Exercise 1.9.2. Length of a curve γ in a manifold M is defined in eq (1.82) as

$L = \int_A^B \sqrt{|g_{ab}\dot{x}^a \dot{x}^b|}\, dt$ where $\gamma = \{x^a(t) : A \leq t \leq B\}$.

show that length is coord indep and parameter indep

a) Let $x^{a'}$ be a primed coord sys for M.

Then $\dot{x}^{a'} = X^{a'}_c \dot{x}^c$, $\dot{x}^{b'} = X^{b'}_d \dot{x}^d$, and $g_{a'b'} = X^a_{a'} X^b_{b'} g_{ab}$.

Recall that $X^a_{a'} X^{a'}_c = \delta^a_c$ and $X^b_{b'} X^{b'}_d = \delta^b_d$.

So $g_{a'b'}\dot{x}^{a'}\dot{x}^{b'} = X^a_{a'} X^b_{b'} g_{ab} X^{a'}_c \dot{x}^c X^{b'}_d \dot{x}^d = \delta^a_c \delta^b_d g_{ab} \dot{x}^c \dot{x}^d = g_{ab}\dot{x}^a \dot{x}^b$ ✓

b) If γ is also parameterized by u: $\gamma = \{x^a(u) : C \leq u \leq D\}$

We assume that γ is a simple curve (no knots) so that u is either an increasing fn or a decreasing fn. wlog we assume t is an increasing fn. If u is increasing,

we wish to show $\int_A^B \sqrt{|g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}|}\, dt = \int_C^D \sqrt{|g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}|}\, du$

(If u is decreasing, then we integrate $\int_D^C$.)

In principle, $u = f(t)$. So $\frac{dx^a}{dt} = \frac{dx^a}{du}\frac{du}{dt}$ and $\frac{dx^b}{dt} = \frac{dx^b}{du}\frac{du}{dt}$

$\int_A^B \sqrt{|g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}|}\, dt = \int_A^B \sqrt{\left(\frac{du}{dt}\right)^2 |g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}|}\, dt = \int_C^D \sqrt{|g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}|}\, du$ ✓

Exercise 1.9.3. Let $g_{\mu\nu}$ be the Schwarzchild metric. Let $\lambda^\mu \equiv c\delta_0^\mu$, $\mu^\mu \equiv \delta_1^\mu$, and $\nu^\mu \equiv \lambda^\mu + c(1-\frac{2m}{r})\delta_1^\mu$. (1) Find the lengths of these vectors and (2) the angles between them. Are any of the vectors null? Are any pairs orthogonal?

Solution. Let $K = 1-\frac{2m}{r}$. In the Schwarzchild metric, $g_{00} = K$, $g_{11} = -\frac{1}{K}$, $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\theta$, the other $g_{\mu\nu} = 0$; $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$; $r > 0$, $\theta > 0$, $m \geq 0$, and $\frac{2m}{r} < 1$. So $0 < K \leq 1$. We can rewrite $\nu^\mu = c(\delta_0^\mu + K\delta_1^\mu)$.

(1) $L^2 \equiv |g_{\mu\nu}\lambda^\mu\lambda^\nu| = |g_{00}(\lambda^0)^2 + g_{11}(\lambda^1)^2|$

λ^μ: $L^2 = |g_{00}c^2(\delta_0^0)^2| = c^2K \Rightarrow L = c\sqrt{1-\frac{2m}{r}}$ ✓

μ^μ: $L^2 = |g_{11}(\delta_1^1)^2| = \frac{1}{K} \Rightarrow L = \frac{1}{\sqrt{1-\frac{2m}{r}}}$ ✓

ν^μ: $L^2 = c^2|g_{00}(\delta_0^0)^2 + g_{11}(K\delta_1^1)^2| = c^2|K - \frac{1}{K}K^2| = 0$ ✓ So ν^μ is null ✓

(2) $\cos\theta = \dfrac{g_{\mu\nu}\lambda^\mu\mu^\nu}{L_\lambda L_\mu}$ where L_λ = length of λ

$\lambda^\mu\mu^\mu$: $g_{\mu\nu}\lambda^\mu\mu^\nu = g_{\mu\nu}\delta_0^\mu\delta_1^\nu = g_{01} = 0 \Rightarrow \cos\theta = \frac{0}{L_\lambda L_\mu} = 0 \Rightarrow \theta = \frac{\pi}{2}$ ✓

So λ^μ is orthogonal to μ^μ ✓

Since $L_\nu = 0$, the angle between ν and either λ or μ is undefined ✓

ν is orthogonal to μ if $g_{\mu\nu}\lambda^\mu\nu^\nu = 0$. $\nu^0 = c(\delta_0^0 + K\delta_1^0) = c$; $\nu^1 = c(\delta_0^1 + K\delta_1^1) =$

$\lambda^\mu\nu^\mu$: $g_{\mu\nu}\lambda^\mu\nu^\nu = g_{00}\lambda^0\nu^0 + g_{11}\lambda^1\nu^1 = K(c)(c) - \frac{1}{K}(0)(cK) = Kc^2 \neq 0$

$\mu^\mu\nu^\mu$: $g_{\mu\nu}\mu^\mu\nu^\nu = g_{11}\mu^1\nu^1 = -\frac{1}{K}(1)(cK) = -c \neq 0$

So ν^μ is not orthogonal to either λ^μ or μ^μ. ✓