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The existence of generalized synchronization of chaotic systems in complex networks

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The paper studies the existence of generalized synchronization in complex networks, which consist of chaotic systems. When a part of modified nodes are chaotic, and the others have asymptotically stable equilibriums or orbital asymptotically stable periodic solutions, under certain conditions, the existence of generalized synchronization can be turned to the problem of contractive fixed point in the family of Lipschitz functions. In addition, theoretical proofs are proposed to the exponential attractive property of generalized synchronization manifold. Numerical simulations validate the theory. © *2010 American Institute of Physics*. [doi:10.1063/1.3309017]

Chaos synchronization has been extensively studied in the past 20 years, including complete synchronization (CS), phase synchronization, lag synchronization, anticipated synchronization, and generalized synchronization (GS). GS extends the notion of CS to cases of coupled nonidentical systems, but the mathematical mechanism of the GS has been unclear over the years for its complexity. Networks exist everywhere, for instance, we, as individuals, are the units of a network of social relationships. However, theoretical analysis of GS in complex networks consisting of chaotic systems is also insufficient. To this question, this paper will give strict proof to demonstrate the existence of three types of GS based on the modified system approach.

I. INTRODUCTION

Since the pioneer work of Pecora and Carroll,[1] chaos synchronization has attracted much attention due to its potential applications in various fields.[2–7] Meanwhile, a wide variety of methods have been proposed to synchronize chaotic systems.[8–13] It is well known that synchronization between coupled chaotic systems can be described in terms of invariant manifolds. At present, several types of chaos synchronization have been revealed, such as complete synchronization (CS), phase synchronization, lag synchronization, anticipated synchronization, and generalized synchronization (GS). GS therein is an interesting and more important topic, which includes many synchronization phenomena observed in laboratory experiments.[14–17]

In 1995, Rulkov *et al.*[18] first described the GS phenomenon and presented the idea of mutual false nearest neighbors to detect the GS. In 1996, Abarbanel *et al.*[19] suggested the auxiliary system method to study GS in master-slave systems and the theory about this method was given in Ref. 20. In 1997, Hunt *et al.*[21] considered the differentiable GS (DGS) and when DGS does not hold, they quantified the degree of nondifferentiability using the Hölder exponent. In 1999, Stark[22] studied a special case of inertial manifold for more general systems. In 2001, Rulkov *et al.*[23] proposed that GS in a regime where the synchronization mapping can become a multivalued function. Additionally, Afraimovich *et al.*[24] proved that under some general assumptions, the master-slave synchronization implies GS. Moreover, the synchronization function may be Lipschitz continuous and even less "smooth," that is only Hölder continuous, depending on the coupling strength. In 2002, Ref. 25 described and illustrated three typical complications that can arise in synchronization set. In 2003, Barreto *et al.*[26] proved that synchronization sets can in general become nondifferentiable, and in the more severe case of noninvertible dynamics, they might even be multivalued. In the same year, Rulkov *et al.*[27] illustrated that the nondifferentiable GS can be revealed in many practical cases. In 2005, Hramov *et al.*[28] proposed a modified system approach to study GS. Based on this method, we have presented some theoretical results about GS in Refs. 29–35. For example, we proved the existence of the GS manifold of two unidirectionally coupled systems,[29–31] and the existence of Hölder continuity of the GS manifold.[32–34] Additionally, Ref. 35 investigated a model consisting of three bidirectionally coupled chaotic systems.

In this paper, we will further prove the existence of GS of chaotic systems in complex networks. Complex dynamical networks exist everywhere in the real world, such as the Internet, which is a huge-scale network of routers and computers connected by various physical or wireless links; the World Wide Web, which is an enormous virtual network of websites connected by hyperlinks; and food webs, biological neural networks, telephone cell graphs, etc. For over a century, the question of how to model complex networks is a point of great interest. In 1960, the theory of random graph was introduced by Erdös and Rényi,[36] which made a breakthrough in the completely regular graph theory. In 1998, Watts and Strogatz[37] introduced the so called small world network, which exhibits a high degree of clustering as in the

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regular networks and a small average distance among nodes as in the random networks. In 1999, scale free network was first pointed out by Barabási and Albert.[38] Scale free networks are inhomogeneous in nature, that is, most nodes have very few connections but a small number of particular nodes have many connections. The network model considered here is very universal, which can include the small world network, the scale free network, and so on. Consequently, the results proposed in the paper have generality, and maybe they can provide foundation for explaining the phenomenon of self-organization in complex networks. Actually, some references have considered the GS in complex networks, such as Ref. 39; however, only numerical results were given.

II. THE EXISTENCE OF GS MANIFOLD

Consider a complex network consisting of N chaotic systems described by

$$\frac{dx_i}{dt} = A_i x_i + f_i(x_i) - K_i \sum_{j=1}^{N} c_{ij} x_j, \quad i = 1,2,\ldots,N, \tag{1}$$

where $x_i \in R^n$, A_i denotes an $n \times n$ constant matrix, $f_i(x_i)$ is a smooth map, $K_i \in R^{n\times n}$ represents the coupling strength, $C = (c_{ij}) \in R^{N\times N}$ is the linking matrix, i.e., if there is a connection between node i and node $j (i \neq j)$, then $c_{ij} = c_{ji} = -1$, otherwise $c_{ij} = c_{ji} = 0$, and $c_{ii} = \Sigma_{j=1, j\neq i}^{N} c_{ij} = \Sigma_{j=1, j\neq i}^{N} c_{ji}$.

We divide the nodes of the network (1) into two parts X and Y, and consider the GS between them. Without loss of generality, suppose X consists of the first p nodes of the network (1) and the others compose Y.

Definition 1: Given two dynamical systems X and Y, if there exists a manifold $S = \{(X,Y) | X = \Phi(Y)\}$, which includes at least one attractor, then X and Y carry out GS, and Φ is called the GS map.

In order to simplify discussion, first of all, we classify GS between X and Y into several types according to their modified systems' dynamical behavior. For each node of the network (1), the modified system can be written as Eq. (2)

$$\frac{dx_i}{dt} = A_i x_i + f_i(x_i) - K_i c_{ii} x_i, \quad i = 1,2,\ldots,N. \tag{2}$$

Consider a modified system in Eq. (2), there can exist four kinds of dynamical behavior, i.e., asymptotically stable equilibrium, orbital asymptotically stable periodic or quasiperiodic solution, and chaotic attractor.

This paper mainly investigates three types GS, which is described as follows:

The first kind of GS: When the first p systems in Eq. (2) have asymptotically stable equilibriums, and the other systems in Eq. (2) are chaotic, the GS between X and Y in Eq. (1) is called the first one.

The second kind of GS: When the first p systems in Eq. (2) have orbital asymptotically stable periodic solutions, and the other systems in Eq. (2) are chaotic, the GS between X and Y in Eq. (1) is called the second one.

