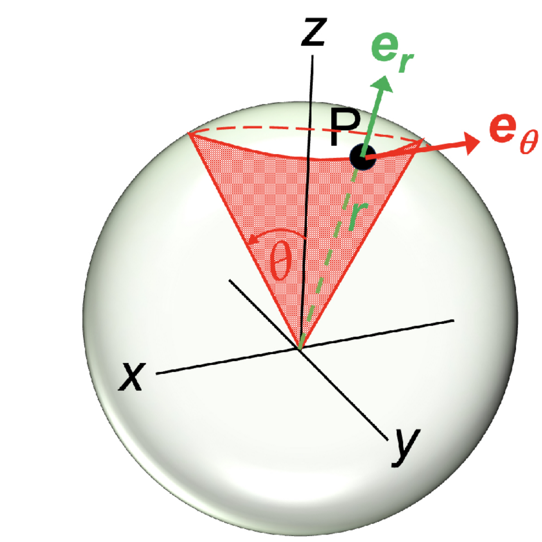
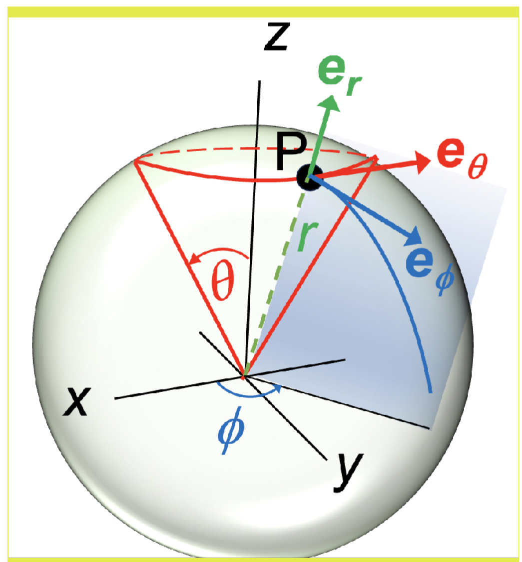
Spherical and cylindrical coordinates in General Relativity

1.1 Euclidean Coordinates / 1.2 Index Notation

Spherical





The RH figure shows three surfaces that intersect at a point P. The LH figure shows two of those surfaces, a green sphere of radius *r* about the origin and a red cone around the *z*-axis at angle θ. The intersection is the circle passing through P. The basis vector $\boldsymbol{e}_\theta$ at point P is shown tangent to the circle. The basis vector $\boldsymbol{e}_r$ lies on the line that passes through the origin and P (because the tangent to a line is the line itself). The RH figure adds the half-plane generated by angle ϕ, and the blue arc is where it intersects the sphere. $\boldsymbol{e}_\phi$ is shown tangent to that arc at P.

The position vector is

$$\boldsymbol{r} = \boldsymbol{r}\,(r,\phi,\theta\,) \overset{(1.3)}{=} x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} = r\sin\theta\cos\phi\,\boldsymbol{i} + r\sin\theta\;\sin\phi\,\boldsymbol{j} \;+ r\cos\theta\,\boldsymbol{k}$$

and

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\;\sin\phi \qquad z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arccos\frac{z}{r} \qquad \phi = \arctan\frac{y}{x}\;.$$

The natural basis is:

$$\mathbf{e}_1 = \mathbf{e}_r = \frac{\partial \boldsymbol{r}}{\partial r} = \frac{\partial x}{\partial r}\boldsymbol{i} + \frac{\partial y}{\partial r}\boldsymbol{j} + \frac{\partial z}{\partial r}\boldsymbol{k} = \sin\theta\,\cos\phi\,\boldsymbol{i} + \sin\theta\,\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} = \frac{\partial x}{\partial \theta}\boldsymbol{i} + \frac{\partial y}{\partial \theta}\boldsymbol{j} + \frac{\partial z}{\partial \theta}\boldsymbol{k} = r\cos\theta\,\cos\phi\,\boldsymbol{i} + r\cos\theta\,\sin\phi\,\boldsymbol{j} - r\sin\theta\,\boldsymbol{k}$$

$$\mathbf{e}_3 = \mathbf{e}_\phi = \frac{\partial \boldsymbol{r}}{\partial \phi} = \frac{\partial x}{\partial \phi}\boldsymbol{i} + \frac{\partial y}{\partial \phi}\boldsymbol{j} + \frac{\partial z}{\partial \phi}\boldsymbol{k} = -r\sin\theta\,\sin\phi\,\boldsymbol{i} + r\sin\theta\,\cos\phi\,\boldsymbol{j}\,,$$

where $r > 0$ and $0 < \theta < \pi$. The natural basis is undefined in spherical coordinates for points on the z-axis because ϕ does not have a unique value for z-axis points and, so, $\mathbf{e}_\theta = r\,(\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j})$ is ambiguously defined for those points.

