Exercise 2.9.1 Let K be an inertial (non-rotating) system with coordinates (T,X,Y,Z) and line element

$$c^2\, \mathrm{d}\tau^2 = c^2\, \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2$$

(2.84)

Denote $X^0 = cT$, $X^1 = X$, $X^2 = Y$, $X^3 = Z$.

Let K′ be a rotating system with coordinates (t,x,y,z), implicitly defined by

$$T = t$$
$$X = x \cos\omega t - y \sin\omega t$$
$$Y = x \sin\omega t + y \cos\omega t$$

(2.85)

$$Z = z\,.$$

Show that the line element in terms of K′ is

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\, \mathrm{d}\tau^2$$
$$= [c^2 - \omega^2(x^2 + y^2)]\, \mathrm{d}t^2 + 2\omega y\, \mathrm{d}x\, \mathrm{d}t - 2\omega x\, \mathrm{d}y\, \mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

(2.86)

Solution:

Denote $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$; $\dot{x}^0 = c\,\dot{t}$, $\dot{x}^1 = \dot{x}$, $\dot{x}^2 = \dot{y}$, $\dot{x}^3 = \dot{z}$; and $\ddot{x}^1 = \ddot{x}$, $\ddot{x}^2 = \ddot{y}$, $\ddot{x}^3 = \ddot{z}$, where dot represents differentiation by τ.

$\mathrm{d}T = \mathrm{d}t$

$dX = \mathrm{d}x \cos \omega t - \mathrm{d}y \sin \omega t - \omega\,(x \sin \omega t + y \cos \omega t)\, \mathrm{d}t$

$dY = \mathrm{d}x \sin \omega t + \mathrm{d}y \cos \omega t + \omega\,(x \cos \omega t - y \sin \omega t)\, \mathrm{d}t$

$\mathrm{d}Z = \mathrm{d}z$

$\mathrm{d}T^2 = \mathrm{d}t^2$

$\mathrm{d}X^2 = \mathrm{d}x^2 \cos^2\omega t + \mathrm{d}y^2 \sin^2\omega t + \omega^2(x^2 \sin^2\omega t + y^2 \cos^2\omega t + 2xy \sin \omega t \cos \omega t)\, \mathrm{d}t^2$

$\quad - 2\, \mathrm{d}x\, \mathrm{d}y \sin \omega t \cos \omega t$

$\quad - 2\omega\,(x \sin \omega t \cos \omega t + y \cos^2\omega t)\, \mathrm{d}x\, \mathrm{d}t + 2\omega\,(x \sin^2 \omega t + y \sin \omega t \cos \omega t)\, \mathrm{d}y\, \mathrm{d}t$

$\mathrm{d}Y^2 = \mathrm{d}x^2 \sin^2\omega t + \mathrm{d}y^2 \cos^2\omega t + \omega^2(x^2 \cos^2\omega t + y^2 \sin^2\omega t - 2xy \sin \omega t \cos \omega t)\, \mathrm{d}t^2$

$+ 2\,\mathrm{d}x\,\mathrm{d}y \sin \omega t \cos \omega t$

$+ 2\omega\,(x \sin \omega t \cos \omega t - y \sin^2\omega t)\,\mathrm{d}x\,\mathrm{d}t + 2\omega\,(x \cos^2 \omega t - y \sin \omega t \cos \omega t)\,\mathrm{d}y$
$\mathrm{d}t$

$\mathrm{d}Z^2 = \mathrm{d}z^2$

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\begin{aligned}
c^2\,\mathrm{d}\tau^2 &\overset{2.84}{=} c^2\,\mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2 \\
&= c^2\,\mathrm{d}t^2 - \Big[\,(\mathrm{d}x^2 + \mathrm{d}y^2)\,(\sin^2\omega t + \cos^2\omega t) \\
&\qquad + \omega^2\,(x^2 + y^2)\,(\sin^2\omega t + \cos^2\omega t)\,\mathrm{d}t^2 \\
&\qquad + (-2\omega y\,\mathrm{d}x\,\mathrm{d}t + 2\omega x\,\mathrm{d}y\,\mathrm{d}t)\,(\sin^2\omega t + \cos^2\omega t) \\
&\qquad + \mathrm{d}z^2\,\Big] \\
&= c^2\,\mathrm{d}t^2 - [\,(\mathrm{d}x^2 + \mathrm{d}y^2) + \omega^2\,(x^2 + y^2)\,\mathrm{d}t^2 - 2\omega y\,\mathrm{d}x\,\mathrm{d}t + 2\omega x\,\mathrm{d}y\,\mathrm{d}t\,] - \mathrm{d}z^2 \\
&= [c^2 - \omega^2(x^2 + y^2)]\,\mathrm{d}t^2 + 2\omega y\,\mathrm{d}x\,\mathrm{d}t - 2\omega x\,\mathrm{d}y\,\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2
\end{aligned}
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