Lorentz general & homogeneous transformations

Foster & Nightingale's derivation the special case of the Lorentz transformation, a boost in the x-direction, is excellent and I repeat it here and then extend it to generate the full Lorentz transformation which, strangely, is much simpler.

Appendix A.1 Lorentz transformations

Definition A **Lorentz transformation** is a coordinate transformation connecting two inertial frames:

$$\mathcal{L} : \mathrm{K} \to \mathrm{K}' : \mathcal{L}(x^{\mu}) = x^{\mu'}$$

In matrix form, we could represent $\mathcal{L}$ as

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu} \ . \qquad \text{(A1-1)}$$

By definition (Section A.0), an inertial frame K′ in flat spacetime moves with respect to inertial frame K at constant velocity and without rotation. (The space portion of K′ may be initially rotated, but when it moves there is no further rotation.) So, $x^{\mu'}$ is obtained from x^{ν} by an affine transformation (i.e., linear transformation plus offset)

$$x^{\mu'} = \tilde{\Lambda}^{\mu'}_{\nu} x^{\nu} + a^{\mu} \qquad \text{(A.7)}$$

where $\tilde{\Lambda}^{\mu'}_{\nu}$ represents a *linear* transformation and $a^{\mu} = (0, a^1, a^2, a^3)$ is a constant. Compare to equation (A1-1).

Observe that (0,0,0) is the coordinates of the center O of frame K at $t = 0$, and (0,0,0) is the coordinates of the center O′ of frame K′ at $t' = 0$. So, if $a^{\mu} = 0$, then the centers O and O′ coincide at $t = t' = 0$, and the Lorentz transformation is called **homogeneous**. Otherwise it is **inhomogeneous**. Some books call the latter transformation a **Poincare transformation** in which case the former is just called a **Lorentz transformation**.

Using this notation, we could call $\tilde{\Lambda}^{\mu'}_{\nu}$ a *homogeneous* matrix whereas $\Lambda^{\mu'}_{\nu}$ is generally inhomogeneous.

The book does not use the tilde notation and it confuses the two concepts. It only uses $\Lambda^{\mu'}_{\ \nu}$, at times to represent the homogeneous matrix (e.g., A.7, $x^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^{\nu} + a^{\mu}$) and at other times to represent the general matrix (e.g., A1-3, the definition that a spacetime vector is an object that satisfies $\lambda^{\mu'} = \Lambda^{\mu'}_{\ \nu} \lambda^{\nu}$). This distinction is important because when performing Lorentz transformations, we insert expressions for $\Lambda^{\mu'}_{\ \nu}$, and in equation (A1.2), below, we show that $\tilde{\Lambda}^{\mu'}_{\nu}$ has a very simple expression, but $\Lambda^{\mu'}_{\ \nu}$ does not.

Caution: In tensor equations, the indices on LHS generally must match the indices on RHS. That would lead to $a^{\mu'}$ on RHS of equation (A.7), which is not correct because an exception to the rule is that *transformation equations* change primed tensors on LHS to unprimed tensors on RHS.

Formula (A.7) can be formally proven using equation (2.32) for how connection coefficients transform:

Since $g_{\mu\nu} = g_{\mu'\nu'} = \eta_{\mu\nu} = 0$, 1, or -1, a constant, then $\partial_{\sigma} g_{\mu\nu} = 0$.

So, $\Gamma^{\alpha}_{\nu\sigma} \overset{(2.9)}{=} \Gamma^{\alpha'}_{\nu'\sigma'} = 0$ for all α, ν, and σ.

Thus, $0 = \Gamma^{\alpha'}_{\nu'\sigma'} \overset{(2.32)}{=} \Gamma^{\mu}_{\beta\gamma} X^{\alpha'}_{\mu} X^{\beta}_{\nu'} X^{\gamma}_{\sigma'} + X^{\mu}_{\sigma'\nu'} X^{\alpha'}_{\mu} = 0 + X^{\mu}_{\sigma'\nu'} X^{\alpha'}_{\mu} = X^{\mu}_{\sigma'\nu'} X^{\alpha'}_{\mu}$

$\Rightarrow X^{\mu}_{\sigma'\nu'} = 0$ for all μ, σ', and ν'.

Swapping primed and unprimed indices gives $X^{\mu'}_{\sigma\nu} = 0$.

