Advanced Algorithms

Lecture 6

Computational Geometry Algorithms:

Sweeping Techniques

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ILO of Lecture 6

- Computational Geometry: sweeping techniques
  - to understand how the basic geometric operations (such as determining how two line segments are oriented and whether they intersect) are performed;
  - to understand the basic idea of the sweeping algorithm design technique;
  - to understand and be able to analyze the sweeping-line algorithm to determine whether any pair of line segments intersect
  - to understand and be able to analyze the Graham's scan algorithm for identifying convex hulls.

Agenda

- Computational geometry
- Basic geometric operations
- Sweeping techniques
- Graham's scan

Computational geometry

- Computational geometry studies algorithms for solving geometric problems.
- Algorithmic basis for many scientific and engineering disciplines:
  - Geographic Information Systems (GIS)
  - Robotics
  - Computer graphics
  - Computer vision
  - Computer Aided Design/Manufacturing (CAD/CAM),
  - Very-large-scale integration (VLSI) design.

Computational geometry problems

- Input: a description of a set of geometric objects.
  - A set of points
  - A set of line segments
  - Vertices of a polygon
- Output:
  - a response to a query about the objects
    - E.g., whether any of the lines intersect
  - a new geometric object,
    - E.g., convex hull of the set of points

- We will deal with *points* and *line segments* in **2D** space.

Agenda

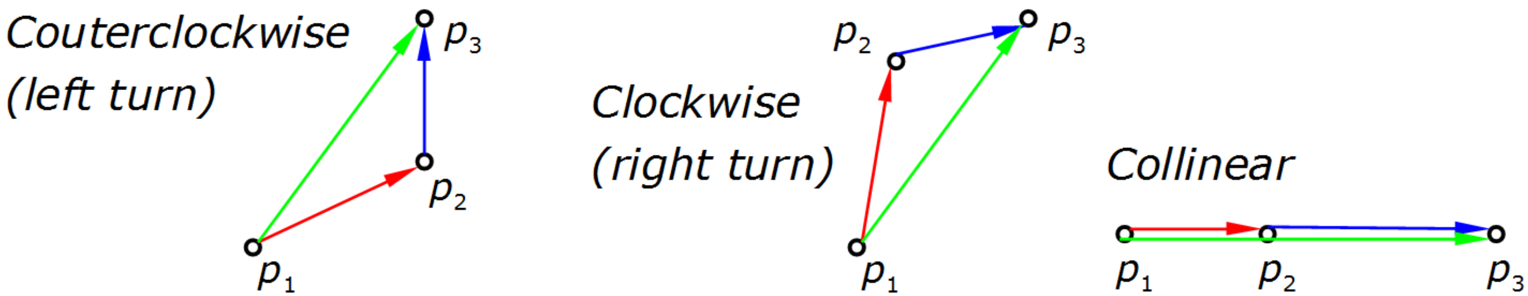
- Computational geometry
- Basic geometric operations
- Sweeping techniques
- Graham's scan

Line-segment properties

- What is the line-segment $\overline{p_1 p_2}$ between $p_1=(x_1, y_1)$ and $p_2=(x_2, y_2)$?
  - It contains any point p_3 that is on the line passing through p_1 and p_2 and is on or between p_1 and p_2 on the line.
  - The set of ***convex combinations*** of $p_1=(x_1, y_1)$ and $p_2=(x_2, y_2)$.
    - $p_3 = \alpha\, p_1 + (1- \alpha)\, p_2$ where $0 \le \alpha \le 1$
  - $p_1=(0, 0)$ and $p_2=(10, 10)$
    - $\alpha = 0$, p_2.
    - $\alpha = 1$, p_1.
    - $\alpha = 0.5$, $(5, 5)$
    - $\alpha = 0.02$, $(9.8, 9.8)$
  - We call p_1 and p_2 as the endpoints of the line-segment $\overline{p_1 p_2}$.
- Directed line-segment $\overrightarrow{p_1 p_2}$ from p_1 to p_2.

Basic operation

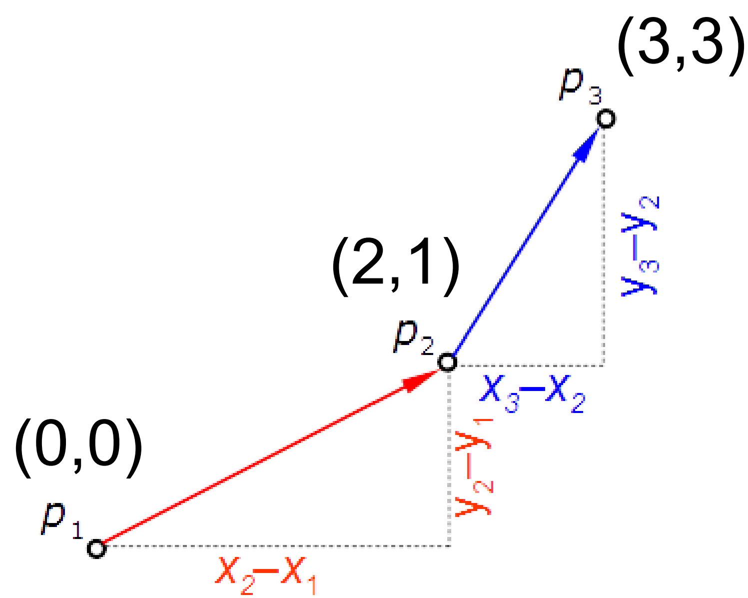
- How to find "orientation" of two line segments?
  - Three points: $p_1(x_1, y_1)$, $p_2(x_2, y_2)$, $p_3(x_3, y_3)$
  - Is segment $\overrightarrow{\mathrm{p_1p_3}}$ **clockwise** or **counterclockwise** from $\overrightarrow{\mathrm{p_1p_2}}$?
  - Going from segment $\overline{\mathrm{p_1p_2}}$ to segment $\overline{\mathrm{p_2p_3}}$, do we make a **right** or a **left** turn?



  - For simplicity, use p_1p_2 to denote $\overline{p_1p_2}$.

Computing the orientation

- Compute the slopes of the two line-segments:
  - Slope of segment p_1p_2: $a = (y_2 - y_1)/(x_2 - x_1)$
  - Slope of segment p_2p_3: $b = (y_3 - y_2)/(x_3 - x_2)$
- How do you compute the orientation then?
  - When $a \geq 0$ and $b \geq 0$
    - counterclockwise (left turn): $a<b$
    - clockwise (right turn): $a>b$
    - collinear (no turn): $a=b$
- p1(0,0), p2(2,1), p3(3, 3)
  - p1p2: (1-0)/(2-0)=0.5
  - p2p3: (3-1)/(3-2)=2
  - 0.5<2, thus p1p2 left turn to p2p3.
  - p3p2: (1-3)/(2-3)=2
  - p2p1: (0-1)/(0-2)=0.5
  - 2>0.5, thus p3p2 right turn to p2p1.



Problem of using slopes

- When computing slopes, we need the division operation.
- When segments are nearly parallel, this method is very sensitive to the precision of the division operation on real computers.
- p1(0,0), p2(2,1), p3(4.0000001, 2.0000001)
  - p1p2: (1-0)/(2-0)=0.5
  - p2p3: (2.0000001-1)/(4.0000001-2)=1.0000001/2.0000001
    - 0.5, collinear
    - 0,50000001, left turn.

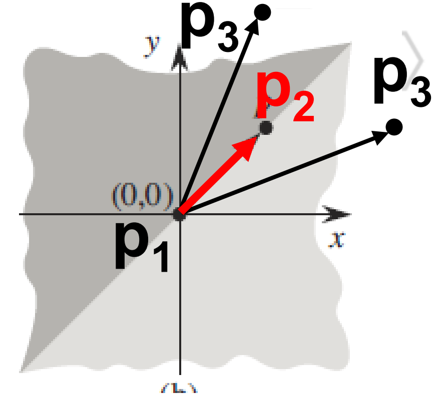
- If a method avoids division, it is much more accurate.

