General Relativity Crib Sheet

Section 2.5 Spacetime

2.66 $g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2}\,(\partial_\rho\,\partial_\sigma\,g_{\mu\nu})_{\mathrm{P}}\,x^\rho x^\sigma$ **1st order approximation of $g_{\mu\nu}$**

Rules: Special Relativity → General Relativity:

Tensor: No change

Partial differentiation (comma) → Covariant differentiation (semicolon) (eq 2.53)

Total derivative (d/du) → Absolute derivative (D/du)

$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$

2.69 $c^2\,(\mathrm{d}\tau)^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ Revised definition of **proper time**

2.70 $f^\mu = \dfrac{\mathrm{D}\,p^\mu}{\mathrm{d}\tau}$ Newton's **2nd law of motion** (formerly, F = ma)

2.71 $\dfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\sigma}\dfrac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$ Equation of motion of a free particle

2.72 $\dfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\sigma}\dfrac{\mathrm{d}x^\sigma}{\mathrm{d}u}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}u} \overset{(2.12)}{=} 0$ Photon eq of motion, *u* affine parameter

2.73 $g_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}u}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}u} = 0 \quad \Leftrightarrow$ Speed of photon = *c*

Section 2.7 Gravitational potential and the geodesic

2.74 $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$

To approximate flat, Special Relativity spacetime, assume:

2.7-2 $h_{\mu\nu} \approx 0$ to the 1st order in 2.74 (Compare with 2.66)

2.73 $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ to first order

2.78 $\Gamma^\mu_{\nu\sigma} \approx \frac{1}{2}\,\eta^{\mu\rho}(\partial_\nu h_{\rho\sigma} + \partial_\sigma h_{\nu\rho} - \partial_\rho h_{\nu\sigma}) \approx 0$ to the 1st order

2.7-5 $\left|\dfrac{\mathrm{d}x^i}{\mathrm{d}t}\right| \ll c$ slow-moving particles

2.7-6 $|\,\partial_0 h_{\mu i}\,| \ll |\,\partial_i h_{\mu 0}\,|$ for all i and μ **quasi-static**

2.83 $g_{00} \approx 1 + \dfrac{2V}{c^2}$ Relation between g_{00} and acceleration potential V

Section 2.8 Newton's law of universal gravitation

Schwarzschild solution:

2.8-1 $$c^2\,d\tau^2 = \left(1 - \frac{2\,GM}{r\,c^2}\right) c^2 dt^2 - \left(1 - \frac{2\,GM}{r\,c^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta$$

$\Rightarrow$ $\boldsymbol{F} = m\boldsymbol{a} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}$ for small vlaues of $\frac{GM}{r\,c^2}$

$\equiv$ Newton's eq for universal gravitation

Section 2.9 A rotating reference system

Euler Force $-m\frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r}'$ (Merry-go-round))

Coriolis Force $-2m\,(\boldsymbol{\omega}\times\boldsymbol{v}')$ (Coordinate center on equator)

Centrifugal Force $-m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$ (Outward force of ball on a string)

where the primed reference system is a rotating coordinate system, and
- $\boldsymbol{\omega}$ is the angular velocity of the rotating frame relative to the inertial system,
- $\boldsymbol{v}'$ is the velocity of an object in the primed system,
- $\boldsymbol{r}'$ is the position vector of an object in the primed system,
- m is the mass of the object.

2.84 $c^2\,d\tau^2 = c^2\,dT^2 - dX^2 - dY^2 - dZ^2$ Line element for non-rotating frame

2.85 Coordinates for Rotating inertial frame K′
$$T = t$$
$$X = x\cos\omega t - y\sin\omega t$$
$$Y = x\sin\omega t + y\cos\omega t$$
$$Z = z$$

2.87 Equations of motion in K′ for free particle
$$\ddot{t} = 0$$
$$\ddot{x} - \omega^2 x\,\dot{t}^2 - 2\omega\,\dot{y}\,\dot{t} = 0$$
$$\ddot{y} - \omega^2 y\,\dot{t}^2 + 2\omega\,\dot{x}\,\dot{t} = 0$$
$$\ddot{z} = 0$$

2.89 $$m\frac{d^2\boldsymbol{r}}{dt^2} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - 2m\boldsymbol{\omega}\times\frac{d\boldsymbol{r}}{dt}$$ Breakout of 2 of the forces

Section 3.1 The stress tensor and fluid motion

Flat spacetime

3.2 $\boxed{T^{\mu\nu} \equiv \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P\eta^{\mu\nu}}$ **Energy-momentum-stress tensor**

3.1-10 $T^{\mu\nu} u_\nu = c^2 \rho u^\mu$ $T^{\mu\nu}$ is symmetric and ρ, P, and u^μ are the fluid fields

3.5 $(\rho\, u^\mu)_{,\mu} + \frac{P}{c^2}\, u^\mu{}_{,\mu} = 0$ **Relativistic continuity eq for a perfect fluid**

3.6 $\left(\rho + \frac{P}{c^2}\right) u^\nu{}_{,\mu}\, u^\mu = \left(\eta^{\mu\nu} - \frac{1}{c^2}\, u^\mu u^\nu\right) P_{,\mu}$ **Eq of motion of a perfect fluid**

