integrable with the same integral.

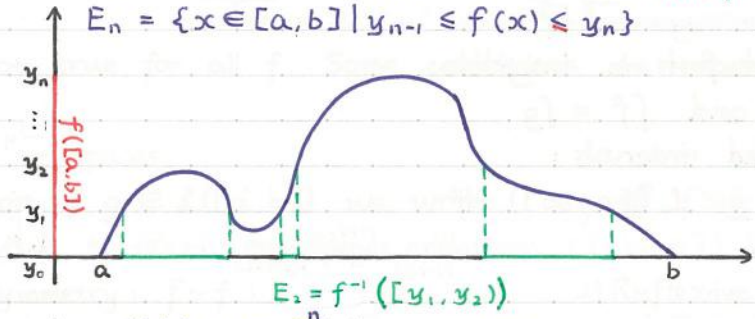
Thm. Let $f \in L^\infty([a,b])$. Then f is Lebesgue integrable iff f is Lebesgue measurable. Not trivial proof

Remark 1: If f is Lebesgue measurable on $[a,b]$, consider a partition of $f([a,b])$.

namely $y_0 = \inf f([a,b]) < y_1 < y_2 < \cdots < y_n = \sup f([a,b])$

and set $E_j := \{x \in [a,b] \mid y_{j-1} \leq f(x) < y_j\}$ L. measurable since f is L.m. for $j = 0, 1, \cdots, n-1$

⚠ $E_n = \{x \in [a,b] \mid y_{n-1} \leq f(x) \leq y_n\}$



Consider $U(f, Y) = \sum_{j=1}^{n} y_j \, m(E_j)$; ...

$\{E_j\}_{j=1}^{n} =:$

$L(f, Y) = \sum_{j=1}^{n} y_{j-1} \, m(E_j)$

By taking finer partitions of $f([a,b])$ we converge to $\int_a^b f(x)\,dx$!

Remark 2: We can define $\sum_{j=1}^{n} y_j \chi_{E_j}$ and a family of such approximations converging to f.

Picture:

Lebesgue

5

+

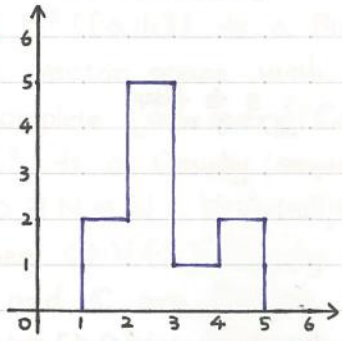
2 × 2

+

1

=

10



= 2 + 5 + 1 + 2 Riemann

Remark:

1) Suppose $f \geq 0$ but not bounded from above. Then we can consider $N \in \mathbb{N}$,

$$f_N(x) = \begin{cases} f(x), & \text{if } f(x) < N \\ N, & \text{if } f(x) \geq N \end{cases} \quad (*)$$

Then if f is L.m. on $[a,b]$, f_N is also L.m. on $[a,b]$ and we consider

$$\int_a^b f_N(x)\,dx \xrightarrow{N \to \infty} \int_a^b f(x)\,dx$$

if this limit exists.

2) If $f \leq 0$ but not bounded from below, Replace N by $(-N)$ in $(*)$

3) If f is not of a definite sign and unbounded from below and above, we need to consider $f = f_+ - f_-$ with $f_+ \geq 0$ and $f_- \geq 0$ and we do separately f_+ and f_-.

Def. The set of all Lebesgue integrable functions on $[a,b]$ is is denoted by $\mathcal{L}([a,b])$

⚠ We do not impose boundedness:

$\int_a^b f_N(x)\,dx \xrightarrow{N\to\infty}$ converge to "$\int_a^b f(x)\,dx$"

Remark: If $f=g$ a.e. and f is L. integrable the g is L. integrable and $\int f = \int g$.

For integration on unbounded intervals:

If $f \geq 0$, we say that $f \in \mathcal{L}([a,\infty))$ if

$\forall N > a: f|_{[a,N]} \in \mathcal{L}([a,N])$; and $\lim_{N\to\infty} \int_a^N f|_{[a,N]}(x)\,dx$

Then we write

$$\int_a^\infty f(x)\,dx := \lim_{N\to\infty} \int_a^N f(x)\,dx$$

For general $f = f_+ - f_-$ with $f_+, f_- \geq 0$:

$f \in \mathcal{L}([a,\infty))$ iff $f_+, f_- \in \mathcal{L}([a,\infty)) \overset{\text{in } \mathbb{R}}{\Longleftrightarrow} |f| \in \mathcal{L}([a,\infty))$

use this for complex-valued f

Similarly one defines $f \in \mathcal{L}((-\infty,a])$ and $f \in \mathcal{L}(\mathbb{R})$.

Thm. (Lebesgue dominated convergence theorem)

Let $\{f_n\}$ be sequence of L. m. functions on $[a,b]$ s.t.

$\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Suppose $\exists g \in \mathcal{L}([a,b])$ dominating function such that

$\forall n: |f_n| \leq g \; [\Leftrightarrow |f_n(x)| \leq g(x) \; \forall x \in [a,b]]$ g is indep of n!

Then f is L. integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

A very useful result!

Example:

For $n \in \mathbb{N}^*$, $x \in [0,1]$, set

$$f_n(x) = \frac{n\sin(x)}{1+n^2\sqrt{x}} + 2e^{x/n}$$

Observe

$f_n(x) \xrightarrow{n\to\infty} 2$ for any x fixed.

And for $x \neq 0$:

$$|f_n(x)| \leq \left|\frac{n}{1+n^2\sqrt{x}}\right| + 2e \leq \frac{1}{n\sqrt{x}} + 2e \leq \frac{1}{\sqrt{x}} + 2e$$

The function $g: [0,1] \mapsto \mathbb{R}$,

$$g(x) = \begin{cases} 2, & x = 0 \\ \frac{1}{\sqrt{x}} + 2e, & 0 < x \leq 1 \end{cases}$$

is L. integrable.

Then by LDCT,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 2\,dx = 2$$

Remark: this theory generalizes to measurable functions

$f: I \longmapsto \mathbb{R}$.

box in $\mathbb{R}^n$

But there exists one important thm in n dimensions.

Thm. (Fubini thm) (in $\mathbb{R}^2$) ← report on this

$$\iint f(x,y)\,\underbrace{dx\,dy}_{\text{measure in 2D}} = \int\Big[\int f(x,y)\,\underbrace{dx}\Big]\underbrace{dy} = \int\Big[\int f(x,y)\,\underbrace{dy}\Big]\underbrace{dx}$$

measures in 1D

⚠ Not true for all f. Some conditions are necessary.

4) L^p - spaces

For $f, g \in \mathcal{L}([a,b])$ we write $f \sim g$ iff $f = g$ a.e.

Exercise: $\sim$ is an equivalence relation.

1) Symmetry: $f \sim f$; 2) Reflexive: $f \sim g \Leftrightarrow g \sim f$;

3) Transitivity: $f \sim g \wedge g \sim h \Rightarrow f \sim h$

Def. $L^1([a,b]) := \mathcal{L}([a,b])/\sim$ (quotion)

We should write $[f] \in L^1([a,b])$ but we simply write $f \in L^1([a,b])$

We set $\|f\|_1 := \int_a^b |f(x)|\,dx$

Exercise: $\|\cdot\|_1$ defines a norm on $L^1([a,b])$.

1) $\|f\|_1 \geq 0$ with equality iff $f = 0$; 2) $\|\lambda f\|_1 = |\lambda|\,\|f\|_1 \quad \forall \lambda \in \mathbb{R}$;

3) $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

Proposition: $L^1([a,b])$ is a Banach space with the norm $\|\cdot\|_1$.

1) L^1 is a vector space with a norm;

2) L^1 is complete [$\Leftrightarrow$ every Cauchy sequence is converging in L^1]

$\{f_n\} \subset L^1$ is a Cauchy sequence iff

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}: \forall n, m \geq N: \|f_n - f_m\|_1 \leq \varepsilon$

Completeness $\Leftrightarrow \forall \{f_n\}$ Cauchy sequence $\exists f_\infty \in L^1: f_n \xrightarrow[\text{in } L^1\text{-norm}]{n\to\infty} f_\infty$

$\Updownarrow$

$\|f_n - f_\infty\|_1 \xrightarrow{n\to\infty} 0$

Remark: $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

⚠ There exists a big difference between pointwise convergence and convergence in L^1-norm.

Example 1:

For $n \in \mathbb{N}^*$, set $f_n: [0,1] \mapsto \mathbb{R}$,

$$f_n(x) = \begin{cases} 0, & x = 0 \\ n, & x \in (0, \frac{1}{n}) \\ \frac{1}{x}, & x \in [\frac{1}{n}, 1] \end{cases}$$

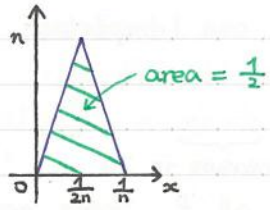
Now $\forall x \in [0,1]$ fixed:

$$f_n(x) \xrightarrow{n\to\infty} \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \in (0,1] \end{cases} =: f_\infty$$

$$\Rightarrow \underset{L^1([0,1])}{\underset{\cap}{f_n}} \xrightarrow[\text{pointwise}]{n\to\infty} \underset{L^1([0,1])}{\underset{\not\cap}{f_\infty}}$$

Example 2:

$$f_n(x) = \begin{cases} 2n^2 x, & 0 \le x \le \frac{1}{2n} \\ -2n^2\left(x - \frac{1}{n}\right), & \frac{1}{2n} < x \le \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$



One has $f_n(x) \xrightarrow{n\to\infty} 0 \ \forall x$ fixed,

and $0 \in L^1([0,1]) \ni f_n$, but $\|f_n - 0\|_1 = \|f_n\|_1 = \frac{1}{2}$.

