Exercise 2.2.5. Given $\Gamma^{a'}_{b'c'} \overset{(2.32)}{=} \Gamma^{d}_{fg} X^{a'}_{d} X^{f}_{b'} X^{g}_{c'} + X^{d}_{c'b'} X^{a'}_{d}$, show

$$\Gamma^{a'}_{b'c'} = \Gamma^{d}_{ef} X^{a'}_{d} X^{e}_{b'} X^{f}_{c'} - X^{e}_{b'} X^{f}_{c'} X^{a'}_{ef} .$$

Solution:

In 1st term, replace $g \mapsto f \to e$:

$$\Gamma^{d}_{fg} X^{a'}_{d} X^{f}_{b'} X^{g}_{c'} \mapsto \Gamma^{d}_{ef} X^{a'}_{d} X^{e}_{b'} X^{f}_{c'} \qquad \checkmark$$

In 2nd term:

Claim $X^{d}_{c'b'} X^{a'}_{d} = - X^{d}_{b'} X^{a'}_{c'd}$:

$$0 = \partial_{c'} \delta^{a}_{b} = \partial_{c'} (X^{d}_{b'} X^{a'}_{d}) = X^{d}_{c'b'} X^{a'}_{d} + X^{d}_{b'} X^{a'}_{c'd} \qquad \checkmark$$

So,

$$X^{d}_{c'b'} X^{a'}_{d} = - X^{d}_{b'} X^{a'}_{c'd} = - X^{d}_{b'} \frac{\partial X^{a'}_{d}}{\partial c'} \overset{\text{(Chain Rule)}}{=} - X^{d}_{b'} \frac{\partial X^{a'}_{d}}{\partial x^{f}} \frac{\partial x^{f}}{\partial c'}$$

$$= - X^{d}_{b'} X^{f}_{c'} X^{a'}_{df} \overset{d \mapsto e}{=} - X^{e}_{b'} X^{f}_{c'} X^{a'}_{ef} \qquad \checkmark$$

∎