Exercise 4.8.1 Show that the Schwarzschild line element (3.59) for the Ingoing Eddington-Finkelstein coordinate system becomes

$$c^2\,\mathrm{d}\tau^2 = \left(1 - \frac{2\,m}{r}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2$$

Solution The Schwarzschild line element is

$$\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 = \left(1 - \frac{2\,m}{r}\right)\mathrm{d}(ct)^2 - \left(1 - \frac{2\,m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \qquad (3.59)$$

$$v = ct + r + 2m\ln\left|\frac{r}{2\,m} - 1\right| = ct + r + 2m\ln\left|\frac{r-2\,m}{2\,m}\right| \qquad (4.60)$$

$$ct = v - r - 2m\ln\left|\frac{r-2\,m}{2\,m}\right|$$

$$\mathrm{d}(ct) = \mathrm{d}v - \mathrm{d}r - 2m\,\frac{2\,m}{r-2\,m}\,\frac{\mathrm{d}r}{2\,m} = \mathrm{d}v - \left(1 + \frac{2\,m}{r-2\,m}\right)\mathrm{d}r = \mathrm{d}v - \frac{r}{r-2\,m}\,\mathrm{d}r$$

$$\mathrm{d}(ct)^2 = \mathrm{d}v^2 - \frac{2\,r}{r-2\,m}\,\mathrm{d}v\,\mathrm{d}r + \left(\frac{r}{r-2\,m}\right)^2\mathrm{d}r^2$$

$$\begin{aligned}
c^2\,\mathrm{d}\tau^2 &\overset{3.59}{=} \left(1 - \frac{2\,m}{r}\right)\mathrm{d}v^2 - \frac{r-2\,m}{r}\,\frac{2\,r}{r-2\,m}\,\mathrm{d}v\,\mathrm{d}r + \frac{r-2\,m}{r}\left(\frac{r}{r-2\,m}\right)^2\mathrm{d}r^2 \\
&\quad - \left(1 - \frac{2\,m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \\
&= \left(1 - \frac{2\,m}{r}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r + \frac{r}{r-2\,m}\,\mathrm{d}r^2 \\
&\quad - \left(1 - \frac{2\,m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2
\end{aligned}$$

$$\frac{r}{r-2\,m} = \left(\frac{r-2\,m}{r}\right)^{-1} = \left(1 - \frac{2\,m}{r}\right)^{-1}$$

So,

$$\begin{aligned}
c^2\,\mathrm{d}\tau^2 &= \left(1 - \frac{2\,m}{r}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r + \left(1 - \frac{2\,m}{r}\right)^{-1}\mathrm{d}r^2 - \left(1 - \frac{2\,m}{r}\right)^{-1}\mathrm{d}r^2 \\
&\quad - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \\
&= \left(1 - \frac{2\,m}{r}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2
\end{aligned}$$

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