Exercise 2.7.4 Show that $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$

Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is small, we assume there is a small $f^{\mu\nu}$ such that
$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}.$$

Observe that
$$\delta^{\mu}_{\nu} = g^{\mu\sigma} g_{\sigma\nu} = (\eta^{\mu\sigma} + f^{\mu\sigma})(\eta_{\sigma\nu} + h_{\sigma\nu})$$
$$= \eta^{\mu\sigma}\eta_{\sigma\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu} + f^{\mu\sigma}h_{\sigma\nu}$$
$$\approx \delta^{\mu}_{\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu} \quad \text{since } f^{\mu\sigma}h_{\sigma\nu} \text{ is 2nd order.}$$

This implies that
$$f^{\mu\sigma}\eta_{\sigma\nu} \approx - \eta^{\mu\sigma}h_{\sigma\nu}.$$

Changing dummy variable ν to ρ, we get
$$f^{\mu\sigma}\eta_{\mu\rho} \approx - \eta^{\mu\sigma}h_{\sigma\rho}.$$

So,
$$f^{\mu\nu} = f^{\mu\sigma}\delta^{\nu}_{\sigma} \approx f^{\mu\sigma}\eta_{\sigma\rho}\eta^{\nu\rho} \approx - \eta^{\mu\sigma}h_{\sigma\rho}\eta^{\nu\rho} \approx - h^{\mu\nu}$$

which gives
$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \qquad \blacksquare$$