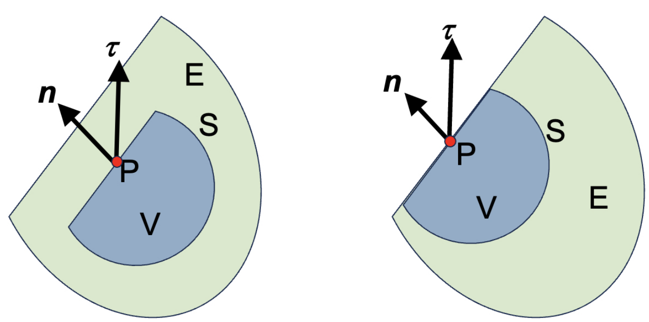
Newtonian versions of General Relativity objects

1.5 Newtonian stress tensor

Einstein's field equations are expressed in terms of metric tensors and the stress tensor, so in this section we develop the stress tensor for Euclidean space.

Suppose we have a 3-dimensional elastic body E that is placed under stress by both external and internal forces. It might be helpful to imagine a moon-sized balloon filled with squish able gel. If the gel is lumpy and massive, then there will be internal gravitational force. If we place the balloon on the Earth, then there will be an external gravitational force as well. If the balloon contains charged particles and it is placed in an electromagnetic field, there will also be internal and external electromagnetic forces. In our development, below, we allow any and all forces.



Let V be a small part of E, let *S* be the surface of V, and let P be a point of *S*. If P is internal to E, we postulate that the forces on P are all internal. If P is on the surface of E, we postulate that all the forces on P are external.

Forces on P can come from all directions. We can imagine that if the surface *S* were flat, a shear force (i.e., a force that hits *S* at an oblique angle) would have less effect than if the force were perpendicular. Let $\boldsymbol{\tau}$ be the sum of all the pressures (i.e., force per unit area) on *S*. The force at P is a function not only of $\boldsymbol{\tau}$ but also of the unit normal $\boldsymbol{n}$ to S at P. We express this as

$$\mathbf{f} = \boldsymbol{\tau}(\boldsymbol{n}). \tag{1.5-1}$$

We assume $\boldsymbol{\tau}$ to be a linear function so that the total force on V due to stresses can be found by "adding" all the differential forces. That is, total force is defined as

$$\iint_S \boldsymbol{f}\, \mathrm{d}S \equiv \iint_S \boldsymbol{\tau}(\boldsymbol{n})\, \mathrm{d}S. \qquad (1.5\text{-}2)$$

Since $\boldsymbol{\tau}$ is a linear function, we also have that for vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and scalars α and β

$$\boldsymbol{\tau}(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\, \boldsymbol{\tau}(\boldsymbol{u}) + \beta\, \boldsymbol{\tau}(\boldsymbol{v}). \qquad (1.5\text{-}3)$$

Suppose we use curvilinear coordinates u^i to label points of E. Then

$$\boldsymbol{f} = f^i \mathbf{e}_i \qquad (1.5\text{-}4)$$

and

$$\boldsymbol{n} = n^j \mathbf{e}_j. \qquad (1.5\text{-}5)$$

Hence,

$$f^i \mathbf{e}_i = n^j\, \boldsymbol{\tau}(\mathbf{e}_j) : \qquad (1.54)$$

$$f^i \mathbf{e}_i \overset{(1.5-4)}{=} \boldsymbol{f} \overset{(1.5-1)}{=} \boldsymbol{\tau}(\boldsymbol{n}) \overset{(1.5-5)}{=} \boldsymbol{\tau}(n^j\, \mathbf{e}_j) \overset{(1.5-3)}{=} n^j\, \boldsymbol{\tau}(\mathbf{e}_j) \quad \checkmark$$

$\boldsymbol{\tau}(\mathbf{e}_j)$ is a vector so it can be expressed as a linear combination of $\{\mathbf{e}_i\}$:

$$\boldsymbol{\tau}(\mathbf{e}_j) \equiv \tau^i_j\, \mathbf{e}_i : \qquad (1.55)$$

Note: we cannot just express $\boldsymbol{\tau}(\mathbf{e}_j)$ as $\boldsymbol{\tau}(\mathbf{e}_j) = \tau^i \mathbf{e}_i$. This would result in $\boldsymbol{\tau}(\mathbf{e}_1) = \tau^i \mathbf{e}_i$ and also $\boldsymbol{\tau}(\mathbf{e}_2) = \tau^i \mathbf{e}_i$, which would make them equal. Thus, τ^i_j requires two indices.