Equation (A.7) now follows because if the 2nd derivative of $x^{\mu'}$ is zero, then the first derivative is a constant, and so

$x^{\mu'} =$ (constant) x^{ν} + (another constant). ✓

More formally, performing the last step using the Fundamental Theorem of calculus, $\int f'(x)\, dx = f(x) + C$:

$$0 = \int 0\, dx^{\sigma} = \int X^{\mu'}_{\sigma\nu}\, dx^{\sigma} = \int \frac{\partial}{\partial x^{\sigma}} \left(X^{\mu'}_{\nu} \right) dx^{\sigma} \overset{\text{(Fund Th)}}{=} X^{\mu'}_{\nu} - \tilde{\Lambda}^{\mu'}_{\nu} \quad \Rightarrow \quad X^{\mu'}_{\nu} = \tilde{\Lambda}^{\mu'}_{\nu}$$

$$\Rightarrow \quad x^{\mu'} \overset{\text{(Fund Th)}}{=} \int \frac{\partial}{\partial x^{\nu}} \left(x^{\mu'} \right) dx^{\nu} = \int X^{\mu'}_{\nu}\, dx^{\nu} = \int \tilde{\Lambda}^{\mu'}_{\nu}\, dx^{\nu} \overset{\text{(Fund Th)}}{=} \tilde{\Lambda}^{\mu'}_{\nu} x^{\nu} + a^{\mu} . \quad \blacksquare$$

As a corollary, we have confirmed that $\tilde{\Lambda}^{\mu'}_{\nu}$ is, in fact, the Jacobian, $X^{\mu'}_{\nu}$. In fact,

$$\tilde{\Lambda}^{\mu'}_{\nu} = X^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}, \quad \text{and} \quad \Lambda^{\mu'}_{\nu} = X^{\mu'}_{\nu} \ \text{ iff } \ a^{\mu} = 0: \qquad \text{(A1-2)}$$

$$\Lambda^{\mu'}_{\nu} x^{\nu} \overset{\text{(A1-1)}}{=} x^{\mu'} \overset{\text{(A.7)}}{=} \tilde{\Lambda}^{\mu'}_{\nu} x^{\nu} + a^{\mu} \ \text{ for all } x^{\mu'}$$

$$\Rightarrow \quad \Lambda^{\mu'}_{\nu} x^{\nu} = \tilde{\Lambda}^{\mu'}_{\nu} x^{\nu} \ \text{ iff } a^{\mu} = 0 \ \text{ for all choices of } x^{\nu},\ \nu = 1 - N$$

Choose $x^{\nu} = 0$ for $\nu > 1$. Then $\Lambda^{\mu'}_{\nu} x^{\nu} = \Lambda^{\mu'}_{1} x^{1}$ and $\tilde{\Lambda}^{\mu'}_{\nu} x^{\nu} = \tilde{\Lambda}^{\mu'}_{1} x^{1}$

$$\Rightarrow \quad \Lambda^{\mu'}_{1} x^{1} = \tilde{\Lambda}^{\mu'}_{\nu} x^{1} \ \text{ iff } a^{\mu} = 0 \ \text{ for all choices of } x^{1}$$

$$\Rightarrow \quad \Lambda^{\mu'}_{\nu} = \tilde{\Lambda}^{\mu'}_{\nu} \ \text{ iff } a^{\mu} = 0 \ \checkmark$$

We take a small digression to clarify the difference between Lorentz and Jacobian transformations. First, we haven't yet defined, or even used, the term "Jacobian transformation". Rather, we have until now limited our terminology to "Jacobian transformation matrix". However, we showed in Section 0 that every matrix has an associated linear transformation. This association is a natural way to define a Jacobian transformation (even though it is rarely done). We do it now.

Definition A **Jacobian transformation** is a linear transformation T associated with a Jacobian matrix $(X^{\mu'}_{\nu})$.

Since a Jacobian transformation matrix maps a coordinate system at a point P to a coordinate system at a point Q, a Jacobian transformation T is a mapping from the tangent space at P to the tangent space at Q. Recall that a tangent space is not located in any manifold; it is an abstract space associated with a point.

A Lorentz transformation $\mathcal{L}$, however, maps flat spacetime to flat spacetime. Even though we could think of the the tangent "space" at a point in flat spacetime as flat spacetime itself, it is not; it is an associated abstract tangent spacetime, $\mathbf{T}_{\mathrm{P}}$. So, technically, a Lorentz transformation $\mathcal{L}$ cannot equal a Jacobian transformation T because, as functions, they do not have the same domains and ranges:

$$\mathcal{L} : \mathrm{K} \to \mathrm{K}'$$
$$T : \mathbf{T}_{\mathrm{P}} \to \mathbf{T}_{\mathrm{Q}}$$

Notation In Special and General Relativity, we use $\Lambda^{\mu'}_{\nu}$ to represent (Lorentz) transformations rather than $U^{i'}_{j}$ (Euclidean space) or $X^{a'}_{b}$ (manifolds).

Definition An object $\lambda^\mu = (\lambda^0, \boldsymbol{\lambda})$ in spacetime is called a **contravariant vector** if it satisfies the Lorentz transformation formula

$$\lambda^{\mu'} = \Lambda^{\mu'}_{\ \nu} \lambda^\nu \tag{A1-3}$$

for every pair of inertial frames K and K′, where $\lambda^{\mu'}$ represents the same object but in primed coordinates.

