Exercise 2.7.4 Show that $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$

Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is small, we assume there is a small (1st order) term, $f^{\mu\nu}$, such that

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}.$$

Since $h_{\sigma\rho}$ is also of 1st order, the product $f^{\mu\nu}h_{\sigma\rho}$ is of 2nd order.

Since g is invertible and η raises subscripts,

$$\delta^{\mu}_{\nu} = g^{\mu\sigma}g_{\sigma\nu} = (\eta^{\mu\sigma} + f^{\mu\sigma})\,(\eta_{\sigma\nu} + h_{\sigma\nu})$$

$$= \eta^{\mu\sigma}\eta_{\sigma\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu} + f^{\mu\sigma}h_{\sigma\nu}$$

$$\approx \delta^{\mu}_{\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu}$$ to first order since $f^{\mu\sigma}h_{\sigma\nu}$ is of 2nd order.

$$\Rightarrow \quad f^{\mu\sigma}\eta_{\sigma\nu} \approx - \eta^{\mu\sigma}h_{\sigma\nu}$$

$$\Rightarrow \quad f^{\mu\rho} = f^{\mu\sigma}\delta^{\rho}_{\sigma} = f^{\mu\sigma}\eta^{\nu\rho}\eta_{\sigma\nu} \approx - \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\nu} = - h^{\mu\rho}, \quad \text{or} \quad f^{\mu\nu} \approx - h^{\mu\nu}$$

$$\Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \qquad ✓$$