The third kind of GS: When the first p systems in Eq. (2) are divided further into two parts, one part of systems has asymptotically stable equilibriums, while the other part has orbital asymptotically stable periodic solutions; in addition, the last $N-p$ systems in Eq. (2) are chaotic, the GS between X and Y in Eq. (1) is called the third one.

A. The first kind of GS

Consider in Eq. (2), the modified systems of Y are chaotic, and the modified systems of X have equilibrium points $x_i = \mathbf{0}$ $(i = 1,2,\ldots,p)$ (if $x_i \neq \mathbf{0}$, we will use linear transformation to make $\mathbf{0}$ to be their equilibrium points). For reason of clarity, we introduce y_i $(i = p+1, p+2, \ldots, N)$ to replace x_i $(i = p+1, p+2, \ldots, N)$. Then the first p systems in Eq. (2) can be rewritten as Eq. (3)

$$\begin{aligned}\frac{dx_i}{dt} &= A_i x_i - K_i c_{ii} x_i + f_i(x_i) + f_i'(\mathbf{0}) x_i - f_i'(\mathbf{0}) x_i \\ &= B_i x_i + F_i(x_i), \quad i = 1,2,\ldots,p,\end{aligned} \tag{3}$$

where $B_i = A_i - K_i c_{ii} + f_i'(\mathbf{0})$, $F_i(x_i) = f_i(x_i) - f_i'(\mathbf{0}) x_i$. In addition, all the eigenvalues of matrix B_i have the negative real part.

Now, the network (1) can be rewritten as

$$\begin{aligned}Y: \frac{dy_i}{dt} &= A_i y_i + f_i(y_i) - K_i \sum_{j=p+1}^{N} c_{ij} y_j - K_i \sum_{j=1}^{p} c_{ij} x_j, \\ & i = p+1, p+2, \ldots, N,\end{aligned} \tag{4}$$

$$\begin{aligned}X: \frac{dx_i}{dt} &= B_i x_i + F_i(x_i) - K_i \sum_{j=1, j\neq i}^{p} c_{ij} x_j - K_i \sum_{j=p+1}^{N} c_{ij} y_j, \\ & i = 1,2,\ldots,p.\end{aligned} \tag{5}$$

We first introduce a useful lemma given in Ref. 40.

Lemma 1: Suppose $\phi(t), \psi(t)$, and $\omega(t)$ are continuous functions defined on $[a,b]$, $\omega(t) > 0$, and $\psi(t)$ is a monotonous non-negative and nondecreasing function, if $\phi(t) \leq \psi(t) + \int_a^t \omega(s)\phi(s)ds$, then $\phi(t) \leq \psi(t) e^{\int_a^t \omega(s)ds}$.

Based on Lemma 1, Lemma 2 can be proved.

Lemma 2: Suppose f_i in the network (1) satisfies Lipschitz condition, i.e., $|f_i(u_i) - f_i(u_i')| \leq L|u_i - u_i'|$, where L is a positive constant, $|\bullet|$ denotes vector norm; $\|e^{(A_i - K_i c_{ii})t}\| \leq M_1 e^{\zeta t}$, where $\zeta > 0$, $t > 0$, and $\|\bullet\|$ denotes matrix or operator norm. When $t \leq \tau$, $u_i(\tau) = \eta_i$, the solutions $y_i(t) = y_i(t;\tau,\eta_i)$ $(i = p+1, p+2, \ldots, N)$ of Eq. (4) exist for any continuous function $x_j : (-\infty, \tau] \to U \subset R_x^n$ on $(-\infty, \tau]$, and for any η, η', x_j, x_j' $(j = 1,2,\ldots,p)$, the following is satisfied:

$$\begin{aligned}&|y_i(t;\tau,\eta_i) - y_i'(t;\tau,\eta_i')| \\ &\quad \leq M_1 e^{\zeta(\tau - t)} |\eta_i - \eta_i'| + M_1 \int_t^{\tau} e^{\zeta(s-t)} \Bigg\{ L|y_i - y_i'| \\ &\qquad + \|K_i\| \sum_{j=1}^{p} |c_{ij}(x_j - x_j')| + \|K_i\| \sum_{j=p+1}^{N} |c_{ij}(y_j - y_j')| \Bigg\} ds.\end{aligned} \tag{6}$$

Proof: Due to

$$\frac{dy_i}{dt} = A_i y_i + f_i(y_i) - K_i \sum_{j=p+1}^{N} c_{ij} y_j - K_i \sum_{j=1}^{p} c_{ij} x_j,$$
$$i = p+1, p+2, \ldots, N,$$

$$\frac{dy_i'}{dt} = A_i y_i' + f_i(y_i') - K_i \sum_{j=p+1}^{N} c_{ij} y_j' - K_i \sum_{j=1}^{p} c_{ij} x_j',$$
$$i = p+1, p+2, \ldots, N,$$

then

$$\frac{dy_i - dy_i'}{dt} = A_i(y_i - y_i') + f_i(y_i) - f_i(y_i') - K_i \times \sum_{j=p+1}^{N} c_{ij}(y_j - y_j') - K_i \times \sum_{j=1}^{p} c_{ij}(x_j - x_j'), \quad i = p+1, p+2, \ldots, N.$$

The solution of the equation can be described as

$$y_i - y_i' = e^{(A_i - K_i c_{ii})(\tau - t)}(\eta_i - \eta_i') + \int_t^\tau e^{(A_i - K_i c_{ii})(s-t)} \left[(f_i(y_i) - f_i(y_i')) - K_i \sum_{j=p+1, j\neq i}^{N} c_{ij}(y_j - y_j') - K_i \sum_{j=1}^{p} c_{ij}(x_j - x_j') \right] ds,$$
$$i = p+1, p+2, \ldots, N.$$

Consequently, the inequality (6) holds.

Based on the above, we first present the theory result about the existence of the first kind of GS manifold as the Theorem 1.

Theorem 1: In systems (4) and (5), $f_i : R^n \to R^n$ $(i = 1, 2, \ldots, N)$ and $F_l : R^n \to R^n$ $(l = 1, 2, \ldots, p)$ are smooth functions, suppose the following conditions are satisfied:

$$\left| F_l(x_l) - K_l \sum_{j=1, j\neq l}^{p} c_{lj} x_j - K_l \sum_{j=p+1}^{N} c_{lj} y_j - F_l(x_l') - K_l \sum_{j=1, j\neq l}^{p} c_{lj} x_j' - K_l \sum_{j=p+1}^{N} c_{lj} y_j' \right| \leq \lambda \left[\sum_{j=1}^{p} |x_j - x_j'| + \sum_{j=p+1}^{N} |y_j - y_j'| \right], \tag{7}$$

$$\left| F_l(x_l) - K_l \sum_{j=1, j\neq l}^{p} c_{lj} x_j - K_l \sum_{j=p+1}^{N} c_{lj} y_j \right| \leq \tilde{N}, \tag{8}$$