(convergent pointwise but not in L^1-norm)

Def. For any $p \ge 1$ we set $\quad p \in \{p \in \mathbb{R} \mid p \ge 1\}$

$$L^p([a,b]) := \{f \text{ measurable on } [a,b] \mid |f|^p \text{ is L. integrable}\} / \sim$$

We define for $f \in L^p([a,b])$

$$\|f\|_p := \left[\int_a^b |f(x)|^p \, dx\right]^{1/p}$$

Prop. $L^p([a,b])$ with $\|\cdot\|_p$ is a Banach space.

$$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Thm. (Hölder inequality)

If $f \in L^p([a,b])$ and $g \in L^q([a,b])$ with $p \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$

Then $fg \in L^1([a,b])$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Trick for proof:

$\forall a, b > 0$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$:

$$ab \le \alpha a^{1/\alpha} + \beta b^{1/\beta}$$

Remark: $L^\infty([a,b])$ with $\|f\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)|$ is again a Banach space.

Remark: For $p = 2$, $L^2([a,b])$ has also an inner product

$$\langle f, g\rangle := \int_a^b \overline{f}(x)\, g(x)\, dx$$

By Hölder inequality,

$$|\langle f, g\rangle| = \|fg\|_1 \le \|f\|_2 \|g\|_2$$

Remark: extension naturally to $L^p([a,\infty))$ and $L^p(\mathbb{R})$.

Operator Theory on Hilbert spaces

Chapter I: Hilbert spaces and bounded operators

I.1: Hilbert spaces

Def. A complex Hilbert space (denoted by $\mathcal{H}$) is a vector space over $\mathbb{C}$ (stable for addition, and for multiplication by complex numbers) with a scalar product (inner product)

$$\mathcal{H} \times \mathcal{H} \ni (f, g) \longmapsto \langle f, g \rangle \in \mathbb{C}$$

which is linear in the second argument, and anti-linear in the first one, which is complete for the norm defined by $\|f\| := \langle f, f \rangle^{\frac{1}{2}}$ see p. 17

and seperable (existence of a countable basis).

Linearity and anti-linearity above:

$$\langle f, \lambda g + h \rangle = \lambda \langle f, g \rangle + \langle f, h \rangle$$
$$\langle \lambda f + h, g \rangle = \overline{\lambda} \langle f, g \rangle + \langle h, g \rangle$$

The scalar product must satisfy

$$\langle f, g \rangle = \overline{\langle g, f \rangle} \quad \text{and} \quad \langle f, f \rangle \geq 0 \text{ with equality iff } f = 0.$$

Examples

1) $\mathbb{C}$ with $\langle \alpha, \beta \rangle = \overline{\alpha} \beta \quad \forall \alpha, \beta \in \mathbb{C}$ 2) $\mathbb{C}^n$ with $\langle \alpha, \beta \rangle = \sum_{j=1}^{n} \overline{\alpha_j} \beta_j \quad \forall \alpha, \beta \in \mathbb{C}^n$

3) $\ell^2(\mathbb{Z}) := \{(a_j)_{j \in \mathbb{Z}} \mid a_j \in \mathbb{C}, \sum_{j \in \mathbb{Z}} |a_j|^2 < \infty$ with $\langle (a_j), (b_j) \rangle = \sum_{j \in \mathbb{Z}} \overline{a_j} b_j$

4) $L^2(\mathbb{R}^n) := \{[f] : \mathbb{R}^n \longmapsto \mathbb{C}$, measurable, $\int_{\mathbb{R}^n} |f(x)|^2 dx < \infty\}$

with $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) \, dx$

⚠ For $k \in \mathbb{R}^n$, $\mathbb{R}^n \ni x \longmapsto \exp(-ik \cdot x) \in \mathbb{C}$ (scalar product in $\mathbb{R}^n$) does NOT belong to $L^2(\mathbb{R}^n)$ but if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space) then $x \longmapsto \int_{\mathbb{R}^n} \exp(-ik \cdot x) \varphi(k) dk$ belongs to $L^2(\mathbb{R}^n)$

(avarage over some k)

Remarks:

1) All Hilbert spaces are isomorphic either to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, or to $\ell^2(\mathbb{Z})$.

2) One has the polarization identity:

$$4 \langle f, g \rangle = \|f+g\|^2 - \|f-g\|^2 - i \|f+ig\|^2 + i \|f-ig\|^2$$

Exercise: check it

3) $_{\text{def}}$ $f \perp g$ iff $\langle f, g \rangle = 0$.

Standard inequalities (proof as an exercise)

- $|\langle f, g \rangle| \leq \|f\| \|g\|$ Schwarz inequality
- $\|f+g\| \leq \|f\| + \|g\|$ triangle inequality
- $\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$
- $\big| \|f\| - \|g\| \big| \leq \|f\| - \|g\|$

Def. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. We write

$$\underset{n\to\infty}{\text{s-lim}}\, f_n = f_\infty (\in \mathcal{H}) \text{ if } \lim_{n\to\infty} \|f_n - f_\infty\| = 0 ; \qquad \text{strong limit}$$

$$\underset{n\to\infty}{\text{w-lim}}\, f_n = f_\infty (\in \mathcal{H}) \text{ if } \lim_{n\to\infty} \langle g, f_n - f_\infty \rangle = 0 \ \forall g \in \mathcal{H}. \qquad \text{weak limit}$$

Example: in $\ell^2(\mathbb{Z}) \ni f_n$,

Consider $f_n(j) = \delta_{jn} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$ then

$\underset{n\to\infty}{\text{w-lim}}\, f_n = \mathbf{0}$; $\underset{n\to\infty}{\text{s-lim}}\, f_n$ does not exist. (exercise)

Remark: when these limits exist, they are unique.

(use uniform boundedness principle)

Lemma: for $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, one has

$$\underset{n\to\infty}{\text{s-lim}}\, f_n = f_\infty \iff \underset{n\to\infty}{\text{w-lim}}\, f_n = f_\infty \text{ and } \lim_{n\to\infty} \|f_n\| = \|f_\infty\|$$

Sketch of the proof:

「$\Rightarrow$」 $|\langle g, f_n - f_\infty \rangle| \leq \|g\| \|f_n - f_\infty\|$ and $\big| \|f_n\| - \|f_\infty\| \big| \leq \|f_n - f_\infty\|$

「$\Leftarrow$」 $\|f_n - f_\infty\|^2 = \langle f_n - f_\infty, f_n - f_\infty \rangle = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty \rangle - \langle f_\infty, f_n \rangle$

Exercise: complete this proof. $\cdots \square$

Def. A linear manifold is a subset of $\mathcal{H}$ which is stable for addition, and for multiplication by complex numbers. sometimes called a subspace

A subspace is a closed (any Cauchy sequence converges inside) linear manifold.

$\rightsquigarrow$ A subspace is a Hilbert space with the inherited scalar product.

Examples

1) $\mathcal{S}(\mathbb{R}^n)$ is a linear manifold in $L^2(\mathbb{R}^n)$, but not a subspace.

(the closure of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ is $L^2(\mathbb{R}^n)$)

2) If $\{f_1, \cdots, f_N\} \subset \mathcal{H}$, then

$$\mathrm{Vect}\{f_1, \cdots, f_N\} \equiv \mathrm{Span}\{f_1, \cdots, f_N\} := \left\{ \sum_{j=1}^{N} a_j f_j \,\middle|\, a_j \in \mathbb{C} \right\}$$

is a subspace isomorphic to $\mathbb{C}^m$ for some $m \in \{1, \cdots, N\}$

3) If $M \subset \mathcal{H}$ is any subset of $\mathcal{H}$, then

$M^\perp := \{ f \in \mathcal{H} \mid \langle f, g \rangle = 0 \ \forall g \in M \}$ is a subspace of $\mathcal{H}$ (exercise)

$M^\perp$ is called the orthocomplement of M.

Def. M is dense in $\mathcal{H}$ iff

$$\forall f \in \mathcal{H} \ \forall \varepsilon > 0 \ \exists g \in M : \|f - g\| \leq \varepsilon$$

Lemma: A linear manifold M is dense in $\mathcal{H}$ iff $M^\perp = \{0\}$. (exercise)

Proposition: (Projection Theorem)

Let M be a subspace of $\mathcal{H}$. Then

$$\forall f \in \mathcal{H} \ \exists!\, f_1 \in M, f_2 \in M^\perp : f = f_1 + f_2 \text{ and } \|f\|^2 = \|f_1\|^2 + \|f_2\|^2.$$

Def. (Dual of $\mathcal{H}$)

$$\mathcal{H}^* := \{\text{bounded linear functional on } \mathcal{H}\}$$

$$\underset{\varphi}{\rotatebox{90}{$\in$}} : \mathcal{H} \longmapsto \mathbb{C} \text{ with}$$

$$\varphi(f+\lambda g) = \varphi(f) + \lambda\varphi(g) \qquad \text{(linearity)}$$

$$|\varphi(f)| \leq c\,\|f\| \text{ for some } c \geq 0 \text{ independent of } f \in \mathcal{H} \qquad \text{(bounded)}$$

Example :

Take $g \in \mathcal{H}$ and set $\varphi_g(f) := \langle g, f \rangle$

Then one has automatically the linearity on f and

$$|\varphi_g(f)| = |\langle g, f\rangle| \leq \underbrace{\|g\|}_{c}\,\|f\| \quad \text{(boundedness)}$$

Def. $\|\varphi\|_{\mathcal{H}^*} = \sup_{\substack{f\in\mathcal{H}^* \\ f\neq 0}} \frac{|\varphi(f)|}{\|f\|} < \infty$

Lemma: (Rietz Lemma)

$\forall \varphi \in \mathcal{H}^* \; \exists! g \in \mathcal{H} : \varphi(f) = \langle g, f\rangle \; \forall f \in \mathcal{H}.$

$\rightsquigarrow$ identification of $\mathcal{H}$ with $\mathcal{H}^*$ anti-linear

I.2 Bounded linear operators

If M, N are vector spaces, $B : M \longmapsto N$ is a linear operator (= map) if

$B(\alpha f + g) = \alpha B(f) + B(g)$

Def. A linear map $B : \mathcal{H} \mapsto \mathcal{H}$ is bounded if

$\exists c > 0 : \|B(f)\| \leq c\|f\| \quad \forall f \in \mathcal{H}$

We set $\mathcal{B}(\mathcal{H})$ [$\equiv \mathcal{L}(\mathcal{H})$ sometimes in old books] for the set of all bounded linear maps on $\mathcal{H}$

The norm of $B \in \mathcal{B}(\mathcal{H})$ is

$\|B\| := \sup_{\substack{f\in\mathcal{H} \\ f\neq 0}} \frac{\|Bf\|}{\|f\|} = \sup_{\|f\|=1} \|Bf\|$

Remark:

- Range of B is

  $\mathrm{Ran}(B) := \{g \in \mathcal{H} \mid g = Bf \; \exists f \in \mathcal{H}\} =: B\mathcal{H}$

- $\|B\| = \sup_{\substack{f\in M_1, g\in M_2 \\ \|f\|=\|g\|=1}} |\langle f, Bg\rangle|$ for M_1, M_2 dense linear manifolds.

