Exercise 2.9.2 Obtain the geodesic equations (2.87) in three different ways:

(a) By using the Euler-Lagrange equations (and [$g^{\mu\nu}$] from Exercise 1.

(b) By extracting [$g^{\mu\nu}$] from the line element (2.86), calculating the $\Gamma^{\mu}_{\nu\sigma}$ from $g^{\mu\nu}$ and $g_{\mu\nu}$ using equation (2.13), and then calculating $\ddot{x}^{\mu}$ using equation (2.71).

(c) By substituting for T, X, Y, Z in $\ddot{T} = \ddot{X} = \ddot{Y} = \ddot{Z} = 0$, using equations (2.85)

Solution. Let K be an inertial (non-rotating) system with coordinates (T,X,Y,Z) and line element

$$c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2 \tag{2.84}$$

Denote $X^0 = cT$, $X^1 = X$, $X^2 = Y$, $X^3 = Z$.

Let K′ be a rotating system with coordinates (t,x,y,z), defined by

$$\begin{aligned} t &= T & T &= t \\ x &= X\cos\omega t + Y\sin\omega t & X &= x\cos\omega t - y\sin\omega t \\ y &= X\sin\omega t + Y\cos\omega t & Y &= x\sin\omega t + y\cos\omega t \\ z &= Z & Z &= z \end{aligned} \tag{2.85}$$

Denote $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and $\dot{x}^0 = c\,\dot{t}$, $\dot{x}^1 = \dot{x}$, $\dot{x}^2 = \dot{y}$, $\dot{x}^3 = \dot{z}$, $\ddot{x}^1 = \ddot{x}$, $\ddot{x}^2 = \ddot{y}$, $\ddot{x}^3 = \ddot{z}$, where dot represents differentiation by τ. The spacetime geodesic equations of motion are

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\nu\sigma}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} = 0 \tag{2.71}$$

We are to show that for a free particle with mass in the rotating frame K′, these equations may be expressed as

$$\begin{aligned} &\ddot{t} = 0 \\ &\ddot{x} - \omega^2 x\,\dot{t}^{\,2} - 2\,\omega\,\dot{y}\,\dot{t} = 0 \\ &\ddot{y} - \omega^2 y\,\dot{t}^{\,2} + 2\,\omega\,\dot{x}\,\dot{t} = 0 \\ &\ddot{z} = 0 \end{aligned} \tag{2.87}$$

Part (a): The Euler-Lagrange equations (2.17) are $\dfrac{\mathrm{d}}{\mathrm{d}\tau}\dfrac{\partial L}{\partial \dot{x}^{\sigma}} - \dfrac{\partial L}{\partial x^{\sigma}} = 0$ where the Lagrangian is $L \overset{(2.1.d)}{=} \dfrac{1}{2}\, g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}$ and we regard x^{μ} and $\dot{x}^{\mu}$ as independent variables. Keep in mind that $g_{\mu\nu}$ is a function of x^{μ}.

First,

$$\dot{x}^0 = c\,\dot{t} = \frac{c\,\mathrm{dt}}{\mathrm{d}\tau} \overset{(A.6)}{=} \frac{c}{\gamma}\text{, a constant } \Rightarrow \ddot{x}^0 = c\,\ddot{t} = 0 \;\Rightarrow\; \ddot{t} = 0 \quad ✓$$

In terms of K′coordinates, the line element was computed in Exercise 2.9.1 as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\,\mathrm{d}\tau^2$$
$$= [c^2 - \omega^2(x^2 + y^2)]\,\mathrm{d}t^2 + 2\omega y\,\mathrm{d}x\,\mathrm{d}t - 2\omega x\,\mathrm{d}y\,\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2\,. \qquad (2.86)$$

