Exercise 2.2.7 (not in book, but needed). Let Δ = det G be the **determinant of *G*,** where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab}, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b} m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\, \partial_c g_{ab} = \Delta g^{ab}\, \partial_c g_{ab}}\,.$$

Solution Δ is defined as a sum of signed products of permutations of the elements g_{ab} in the following way. Rows are labeled "a" and columns are labeled "b". G has N rows and N columns. If $b_1, b_2, \ldots, b_n$ are the numbers 1–N in some (possibly) random order, this constitutes a **permutation** of the numbers 1, 2, ... , N. Let $\mathcal{P}$ denote the set of all such permutations. A **permutation is even (odd)** if it takes an even (odd) number of pairwise swaps of adjacent numbers to restore the natural order. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation. If π is a permutation, define **sign(π)** as +1 if π is even, and -1 when π is odd.

The definition of **determinant of *G*** is given by the equation

$$\begin{aligned} \Delta &= \textstyle\sum_{\pi\in\mathcal{P}} \operatorname{sign}(\pi)\, g_{1,\pi(1)}\, g_{2,\pi(2)} \cdots g_{N,\pi(N)} \\ &= \textstyle\sum_{\pi\in\mathcal{P}} \operatorname{sign}(\pi)\, g_{1,b_1}\, g_{2,b_2} \cdots g_{N,b_N}, \end{aligned} \tag{A}$$

where $\pi = b_1, b_2, \ldots, b_N$, and so we have denoted $b_1 = \pi(1), \ldots, b_N = \pi(N)$.

Note that the first subscripts are 1, 2, ... , N, representing rows 1 – N, and the second subscripts constitute a permutation of columns 1 – N.

From this definition, various other formulas for computing the determinant can be derived and are commonly used. We use two formulas here. The first one is the sum of cofactors across row a. We have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab} c^{ab}.$$

So,

$$\sum_{a=1}^{N} \sum_{b=1}^{N} g_{ab} c^{ab} = \sum_{a=1}^{N} \Delta = N\,\Delta.$$

In Einstein notation this is denoted

$$g_{ab}\, c^{ab} = N\,\Delta. \tag{a}$$

Similarly, in non-Einstein notation,

$$\sum_{a=1}^{N}\ \sum_{b=1}^{N} g_{a\,b} g^{a\,b} = \sum_{a=1}^{N}\ \sum_{b=1}^{N} g_{a\,b} g^{b\,c} \delta_c^a = \sum_{a=1}^{N} \delta_a^c\, \delta_c^a = \sum_{a=1}^{N} (1) = N.$$

In Einstein notation this is

$$g_{a\,b}\, g^{a\,b} = N. \qquad \text{(b)}$$

Putting formulas (a) and (b) together we get

$$c^{a\,b} = \Delta\ g^{a\,b}: \qquad \text{(c)}$$

$$g_{a\,b}\, c^{a\,b} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{a\,b}\, g^{a\,b} \Delta \quad \Rightarrow \quad c^{a\,b} = g^{a\,b} \Delta \qquad ✓$$

The second formula we use for Δ is the definition of determinant as given in definition (A) but expressed using Einstein notation to sum over all of the different combinations of indices $b_1, b_2, \ldots, b_n$:

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\,b_1} \ldots g_{N\,b_N}, \qquad \text{(d)}$$

Equation (d) utilizes the **Levi-Civita symbol, $\epsilon^{b_1 \ldots b_N}$,** which we now explain.

Each index b_k range over the integers 1 – N. The symbol $\epsilon^{b_1 \ldots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless $b_1, \ldots, b_N$ is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for even permutations and -1 for odd permutations. That is, for a permutation $\pi = b_1, b_2, \ldots, b_n$, the Levi-Civita symbol is $\epsilon^{b_1 \ldots b_N} = \text{sign}(\pi)$. For example, for $N = 3$, $\Delta = \epsilon^{b_1\, b_2\, b_3} g_{1\,b_1}\, g_{2\,b_2}\, g_{3\,b_3}$. Consider b_1=1, b_2=2, b_3=3. That term in equation (d) is $\epsilon^{123} g_{11}\, g_{22}\, g_{33} = (-1)^0\, g_{11}\, g_{22}\, g_{33}$ because zero permutations are required to restore the natural order. But, for b_1=1, b_2=3, b_3=2, the term is $\epsilon^{132} g_{11}\, g_{23}\, g_{32} = (-1)^1 g_{11}\, g_{23}\, g_{32}$ because one permutation is needed.

