Exercise 3.2.1 Prove $R^d_{abc} = 0$ is a necessary and sufficient condition for being able to interchange the order of covariant differentiation of fields

(a) For a contravariant vector field λ^a, show $\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = -R^a_{dbc}\,\lambda^d$

$\lambda^a{}_{;b} \overset{(2.55)}{=} \partial_b\lambda^a + \Gamma^a_{eb}\,\lambda^e$. Set $\tau^a{}_b \equiv \lambda^a{}_{;b}$. Then

$$\lambda^a{}_{;bc} = \tau^a{}_{b;c} \overset{(2.59)}{=} \partial_c\tau^a{}_b + \Gamma^a_{ec}\,\tau^e{}_b - \Gamma^e_{bc}\tau^a{}_e = \partial_c\lambda^a{}_{;b} + \Gamma^a_{ec}\,\lambda^e{}_{;b} - \Gamma^e_{bc}\,\lambda^a{}_{;e}$$
$$= \partial_c\,\partial_b\lambda^a + (\partial_c\Gamma^a_{db})\,\lambda^d + \Gamma^a_{db}\,\partial_c\lambda^d$$
$$+\ \Gamma^a_{ec}\,[\,\partial_b\lambda^e + \Gamma^e_{db}\,\lambda^d] - \Gamma^e_{bc}\,[\,\partial_e\lambda^a + \Gamma^e_{de}\,\lambda^d]$$

$$\lambda^a{}_{;cb} = \partial_b\,\partial_c\lambda^a + (\partial_b\Gamma^a_{dc})\,\lambda^d + \Gamma^a_{dc}\,\partial_b\lambda^d$$
$$+\ \Gamma^a_{eb}\,[\,\partial_c\lambda^e + \Gamma^e_{dc}\,\lambda^d] - \Gamma^e_{cb}\,[\,\partial_e\lambda^a + \Gamma^e_{de}\,\lambda^d]$$

The colored items subtract out, leaving

$$\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = (\partial_c\Gamma^a_{db} - \partial_b\Gamma^a_{dc} + \Gamma^a_{ec}\,\Gamma^e_{db} - \Gamma^a_{eb}\,\Gamma^e_{dc})\,\lambda^d$$

Swapping a and d in equation (3.13) yields

$$R^a{}_{dbc} \equiv \partial_b\Gamma^a_{dc} - \partial_c\Gamma^a_{db} + \Gamma^e_{dc}\,\Gamma^a_{eb} - \Gamma^e_{db}\,\Gamma^a_{ec}$$

$$\Rightarrow\quad -R^a_{dbc}\,\lambda^d \overset{(3.13)}{=} -(\partial_b\Gamma^a_{dc} - \partial_c\Gamma^a_{db} + \Gamma^e_{dc}\,\Gamma^a_{eb} - \Gamma^e_{db}\,\Gamma^a_{ec})\,\lambda^d$$
$$= \lambda^a{}_{;bc} - \lambda^a{}_{;cb} \quad ✓$$

(b) For a type (2,0) tensor field $\tau^{a\,b}$, show $\tau^{a\,b}{}_{;c\,d} - \tau^{a\,b}{}_{;d\,c} = -R^a_{e\,c\,d}\,\tau^{e\,b} - R^b_{e\,c\,d}\,\tau^{a\,e}$

$$R^a_{e\,c\,d}\,\tau^{e\,b} = (\partial_c\Gamma^a_{e\,d} - \partial_d\Gamma^a_{e\,c} + \Gamma^f_{e\,d}\,\Gamma^a_{f\,c} - \Gamma^f_{e\,c}\,\Gamma^a_{f\,d})\,\tau^{e\,b}$$
$$R^b_{e\,c\,d}\,\tau^{a\,e} = (\partial_c\Gamma^b_{e\,d} - \partial_d\Gamma^b_{e\,c} + \Gamma^f_{e\,d}\,\Gamma^b_{f\,c} - \Gamma^f_{e\,c}\,\Gamma^b_{f\,d})\,\tau^{a\,e}$$
$$\tau^{a\,b}{}_{;c} \overset{(2.57)}{=} \partial_c\tau^{a\,b} + \Gamma^a_{e\,c}\,\tau^{e\,b} + \Gamma^b_{e\,c}\,\tau^{a\,e}$$

Define $\sigma^{a\,b}{}_c \equiv \tau^{a\,b}{}_{;c}$

$$\begin{aligned}
\tau^{a\,b}{}_{;c\,d} = \sigma^{a\,b}{}_{c\,;d} &\overset{(2.59\,a)}{=} \partial_d\sigma^{a\,b}{}_c + \Gamma^a_{e\,d}\sigma^{e\,b}{}_c + \Gamma^b_{e\,d}\sigma^{e\,a}{}_c - \Gamma^e_{c\,d}\sigma^{a\,b}{}_e \\
&= \partial_d\tau^{a\,b}{}_{;c} + \Gamma^a_{e\,d}\tau^{e\,b}{}_{;c} + \Gamma^b_{e\,d}\tau^{e\,a}{}_{;c} - \Gamma^e_{c\,d}\tau^{a\,b}{}_{;e} \\
&= \partial_d(\partial_c\tau^{a\,b} + \Gamma^a_{e\,c}\,\tau^{e\,b} + \Gamma^b_{e\,c}\,\tau^{a\,e}) + \Gamma^b_{e\,d}\tau^{e\,a}{}_{;c} - \Gamma^e_{c\,d}\tau^{a\,b}{}_{;e}
\end{aligned}$$