The covariant metric tensor is

$$G = (g_{ij}) = (\mathbf{e}_i \bullet \mathbf{e}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \quad ✓$$

The dual basis is:

$$\mathbf{e}^1 = \mathbf{e}^r = \nabla \boldsymbol{r} = \frac{\partial r}{\partial x}\,\boldsymbol{i} + \frac{\partial r}{\partial y}\,\boldsymbol{j} + \frac{\partial r}{\partial z}\,\boldsymbol{k}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{k}$$

$$= \sin\theta\,\cos\phi\;\boldsymbol{i} + \sin\theta\,\sin\phi\;\boldsymbol{j} + \cos\theta\;\boldsymbol{k}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = \frac{\partial\theta}{\partial x}\,\boldsymbol{i} + \frac{\partial\theta}{\partial y}\,\boldsymbol{j} + \frac{\partial\theta}{\partial z}\,\boldsymbol{k}$$

$$= \frac{x\,z}{(x^2+y^2+z^2)\,\sqrt{x^2+y^2}}\,\boldsymbol{i} + \frac{y\,z}{(x^2+y^2+z^2)\,\sqrt{x^2+y^2}}\,\boldsymbol{j} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{k}$$

$$= \frac{\cos\theta\,\cos\phi}{r}\,\boldsymbol{i} + \frac{\cos\theta\,\sin\phi}{r}\,\boldsymbol{j} - \frac{\sin\theta}{r}\,\boldsymbol{k}$$

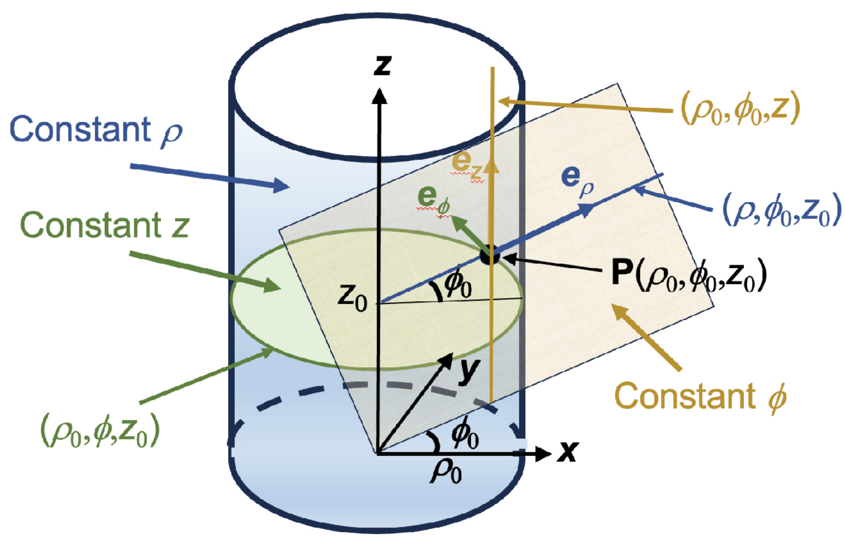
$$\mathbf{e}^3 = \mathbf{e}^\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k} = -\frac{y}{x^2+y^2}\,\boldsymbol{i} + \frac{x}{x^2+y^2}\,\boldsymbol{j}$$

$$= -\frac{\sin\phi}{r\sin\theta}\,\boldsymbol{i} + \frac{\cos\phi}{r\sin\theta}\,\boldsymbol{j}$$

The contravariant metric tensor is

$$\hat{G} = \left(g^{ij}\right) = (\mathbf{e}^i \cdot \mathbf{e}^j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}. \checkmark$$

Cylindrical



The figure shows a cylinder, the surface of constant $\rho = \rho_0$, shaded in blue; the disc of constant $z = z_0$ in green; and the half-plane of constant $\phi = \phi_0$ in tan. The point P is at the intersection of the three surfaces. The boundary of the green disc is a circle having ϕ for its parameter, and, so, the natural basis vector $\mathbf{e}_\phi$ lies along a tangent to the circle at P. The blue line through P is parameterized by ρ, and, so $\mathbf{e}_\rho$ points outward from P as shown. It is on the blue line because the line is the tangent to itself. The vertical tan line through P is parameterized by z, and $\mathbf{e}_z$ points upward from P as shown, tangent to the line. The dual basis vectors (not shown) lie normal to the three surfaces and are parallel to their respective natural basis vectors.

