Exercise 1.6.2 Find the line element $ds^2$ for the following surfaces: (a) Sphere of radius $a$ , (b) Cylinder of radius $a$, (c) Hyperbolic paraboloid of Example 1.6.1

Solution.
One way to find the line element is to directly use the line element formula for a surface:

$$ds^2 = g_{AB}\, du^A\, du^B \text{ for } A, B = 1, 2$$

$u^A$ is known in all three cases:

(a) $u^1 = \theta$ and $u^2 = \phi$
(b) $u^1 = \theta$ and $u^2 = z$
(c) $u^1 = u$ and $u^2 = v$

If we don't already have the metric tensor components, we get it from

$$g_{AB} = \boldsymbol{e}_A \bullet \boldsymbol{e}_B$$

by computing

$$\boldsymbol{e}_A = \frac{\partial \boldsymbol{r}}{\partial u^A} = \frac{\partial x}{\partial u^A}\boldsymbol{i} + \frac{\partial y}{\partial u^A}\boldsymbol{j} + \frac{\partial z}{\partial u^A}\boldsymbol{k}$$

where

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}.$$

Here we use a direct approach, starting with $\boldsymbol{r}$, computing $d\boldsymbol{r}$, and then

$$ds^2 = d\boldsymbol{r} \bullet d\boldsymbol{r}$$

(a) Sphere of radius $a$
Fixing $r = a$, then from Example 1.1.4,

$$\boldsymbol{r} = a \operatorname{Sin}\theta \operatorname{Cos}\phi\, \boldsymbol{i} + a \operatorname{Sin}\theta \operatorname{Sin}\phi\, \boldsymbol{j} + a \operatorname{Cos}\theta\, \boldsymbol{k}$$

$$\begin{aligned} d\boldsymbol{r} &= a\,(\operatorname{Cos}\theta \operatorname{Cos}\phi\, d\theta - \operatorname{Sin}\theta \operatorname{Sin}\phi\, d\phi) \\ &\quad + a\,(\operatorname{Cos}\theta \operatorname{Sin}\phi\, d\theta + \operatorname{Sin}\theta \operatorname{Cos}\phi\, d\phi) \\ &\quad - a \operatorname{Sin}\theta\, d\theta \end{aligned}$$

$$\begin{aligned} ds^2 &= d\boldsymbol{r} \bullet d\boldsymbol{r} \\ &= a^2\, [\, (\operatorname{Cos}^2\theta \operatorname{Cos}^2\phi + \operatorname{Cos}^2\theta \operatorname{Sin}^2\phi + \operatorname{Sin}^2\theta)\, d\theta^2 \\ &\qquad + (\operatorname{Sin}^2\theta \operatorname{Sin}^2\phi + \operatorname{Sin}^2\theta \operatorname{Cos}^2\phi)\, d\phi^2 ] \\ &= a^2\, (d\theta^2 + \operatorname{Sin}^2\theta\, d\phi^2) \qquad ✔ \end{aligned}$$

(b) Cylinder of radius $a$

Fixing $r = a$, then from Exercise 1.1.1,

$$\boldsymbol{r} = a \operatorname{Cos}\phi\, \boldsymbol{i} + a \operatorname{Sin}\phi\, \boldsymbol{j} + z\, \boldsymbol{k},$$

$$\mathrm{d}\boldsymbol{r} = -a \operatorname{Sin}\phi\, \mathrm{d}\phi\, \boldsymbol{i} + \mathrm{a} \operatorname{Cos}\phi\, \mathrm{d}\phi\, \boldsymbol{j} + \mathrm{d}z\, \boldsymbol{k}$$

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \bullet \mathrm{d}\boldsymbol{r}$$

$$= a^2 (\operatorname{Sin}^2\phi + \operatorname{Cos}^2\phi)\, \mathrm{d}\phi^2 + \mathrm{d}z^2 = a^2\, \mathrm{d}\phi^2 + \mathrm{d}z^2 \quad ✔$$

(C) Hyperbolic paraboloid

$$\boldsymbol{r} = (u + v)\, \boldsymbol{i} + (u - v)\, \boldsymbol{j} + 2\, u\, v\, \boldsymbol{k}$$

$$\mathrm{d}\boldsymbol{r} = (\mathrm{d}u + \mathrm{d}v)\, \boldsymbol{i} + (\mathrm{d}u - \mathrm{d}v)\, \boldsymbol{j} + 2(v\, \mathrm{d}u + u\, \mathrm{d}v)\, \boldsymbol{k}$$

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \bullet \mathrm{d}\boldsymbol{r}$$

$$= 2(1 + 2\, v^2)\, \mathrm{d}u^2 + 2(1 + 2\, u^2)\, \mathrm{d}v^2 + 8uv\, \mathrm{d}u\, \mathrm{d}v \quad ✔$$