3.7 $\partial\rho/\partial t + \nabla \cdot (\rho v) = 0$ **Classical continuity eq for a perfect fluid**

3.9 $\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = -\nabla P$ **Classical eq of motion of a perfect fluid**

3.1-11 **div $\boldsymbol{T^{\mu\nu}}$** $\equiv \nabla \cdot T^{\mu\nu} \overset{(2.61\text{ D})}{\equiv} T^{\mu\nu}{}_{,\mu}$ **Divergence**

Th 3.1.1 The divergence $T^{\mu\nu}{}_{,\mu} = 0$ iff equations (3.5) and (3.6) hold

Curved spacetime

3.1-17 $(\rho\, u^\mu)_{;\mu} + \frac{P}{c^2}\, u^\mu{}_{;\mu} = 0$ **Continuity equation**

3.1-18 $\left(\rho + \frac{P}{c^2}\right) u^\nu{}_{;\mu}\, u^\mu = \left(g^{\mu\nu} - \frac{1}{c^2}\, u^\mu u^\nu\right) P_{;\mu}$ **Eq of motion of a perfect fluid**

3.10 $T^{\mu\nu} \equiv \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu}$ **Stress tensor**

3.1.19 $T_{\mu\nu} = (\rho + \frac{P}{c^2})\, u_\mu u_\nu - P g_{\mu\nu}$ and

$T_\mu{}^\nu = (\rho + \frac{P}{c^2})\, u_\mu u^\nu - P \delta_\mu^\nu$ **Associated Stress Tensors**

3.11 $T^{\mu\nu}{}_{;\mu} = 0$ **Vanishing divergence**

Section 3.2 The curvature tensor and related tensors

3.2-1 $\lambda_{a\,;\,b} \overset{(2.56)}{=} \partial_b \lambda_a - \Gamma^d_{a\,b} \lambda_d$ **Covariant derivative of a vector**

3.2-2 $\lambda_{a\,;\,b\,c} = \partial_c(\lambda_{a\,;b}) - \Gamma^e_{a\,c} \lambda_{e\,;\,b} - \Gamma^e_{b\,c} \lambda_{a\,;\,e}$ **2nd covariant derivative**

3.13 $R^d{}_{a\,b\,c} \equiv \partial_b \Gamma^d_{a\,c} - \partial_c \Gamma^d_{a\,b} + \Gamma^e_{a\,c} \Gamma^d_{e\,b} - \Gamma^e_{a\,b} \Gamma^d_{e\,c}$ **Curvature tensor**

3.2-4 $R_{a\,b\,c\,d} \equiv g_{a\,e}\, R^e{}_{b\,c\,d}$ and $R^{a\,b}{}_{c\,d} \equiv g^{b\,e}\, R^a{}_{e\,c\,d}$ **Assoc curvature tensors**

Th 3.2.3 Can interchange order of covariant diff of a tensor iff $R^d{}_{a\,b\,c} = 0$

3.14 $R^a{}_{b\,c\,d} + R^a{}_{c\,d\,b} + R^a{}_{d\,b\,c} = 0$ **Cyclic identity**

3.15 $R_{a\,b\,c\,d} = \frac{1}{2}\, [\partial_d \partial_a g_{b\,c} - \partial_d \partial_b g_{a\,c} + \partial_c \partial_b g_{a\,d} - \partial_c \partial_a g_{b\,d}]$

$$- g^{ef}\,[\Gamma_{eac}\,\Gamma_{fbd} - \Gamma_{ead}\,\Gamma_{fbc}]$$

3.16 $R_{abcd} = -R_{bacd}$

3.17 $R_{abcd} = -R_{abdc}$

3.2-5 $R_{abcd} = R_{badc}$

3.18 $R_{abcd} = R_{cdab}$

3.19 $R^a{}_{acd} = 0$

3.20 $R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$ **Bianchi identity**

3.21 $R_{ab} \equiv R^c{}_{abc}$ **Ricci tensor**

3.2-9 $R_{ab} = R_{ba}$ Symmetric in a and b

3.2-12 $R^a_b = R^{ca}{}_{bc}$

3.22 $R \equiv R^a_a = g^{ab}\,R_{ba}$ **Curvature scalar**

3.23 $G_{ab} \equiv R_{ab} - \frac{1}{2}\,R\,g_{ab}$ **Einstein tensor**

3.2-13 $G^a_b = R^a_b - \frac{1}{2}\,R\,\delta^a_b$ **Associated Einstein tensor**

3.2-14 $G^{ab} = G^a_c\,g^{cb} = R^{ab} - \frac{1}{2}\,R\,g^{ab}$ **Associated Einstein tensor**