$$|e^{B_l t}| \leq M e^{-\beta t}, \tag{9}$$

when $t \leq \tau$, $y_i(\tau) = \eta_i \in R^n$, based on Lemma 2, the solutions of the systems (4) and (5) exist on $(-\infty, \tau)$. For any η_i, η_i', the inequality (6) holds. Additionally,

$$|x_l| \leq D, \quad \frac{M\tilde{N}}{\beta} \leq D,$$

$$\frac{\lambda M\sqrt{2p}}{\beta} + \frac{\lambda M(1+\Delta p)2M_1(N-p-1)\|K\|c\sqrt{2p}/\zeta}{\beta - \zeta - M_1(L + \|K\|c(N-p-1)(\Delta p+1))} < 1, \tag{10}$$

$$\frac{\lambda M(1+\Delta p)M_1}{\beta - \zeta - M_1(L + \|K\|c(N-p-1)(\Delta p+1))} \leq \Delta,$$

where $|c_{ij}| \leq c$, $\|K_i\| \leq \|K\|$, λ, M, M_1, $\tilde{N}$, β, ζ, D, and Δ are non-negative constants. Then there exists a GS manifold between (4) and (5):

$$S = \{(x_1, x_2, \ldots, x_p, y_{p+1}, y_{p+2}, \ldots, y_N) | (x_1, x_2, \ldots, x_p) = \sigma(y_{p+1}, y_{p+2}, \ldots, y_N), -\infty < t < +\infty\},$$

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p).$$

Moreover,

$$|\sigma_l(y_{p+1}, y_{p+2}, \ldots, y_N)| \leq D$$

and

$$|\sigma_l(y_{p+1}, y_{p+2}, \ldots, y_N) - \sigma_l(y_{p+1}', y_{p+2}', \ldots, y_N')| \leq \Delta \left(\sum_{i=p+1}^{N} |y_i - y_i'| \right)$$

are satisfied.

Proof: Suppose $F_{D,\Delta}$ is a family of Lipschitz function satisfying

$$\begin{aligned} &|\sigma_l(y_{p+1}, y_{p+2}, \ldots, y_N)| \leq D, \\ &|\sigma_l(y_{p+1}, y_{p+2}, \ldots, y_N) - \sigma_l(y_{p+1}', y_{p+2}', \ldots, y_N')| \leq \Delta \left(\sum_{i=p+1}^{N} |y_i - y_i'| \right). \end{aligned} \tag{11}$$

Define the distance

$$\|\sigma - \tilde{\sigma}\| = \sup_{y_i \in R^n} \left\{ \sum_{l=1}^{p} |\sigma_l(y_{p+1}, y_{p+2}, \ldots, y_N) - \sigma_l(y_{p+1}', y_{p+2}', \ldots, y_N')|^2 \right\}^{1/2},$$

it is easy to verify that $F_{D,\Delta}$ is a complete metric space. Let $t \leq \tau$, $y_i(\tau) = \eta_i \in R^n$, then the solution $y_i(t) = y_i(t; \tau, \eta_i)$ of the system (4) exists on $(-\infty, \tau)$.

The operator $G : G = (G_1, G_2, \ldots, G_p)$ is defined as follows:

$$G_l(\sigma)(\tau,\eta_i)=\int_{-\infty}^{\tau}e^{B_l(\tau-s)}\left[F_l(\sigma_l(y_{p+1}(s),y_{p+2}(s),\ldots,y_N(s)))-K_l\sum_{j=1,j\neq l}^{p}c_{lj}x_j(s)-K_l\sum_{j=p+1}^{N}c_{lj}y_j(s)\right]ds. \tag{12}$$

We will prove that G maps $F_{D,\Delta}$ on $F_{D,\Delta}$, and it is a contraction map: $|G_l(\sigma)(\tau,\eta_i)|\leq\int_{-\infty}^{\tau}e^{-\beta(\tau-s)}M\widetilde{N}ds=M\widetilde{N}/\beta\leq D$, suppose σ and σ' are two maps satisfying the inequality (11), η_i and $\eta_i'\in R^n$. Let $y_i(t)=y_i(t;\tau,\eta_i)$, $y_i'(t)=y_i'(t;\tau,\eta_i')$, from Eq. (6), we have

$$\begin{aligned}
&|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\\
&\leq M_1e^{\zeta(\tau-t)}|\eta_i-\eta_i'|+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L|y_i-y_i'|+\|K\|\sum_{j=1}^{p}|c_{ij}(x_j-x_j')|+\|K\|\sum_{j=p+1}^{N}|c_{ij}(y_j-y_j')|\right\}ds\\
&\leq M_1e^{\zeta(\tau-t)}|\eta_i-\eta_i'|+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L|y_i-y_i'|+\|K\|\sum_{j=1}^{p}|c_{ij}|\|\sigma_j(y_{p+1},y_{p+2},\ldots,y_N)-\sigma_j'(y_{p+1}',y_{p+2}',\ldots,y_N')\|\right.\\
&\quad\left.+\|K\|\sum_{j=p+1}^{N}|c_{ij}(y_j-y_j')|\right\}ds,
\end{aligned}$$

then

$$\begin{aligned}
&|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\\
&\leq M_1e^{\zeta(\tau-t)}|\eta_i-\eta_i'|+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L|y_i-y_i'|+\|K\|\left[\sum_{j=1}^{p}c\|\sigma_j-\sigma_j'\|+\Delta\sum_{j=p+1}^{N}|y_j-y_j'|\right]+\|K\|\sum_{j=p+1}^{N}c|y_j-y_j'|\right\}ds\\
&\leq M_1e^{\zeta(\tau-t)}|\eta_i-\eta_i'|+\frac{M_1e^{\zeta(\tau-t)}\|K\|c\sqrt{2}\|\sigma-\sigma'\|}{\zeta}+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L|y_i-y_i'|+\|K\|\left[\Delta p\sum_{j=p+1}^{N}|y_j-y_j'|\right]+\|K\|\sum_{j=p+1}^{N}c|y_j-y_j'|\right\}ds,
\end{aligned}$$

and then

$$\begin{aligned}
&e^{\zeta(\tau-t)}|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\\
&\leq M_1|\eta_i-\eta_i'|+\frac{M_1\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta}+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L|y_i-y_i'|+\|K\|\left[c\Delta p\sum_{j=p+1}^{N}|y_j-y_j'|\right]+\|K\|\sum_{j=p+1}^{N}c|y_j-y_j'|\right\}ds,
\end{aligned}$$

$$\begin{aligned}
\sum_{i=p+1}^{N}e^{\zeta(\tau-t)}|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\leq&\sum_{i=p+1}^{N}M_1|\eta_i-\eta_i'|+\frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta}\\
&+M_1\int_t^{\tau}e^{\zeta(s-t)}\left\{L\sum_{i=p+1}^{N}|y_i-y_i'|+\|K\|c(N-p-1)\left[\Delta p\sum_{j=p+1}^{N}|y_j-y_j'|+\sum_{j=p+1}^{N}|y_j-y_j'|\right]\right\}ds.
\end{aligned}$$