As a result, we get

$$f^i = \tau^i_j\, n^j : \qquad (1.56)$$

$$\forall\, i,\ \ f^i \mathbf{e}_i \overset{(1.54)}{=} n^j\, \boldsymbol{\tau}(\mathbf{e}_j) \overset{(1.55)}{=} \tau^i_j\, n^j\, \mathbf{e}_i \qquad \checkmark$$

[The matrix version of this is $\boldsymbol{F} = \boldsymbol{TN}$, where $\boldsymbol{F} = (f^i)$, $\boldsymbol{T} = (\tau^i_j)$, and $\boldsymbol{N} = (n^j)$].

Definition The linear function $\boldsymbol{\tau}$ is called the **stress tensor**. It has components τ^i_j defined by equation (1.55).

2.6 Newtonian gravitation and fluid dynamics

In this section we derive several fundamental equations in Newtonian physics to which corresponding equations of General Relativity must reduce. The equations are

1. Poisson’s equation (the field equation for Newtonian gravitation) (2.6-11)
2. Classical continuity equations for a perfect fluid (2.6-12)
3. Euler’s classical equation of motion for a perfect fluid (2.6-15)

We begin with development of gravitational potentials in one dimension. Consider a particle moving from point A to point B, and denote points in between as x. Set

K = Kinetic energy at x
U = Potential energy at x
E = Total energy at x
W = Work performed to move the particle from point A to point B
V = Potential energy per unit mass at x

Law of Conservation of Energy Total energy is preserved in moving a particle from point A to point B.

Think of moving a particle directly upwards from its resting spot on the Earth. The x-direction is then “up” (or radially outward), so

$E = K + U$ and $\Delta E = 0$, or $\Delta U = -\Delta K$.

The work performed is the product of the magnitude of the force applied and the distance moved: $W = F\,\Delta\mathrm{x}$. F can represent the the conglomeration of many different types of forces, or it can represent just a single type of force like gravitational force or electromagnetic force.

The gravitational force is **conservative**, meaning that total work performed in moving an object from A to B depends only on the location of points A and B and not on the path between them. We restrict our attention to conservative forces. Let m be the mass of the point being moved, v be its velocity, and a its acceleration. Let x be the direction of movement from A to B. Then

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial v}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = v\,\frac{\mathrm{d}v}{\mathrm{d}x},$$

$$\int_A^x F\,\mathrm{d}x = \int_A^x ma\,\mathrm{d}x = \int_A^x mv\,\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_{v_A}^v mv\,\mathrm{d}v = \frac{1}{2}mv^2 - \frac{1}{2}mv_A^2$$

$$= \Delta K = -\Delta U = -[U(x) - U(A)]$$

Let *A* be a reference point where we assign $U(A) = 0$. Then

$$\int_A^x F\, \mathrm{d}x = -\, U(x) \quad \text{and} \quad F(x) = -\, \frac{\mathrm{d}}{\mathrm{d}x} U(\mathrm{x})\,. \tag{2.6-1}$$

Since $a = \frac{F}{m}$, then $\int_A^x a\, \mathrm{d}x = -\, \frac{U(x)}{m}$. Define the potential energy per unit mass by

$$V(x) \equiv \frac{U(x)}{m}.$$

Then

$$\boxed{\int_A^x a\, dx \; = \; -V \;\; \text{and} \;\; a \; = -\frac{\mathrm{d}V}{\mathrm{d}x}}\,. \tag{2.6-2}$$

Formula (2.6-2) tells us that for an acceleration field *a*, there is a *potential function V* such that $a = -\, \frac{\mathrm{d}V}{\mathrm{d}x}$. Compare this to equation (2.6–-1) that tells us that for a force field *F* there is a *potential function U* such that $F = -\, \frac{\mathrm{d}U}{\mathrm{d}x}$. If the field is the gravitational field, then ***U*** is the **gravitational potential** and ***V*** is the **gravitational potential per unit mass** or the **gravitational acceleration potential**.

Like a force field, *a* can represent acceleration due to an aggregation of many different types of forces. If we ignore all forces except gravity, then we can write $g = -\, \frac{\mathrm{d}V}{\mathrm{d}x}$.

