The Phase Plane Inverse Problem

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1 Objective

Given a set of equilibria and their stability, construct a system of equations that produces "similar" dynamics to those determined by the equilibria.

2 Introduction

We will attempt to address this in two dimensions. Take a general planar system of the following form.

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

What form should $f(x,y)$ and $g(x,y)$ take? We will first show that any single equilibrium in two dimensions can be reproduced by a coupled system of polynomials.

$$\begin{aligned}f(x,y) &= \mathrm{P}_1(x) + y\\ g(x,y) &= -x + \mathrm{P}_2(y)\end{aligned} \tag{1}$$

where $\mathrm{P}_1(x)$ and $\mathrm{P}_2(y)$ are arbitrary polynomials of degree greater than 1. This form was chosen so that the nullclines could be explicitly defined and would be conducive to analysis. The nullclines are given by

$$y = -\mathrm{P}_1(x) \tag{2}$$
$$x = \mathrm{P}_2(y) \tag{3}$$

and the Jacobian of the system is given by

$$J(x,y) = \begin{bmatrix} p_1 & 1\\ -1 & p_2\end{bmatrix} \tag{4}$$

where for notational brevity we are referring to the derivatives as $\frac{d\mathrm{P}_1}{dx}(x) \equiv p_1$ and $\frac{d\mathrm{P}_2}{dy}(y) \equiv p_2$. We can quickly see the trace and determinant.

$$\mathrm{Trace}(J) = \tau(p_1,p_2) = p_1 + p_2 \tag{5}$$
$$\mathrm{Det}(J) = \Delta(p_1,p_2) = p_1p_2 + 1 \tag{6}$$

The eigenvalues of the Jacobian determine the type and stability of nonhyperbolic equilibria. The characteristic for this system is given by

$$\begin{vmatrix} p_1-\lambda & 1\\ -1 & p_2-\lambda\end{vmatrix} = (p_1-\lambda)(p_2-\lambda)+1$$

$$\Rightarrow \lambda^2 - (p_1 + p_2)\lambda + p_1 p_2 + 1$$

$$\Rightarrow \lambda^2 - \tau\lambda + \Delta \tag{7}$$

and given the characteristic we find the eigenvalues.

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

In conventional dynamics theory the type of equilibria for $\Delta > 0$ is determined by the sign of the radicand $\tau^2 - 4\Delta$ and their stability is determined by the sign of τ and Δ. This relationship is usually depicted in a τ vs Δ plot such as the one shown below in figure 1.