  If $\mathcal{H} = L^2(\mathbb{R})$, $M_1 = \mathcal{S}(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ or $C_c(\mathbb{R})$ with compact support

Lemma: For any $B \in \mathcal{B}(\mathcal{H})$ $\exists!\, B^* \in \mathcal{B}(\mathcal{H})$ called the adjoint satisfying

$\langle B^* f, g\rangle = \langle f, Bg\rangle.$

In addition, C^*-property $\rightsquigarrow$ C^*-algebra

$\|B^*\| = \|B\|$; $(B^*)^* = B$; $\|B^*B\| = \|B\|^2$; $(AB)^* = B^*A^*$.

Hint: Consider for fixed $f \in \mathcal{H}$,

$g \xmapsto[\text{linear}]{\text{bounded}} \langle f, Bg\rangle \overset{\text{Rietz}}{\underset{\text{lemma}}{=}} \langle f^*, g\rangle$

Now we define

$B^* : \mathcal{H} \mapsto \mathcal{H}, \; f \longmapsto f^*$

$\Rightarrow \mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Remark: For $\mathcal{H} = \mathbb{C}^n$, and $B = (b_{ij}) \in M^n(\mathbb{C})$, then $B^* = (\overline{b_{ji}})$

Counterexample: ~~$\mathcal{B}(\mathbb{C}^n) =$~~ $M^n(\mathbb{C})$ with the norm $|||a_{ij}||| = \sup_{i,j} |a_{ij}|$ is not a C^*-algebra.

About convergences:

Def. Let $\{B_n\} \in \mathcal{B}(\mathcal{H})$.

$$\underset{n\to\infty}{u\text{-}\lim}\, B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \|B_n - B_\infty\| = 0 \qquad \text{(uniform)}$$

$$\Downarrow \not\Uparrow$$

$$\underset{n\to\infty}{s\text{-}\lim}\, B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \|(B_n - B_\infty) f\| = 0 \quad \forall f \in \mathcal{H} \qquad \text{(strong)}$$

$$\Downarrow \not\Uparrow$$

$$\underset{n\to\infty}{w\text{-}\lim}\, B_n = B_\infty \in \mathcal{B}(\mathcal{H}) \text{ iff } \lim_{n\to\infty} \langle g, (B_n - B_\infty) f \rangle = 0 \quad \forall f, g \in \mathcal{H} \qquad \text{(weak)}$$

Exercise: examples

Remark:

- $\underset{n\to\infty}{w\text{-}\lim}\, B_n = B_\infty \Rightarrow \underset{n\to\infty}{w\text{-}\lim}\, B_n^* = B_\infty^*$
- $\underset{n\to\infty}{s\text{-}\lim}\, A_n = A_\infty$ and $\underset{n\to\infty}{s\text{-}\lim}\, B_n = B_\infty \Rightarrow \underset{n\to\infty}{s\text{-}\lim}\, A_n B_n = A_\infty B_\infty$ tricky to prove; based on uniform boundedness principle.

Def. (Invertibility)

$B \in \mathcal{B}(\mathcal{H})$ is invertible if $Bf = 0 \Rightarrow f = 0$ (injectivity)

Indeed $Bf = Bg \Leftrightarrow B(f-g) = 0 \Rightarrow f - g = 0 \Leftrightarrow f = g$

Then we set

$B^{-1} : \mathrm{Ran}(B) \longmapsto \mathcal{H}$,

satisfying

$B^{-1}Bf = f \ \forall f \in \mathcal{H}$; $BB^{-1}g = g \ \forall g \in \mathrm{Ran}(B)$.

B is boundedly invertible if

$\|B^{-1}g\| \leq c\|g\| \ \forall g \in \mathrm{Ran}(B)$

In particular,

$\mathrm{Ran}(B) = \mathcal{H} \Rightarrow B^{-1} \in \mathcal{B}(\mathcal{H}) \Leftrightarrow B$ is boundedly invertible based on closed graph thm.

⚠ If $\mathrm{Ran}(B) \neq \mathcal{H}$, it is possible that B^{-1} is an unbounded operator.

Example:

Consider $\mathcal{H} \in L^2(\mathbb{R})$, and for $\varphi : \mathbb{R} \mapsto \mathbb{C}$, bounded (more precisely $\varphi \in L^\infty(\mathbb{R})$)

set $M_\varphi \equiv \varphi(X) \equiv \varphi(Q)$ multiplication operator $\in \mathcal{B}(\mathcal{H})$ with

$[M_\varphi f](x) := \varphi(x) f(x) \quad \forall f \in \mathcal{H}, x \in \mathbb{R}$.

Clearly M_φ is linear and

$$\|M_\varphi f\|_{L^2}^2 = \int_{\mathbb{R}} |[M_\varphi f](x)|^2 dx = \int_{\mathbb{R}} |\varphi(x) f(x)|^2 dx = \int_{\mathbb{R}} |\varphi(x)|^2 |f(x)|^2 dx$$
$$\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\varphi(x)|^2 \int_{\mathbb{R}} |f(x)|^2 dx = \|\varphi\|_\infty^2 \|f\|_{L^2}^2$$

$\Rightarrow M_\varphi \in \mathcal{B}(\mathcal{H})$; $\|M_\varphi\| = \|\varphi\|_\infty$

One has $M_\varphi^* = M_{\overline{\varphi}}$.

Indeed $\langle f, M_\varphi g\rangle = \int \overline{f(x)}\, \overline{\overline{\varphi(x)}}\, g(x)\, dx = \int \overline{f(x)\,\overline{\varphi(x)}}\, g(x)\, dx = \langle M_{\overline{\varphi}} f, g\rangle$

Example:

Consider $\varphi(x) := \tanh(x)$. Then $M_{\tanh} \in \mathcal{B}(\mathcal{H})$.

One has $M_{\tanh}^{-1} = M_{\frac{1}{\tanh}}$ because

$M_{\tanh} M_{\frac{1}{\tanh}} = M_{\frac{1}{\tanh}} M_{\tanh} = 1$.

But $\frac{1}{\tanh}$ is not bounded. $\Rightarrow M_{\frac{1}{\tanh}} \neq \mathcal{B}(\mathcal{H})$

One says that $M_{\tanh}$ is not invertible in $\mathcal{B}(\mathcal{H})$ or not boundedly invertible.

Remark:

$\mathbf{0} \in \mathcal{B}(\mathcal{H})$, $\mathbf{0} f = 0$;

$1 \in \mathcal{B}(\mathcal{H})$, $1 f = f$

Lemma: (Newmann series)

If $B \in \mathcal{B}(\mathcal{H})$ with $\|B\| < 1$ then

$(1-B)^{-1} = \sum_{n=0}^{\infty} B^n$ with $B^0 = 1$ and $\|(1-B)^{-1}\| \leq (1-\|B\|)^{-1}$

I.3 Special classes of operators

Def. $B \in \mathcal{B}(\mathcal{H})$ is self-adjoint (or Hermitian) if $B^* = B$.

In the previous example, $M_\varphi^* = M_\varphi$ iff φ is real valued.

Remark:

For self-adjoint operators

$\|B\| = \sup_{\substack{f \in M \\ f \neq 0}} |\langle f, Bf\rangle|$ (M is dense linear manifold)

Def. $P \in \mathcal{B}(\mathcal{H})$ is a projection (orthogonal projection) if $P^2 = P = P^*$.

In the previous example,

$M_\varphi^2 = M_\varphi M_\varphi = M_{\varphi^2} \overset{\text{projection}}{=\!=} M_\varphi = M_{\overline{\varphi}} \Leftrightarrow \varphi^2 = \varphi = \overline{\varphi}$

φ is a characteristic function.

Remark about projections:

There exists a bijective relation between projections and subspaces of $\mathcal{H}$.
It means $\mathrm{Ran}(P)$ is a subspace of $\mathcal{H}$,
and there exists a unique projection on any subspace.
We write $P = P_M$ ← underlying subspace

⚠ $P_M P_N \neq P_N P_M$ in general $\Rightarrow P_M P_N$ is not a projection in general.

Def. $U \in \mathcal{B}(\mathcal{H})$ is unitary if $UU^* = 1$ and $U^*U = 1$.

⚠ When $\dim \mathcal{H} = \infty$, these 2 conditions are not equivalent.

Note that for U unitary, $U^{-1} = U^*$ (in particular $U^{-1} \in \mathcal{B}(\mathcal{H})$)

Def. $V \in \mathcal{B}(\mathcal{H})$ is an isometry if $V^*V = 1$.