Based on Lemma 1, the following is satisfied:

$$\begin{aligned}
&\sum_{i=p+1}^{N}e^{\zeta(\tau-t)}|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\\
&\leq\left(\sum_{i=p+1}^{N}M_1|\eta_i-\eta_i'|+\frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta}\right)\\
&\quad\times\exp\left\{M_1[L+\|K\|c(N-p-1)(\Delta p+1)](\tau-t)\sum_{i=p+1}^{N}|y_i(t;\tau,\eta_i)-y_i'(t;\tau,\eta_i')|\right\}\\
&\leq\left(\sum_{i=p+1}^{N}M_1|\eta_i-\eta_i'|+\frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta}\right)\exp\{M_1[L+\|K\|c(N-p-1)(\Delta p+1)+\zeta](\tau-t)\}.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&|G_l(\sigma)(\tau,\eta_i) - G_l(\sigma')(\tau,\eta_i')| \\
&= \left| \int_{-\infty}^{\tau} e^{B_l(\tau-s)}\left[F_l(\sigma_l(y_{p+1},y_{p+2},\dots,y_N)) - K_l \sum_{j=1,j\neq l}^{p} c_{lj}x_j - K_l \sum_{j=p+1}^{N} c_{lj}y_j - F_l(\sigma_l'(y_{p+1}',y_{p+2}',\dots,y_N')) \right.\right. \\
&\quad \left.\left. - K_l \sum_{j=1,j\neq l}^{p} c_{lj}x_j' - K_l \sum_{j=p+1}^{N} c_{lj}y_j' ds \right]\right| \\
&\leq M\lambda \left| \int_{-\infty}^{\tau} e^{-\beta(\tau-s)}\left[\sum_{j=1}^{p} (\sigma_j(y_{p+1},y_{p+2},\dots,y_N)) - (\sigma_j'(y_{p+1}',y_{p+2}',\dots,y_N')) + \sum_{j=p+1}^{N} |y_j - y_j'| \right] ds \right| \\
&\leq \frac{\lambda M\sqrt{2}}{\beta}\|\sigma-\sigma'\| + \lambda M(1+\Delta p)\int_{-\infty}^{\tau} e^{-\beta(\tau-s)}\left(\sum_{i=p+1}^{N} M_1|\eta_i - \eta_i'| + \frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta} \right) \\
&\quad \times \exp\{M_1[L+\|K\|c(N-p-1)(\Delta p+1)+\zeta](\tau-t)\}ds \\
&\leq \frac{\lambda M\sqrt{2}}{\beta}\|\sigma-\sigma'\| + \frac{\lambda M(1+\Delta p)}{\beta-\zeta-M_1(L+\|K\|c(N-p-1)(\Delta p+1))}\left\{ M_1\sum_{i=p+1}^{N} |\eta_i-\eta_i'| + \frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta} \right\},
\end{aligned} \tag{13}
$$

then

$$
\|G_l(\sigma) - G_l(\sigma')\| \leq \frac{\lambda M\sqrt{2}}{\beta}\|\sigma-\sigma'\| + \frac{\lambda M(1+\Delta p)}{\beta-\zeta-M_1(L+\|K\|c(N-p-1)(\Delta p+1))}\frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta},
$$

and

$$
\begin{aligned}
\|G(\sigma) - G(\sigma')\| &= \left(\sum_{l=1}^{p} \|G_l(\sigma) - G_l(\sigma')\|^2 \right)^{1/2} \\
&\leq \frac{\lambda M\sqrt{2}}{\beta}\sqrt{p}\|\sigma-\sigma'\| + \frac{\lambda M(1+\Delta p)\sqrt{p}}{\beta-\zeta-M_1(L+\|K\|c(N-p-1)(\Delta p+1))}\frac{M_1(N-P-1)\|Kc\|\sqrt{2}\|\sigma-\sigma'\|}{\zeta}.
\end{aligned}
$$

Then the operator G is a Lipschitz contraction map on $F_{D,\Delta}$, and has a fixed point, i.e., there exists a GS manifold,

$$
\begin{aligned}
S = \{&(x_1,x_2,\dots,x_p,y_{p+1},y_{p+2},\dots,y_N)|(x_1,x_2,\dots,x_p) \\
&= \sigma(y_{p+1},y_{p+2},\dots,y_N), -\infty < t < +\infty\}.
\end{aligned}
$$

In addition, the manifold S is invariant. Suppose $(x_{10},x_{20},\dots,x_{p0},y_{(p+1)0},y_{(p+2)0},\dots,y_{N0}) \in S$, $x_{l0}=\sigma_l(y_{(p+1)0}, y_{(p+2)0},\dots,y_{N0})$, for equation

$$
\begin{aligned}
\frac{dx_l}{dt} &= B_l x_l + F_l(x_l) - K_l \sum_{j=1,j\neq l}^{p} c_{lj}\sigma_j(y_{p+1},y_{p+2},\dots,y_N) \\
&\quad - K_l \sum_{j=p+1}^{N} c_{lj}y_j, \quad x_l(t_0) = x_{l0}; \quad l=1,2,\dots,p,
\end{aligned}
$$

let $x_l(t)=\sigma_l(y_{p+1}(t),y_{p+2}(t),\dots,y_N(t))$, there exists the unique solution of Eq. (5)

$$
\begin{aligned}
x_l(t) = \int_{-\infty}^{t} e^{B_l(t-s)}&\left[F_l(\sigma_l(y_{p+1},y_{p+2},\dots,y_N)) \right. \\
&\left. - K_l \sum_{j=1,j\neq l}^{p} c_{lj}\sigma_j(y_{p+1},y_{p+2},\dots,y_N) - K_l \sum_{j=p+1}^{N} c_{lj}y_j \right] ds.
\end{aligned}
$$

Moreover, $x_l(t)$ is bounded when $t\to\infty$. It means that S is invariant.

Theorem 2: Under the conditions in the Theorem 1, GS manifold is exponential attractive. Especially, when $x_l(t)$ is the solution of the system (5), then

$$
\begin{aligned}
&|x_l(t) - \sigma_l(y_{p+1}(t),y_{p+2}(t),\dots,y_N(t))| \\
&\quad \leq M|x_l(t_0) - \sigma_l(y_{p+1}(t_0),y_{p+2}(t_0),\dots,y_N(t_0))| \\
&\qquad \times e^{(-\beta+M\lambda(1+p))(t-t_0)},
\end{aligned} \tag{14}
$$

where $M\lambda(1+p)<\beta$.