We call $\epsilon^{b_1 \ldots b_N}$ the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, should also be able to be expressed in Levi-Civita terminology since it is a determinant. Consider the following proposed definition:

$$c^{a\,b} \equiv \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \ldots g_{N\,k_N} \qquad \text{(e)}$$

There are several things to confirm.

1. Non-permutations have zero value
2. Every row except row a appears
3. Every column except column b appears
4. $\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}$ is the correct sign for the permutation $k_1 \ldots k_{a-1}\, k_{a+1} \ldots k_N$
5. The equation adheres to tensor notation.

#1 is true by definition of the Levi-Civita symbol. ✓

2 is true since the rows, represented by the first subscripts of g, are
$1, \ldots, a-1, a+1, \ldots, N$. ✓

#3 The exponent of ϵ for a permutation $k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N$ means that the k_a's represent every column except b. Thus, the 2nd subscripts of g represent every column except b. ✓

#4 Since $c^{a\,b} = (-1)^{a+b}\, m^{a\,b}$, the correct value for $\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}$ is $(-1)^{a+b}$. The superscripts of ϵ change for each term of $c^{a\,b}$, so we must show $\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N} = (-1)^{a+b}$ holds for every term. Observe that $(-1)^{b-a} = (-1)^{a+b}$ since $b - a$ is even or odd according to whether $a + b$ is even or odd. So, it is sufficient to show that

$$\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N} = (-1)^{b-a} \tag{f}$$

for every term of $c^{a\,b}$.

We will show this for the case that $1 < a < b < N$. The proof is the same for the cases when $a = 1$, $a = b$, $a > b$, $a = N$, and $b = N$, but the expressions for the Levi-Civita ϵ and $m^{a\,b}$ are slightly different. Since $1 < a < b < N$,

$$m^{a\,b} = g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_a} \cdots g_{b\,k_{b-1}}\, g_{b+1\,k_{b+1}} \cdots g_{N\,k_N}$$

and we can expand

$$\text{LHS of (f)} = \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_{b-1}\, k_{b+1} \ldots k_N}.$$

We first prove equation (f) for the natural term of the matrix $m^{a\,b}$; i.e., for the term

$$g_{11} \cdots g_{a-1\,a-1}\, g_{a+1\,a} \cdots g_{b\,b-1}\, g_{b+1\,b+1} \cdots g_{N\,N}.$$

Since it requires $b - a$ pairwise swaps of adjacent indices to move b from position a to the natural position b,

$$\text{LHS of (f)} = \epsilon^{1 \;\cdots\; a-1\; b\; a+1 \cdots\; b-1\; b+1\; \cdots\; N} = (-1\,)^{b-a}\, \epsilon^{1\;\cdots\; a-1\; a+1\;\cdots\; b-1\;\; b\;\; b+1\;\cdots\; N}$$
$$= \;(-1)^{b-a}\,(1) = \;(-1)^{b-a} \qquad ✓$$

Claim: if we swap of a pair of adjacent column indices in any term of $m^{a\,b}$, then ϵ will change sign:

- If one superscript is just left of b and the other just right of b, then the Levi-Civita symbol requires 3 swaps (swap the left superscript with b, swap the two superscripts, then swap b with the superscript to its right), which is a change of sign $(-1)^3$. ✓

- If the adjacent indices are both smaller or both larger than b, then only one swap occurs in ϵ. This also represents a change of sign. ✓

Every term of $m^{a\,b}$ can be built up from the natural order by a series of pairwise swaps of adjacent indices. Since the natural order has the correct sign, and since each swap implements the correct sign, the final permutation has the correct sign. ✓

#5 Tensor notation rules state that indices can appear at most twice in a term, and then one must be a superscript and the other a subscript. The indices being summed over are the **dummy indices** and all other indices are **free indices**. The rules also specify that any free indices must appear in every term and maintain consistent superscript or subscript status.

The superscript a appears as a free index on LHS of equation (e) but not on RHS. In the steps we take to fix this, and in the remaining steps, we must be very, very careful to adhere to these rules at each step.

To correct superscript a, we could try using a Kronecker delta as follows:

$$c^{a\,b} = \delta^a_s\, \epsilon^{k_1 \ldots k_{a-1}\; b\; k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N},$$

but this fails because we have introduced a new free index, s, that appears on RHS but not LHS. (Observe that k_{a-1} and k_{a+1} are dummy indices and do not have to be changed.)

