Exercise 3.5.1

Given
$$R^{\mu\nu} - \frac{1}{2}R\, g^{\mu\nu} = \kappa\, T^{\mu\nu} \tag{3.38}$$
Show
$$R^{\mu\nu} = \kappa\, (T^{\mu\nu} - \frac{1}{2}T\, g^{\mu\nu}) \tag{3.39}$$
where $T = T^{\mu}_{\nu}$.

Solution
Recall $R = R^{\mu}_{\mu}$ \hfill (3.22)

$$R^{\mu}_{\nu} - \frac{1}{2}R\, \delta^{\mu}_{\nu} = g_{\nu\sigma}\,(R^{\mu\sigma} - \frac{1}{2}R\, g^{\mu\sigma}) \overset{(3.38)}{=} g_{\nu\sigma}\,\kappa\, T^{\mu\sigma} = \kappa\, T^{\mu}_{\nu} \tag{a}$$

$$\delta^{\mu}_{\mu} = 4 \tag{b}$$

$$-R = R - 2R \overset{(3.22)}{=} R^{\mu}_{\mu} - \frac{1}{2}R\,(4) \overset{(b)}{=} R^{\mu}_{\mu} - \frac{1}{2}R\,\delta^{\mu}_{\mu} \overset{(a)}{=} \kappa\, T^{\mu}_{\mu} = \kappa\, T \tag{c}$$

$$R^{\mu\nu} \overset{(3.38)}{=} \kappa\, T^{\mu\nu} + \frac{1}{2}R\, g^{\mu\nu} \overset{(c)}{=} \kappa\, T^{\mu\nu} - \frac{1}{2}\kappa\, T\, g^{\mu\nu} = \kappa\,(T^{\mu\nu} - \frac{1}{2}T\, g^{\mu\nu}) \qquad \blacksquare$$