Remark: for $f \in \mathcal{H}$

$$\|Vf\|^2 = \langle Vf, Vf\rangle = \langle V^*Vf, f\rangle = \langle f, f\rangle = \|f\|^2$$

$\Rightarrow \|Vf\| = \|f\|$ (isometry)

Also $\langle Vf, Vg\rangle = \langle f, g\rangle$

Question: give an example (e.g. in $\ell^2(\mathbb{N})$)

Remark: $P := VV^*$ is a projection on $\mathrm{Ran}(V)$. Indeed

- $P^2 = (VV^*)(VV^*) = V(V^*V)V^* = VV^* = P$;
  $P^* = (VV^*)^* = (V^*)^*V^* = VV^* = P$.
- $\mathrm{Ran}(P) = \mathrm{Ran}(VV^*) \subset \mathrm{Ran}(V)$;
  $\forall g \in \mathrm{Ran}(V): \exists f \in \mathcal{H}: g = Vf$
  $\Rightarrow Pg = VV^*g = VV^*Vf = Vf = g$

Thus P acts as identity on $\mathrm{Ran}(V)$, which means $\mathrm{Ran}(P) = \mathrm{Ran}(V)$.

$\Rightarrow P$ is a projection on $\mathrm{Ran}(V)$.

V is injective since $Vf = \mathbf{0} \Rightarrow f = \mathbf{0}$.

Hence V is invertible. But its inverse is not always bounded.

More generally:

Def. $W \in \mathcal{B}(\mathcal{H})$ is a partial isometry if $W^*W =: Q$ is a projection.

Lemmas: (as exercises)

- $WQ = W$
- $PW = W$
- $WW^* =: P$

Reference: [Amr] Prop. 2.1

Q is called the initial set projection, and P the final set projection.

Let $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$, and for any $f \in \mathcal{H}$ we define

$$Af := \sum_{j=1}^N |h_j\rangle\langle g_j| f \equiv \sum_{j=1}^N \langle g_j, f\rangle h_j \in \text{Vect}\{h_j\}_j$$

$= 0$ if $f \in \{g_j\}_j^\perp$

Af is clearly linear in f; and

$$\|Af\| \leq \sum_{j=1}^N \|\underbrace{\langle g_j, f\rangle}_{\in \mathbb{C}} h_j\| = \sum_{j=1}^N |\langle g_j, f\rangle| \|h_j\| \leq \sum_{j=1}^N \|f\| \|g_j\| \|h_j\|$$

$$\Rightarrow \|A\| = \sup_{f \in \mathcal{H}; f \neq 0} \frac{\|Af\|}{\|f\|} \leq \sum_{j=1}^N \|g_j\| \|h_j\|$$

$\Rightarrow \|A\| \in \mathcal{B}(\mathcal{H})$

Such A is called a finite rank operator. Its set is denoted as $\mathcal{F}(\mathcal{H})$.

Remark:

- $\underset{\in}{h_j} \otimes \underset{\in}{g_j} := |h_j\rangle\langle g_j|$ ($h_j \in \mathcal{H}$, $g_j \in \mathcal{H}^*$)
- $Af = 0$ if $f \notin \text{Vect}\{g_j\}$; $\text{Ran}(Af) \subset \text{Vect}\{h_j\}$. Both are finite dimensional

Def. An element $K \in \mathcal{B}(\mathcal{H})$ is compact if

$\forall \varepsilon > 0 : \exists A$ finite rank operator s.t. $\|K - A\| \leq \varepsilon$.

We write $\mathcal{K}(\mathcal{H})$ for the set of all compact operators. subset of $\mathcal{B}(\mathcal{H})$

Remark:

$\overline{\mathcal{F}(\mathcal{H})}^u = \mathcal{K}(\mathcal{H})$ in the norm topology; (closure)

$\overline{\mathcal{K}(\mathcal{H})}^u = \mathcal{K}(\mathcal{H})$ in the norm topology;

$\overline{\mathcal{K}(\mathcal{H})}^s = \mathcal{B}(\mathcal{H}) = \overline{\mathcal{K}(\mathcal{H})}^w$ strong or weak topologies.

Example: Consider $\mathcal{H} = L^2(\mathbb{R})$, and

$$\varphi_n : \mathbb{R} \mapsto \mathbb{R}, \quad \varphi_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases} \quad \text{with } n \in \mathbb{N}.$$

Consider $B_n = M_{\varphi_n}$. Then

$\underset{n\to 0}{\text{s-lim}}\, B_n = \mathbf{0}$ but $\underset{n\to 0}{\text{u-lim}}\, B_n$ does not exist.

For s-lim, consider a fixed $f \in \mathcal{H}$ and observe that

$$\|(B_n - \mathbf{0})f\|^2 = \int |[B_n f](x)|^2 dx = \int_n^{n+1} |f(x)|^2 dx \xrightarrow[n\to\infty]{\text{by contradiction in } L^2(\mathbb{R})} 0$$

For u-lim, we cannot fix f, and

$\forall n \in \mathbb{N} : \exists f_n \in \mathcal{H}, \|f_n\| = 1$, $\text{supp}\, f_n \subset [n, n+1]$:

$$\|(B_n - \mathbf{0})f_n\|^2 = \int |[B_n f_n](x)|^2 dx = \int_n^{n+1} |f_n(x)|^2 dx = 1$$

which implies that $\underset{n\to 0}{\text{u-lim}}\, B_n$ does not exist.

$\mathcal{K}(\mathcal{H})$ is already a very rich set:

non-commutative integration theory, trace class op.,

Hilbert-Schmidt, Schatten classes are under $\mathcal{K}(\mathcal{H})$.

Prop. Let $K \in \mathcal{K}(\mathcal{H})$, $\{f_n\}_n \subset \mathcal{H}$, $\{B_n\}_n \subset \mathcal{B}(\mathcal{H})$.

1) $\underset{n\to\infty}{\text{w-lim}}\, f_n = f \in \mathcal{H} \Rightarrow \underset{n\to\infty}{\text{s-lim}}\, Kf_n = Kf$

2) $\underset{n\to\infty}{\text{s-lim}}\, B_n = B \in \mathcal{B}(\mathcal{H}) \Rightarrow \underset{n\to\infty}{\text{u-lim}}\, B_n K = BK$ and $\underset{n\to\infty}{\text{u-lim}}\, K B_n^* = K B^*$.

I.4 Vector valued and operator valued functions

Consider $[a,b] \subset \mathbb{R}$, and

$\varphi : [a,b] \longmapsto \mathcal{H}$, $\varphi(x) \in \mathcal{H}$ a vector valued function

or :

$\psi : [a,b] \longmapsto \mathcal{B}(\mathcal{H})$, $\varphi(x) \in \mathcal{B}(\mathcal{H})$ an operator valued function

Continuity or differentiability can be associated to the different notions of convergence (= different topology).

Example :

φ is strongly continuous at x if

$$\| \varphi(x+\varepsilon) - \varphi(x) \| \xrightarrow{\varepsilon \to 0} 0$$

or ψ is uniformly differenciable at x if $\exists \psi'(x) \in \mathcal{B}(\mathcal{H})$ with

$$\left\| \frac{\psi(x+\varepsilon) - \psi(x)}{\varepsilon} - \psi'(x) \right\| \xrightarrow{\varepsilon \to 0} 0$$

Also, Riemann's type integral exists.

$$\int_a^b \varphi(x)\,dx := \underset{\substack{\text{finite partitions} \\ \text{of } [a,b]}}{\text{s-lim}} \sum_{j=1}^{N} (x_k - x_{k-1})\, \varphi(y_k) \in \mathcal{H} \text{ if the limit exists.}$$

$$x_{k-1} < x_k \,;\; x_{k-1} \leq y_k \leq x_k$$

Then

$\varphi \longmapsto \int_a^b \varphi(x)\,dx \in \mathcal{H}$ is a linear map ;

and one has

$$\left\| \int_a^b \varphi(x)\,dx \right\| \leq \int_a^b \|\varphi(x)\|\,dx$$

Similarly one can define

$$\int_a^b \psi(x)\,dx \in \mathcal{B}(\mathcal{H})$$

Question :

If $\psi(x) \in \mathcal{K}(\mathcal{H})$ $\forall x \in [a,b]$, when does $\int_a^b \psi(x)\,dx \in \mathcal{K}(\mathcal{H})$?

Depends on the property of

$$[a,b] \ni x \longmapsto \psi(x) \in \mathcal{K}(\mathcal{H})$$

Chapter II : Unbounded and self-adjoint operators

Reminder : Special classes in $\mathcal{B}(\mathcal{H})$

- Self-adjoint op. — Hamiltonian quantum system
- Projection — Energy region
- unitary op. — Evolution
- isometries — Non-reversible systems
- partial isometries — Scattering theory
- Finite rank op. — Easy perturbation theory
- compact op. — More advanced perturbation theory

II.1 : Some defination

Def. A linear operator on $\mathcal{H}$ is a pair $(A, D(A))$
with $D(A) \subset \mathcal{H}$ a linear manifold,
and $A : D(A) \longmapsto \mathcal{H}$ is a linear map. $A(f+\lambda g) = Af + \lambda Ag$
We say that $(A, D(A))$ is densely defined if $D(A)$ is dense in $\mathcal{H}$.

$$\mathrm{Ran}(A) = \{f \in \mathcal{H} \mid \exists g \in D(A) : f = Ag\} = AD(A)$$

$$\mathrm{Ker}(A) = \{f \in D(A) \mid Af = 0\}$$

Clearly, if A, B are linear operators then

- $A+B$ is only defined on $D(A) \cap D(B)$
- AB is only defined on $\{f \in D(B) \mid Bf \in D(A)\}$
- $[A,B] = AB - BA$ only defined on a suitable domain.

Example:

$\mathcal{H} = L^2(\mathbb{R}) \ni f$.

Let $X\ (= Q = M_{id})$ be the operator given by $[Xf](x) = xf(x)$

Some domains are $C_c^\infty(\mathbb{R})$, $S(\mathbb{R})$, or

$\{f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |xf(x)|^2 dx < \infty\} =: \mathcal{H}_1(\mathbb{R})$: weighted Hilbert space.

Let us check that $X \notin \mathcal{B}(\mathcal{H})$.