In 3 dimensions, equation (2.6-2) generalizes to

$$\boldsymbol{a}(x,y,z) = -\, \left(\frac{\partial V}{\mathrm{d}x}\boldsymbol{i} \; + \; \frac{\partial V}{\mathrm{d}y}\boldsymbol{j} \; + \; \frac{\partial V}{\mathrm{d}z}\boldsymbol{k}\right) = -\, \nabla V,$$

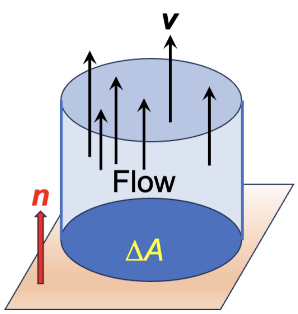
which can be expressed either in terms of components or as a vector:

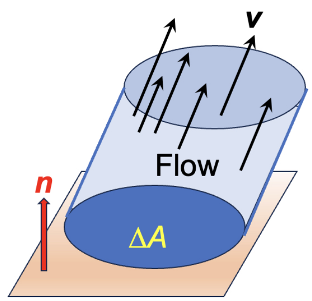
$$\boxed{\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\,\partial_i V} \qquad \text{or} \qquad \boxed{\boldsymbol{a} = \, -\, \nabla V} \tag{2.6-3}$$

Equation (2.6-3) is the **Newtonian equation of motion for a particle moving in a gravitational field of potential *V***.

Next we develop equations of motion for fluids, which include liquids, gases, and plasma (a mixture of electrons and isotopes, produced, for example, by the sun).

For a fluid, **flux** Φ is defined as the rate of fluid that flows through a surface; i.e., the amount of fluid mass that flows through during a unit of time. We start by finding flux for a small part of a surface and then integrate to generate flux for the full surface. For a small surface, δS, we can consider the surface to be flat and the velocity of the fluid to be the same at every point of the surface. Flux can be defined in terms of either volume or mass. Let the fluid have density ρ and speed v. Let ΔA be the area of δS and $\hat{\boldsymbol{n}}$ the unit normal vector to δS that has the same orientation as the flow, as shown below. We first consider the case where the velocity of the fluid is in the direction $\hat{\boldsymbol{n}}$, the figure on the left.





$V_\Phi = v \times$ (1 unit of time) $\times \Delta A$ is the volume of fluid that flows in a unit of time

$\Delta\Phi = \rho V_\Phi = \rho v\, \Delta A$ is the mass of the fluid that flows through δS in a unit of time

If the direction of flow is not the same as $\hat{\boldsymbol{n}}$, we have the skewed cylinder on the right with base ΔA and height $v\, cos\theta$ where θ is the angle between the velocity $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$.

$$V_\Phi = v\, \Delta A\, cos\theta = \Delta A\, \boldsymbol{v} \bullet \hat{\boldsymbol{n}}$$

$$\Delta\Phi = \rho\, \Delta A\, \boldsymbol{v} \bullet \hat{\boldsymbol{n}}$$

If we define a vector $\boldsymbol{\delta A}$ whose magnitude is the area ΔA and whose direction is normal to δS, then $\boldsymbol{\delta A} = \Delta A\, \hat{\boldsymbol{n}}$. So, we can rewrite $\Delta\Phi$ as

$$\Delta\Phi = \rho\, \boldsymbol{v} \bullet (\Delta A\, \hat{\boldsymbol{n}}) = \rho \boldsymbol{v} \bullet \boldsymbol{\delta A}$$

Just as breaking the full surface S into small, flat surfaces δS, adding them up, and taking the limit gives the surface area of S,

$$A = \iint_S \mathrm{d}\boldsymbol{A},$$

adding up the small fluxes, and taking the limit gives

$$\Phi = \iint_S \rho \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{A}.$$

If we define the vector field as ***F***, this can be written

$$\boxed{\Phi = \iint_S \boldsymbol{F} \bullet \mathrm{d}\boldsymbol{A}}\ . \tag{2.6-4}$$

Any integral of this form is called a **flux integral**. (This ***F*** has units of momentum but we can consider the concept of flux when ***F*** has other units, like force, below.)