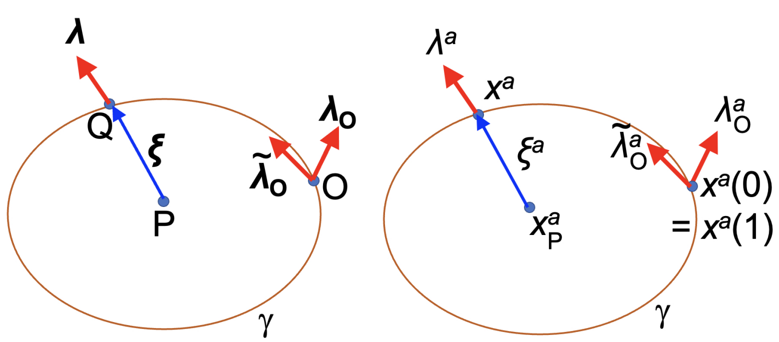
Th 3.2.5 $G^{ab}{}_{;a} = 0$ **Vanishing divergence**

3.2-20 $G^b_{c;b} = 0$ **Vanishing divergence**

3.2-21 $G^{ba}{}_{;b} = 0$ **Vanishing divergence**

Section 3.3 Curvature and parallel transport

Parallel transport enables computation of the curvature components $R^a{}_{bcd}$

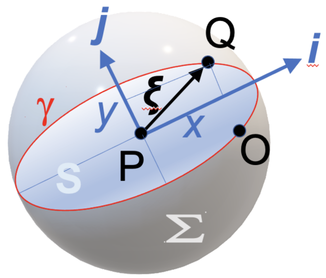


$P = (x^a_P)$	x^a_P
$Q = (x^a)$	$x^a = x^a(t)$
O	$x^a(0) = x^a(1)$
$\boldsymbol{\lambda} = (\lambda^a)$	$\lambda^a = \lambda^a(t)$
$\boldsymbol{\lambda}_O$	$\lambda^a_O = \lambda^a(0)$
$\tilde{\boldsymbol{\lambda}}_O$	$\tilde{\lambda}^a_O = \lambda^a(1)$
$\boldsymbol{\xi} = (\xi^a)$	$\xi^a = \xi^a(t)$

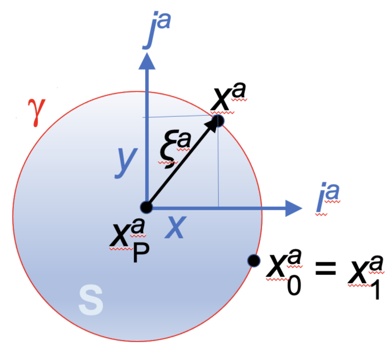
3.24 $\dfrac{d\lambda^a}{dt} = -\Gamma^a_{bc}\,\lambda^b\,\dfrac{dx^c}{dt}$ Defn of parallel transport of λ^a_O

3.29 $f^{cd} \equiv \oint \xi^c\,d\xi^d = \frac{1}{2}\oint[\xi^c\,d\xi^d - \xi^d\,d\xi^c]$ where $\xi^a \equiv x^a - x^a_P$

3.30 $\Delta\lambda^a \equiv \tilde{\lambda}^a_O - \lambda^a_O = -\frac{1}{2}\,(R^a{}_{bcd})_P\,\lambda^b_O\,f^{cd}$ 2nd order approx of $\Delta\boldsymbol{\lambda}$

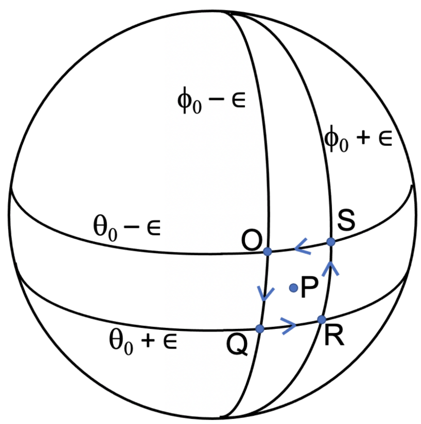


Small sphere Σ of radius *a*



Cross section *S* with area *A*

3.32 $\quad \dfrac{\Delta\lambda^a}{A} = -\,(R^a{}_{b\,c\,d})_{\mathrm{P}}\ \lambda^b_{\mathrm{O}}\ i^c j^d$ Geometric version of (3.30)



3.3-14 $\quad \Delta\lambda^a \approx \dfrac{4\,\epsilon^2}{a}\,\delta^a_2$

3.3-15 $\quad A \approx 4\epsilon^2 a^2 \sin\theta_0,$

3.3-18 $\quad -\,(R^a{}_{b\,c\,d})_{\mathrm{P}}\ \lambda^b_{\mathrm{O}}\ i^c j^d = -\,(R^a{}_{112})_{\mathrm{P}}\ \dfrac{1}{a^3 \sin\theta_0}$

3.3-19 $\quad (R^a{}_{112})_{\mathrm{P}} = -\,\delta^a_2$

3.3-21 $\quad (R_{1\,b\,12})_{\mathrm{P}} = g_{2\,b}$

$\Rightarrow (R_{1212})_{\mathrm{P}} = (R_{2121})_{\mathrm{P}} = -(R_{2112})_{\mathrm{P}} = -(R_{1221})_{\mathrm{P}} = g_{22}$ and the rest are zero