Special Relativity Crib Sheet

Section A.0 Introduction

A.1 $c = \dfrac{dr}{dt} = \dfrac{dr'}{dt'}$ where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is position vector and $r^2 = \boldsymbol{r} \bullet \boldsymbol{r}$

A.2 $\pm\, ds^2 \equiv c^2\, dt^2 - dr^2 = c^2\, dt'^2 - dr'^2$ **Invariant interval** between events

A0-1 $ds^2 \equiv c^2\, dt^2 - dx^2 - dy^2 - dz^2 = c^2\, dt'^2 - dx'^2 - dy'^2 - dz'^2$

A0-2 $c\, d\tau \equiv ds$

A.3 $\boxed{x^0 \equiv ct, \quad x^1 \equiv x, \quad x^2 \equiv y, \quad x^3 \equiv z}$, or $\boxed{x^\mu = (c\,t,\, x,\, y,\, z)}$

A.4 $\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = \eta_{\mu'\,\nu'}\, dx^{\mu'}\, dx^{\nu'}} = c^2\, dt^2 - dx^2 - dy^2 - dz^2$

A0-3 $(\eta_{\mu\nu}) \equiv (\eta_{\mu'\,\nu'}) \equiv \begin{pmatrix} 1 & \\ & -\mathrm{I} \end{pmatrix}$ **Covariant** Cartesian **metric tensor**, I=3x3

Example A.0.1(a) if $\lambda^\mu = (\lambda^0, \lambda^1, \lambda^2, \lambda^3)$ then $\lambda_\mu = (\lambda^0, -\lambda^1, -\lambda^2, -\lambda^3)$

A.5 $\boxed{c^2\, (d\tau)^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = \eta_{\mu'\,\nu'}\, dx^{\mu'}\, dx^{\nu'}}$ Special Rel **line element**

A0-6 $\eta^{\mu\nu} \equiv {\eta_{\mu\nu}}^{-1} = \eta_{\mu\nu}$ **Contravariant** Cartesian **metric tensor**

A0-7 $v^2 = \boldsymbol{v} \bullet \boldsymbol{v} = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2$ and $v'^2 = \left(\dfrac{dx'}{dt'}\right)^2 + \left(\dfrac{dy'}{dt'}\right)^2 + \left(\dfrac{dz'}{dt'}\right)^2$

A0-8 $v^2\, dt^2 = dx^2 + dy^2 + dz^2$ and $v'^2 dt'^2 = dx'^2 + dy'^2 + dz'^2$

A.6 $\boxed{d\tau = \sqrt{1 - \dfrac{v^2}{c^2}}\; dt = \sqrt{1 - \dfrac{v'^2}{c^2}}\; dt'}$

Section A.1 Lorentz transformations

A1-1 $x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^\nu$ **Lorentz transformation**

A.7 $x^{\mu'} = \tilde{\Lambda}^{\mu'}_{\ \nu}\, x^\nu + a^\mu$ **Lorentz transformation matrix**

A1-2 $\tilde{\Lambda}^{\mu'}_{\ \nu} = X^{\mu'}_{\nu} = \dfrac{\partial x^{\mu'}}{\partial x^\nu}$, and $\Lambda^{\mu'}_{\ \nu} = X^{\mu'}_{\nu}$ iff $a^\mu = 0$ **Jacobian matrix**

A1-3 $\lambda^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, \lambda^\nu$ **Transformation for contravariant vectors**

A.8 $\boxed{\eta_{\mu\nu} = \Lambda^{\sigma'}_{\mu}\, \Lambda^{\rho'}_{\nu}\, \eta_{\sigma'\,\rho'}}$ **Transformation for covariant metric tensor**

$\boxed{\tau^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} = \Lambda^{\mu_1}_{\sigma_1'} \cdots \Lambda^{\mu_r}_{\sigma_r'}\, \Lambda^{\rho_1'}_{\nu_1} \cdots \Lambda^{\rho_s'}_{\nu_s}\, \tau^{\sigma_1' \ldots \sigma_r'}_{\rho_1' \ldots \rho_s'}}$ **Type (r,s) tensor**

A1-4 $\frac{dx}{dt} = v$ **Frame K' x-boost speed**

A.11 $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{c}{\sqrt{c^2-v^2}}}$ Note that $\gamma \geq 1$.

A.12 $t' = \gamma\,(t - \frac{v}{c^2}\,x)$ or $ct' = \gamma\, ct - \frac{\gamma\, v}{c}\, x$

$x' = \gamma\,(x - v\,t)$ $y' = y$ $z' = z$ **Spacetime boost in x-direction**

A1-5 $\frac{dx'}{dt'} = -\mathrm{v}$ **Frame K x'-boost speed**

A1-7 $t' = B\,t + \boldsymbol{C} \bullet \boldsymbol{x}$ $\boldsymbol{x}' = A\,(\boldsymbol{x} - \boldsymbol{v}\,t)$