Method without division: cross product

- Finding orientation without division to avoid numerical problems on different computers.
- Whether p_1p_3 is clockwise or counter-clockwise from $\boldsymbol{p_1p_2}$.
- Cross product
  - $(p_3-p_1)\times(p_2-p_1)=(x3-x1)(y2-y1)-(x2-x1)(y3-y1)$
- Or determinant of the following matrix
  - $\begin{pmatrix} x3-x1 & x2-x1 \\ y3-y1 & y2-y1 \end{pmatrix}$
- Positive – p_1p_3 is clockwise from $\boldsymbol{p_1p_2}$
- Negative – p_1p_3 is counterclockwise from $\boldsymbol{p_1p_2}$
- Zero – collinear

Example

$$\begin{pmatrix} x3-x1 & x2-x1 \\ y3-y1 & y2-y1 \end{pmatrix}$$



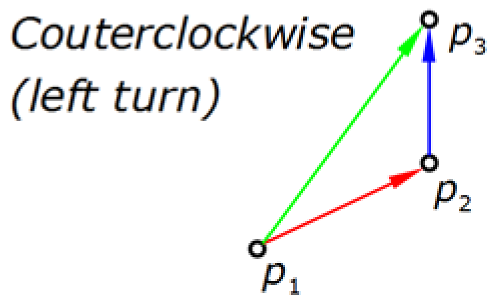
- Assume p_1=(0, 0) p_2=(1,1)

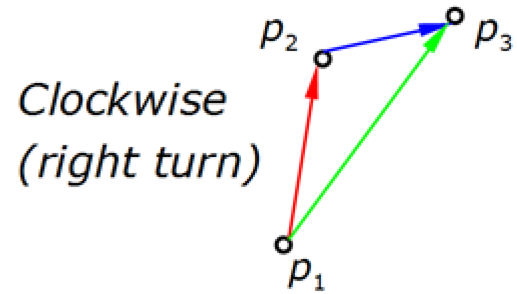
- If p_3 is in the lightly shaded region, p_1p_3 is clockwise from p_1p_2
  - E.g., p_3=(2, 1), we have $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$=2-1=1

- If p_3 is in the darkly shaded region, p_1p_3 is counter-clockwise from p_1p_2
  - E.g., p_3=(1, 2), we have $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$=1-2=-1

Determine left/right turn

- Determine whether two consecutive segments p_1p_2 and p_2p_3 turn left or right at p_2.

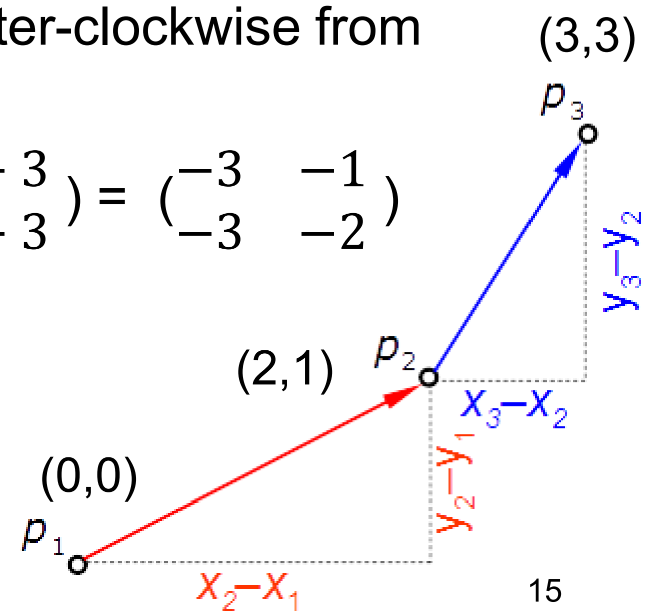




- Segment p_1p_3 is clockwise or counterclockwise relative to segment $\boldsymbol{p_1p_2}$
  - Counterclockwise: left turn. Clockwise: right turn.

Examples

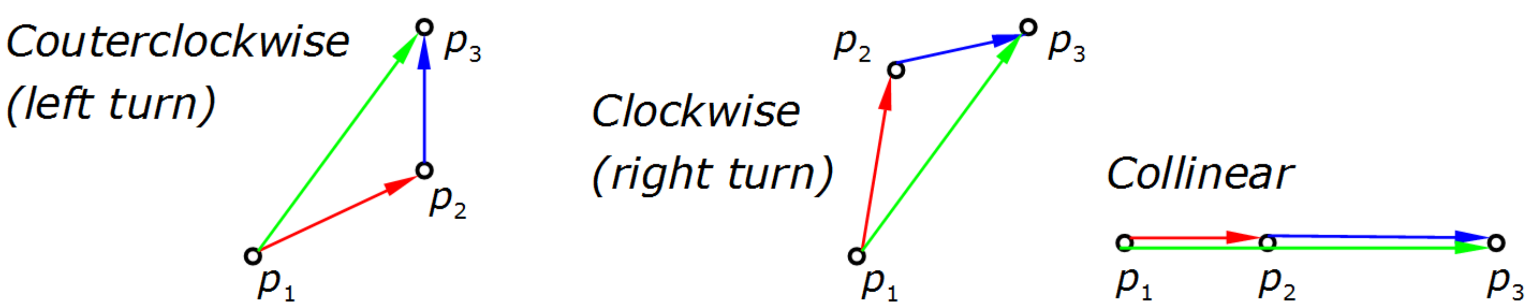
- p1(0,0), p2(2,1), p3(3, 3)
- From p1p2 and p2p3, left or right turn?
  - Determine whether p1p3 is clockwise/counter-clockwise from ***p1p2***.
  - p1p3, ***p1p2***: (p3-p1) × (p2-p1)=$\begin{pmatrix} 3-0 & 2-0 \\ 3-0 & 1-0 \end{pmatrix}$ = $\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$=3-6=-3, counter-clockwise, thus left turn.
- From p3p2 to p2p1, left or right turn?
  - Determine whether p3p1 is clockwise/counter-clockwise from ***p3p2***.
  - p3p1, ***p3p2***: (p1-p3) × (p2-p3)= $\begin{pmatrix} 0-3 & 2-3 \\ 0-3 & 1-3 \end{pmatrix}$ = $\begin{pmatrix} -3 & -1 \\ -3 & -2 \end{pmatrix}$ =6-3=3, clockwise, thus right turn.



A quick summary

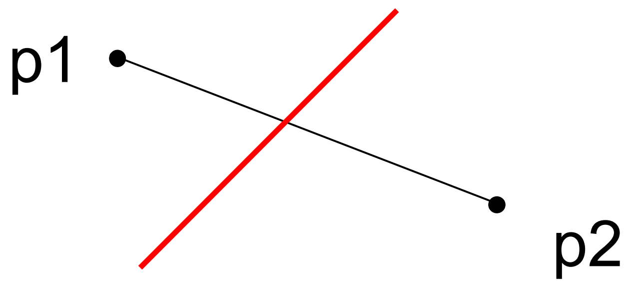
- Q1: Whether p_1p_3 is clockwise/counter-clockwise from p_1p_2?
- Q2: From p_1p_2 to p_2p_3, do you need to turn right/left at p_2?
- Compute the cross product
  - $(p_3-p_1)\times(p_2-p_1)=(x3-x1)(y2-y1)-(x2-x1)(y3-y1)$
  - $= \begin{pmatrix} x3-x1 & x2-x1 \\ y3-y1 & y2-y1 \end{pmatrix}$

Cross product	Q1	Q2
Negative	Counterclockwise	Left turn
Positive	Clockwise	Right turn
Zero	Collinear	Go straight



Whether two line segments intersect

- A segment p1p2 ***straddles*** a line (e.g., the red line) if point p1 lies on one side of the line but p2 lies on the other side.