Gravitation is related to flux and potential. Assume the mass M that generates the gravitational force is a point mass located at the origin. Newton's formula for the force extended on a small particle of mass m located at a distance r from the origin is

$$F = \frac{G\,M\,m}{r^2}\ , \tag{2.6-5}$$

where G is **Newton's universal gravitational constant**. Equation (2.6-5) is known as **Newton's equation for universal gravitation**. Note that $G\,M\,/\,r^2$ must have units of acceleration. The **gravitational potential energy** U is the work required to move the particle from some reference distance to a given distance, r. The reference point for gravitational potential energy is set to ∞, and the reference value is set to zero. That is, $U(\infty) \equiv 0$. Since positive work is required to move the particle from r to ∞, we have that, for all x, $U(x) < 0$. We express U as

$$U = -\frac{G\,M\,m}{r}:$$

$$U \overset{(2.6-1)}{=} \int_\infty^r F\,\mathrm{d}r = \int_\infty^r \frac{GMm}{r^2}\,\mathrm{d}r = -GMm\left(\frac{1}{r} - 0\right) = -\frac{G\,M\,m}{r}\,.$$

The force on a point mass located on a sphere of radius r about the central mass M is expressed as

$$\boldsymbol{a}(\boldsymbol{r}) \equiv -\frac{F}{m}\ \boldsymbol{e}_r = -\frac{G\,M}{r^2}\ \boldsymbol{e}_r\ ,$$

where $\boldsymbol{e}_r$ is a radial unit vector. ***a*** is called the **gravitational field** or, sometimes, the **gravitational acceleration** (because $G\,M\,/\,r^2$ has units of acceleration)**.** The "field" symbol ***F*** is often used instead of ***a***.

Let S be a solid ball of radius r about the origin. Let its volume be denoted by V_S and let ∂S be its surface. ∂S is a sphere of radius r with surface area $A = 4\pi\,r^2$. Let ***A*** be a vector of magnitude A that is normal to ∂S. Its unit direction vector is $\boldsymbol{e}_r$. By equation (2.6-4), the total flux of the gravitational field ***F*** across the surface ∂S is $\iint_{\partial S} \boldsymbol{a} \bullet \mathrm{d}\boldsymbol{A}$.

Claim $\iint_{\partial S} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{A} = -4\pi G M$: (2.6-6)

$$\iint_{\partial S} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{A} = \iint_{\partial S} \left(-\frac{G M}{r^2}\, \boldsymbol{e}_r\right) \cdot (\boldsymbol{e}_r\, \mathrm{d}A) = -\frac{G M}{r^2} \iint_{\partial S} \boldsymbol{e}_r \cdot \boldsymbol{e}_r\, \mathrm{d}A = -\frac{G M}{r^2} \iint_{\partial S} \mathrm{d}A$$
$$= -\frac{G M}{r^2}\, 4\pi\, r^2 = -4\pi G M \quad ✓$$

By the Divergence Theorem, certain integrals over a surface can be expressed as integrals over its enclosed volume:

$$\iint_{\partial S} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{A} = \iiint_S \nabla \cdot \boldsymbol{a}\, \mathrm{d}V_S\,. \qquad (2.6\text{-}7)$$

Let ρ be the **mass density** (i.e., mass per unit volume) of a point of S. Then the gravitational mass M can be expressed

$$M = \iiint_S \rho\, \mathrm{d}V_S\,. \qquad (2.6\text{-}8)$$

Claim $\nabla \cdot \boldsymbol{a} = -4\pi G \rho$: (2.6-9)

$$\iiint_S \nabla \cdot \boldsymbol{a}\, \mathrm{d}V_S \overset{(2.6-7)}{=} \iint_{\partial S} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{A} \overset{(2.6-6)}{=} -4\pi G M \overset{(2.6-8)}{=} -4\pi G \iiint_S \rho\, \mathrm{d}V_S$$
$$= -\iiint_S 4\pi G \rho\, dV_S$$
$$\Rightarrow \quad \nabla \cdot \boldsymbol{a} = -4\pi G \rho \qquad ✓$$