Take $y \in \mathbb{R}$ and consider $f_y : \mathbb{R} \longmapsto \mathbb{R}$

$$f_y(x) = \begin{cases} 1 & \text{if } x \in [y, y+1] \\ 0 & \text{otherwise} \end{cases} \Rightarrow \|f_y\| = 1$$

$$\Rightarrow \|Xf_y\|^2 = \int_{\mathbb{R}} |xf_y(x)|^2 dx = \int_y^{y+1} x^2 dx = \frac{1}{3}x^3 \Big|_y^{y+1} = y^2 + y + \frac{1}{3}$$

If $X \in \mathcal{B}(\mathcal{H})$ then

$$\|X\| = \sup_{f \in \mathcal{H}, \|f\|=1} \|Xf\| = \infty \text{ by taking } f = f_y \text{ with } y \to \infty$$

$\Rightarrow X \notin \mathcal{B}(\mathcal{H})$

Remark : $C_c^\infty(\mathbb{R})$, $S(\mathbb{R})$, $\mathcal{H}_1(\mathbb{R})$ are dense in $\mathcal{H}$.

Def. Let $(A, D(A))$ and $(B, D(B))$ be linear operators in $\mathcal{H}$.
If $D(A) \subset D(B)$ and $Af = Bf \ \forall f \in D(A)$ then we say that
$(B, D(B))$ is an extension of $(A, D(A))$, or
$(A, D(A))$ is an restriction of $(B, D(B))$ to $D(A)$.

For example

$(X, C_c^\infty(\mathbb{R})) \subset (X, \mathcal{S}(\mathbb{R})) \subset (X, \mathcal{H}_1(\mathbb{R}))$.

Remark: If $(A, D(A))$ is a densely defined linear operator
and if $\exists\, c > 0$ with $\|Af\| \leq c\|f\| \ \forall f \in D(A)$
then A can be continuously extended to an element of $B(\mathcal{H})$.

$\rightsquigarrow$ closed operators: natural generalization of bounded operators.

If such a c does not exist, then we say
that $(A, D(A))$ is a densely defined unbounded operators.

Def. Let $(A, D(A))$ be densely defined. Set $\longrightarrow$ implying that A^*f is unique

$$D(A^*) := \{ f \in \mathcal{H} \mid \exists! f^* \in \mathcal{H} \text{ with } \langle f^*, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A) \}$$

$$A^* f := f^* \quad \forall f \in D(A^*).$$

Clearly $\langle A^* f, g \rangle = \langle f, Ag \rangle \ \forall g \in D(A),\ f \in D(A^*)$ $D(A^*)$ is dense if A is closable

Exercise:

- $\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp$ 2~3 lines
- If $(A, D(A)) \subset (B, D(B))$ with $D(A)$ dense,
  then $(B^*, D(B^*)) \subset (A^*, D(A^*))$ be careful about the def of A^*, B^*

Def. $(A, D(A))$ densly defined is self-adjoint if

$$D(A^*) = D(A) \text{ and } A^* f = Af \ \forall f \in D(A)$$

Remark: For any $(A, D(A))$ self-adjoint,

$$\langle Af, g \rangle = \langle f, Ag \rangle \ \forall f, g \in D(A) \qquad (*)$$

⚠ But this equality does not imply that A is self-adjoint.

Def. A densely operator $(A, D(A))$ satisfying $(*)$ is called symmetric.

$$\{\text{symmetric op.}\} \supset \{\text{self-adjoint op.}\}$$

Def. A symmetric operator $(A, D(A))$ is essentially self-adjoint
if it has a unique self-adjoint extension.
In this case, we call $D(A)$ a core for A.

II.2 Resolvent and spectrum

Let $(A, D(A))$ be a densely defined linear operator.

A is invertible if $\mathrm{Ker}(A) = \{0\}$, and then $A^{-1} : R(A) \longmapsto D(A)$ is bijectiv (range)

If $R(A) = \mathcal{H}$, then $A^{-1} \in \mathcal{B}(\mathcal{H})$.

Def. Let $(A, D(A))$ be self-adjoint. This theory is valid also for closed operators.

We set the resolvent set

$$\rho(A) = \{ z \in \mathbb{C} \mid (A-z)^{-1} \in \mathcal{B}(\mathcal{H}) \} \subset \mathbb{C}$$
$$= \{ z \in \mathbb{C} \mid \mathrm{Ker}(A-z) = \{0\} \text{ and } \mathrm{Ran}(A-z) = \mathcal{H} \} \subset \mathbb{C}$$

and the spectrum of A

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$ For Hermitian matrices this is the set of eigenvalues

Example :

Consider $\mathcal{H} = L^2(\mathbb{R}) \ni f$,

$\varphi \in L^\infty(\mathbb{R}, \mathbb{R})$, $\varphi(X) \equiv M_\varphi \equiv \varphi(Q)$ (real-valued)

$[\varphi(X) f](x) = \varphi(x) f(x)$

One has $\sigma(\varphi(X)) = \overline{\mathrm{Ran}(\varphi)} = \overline{\varphi(\mathbb{R})}$ (closure in $\mathbb{R}$)

Indeed $\varphi(X) - z$ is boundedly invertible if

$\varphi(x) - z \neq 0 \quad \forall x \in \mathbb{R}$

Properties :

- $\rho(A)$ is an open set in $\mathbb{C} \Leftrightarrow \sigma(A)$ is closed in $\mathbb{C}$.
- (First resolvent equation) if $z_1, z_2 \in \rho(A)$ then
  $$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}$$ one-line proof
- $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute.

Def. Let $z \in \mathbb{C}$ and suppose that

$$\exists f \in D(A),\ f \neq 0 : \ Af = zf$$

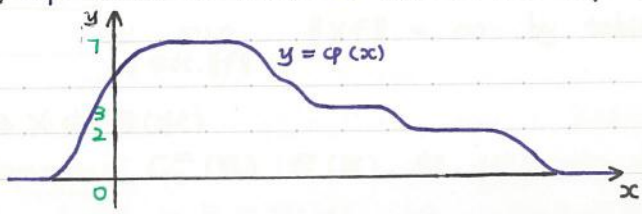
Then z is called an eigenvalue of A, with the set of all eigenvalues $\sigma_p(A)$, and f an associated eigenfunction or eigenvector.

Remark :

In the previous example with $\varphi = \tanh$, then

$\sigma(\tanh(X)) = [-1, 1]$ but $\sigma_p(\tanh(X)) = \phi$.

If φ takes a value c on a set of measure $\neq 0$, then $c \in \sigma_p(\varphi(X))$.



$\sigma(\varphi(X)) = [0, 7]$

$\sigma_p(\varphi(X)) = \{0, 2, 3, 7\}$

Lemma: Let $A \in \mathcal{B}(\mathcal{H})$. Then (proof as exercise)

$\sigma(A) \subset B_{\|A\|}(0) \subset \mathbb{C}$ $\Leftrightarrow$ If $z \notin B_{\|A\|}(0)$ then $z \in \rho(A)$ easier to prove

↳ ball centered at 0 with radius $\|A\|$ (using Neumann series)

Prop. Let $(A, D(A))$ be self-adjoint. Then (très important)

1) $\sigma_p(A) \subset \mathbb{R}$
2) $\sigma(A) \subset \mathbb{R}$
3) If λ, μ are eigenvalues of A, with $\mu \neq \lambda$ and if f_λ, f_μ are the corresponding eigenfunctions, then f_λ and f_μ are orthogonal.

Proof:

1) Let $0 \neq f \in D(A)$ and $z \in \mathbb{C}$ s.t. $Af = zf$. Then

$$z\|f\|^2 = z\langle f, f\rangle = \langle f, zf\rangle = \langle f, Af\rangle \overset{\text{self-}}{\underset{\text{adjoint}}{=}} \langle Af, f\rangle = \langle zf, f\rangle = \bar{z}\langle f, f\rangle$$
$$= \bar{z}\|f\|^2$$

$\Rightarrow z = \bar{z} \Rightarrow z \in \mathbb{R}$.

2) Set $z = \lambda + i\varepsilon$ with $\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$ and let us show that $z \in \rho(A)$.
(it then implies that $\sigma(A) \in \mathbb{R}$) resolvent set

- $\forall f \in D(A)$ one has

$$\|(A-z)f\|^2 = \|(A-\lambda)f - i\varepsilon f\|^2 = \langle (A-\lambda)f - i\varepsilon f, (A-\lambda)f - i\varepsilon f\rangle$$
$$= \underbrace{\|(A-\lambda)f\|^2}_{>0} + \varepsilon^2\|f\|^2$$

$$\Rightarrow \|(A-z)f\| \geq |\varepsilon|\|f\| \begin{cases} >0, & \text{for } f \neq 0 \\ =0, & \text{for } f = 0 \end{cases}$$

$\Rightarrow A - z$ is injective $\Rightarrow A - z$ is invertible on $\mathrm{Ran}(A-z)$

- $\forall g \in \mathrm{Ran}(A-z)$, one has

$$\|g\| = \|(A-z)\underbrace{(A-z)^{-1}g}_{\in D(A-z) = D(A)}\| \overset{\text{by the previous point}}{\geq} |\varepsilon|\|(A-z)^{-1}g\|$$

$\Leftrightarrow \|(A-z)^{-1}g\| \leq \frac{1}{|\varepsilon|}\|g\|$ $\Rightarrow (A-z)^{-1}$ is bounded on $\mathrm{Ran}(A-z)$

- Let us show that $\mathrm{Ran}(A-z)$ is dense in $\mathcal{H}$. Indeed

$$\mathrm{Ran}(A-z)^\perp = \mathrm{Ker}((A-z)^*) = \mathrm{Ker}(A^* - \bar{z}) \overset{\text{self-}}{\underset{\text{adjoint}}{=}} \mathrm{Ker}(A - \bar{z})$$
$$= \{0\} \text{ since all eigenvalues are real}$$

$\Rightarrow \mathrm{Ran}(A-z)$ is dense in $\mathcal{H}$

$\Rightarrow (A-z)^{-1}$ is a densely defined bounded linear operator in $\mathcal{H}$.
By the first Remark in p. 29, $(A-z)^{-1}$ admits a continuous bounded extension on $\mathcal{H}$
$\Leftrightarrow (A-z)^{-1} \in \mathcal{B}(\mathcal{H})$ $\Leftrightarrow z \in \rho(A)$.