$$\begin{aligned}
\tau^{a\,b}{}_{;c\,d} = {} & \partial_d\partial_c\tau^{a\,b} + (\partial_d\Gamma^a_{e\,c})\tau^{e\,b} + \Gamma^a_{e\,c}\,\partial_d\tau^{e\,b} + (\partial_d\Gamma^b_{e\,c})\tau^{a\,e} + \Gamma^b_{e\,c}\,\partial_d\tau^{a\,e} \\
& + \Gamma^a_{e\,d}\,[\partial_c\tau^{e\,b} + \Gamma^e_{f\,c}\,\tau^{f\,b} + \Gamma^b_{f\,c}\,\tau^{e\,f}] + \Gamma^b_{e\,d}\,[\partial_c\tau^{e\,a} + \Gamma^e_{f\,c}\,\tau^{f\,a} + \Gamma^a_{f\,c}\,\tau^{e\,f}] \\
& - \Gamma^e_{c\,d}\,[\partial_e\tau^{a\,b} + \Gamma^a_{f\,e}\,\tau^{f\,b} + \Gamma^b_{f\,e}\,\tau^{a\,f}]
\end{aligned}$$

$$\begin{aligned}
\tau^{a\,b}{}_{;d\,c} = {} & \partial_c\partial_d\tau^{a\,b} + (\partial_c\Gamma^a_{e\,d})\tau^{e\,b} + \Gamma^a_{e\,d}\,\partial_c\tau^{e\,b} + (\partial_c\Gamma^b_{e\,d})\tau^{a\,e} + \Gamma^b_{e\,d}\,\partial_c\tau^{a\,e} \\
& + \Gamma^a_{e\,c}\,[\partial_d\tau^{e\,b} + \Gamma^e_{f\,d}\,\tau^{f\,b} + \Gamma^b_{f\,d}\,\tau^{e\,f}] + \Gamma^b_{e\,c}\,[\partial_d\tau^{e\,a} + \Gamma^e_{f\,d}\,\tau^{f\,a} + \Gamma^a_{f\,d}\,\tau^{e\,f}] \\
& - \Gamma^e_{d\,c}\,[\partial_e\tau^{a\,b} + \Gamma^a_{f\,e}\,\tau^{f\,b} + \Gamma^b_{f\,e}\,\tau^{a\,f}]
\end{aligned}$$

$$\begin{aligned}
\tau^{a\,b}{}_{;c\,d} - \tau^{a\,b}{}_{;d\,c} &= (\partial_d\Gamma^a_{e\,c} - \partial_c\Gamma^a_{e\,d})\tau^{e\,b} + (\Gamma^a_{e\,d}\,\Gamma^e_{f\,c} - \Gamma^a_{e\,c}\,\Gamma^e_{f\,d})\,\tau^{f\,b} \\
&\quad + (\partial_d\Gamma^b_{e\,c} - \partial_c\Gamma^b_{e\,d})\tau^{a\,e} + (\Gamma^b_{e\,d}\,\Gamma^e_{f\,c} - \Gamma^b_{e\,c}\,\Gamma^e_{f\,d})\,\tau^{f\,a} \\
&= [\partial_d\Gamma^a_{e\,c} - \partial_c\Gamma^a_{e\,d} + \Gamma^a_{f\,d}\,\Gamma^f_{e\,c} - \Gamma^a_{f\,c}\,\Gamma^f_{e\,d}]\,\tau^{e\,b} \\
&\quad + [\partial_d\Gamma^b_{e\,c} - \partial_c\Gamma^b_{e\,d} + \Gamma^b_{f\,d}\,\Gamma^f_{e\,c} - \Gamma^b_{f\,c}\,\Gamma^f_{e\,d}]\,\tau^{a\,e} \\
&= -R^a_{e\,c\,d}\,\tau^{e\,b} - R^b_{e\,c\,d}\,\tau^{a\,e} \qquad \checkmark
\end{aligned}$$

(c) Guess the form of $\tau^{ab}{}_{c;de} - \tau^{ab}{}_{c;ed}$

First, it is clear that $\tau^{abc}{}_{;de} - \tau^{abc}{}_{;de} = -R^a_{fde}\,\tau^{fbc} - R^b_{fde}\,\tau^{afc} - R^c_{fde}\,\tau^{abf}$.
Next, the mnemonic "co-below and minus" suggests that $\lambda_{a;bc} - \lambda_{a;cb} = R^d_{abc}\,\lambda_d$.
So a guess is that $\tau^{ab}{}_{c;de} - \tau^{ab}{}_{c;ed} = -R^a_{fde}\,\tau^{fb}{}_c - R^b_{fde}\,\tau^{af}{}_c + R^f_{cde}\,\tau^{ab}{}_f$.

(d) Guess the form of $\tau^{a_1\ldots a_r}_{b_1\ldots b_s;cd} - \tau^{a_1\ldots a_r}_{b_1\ldots b_s;dc}$

From (c), the general form is clear:

$$\tau^{a_1\ldots a_r}_{b_1\ldots b_s;cd} - \tau^{a_1\ldots a_r}_{b_1\ldots b_s;dc} = -\sum_{k=1}^{r} R^{a_k}_{ecd}\,\tau^{a_1\ldots a_{k-1}\,e\,a_{k+1}\ldots a_r}_{b_1\ldots b_s} + \sum_{k=1}^{s} R^e_{b_k cd}\,\tau^{a_1\ldots a_r}_{b_1\ldots b_{k-1}\,e\,b_{k+1}\ldots b_s} \qquad \text{(a)}$$

(e) Prove the claim that $R^d_{abc} = 0$ is a necessary and sufficient condition for being able to interchange the order of covariant differentiation of fields

If $R^d_{abc} = 0$ then then (a) shows that the order of differentiation can be reversed, and if the order can be reversed, then (a) shows that

$$R^d_{abc} = -R^d_{abc} \Leftrightarrow R^d_{abc} = 0.$$

This proves the claim.