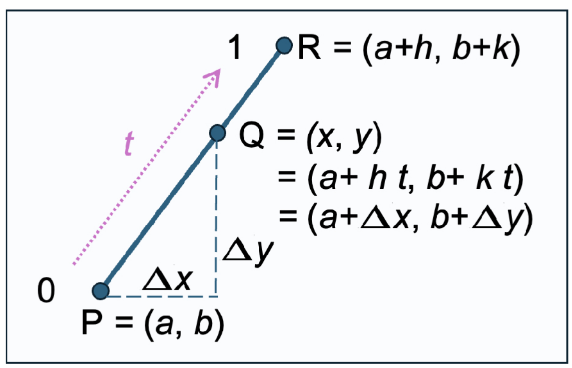
0.2 Taylor Series of Functions of Several Variables

Taylor series in several variables are important for generating first and second order approximations. Tensor notation, which is developed later, is used to express functions of several variables, but we first develop multivariate Taylor Series using familiar notation. Only at the end do we convert it to tensor notation. The reader can return to this last part after tensor notation has been developed.

Let $f(x,y)$ have partials in an open region U containing points P = (a,b) and R = $(a+h,\ b+k)$. We parameterize the line segment $\overline{\text{PR}}$ with t:

$$\left.\begin{aligned} x &= a + t\,h \\ y &= b + t\,k \end{aligned}\right\}\ t \in [0,\ 1] \tag{0.2-1}$$



$$\Rightarrow \quad \left.\begin{aligned} \frac{\text{dx}}{\text{dt}} &= \frac{d}{\text{dt}}(a+t\,h) = h \\ \frac{\text{dy}}{\text{dt}} &= \frac{d}{\text{dt}}(b+t\,k) = k \end{aligned}\right\} \tag{0.2-2}$$

Let Q = (x,y) be a point on the line segment. Define

$$\Delta x \equiv x - a \qquad \text{and} \qquad \Delta y = y - b \tag{0.2-3}$$

Then

$$\Delta x \overset{(0.2-1)}{=} t\,h \qquad \text{and} \qquad \Delta y = t\,k \tag{0.2–4}$$

Define

$$F(t) \equiv f(a + t\,h,\ b + t\,k) \tag{0.2-5}$$

Then

$$F(t) \overset{(0.2-1)}{=} f(x,y) \qquad \text{and} \qquad F(0) \overset{(0.2-5)}{=} f(a,b) \tag{0.2-6}$$

The standard Taylor series for F is

$$F(t) = F(t_0) + F'(t_0)\,(t - t_0) + \frac{1}{2!}\,F''(t_0)\,(t - t_0)^2 + \ldots$$

In this case, $t_0 = 0$, so

$$F(t) = F(0) + F'(0)\,t + \frac{1}{2!}F''(0)\,t^2 + \ldots \tag{0.2-7}$$

$$\Rightarrow \quad F'(0) = \frac{\mathrm{d}F(0)}{\mathrm{d}t} = \frac{\partial F(0)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F(0)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \overset{(0.2-6,\,0.2-2)}{=} \frac{\partial f(a,b)}{\partial x}h + \frac{\partial f(a,b)}{\partial y}k \tag{0.2-8}$$

$$\Rightarrow \quad F'(0)\,t = \frac{\partial f(a,b)}{\partial x}\Delta x + \frac{\partial f(a,b)}{\partial y}\Delta y \tag{0.2-9}$$

and

$$F''(0) = \frac{\mathrm{d}F'(0)}{\mathrm{d}t} = \frac{\partial F'(0)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F'(0)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\overset{(0.2-8,\,0.2-2)}{=} \frac{\partial^2 f(a,b)}{\partial x^2}h^2 + \frac{\partial f(a,b)}{\partial x\,\partial y}hk + \frac{\partial^2 f(a,b)}{\partial x\,\partial y}h\mathrm{k} + \frac{\partial^2 f(a,b)}{\partial y^2}k^2$$

$$= \frac{\partial^2 f(a,b)}{\partial x^2}h + 2\frac{\partial f(a,b)}{\partial x\,\partial y}k + \frac{\partial^2 f(a,b)}{\partial y^2}k \tag{0.2-10}$$

$$\Rightarrow \quad F''(0)\,t^2 = \frac{\partial^2 f(a,b)}{\partial x^2}h^2 t^2 + 2\frac{\partial f(a,b)}{\partial x\,\partial y}hkt^2 + \frac{\partial^2 f(a,b)}{\partial y^2}k^2 t^2$$

$$= \frac{\partial^2 f(a,b)}{\partial x^2}\,(\Delta x)^2 + 2\frac{\partial f(a,b)}{\partial x\,\partial y}\,\Delta x\,\Delta y + \frac{\partial^2 f(a,b)}{\partial y^2}\,(\Delta y)^2 \tag{0.2-11}$$

Plugging equations (0.2-6), (0.2-9), and (0.2-11) for $F(0)$, $F'(0)\,t$, and $F''(0)\,t^2$, respectively, into equation (0.2-7) yields the **Taylor series for a function of 2 variables**:

$$f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}\Delta x + \frac{\partial f(a,b)}{\partial y}\Delta y$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f(a,b)}{\partial x^2}\,(\Delta x)^2 + 2\frac{\partial f(a,b)}{\partial x\,\partial y}\,\Delta x\,\Delta y + \frac{\partial^2 f(a,b)}{\partial y^2}\,(\Delta y)^2\right] + \ldots \tag{0.2-12}$$

Equation (0.2-12) easily extends to $f(x,y,z, \ldots, w)$. It is simpler to express if we use index notation:

$F(t) = f(x,y,z, \ldots, w)$ as $f(x^1, x^2, x^3, \ldots, x^N)$

$F(0) = f(a, b, c, \ldots, d) = f(x_0^1, x_0^2, x_0^3, \ldots, x_0^N)$

$\Delta x^i = x^i - x_0^i$ (for example, $\Delta x^2 = x^2 - x_0^2 = y - b = \Delta y$)

Then the Taylor series generalizes to

$$f(x^1, x^2, x^3, ..., x^N) = f(x_0^1, x_0^2, x_0^3, ..., x_0^N) + \sum_i \frac{\partial f(x_0^1, x_0^2, x_0^3, ..., x_0^N)}{\partial x^i} \Delta x^i$$

$$+ \frac{1}{2!}\left[\sum_i \sum_j \frac{\partial^2 f(x_0^1, x_0^2, x_0^3, ..., x_0^N)}{\partial x^i\, \partial x^j} \Delta x^i \Delta x^j\right] + ... \qquad (0.2\text{-}13)$$

In tensor notation (which will be developed later), we write $f(x^a) \equiv f(x^1, x^2, x^3, ..., x^N)$, $f(x_0^a) \equiv f(x_0^1, x_0^2, x_0^3, ..., x_0^N)$, and $\xi^a \equiv \Delta x^a$, and this shortens to

$$\boxed{f(x^a) = f(x_0^a) + \frac{\partial f(x_0^a)}{\partial x^b}\, \xi^b + \frac{1}{2!}\, \frac{\partial^2 f(x_0^a)}{\partial x^b\, \partial x^c}\, \xi^b\, \xi^c + ...} \qquad (0.2\text{-}14)$$

because when an index like “b” (or “c”) appears in both numerator and denominator, it means to sum over “b” (or double sum over “b” and “c”). This is called the Einstein summation convention.