  - Along the line, one needs to turn different directions to go to p1 and p2.
- Two line segments intersect if and only if either of the following two conditions holds:
  - Each segment straddles the line containing the other.
  - An endpoint of one segment lies on the other segment.

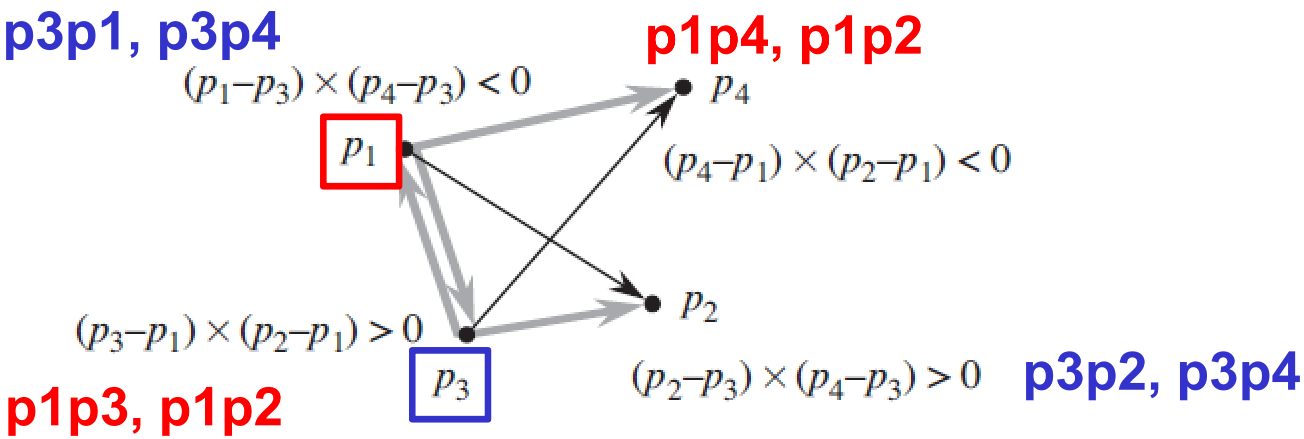
Pseudo code

```
SEGMENTS-INTERSECT(p1, p2, p3, p4)
  d1 = DIRECTION(p3, p4, p1)
  d2 = DIRECTION(p3, p4, p2)
  d3 = DIRECTION(p1, p2, p3)
  d4 = DIRECTION(p1, p2, p4)
  if ((d1 > 0 and d2 < 0) or (d1 < 0 and d2 > 0)) and
         ((d3 > 0 and d4 < 0) or (d3 < 0 and d4 > 0))
         return TRUE
  elseif d1 == 0 and ON-SEGMENT(p3, p4, p1)
         return TRUE
  elseif d2 == 0 and ON-SEGMENT(p3, p4, p2)
         return TRUE
  elseif d3 == 0 and ON-SEGMENT(p1, p2, p3)
         return TRUE
  elseif d4 == 0 and ON-SEGMENT(p1, p2, p4)
         return TRUE
  else return FALSE
```

Check if each segment straddles the line containing the other.
If p1p2 straddles p3p4 and p3p4 straddles p1p2

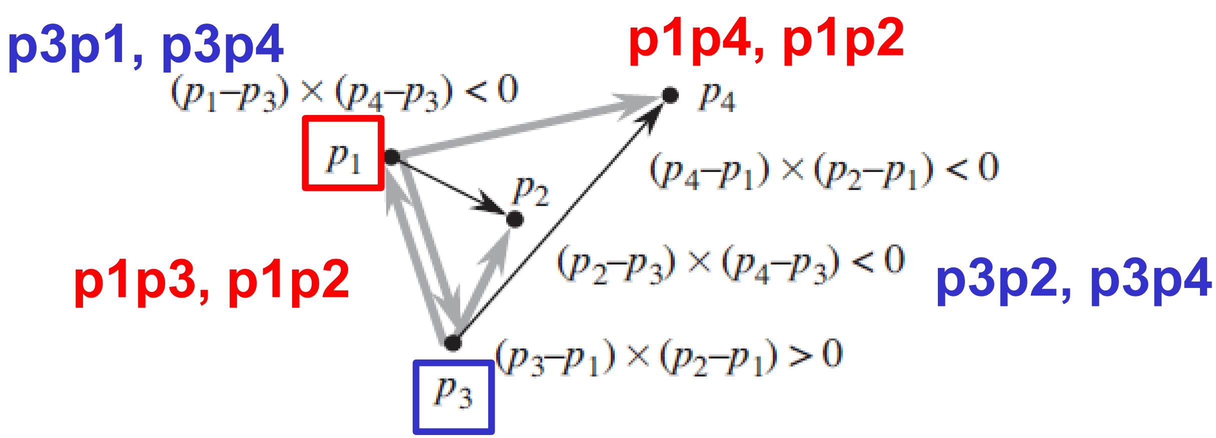
An endpoint of one segment lies on the other segment.

Example



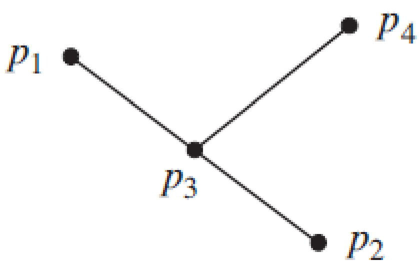
- Check if p1p2 straddles p3p4
  - The turn from p3p4 to p4p1 vs. the turn from p3p4 to p4p2.
  - p3p1, p3p4: (p1-p3) × (p4-p3) < 0
  - p3p2, p3p4: (p2-p3) × (p4-p3) > 0, so yes.
- Then, check if p3p4 straddles p1p2
  - The turn from p1p2 to p2p3 vs. the turn from p1p2 to p2p4.
  - p1p3, p1p2: (p3-p1) × (p2-p1) >0
  - p1p4, p1p2: (p4-p1) × (p2-p1) <0, so yes.
- Yes, they intersect.

Example 2

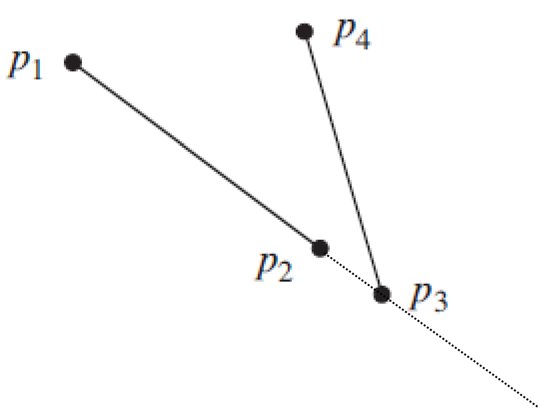


- Check if p1p2 straddles p3p4
  - The turn from p3p4 to p4p1 vs. the turn from p3p4 to p4p2.
  - p3p1, p3p4: (p1-p3) × (p4-p3) < 0
  - p3p2, p3p4: (p2-p3) × (p4-p3) < 0, so no.
- Then, check if p3p4 straddles p1p2
  - The turn from p1p2 to p2p3 vs. the turn from p1p2 to p2p4.
  - p1p3, p1p2: (p3-p1) × (p2-p1) >0
  - p1p4, p1p2: (p4-p1) × (p2-p1) <0, so yes.
- No, they do not intersect.

Example 3



- p1p2, p2p3: collinear.
- p3 is on segment p1p2.
- So they intersect.



- p1p2, p2p3: collinear.
- But p3 is not on segment p1p2.
- So they do not intersect.

