Chapter 13. Symmetry Groups

Groups

Definitions:
A **group** is a set G with an operation $\circ$ that is closed and associative, has an identity e, and every element g has an inverse g^{-1} such that $g \circ g^{-1} = e = g^{-1} \circ g$.$\circ$

A group G is **Abelian** if it is commutative: $g \circ h = h \circ g$ for all g, h in G.

A **subgroup** is a subset of G that is a group under $\circ$.

Let H be a subgroup of G. A **coset of H** is a set $H \circ g = \{h \circ g: h \in H\}$, where $g \in$ G. The only coset of H that is a group is the set H itself: $H = H \circ e$ where e is the identity element. The cosets of H form a partition of G.

A **normal subgroup** is a subgroup H that satisfies $g \circ H = H \circ g$ for all g in G, or equivalently $H = g^{-1} \circ H \circ g$.

A group is **simple** if it contains no non-trivial normal subgroup. The simple groups are the fundamental "building blocks" of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

- Classical Families: A_m, B_m, C_m, D_m having dimensions $m(m+2)$, $m(2m+1)$, $m(2m+1)$, and $m(2m-1)$, respectively where $m \in \mathbb{Z}^+$.
- Exceptional Groups: E_6, E_7, E_8, F_4, G_2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has ≈ 10^{60} elements and is known as **the monster.**

Definition. The **Product Group** of groups G and H is **GxH** = $\{(g,h): g \in G, h \in H\}$ with group operation $(g_1,h_1) \circ (g_2,h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.

Definition. Let N be a subgroup of G. The **Factor Space G/N** is the collection of cosets $N \circ g$ along with the operation $(N \circ g_1) \circ (N \circ g_2) = N \circ (g_1 \circ g_2)$.

Theorem. If N is normal then G/N is a group, called the **Factor Group.**

Theorem. [13.10] $H \cong (G \times H)/G$.

The group operation is a function, $\circ: G \times G \to G$. If G is also a topological space, then $\circ$ can either be continuous or not.

Definition. A **group (G,$\circ$) is continuous** if $\circ: G \times G \to G$ is a continuous function when G is considered as a topological space. A **Lie group** is a continuous group where the inversion operation is also continuous.

The **dimension of a group** is its dimension as a topological space, which we now define. Intuitively the dimension of a space is 1 + dim (boundary of space). For example, a disk has dimension 2 and its boundary, a circle, has dimension 1. A line segment has dimension 1 and its boundary, 2 points, has dimension zero. Since a point has dimension 0, its boundary, the empty set $\varnothing$, is defined to have dimension -1. It is standard to start with $\varnothing$ and define topological dimension inductively.

There are several definitions of dimension. We give the **Small Inductive Dimension** and briefly two others.

Definition. A collection $\mathcal{U} = \{U_\alpha\}$ is an **open basis for a topological space** X if every open set is a union of sets from $\mathcal{U}$. We first give a few preliminaries.

Examples. In $\mathbb{R}$, the collection of open intervals forms a basis. In $\mathbb{R}^3$, the collection of open balls forms a basis.

Definition. The **closure** $\overline{A}$ of a set A is the smallest closed set that contains A. The **boundary** of A is $\partial A = \overline{A} - A$.

Definition. Let X be a topological space and $\mathcal{U}$ an open basis for X.

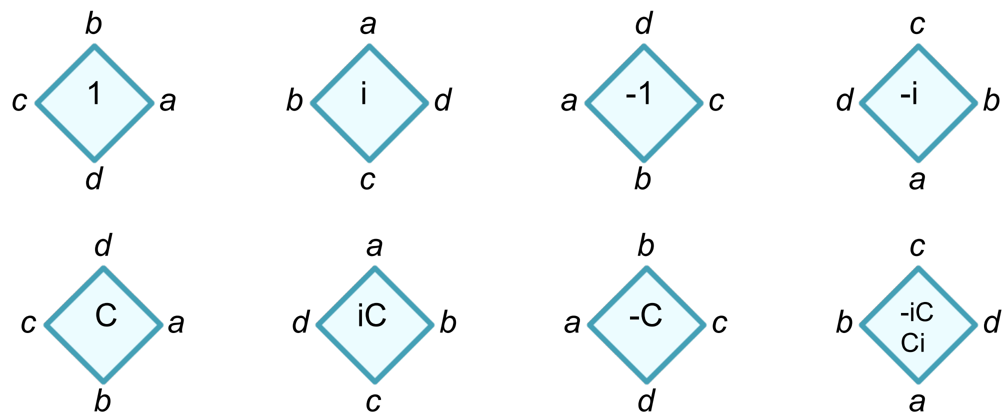
(1) **dim X = -1** if X = $\varnothing$
(2) dim X ≤ *n* if for all points *x* and open sets W such that $x \in W$ there exists $U \in \mathcal{U}$ such that $x \in U \subseteq \overline{U} \subseteq W$ and dim $\partial U \leq n - 1$
(3) **dim X = *n*** if (2) is true for *n* but false for *n* – 1
(4) **dim X =** ∞ if for every *n*, dim X ≤ *n* is false

The **Large Inductive Dimension** is similar but replaces points with closed sets.