Dividing by c^2 yields

$$\mathrm{d}\tau^2 = [1 - \frac{(\mathrm{x}^2+\mathrm{y}^2)\,\omega^2}{\mathrm{c}^2}]\,\mathrm{d}t^2 + 2\frac{\mathrm{y}\,\omega}{\mathrm{c}}\,\mathrm{d}x\,\mathrm{d}t - 2\frac{\mathrm{x}\,\omega}{\mathrm{c}}\,\mathrm{d}y\,\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2\,,$$

from which we can extract

$$g_{00} = \frac{\mathrm{c}^2 - (\mathrm{x}^2+\mathrm{y}^2)\,\omega^2}{\mathrm{c}^2},\; g_{11} = g_{22} = g_{33} = -1,\; g_{01} = g_{10} = \omega\,y,\;\; g_{02} = g_{20} = -x\,\omega,$$

and all others are zero.

Thus, the Lagrangian, L, is

$$L = \frac{1}{2}\,g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu$$
$$= \frac{1}{2}\{\frac{c^2-(x^2+y^2)\,\omega^2}{c^2}\;c^2\,\dot{t}^2 + 2\,y\,\omega\,\dot{x}\,\dot{t} - 2\,x\,\omega\,\dot{y}\,\dot{t} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\}$$
$$= \frac{1}{2}\,[c^2 - (x^2 + y^2)\,\omega^2]\,\dot{t}^2 + \omega\,(y\,\dot{x} - x\,\dot{y})\,\dot{t} - \frac{1}{2}(\,\dot{x}^2 - \dot{y}^2 - \dot{z}^2\,)$$

So,

$$\frac{\partial L}{\partial x^1} = \frac{\partial L}{\partial x} = -\,\omega^2\,x\,\dot{t}^2 - \omega\,\dot{y}\,\dot{t}$$
$$\frac{\partial L}{\partial x^2} = \frac{\partial L}{\partial y} = -\,\omega^2\,y\,\dot{t}^2 + \omega\,\dot{x}\,\dot{t}$$
$$\frac{\partial L}{\partial x^3} = \frac{\partial L}{\partial z} = 0$$

$$\frac{\partial L}{\partial \dot{x}^1} = \frac{\partial L}{\partial \dot{x}} = \omega\,y\,\dot{t} - \dot{x}$$
$$\frac{\partial L}{\partial \dot{x}^2} = \frac{\partial L}{\partial \dot{y}} = -\omega\,x\,\dot{t} - \dot{y}$$

$$\frac{\partial L}{\partial \dot{x}^3} = \frac{\partial L}{\partial \dot{z}} = -\,\dot{z}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^1} = \omega\,\dot{y}\,\dot{t} - \ddot{x}$$
$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^2} = -\omega\,\dot{x}\,\dot{t} - \ddot{y}$$
$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^3} = -\,\ddot{z}$$

and Euler-Lagrange equations $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\sigma}\right) - \frac{\partial L}{\partial x^\sigma} = 0$ become:

(1) $0 = \omega\,\dot{y}\,\dot{t} - \ddot{x} + \omega^2\,x\,\dot{t}^2 + \omega\,\dot{y}\,\dot{t} \;\Rightarrow\; \ddot{x} - \omega^2 x\,\dot{t}^{\,2} - 2\,\omega\,\dot{y}\,\dot{t} = 0$ ✓

(2) $0 = -\omega\,\dot{x}\,\dot{t} - \ddot{y} + \omega^2\,y\,\dot{t}^2 - \omega\,\dot{x}\,\dot{t} \;\Rightarrow\; \ddot{y} - \omega^2 y\,\dot{t}^{\,2} + 2\,\omega\,\dot{x}\,\dot{t} = 0$ ✓