Since gravity is a conservative force, by equation (2.6-2) there is a **gravitational acceleration potential** V such that

$$\boldsymbol{a} = -\nabla V\,. \qquad (2.6\text{-}10)$$

Thus, the gravitational acceleration potential, V, satisfies

$$\boxed{\nabla^2 V = 4\,\pi G \rho} \; : \qquad (2.6\text{-}11)$$

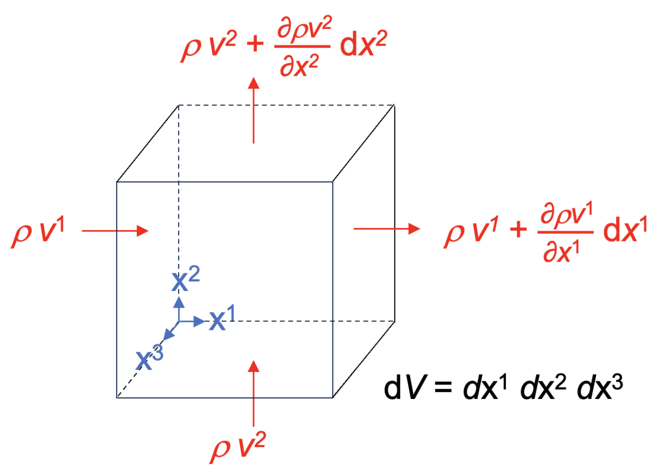
$$\nabla^2 V = \nabla \cdot \nabla V \overset{(2.6-10)}{=} -\nabla \cdot \boldsymbol{a} \overset{(2.6-9)}{=} 4\pi G \rho \qquad ✓$$

Equation (2.6-11) is known as **Poisson’s equation**. It is the **field equation for Newtonian gravitation** to which Einstein’s 16 field equations must reduce.

Next, we examine fluid flow to generate two more classical equations to which curved spacetime equations must reduce.

The most precise treatment of fluid flow, called Lagrangian analysis, is to track each individual particle over time. A simpler treatment, Eulerian analysis, is to focus on the properties over time of each location (x,y,z) within a fluid, and to track particles for only a short time interval dt. We begin with the continuity equation.

The continuity equation pertains to perfect fluids. It reflects the fact that mass is conserved in conventional Newtonian mechanics. The equation is developed by adding up the rate at which mass is flowing into and out of a control volume (depicted below as a cube) and setting the net mass in-flow equal to the net-outflow. In the figure below, ρ is the (time-dependent) density of the fluid and V_i is the volume of mass flowing in the i-direction.



First consider fluid flowing through the box with speed v^1 in the x^1-direction, from left to right. The area of the left face, and also the right face, is $dx^2 dx^3$. The amount of mass that flows into the box through the left face in a unit of time is $\rho\, v^1 dx^2 dx^3$, and the amount that flows out of the box through the right face is the flow in plus the change, $\rho\, v^1 dx^2 dx^3 + \frac{\partial(\rho\, v^1)}{\partial x^1} dx^1 (dx^2 dx^3)$. The change in mass flow, in minus out, is $-\frac{\partial(\rho\, v^1)}{\partial x^1} dx^1 dx^2 dx^3$. Similar formulas apply to flow in the x^2 and x^3 directions. Keeping in mind that the volume of the box is $dx^1 dx^2 dx^3$, the total change is mass is

$$\frac{\partial \rho}{\partial t} dx^1 dx^2 dx^3 = \left[-\frac{\partial(\rho\, v^1)}{\partial x^1} - \frac{\partial(\rho\, v^2)}{\partial x^2} - \frac{\partial(\rho\, v^3)}{\partial x^3}\right] dx^1 dx^2 dx^3$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v^1)}{\partial x^1} + \frac{\partial(\rho\, v^2)}{\partial x^2} + \frac{\partial(\rho\, v^3)}{\partial x^3} = 0.$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x^1}\boldsymbol{i} + v^2 \frac{\partial}{\partial x^2}\boldsymbol{j} + v^3 \frac{\partial}{\partial x^2}\boldsymbol{k}\right) \bullet \rho\,(\, v^1 \boldsymbol{i} + v^2 \boldsymbol{j} + v^3 \boldsymbol{k}\,)$$

This is **the classical continuity equation for a perfect fluid**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \qquad (2.6\text{-}12)$$

Euler's Equation is the other main equation in Newtonian mechanics for computing fluid flow. This equation describes flow in a perfect fluid. A fluid that can be completely characterized by its rest frame mass density ρ and isotropic pressure P (same in any direction) is called a **perfect fluid**. Real fluids are "sticky" and contain (and conduct) heat. In perfect fluids these possibilities are neglected. Specifically, perfect fluids have no shear stresses, viscosity, or heat conduction.