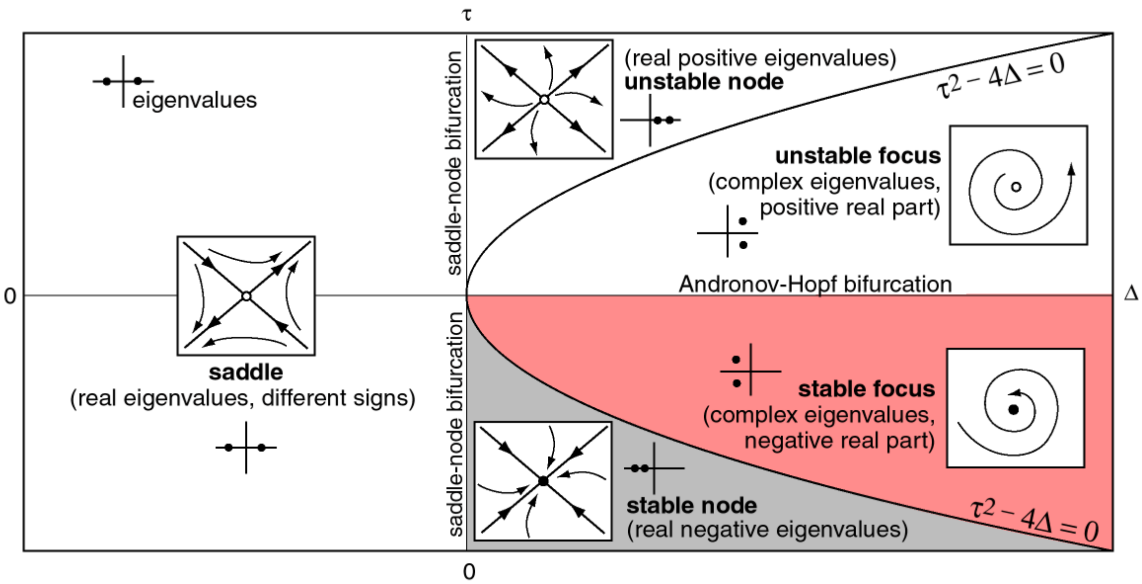

Figure 1: Source: http://www.intechopen.com/source/html/40762/media/image13.png

In general τ and Δ can take on all values and collectively describe all hyperbolic equilibria in two dimensions. Given that $\mathrm{P}_1(x)$ and $\mathrm{P}_2(y)$ are polynomials we know that their derivatives can take on any value $p_1, p_2 \in (-\infty, \infty)$. Also, because our general system above produces continuous τ and Δ as seen in equations 5 and 6 we see that it is capable of reproducing all equilibria in two dimensions. Or in simpler terms, we know that two linear polynomials are sufficient to produce any equilibria and the form presented in this document reduces to a linear system, therefore the system 1 can also produce any equilibria in the plane.

To make this more concrete let us look at a p_1 vs. p_2 plot to examine the relationships which produce the various types of equilibria. There are three things we will use, the trace and determinant, equations 5 and 6, and the radicand shown below.

$$\tau^2 - 4\Delta \tag{8}$$

The idea is to express each in terms of p_1, p_2 and then examine the collection of inequalities, much like in figure 1. We start by looking at the trace, τ.

$$\begin{aligned} \tau(p_1, p_2) = p_1 + p_2 = 0 \\ p_1 = -p_2 \end{aligned} \tag{9}$$

We know that the trace determines stability. If the trace is positive, the equilibrium is unstable. If the trace is negative, the equilibrium is stable. Therefore equation 9 will represent the partition between stablility and instability. Any point along the line produces a neutrally stable center point which is where a Hopf bifurcation occurs.

Next we look at the determinant.

$$\begin{aligned} \Delta(p_1, p_2) &= p_1 p_2 + 1 = 0 \\ p_1 &= -\frac{1}{p_2} \end{aligned} \tag{10}$$

We know that when the determinant is negative, we have a saddle node. Therefore equation 10 will represent the partition between saddle nodes and all other equilibria. All points along the curve are where nodes and saddles fuse or split, also known as a Saddle Node bifurcation.

Lastly, we look at the radicand. Like before we will use equations 5 and 6 to express the radicand in terms of p_1 and p_2.

$$\begin{aligned} \tau^2(p_1, p_2) - 4\Delta(p_1, p_2) &= 0 \\ (p_1 + p_2)^2 - 4(p_1 p_2 + 1) &= 0 \\ p_1^2 + 2p_1 p_2 + p_2^2 - 4p_1 p_2 - 4 &= 0 \\ p_1^2 - 2p_1 p_2 + p_2^2 - 4 &= 0 \\ (p_1 - p_2)^2 - 4 &= 0 \\ (p_1 - p_2 - 2)(p_1 - p_2 + 2) &= 0 \\ p_1 &= p_2 \pm 2 \end{aligned} \tag{11}$$

The sign of the radicand determines the type of equilibria. If the radicand is positive we have real eigenvalues and the equilibrium is a node. If the radicand is negative we have imaginary eigenvalues and the equilibrium is a spiral. Therefore the two lines in equation 11 are the partitions between nodes and spirals. In other words $|p_1 - p_2| < 2$ are the parameters that create a spiral and $|p_1 - p_2| > 2$ are the parameters that create a node. Well it is only a node if $p_1 p_2 > -1$ is also satisfied, i.e. it is not a saddle node. All p_1 and p_2 satisfying $|p_1 - p_2| = 0$ create degenerate nodes, or stars. The plot of p_1 vs p_2 is shown in figure 2.

