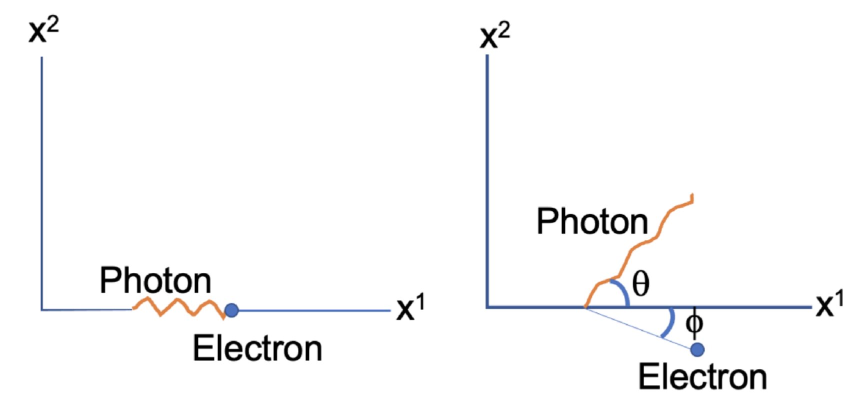
Exercise A.6.3 Confirm equation (A.36) for Compton scattering frequency of the photon and also the resulting velocity of the electron.

Solution We are given the equations for conservation of system momentum:

(1) $\dfrac{h\nu}{c} + mc = \dfrac{h\bar{\nu}}{c} + \gamma mc$

(2) $\dfrac{h\nu}{c} = \dfrac{h\bar{\nu}}{c}\cos\theta + \gamma mv\cos\phi \quad \Leftrightarrow \quad \cos\phi = \dfrac{h}{\gamma mvc}\,(\nu - \bar{\nu}\cos\theta)$

(3) $0 = \dfrac{h\bar{\nu}}{c}\sin\theta - \gamma mv\sin\phi \quad \Leftrightarrow \quad \sin\phi = \dfrac{h\bar{\nu}\sin\theta}{\gamma mvc}.$



Notice that this constitutes 3 equations in 2 unknowns [$\bar{\nu}$ and v (or γ)]. Presumably these equations are consistent and lead to correct values for v and $\bar{\nu}$.

For the photon after-collision frequency, $\bar{\nu}$, we desire a formula dependent only on the pre-collision parameters along with the photon's after-collision deflection angle, θ. It should not include the electron's post-collision speed v or deflection angle, ϕ. Similarly for v, we seek a formula independent of θ and $\bar{\nu}$.

We can replace equations (2) and (3) by a new equation (4) that does not involve ϕ :

$$1 = \sin^2\phi + \cos^2\phi = \left(\frac{h}{\gamma mvc}\right)^2 [\,\bar{\nu}^2(\sin^2\theta + \cos^2\theta) - 2\,\nu\,\bar{\nu}\cos\theta + \nu^2\,]$$

$$\Rightarrow \quad \gamma^2 m^2 v^2 = \frac{h^2}{c^2}\,(\nu^2 - 2\,\nu\bar{\nu}\cos\theta + \bar{\nu}^2)\,. \qquad (4)$$

Equations (1) and (4) constitute two equations in two unknowns. We next wish to eliminate v. Equation (1) has γ (which is a function of v) and equation (4) has v. We first express v in terms of γ:

$$\gamma^2 v^2 = \frac{v^2}{1-\frac{v^2}{c^2}} = \frac{c^2\,\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = -\,c^2\,\frac{\left(1-\frac{v^2}{c^2}\right)-1}{1-\frac{v^2}{c^2}}$$

$$= -\,c^2\left(1-\frac{1}{1-\frac{v^2}{c^2}}\right) = c^2\,(\gamma^2-1) \qquad \text{(a)}$$

$$\Rightarrow \quad \gamma^2 m^2 v^2 = m^2 c^2\,(\gamma^2-1) \qquad \text{(b)}$$

We can rearrange equation (1) to solve for $m^2c^2\,(\gamma^2-1)$:

$$\gamma m c = \frac{h}{c}(\nu-\overline{\nu}) + mc \quad \Rightarrow \quad \gamma^2 m^2 c^2 = \frac{h^2}{c^2}(\nu-\overline{\nu})^2 + 2\,hm\,(\nu-\overline{\nu}) + m^2c^2$$