Plugging $dx^{\mu'} = \Lambda^{\mu'}_{\rho} dx^\rho$ into equation (A.4):

$$\eta_{\mu\nu}\, dx^\mu\, dx^\nu \overset{(A.4)}{=} \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'} \overset{(A.7)}{=} \eta_{\mu'\nu'}\, \Lambda^{\mu'}_{\rho}\, \Lambda^{\nu'}_{\sigma}\, dx^\rho\, dx^\sigma$$

$$\overset{(\mu\leftrightarrow\rho,\, \nu\leftrightarrow\sigma)}{=} \eta_{\rho'\sigma'}\, \Lambda^{\rho'}_{\mu}\, \Lambda^{\sigma'}_{\nu}\, dx^\mu\, dx^\nu$$

$$\Rightarrow \quad \boxed{\eta_{\mu\nu} = \Lambda^{\rho'}_{\mu}\, \Lambda^{\sigma'}_{\nu}\, \eta_{\rho'\sigma'}}\ . \tag{A.8}$$

This is the necessary and sufficient condition for $\eta_{\mu\nu}$ to be a type (0,2) spacetime tensor. This pattern can be extended as was done in Chapter 2 to generate definition (1.72) of a manifold type (r, s) tensor. The **Lorentz transformation equations for a type (*r*, *s*) spacetime tensor** are:

$$\boxed{\tau^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} = \Lambda^{\mu_1}_{\sigma_1{}'} \cdots \Lambda^{\mu_r}_{\sigma_r{}'}\, \Lambda^{\rho_1{}'}_{\nu_1} \cdots \Lambda^{\rho_s{}'}_{\nu_s}\, \tau^{\sigma_1{}' \ldots \sigma_r{}'}_{\rho_1{}' \ldots \rho_s{}'}}\ .$$

Application of this will be made shortly not only for the laws of motion but also for Maxwell's equations of electromagnetism.

While spacetime mechanics is invariant under Lorentz transformations, Newtonian mechanics is not. Rather, Newtonian mechanics is invariant under **Galilean transformations** to which Lorentz transformations reduce when v/c is negligible. We now expand the development of Lorentz transformations, first for motion along the x-axis, followed by motion in 3-dimensions, and then we define and compare the corresponding Galilean transformation.

Definition A **spacetime boost** (in the x-direction) is an affine transformation such as the following linear (i.e., homogeneous) transformation:

$$t' = B\,t + C\,x$$
$$x' = A\,(x - v\,t)$$
$$y' = y$$
$$z' = z \tag{A.9}$$

where A, B, $C \neq 0$. The first two equations in (A.9) represent a linear mixing of space and time due to non-zero velocity in the x-direction.

Claim equations (A.9) can be considered to represent the spatial origin O' of K' moving along the x-axis of K with a velocity v, and the axes coinciding when $t = t' = 0$:

- Along the x-axis , $y = z = 0$. Equations (3) and (4) then cause $y' = 0$ and $z' = 0$.
- Equation (2) for the x'component of O' is $0 = A\,(x - v\,t) \Leftrightarrow x = v\,t$.
  Thus, O', and hence all of K', moves in the $+x$ direction with speed v. ✓
  $$\frac{\mathrm{d}x}{\mathrm{d}t} = v \tag{A1-4}$$
- When $t = t' = 0$, the first equation in (A.9) makes $x = 0$, and then the second equation makes $x' = 0$; that is, the origins coincide. ✓

Thus, we have shown that, as claimed, equations (A.9) describe a homogeneous transformation (i.e., the centers coincide at time $t = t' = 0$). A more general spacetime boost in the x-direction would include an offset at time 0.

We can solve for A, B, and C. From equation (A.9) we get

$$\begin{aligned}
&\mathrm{d}t' = B\,\mathrm{d}t + C\,\mathrm{d}x && (\mathrm{d}t')^2 = B^2\,\mathrm{d}t^2 + 2BC\,\mathrm{d}t\,\mathrm{d}x + C^2\,\mathrm{d}x^2\\
&\mathrm{d}x' = \mathrm{A}\,(\mathrm{d}x - \mathrm{v}\,\mathrm{d}t) && (\mathrm{d}x')^2 = A^2\left(\mathrm{d}x^2 - 2v\,\mathrm{d}t\,\mathrm{d}x + v^2\,\mathrm{d}t^2\right)\\
&\mathrm{d}y' = \mathrm{d}y && (\mathrm{d}y')^2 = \mathrm{d}y^2\\
&\mathrm{d}z' = \mathrm{d}z && (\mathrm{d}z')^2 = \mathrm{d}z^2
\end{aligned}$$

Plugging into equation (A0.1) gives

$$c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \overset{(\mathrm{A0-1})}{=} c^2\,\mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2$$
$$= c^2B^2\,\mathrm{d}t^2 + 2c^2BC\,\mathrm{d}t\,\mathrm{d}x + c^2C^2\,\mathrm{d}x^2 - A^2\,\mathrm{d}x^2 + 2\,A^2v\,\mathrm{d}t\,\mathrm{d}x - A^2v^2\,\mathrm{d}t^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

