Tensor Crib Sheet

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$\lambda^{a'} = X^{a'}_{b}\,\lambda^{b}$ (1.70)

$\lambda_{a'} = X^{b}_{a'}\,\lambda_{b}$ (1.71)

$\mathbf{e}_j = U^{i'}_{j}\,\mathbf{e}_{i'}$ (1.44)

$g^{ab} g_{bc} = \delta^{a}_{c} = \delta^{c}_{a} = g_{ab}\, g^{bc}$ (1.78)

$g_{ab}\,\lambda^{a}\,\mu^{b} = g^{ab}\,\lambda_{a}\,\mu_{b} = \lambda_{a}\,\mu^{a} = \lambda^{a}\,\mu_{a}$ (1.79)

$X^{a}_{c} = \frac{\mathrm{d}x^{a}}{\mathrm{d}x^{c}} = \delta^{a}_{c}$

$X^{a'}_{b}\,X^{b}_{c'} = \delta^{a'}_{c'} = \delta^{a}_{c}$ (1.69)

$X^{a}_{b'} X^{b'}_{c} = \delta^{a}_{c}$ (1.68)

$\dot{x}^{a'} = X^{a'}_{b}\,\dot{x}^{b}$ for a curve $\gamma(t)$ where $x = x(t)$ and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$

$g_{a'b'} = X^{a}_{a'} X^{b}_{b'} g_{ab}$

$g^{a'b'} = X^{a'}_{a} X^{b'}_{b}\, g^{ab}$

$\tau^{a'}_{b'c'} = X^{a'}_{d}\; X^{e}_{b'}\; X^{f}_{c'}\; \tau^{d}_{ef}$ (1.75)

$\tau^{a'}_{b'c'}\; \lambda^{c'} = X^{a'}_{d}\; X^{e}_{b'}\; \tau^{d}_{ef}\; \lambda^{f}$ (1.76)

$\tau^{a'}_{b'c'}\; X^{c'}_{f} = X^{a'}_{d}\; X^{e}_{b'}\; \tau^{d}_{ef}$ (1.77)

$\tau^{a_1{}'\dots a_r{}'}_{b_1{}'\dots b_s{}'} = X^{a_1{}'}_{c_1} \cdots X^{a_r{}'}_{c_r}\; X^{d_1}_{b_1{}'} \cdots X^{d_s}_{b_s{}'}\; \tau^{c_1\dots c_r}_{d_1\dots d_s}$ (1.73)

$X^{\;a}_{b'c'} \equiv \partial_{b'}\, X^{\;a}_{c'} = \frac{\partial^2 x^{a}}{\partial x^{b'}\,\partial x^{c'}} = X^{\;a}_{c'b'}$ (Defn)

$X^{\;a'}_{b'c} = X^{\;d}_{b'}\, X^{\;a'}_{cd}$ (Th 4.2.4)

$\Gamma^{a}_{bc} = \frac{1}{2}\, g^{ad}\,(\partial_b\, g_{dc} + \partial_c\, g_{bd} - \partial_d\, g_{bc})$ (2.13)

$\Gamma_{dbc} \equiv \frac{1}{2}\,(\partial_b\, g_{dc} + \partial_c\, g_{bd} - \partial_d\, g_{bc})$ (2.33)

$\Gamma^{a}_{bc} = g^{ad}\,\Gamma_{dbc}$ (2.34)

$\Gamma_{abc} = g_{ad}\,\Gamma^{d}_{bc}$ (2.35)

$\partial_c\, g_{ab} = \Gamma_{abc} + \Gamma_{bac}$ (2.36)

$\Gamma_{d'e'c'} = X^{\;b}_{d'}\, X^{\;f}_{e'}\, X^{\;a}_{c'}\, \Gamma_{bfa} + X^{\;a}_{e'c'}\, X^{\;b}_{d'}\, g_{ab}$ (2.32b)

$\Gamma^{a'}_{b'c'} = \Gamma^{d}_{fg}\, X^{\;a'}_{d}\, X^{\;f}_{b'}\, X^{\;g}_{c'} + X^{\;d}_{c'b'}\, X^{\;a'}_{d}$ (2.32)