Proof: The solution of the system (5) can be written as

$$x_l(t) = e^{B_l(t-t_0)}x_l(t_0) + \int_{t_0}^{t} e^{B_l(t-s)}\left[F_l(x_l) - K_l\sum_{j=1}^{p} c_{lj}x_j - K_l\sum_{j=p+1}^{N} c_{lj}y_j\right]ds,$$

and on S,

$$\sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t)) = \int_{-\infty}^{t} e^{B_l(t-s)}\left[F_l(\sigma_l(y_{p+1},y_{p+2},\ldots,y_N)) - K_l\sum_{j=1}^{p} c_{lj}\sigma_j(y_{p+1},y_{p+2},\ldots,y_N) - K_l\sum_{j=p+1}^{N} c_{lj}y_j\right]ds,$$

then

$$\begin{aligned}\xi_l(t) &= x_l(t) - \sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t)) \\ &= e^{B_l(t-t_0)}x_l(t_0) + \int_{t_0}^{t} e^{B_l(t-s)}\left[F_l(x_l) - K_l\sum_{j=1}^{p} c_{lj}x_j - K_l\sum_{j=p+1}^{N} c_{lj}y_j\right]ds \\ &\quad - \int_{-\infty}^{t} e^{B_l(t-s)}\left[F_l(\sigma_l(y_{p+1},y_{p+2},\ldots,y_N)) - K_l\sum_{j=1}^{p} c_{lj}\sigma_j(y_{p+1},y_{p+2},\ldots,y_N) - K_l\sum_{j=p+1}^{N} c_{lj}y_j\right]ds,\end{aligned}$$

$$\xi_l(t) - e^{B_l(t-t_0)}\xi_l(t_0) = \int_{t_0}^{t} e^{B_l(t-s)} \times\left[F_l(x_l(s)) - F_l(\sigma_l(y_{p+1}(s),y_{p+2}(s),\ldots,y_N(s))) - K_l\sum_{j=1}^{p} c_{lj}x_j + K_l\sum_{j=1}^{p} c_{lj}\sigma_j(y_{p+1},y_{p+2},\ldots,y_N)\right]ds.$$

Consequently,

$$|\xi_l(t)| \le e^{-\beta(t-t_0)}M|\xi_l(t_0)| + \lambda M\int_{t_0}^{t} e^{-\beta(t-s)} \times\left[|x_l(s) - \sigma_l(y_{p+1}(s),y_{p+2}(s),\ldots,y_N(s))| + \sum_{j=1}^{p}|x_l(s) - \sigma_l(y_{p+1}(s),y_{p+2}(s),\ldots,y_N(s))|\right]ds,$$

$$e^{\beta(t-t_0)}\sum_{l=1}^{p}|\xi_l(t)| \le M\sum_{l=1}^{p}|\xi_l(t_0)| + \lambda M\int_{t_0}^{t} e^{-\beta(t-s)}(1+p)\sum_{l=1}^{p}|\xi_l(s)|ds.$$

Then based on Lemma 1, we have

$$e^{\beta(t-t_0)}\sum_{l=1}^{p}|\xi_l(t)| \le M\sum_{l=1}^{p}|\xi_l(t_0)|e^{\lambda M(1+p)(t-t_0)},$$

$$\sum_{l=1}^{p}|\xi_l(t)| \le M\sum_{l=1}^{p}|\xi_l(t_0)|e^{(-\beta+\lambda M(1+p))(t-t_0)}.$$

B. The second kind of GS

In this subsection, consider the first p systems of Eq. (2) have asymptotically stable periodic solutions $\bar{x}_i(t)=\bar{x}_i(t+T_i)$, $T_i>0$, denote $x(t)=(x_1(t),x_2(t),\ldots,x_p(t))$, $\bar{x}(t)=(\bar{x}_1(t+T_1),\bar{x}_2(t+T_2),\ldots,\bar{x}_p(t+T_p))$, we introduce another two lemmas in Ref. 41 to prove the existence of this kind of GS manifold.

Lemma 3: For a linear periodic differential equation described as

$$\frac{dx}{dt} = A(t)x, \tag{15}$$

where $A(t)$ is a continuous function, $A(t+T)=A(t)$, $T>0$. Then there exists a continuous function $Z(t)$ with period T, and the Eq. (15) can be transformed into Eq. (16) using $w(t)=Z(t)x$,

$$\frac{dw}{dt} = Bw, \tag{16}$$

where B is a constant matrix.

Lemma 4: Suppose the real parts of the eigenvalues of the matrix B in the equality (16) are $\beta_1,\beta_2,\ldots,\beta_n$, the eigenvalues of $(1/2)[A(t)+A^*(t)]$ are $\alpha_1(t),\alpha_2(t),\ldots,\alpha_n(t)$. Then there exists a unitary matrix $S(t)=[S_{i,j}(t)]$ such that $\beta_i=(1/T)\int_0^T\sum_{j=1}^n|S_{i,j}(t)|^2\alpha_{i,j}(t)dt$, $i=1,2,\ldots,n$.

Let $e_l(t)=x_l(t)-\bar{x}_l(t)$, $l=1,2,\ldots,p$, then

$$\begin{aligned}\frac{de_l}{dt} &= A_lx_l + f_l(x_l) - K_l\sum_{j=1}^{p} c_{lj}x_j - [A_l\bar{x}_l + f_l(\bar{x}_l) - K_lc_{ll}\bar{x}_l] - K_l\sum_{j=p+1}^{N} c_{lj}y_j \\ &= A_le_l - K_lc_{ll}e_l + Df_l(\bar{x}_l)e_l + f_l(x_l) - f_l(\bar{x}_l) - Df_l(\bar{x}_l)e_l - K_l\sum_{j=1}^{p} c_{lj}(\bar{x}_j+e_j) - K_l\sum_{j=p+1}^{N} c_{lj}(\bar{y}_j+e_j) \\ &= A_l(t)e_l + F_l(x_l,\bar{x}_l) - K_l\sum_{j=1}^{p} c_{lj}(\bar{x}_j+e_j) - K_l\sum_{j=p+1}^{N} c_{lj}(\bar{y}_j+e_j),\end{aligned} \tag{17}$$

where $A_l(t)=A_l-K_lc_{ll}+Df_l(\bar{x}_l)$, $F_l(x_l,\bar{x}_l)=f_l(x_l)-f_l(\bar{x}_l)-Df_l(\bar{x}_l)e_l$. By using the Lemma 3 and $e_l(t)=Z_l^{-1}(t)w_l(t)$, Eq. (17) is transformed into

$$\frac{dw_l}{dt} = B_l w_l + Z_l(t)F_l(t,Z_l^{-1}(t)w_l(t)) + Z_l(t) \times \left[-K_l\sum_{j=1}^{p} c_{lj}(\bar{x}_j + Z_j^{-1}w_j) - K_l \sum_{j=p+1}^{N} c_{lj}(\bar{y}_j + Z_j^{-1}w_j) \right]. \tag{18}$$

Then the corresponding GS manifold $x_l(t)=\bar{x}_l(t) + \sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t))$ near $\bar{x}_l(t)$ is transformed to $e_l(t)=Z_l^{-1}(t)w_l(t)=\sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t))$, $w_l(t) = Z_l(t)\sigma_l(t,y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t))=\Sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots, y_N(t))$.

