Problem 3.1

a. Show that in a 2-dimensional Riemannian manifold that all components of R_{ABCD} are either zero or $\pm R_{1212}$.

b. Show that in spherical coordinates, for a sphere of radius *a,* that

$$R_{1212} = a^2 \sin^2\theta \quad \text{and } \mathrm{R} = -\frac{2}{a^2}.$$

Solution.

a.

$$R_{AACD} \overset{(3.16)}{=} -R_{AACD} \quad \Rightarrow \quad R_{AACD} = 0 \text{ for all } A$$

$$R_{ABCC} \overset{(3.17)}{=} -R_{ABCC} \quad \Rightarrow \quad R_{ABCC} = 0 \text{ for all C}$$

$\Rightarrow$ if $R_{ABCD} \neq 0$ then $A \neq B$ and $C \neq D$. ✓

Thus, the only non-zero possibilities are R_{1212}, R_{1221}, R_{2112}, and R_{2121}, and

$$R_{1221} \overset{(3.17)}{=} -R_{1212}, \qquad R_{2112} \overset{(3.16)}{=} -R_{1212}, \qquad R_{2121} \overset{(3.17)}{=} -R_{2112} \overset{(3.16)}{=} R_{1212} \quad ✓$$

In summary,

$$R_{ABCD} = \begin{cases} R_{1212} & \text{If } ABCD = 2121 \\ -R_{1212} & \text{If } ABCD = 2112 \text{ or } 1221 \\ 0 & \text{If } A = B \text{ or } C = D \end{cases} \tag{a}$$

b.

The sphere of radius a is parameterized by θ and ϕ.

The natural basis is

$$\mathbf{e}_1 = \mathbf{e}_\theta = a\cos\theta\cos\phi\, \boldsymbol{i} + a\cos\theta\sin\phi\, \boldsymbol{j} - a\sin\theta\, \boldsymbol{k}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = -a\sin\theta\sin\phi\, \boldsymbol{i} + a\sin\theta\cos\phi\, \boldsymbol{j}.$$

The dual basis is

$$\mathbf{e}^1 = \mathbf{e}^\theta = \frac{1}{a}\cos\theta\cos\phi\, \boldsymbol{i} + \frac{1}{a}\cos\theta\sin\phi\, \boldsymbol{j} - \frac{1}{a}\sin\theta\, \boldsymbol{k}$$

$$\mathbf{e}^2 = \mathbf{e}^\phi = \frac{1}{a}\frac{\sin\phi}{\sin\theta}\boldsymbol{i} + \frac{1}{a}\frac{\cos\phi}{\sin\theta}\boldsymbol{j}.$$

The metric tensor are

$$g_{AB} = (\, \mathbf{e}_A \cdot \mathbf{e}_B\,) = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} \tag{b}$$

$$g^{AB} = (\, \mathbf{e}^A \cdot \mathbf{e}^B\,) = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2\sin^2\theta} \end{pmatrix}. \tag{c}$$

$$R_{ABCD} \overset{(3.15)}{=} \frac{1}{2}\left[\partial_D\partial_A g_{BC} - \partial_D\partial_B g_{AC} + \partial_C\partial_B g_{AD} - \partial_C\partial_A g_{BD}\right]$$
$$- g^{EF}\left[\Gamma_{EAC}\,\Gamma_{FBD} - \Gamma_{EAD}\,\Gamma_{FBC}\right]$$
$$= \frac{1}{2}\left[\partial_D\partial_A g_{BC} - \partial_D\partial_B g_{AC} + \partial_C\partial_B g_{AD} - \partial_C\partial_A g_{BD}\right]$$
$$- \Gamma^F_{AC}\,\Gamma_{FBD} - \Gamma^F_{AD}\Gamma_{FBC}$$

In Exercise 2.15 we showed that the only non-zero connection coefficients are

$$\Gamma^1_{22} = -\sin\theta\cos\theta \qquad \text{and} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$$

So,

$$R_{1212} = \frac{1}{2}\left[\partial_2\partial_1 g_{21} - \partial_2\partial_2 g_{11} + \partial_1\partial_2 g_{12} - \partial_1\partial_1 g_{22}\right]$$
$$- \Gamma^1_{11}\,\Gamma_{122} - \Gamma^1_{12}\Gamma_{121} - \Gamma^2_{11}\,\Gamma_{222} - \Gamma^2_{12}\Gamma_{221}$$

$$\partial_1\partial_1 g_{22} \overset{(b)}{=} \partial_\theta{}^2(a^2\sin^2\theta) = 2a^2\,\partial_\theta(\sin\theta\cos\theta) = 2a^2(\cos^2\theta - \sin^2\theta) \qquad \text{(d)}$$

$$\Gamma_{221} = \frac{1}{2}\left[\partial_\phi g_{21} - \partial_\theta g_{22} - \partial_\phi g_{21}\right] \overset{(b)}{=} \frac{1}{2}\partial_\theta a^2\sin^2\theta = a^2\sin\theta\cos\theta$$

$$\Gamma^2_{12}\Gamma_{221} = \cot\theta\;\; a^2\sin\theta\cos\theta = a^2\cos^2\theta \qquad \text{(e)}$$

$$R_{1212} = -\frac{1}{2}\partial_1\partial_1 g_{22} - \Gamma^2_{12}\Gamma_{221} \overset{(d,\,e)}{=} a^2(\cos^2\theta - \sin^2\theta) - a^2\cos^2\theta$$
$$= a^2\sin^2\theta \quad ✓ \qquad \text{(f)}$$

$$R_{BA} \overset{(3.21)}{=} R^C_{BAC} = g^{DC}R_{DBAC} = g^{11}R_{1\,BA1} + g^{12}R_{1\,BA2} + g^{21}R_{2\,BA1} + g^{22}R_{2\,BA2}$$
$$= \frac{1}{a^2}R_{1\,BA1} + \frac{1}{a^2\sin^2\theta}R_{2\,BA2} \qquad \text{(g)}$$

$$\Rightarrow \quad R_{11} \overset{(g)}{=} \frac{1}{a^2}R_{1111} + \frac{1}{a^2\sin^2\theta}R_{2112} \overset{(a)}{=} -\frac{1}{a^2\sin^2\theta}R_{1212} \overset{(f)}{=} -\frac{1}{a^2\sin^2\theta}a^2\sin^2\theta$$
$$= -1 \qquad \text{(h)}$$

$$R_{22} \overset{(g)}{=} \frac{1}{a^2}R_{1221} + \frac{1}{a^2\sin^2\theta}R_{2222} \overset{(a)}{=} -\frac{1}{a^2}R_{1212} \overset{(f)}{=} -\frac{1}{a^2}a^2\sin^2\theta$$
$$= -\sin^2\theta \qquad \text{(i)}$$

$$\Rightarrow \quad R \overset{(3.22)}{=} g^{AB}R_{BA} = g^{11}R_{11} + g^{12}R_{21} + g^{21}R_{12} + g^{22}R_{22}$$
$$\overset{(c,\,h,\,i)}{=} -\frac{1}{a^2} - \frac{1}{a^2\sin^2\theta}\sin^2\theta = -\frac{2}{a^2} \qquad ✓$$