Comparing coefficients of $\mathrm{d}t^2$, $dt\,dx$, and $\mathrm{d}x^2$ yields

$$c^2B^2 - A^2v^2 = c^2 \quad\Leftrightarrow\quad c^2B^2 = A^2v^2 + c^2 \tag{1}$$
$$c^2BC + A^2v = 0 \quad\Leftrightarrow\quad c^2BC = -A^2v \tag{2}$$
$$c^2C^2 - A^2 = -1 \quad\Leftrightarrow\quad c^2C^2 = A^2 - 1 \tag{3}$$

Multiplying equation (1) by (3), and squaring both sides of (2), is solvable, but introduces extraneous solutions that must be discarded:

$c^4 B^2 C^2 = A^4 v^2 + A^2 c^2 - A^2 v^2 - c^2$ (LHS1 LHS3 = RHS1 RHS3)

$c^4 B^2 C^2 = A^4 v^2$ $\left(\text{LHS2}^2 = \text{RHS2}^2\right)$

Subtracting yields:

$$0 = A^2 c^2 - A^2 v^2 - c^2 \iff A^2 (c^2 - v^2) = c^2$$

$$\iff A^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}} \iff A = \frac{\pm 1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Plugging A into (1):

$$B^2 = \frac{A^2 v^2 + c^2}{c^2} \iff B = \frac{\pm 1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Plugging A into (3):

$$C^2 = \frac{A^2 - 1}{c^2} \iff C = \frac{\pm \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In equations (1) and (3), both “+“ and “-“ work for A, B, and C. In equation (2), A can still have either sign, so we choose “+“. Constants B and C must have opposite signs in equation (2), so we are free to choose “+“ for B and “-“ for C. A solution is thus

$$A = B = \frac{\pm 1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad C = \frac{-\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ✓ \tag{A.10}$$

Definition $$\boxed{\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c}{\sqrt{c^2 - v^2}}}\ . \tag{A.11}$$

We can choose $A = B = \gamma$ and $C = -\frac{\gamma v}{c^2}$, and the boost equations (A.9) becomes:

$$t' = \gamma \left(t - \frac{v}{c^2} x\right) \quad \text{or} \quad ct' = \gamma\, ct - \frac{\gamma v}{c} x$$

$$x' = \gamma (x - v t)$$

$$y' = y$$

$$z' = z\,. \tag{A.12}$$

In matrix form, we write this as

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma\,v}{c} & 0 & 0 \\ -\frac{\gamma\,v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma\,c\,t - x\,\frac{\gamma\,v}{c} \\ -\gamma\,v\,t + x\,\gamma \\ y \\ z \end{pmatrix}. \tag{A.13}$$

Observation 1 When $x = 0$, the second equation gives $x' = -\gamma\,v\,t$. Then the first equation gives $t' = \gamma\,t$, so that $x' = -v\,t'$. This shows that K moves with speed $-v$ with respect to K′, as expected. That is,

$$\frac{\mathrm{dx}'}{\mathrm{d}t'} = -\,\mathrm{v} \tag{A1-5}$$

Observation 2 $\gamma \geq 1$. From the next observation, this means that $\Delta t > \Delta t'$.

Observation 3 If we take the dt' derivative of both sides of equation 1 of (A.12) we get

$$1 = \frac{\mathrm{d}t'}{\mathrm{d}t'} = \gamma\left(\frac{\mathrm{d}t}{\mathrm{d}t'} - \frac{v}{c^2}\frac{\mathrm{dx}}{\mathrm{d}t'}\right) = \gamma\frac{\mathrm{d}t}{\mathrm{d}t'}\left[1 - \left(\frac{v}{c^2}\frac{\mathrm{dx}}{\mathrm{d}t'}\right)\frac{\mathrm{d}t'}{\mathrm{d}t}\right] = \gamma\frac{\mathrm{d}t}{\mathrm{d}t'}\left(1 - \frac{v}{c^2}\frac{\mathrm{dx}}{\mathrm{d}t}\right)$$

$$\overset{(A1-4)}{=} \gamma\frac{\mathrm{d}t}{\mathrm{d}t'}\left(1 - \frac{v^2}{c^2}\right) = \gamma\frac{\mathrm{d}t}{\mathrm{d}t'}\frac{1}{\gamma^2} = \frac{1}{\gamma}\frac{\mathrm{d}t}{\mathrm{d}t'}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}t}{\mathrm{d}t'} = \gamma} \tag{A1-6}$$

The book’s development of spacetime boost is limited to the x-direction. The approach generalizes to a boost in an arbitrary space direction, developed now.

Definition The spacetime boost (A.12), above, is called the **1+1 homogeneous Lorentz transformation with speed** v. The **3+1 Lorentz homogeneous transformation with velocity** $\boldsymbol{v}$ is a transformation where motion is in an arbitrary space direction. It has the equation

$$\begin{aligned} t' &= B\,t + \boldsymbol{C} \cdot \boldsymbol{x} \\ \boldsymbol{x}' &= A\,(\boldsymbol{x} - \boldsymbol{v}\,t) \end{aligned} \tag{A1-7}$$

where we denote coordinates $x^\mu = (ct, x, y, z) = (x^0, \boldsymbol{x})$. We use boldface to denote $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{v} = (v_x, v_y, v_z)$. Note that A and B must be scalars and $\boldsymbol{C}$ must be a 3-vector in order that the RHS of the first equation be a scalar and the RHS of the second equation be a space 3-vector.