Obviously, the existence of the first kind of GS manifold between the systems (4) and (18) means the existence of the second kind of GS manifold between the systems (4) and (5).

We derive the following theorems similar to the Theorems 1 and 2:

Theorem 3: Suppose in systems (4) and (18), the following conditions are satisfied:

$$\left| Z_lF_l(Z_l^{-1}w_l) - K_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{x}_j + Z_j^{-1}w_j) - K_l \sum_{j=p+1}^{N} c_{lj}y_j - Z_lF_l(Z_l^{-1}w_l') + K_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{x}_j + Z_j^{-1}w_j') + K_l \sum_{j=p+1}^{N} c_{lj}y_j' \right| \leq \lambda\left[\sum_{j=1}^{p} |w_j - w_j'| + \sum_{j=p+1}^{N} |y_j - y_j'| \right],$$

$$\left| Z_lF_l(Z_l^{-1}w_l) - Z_lK_l \sum_{j=1,j\neq l}^{p} c_{lj}x_jZ_j^{-1}w_j - Z_lK_l \sum_{j=p+1}^{N} c_{lj}y_j \right| \leq \tilde{N},$$

$$|e^{B_lt}| \leq Me^{-\beta t},$$

when $t\leq\tau$, $y_i(\tau)=\eta_i\in R^n$, there exist solutions $y_i(t) = y_i(t;\tau,\eta_i)$ $(i=p+1,p+2,\ldots,N)$ of Eq. (4) on $(-\infty,\tau)$. Moreover, for any η_i and η_i', the following are satisfied:

$$|x_l| \leq D, \quad \frac{M\tilde{N}}{\beta} \leq D,$$

$$\frac{\lambda M\sqrt{2p}}{\beta} + \frac{\lambda M(1+\Delta p)2M_1(N-p-1)\|Kc\|\sqrt{2p/\zeta}}{\beta-\zeta-M_1(L+\|Kc\|(N-p-1)(\Delta p+1))} < 1,$$

$$\frac{\lambda M(1+\Delta p)M_1}{\beta-\zeta-M_1(L+\|Kc\|(N-p-1)(\Delta p+1))} \leq \Delta,$$

where λ, M, M_1, $\tilde{N}$, β, ζ, D, and Δ are non-negative constants. Then a first kind of GS manifold

$$S=\{(w_1,w_2,\ldots,w_p,y_{p+1},y_{p+2},\ldots,y_N)|(w_1,w_2,\ldots,w_p) = \Sigma(y_{p+1},y_{p+2},\ldots,y_N), -\infty < t < +\infty\}$$

exists between systems (4) and (18), i.e., there exists a second kind of GS manifold between systems (4) and (5),

$$x_l(t) = \bar{x}_l(t) + \sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t)) = \bar{x}_l(t) + Z_l^{-1}(t)\Sigma_l(y_{p+1}(t),y_{p+2}(t),\ldots,y_N(t)).$$

C. The third kind of GS

We consider that the first p systems of Eq. (2) are divided into two parts, $x_1,x_2,\ldots,x_q$ and $z_{q+1},z_{q+2},\ldots,z_p$. Suppose that the solutions of x_l $(l=1,2,\ldots,q)$ have asymptotically stable equilibriums, z_m $(m=q+1,q+2,\ldots,p)$ have orbital asymptotically stable periodic solutions, and y_i $(i=p+1,p+2,\ldots,N)$ are still chaotic.

Now based on the above analysis, we can get the following easily:

(a) If the modified systems of z_m satisfy the conditions in Lemma 3 and Lemma 4, there will exist a GS manifold $\{(x_1,x_2,\ldots,x_q,z_{q+1},z_{q+2},\ldots,z_p)=\sigma(y_{p+1},y_{p+2},\ldots,y_N)\}$ between systems (4) and (5) according to Theorem 3.

(b) Otherwise, a GS manifold $\{(x_1,x_2,\ldots,x_q) =\sigma(z_{q+1},z_{q+2},\ldots,z_p,y_{p+1},y_{p+2},\ldots,y_N)\}$ will exist between systems (4) and (5) according to Theorem 1.

III. NUMERICAL SIMULATIONS

In order to illustrate the aforementioned theoretical analysis clearly, we take a dynamical network with nine nodes, for instance, and the first six nodes are Rössler systems, the other three nodes are Lorenz systems. The network is described as Eqs. (19) and (20)

$$\frac{dy_i}{dt} = \begin{bmatrix} -a_2 & a_2 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix} y_i + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix} - K_i\sum_{j=7}^{9} c_{ij}y_j - K_i\sum_{j=1}^{6} c_{ij}x_j, \tag{19}$$

where $i=7,8,9$, $y_i=(y_{i1},y_{i2},y_{i3})\in R^3$, the matrix K_i denotes coupling strength, and c_{ij} is the linking parameter. We know that when $a_2=10$, $b_2=28$, and $c_2=8/3$, the Lorenz system is chaotic.

$$\frac{dx_l}{dt} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix} x_l + \begin{bmatrix} 0 \\ 0 \\ x_{i1}x_{i3}+b_1 \end{bmatrix} - K_l\sum_{j=1}^{6} c_{lj}x_j - K_l\sum_{j=7}^{9} c_{lj}y_j, \tag{20}$$

where $l=1,2\ldots,6$, $x_l=(x_{l1},x_{l2},x_{l3})\in R^3$, when $a_1=0.2$, $b_1=0.2, c_1=5.7$, the Rössler system is also chaotic.

From Eqs. (19) and (20), we have two parts of modified systems

$$\frac{dy_i}{dt} = \begin{bmatrix} -a_2 & a_2 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix} y_i + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix} - K_ic_{ii}y_i, \tag{21}$$

$$\frac{dx_l}{dt} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix} x_l + \begin{bmatrix} 0 \\ 0 \\ x_{i1}x_{i3}+b_1 \end{bmatrix} - K_lc_{ll}x_l. \tag{22}$$

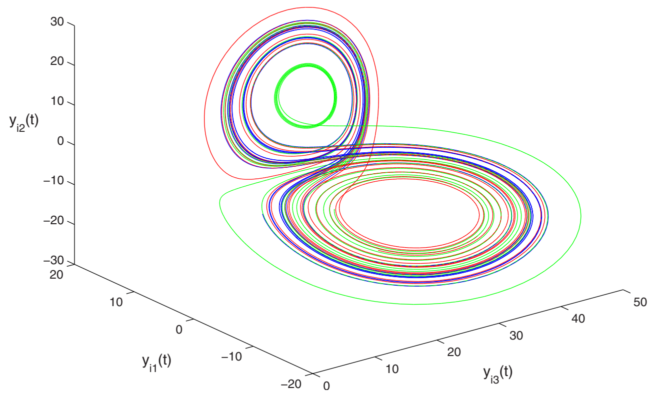

FIG. 1. (Color online) Graphical representations of the modified systems (21) ($i=7,8,9$ and there exist three chaotic attractors).