3) One has (for $\lambda, \mu \in \sigma_p(A) \subset \mathbb{R}$, $\lambda \neq \mu$)

$$\lambda \langle f_\lambda, f_\mu \rangle = \langle \lambda f_\lambda, f_\mu \rangle = \langle A f_\lambda, f_\mu \rangle \overset{\text{self-}}{\underset{\text{adjoint}}{=}} \langle f_\lambda, A f_\mu \rangle = \langle f_\lambda, \mu f_\mu \rangle$$
$$= \mu \langle f_\lambda, f_\mu \rangle \Rightarrow \langle f_\lambda, f_\mu \rangle = 0$$

□

II.3 Perturbation theory

Let $(A, D(A))$ be a self-adjoint op., and consider $(B, D(B))$ symmetric op.,

Question: when is $A+B$ self-adjoint?

1) easy case: if $B = B^* \in \mathcal{B}(\mathcal{H})$, then $(A+B, D(A))$ is self-adjoint (exercise)

What about $B \notin \mathcal{B}(\mathcal{H})$?

Def. In the above framework, B is A-bounded (or bounded with respect to A) if $D(A) \subset D(B)$ and $\exists\, \alpha, \beta \geq 0$:

$$\|Bf\| \leq \alpha \|Af\| + \beta \|f\| \quad \forall f \in D(A)$$

The infimum of α is called the A-bound of B.

Remark:

- If $B \in \mathcal{B}(\mathcal{H})$, the A-bound is 0.
- One can have an A-bound equal to 0 even if $B \notin \mathcal{B}(\mathcal{H})$ $\quad \alpha_n \xrightarrow{n\to\infty} 0$, $\beta_n \longrightarrow \infty$

Thm. (Kato-Rellich thm)

Let $(A, D(A))$ be self-adjoint,

and $(B, D(B))$ be symmetric and A-bounded with A-bound < 1. Then

1) $(A+B, D(A))$ is self-adjoint;
2) B is also $(A+B)$-bounded.

Remark: The statement is not correct if the A-bound $= 1$.

Take $B = -A$ but then $(\mathbf{0}, D(A))$ is not self-adjoint. the domain of $\mathbf{0}^*$ is not $D(A)$

Thm. (The second resolvent equation)

In the setting of the previous thm, one has for $z \in \rho(A) \cap \rho(A+B)$:

$$(A-z)^{-1} - (A+B-z)^{-1} = (A-z)^{-1} B (A+B-z)^{-1} = (A+B-z)^{-1} B (A-z)^{-1}$$

It is not assumed that B is bounded, but for $B(A+z)^{-1}$

$\mathcal{H} \xrightarrow{(A-z)^{-1}} D(A) \subset D(B) \xrightarrow{B}$ meaningful;

same for $B(A+B-z)^{-1}$ because of Kato-Rellich thm.

Formal proof:

$$(A-z)^{-1} - (A+B-z)^{-1} = (A-z)^{-1} \underline{(A+B-z)(A+B-z)^{-1}} - \underline{(A-z)^{-1}(A-z)}(A+B-z)^{-1}$$
$$= (A-z)^{-1} \left[(A+B-z) - (A-z)\right] (A+B-z)^{-1}$$
$$= (A-z)^{-1} B (A+B-z)^{-1}$$

II.4 Examples

Let $\mathcal{H} = L^2(\mathbb{R}^n) \ni f$.

If $\varphi : \mathbb{R}^n \longmapsto \mathbb{C}$ we set $M_\varphi \equiv \varphi(X) \equiv \varphi(Q)$ acting as $(M_\varphi f)(x) = \varphi(x) f(x)$ (multiplication operators)

One has $D(M_\varphi) = \{ f \in \mathcal{H} \mid M_\varphi f \in \mathcal{H} \}$

Examples

X_j, or $X_j X_k$, with $j, k \in \{1, 2, \cdots, n\}$

More generally $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ with $\alpha \in \mathbb{N}^n$

E.g. $X^2 := X_1^2 + X_2^2 + \cdots + X_n^2$

Fourier transform (in Schwarz space)

For $f \in \mathcal{S}(\mathbb{R}^n)$ we set

$$[\mathcal{F}^{-1} f](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\pm i \xi \cdot x} f(x)\, dx$$

$\mathcal{F}$ is a unitary operator : $\mathcal{F}^* = \mathcal{F}^{-1}$

- For the multiplication operators

$$\begin{aligned}
(2\pi)^{n/2} [X_j \mathcal{F} f](\xi) &= (2\pi)^{n/2} \xi_j [\mathcal{F} f](\xi) \\
&= \xi_j \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx = \int_{\mathbb{R}^n} i(\partial_{x_j} e^{-i\xi\cdot x}) f(x)\, dx \\
\text{(integate by part)} \quad &= \int_{\mathbb{R}^n} e^{-i\xi\cdot x} [-i\partial_j f](x)\, dx \\
&= (2\pi)^{n/2} [\mathcal{F}(-i\partial_j f)](\xi)
\end{aligned}$$

$\Rightarrow X_j \mathcal{F} = \mathcal{F}(-i\partial_j)$

$\Leftrightarrow \mathcal{F}^* X_j \mathcal{F} = -i\partial_j =: P_j \equiv D_j$ (differential operator)

In particular we consider

$\Delta := D_1^2 + D_2^2 + \cdots + D_n^2$ (Laplacian)

one gets

$\mathcal{F} \Delta \mathcal{F}^* = X^2 = X_1^2 + X_2^2 + \cdots + X_n^2$

Hence since $\mathcal{F}$ is unitary ($\Rightarrow$ preserving the norm),

$\sigma(\Delta) = \sigma(X^2) = \mathrm{Ran}(x \mapsto x^2) = [0, \infty)$

For $\varphi : \mathbb{R}^n \longmapsto \mathbb{R}$, we can also define :

$\varphi(D) := \mathcal{F}^* \underset{\substack{\| \\ M_\varphi}}{\varphi(X)} \mathcal{F}$ (convolution operator)

For $f \in L^2(\mathbb{R}^n)$ observe that

$$\varphi(D) f = \mathcal{F}^* \varphi(X) \mathcal{F} f = \mathcal{F}^* \big(\varphi(X) \underbrace{\hat{f}}_{\text{Fourier transfm.}}\big) = \underbrace{\check{\varphi} * f}_{\text{Inverse Fourier transfm.}}(x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \check{\varphi}(y) f(x-y)\, dy =: \text{convolution}$$

Exercise:

In $L^2(\mathbb{R}^n)$, recall $D_j := -i\partial_j$ and $[X_k f](x) := x_k f(x)$. Then prove:

$$[X_j, X_k] = 0; \quad [D_j, D_k] = 0; \quad [iD_j, X_k] = \delta_{jk} \quad \text{on } \mathcal{S}(\mathbb{R}^n)$$

Harmonic oscillator in $L^2(\mathbb{R}^d)$

$$H := \underset{-(\partial_1^2 + \cdots + \partial_n^2)}{\underset{\parallel}{\Delta}} + \omega \underset{(X_1^2 + \cdots + X_n^2)}{\underset{\parallel}{X^2}} \quad \text{with } \omega \in \mathbb{R}$$

Exercise: its spectrum

This operator has to be defined on a suitable domain, and then it is self-adjoint.

Schrödinger operator in $L^2(\mathbb{R}^n)$

Consider $h: \mathbb{R}^n \longmapsto \mathbb{R}$ as e.g.

- $h(\xi) = \xi^2$ — free Laplace operator Δ
- $h(\xi) = |\xi|$ (Euclidean norm) — free relativistic Schödinger op.
- $h(\xi) = \sqrt{|\xi|^2 + 1} - 1$ — free relativistic Schödinger op. of mass 1

The corresponding operators $h(D)$ are

Then we usually consider

$$H := h(D) + V(X) \quad \text{with } V: \mathbb{R}^n \longmapsto \mathbb{R}$$

($V(x)$ called the potential)

⚠ if V is not bounded we need a suitable domain for H.

Hydrogen atom in $L^2(\mathbb{R}^3)$

$$H := \Delta - \gamma \frac{1}{|X|} \quad \gamma \in \mathbb{R}$$

($\frac{1}{|X|}$: multiplication op. by an unbounded function)

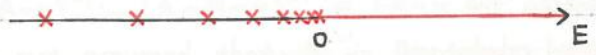
with domain $D(H) = D(-\Delta)$ is self-adjoint.

In fact $\frac{1}{|X|}$ is Δ-bounded with relative bound 0.

$\Rightarrow D(H) = D(\Delta)$ for any coupling constant γ.

One has

$$\sigma(H) = \mathbb{R}_+ \sqcup \left\{ -\left(\frac{\gamma}{2(n+1)}\right)^2 \,\middle|\, n \in \mathbb{N} \right\} \quad \text{if } \gamma > 0$$



Weyl calculus

What about $h: \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{C}$ and $h(X, D)$?

How can we define $h(X, D)$?

More than one solutions, which are not equivalent.

One of them is Weyl calculus.

We would like to define a product $\circ$ and an involution $^\circ$ such that

$$h_1(X,D)\, h_2(X,D) \underset{\text{two operators}}{\overset{\text{product of}}{=}} (h_1 \circ h_2)(X,D) \text{ product of two functions}$$

and

$$h(X,D)^* \underset{\text{operator}}{\overset{\text{adjoint}}{=}} h^\circ(X,D) \text{ involution of a function}$$

For $h : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{C}$ and $u \in L^2(\mathbb{R}^n)$, one sets

$$[h(X,D)u](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, h\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi$$

called a quantization

scalar product in $\mathbb{R}^n$

Exercise: if h depends only on the first n variables (or the last n variables) one gets $h(X)$ or $h(D)$.