Perfect fluid flow is **steady**, meaning that velocity $\boldsymbol{v}$ at location (x,y,z) is always the same. Of course, a *particle* can have different velocities during its flow, but once the particle reaches a location (x,y,z), it will at that point have the same velocity as any other particle that later (or earlier) reaches this location.

The parameters describing Eulerian fluid motion at each location are its density ρ and velocity $\boldsymbol{v}$. The cause of changes in fluid motion is force.

A force cannot be sustained by a single particle of a fluid but only by a surface. Furthermore, if the fluid is at rest and is to remain at rest, the force must be applied at a right angle to the surface (or particles will slide). Thus, the parameter for force in Euler's equation is pressure, P, which is defined as the magnitude of the normal force per unit area.

Derivation of Euler's equation starts with the same first step as the continuity equation except that pressure P is substituted for mass ρA. That is, the force due to constant pressure on a surface is simply the pressure times the area of the surface. So, the magnitude of the force on the left face in the figure above is $P\, \mathrm{d}x^2 \mathrm{d}x^3$, and on the right face it is $[P + \frac{\partial P}{\partial x^1} \mathrm{d}x^1]\, (\mathrm{d}x^2 \mathrm{d}x^3)$. Therefore, the net force in the x-direction is $-\frac{\partial P}{\partial x^1} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \boldsymbol{i}$, where $\boldsymbol{i}$ is the unit vector pointing along the positive x-axis. The pressure per unit volume in the x-direction is $-\frac{\partial P}{\partial x^1} \boldsymbol{i}$, the net force divided by the volume. Repeating this in the y and z directions yields a net force per unit volume

$$\boldsymbol{F} = -\left(\frac{\partial P}{\partial x^1} \boldsymbol{i} + \frac{\partial P}{\partial x^2} \boldsymbol{j} + \frac{\partial P}{\partial x^3} \boldsymbol{k}\right) = -\nabla P.$$

From Newton's Second Law, we have force per unit volume = mass per unit volume times acceleration:

$$\boldsymbol{F} = \rho \, \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}.$$

Combining these yields

$$\rho \, \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = - \nabla P. \qquad (2.6\text{-}13)$$

The last step is to calculate the acceleration, $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$, of the fluid particles. This is a bit tricky because a particle in the control volume can be subject to both internal and external forces. There could be acceleration $\frac{\partial \mathbf{v}}{\partial t}$ (note the partial derivative notation) as a particle moves within the inertial frame coordinate system. There could be further acceleration if the box itself is moving. For example, if the box is immersed in a circular flow, there will be centripetal acceleration.

To compute the acceleration of a particle, we need to find the rate of change of the velocity due to both internal and external forces:

$$\frac{d\boldsymbol{v}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}\left(t + \Delta t\,,\, x + v^1\,\Delta t,\; y + v^2\,\Delta t,\; z + v^3\,\Delta t\right)}{\Delta t} = \frac{\partial \mathbf{v}}{\partial t} + v^1 \frac{\partial \boldsymbol{v}}{\partial x^1} + v^2 \frac{\partial \boldsymbol{v}}{\partial x^2} + v^3 \frac{\partial \boldsymbol{v}}{\partial x^2}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + [\,(\,v^1\boldsymbol{i} + v^2\boldsymbol{j} + v^3\boldsymbol{k}\,) \bullet (\frac{\partial}{\partial x^1}\boldsymbol{i} + v^2 \frac{\partial}{\partial x^2}\boldsymbol{j} + v^3 \frac{\partial}{\partial x^2}\boldsymbol{k})\,]\,\boldsymbol{v}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + (\boldsymbol{v} \bullet \nabla)\,\boldsymbol{v}. \qquad (2.6\text{-}14)$$

The first term reflects acceleration of the particle within the inertial frame and the second term reflects acceleration due to motion of the fluid body. The second term is nonlinear and is the source of many difficulties in fluid mechanics.

Combining equations (2.6-13) and (2.6-14) yields **Euler's classical equation of motion for a perfect fluid:**

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\boldsymbol{v} \bullet \nabla)\,\boldsymbol{v}\right] = - \nabla P \qquad (2.6\text{-}15)$$