Exercise 4.1 Check the integral (4.7)

Equation (4.7) follows from the equation

$$\int \frac{\sqrt{x+a}}{\sqrt{x+b}}\, dx = \sqrt{x+a}\ \sqrt{x+b} + (a-b)\ln\left|\sqrt{x+a} + \sqrt{x+b}\right|, \qquad (1)$$

where $x = r$, $a = 0$, and $b = -\frac{2\,GM}{c^2}$.

Equation (1) follows immediately from

$$\int \frac{\sqrt{x+a}}{\sqrt{x+b}}\, dx = \sqrt{x+a}\ \sqrt{x+b} + \frac{a-b}{2}\int \frac{1}{\sqrt{x+a}\ \sqrt{x+b}}\, dx \qquad (2)$$

since $\frac{d}{dx}\ln\left|\sqrt{x+a} + \sqrt{x+b}\right| = \dfrac{\frac{1}{2\sqrt{x+a}} + \frac{1}{2\sqrt{x-b}}}{\sqrt{x+a}+\sqrt{x-b}} = \dfrac{1}{2}\ \dfrac{\sqrt{x-b}+\sqrt{x+a}}{\left(\sqrt{x+a}+\sqrt{x-b}\right)\sqrt{x+a}\ \sqrt{x-b}}$.

One way to derive equation (2) is to implement a natural sequence of substitutions followed by a partial fraction expansion. I could not find a simpler approach like integration by parts.

For the record, the RHS integral in equation (2) can be solved as follows:

First, substitute $u = \sqrt{x+a}$. Then $du = \frac{1}{2\sqrt{x+a}}dx = \frac{1}{2u}dx$, or $dx = 2u\, du$.

The integral then becomes $\int \frac{2u^2}{\sqrt{u^2+b-a}}\, du$.

A natural next substitution is to set $s = \frac{1}{u^2}$ to get $-\int \frac{1}{s^2\sqrt{s(b-a)+1}}\, ds$.

Then substitute $p = s(b-a) + 1$ to get $(a-b)\int \frac{1}{(p-1)^2\sqrt{p}}\, dp$.

Then substitute $w = \sqrt{p}$ to get $(2a - 2b)\int \frac{1}{(w^2-1)^2}\, dw$,

which by partial fraction expansion equals

$(2a-2b)\int\left(\frac{1}{4(w+1)} + \frac{1}{4(w+1)^2} - \frac{1}{4(w-1)} + \frac{1}{4(w-1)^2}\right) dw$.

Each of these integrals is easy to compute, and then you back substitute from w to p to s to u to x to get the result.