The position vector is

$$\boldsymbol{r} = \boldsymbol{r}\,(\rho,\phi,z) \overset{(1.3)}{=} x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} = \rho\cos\phi\,\boldsymbol{i} + \rho\sin\phi\,\boldsymbol{j} + z\,\boldsymbol{k},$$

where

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \arctan\left[\frac{y}{x}\right] \qquad z = z\,.$$

The natural basis consists of the tangent vectors of $\boldsymbol{r}$:

$$\boldsymbol{e}_\rho = \frac{\partial \boldsymbol{r}}{\partial \rho} = \frac{\partial x}{\partial \rho}\,\boldsymbol{i} + \frac{\partial y}{\partial \rho}\,\boldsymbol{j} + \frac{\partial z}{\partial \rho}\,\boldsymbol{k} = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}$$

$$\boldsymbol{e}_\phi = \frac{\partial \boldsymbol{r}}{\partial \phi} = \frac{\partial x}{\partial \phi}\,\boldsymbol{i} + \frac{\partial y}{\partial \phi}\,\boldsymbol{j} + \frac{\partial z}{\partial \phi}\,\boldsymbol{k} = -\rho\sin\phi\,\boldsymbol{i} + \rho\cos\phi\,\boldsymbol{j}$$

$$\boldsymbol{e}_z = \frac{\partial \boldsymbol{r}}{\partial z} = \frac{\partial x}{\partial z}\,\boldsymbol{i} + \frac{\partial y}{\partial z}\,\boldsymbol{j} + \frac{\partial z}{\partial z}\,\boldsymbol{k} = \boldsymbol{k}\,,$$

where $\rho > 0$ and $0 \le \phi < 2\pi$. The natural basis in cylindrical coordinates is undefined on the z-axis because ϕ does not have unique values for z-axis points and, so, the values of the basis vectors $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ are ambiguous.

The covariant metric tensor is

$$G = (g_{ij}) = (\boldsymbol{e}_i \bullet \boldsymbol{e}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To find the dual basis, first convert the partial derivatives from x, y, z to ρ, ϕ, and z:

$$\frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{\rho\cos\phi}{\sqrt{\rho^2}} = \cos\phi, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{\rho\sin\phi}{\sqrt{\rho^2}} = \sin\phi, \quad \frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\rho\sin\phi}{\rho^2} = \frac{-\sin\phi}{\rho}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2} = \frac{\rho\cos\phi}{\rho^2} = \frac{\cos\phi}{\rho}, \quad \frac{\partial \phi}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = 0, \qquad \frac{\partial z}{\partial z} = 1$$

The dual basis is composed of the gradient vectors, defined in terms of the partials:

$$\mathbf{e}^{\rho} = \nabla\rho = \frac{\partial\rho}{\partial x}\,\mathbf{i} + \frac{\partial\rho}{\partial y}\,\mathbf{j} + \frac{\partial\rho}{\partial z}\,\mathbf{k} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$$

$$\mathbf{e}^{\phi} = \nabla\phi = \frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k} = -\frac{\sin\phi}{\rho}\,\mathbf{i} + \frac{\cos\phi}{\rho}\,\mathbf{j}$$

$$\mathbf{e}^{z} = \nabla z = \frac{\partial z}{\partial x}\,\mathbf{i} + \frac{\partial z}{\partial y}\,\mathbf{j} + \frac{\partial z}{\partial z}\,\mathbf{k} = \mathbf{k}$$

The dual basis in cylindrical coordinates is also undefined on the *z*-axis.

Observe further that the corresponding natural and dual basis vectors have the same direction, only differing by up to a scalar factor. Further, observe that the basis vectors are orthogonal. For example, $\mathbf{e}_{\rho} \bullet \mathbf{e}_{\phi} = 0$.

The contravariant metric tensor is

$$\hat{G} = \left(g^{ij}\right) = \left(\mathbf{e}^{i} \bullet \mathbf{e}^{j}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$