3 Angle Dependent Dynamics

Using implicit differentiation we can find the slope of each nullcline in phase space (y vs. x) as

$$-\frac{f_x}{f_y} = -p_1 \tag{12}$$

$$-\frac{g_x}{g_y} = \frac{1}{p_2} \tag{13}$$

In the y vs x phase space we'll define a vector tangent to each nullcline as

$$\vec{T_x} = 1\hat{i} + -p_1\hat{j} \tag{14}$$

$$\vec{T_y} = p_2\hat{i} + 1\hat{j} \tag{15}$$

We are interested in the angle between these vectors at an equilibria, therefore we will use the following formula…

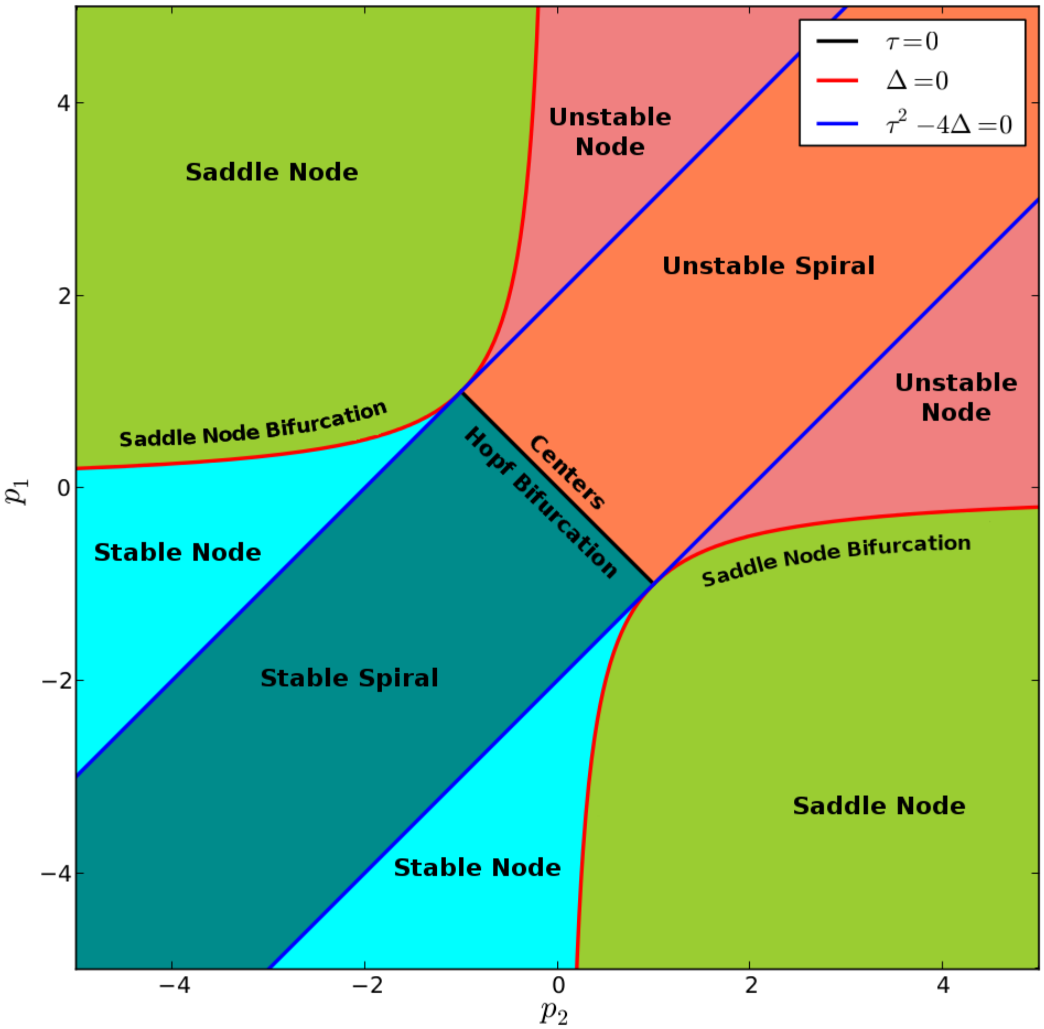

Figure 2: We plot the three partitions in equations 9, 10, 11 and label the different regions of the parameter space with their associated equilibria types and stability. The black line in the middle is $\tau = 0$. As this is where the trace transitions from negative to positive, it divides the plot into stable and unstable regions. The red line is where $\Delta = 0$. This divides the region between saddle nodes, and everything else. Lastly, the blue lines are where $\tau^2 - 4\Delta = 0$. They divide the regions between where nodes and spirals occur.

$$\theta = \cos^{-1}\left(\frac{\vec{T_x} \cdot \vec{T_y}}{\|T_x\|\|T_y\|}\right)$$

Starting with the numerator we get...

$$\vec{T_x} \cdot \vec{T_y} = p_2 - p_1$$

And in the denominator

$$\|T_x\|\|T_y\| = \sqrt{1 + p_1^2}\sqrt{1 + p_2^2}$$

The objective here is to find the angle between the tangent vectors in terms of the eigenvalues of the system. We will need to convert the numerator and denominator into expressions of only eigenvalues. For this we will be using the handy fact from linear algebra that the trace of a matrix is equal to the sum of its eigenvalues, $\tau = \lambda_1 + \lambda_2$, and the determinant of a matrix is equal to the product of its eigenvalues, $\Delta = \lambda_1 \lambda_2$.

TO DO: elegant algebra leading to the following formulas (don't have time now)