Observe that when $\boldsymbol{x}' = 0$, then $\boldsymbol{x} = \boldsymbol{v}\, t$, which shows that the center O′, and hence all of frame K′, moves with 3-space velocity $\boldsymbol{v}$. When $t = t' = 0$, the first equation shows that $\boldsymbol{x} = 0$ and the second equation shows that $\boldsymbol{x}' = 0$. Thus, Equations (A1-7) describe an inertial frame K′ having velocity $\boldsymbol{v}$ whose center O′ coincides with center O at time zero, confirming that equations (A.17) describe a *homogeneous* Lorentz transformation.

As in the 1+1 case, we can solve for A, B, and $\boldsymbol{C}$ by using the equation

$$c^2\,(\mathrm{d}t')^2 - (\mathrm{d}\boldsymbol{x}')^2 = c^2\,(\mathrm{d}t)^2 - (\mathrm{d}\boldsymbol{x})^2. \tag{A1-8}$$

$$\mathrm{d}t' = B\,\mathrm{d}t + \boldsymbol{C}\bullet\mathrm{d}\boldsymbol{x} \qquad (\mathrm{d}t')^2 = B^2\,\mathrm{d}t^2 + 2B\boldsymbol{C}\,\mathrm{d}t\bullet\mathrm{d}\boldsymbol{x} + (\boldsymbol{C}\bullet\mathrm{d}\boldsymbol{x})^2$$

$$\mathrm{d}\boldsymbol{x}' = A\,(\mathrm{d}\boldsymbol{x} - \boldsymbol{v}\,\mathrm{d}t) \qquad (\mathrm{d}x')^2 = A^2\left(\mathrm{d}x^2 - 2\boldsymbol{v}\,\mathrm{d}t\bullet\mathrm{d}\boldsymbol{x} + v^2\,\mathrm{d}t^2\right)$$

Plugging into equation (A0.1) gives

$$c^2\,(\mathrm{d}t)^2 - (\mathrm{d}x)^2 \overset{(A0-1)}{=} c^2\,(\mathrm{d}t')^2 - (\mathrm{d}x')^2$$
$$= c^2B^2\,\mathrm{d}t^2 + 2c^2B\boldsymbol{C}\bullet\mathrm{d}t\,\mathrm{d}\boldsymbol{x} + c^2\,(\boldsymbol{C}\bullet\mathrm{d}\boldsymbol{x})^2 - A^2\left(\mathrm{d}x^2 - 2\boldsymbol{v}\bullet\mathrm{d}t\,\mathrm{d}\boldsymbol{x} + v^2\,\mathrm{d}t^2\right)$$

Comparing coefficients of $(\mathrm{d}t)^2$ and $(\bullet\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x})$ yields

$$c^2B^2 - A^2v^2 = c^2 \quad\Leftrightarrow\quad c^2B^2 = A^2v^2 + c^2 \tag{1}$$
$$c^2B\boldsymbol{C} + A^2v = 0 \quad\Leftrightarrow\quad c^2B\boldsymbol{C} = -\,A^2v \tag{2}$$

There is a 3rd equation, a vector equation, for the coefficients of $(\mathrm{d}x)^2$. It can be broken down into three pieces: $(\mathrm{d}x^1)^2$, $(\mathrm{d}x^2)^2$, and $(\mathrm{d}x^3)^2$, and is complicated to express. Nonetheless, in principle, this process generates 5 equations in 5 unknowns, the unknowns being A, B, and C^1, C^2, and C^3. In principle, we can generate the remaining three equations, and solve the system of equations. In practice, it is easier to guess the solution and then work backwards to check it. Modeled after the 1+1 solution, we guess that

$$A = B = \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{c}{\sqrt{c^2-v^2}} \quad\text{and}\quad \boldsymbol{C} = -\frac{\gamma}{c^2}\boldsymbol{v}\,. \tag{A1-9}$$

To confirm this solution, we show that equation (A1-8) holds.

