[13.39] Let V be an n-dimensional vector space. Let $\mathcal{V}$ be the vector space of $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors on V. Show that a linear transformation T on V induces a linear transformation $\mathcal{T}:\ Q^{f\cdots h}_{a\cdots c} \mapsto S^{f}{}_{f'}\cdots S^{h}{}_{h'}\, T^{a'}{}_{a}\cdots T^{c'}{}_{c}\, Q^{f'\cdots h'}_{a'\cdots c'}$ on $\mathcal{V}$.

Proof. To show that $\mathcal{T}$ is linear, let P and Q be $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors, α a scalar, and $R = P + Q$. Then

$$\begin{aligned}
\mathcal{T}\left(P^{f\cdots h}_{a\cdots c}+Q^{f\cdots h}_{a\cdots c}\right) &= \mathcal{T}\left(R^{f\cdots h}_{a\cdots c}\right) = S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}R^{f\cdots h}_{a\cdots c} \\
&= S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}\left(P^{f\cdots h}_{a\cdots c}+Q^{f\cdots h}_{a\cdots c}\right) \\
&= S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}P^{f\cdots h}_{a\cdots c}+S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}Q^{f\cdots h}_{a\cdots c} \\
&= \mathcal{T}\left(P^{f\cdots h}_{a\cdots c}\right)+\mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}\left(\alpha\, Q^{f\cdots h}_{a\cdots c}\right) &= \mathcal{T}\left(\left(\alpha\, Q\right)^{f\cdots h}_{a\cdots c}\right) = S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}\left(\alpha\, Q\right)^{f\cdots h}_{a\cdots c} \\
&= \alpha S^{f}{}_{f'}\cdots S^{h}{}_{h'}T^{a'}{}_{a}\cdots T^{c'}{}_{c}Q^{f\cdots h}_{a\cdots c} \\
&= \alpha\, \mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right)
\end{aligned}$$