Mini-quiz

- What is the time complexity of the algorithm that tests if two segments intersect?

SEGMENTS-INTERSECT(p_1, p_2, p_3, p_4)

$d_1 =$ DIRECTION(p_3, p_4, p_1)
$d_2 =$ DIRECTION(p_3, p_4, p_2)
$d_3 =$ DIRECTION(p_1, p_2, p_3)
$d_4 =$ DIRECTION(p_1, p_2, p_4)
if $((d_1 > 0$ and $d_2 < 0)$ or $(d_1 < 0$ and $d_2 > 0))$ and
    $((d_3 > 0$ and $d_4 < 0)$ or $(d_3 < 0$ and $d_4 > 0))$
    return TRUE
elseif $d_1 == 0$ and ON-SEGMENT(p_3, p_4, p_1)
    return TRUE
elseif $d_2 == 0$ and ON-SEGMENT(p_3, p_4, p_2)
    return TRUE
elseif $d_3 == 0$ and ON-SEGMENT(p_1, p_2, p_3)
    return TRUE
elseif $d_4 == 0$ and ON-SEGMENT(p_1, p_2, p_4)
    return TRUE
else return FALSE

Constant time.

A quick summary

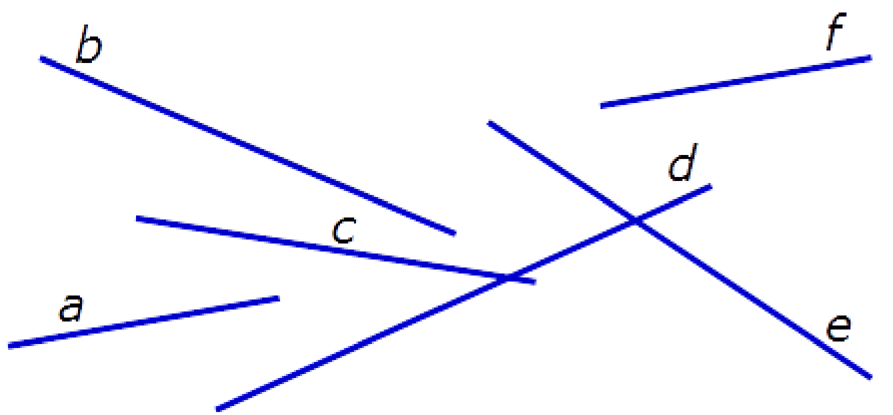
- Cross product is a fundamental operation in computational geometry.
- Checking whether two segments intersect is based on cross products.
- The complexity of checking whether two segments intersect is constant.

Agenda

- Computational geometry
- Basic geometric operations
- Sweeping techniques
- Graham's scan

Intersections in a set of line segments

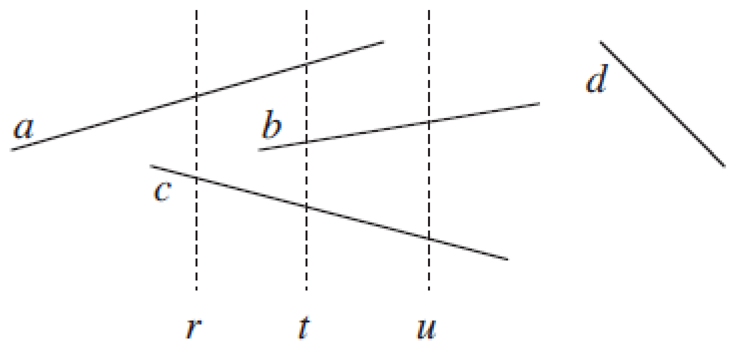
- Given a set of n line segments, determine whether any two line segments intersect.
  - Note: not asking to report all intersections, but just true or false.
  - *What would be the brute force algorithm and what is its worst-case complexity in terms of n, i.e., the number of line segments?*



  - *We will see a O(nlgn) algorithm using the sweeping technique.*

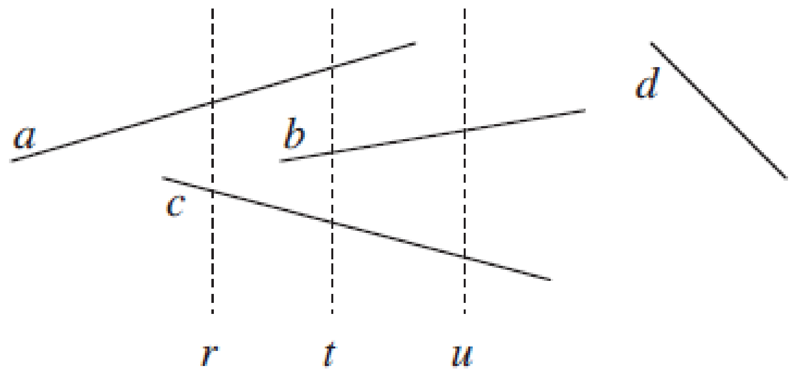
Sweeping

- Image a vertical sweep line passes through the given set of geometric objects, usually from left to right.
- Sweeping provides a method for ***ordering*** geometric objects, usually by placing them into a dynamic data structure, and for taking advantage of the relationships among them.



Ordering segments

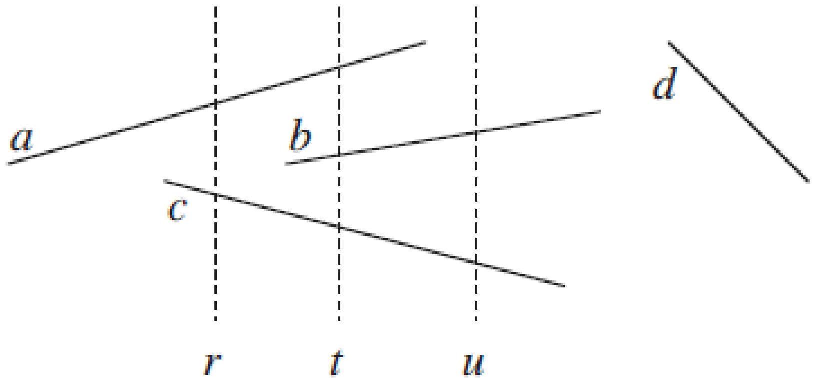
- We order the given segments that intersects a vertical sweep line according to the y-coordinates of the points of intersection.



- Given two segments s1 and s2. They are comparable at x if the vertical sweep line with x-coordinate being x intersects both of them.
  - E.g., Segments a and c are comparable at r.

Ordering segments

- We say that s1 is above s2 at x, denoted as $s1 \geq_x s2$.
  - if s1's y coordinate of the intersection is higher than that of s2's.
  - or if s1 and s2 intersect at the sweep line.



  - At r: $a \geq_r c$
  - At t: $a \geq_t b$, $a \geq_t c$, $b \geq_t c$
  - At u: $b \geq_u c$

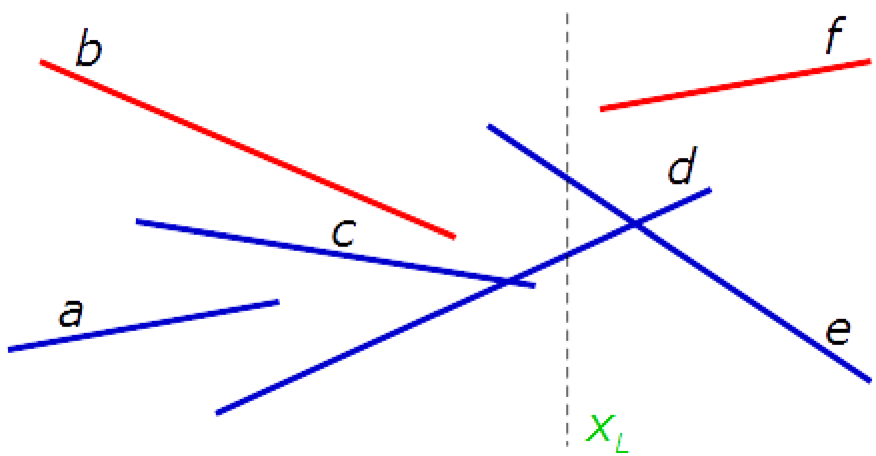
Moving the sweep line

- Sweeping algorithms typically manage two sets of data:
- **Sweep-line status:** the relationships among the objects that the sweep line intersects.
- **Event-point schedule:** is a sequence of points where updates to the sweep-line status are required.