3+1 homogeneous spacetime boost/Lorentz transformation

A1-8 $c^2\,(dt')^2 - (d\boldsymbol{x}')^2 = c^2\,(dt)^2 - (d\boldsymbol{x})^2$ where $\boldsymbol{x} = (x, y, z)$

A1-10 $d\boldsymbol{x} = \boldsymbol{v}\,dt$, $\boldsymbol{v} \bullet d\boldsymbol{x} = v^2\,dt$, and $(d\boldsymbol{x})^2 = d\boldsymbol{x} \bullet d\boldsymbol{x} = v^2\,dt^2$

A1-13 $\boxed{\frac{dt}{dt'} = \gamma}$

A-15 $t' = \gamma\,(t - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c^2})$ or $ct' = \gamma\,(ct - \frac{\boldsymbol{v}}{c} \bullet \boldsymbol{x})$ $\boldsymbol{x}' = \gamma\,(\boldsymbol{x} - \boldsymbol{v}\,t)$

3+1 Homogeneous Lorentz transformation = Spacetime boost

A1-17 $$\begin{pmatrix} c\,t' \\ \boldsymbol{x}' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\boldsymbol{v}}{c} & \mathrm{I} \end{pmatrix} \begin{pmatrix} c\,t \\ \boldsymbol{x} \end{pmatrix} = \gamma \begin{pmatrix} c\,t - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c} \\ -\boldsymbol{v}\,t + \boldsymbol{x} \end{pmatrix}$$ **Homogeneous form**

A1-18, A1-19 $$\tilde{\Lambda} \equiv \gamma \begin{pmatrix} 1 & -\frac{\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\boldsymbol{v}}{c} & \mathrm{I} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v_x}{c} & -\frac{v_y}{c} & -\frac{v_z}{c} \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ -\frac{v_z}{c} & 0 & 0 & 1 \end{pmatrix}$$ **Homogeneous**

A1-23 $\gamma = 1 + \frac{v^2}{c^2}\frac{\gamma^2}{\gamma+1}$ Identity to convert from A1-17 to A1-18

A1-20 $$\Lambda = \begin{pmatrix} \gamma & -\frac{\gamma\,\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\gamma\,\boldsymbol{v}}{c} & \mathrm{I} + \frac{\gamma^2}{c^2\,(\gamma^2+1)}\,\mathbf{v}\mathbf{v}^{\mathrm{T}} \end{pmatrix}$$ **Frobenius form**

A1-21 $$\Lambda = \begin{pmatrix} \gamma & -\frac{\gamma v_x}{c} & -\frac{\gamma v_y}{c} & -\frac{\gamma v_z}{c} \\ -\frac{\gamma v_x}{c} & 1+(\gamma-1)\frac{v_x{}^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\ -\frac{\gamma v_y}{c} & (\gamma-1)\frac{v_y v_x}{v^2} & 1+(\gamma-1)\frac{v_y{}^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\ -\frac{\gamma v_z}{c} & (\gamma-1)\frac{v_z v_x}{v^2} & (\gamma-1)\frac{v_z v_y}{v^2} & 1+(\gamma-1)\frac{v_z{}^2}{v^2} \end{pmatrix}$$ **Wikipedia**

A1-26 $\boxed{x^{\mu'} = \Lambda^{\mu'}_{\nu}\, x^{\nu} = \tilde{\Lambda}^{\mu'}_{\nu}\, x^{\nu} + a^{\mu}}$ Lorentz = Homogeneous Lorentz + offset

A1-27 $\tanh \psi \equiv \frac{v}{c}$ **Definition of angle ψ**

A1-32 $$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix}$$ **Boost in *x*-direction**

A1-34 $$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix}$$ **Spacetime *xy*-rotation**

Section A.2 Relativistic additon of velocities

A.17 $\boxed{u = \frac{v + w}{1 + \frac{v\,w}{c^2}}}$ **Relativistic addition of velocities**

Section A.4 Time dilation and length contraction

A.18 $\boxed{\Delta\tau = \sqrt{1 - \frac{v^2}{c^2}}\;\Delta t}$ **Time dilation:** moving clocks run more slowly

A.19 $\boxed{\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}}}$ **Length contraction** (in direction of motion)