(3) $0 = -\,\ddot{z}$ ✓

Part (b): In part (a) we found that

$$(g_{\mu\nu}) = \begin{pmatrix} 1 - \frac{(x^2+y^2)\,\omega^2}{c^2} & \frac{y\,\omega}{c} & -\frac{x\,\omega}{c} & 0 \\ \frac{y\,\omega}{c} & -1 & 0 & 0 \\ -\frac{x\,\omega}{c} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The contravariant metric tensors matrix is

$$(g^{\mu\nu}) = (g_{\mu\nu})^{-1} = \begin{pmatrix} 1 & \frac{y\,\omega}{c} & -\frac{x\,\omega}{c} & 0 \\ \frac{x\,\omega}{c} & -1 + \frac{y^2\,\omega^2}{c^2} & -\frac{x\,y\,\omega^2}{c^2} & 0 \\ -\frac{x\,\omega}{c} & -\frac{x\,y\,\omega^2}{c^2} & -1 + \frac{x^2\,\omega^2}{c^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Christoffel coefficients are computed from the covariant and contravariant metric tensors. There are 64 connection coefficients:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}\,g^{\mu\rho}\;(\partial_\nu\,g_{\rho\sigma} + \partial_\sigma\,g_{\nu\rho} - \partial_\rho\,g_{\nu\sigma}) = g^{\mu\delta}\,\Gamma_{\delta\nu\gamma}.$$

Using Mathematica, we find that the only non-zero ones are:

$$\Gamma^1_{00} = -\frac{x\,\omega^2}{c^2}, \quad \Gamma^1_{02} = \Gamma^1_{20} = -\frac{\omega}{c},$$
$$\Gamma^2_{00} = -\frac{y\,\omega^2}{c^2}, \quad \Gamma^2_{01} = \Gamma^2_{01} = \frac{\omega}{c}.$$

For reference, the non-zero coefficients are computed from $g^{\mu\delta}$ and the non-zero $\Gamma_{\delta\nu\gamma}$ coefficients, as shown below. Take care in the computations for $\Gamma_{\delta\nu\gamma}$ that terms like $\frac{\partial x}{\partial t}$ equal zero because the coordinates are considered to be independent of each other.

μ	δ	$\mathbf{g^{\mu\delta}}$
0	0	1
0	1	$\frac{y\,\omega}{c}$
0	2	$-\frac{x\,\omega}{c}$
0	3	0
1	0	$\frac{y\,\omega}{c}$
1	1	$-1+\frac{y^2\,\omega^2}{c^2}$
1	2	$-\frac{x\,y\,\omega^2}{c^2}$
1	3	0
2	0	$-\frac{x\,\omega}{c}$
2	1	$-\frac{x\,y\,\omega^2}{c^2}$
2	2	$-1+\frac{x^2\,\omega^2}{c^2}$
2	3	0
3	0	0
3	1	0
3	2	0
3	3	-1

&

δ	ν	γ	$\Gamma_{\delta\nu\gamma}$
0	0	1	$-\frac{x\,\omega^2}{c^2}$
0	0	2	$-\frac{y\,\omega^2}{c^2}$
0	1	0	$-\frac{x\,\omega^2}{c^2}$
0	2	0	$-\frac{y\,\omega^2}{c^2}$
1	0	0	$\frac{x\,\omega^2}{c^2}$
1	0	2	$\frac{\omega}{c}$
1	2	0	$\frac{\omega}{c}$
2	0	0	$\frac{y\,\omega^2}{c^2}$
2	0	1	$-\frac{\omega}{c}$
2	1	0	$-\frac{\omega}{c}$

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μ	ν	γ	$\Gamma^{\mu}_{\nu\gamma}$
1	0	0	$-\frac{x\,\omega^2}{c^2}$
1	0	2	$-\frac{\omega}{c}$
1	2	0	$-\frac{\omega}{c}$
2	0	0	$-\frac{y\,\omega^2}{c^2}$
2	0	1	$\frac{\omega}{c}$
2	1	0	$\frac{\omega}{c}$