- Let’s see two algorithms using the sweeping techniques which both are able to identify whether any two line segments intersect.

Algorithm 1

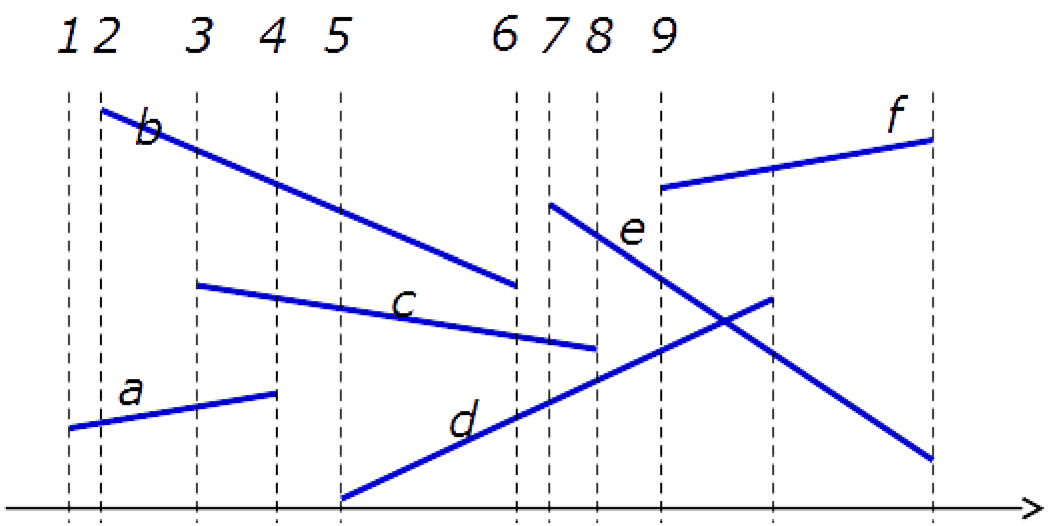
- Observations:
  - *Two segments definitely* ***do not*** *intersect if their projections to the x axis do not intersect.*
  - In other words: *If segments intersect, there is some x_L such that a vertical line* at x_L *intersects both segments.*



  - *b and f cannot intersect since their projects to x axis do not intersect.*

Algorithm 1

- **Event-point schedule:**
  - Each segment's end points are event points.
  - Order them from left to right.
- **Sweep-line status:**
  - At an event point, update the status of the sweep line and perform intersection tests.
  - Left end point: a new segment is added to the status and it needs to be checked against ***all*** the existing segments in the status
  - Right end point: the corresponding segment is deleted from the status.



1. {a}
2. {a, b}
3. {a, b, c}
4. {b, c}
5. {b, c, d}
6. {c, d}
7. {c, d, e}

d and e intersect, stop

Mini quiz

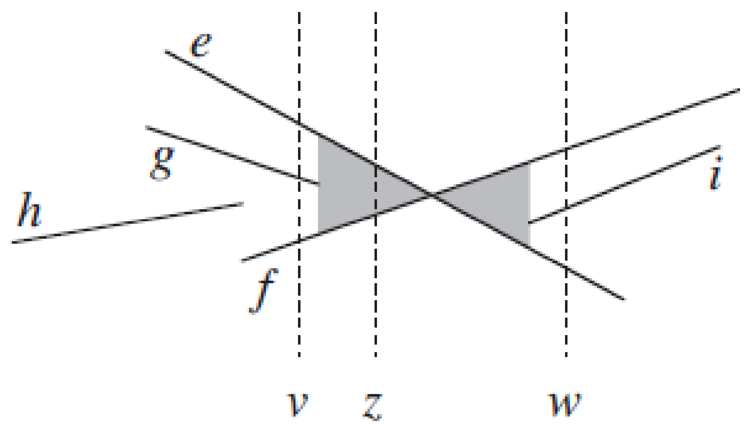
- What is the worst case example?
- What is the worst case complexity?
- Is it better than brute-force?

- $O(n^2)$

- Why can we remove a segment from the status when we see its right end point?
- Why do we need to insert a segment into the status when we see its left end point?

Algorithm 2

- More insightful observations:
  - For a specific position of the sweep line, there is an **order of** segments in the y-axis;
  - If two segments intersect, there is a position of the sweep-line such that the two segments are **adjacent** in this order;
  - Order does not change in-between event points until the first intersection point.



  - We do not need to check ***all*** segments in the sweep-line status, but only the ***adjacent*** ones (which are ***at most 2 segments***).

Algorithm 2

- Sweep-line status data structure:
  - Operations:
    - Insert(T, s): insert segment s into the status T.
    - Delete(T, s): delete segment s from the status T.
    - Above(T, s): return the segment immediately above s in T, predecessor.
    - Below(T, s): return the segment immediately below s in T, successor.
  - Balanced binary search tree T (e.g., red-black tree)
    - All operations can be done in O(lgn).

Algorithm 2

```
ANY-SEGMENTS-INTERSECT(S)
T = ∅
sort the endpoints of the segments in S from left to right,
    breaking ties by putting left endpoints before right endpoints
    and breaking further ties by putting points with lower
    y-coordinates first
for each point p in the sorted list of endpoints
    if p is the left endpoint of a segment s
        INSERT(T, s)
        if (ABOVE(T, s) exists and intersects s)
            or (BELOW(T, s) exists and intersects s)
            return TRUE
    if p is the right endpoint of a segment s
        if both ABOVE(T, s) and BELOW(T, s) exist
            and ABOVE(T, s) intersects BELOW(T, s)
            return TRUE
        DELETE(T, s)
return FALSE
```

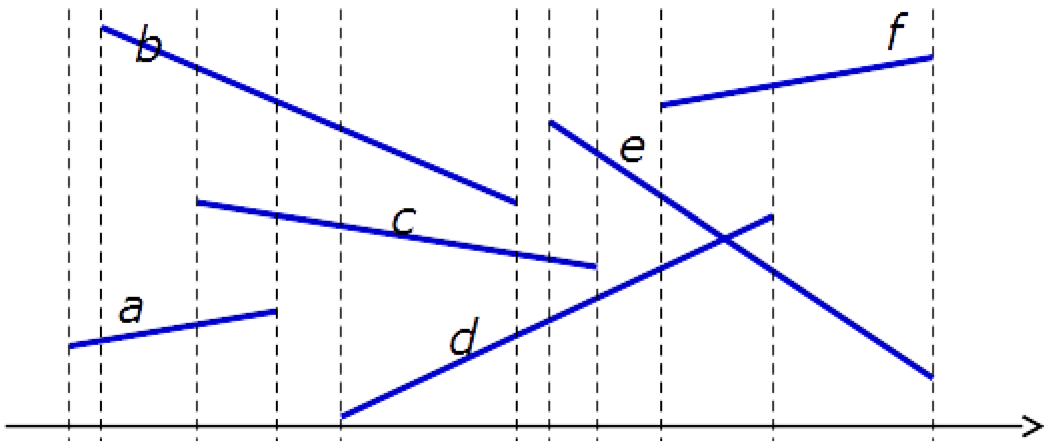
Event-point schedule

A new segment comes in. Check it with its predecessor and its successor.