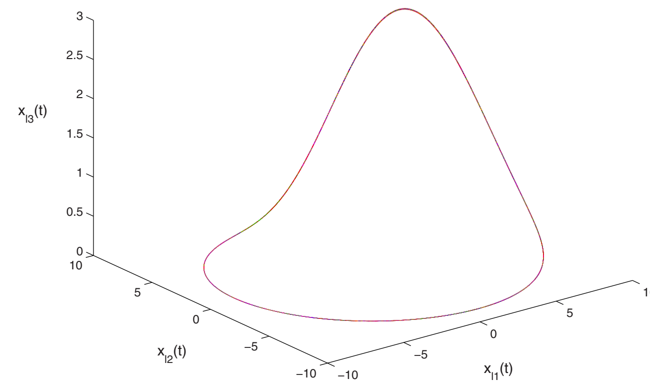

FIG. 3. (Color online) Graphical representations of the modified systems (22) ($l=1,2,\ldots,6$ and six curves coincide with each other).

We use the auxiliary system approach to verify GS between systems (19) and (20), as a result, the auxiliary system of Eq. (20) is introduced as

$$\frac{dX_l}{dt} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix} X_l + \begin{bmatrix} 0 \\ 0 \\ X_{i1}X_{i3}+b_1 \end{bmatrix} - K_l \sum_{j=1}^{6} c_{lj} X_j - K_l \sum_{j=7}^{9} c_{lj} y_j. \tag{23}$$

Let the diagonal elements of the matrix $C=(c_{ij})_{9\times 9}$ be $(1,2,1,1,2,2,3,4,4)$, the initial values of systems (20) and (22) are chosen as $x(0)=(x_1(0),x_2(0),\ldots,x_6(0))=((0.2,0.4,0.6),(0.8,1.0,1.2),\ldots,(3.2,3.4,3.6))$, the initial values of systems (19) and (21) are $y(0)=(y_7(0),y_8(0),y_9(0))=((3.8,4.0,4.2),(4.4,4.6,4.8),(5.0,5.2,5.4))$, and for system (23), $X(0)=(X_1(0),X_2(0),\ldots,X_6(0))=((0.1,0.2,0.3),(0.4,0.5,0.6),\ldots,(1.6,1.7,1.8))$.

The first kind of GS: In this case, solutions of Eq. (22) have asymptotically stable equilibriums and systems in Eq. (21) are chaotic.

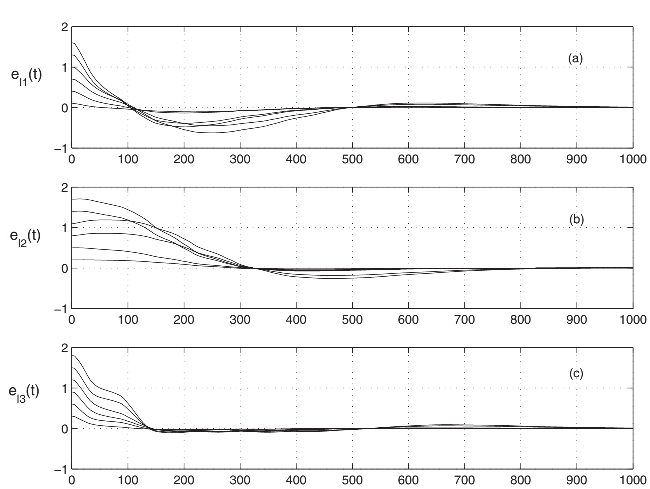

FIG. 2. Graphical representations of errors between systems (20) and (23) [$l=1,2,\ldots,6$; (a) $e_{l1}(t)=x_{l1}(t)-X_{l1}(t)$; (b) $e_{l2}(t)=x_{l2}(t)-X_{l2}(t)$; and (c) $e_{l3}(t)=x_{l3}(t)-X_{l3}(t)$].

We select $K_l=\text{diag}\{0.5,0.5,0.5\}$ ($l=1,2,\ldots,6$), and $K_i=\text{diag}\{0.05,0.05,0.05\}$ ($i=7,8,9$). Then we find that systems in Eq. (22) approach asymptotically stable equilibriums, which are $(-0.0084, -0.0280, 0.0322)$, $(-0.0132, -0.0166, 0.0298)$, $(-0.0084, -0.0280, 0.0322)$, $(-0.0084, -0.0280, -0.0322)$, $(-0.0132, -0.0166, 0.0298)$, and $(-0.0132, -0.0166, 0.0298)$, respectively, while systems in Eq. (21) are still chaotic (see Fig. 1). Systems in Eq. (20) and auxiliary systems in Eq. (23) carry out CS, which is shown in Fig. 2. Based on the auxiliary system method, we have that systems (19) and (20) realize GS.

The second kind of GS: In this case, systems in Eq. (22) collapse to orbital asymptotically stable periodic solutions, and systems in Eq. (21) are still chaotic.

We choose $K_l=\text{diag}\{0.06,0.06,0.6\}$ ($l=1,3,4$), $K_l=\text{diag}\{0.03,0.03,0.3\}$ ($l=2,5,6$), and $K_i=\text{diag}\{0.05,0.05,0.05\}$ ($i=7,8,9$). The results are shown in Figs. 3–5. Especially, in Fig. 3, we can see a colorful curve, in the fact, it is coincided with six curves, which have different colors and represent the periodic solutions of the modified systems (22).

The third kind of GS: Now we consider that systems in Eq. (22) are divided to two parts, i.e., $x_l(l=1,2,3)$ and $z_m(m=4,5,6)$. In this type of GS, $x_l(l=1,2,3)$ will have asymptotically stable equilibriums, $z_m(m=4,5,6)$ will collapse

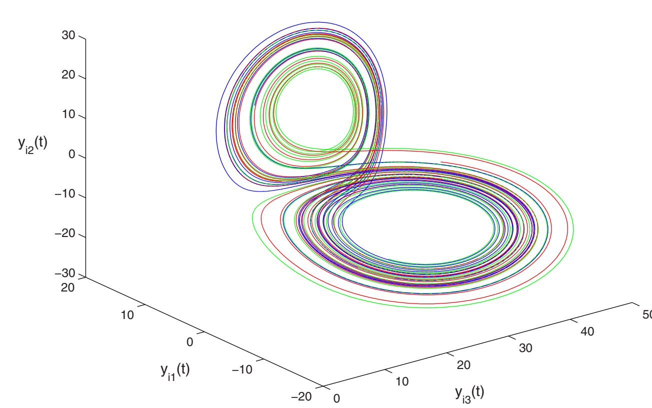

FIG. 4. (Color online) Graphical representations of the modified systems (21) ($i=7,8,9$ and there exist three chaotic attractors).