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This yields the geodesic equations (2.87): $\ddot{x} + \Gamma^{\mu}_{\nu\gamma}\dot{x}^{\nu}\dot{x}^{\gamma} = 0$

$$0 = \ddot{x}^{\,0} + \Gamma^{0}_{\nu\ \sigma}\,\dot{x}^{\nu}\,\dot{x}^{\sigma} = c\,\ddot{t}^{\,0} + 0 = c\,\ddot{t} \Rightarrow\ 0 = \ddot{t}\ ✓$$

$$0 = \ddot{x}^1 + \Gamma^1_{\nu\sigma}\,\dot{x}^\nu \dot{x}^\sigma = \ddot{x}^1 + \Gamma^1_{00}\,(\dot{x}^0)^2 + 2\,\Gamma^1_{02}\,\dot{x}^0\,\dot{x}^2$$
$$= \ddot{x}^1 + \Gamma^1_{00}\,c^2\,\dot{t}^2 + 2\,\Gamma^1_{02}\,c\,\dot{y}\,\dot{t}$$
$$= \ddot{x} - \left(\frac{x\,\omega^2}{c^2}\right) c^2\,\dot{t}^2 + 2\left(-\frac{\omega}{c}\right) c\,\dot{y}\,\dot{t} = \ddot{x} - \omega^2 x\,\dot{t}^2 - 2\,\omega\,\dot{y}\,\dot{t} \quad ✓$$
$$0 = \ddot{x}^2 + \Gamma^2_{\nu\sigma}\,\dot{x}^\nu\,\dot{x}^\sigma = \ddot{x}^2 + \Gamma^2_{00}\,(\dot{x}^0)^2 + 2\,\Gamma^2_{01}\,\dot{x}^0\,\dot{x}^1$$
$$= \ddot{x}^2 + \Gamma^2_{00}\,c^2\,\dot{t}^2 + 2\,\Gamma^2_{01}\,c\,\dot{x}\,\dot{t}$$
$$= \ddot{y} - \frac{\omega^2}{c^2}\,y\,c^2\,\dot{t}^2 + 2\,\frac{\omega}{c}\,c\,\dot{x}\,\dot{t} = \ddot{y} - \omega^2 y\,\dot{t}^2 + 2\,\omega\,\dot{t}\,\dot{x} \quad ✓$$
$$0 = \ddot{x}^3 + \Gamma^3_{\nu\,\sigma}\,\dot{x}^\nu\,\dot{x}^\sigma = \ddot{x}^3 + 0 = \ddot{z} \quad ✓$$

Part (c): We can rewrite equation (2.84) by dividing both sides by $d\tau^2$:

$$c^2 \overset{(2.84)}{=} c^2\dot{T}^2 - \dot{X}^2 - \dot{Y}^2 - \dot{Z}^2.$$
$$\Rightarrow \quad 0 = \dot{c}^2 = 2\left[c^2\dot{T}\,\ddot{T} - \dot{X}\,\ddot{X} - \dot{Y}\,\ddot{Y} - \dot{Z}\,\ddot{Z}\right]$$
$$\Rightarrow \quad \ddot{T} = \ddot{X} = \ddot{Y} = \ddot{Z} = 0\,.$$

Thus, $\ddot{t} \overset{(2.85)}{=} \ddot{T} = 0$ and $\ddot{z} \overset{(2.85)}{=} \ddot{Z} = 0$ ✓

From the 2nd equation on RHS of equations (2.85) we get

$$\dot{X} = -\,\omega x\,\dot{t}\,\sin\omega t - \omega y\,\dot{t}\,\cos\omega t + \dot{x}\,\cos\omega t - \dot{y}\,\sin\omega t.$$

Then, keeping in mind that $\ddot{t} = 0$, we get

$$0 = \ddot{X} = \cos\omega t\,[\ddot{x} - \omega^2 x\,\dot{t}^2 - 2\omega\,\dot{y}\,\dot{t}\,] - \sin\omega t\,[\ddot{y} - \omega^2 y\,\dot{t}^2 + 2\omega\,\dot{x}\,\dot{t}\,]\,.$$

Since equations (2.85) hold for all t, this implies

$$\ddot{x} - \omega^2 x\,\dot{t}^2 - 2\omega\,\dot{y}\,\dot{t} = 0 \quad \text{and} \quad \ddot{y} - \omega^2 y\,\dot{t}^2 + 2\omega\,\dot{x}\,\dot{t} = 0 \quad ✓$$

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