Section A.6 Some standard 4-vectors

A6-1	$\boxed{u^\mu \equiv \dfrac{dx^\mu}{d\tau}}$	**World velocity**
A.23	$\boxed{v^\mu \equiv \dfrac{dx^\mu}{dt} = (c, \boldsymbol{v})}$	**Coordinate velocity**
A6-2	$\boxed{\dfrac{dt}{d\tau} = \gamma}$	
A.24	$u^\mu = \gamma\, v^\mu = \gamma\,(\mathrm{c}, \boldsymbol{v})$	
A.25	$\boxed{p^\mu \equiv m\, u^\mu}$	**Particle 4-momentum**
A6-4	$p^\mu = (E/c, \boldsymbol{p})$	where E is particle's energy
A.27	$\boxed{k^\mu \equiv \left(\dfrac{2\pi}{\lambda}, \boldsymbol{k}\right)}$	**Wave 4-vector of a photon**, where λ is wavelength
A6-5	$\boldsymbol{k} = \dfrac{2\pi}{\lambda}\boldsymbol{n}$	$\boldsymbol{n}$ is the unit 3-vector in direction of propagation
A6-6	$k^\mu = \dfrac{2\pi}{\lambda}\left(1, \dfrac{\boldsymbol{w}}{c}\right)$	**w** = c**n** is photon's 3-velocity
A6-7	$k^\mu k_\mu = 0$	i.e., k^μ is null (lightlike)
A.28	$\boxed{p^\mu \equiv \hbar\, k^\mu}$	**Photon 4-momentum**
A6-8	$\boxed{p^\mu p_\mu = 0}$	Photon's 4-momentum is null (lightlike)
A6-13	$p^\mu = (E/c, \boldsymbol{p})$	where E is photon's energy and $\boldsymbol{p} = \hbar\,\boldsymbol{k}$,
A6-11	$\nu \equiv \dfrac{c}{\lambda}$	**Photon's frequency and wavelength**
A6-12	$\mathrm{E} = h\,\nu$	
A.29	$dp^\mu / d\tau = f^\mu$	**Newton's (relativistic) second law**
A.30	$\boxed{f^\mu \equiv (f^0, \gamma\boldsymbol{F})}$	where $\boldsymbol{F}$ is the 3-force on the particle
A6-14	$f^0 = \dfrac{\gamma}{c}\,\boldsymbol{F} \cdot \boldsymbol{v}$	
A.32	$\boxed{E = \sqrt{p^2 c^2 + m^2 c^4}}$	Where $E = m\,c^2$ comes from for a particle at rest
A.33	$\boldsymbol{p} = \gamma m \boldsymbol{v}$	Shows $\boldsymbol{p} \approx m\boldsymbol{v}$ when $\boldsymbol{v}$ is small
A.34	$E = \gamma mc^2 = mc^2\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}} = mc^2 + \frac{1}{2}mv^2 + \cdots$	Rest + kinetic energy

A.35 $\sum_{\text{all particles}} p^{\mu} = \text{constant}$ **Conservation of energy & momentum**

A.36 $\boxed{\bar{\nu} = \dfrac{\nu}{1+\dfrac{h\nu}{mc^2}\,(1-\cos\theta)}}$ **Compton scattering formula**

for after-collision frequency

Section A.7 Doppler effect

A7-1 $\boxed{\dfrac{\lambda}{\lambda'} = \gamma\left(1-\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{c^2}\right)}$ **Doppler shift formula** where the primed

frame is the emitter and the unprimed frame is the observer

Section A.8 Electromagnetism

A8-1 $\nabla = \dfrac{\partial}{\partial x}\boldsymbol{i} + \dfrac{\partial}{\partial y}\boldsymbol{j} + \dfrac{\partial}{\partial z}\boldsymbol{k}$

A8-2 $\nabla \cdot \boldsymbol{B} = \dfrac{\partial B^x}{\partial x} + \dfrac{\partial B^y}{\partial y} + \dfrac{\partial B^z}{\partial z}$

A8-3 $\nabla \times \boldsymbol{B} = \left(\dfrac{\partial B^z}{\partial y} - \dfrac{\partial B^y}{\partial z}\right)\boldsymbol{i} + \left(\dfrac{\partial B^x}{\partial z} - \dfrac{\partial B^z}{\partial x}\right)\boldsymbol{j} + \left(\dfrac{\partial B^y}{\partial x} - \dfrac{\partial x}{\partial y}\right)\boldsymbol{k} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B^x & B^y & B^z \end{vmatrix}$

A.43 $\nabla \cdot \boldsymbol{B} = 0$ **Maxwell's equations in free space,** differential form

A.44 $\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$

A.45 $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$

A.46 $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$

The vector fields $\boldsymbol{B}$ and $\boldsymbol{E}$ can be expressed in terms of a vector potential $\boldsymbol{A}$ and a scalar potential φ :

A8-5 $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla\varphi - \dfrac{\partial \boldsymbol{A}}{\partial t}$

A.49 $\nabla \cdot \boldsymbol{A} + \mu_0 \varepsilon_0 \dfrac{\partial \varphi}{\partial t} = 0$ **Lorentz gauge condition**

A.54 $$(F_{\mu\nu}) = \begin{pmatrix} 0 & -\frac{E^1}{c} & -\frac{E^2}{c} & -\frac{E^3}{c} \\ \frac{E^1}{c} & 0 & B^3 & -B^2 \\ \frac{E^2}{c} & -B^3 & 0 & B^1 \\ \frac{E^3}{c} & B^2 & -B^1 & 0 \end{pmatrix}$$

where $\boldsymbol{E} = (E^1, E^2, E^3)$, $\boldsymbol{B} = (B^1, B^2, B^3)$, $j^\mu = (\rho\, c, \boldsymbol{J}) = \rho\, v^\mu = (\gamma\, \rho_0)\, v^\mu = \rho_0\, u^\mu$

A.55 $F^{\mu\nu}{}_{,\nu} = \mu_0\, j^\mu$

A.56 $F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0$