Field	**Absolute Derivative Equation (along a curve $\gamma(u)$)**	
Scalar	$\boxed{D\varphi/du = d\varphi/du}$	(2.44)
Contravariant vector	$\boxed{D\lambda^a/du = \dot{\lambda}^a + \Gamma^a_{cd}\lambda^c \dot{x}^d}$	(2.45)
Covariant vector	$\boxed{D\mu_b/du = \dot{\mu}_b - \Gamma^c_{bd}\mu_c \dot{x}^d}$	(2.46)
Type (2,0) tensor	$\boxed{D\tau^{ab}/du \equiv \dot{\tau}^{ab} + \left[\Gamma^a_{cd}\tau^{cb} + \Gamma^b_{cd}\tau^{ac}\right]\dot{x}^d}$	(2.48)
Type (0,2) tensor	$\boxed{D\tau_{ab}/du \equiv \dot{\tau}_{ab} - [\Gamma^c_{ad}\tau_{cb} - \Gamma^c_{bd}\tau_{ac}]\dot{x}^d}$	(2.49)
Type (1,1) tensor	$\boxed{D\tau^a_b/du \equiv \dot{\tau}^a_b + [\Gamma^a_{cd}\tau^c_b - \Gamma^c_{bd}\tau^a_c]\dot{x}^d}$	(2.50)
(r, s) :	$\boxed{\begin{array}{l} D\tau^{a_1\ldots a_r}_{b_1\ldots b_s}/du = \dot{\tau}^{a_1\ldots a_r}_{b_1\ldots b_s} \\ + \left[\sum_{k=1}^r \Gamma^{a_k}_{cd}\tau^{a_1\ldots a_{k-1}\,c\,a_{k+1}\ldots a_r}_{b_1\ldots b_s} - \sum_{k=1}^s \Gamma^c_{b_k d}\tau^{a_1\ldots a_r}_{b_1\ldots b_{k-1}\,c\,b_{k+1}\ldots b_s}\right]\dot{x}^d \end{array}}$	(2.51a)

Example: $D\tau^{ab}_c/du = \dot{\tau}^{ab}_c + \left[\Gamma^a_{de}\tau^{db}_c + \Gamma^b_{de}\tau^{ad}_c - \Gamma^d_{ce}\tau^{ab}_d\right]\dot{x}^e$

Notation: $\lambda^a{}_{,c} \equiv \dfrac{\partial\lambda^a}{\partial x^c}$ $\qquad \lambda^a{}_{;c} \equiv \lambda^a{}_{,c} + \Gamma^a_{bc}\lambda^b$ (2.53)

Field	**Covariant Derivative Equation (in a manifold)**	
Scalar	$\boxed{\varphi_{;a} = \partial_a\varphi}$	(2.54)
Contravariant vector	$\boxed{\lambda^a{}_{;b} = \partial_b\lambda^a + \Gamma^a_{cb}\lambda^c}$	(2.55)
Covariant vector	$\boxed{\mu_{a;b} = \partial_b\mu_a - \Gamma^c_{ab}\mu_c}$	(2.56)
Type (2,0) tensor	$\boxed{\tau^{ab}{}_{;c} \equiv \partial_c\tau^{ab} + \Gamma^a_{dc}\tau^{db} + \Gamma^b_{dc}\tau^{ad}}$	(2.57)
Type (0,2) tensor	$\boxed{\tau_{ab;c} \equiv \partial_c\tau_{ab} - \Gamma^d_{ac}\tau_{db} - \Gamma^d_{bc}\tau_{ad}}$	(2.58)
Type (1,1) tensor	$\boxed{\tau^a_{b;c} \equiv \partial_c\tau^a_b + \Gamma^a_{dc}\tau^d_b - \Gamma^d_{bc}\tau^a_d}$	(2.59)
Type (r,s) tensor	$\boxed{\begin{array}{l} \tau^{a_1\ldots a_r}_{b_1\ldots b_s;c} = \partial_c\tau^{a_1\ldots a_r}_{b_1\ldots b_s} + \sum_{k=1}^r \Gamma^{a_k}_{dc}\tau^{a_1\ldots a_{k-1}\,d\,a_{k+1}\ldots a_r}_{b_1\ldots b_s} \\ \qquad - \sum_{k=1}^s \Gamma^d_{b_k c}\tau^{a_1\ldots a_r}_{b_1\ldots b_{k-1}\,d\,b_{k+1}\ldots b_s} \end{array}}$	(2.59a)

Type (2,1) tensor $\tau^{ab}{}_{c;d} \equiv \partial_d\tau^{ab}{}_c + \Gamma^a_{ed}\tau^{eb}{}_c + \Gamma^b_{ed}\tau^{ea}{}_c - \Gamma^e_{cd}\tau^{ab}{}_e$