An old segment comes out. Check its predecessor and its successor.

Mini quiz (also on Moodle)

- How many intersection tests do we need to do on the following set of segments?
- At which event an intersection is discovered?
  - Use this format: segment.l or segment.r
    - E.g.: a.l, a.r, b.l, b.r



Mini quiz

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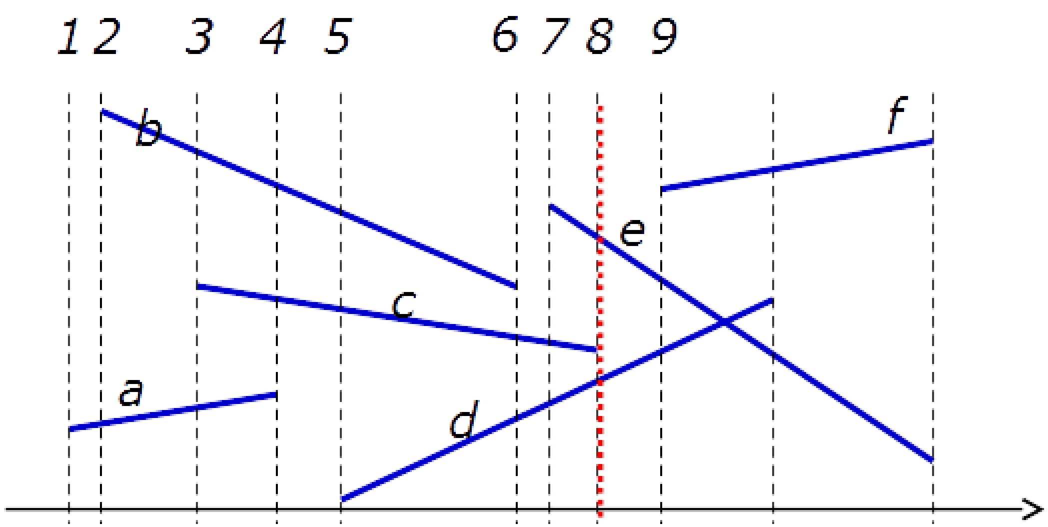

1. <a>, 0 check.
2. <b, a>, 1 check: ba
3. <b, c, a>, 2 checks: bc, ca
4. <b, c>, 0 check.
5. <b, c, d>, 1 check: cd
6. <c, d>, 0 check.
7. <e, c, d>, 1 check: ec
8. <e, d> , 1 check: ed, found!

6 checks.

c.r

Algorithm 2 Complexity

```
ANY-SEGMENTS-INTERSECT(S)
T = Ø
sort the endpoints of the segments in S from left to right,
    breaking ties by putting left endpoints before right endpoints
    and breaking further ties by putting points with lower
    y-coordinates first
for each point p in the sorted list of endpoints
    if p is the left endpoint of a segment s
        INSERT(T, s)
        if (ABOVE(T, s) exists and intersects s)
            or (BELOW(T, s) exists and intersects s)
            return TRUE
    if p is the right endpoint of a segment s
        if both ABOVE(T, s) and BELOW(T, s) exist
            and ABOVE(T, s) intersects BELOW(T, s)
            return TRUE
        DELETE(T, s)
return FALSE
```

Event-point schedule. Sorting O(nlgn)

For loop iterates 2n times.

Each of the insert, delete, above, below operations takes O(lgn).

Intersection test takes constant time.

In total, O(nlgn).

Sweeping technique principles

- Define events and their order.
  - If all the events can be determined in advance – *sort* the events.
  - Otherwise, use a *priority queue* to manage the events.
- Determine which operations have to be performed with the sweep-line status at each event point.
  - Left endpoint: add a new segment into the status.
  - Right endpoint: delete the corresponding new segment from the status.
- Choose a data-structure for the sweep-line status to efficiently support those operations.
  - A balanced binary tree for efficient predecessor and successor operations.

Agenda

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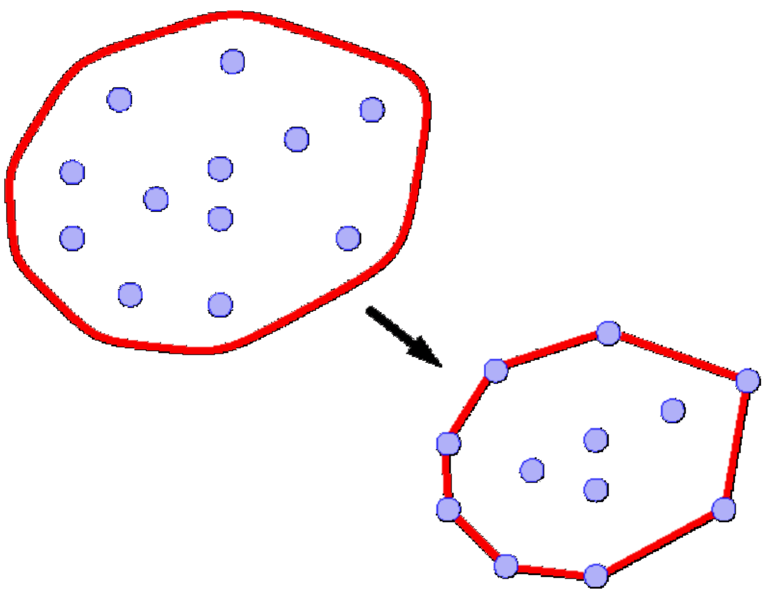
- Graham's scan

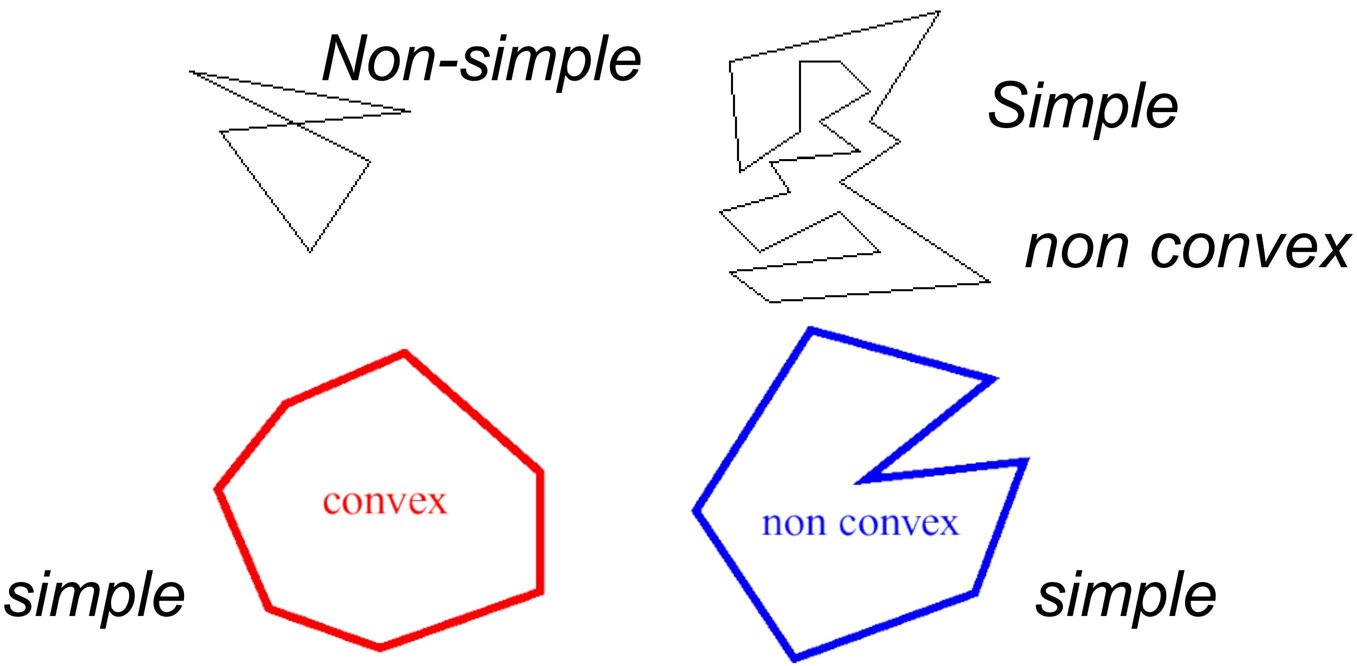
Finding the Convex Hull

- Let S be a set of n points in a plane. Compute the convex hull of these points.
- Intuition :
  - Each point in S is a nail sticking out from a board.
  - The convex hull is a tight rubber band that surrounds all the nails.