$$\Rightarrow \quad m^2c^2\,(\gamma^2-1) = \frac{h^2}{c^2}(\nu-\overline{\nu})^2 + 2\,hm\,(\nu-\overline{\nu})$$

$$= \frac{h^2}{c^2}(\nu^2+\overline{\nu}^2) - 2\,\frac{h^2}{c^2}\nu\,\overline{\nu} + 2\,hm\,(\nu-\overline{\nu})$$

$$= \left(\frac{h\nu}{c} + mc - \frac{h\,\overline{\nu}}{c}\right)^2 - m^2c^2 \qquad (1')$$

Now, combining (1′) and (4) we can eliminate v (and γ):

$$\frac{h^2}{c^2}\,(\nu^2 - 2\,\nu\overline{\nu}\cos\theta + \overline{\nu}^2) \overset{(4)}{=} \gamma^2 m^2 v^2 \overset{(b)}{=} m^2c^2\,(\gamma^2-1)$$

$$\overset{(1')}{=} \left(\frac{h\nu}{c} + mc - \frac{h\,\overline{\nu}}{c}\right)^2 - m^2c^2 = \frac{h^2}{c^2}(\nu^2+\overline{\nu}^2) - 2\,\frac{h^2}{c^2}\nu\,\overline{\nu} + 2\,hm\,(\nu-\overline{\nu})$$

We solve this equation for $\overline{\nu}$ by moving all of the $\overline{\nu}$ terms to the LHS:

$$\overline{\nu}^2\left(\frac{h^2}{c^2} - \frac{h^2}{c^2}\right) + \overline{\nu}\left(-2\,\frac{h^2}{c^2}\nu\cos\theta + 2\,\frac{h^2}{c^2}\nu + 2hm\right) = -\frac{h^2}{c^2}\nu^2 + \frac{h^2}{c^2}\nu^2 + 2\,hm\nu$$

$$\Rightarrow \quad \overline{\nu}\left(-2\,\frac{h^2}{c^2}\nu\cos\theta + 2\,\frac{h^2}{c^2}\nu + 2hm\right) = 2\,hm\nu$$

$$\Rightarrow \quad \overline{\nu} = \frac{2\,hm\,\nu}{2\,hm + 2\,\frac{h^2\,\nu}{c^2}\,(1-\cos\theta)}$$

$$\Rightarrow \quad \boxed{\bar{\nu} = \frac{\nu}{1 + \frac{h\nu}{mc^2}(1-\cos\theta)}} \qquad \checkmark$$

We next wish to solve for *v* after eliminating θ and $\bar{\nu}$. We can eliminate θ by combining equations (2) and (3) into yet another new equation, (5):

$$\text{(2)}\quad \cos\theta = \frac{1}{h\bar{\nu}}(\mathrm{h}\nu - \gamma m v c \cos\phi)$$

$$\text{(3)}\quad \sin\theta = \frac{\gamma m v c \sin\phi}{h\bar{\nu}}$$

$$1 = \sin^2\theta + \cos^2\theta$$

$$= \frac{1}{h^2\bar{\nu}^2}\left(h^2 \nu^2 - 2\gamma m v c h\nu \cos\phi + \gamma^2 m^2 v^2 c^2 (\sin^2\phi + \cos^2\phi)\right)$$

$$\Rightarrow \quad h^2(\bar{\nu}^2 - \nu^2) = \gamma m v c\,(\gamma m v c - 2\,h\nu\cos\phi) \qquad (5)$$

Rearranging equation (1) and squaring gives

$$h^2(\bar{\nu}^2 - \nu^2) = m^2 c^4\,(1-\gamma)^2 \qquad (1'')$$

Setting equation (1″) equal to equation (5) eliminates $\bar{\nu}$:

$$mc^3(1-\gamma)^2 = \gamma v\,(\gamma m v c - 2\,h\nu\cos\phi)$$

However, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and this equation involves γ^2, γ, v^2, and *v*. In order to clear the square roots yields a 4th degree equation in *v* that has a solution in theory but which I can't solve.