Exercise 2.4.1 In the primed coord sys $x^{a'} \overset{2.62}{=} x^a - x_\theta^a + \frac{1}{2}(\Gamma^a_{bc})_\theta (x^b - x_\theta^b)(x^c - x_\theta^c)$,

show that $(g_{a'b'})_\theta = (g_{ab})_\theta$.

Solution. In Th 2.4.1 we showed that $X_d^{a'} \overset{(a)}{=} \delta_d^a + (\Gamma^a_{dc})_\theta (x^c - x_\theta^c)$

$$\text{So } g^{a'b'} = X_c^{a'} X_d^{b'} g^{cd} \overset{(a)}{=} [\delta_c^a + \Gamma^a_{ce}(x^e - x_\theta^e)][\delta_d^b + \Gamma^b_{df}(x^f - x_0^f)]\, g_\theta^{cd}$$

$$\therefore g_\theta^{a'b'} = [\delta_c^a + (\Gamma^a_{ce})_\theta \underbrace{(x_\theta^e - x_\theta^e)}_{0}][\delta_d^b + (\Gamma^b_{df})_\theta \underbrace{(x_\theta^f - x_\theta^f)}_{0}]\, g_\theta^{cd} = \delta_c^a \delta_d^b\, g_\theta^{cd} = g_\theta^{ab}$$

$$(G'_\theta)^{-1} = G_\theta^{-1} \Rightarrow I = (G'_\theta)^{-1} G'_\theta = G_\theta^{-1} G'_\theta \Rightarrow G'_\theta = G_\theta \Rightarrow (g_{a'b'})_\theta = (g_{ab})_\theta \quad ∎$$