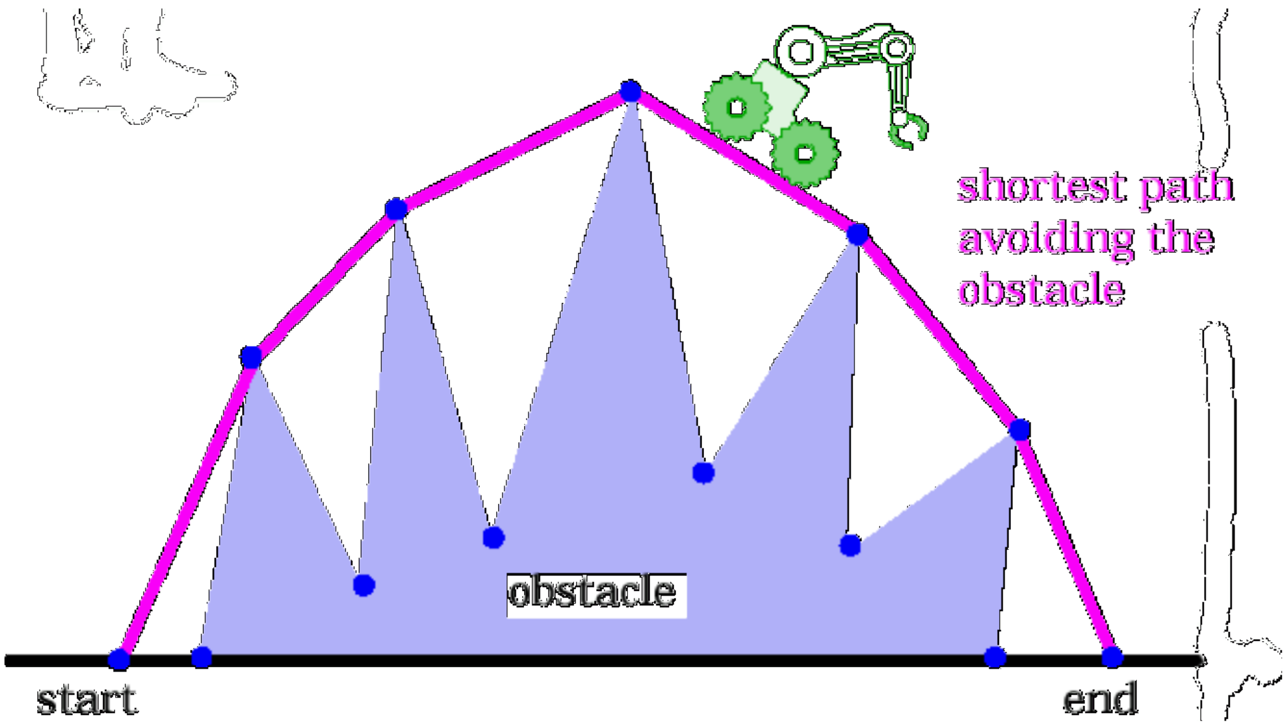
Convex hull

- Formal definition: the convex hull of S is the smallest convex polygon that contains all the points of S.
- *A polygon P is said to be* ***convex*** *if* :
  - *P* is *simple* (boundaries do not intersect in the middle but only at endpoints);
  - And, for any two points *p* and *q* on the boundary of *P* , segment *pq* lies entirely inside *P*



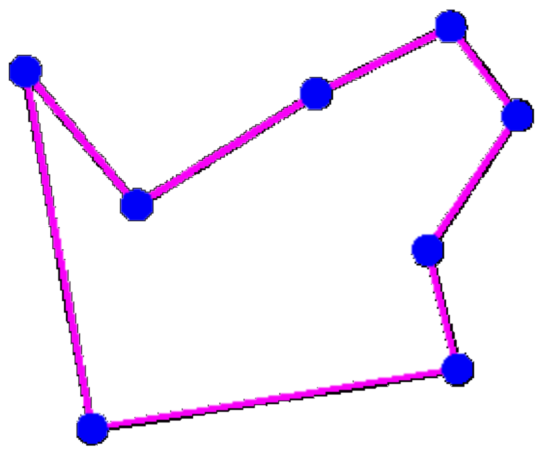
Robot motion planning

- In motion planning for robots, sometimes there is a need to compute convex hulls.



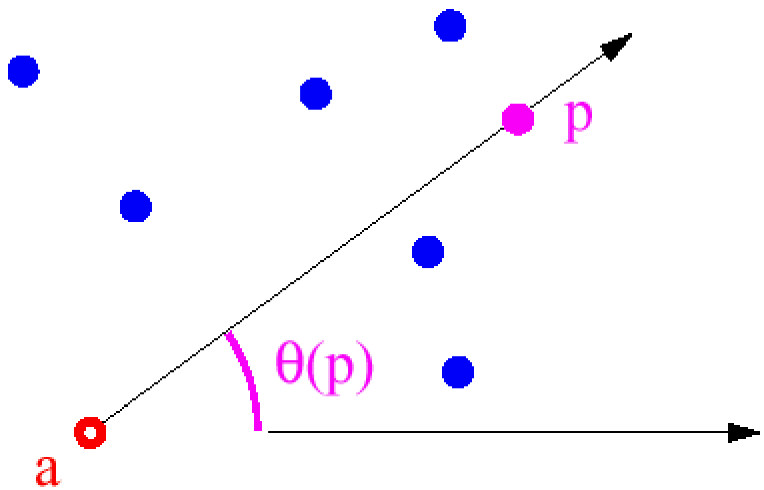
Graham scan

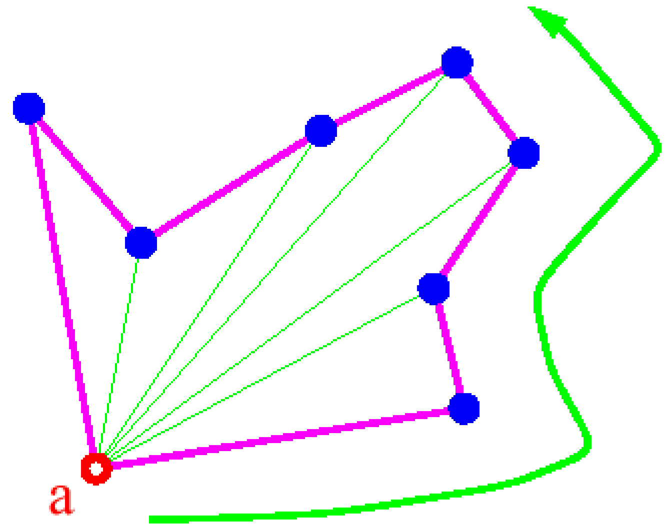
- *Phase 1:* Solve the problem of finding the simple (non-crossing) closed path visiting all points



Finding non-crossing path

- *How do we find such a non-crossing path:*
  - Pick the point ***a*** as the anchor point, where ***a*** has the minimum y-coordinate, or the leftmost such point in case of a tie.
  - For each point p, we have an angle θ (p) of the segment ap with respect to the *x*-axis (i.e., the polar-angle)
  - Traversing the points by **increasing angle** yields a simple closed path.