Thus we find the angle between the tangent vectors at the equilibria is given by

$$\theta(\lambda_1, \lambda_2) = \cos^{-1}\left(\pm \sqrt{\frac{(\lambda_1 - \lambda_2)^2 + 4}{(\lambda_1 + \lambda_2)^2 + (\lambda_1 \lambda_2 - 2)^2}} \right)$$

Or, in terms of the trace and determinant

$$\theta(\lambda_1, \lambda_2) = \cos^{-1}\left(\pm \sqrt{\frac{\tau^2 + 4(1 - \Delta)}{\tau^2 + (\Delta - 2)^2}} \right)$$

We will show that this is defined for all nonzero λ_1 and λ_2. We needn't worry about the degenerate case as the Hartman-Grobman theory tells us the Jacobian is not guaranteed to correctly determine the stability of non-hyperbolic equilibria. We need to show three things.

1. The denominator is never zero
2. The radicand is always positive
3. The magnitude of the argument to $\cos^{-1}$ is bounded by one

The second equation is found by algebraic manipulations and subsitutions of the first. Therefore showing any of the properties for one equation implies it is true for the other.

1. *The denominator is never zero*
   By inspection we see that both terms in the denominator are squared (in both equations). Thus, the denominator is nonzero for nonzero eigenvalues.

2. *The radicand is always positive*
   By inspecting the first equation we see that all terms in the radicand are positive, and we are done.

3. *The magnitude of the argument to* $\cos^{-1}$ *is bounded by one*
   Looking at the second equation...

$$\frac{\tau^2 + 4(1 - \Delta)}{\tau^2 + (\Delta - 2)^2} < 1 \quad \rightarrow \quad \tau^2 + 4(1 - \Delta) < \tau^2 + (\Delta - 2)^2$$

   Recalling that the denominator is nonzero we do not flip the inequality. We expand both sides,

$$\tau^2 - 4\Delta + 4 < \tau^2 + \Delta^2 - 4\Delta + 4 \quad \rightarrow \quad 0 < \Delta^2$$

   which is always true for nonzero eigenvalues.

TO DO: Demonstrate with some small examples