What works is to introduce both a summation over *s* and a Kronecker delta:

$$c^{a\,b} \equiv \sum_{s=1}^{N} \delta_s^a \;\epsilon^{k_1 \dots k_{a-1}\, b\, k_{a+1} \dots k_N}\, g_{1\,k_1} \dots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N}. \tag{g}$$

Index *a* is a superscript on both sides of the equation, index *s* is a dummy index of summation, and k_{a-1} and k_{a+1} are also dummy indices. In this summation we throw away all terms except when $s = a$, making expression (g) identical to expression (e) while adhering to the tensor notation rules.

We next compute $\partial_c \Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1 \dots k_N}\, g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a\,k_a}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}. \tag{h}$$

Using the product rule to compute $\partial_c \Delta$, we get

$$\begin{aligned}\partial_c \Delta = {} & \partial_c g_{1\,k_1}\, \epsilon^{k_1\, k_2 \dots k_N} g_{2\,k_2} \cdots g_{N\,k_N} \\ & + \dots \\ & + \partial_c g_{a\,k_a}\, \epsilon^{k_1 \dots k_{a-1}\, k_a\, k_{a+1} \dots k_N}\, g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N} \\ & + \dots \\ & + \partial_c g_{N\,k_N}\, \epsilon^{k_1 \dots k_{N-1}\, k_N} g_{1\,k_1} \cdots g_{N-1\,k_{N-1}}\end{aligned}$$

Notice that ϵ is the same in all the terms above. We can replace k_a by *b* to get

$$\begin{aligned}\partial_c \Delta = {} & \partial_c g_{1\,b}\, \epsilon^{b\, k_2 \dots k_N} g_{2\,k_2} \cdots g_{N\,k_N} \\ & + \dots \\ & + \partial_c g_{a\,b}\, \epsilon^{k_1 \dots k_{a-1}\, b\, k_{a+1} \dots k_N}\, g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N} \\ & + \dots \\ & + \partial_c g_{N\,b}\, \epsilon^{k_1 \dots k_{N-1}\, b} g_{1\,k_1} \cdots g_{N-1\,k_{N-1}}\end{aligned} \tag{i}$$

Consider the *a*th term. We wish to make it resemble RHS of equation (g), so we insert a summation and a Kronecker delta:

$$\sum_{s=1}^{N} \delta_a^s\, \partial_c g_{s\,b}\, \epsilon^{k_1 \dots k_{a-1}\, b\, k_{a+1} \dots k_N}\, g_{1\,k_1} \dots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N}. \tag{j}$$

Observe that we have made the free index *a* the subscript of δ_a^s . In equation (g), “*a*” is the superscript of δ_s^a so that RHS and LHS side match . But here, in order to claim that expression (j) equals the *a*th term of expression (i), the index must be a subscript.

Next, we want to replace $\partial_c g_{s\,b}$ by $\partial_c g_{a\,b}$:

$$\sum_{s=1}^{N} \delta_a^s\, \partial_c g_{a\,b}\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N}.$$

This does not change the value of the expression since the terms are all zero except when $s = a$. However, this is not a valid expression because free index a appears twice as a subscript.

To correct this, we can move $\partial_c g_{a\,b}$ to the left of the summation sign if we also reverse the indices on δ:

$$\partial_c g_{a\,b} \sum_{s=1}^{N} \delta_s^a\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N} \qquad \text{(k)}$$

In expression (k), index a has become a dummy summation index since it appears twice. This expression is no longer just the ath term of equation (i). Claim expression (k) equals RHS of equation (i):

$a = 1$: $\partial_c g_{1\,b}\, \epsilon^{b\, k_2 \ldots k_N}\, g_{2\,k_2} \cdots g_{N\,k_N}$ ✓

⋮

$a = a$: Expression (k) is $\partial_c g_{a\,b}\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{+1\,k_{a+1}} \cdots g_{N\,k_N}$

Expression (j) is $\partial_c g_{a\,b}\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}$ ✓

⋮

$a = N$: $\partial_c g_{N\,b}\, \epsilon^{k_1 \ldots k_{N-1}\, b}\, g_{1\,k_1} \ldots g_{N-1\,k_{N-1}}$ ✓

Thus,

$$\partial_c \Delta \overset{(h,\,j)}{=} \partial_c g_{a\,b} \sum_{s=1}^{N} \delta_s^a\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{a-1}}\, g_{s+1\,k_{a+1}} \cdots g_{N\,k_N}$$

$$\overset{(g)}{=} \partial_c g_{a\,b}\, c^{a\,b} \quad ✓$$

$$\overset{(c)}{=} \Delta\, g^{a\,b}\, \partial_c g_{a\,b} \quad ✓$$

Based on
https://www.physicsforums.com/threads/i-cant-verify-a-relationship-between-cofactor-and-determinant.970419/#post-6165630