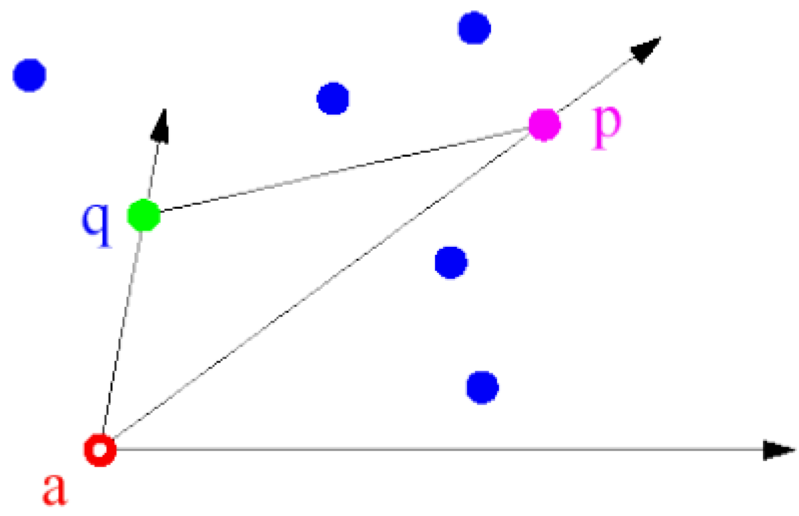




Sorting by angle

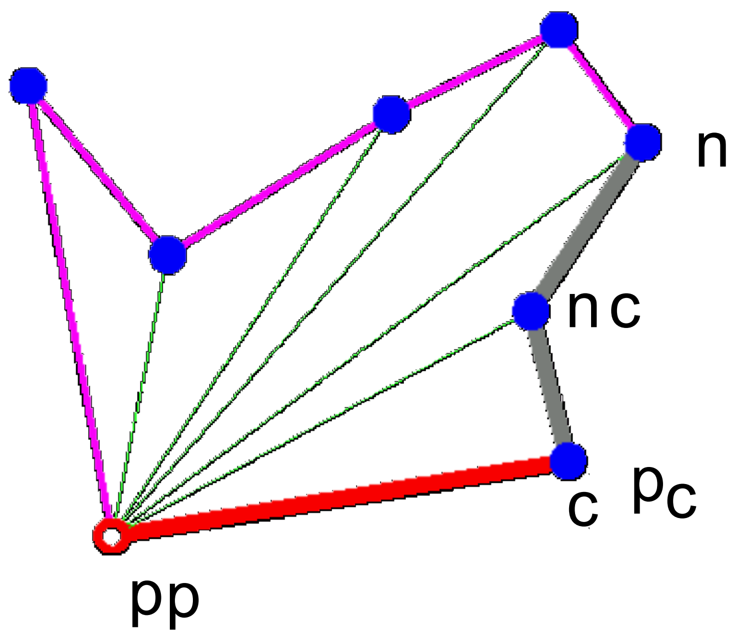
- *How do we sort by increasing angle*?
  - *Observation*: We do not need to compute the actual angle.
  - We just need to compare them for sorting

$\theta(p) < \theta(q)$
$\Leftrightarrow$ orientation(a,p,q) =
counterclockwise



Rotational sweeping

- *Phase 2 of Graham Scan:* ***Rotational sweeping***
- The anchor point and the first point in the polar-angle order have to be in the hull.
- Traverse the remaining points in the sorted order:
  - We denote the current point c, its previous point p, and its next point n.
  - If from segment pc to cn, we need to make a left turn, include c.
  - If not, discard *c* and consider its previous point as a new c.



Pseudo code

Phase 1: sorting, O(nlgn)

GRAHAM-SCAN(Q)

let p_0 be the point in Q with the minimum y-coordinate,
    or the leftmost such point in case of a tie
let $\langle p_1, p_2, \ldots, p_m \rangle$ be the remaining points in Q,
    sorted by polar angle in counterclockwise order around p_0
    (if more than one point has the same angle, remove all but
    the one that is farthest from p_0)
let S be an empty stack
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i = 3$ **to** m
    while the angle formed by points $\boxed{\text{NEXT-TO-TOP}(S)}$, $\boxed{\text{TOP}(S)}$,
            and $\boxed{p_i}$ makes a nonleft turn
        POP(S)
    PUSH(p_i, S)
return S

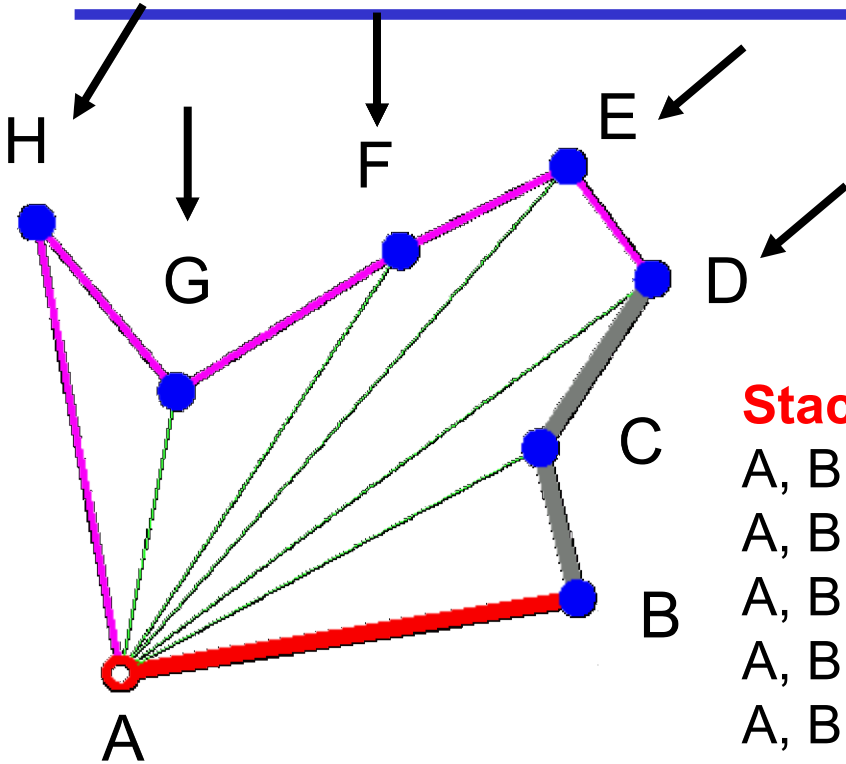
Previous point p

Current point c

Next point n

Phase 2:
Each point is inserted into and removed from the stack at most once. O(n)

In total: O(nlgn)



Stack	Test
A, B, C	B, C, D, right turn, pop
A, B	A, B, D, left turn, push(D)
A, B, D	B, D, E, left turn, push(E)
A, B, D, E	D, E, F, left turn, push(F)
A, B, D, E, F	E, F, G, left turn, push(G)
A, B, D, E, F, G	F, G, H, right turn, pop
A, B, D, E, F	E, F, H, right turn, pop
A, B, D, E	D, E, H, left turn, push(H)
A, B, D, E, H	done

A, B, D, E, H are the vertices on the convex hull.

ILO of Lecture 6

- Computational Geometry: sweeping techniques
  - to understand how the basic geometric operations (such as determining how two line segments are oriented and whether they intersect) are performed;
  - to understand the basic idea of the sweeping algorithm design technique;
  - to understand and be able to analyze the Graham's scan and the sweeping-line algorithm to determine whether any pair of line segments intersect.

Next lecture

- Computational Geometry Algorithms: Divide and Conquer