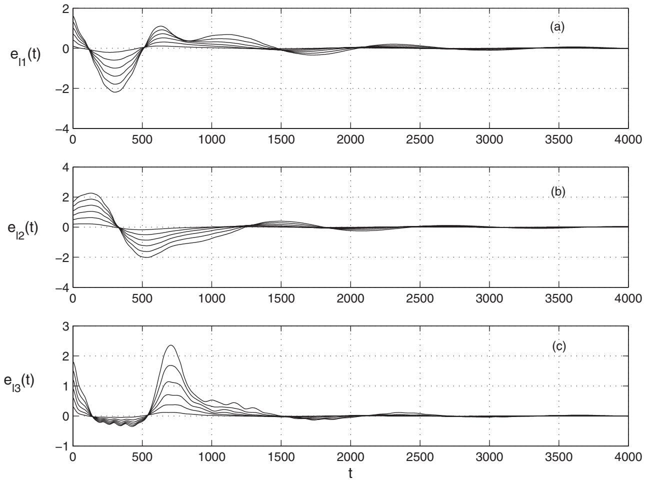

FIG. 5. Graphical representations of errors between the systems (20) and (23) [$l=1,2,\ldots,6$; (a) $e_{l1}(t)=x_{l1}(t)-X_{l1}(t)$; (b) $e_{l2}(t)=x_{l2}(t)-X_{l2}(t)$; and (c) $e_{l3}(t)=x_{l3}(t)-X_{l3}(t)$].

to orbital asymptotically stable periodic solutions, and systems in Eq. (21) are still chaotic. Since z_m satisfy the conditions in Lemma 3 and Lemma 4, there will exist a GS manifold $\{(x_1,x_2,x_3,z_4,z_5,z_6)=\sigma(y_7,y_8,y_9)\}$ between systems (19) and (20) according to Theorem 3.

We choose $K_l=\mathrm{diag}\{0.5,0.5,0.5\}$ $(l=1,2,3)$, $K_4=\mathrm{diag}\{0.06,0.06,0.6\}$, $K_m=\mathrm{diag}\{0.03,0.03,0.3\}$ $(m=5,6)$, and $K_i=\mathrm{diag}\{0.05,0.05,0.05\}$ $(i=7,8,9)$. It is found that there are three equilibriums $(-0.0084,\ -0.0280,\ 0.0322)$, $(-0.0132,\ -0.0166,\ 0.0298)$, and $(-0.0084,\ -0.0280,\ 0.0322)$ for x_l, when $l=1,2,3$, respectively. The solutions of z_m and y_i are shown in Figs. 6 and 7, i.e., Fig. 6 shows the periodic solutions of z_m and Fig. 7 represents the chaotic attractors of y_i. Moreover, the errors between systems (20) and (23) are demonstrated in Fig. 8.

On the whole, from the results of the above simulations, it can be seen that the chaotic systems in the network (1) achieve three kinds of GS.

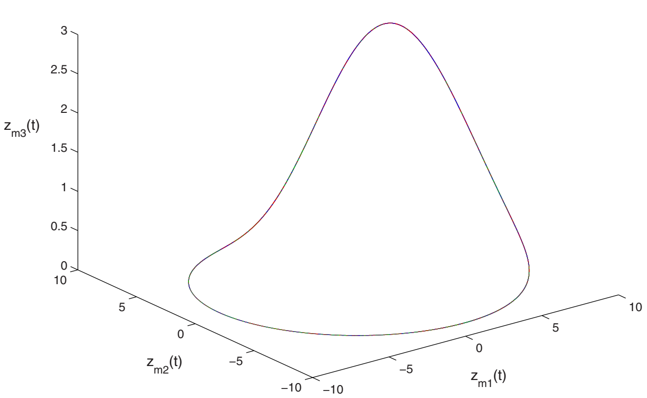

FIG. 6. (Color online) Graphical representations of z_m in the modified systems (22) ($m=4,5,6$ and three curves coincide with each other).

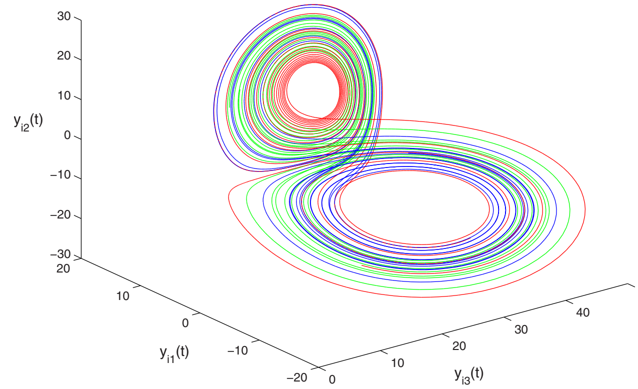

FIG. 7. (Color online) Graphical representations of the modified systems (21) ($i=7,8,9$ and there exist three chaotic attractors).

IV. CONCLUSIONS AND DISCUSSIONS

Although GS phenomenon exists in nature universally, the mathematical mechanism of the GS of chaotic systems has been unclear over the years for its complexity. Theoretical analysis of GS in complex dynamical networks, which are being studied across many fields of science and engineering today, is also insufficient. However, the problem is solved partly in this paper, we present efficient theoretical results for existence of GS manifold based on the modified system method. Theorems derived here propose the sufficient but not necessary conditions. It is valuable to point out that some other kinds of GS can be analyzed as the same, for example, when some modified systems of network collapse orbital asymptotically stable quasiperiod solutions, while the others are chaotic. However, there also exist many problems to be solved, such as if all modified systems are chaotic. We will focus on work regarding this topic in the future, which includes the existence of Hölder continuous GS manifold in complex networks.

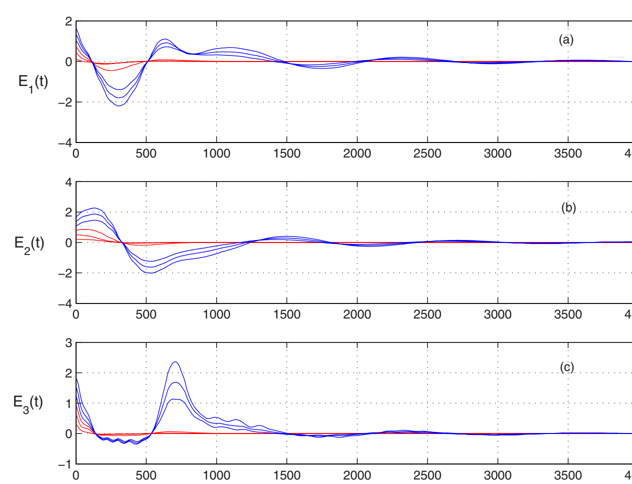

FIG. 8. (Color online) Graphical representations of errors between the systems (20) and (23) $[E_n(t)=(e_{1n}(t),e_{2n}(t),\ldots,e_{6n}(t))$ $(n=1,2,3)$, where $e_{ln}=x_{ln}-X_{ln}$ $(l=1,2,\ldots,6)$. (a) $E_1(t)$; (b) $E_2(t)$; (c) $E_3(t)$].

ACKNOWLEDGMENTS

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