Exercise 3.7.2 If we use the flat spacetime coordinates

$$x^0 = c\,t, \quad x^1 = r \sin\theta \cos\phi, \quad x^2 = \text{r} \sin\theta \sin\phi, \quad x^3 = r \cos\theta, \tag{3..58}$$

what form does $g_{\mu\nu}$ take?

Solution. I used $k_1 = - c^2 k$ rather than k for my derivation. Simplifying the notation of $A(r)$ and $B(R)$ to just A and B, we have

$$\Rightarrow \quad A = c^2 - \frac{k_1}{r} = \frac{c^2\,r - k_1}{r} \quad \text{and} \quad B = \frac{c^2}{A}.$$

Substituting the expressions above for A and B into the line element equation (3.51) yields

$$c^2\,\tau^2 = \left(c^2 - \frac{k_1}{r}\right) \mathrm{d}t^2 - c^2 \left(c^2 - \frac{k_1}{r}\right)^{-1} \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\,sin^2\theta\,\mathrm{d}\phi^2 \tag{3.7-1}$$

or, alternatively,

$$c^2\,\tau^2 = \left(1 - \frac{k_1}{c^2\,r}\right) \mathrm{d}(c\,t)^2 - \left(1 - \frac{k_1}{c^2\,r}\right)^{-1} \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\,sin^2\theta\,\mathrm{d}\phi^2 \tag{3.7-2}$$

As $r \to \infty$, the line element approaches

$$c^2\,\tau^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\,sin^2\theta\,\mathrm{d}\phi^2, \tag{3.7-3}$$

or

$$c^2\,\tau^2 = \mathrm{d}(c\,t)^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\,sin^2\theta\,\mathrm{d}\phi^2, \tag{3.7-4}$$

which is the line element for spherical coordinates in flat spacetime. I express the Cartesian coordinates (3.58) as

$$x^0 = c\,t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{3.7-5}$$

and denote spherical coordinates to be the alternate, primed coordinate system:

$$x^{0'} = c\,t, \quad x^{1'} = r, \quad x^{2'} = \theta, \quad x^{3'} = \phi, \tag{3.7-6}$$

where

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta. \tag{3.7-7}$$

Recall

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \text{arc cos}\,\frac{z}{r}, \quad \phi = \text{arc tan}\,\frac{y}{x}. \tag{3.7-8}$$

From equation (3.7-2) we see that

$$g_{0'\times 0'} = 1 - \frac{k_1}{c^2\, r} = 1 - \frac{k_1}{c^2 \sqrt{x^2+y^2+z^2}}$$

$$g_{1'\times 1'} = -\left(1 - \frac{k_1}{c^2\, r}\right)^{-1} = -\left(1 - \frac{k_1}{c^2 \sqrt{x^2+y^2+z^2}}\right)^{-1} \qquad (3.7\text{-}9)$$

$$g_{2'\times 2'} = -r^2 = -(\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2)$$

$$g_{3'\times 3'} = -r^2\, sin^2\theta = -r^2\,(1 - \cos^2\theta) = -r^2 + z^2$$

and the rest are zero. We solve for the Schwarzfield covariant metric tensors using

$$g_{\mu\nu} = \Lambda_\mu^{\sigma'} \Lambda_\nu^{\rho'} g_{\sigma'\,\rho'} = \frac{\partial x^{\sigma'}}{\partial x^\mu} \frac{\partial x^{\sigma'}}{\partial x^\gamma}\, g_{\sigma'\,\sigma'}. \qquad (3.7\text{-}10)$$

Since $g_{\sigma'\,\rho'} = 0$ unless $\sigma' = \rho'$, this becomes

$$g_{\mu\nu} = \Lambda_\mu^{0'}\, \Lambda_\nu^{0'}\, g_{0'\times 0'} + \Lambda_\mu^{1'}\, \Lambda_\nu^{1'}\, g_{1'\times 1'} + \Lambda_\mu^{2'}\, \Lambda_\nu^{2'}\, g_{2'\times 2'} + \Lambda_\mu^{3'}\, \Lambda_\nu^{3'}\, g_{3'\times 3'}$$
(3.7-11)

$$g_{00} = \frac{\partial(c\,t)}{\partial(c\,t)} \frac{\partial(c\,t)}{\partial(c\,t)}\, g_{0'\times 0'} = (1)\,(1)\,(1 - \frac{k_1}{r}) = 1 - \frac{k_1}{r}\;, \qquad (3.7\text{-}12)$$