Since points in the K′ frame move with velocity $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}$,

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\,\mathrm{d}t\,, \qquad \boldsymbol{v}\bullet\mathrm{d}\boldsymbol{x} = v^2\,\mathrm{d}t\,, \qquad \text{and} \qquad (\mathrm{d}\boldsymbol{x})^2 = \mathrm{d}\boldsymbol{x}\bullet\mathrm{d}\boldsymbol{x} = v^2\,\mathrm{d}t^2\,. \tag{A1-10}$$

Relative to the K′ frame, points $\boldsymbol{x}'$ do not move:

$$\frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} = 0\,. \qquad \text{(A1-11)}$$

By differentiating equations (A1-7) we get

$$\begin{aligned} \mathrm{d}t' &= B\,\mathrm{d}t + \boldsymbol{C}\cdot\mathrm{d}\boldsymbol{x} \\ \mathrm{d}\boldsymbol{x}' &= A\,[\mathrm{d}\boldsymbol{x} - \boldsymbol{v}\,\mathrm{d}t\,] \end{aligned} \qquad \text{(A1-12)}$$

$$\begin{aligned} \Rightarrow\quad (\mathrm{d}t')^2 &= B^2\,(\mathrm{d}t)^2 + 2\,B\,\boldsymbol{C}\cdot\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + (\boldsymbol{C}\cdot\mathrm{d}\boldsymbol{x})^2 \\ &\overset{\text{(A1-9)}}{=} \gamma^2\,(\mathrm{d}t)^2 + 2\gamma\,(-\frac{\gamma}{c^2}\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + (-\frac{\gamma}{c^2}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x})^2 \\ &= \gamma^2\,(\mathrm{d}t)^2 - \frac{2\,\gamma^2}{c^2}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{\gamma^2}{c^4}(\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x})^2 \\ &\overset{\text{(A1-10)}}{=} \gamma^2\,(\mathrm{d}t)^2 - \frac{2\,\gamma^2}{c^2}v^2\,\mathrm{d}t^2 + \frac{\gamma^2}{c^4}v^4\,\mathrm{d}t^2 \\ &= \gamma^2\,(\mathrm{d}t)^2\,(1 - 2\frac{v^2}{c^2} + \frac{v^4}{c^4}) = \gamma^2\,(\mathrm{d}t)^2\,(1 - \frac{v^2}{c^2})^2 \end{aligned}$$

$$\Rightarrow\quad \mathrm{d}t' = \gamma\,\mathrm{d}t\,(1 - \frac{v^2}{c^2}) \overset{(A.11)}{=} \gamma\,\mathrm{d}t\,\frac{1}{\gamma^2} = \frac{\mathrm{d}t}{\gamma}$$

$$\Rightarrow\quad \boxed{\frac{\mathrm{d}t}{\mathrm{d}t'} = \gamma} \qquad \text{(A1-13)}$$

As expected, this agrees with the 1+1 case, equation (A1-6).

Also, from (A1-12), we get

$$(\mathrm{d}\boldsymbol{x}')^2 = 0\,: \qquad \text{(A1-14)}$$

$$\begin{aligned} (\mathrm{d}\boldsymbol{x}')^2 &= A^2\,[\,(\mathrm{d}\boldsymbol{x})^2 - 2\,\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + v^2\,\mathrm{d}t^2\,] \\ &\overset{\text{(A1-10)}}{=} \gamma^2\,[\,v^2 - 2v^2 + v^2\,]\,\mathrm{d}t^2 = 0. \qquad ✓ \end{aligned}$$

Though perhaps a little strange at first, this is consistent with (A1-11) that $\frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} = 0$.

We are now ready to show our guesses for *A*, *B*, and ***C*** are correct, that equation (A1-8) holds:

$$c^2\,(\mathrm{d}t')^2 - (\mathrm{d}\boldsymbol{x}')^2 \overset{(\text{A1-13, A1-14})}{=} c^2\,\frac{(\mathrm{d}t)^2}{\gamma^2} - 0 = c^2\,\frac{c^2 - v^2}{c^2}\,(\mathrm{d}t)^2 = (c^2 - v^2)\,(\mathrm{d}t)^2$$

$$= c^2\,(\mathrm{d}t)^2 - v^2\,\mathrm{d}t^2 \overset{(\text{A1-10})}{=} c^2\,(\mathrm{d}t)^2 - (\mathrm{d}\boldsymbol{x})^2. \qquad ✓$$

Having established values for *A*, *B*, and ***C*** in (A1-9), we plug these values into the 3+1 Lorentz transformation (A1-7) to generate

$$t' = \gamma\,\left(t - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c^2}\right) \quad \text{or} \quad ct' = \gamma\,\left(ct - \frac{\boldsymbol{v}}{c} \bullet \boldsymbol{x}\right)$$

$$\boldsymbol{x}' = \gamma\,(\boldsymbol{x} - \boldsymbol{v}\,t) \tag{A1-15}$$

and, in matrix form,

$$\mathrm{x}^{\mu'} = \tilde{\Lambda}^{\mu'}_{\ \nu}\, x^{\nu}: \tag{A1-16}$$

$$\begin{pmatrix} c\,t' \\ \boldsymbol{x}' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\boldsymbol{v}}{c} & \mathrm{I} \end{pmatrix} \begin{pmatrix} c\,t \\ \boldsymbol{x} \end{pmatrix} = \gamma \begin{pmatrix} c\,t - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c} \\ -\boldsymbol{v}\,t + \boldsymbol{x} \end{pmatrix}, \tag{A1-17}$$

where $\tilde{\Lambda}^{\mu'}_{\ \nu}$ is the *homogeneous* Lorentz matrix, $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_v \\ v_z \end{pmatrix}$, $\boldsymbol{v}^T = (v_x\ \ v_y\ \ v_z)$, and I is the 3x3 identity matrix. We use the transpose of $\boldsymbol{v}$ in the (1,2) element of the matrix because the row vector $\boldsymbol{v}^{\mathrm{T}}$ times the column vector $\boldsymbol{x}$ yields the dot product $\boldsymbol{v} \bullet \boldsymbol{x}$. This completes the development of the (3+1) boost. ■