Also, if h is real-valued then $h(X,D)$ is symmetric (and self-adjoint if we are lucky!)

And we have

$$\begin{cases} h_1(X)\, h_2(X) = (h_1 h_2)(X) \\ h_1(D)\, h_2(D) = (h_1 h_2)(D) \end{cases}$$

Note that there is a form for $h_1 \circ h_2$,

but it is not the pointwise multiplications,

it corresponds to a pointwise product in the first n variables,

and a convolution product in the last n variables.

And

$$h^\circ = \overline{h} \quad \text{complex conjugation of a function}$$

Chapter III: Spectral theory for self-adjoint operators

III.1: Spectral measures

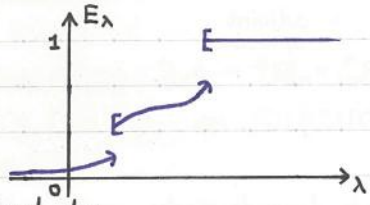
Def: A spectral family, or resolution of the identity, is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ with E_λ a projection in $\mathcal{H}$ $\overset{\text{it means}}{E_\lambda = E_\lambda^2 = E_\lambda^*}$ such that

i) $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}} = E_\mu E_\lambda$

ii) $E_\lambda = E_{\lambda+0} := \underset{\varepsilon\searrow 0}{s\text{-}\lim} E_{\lambda+\varepsilon}$ right continuity

iii) $\underset{\varepsilon\to-\infty}{s\text{-}\lim} E_\lambda = 0$

$\underset{\varepsilon\to\infty}{s\text{-}\lim} E_\lambda = 1$



The support of a spectral family is defined by

$$\mathrm{Supp}\,\{E_\lambda\} := \{\mu\in\mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq 0 \ \forall \varepsilon > 0\}$$

Then we define $E((a,b]) := E_b - E_a$

and we extend this defination to all Borel sets of $\mathbb{R}$.

Def. A Borel set of $\mathbb{R}$ is a subset of $\mathbb{R}$ obtained by countable unions, intersections and complements of $(a,b]$ with $a,b\in\mathbb{R}$.

The set of all Boral sets of $\mathbb{R}$ is denoted as $\mathcal{A}_B$.

$\Rightarrow$ At the end, we obtain a function

$$E : \mathcal{A}_B \longmapsto P(\mathcal{H}) \quad \text{set of all projections on } \mathcal{H}$$

Def. The map $E : \mathcal{A}_B \longmapsto P(\mathcal{H})$ is called the spectral measure associated with the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$.

If $f\in\mathcal{H}$ we can consider

$$F_f(\lambda) := \langle f, E_\lambda f\rangle = \langle E_\lambda f, E_\lambda f\rangle = \|E_\lambda f\|^2 \in \mathbb{R}$$

and the function $\lambda \longmapsto F_f(\lambda)$ satisfies

1) $F_f(\lambda) \geq F_f(\mu)$ if $\lambda \geq \mu$

2) $F_f(\lambda) = F_f(\lambda+0) := \lim_{\varepsilon\searrow 0} F_f(\lambda+\varepsilon)$

3) $F_f(-\infty) = 0$

$F_f(\infty) = \|f\|^2$

and we can define a measure $m_f : \mathcal{A}_B \longmapsto \mathbb{R}$ by

$$m_f(V) = \langle f, E(V) f\rangle \quad \text{for any } V\in\mathcal{A}_B$$

We consider for $\varphi \in C([a,b], \mathbb{C})$ the operator

$$\int_a^b \varphi(\lambda)\, E(d\lambda) \equiv \int_a^b \varphi(\lambda)\, dE(\lambda) \quad \text{by some authors}$$

by a limit of Riemann sums:

$$\int_a^b \varphi(\lambda)\, E(d\lambda) := \underset{\text{finer partition}}{\text{s-lim}} \sum_{j=1}^{n} \varphi(y_j)\, E\big((x_{j-1}, x_j]\big) \quad \text{notations similar to Riemannian integral}$$

where the limit exists. For $\varphi \in C([a,b])$ the limit exists.

$$\int_a^b \varphi(\lambda)\, E(d\lambda) \in \mathcal{B}(\mathcal{H}).$$

⚠ $E(\{y\})$ can be different from 0.

Prop. (spectral integral)

Let $\{E_\lambda\}$ be a spectral family, and let $\varphi \in C([a,b], \mathbb{C})$.

1) $\left\| \int_a^b \varphi(\lambda)\, E(d\lambda) \right\| = \sup_{\varphi \in [a,b] \cap \operatorname{supp}(E_\lambda)} |\varphi(\lambda)|$

2) $\left[\int_a^b \varphi(\lambda)\, E(d\lambda)\right]^* = \int_a^b \overline{\varphi(\lambda)}\, E(d\lambda)$

3) $\forall f \in \mathcal{H}$, $\left\| \int_a^b \varphi(\lambda)\, E(d\lambda) f \right\|^2 = \int_a^b |\varphi(\lambda)|^2\, m_f(d\lambda)$

4) If $\psi \in C([a,b], \mathbb{C})$, $\left(\int_a^b \varphi(\lambda)\, E(d\lambda)\right)\left(\int_a^b \psi(\lambda)\, E(d\lambda)\right) = \int_a^b \varphi(\lambda)\psi(\lambda)\, E(d\lambda)$.

Hints for 3):

$$\left\langle \int_a^b \varphi(\lambda)\, E(d\lambda) f, \int_a^b \varphi(\lambda)\, E(d\lambda) f \right\rangle \overset{②}{=} \left\langle f, \int_a^b \overline{\varphi(\lambda)}\, E(d\lambda) \int_a^b \varphi(\lambda)\, E(d\lambda) f \right\rangle$$

$$\overset{④}{=} \left\langle f, \int_a^b |\varphi(\lambda)|^2\, E(d\lambda) f \right\rangle$$

$$= \int_a^b |\varphi(\lambda)|^2 \underbrace{\langle E(d\lambda) f, f \rangle}_{= m_f(d\lambda)}.$$

How can one generate this to $\int_{\mathbb{R}} \varphi(\lambda)\, E(d\lambda)$?

Easy if $\|\varphi\|_\infty < \infty$ or if $\operatorname{supp}(E_\lambda)$ is bounded (Prop. 1).

But otherwise: by ③ it is natural to define

$$D_\varphi := \left\{ f \in \mathcal{H} \;\middle|\; \int_{-\infty}^{\infty} |\varphi(x)|^2\, m_f(d\lambda) < \infty \right\}$$

and we consider the unbounded operator

$$\left(\int_{-\infty}^{\infty} \varphi(\lambda)\, E(d\lambda), D_\varphi\right) \quad \int_{-\infty}^{\infty} \varphi(\lambda)\, E(d\lambda) f \text{ is well-defined for } f \in D_\varphi \text{ by ③.}$$

Question

Is D_φ dense in $\mathcal{H}$?

Yes since $\{E((-M,M))f \mid f \in \mathcal{H}\ \forall M \in \mathbb{R}_+\}$

is dense in $\mathcal{H}$ and is included in D_φ.

since $\operatorname{s-lim}_{M\to\infty} E((-M,M)) = 1$

Thus

$\left(\int_a^b \varphi(\lambda)\, E(d\lambda),\ D_\varphi\right)$ is a densly defined operator in $\mathcal{H}$.

Lemma:

$\left(\int_a^b \varphi(\lambda)\, E(d\lambda),\ D_\varphi\right)$ is self-adjoint if $\varphi \in C(\mathbb{R}, \underline{\mathbb{R}})$. can be weakened

It is called the self-adjoint operator associated with $\{E_\lambda\}$ and φ.

Note that we can consider the special case

$\varphi(\lambda) = \lambda = id(\lambda)$ identity function

Thm. (Spectral thm)

There exists a bijective correspondence

between spectral families (or spectral measures) and self-adjoint operators.

The correspondense is

$A = \int_{\mathbb{R}} \lambda E(d\lambda)$, $D(A) = D_{id}$.

We often write E^A for the spectral family associated with A.

It is also natural to write

$\varphi(A) := \int_{\mathbb{R}} \varphi(\lambda) E^A(d\lambda)$

Remark: In good situation, $\varphi(A)$ can be constructed

by the Taylor expansion of φ, namely

if $\varphi(x) = \sum_{j=0}^{\infty} c_j x^j$, then $\varphi(A) = \sum_{j=0}^{\infty} c_j A^j$ whenever convergent in the strong topology.

⚠ If A is unbounded, $\sum_{j=0}^{\infty} c_j A^j$ is defined only on $\bigcap_{j=0}^{\infty} D(A^j)$,

which can be quite small.

Exercises:

- $\operatorname{supp}(E^A_\lambda) = \sigma(A)$
- $\|(A-z)^{-1}\| = \operatorname{dist}(z, \sigma(A))^{-1}$ distance on the complex plane

Let $t \in \mathbb{R}$ and consider the function $\varphi_r : \mathbb{R} \ni \lambda \longmapsto e^{-it\lambda}$, which is a bounded function. Then

$$\varphi_r(A) = e^{-itA} \in B(\mathcal{H}), \text{ and it is even a unitary operator.}$$

In addition

$$e^{-itA} e^{-isA} = (e^{-it\cdot} e^{-is\cdot})(A) = (e^{-i(t+s)\cdot})(A) = e^{-i(t+s)A} \qquad \forall s,t \in \mathbb{R}$$

which means that

$\{e^{-itA}\}_{t\in\mathbb{R}}$ is a representation of the group $(\mathbb{R},+)$ by unitary operators

Moreover, it is strongly continuous. It means

$$\circledast := \| e^{-i(t+\varepsilon)A} f - e^{-itA} f \| \xrightarrow[\text{strongly}]{\varepsilon\to 0} 0$$

Indeed,

$$\circledast = \| e^{-itA}(e^{-i\varepsilon A} - 1) f \| \overset{\text{unitary}}{=\!=} \| (e^{-i\varepsilon A} - 1) f \| = \int_{-\infty}^{\infty} \underbrace{|e^{-i\varepsilon\lambda} - 1|^2}_{\leq 4} m^A{}_f(d\lambda)$$

$$\leq 4 \int_{-\infty}^{\infty} m^A{}_f(d\lambda) = 4\|f\|^2$$

By applying Lebesgue dominated convergence thm, one has

$$\lim_{\varepsilon\to 0} \circledast = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} |e^{-i\varepsilon\lambda} - 1|^2 m^A{}_f(d\lambda) = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} |e^{-i\varepsilon\lambda} - 1|^2 m^A{}_f(d\lambda)$$

$$= \int_{\mathbb{R}} 0 \, m^A{}_f(d\lambda) = 0$$

It means

$\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary representation of $\mathbb{R}$.