$$\begin{aligned} g_{0\,i} = {} & \frac{\partial(c\,t)}{\partial(c\,t)} \frac{\partial(c\,t)}{\partial x^i}\, g_{0'\times 0'} + \frac{\partial x^{1'}}{\partial(\mathrm{ct})} \frac{\partial x^{1'}}{\partial x^i}\, g_{1'\times 1'} \\ & + \frac{\partial x^{2'}}{\partial(\mathrm{ct})} \frac{\partial x^{2'}}{\partial x^i}\, g_{2'\times 2'} + \frac{\partial x^{3'}}{\partial(\mathrm{ct})} \frac{\partial x^{3'}}{\partial x^i}\, g_{3'\times 3'} \\ & = 0, \end{aligned} \qquad (3.7\text{-}13)$$

and

$$\begin{aligned} g_{i\,i} = {} & \frac{\partial(c\,t)}{\partial x^i} \frac{\partial(c\,t)}{\partial x^i}\, g_{0'\times 0'} + \frac{\partial x^{1'}}{\partial x^i} \frac{\partial x^{1'}}{\partial x^i}\, g_{1'\times 1'} \\ & + \frac{\partial x^{2'}}{\partial x^i} \frac{\partial x^{2'}}{\partial x^i}\, g_{2'\times 2'} + \frac{\partial x^{3'}}{\partial x^i} \frac{\partial x^{3'}}{\partial x^i}\, g_{3'\times 3'} \\ & = \frac{\partial x^{1'}}{\partial x^i} \frac{\partial x^{1'}}{\partial x^i}\, g_{1'\times 1'} + \frac{\partial x^{2'}}{\partial x^i} \frac{\partial x^{2'}}{\partial x^i}\, g_{2'\times 2'} + \frac{\partial x^{3'}}{\partial x^i} \frac{\partial x^{3'}}{\partial x^i}\, g_{3'\times 3'} \end{aligned} \qquad (3.7\text{-}14)$$

Using Mathematica to solve the change of coordinate equations (3.7-11 to 3.7-14) for $g_{\mu\nu}$ yields:

σ'	μ	$\Lambda^{\sigma'}_{\mu} = \frac{\partial x^{\sigma'}}{\partial x^{\mu}}$
0	0	1
0	1	0
0	2	0
0	3	0
1	0	0
1	1	$\frac{x}{r}$
1	2	$\frac{y}{r}$
1	3	$\frac{z}{r}$
2	0	0
2	1	$\frac{x\,z}{r^2\sqrt{x^2+y^2}}$
2	2	$\frac{y\,z}{r^2\sqrt{x^2+y^2}}$
2	3	$-\frac{\sqrt{x^2+y^2}}{r^2}$
3	0	0
3	1	$-\frac{y}{x^2+y^2}$
3	2	$\frac{x}{x^2+y^2}$
3	3	0

μ	ν	$g_{\mu\nu} = \Lambda^{\sigma'}_{\mu}\Lambda^{\sigma'}_{\nu}\, g_{\sigma'\sigma'}$
0	0	$1-\frac{k_1}{c^2\,r}$
0	1	0
0	2	0
0	3	0
1	0	0
1	1	$-1-\frac{x^2\,k_1}{r^2\,(c^2\,r-k_1)}$
1	2	$-\frac{x\,y\,k_1}{r^2\,(c^2\,r-k_1)}$
1	3	$-\frac{x\,z\,k_1}{r^2\,(c^2\,r-k_1)}$
2	0	0
2	1	$-\frac{x\,y\,k_1}{r^2\,(c^2\,r-k_1)}$
2	2	$-1-\frac{y^2\,k_1}{r^2\,(c^2\,r-k_1)}$
2	3	$-\frac{y\,z\,k_1}{r^2\,(c^2\,r-k_1)}$
3	0	0
3	1	$-\frac{x\,z\,k_1}{r^2\,(c^2\,r-k_1)}$
3	2	$-\frac{y\,z\,k_1}{r^2\,(c^2\,r-k_1)}$
3	3	$-1-\frac{z^2\,k_1}{r^2\,(c^2\,r-k_1)}$

(3.7-15)

Converting k_1 back to k, yields the answer to this exercise:

μ	ν	$g_{\mu\nu} = \Lambda_{\mu}^{\sigma'} \Lambda_{\nu}^{\sigma'} g_{\sigma' \sigma'}$
0	0	$1 + \frac{k}{r}$
0	1	0
0	2	0
0	3	0
1	0	0
1	1	$-1 + \frac{k x^2}{r^2 (k+r)}$
1	2	$\frac{k x y}{r^2 (k+r)}$
1	3	$\frac{k x z}{r^2 (k+r)}$
2	0	0
2	1	$\frac{k x y}{r^2 (k+r)}$
2	2	$-1 + \frac{k y^2}{r^2 (k+r)}$
2	3	$\frac{k y z}{r^2 (k+r)}$
3	0	0
3	1	$\frac{k x z}{r^2 (k+r)}$
3	2	$\frac{k y z}{r^2 (k+r)}$
3	3	$-1 + \frac{k z^2}{r^2 (k+r)}$