As in the 1+1 case, we observe that when $\boldsymbol{x} = 0$, the 2nd equation gives $\boldsymbol{x}' = \gamma\,\boldsymbol{v}\,t$. Then the first equation gives $t' = \gamma\,t$, so that $\boldsymbol{x}' = -\,\boldsymbol{v}\,t'$. This confirms that K moves with velocity $-\,\boldsymbol{v}$ with respect to K′; i.e., $\boldsymbol{v}' = -\,\boldsymbol{v}$.

Using tilde (~) for the homogeneous case, the matrix

$$\tilde{\Lambda} \equiv \gamma \begin{pmatrix} 1 & -\frac{\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\boldsymbol{v}}{c} & \mathrm{I} \end{pmatrix} \tag{A1-18}$$

is the general form for the homogeneous Lorentz transformation matrix.

This can also be expressed in expanded form:

$$\tilde{\Lambda} = \gamma \begin{pmatrix} 1 & -\frac{v_x}{c} & -\frac{v_y}{c} & -\frac{v_z}{c} \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ -\frac{v_z}{c} & 0 & 0 & 1 \end{pmatrix}. \qquad \text{(A1-19)}$$

The general (not necessarily homogeneous) Lorentz transformation matrix is given by Frobenius at StackExchange.com (search for Transformation of 4−velocity) as

$$\boxed{\Lambda = \begin{pmatrix} \gamma & -\frac{\gamma \boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\gamma \boldsymbol{v}}{c} & \mathrm{I} + \frac{\gamma^2}{c^2(\gamma+1)} \mathbf{v}\mathbf{v}^{\mathrm{T}} \end{pmatrix}} \qquad \text{(A1-20)}$$

which equals

$$\Lambda = \begin{pmatrix} \gamma & -\frac{\gamma v_x}{c} & -\frac{\gamma v_y}{c} & -\frac{\gamma v_z}{c} \\ -\frac{\gamma v_x}{c} & 1 + (\gamma - 1)\frac{v_x{}^2}{v^2} & (\gamma - 1)\frac{v_x v_y}{v^2} & (\gamma - 1)\frac{v_x v_z}{v^2} \\ -\frac{\gamma v_y}{c} & (\gamma - 1)\frac{v_y v_x}{v^2} & 1 + (\gamma - 1)\frac{v_y{}^2}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} \\ -\frac{\gamma v_z}{c} & (\gamma - 1)\frac{v_z v_x}{v^2} & (\gamma - 1)\frac{v_z v_y}{v^2} & 1 + (\gamma - 1)\frac{v_z{}^2}{v^2} \end{pmatrix} \qquad \text{(A1-21)}$$

from Wikipedia (search for Lorentz transformation matrix). We easily see that these matrices are equal by using $\frac{\gamma-1}{v^2} = \frac{\gamma^2}{c^2(\gamma+1)}$ from equation (A1-23), below, with the understanding that a column vector $\boldsymbol{v}$ times a row vector $\boldsymbol{v}^{\mathrm{T}}$ is

$$\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} (\, v_x \;\; v_y \;\; v_z \,) = \begin{pmatrix} {v_x}^2 & v_x\, v_y & v_x\, v_z \\ v_y\, v_x & {v_y}^2 & v_y\, v_z \\ v_z\, v_x & v_z\, v_y & {v_z}^2 \end{pmatrix}.$$

The usual method for generating the matrix Λ involves a somewhat complex process of combining a sequence of transformations that includes a 3-space rotation, a spatial translation (i.e., an offset), and a boost in an arbitrary direction. However, we can derive it quite easily from the homogeneous Lorentz matrix $\tilde{\Lambda}$.

Consider the equations $x^{\mu'} = \Lambda^{\mu'}_{\nu}\, x^{\nu}$:

$$t' = \gamma \left(t - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c^2}\right) \quad \text{or} \quad ct' = \gamma \left(ct - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c}\right)$$

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma+1)}\, (\boldsymbol{v} \bullet \boldsymbol{x})\, \boldsymbol{v} - \frac{\gamma\, \boldsymbol{v}}{c}\, c\, t \tag{A1-22}$$

Claim these equations equal equations (A1-16) plus an offset, a^{μ}:

The expressions for t' are identical (and the offset time component is 0). ✓

To see that the expressions for $\boldsymbol{x}'$ only differ by an offset, we first develop the equation for d$\boldsymbol{x}'$:

$$\gamma = 1 + \frac{v^2}{c^2}\, \frac{\gamma^2}{\gamma+1} \; : \tag{A1-23}$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2} \quad \Leftrightarrow \quad c^2\gamma^2 - v^2\gamma^2 = c^2$$

$$\Leftrightarrow \quad c^2\gamma^2 = c^2 + v^2\gamma^2 \tag{A1-24}$$

$$\Rightarrow \quad \gamma\, c^2(\gamma + 1) = c^2\gamma + c^2\gamma^2 \overset{(A1-16)}{=} c^2\gamma + c^2 + v^2\gamma^2 = c^2(\gamma + 1) + v^2\gamma^2.$$

Dividing both sides by $c^2(\gamma + 1)$ gives $\gamma = 1 + \frac{v^2}{c^2}\, \frac{\gamma^2}{\gamma+1}$ ✓

$$\gamma\, \mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma+1)}\, (\boldsymbol{v} \bullet d\boldsymbol{x})\, \boldsymbol{v} \, : \tag{A1-25}$$

$$\gamma\, \mathrm{d}\boldsymbol{x} \overset{(A1-23)}{=} \mathrm{d}\boldsymbol{x} + \frac{v^2}{c^2}\, \frac{\gamma^2}{\gamma+1}\, \mathrm{d}\boldsymbol{x} \overset{(A1-10)}{=} \mathrm{d}\boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma+1)}\, (v^2\, \mathrm{d}t)\, \mathbf{v}$$

$$\overset{(A1-10)}{=} \mathrm{d}\boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma+1)}\, (\boldsymbol{v} \bullet \mathrm{d}\boldsymbol{x})\, \boldsymbol{v} \qquad ✓$$

$$\mathrm{d}\boldsymbol{x}' \overset{\text{(A1–15)}}{=} \gamma\, \mathrm{d}\boldsymbol{x} - \gamma\, \boldsymbol{v}\, \mathrm{d}t \overset{\text{(A1–25)}}{=} \mathrm{d}\boldsymbol{x} + \frac{\gamma^2}{c^2(\gamma+1)}\, (\boldsymbol{v} \bullet \mathrm{d}\boldsymbol{x})\, \boldsymbol{v} - \frac{\gamma\, \boldsymbol{v}}{c}\, c\, \mathrm{d}t \qquad ✓$$

When we integrate the right two terms, we get

$$\boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma+1)}\, (\boldsymbol{v} \bullet \boldsymbol{x})\, \boldsymbol{v} - \frac{\gamma\, \boldsymbol{v}}{c}\, c\, t = \gamma\, (\boldsymbol{x} - \boldsymbol{v}\, t) + \boldsymbol{a} \qquad \text{(A1-26)}$$

where $\boldsymbol{a}$ is the spatial offset.

So, the spacetime offset, composed of the time and space offsets, is $a^\mu = \begin{pmatrix} 0 \\ \boldsymbol{a} \end{pmatrix}$.

Putting this altogether, we confirm the claim, essentially by combining equations (A1-1) and (A.7):

$$\boxed{x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^\nu = \tilde{\Lambda}^{\mu'}_{\ \nu}\, x^\nu + a^\mu} : \qquad \text{(A1-27)}$$

$$x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^\nu = \begin{pmatrix} \gamma & -\frac{\gamma\, \boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\gamma\, \boldsymbol{v}}{c} & \mathrm{I} + \frac{\gamma^2}{c^2\,(\gamma^2+1)}\, \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} c\, t \\ \boldsymbol{x} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma \left(ct - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c}\right) \\ -\gamma\, \boldsymbol{v}\, t + \boldsymbol{x} + \frac{\gamma^2}{c^2\,(\gamma^2+1)}\, (\boldsymbol{v} \bullet \boldsymbol{x})\, \boldsymbol{v} \end{pmatrix} \overset{\text{(A1–26)}}{=} \begin{pmatrix} \gamma \left(ct - \frac{\boldsymbol{v} \bullet \boldsymbol{x}}{c}\right) \\ \gamma\, (\boldsymbol{x} - \boldsymbol{v}\, t) + \boldsymbol{a} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} 1 & -\frac{\boldsymbol{v}^{\mathrm{T}}}{c} \\ -\frac{\boldsymbol{v}}{c} & \mathrm{I} \end{pmatrix} \begin{pmatrix} c\, t \\ \boldsymbol{x} \end{pmatrix} + \begin{pmatrix} 0 \\ \boldsymbol{a} \end{pmatrix} = \tilde{\Lambda}^{\mu'}_{\ \nu}\, x^\nu + a^\mu \qquad ✓$$

That is, the general Lorentz transformation can be expressed as the homogeneous Lorentz transformation plus an offset. I have never seen the homogeneous matrix referenced. However, it is easier to use and can be appropriate in many situations. For example, in Section A.7 I use it to derive the Doppler frequency shift, which is not affected by translation to an offset, and thus only needs the simpler homogeneous transformation.