Thm. (Stone's thm)

There is a bijective correspondence between self-adjoint operators and strongly continuous unitary representation of $\mathbb{R}$.

In particular, if $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unit. rep. of $\mathbb{R}$, we set

$$D(A) = \{ f \in \mathcal{H} \mid \exists \operatorname*{s-lim}_{t\to 0} \tfrac{1}{t}(U_t - 1) f \}$$

and

$$Af = \lim_{t\to 0} \frac{i}{t}(U_t - 1) f$$

In summary : bijective correspondance

spectral families $\Leftrightarrow$ spectral measures $\Leftrightarrow$ self-adjoint operators

$\Leftrightarrow$ strongly continuous unitary groups.

Remark : For suitable φ, the operator $\varphi(A)$ can also be defined by $\{e^{-itA}\}_{t\in\mathbb{R}}$:

$$\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t) e^{-itA} dt$$

Indeed

$$\langle f, \varphi(A) f \rangle = \int \varphi(\lambda) m_f(d\lambda) = \int m_f(d\lambda) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\lambda} \check{\varphi}(t) dt$$

$$\overset{\text{Fubini}}{=\!=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \check{\varphi}(t) \left[\int e^{-it\lambda} m_f(d\lambda) \right] dt$$

$$= \langle f, \frac{1}{\sqrt{2\pi}} \int \check{\varphi}(\lambda) e^{-itA} dt \, f \rangle$$

Then, use polarization identity for

$$\langle f, \varphi(A) g \rangle = \langle f, \frac{1}{\sqrt{2\pi}} \int \check{\varphi}(t) e^{-itA} dt \, g \rangle$$

III.2 Spectral parts of a self-adjoint operators

Reminders about measures on $\mathbb{R}$: 3 types of measures

1) m is a pure point measure if

$$\forall v \in \mathcal{A}_B \ \exists \{x_j\}_j \subset v : \ m(v) = \sum_j m(\{x_j\})$$

2) m is an absolutely continuous measure with respect to the Lebesgue measure if

$$\forall v \in \mathcal{A}_B : m(v) = \int_v f(x)\,dx \quad \text{for some } f \in L^1(v).$$

dx always means Lebesgue integral.

f = Radon Nykodym density ; f does not depend on v.

3) m is a singular continuous measure with respect to the Lebesgue measure if

$$\forall x \in \mathbb{R} : m(\{x\}) = 0 \quad \text{and}$$

$$\exists v \in \mathcal{A}_B : \int_v 1\,dx = 0 \quad \text{and} \quad m(\mathbb{R} \setminus v) = 0$$

It means the support of m is concentrated on a set of Lebesgue measure 0, but which is not countable.

Thm. (Lebesgue decomposition thm)

Any Stieltjes (= regular Borel) measure m admits a unique decomposition

$$m = m_{pp} + m_{ac} + m_{sc}$$

with m_{pp} pure point

$$\left.\begin{array}{ll} m_{ac} & \text{absolutely continuous} \\ m_{sc} & \text{singular continuous} \end{array}\right\} \text{with respect to the Lebesgue measure.}$$

Remark: The measures $m_f = \langle f, E(\bullet) f \rangle$ are Stieltjes measures!

$f \in \mathcal{H}$, $E(\bullet)$: spectral measure

Let A be a self-adjoint op. and let $\{E^A_\lambda\}_{\lambda \in \mathbb{R}}$ be the associated spectral family, and E^A the associated spectral measure.

Recall that

$$\mathrm{Ran}\,(E^A(\{\mu\})) = \{f \in \mathcal{H} \mid E^A(\{\mu\}) f = f\}$$

Observe that

$$\mathrm{Ran}\,(E^A(\{\mu\})) = \{f \in D(A) \mid Af = \mu f\}$$

Indeed this follows from the equality

$$\|(A-\mu)f\|^2 = \int |\lambda - \mu|^2 \, m_f(d\lambda)$$

If f satisfies $Af = \mu f$, then l.h.s $= 0 \iff m_f$ is only supported at μ.

Def. $\mathcal{H}_p(A) := \bigoplus_{\mu} \mathrm{Ran}\,(E^A(\{\mu\})) \subset \mathcal{H}$

μ: eigenvalues of A — this set is countable (because $\mathcal{H}$ is seperable)

$\mathcal{H}_p(A)$ is the subspace generated by all eigenfunctions for all eigenvalues of A.

Remark: This subset can be small,

but for matrices it is all the finite dimensional space.

Def. We set

$\mathcal{H}_{ac}(A) := \{ f \in \mathcal{H} \mid m^A_f \text{ is a. c.} \}$

$\mathcal{H}_{sc}(A) := \{ f \in \mathcal{H} \mid m^A_f \text{ is s. c.} \}$

Thm. (Decomposition thm)

$$A = \begin{pmatrix} A_p & 0 & 0 \\ 0 & A_{ac} & 0 \\ 0 & 0 & A_{sc} \end{pmatrix}$$

These blocks are either 0 or infinite dimensional.

1) $\mathcal{H}_p(A) = \{ f \in \mathcal{H} \mid m^A_f \text{ is pure point} \}$

2) $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A)$ and

$A = A_p \oplus A_{ac} \oplus A_{sc}$ which are self-adjoint restrictions.

3) If $\varphi : \mathbb{R} \mapsto \mathbb{C}$ is bounded and continuous,

$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc})$

4) $\sigma(A) = \sigma(A_p) \cup \sigma(A_{ac}) \cup \sigma(A_{sc})$

$\hookrightarrow = \overline{\sigma_p(A)}$ closure

These thms are not so difficult; they take some time.

Def. (2 more sets)

$\sigma_d(A) = \{ \lambda \in \sigma_p(A) \mid \lambda \text{ is isolated in } \sigma(A) \text{ and of finite multiplicity} \} \subset \mathbb{R}$

discrete

$\hookrightarrow \exists \varepsilon > 0 : (\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(A) = \{\lambda\}$

$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A) \subset \mathbb{R}$

essential

Stability of the essential spectrum important for perturbation theory

Prop. Let A be self-adjoint, B be symmetric and such that

$B(A-z)^{-1} \in \mathcal{K}(\mathcal{H})$ B is A-compact for $z \in \mathbb{C} \setminus \mathbb{R}$

Then

$\sigma_{ess}(A+B) = \sigma_{ess}(A)$

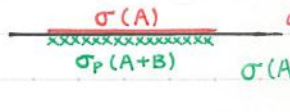
The nature of the spectrum can change a lot!

Prop.

$\forall A$ self-adjoint $\forall \varepsilon > 0 \ \exists B$ self-adjoint, $B \in \mathcal{K}(\mathcal{H})$ and $\|B\| \leq \varepsilon$ s.t.

$A + B$ has only point spectrum

$(\Leftrightarrow \mathcal{H}_{ac}(A+B) = \{0\} = \mathcal{H}_{sc}(A+B))$



$\sigma(A) = \sigma(A+B) = \overline{\sigma_p(A+B)}$

Weyl criterion

$\lambda \in \sigma(A)$ iff

$\exists \{f_n\}_{n \in \mathbb{N}} \subset D(A)$, $\|f_n\| = 1$ and $\operatorname{s-lim}_{n \to \infty} (A - \lambda) f_n = 0$

(A family of approximate eigenfunctions)

About $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$:

Let A be self-adjoint and consider

$$U_t = e^{-itA}$$

Lemma :

1) If $f \in \mathcal{H}_{ac}(A)$ then $\lim_{t \to \pm\infty} \langle g, U_t f \rangle = 0 \quad \forall g \in \mathcal{H}$. It spreads on the entire $\mathcal{H}$.

This is not true in general for $f \in \mathcal{H}_{sc}(A)$ but

2) If $f \in \mathcal{H}_{sc}(A)$ then $\lim_{T \to \infty} \underbrace{\frac{1}{T} \int_0^{\pm T}}_{\text{average on time}} |\langle g, U_t f \rangle|^2 \, dt = 0 \quad \forall g \in \mathcal{H}$.

One can show that the system comes back to an inertial g less and less often, but still an infinite number of times.

Remark :

If f satisfies $Af = \mu f$, then

$$\langle g, e^{-itA} f \rangle = \langle g, e^{-it\mu} f \rangle = e^{-it\mu} \langle g, f \rangle \text{ not convergent}$$

Spectral representation of a self-adjoint op. A in a Hilbert space $\mathcal{H}$:

There exists a Hilbert space

$$\mathcal{K} = \int_{\sigma(A)}^{\oplus} \mathcal{K}(\lambda) \underbrace{\mu(d\lambda)}_{\text{measure on } \mathbb{R}}$$

and a unitary transformation $U : \mathcal{H} \longrightarrow \mathcal{K}$ s.t.

$$U A U^* = \int_{\sigma(A)}^{\oplus} \lambda \, \mu(d\lambda)$$

operator of multiplication by the variable λ

This is the diagonalization of any self-adjoint op.