The **Lebesgue Covering Dimension** defines dim X = *n* if every open cover of X has an open refinement in which no point belongs to more that *n* + 1 sets in the refinement.

All of these definitions agree on spaces like $\mathbb{R}^n$ that are separable and metrizable.

Symmetries of a Square



Definitions:
Non-reflecting Group: < i > = {1, i, -1, -i}
Reflecting Group: < i, C > = {1, i, -1, -i, C, iC, -C, -iC = Ci}
C is complex conjugation: $a+bi \mapsto a-bi$. **1** is the null rotation, which is the group identity element. ***i*** is the 90° counter-clockwise rotation of the square

Convention: ab means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

Examples:

Normal subgroups of < i, C > :

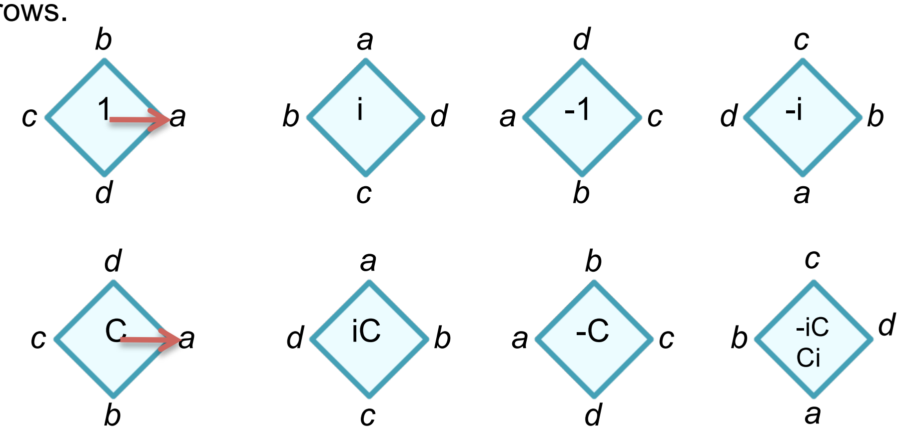
{1, -1, C, -C} , {1, -1}, {1, -1}

Non-normal subgroups of < i, C > :

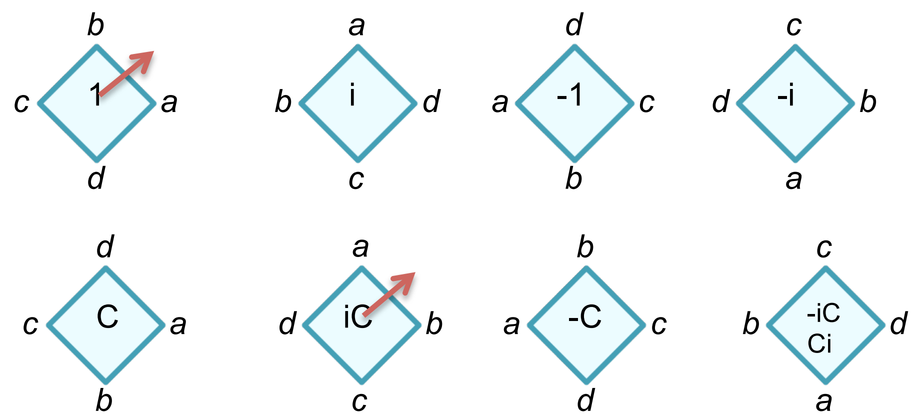
{1, -C}, {1, iC}, {1, C}

For example, {1, C} i = {i, Ci} ≠ {i, -Ci} = i {1, C}

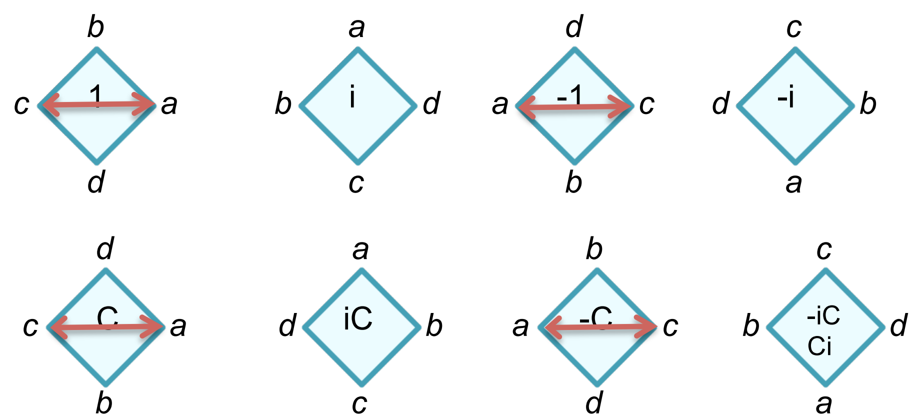
Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.



{1, C} is a reduced symmetry group



{1, iC} is a reduced symmetry group



{1, -1, C, -C} is a reduced symmetry group

Symmetries of a Sphere

Definitions:

A group G whose underlying set is continuous is called a **Lie Group.**
SO(3) is the group of non-reflective symmetries of a 3-sphere
O(3) is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

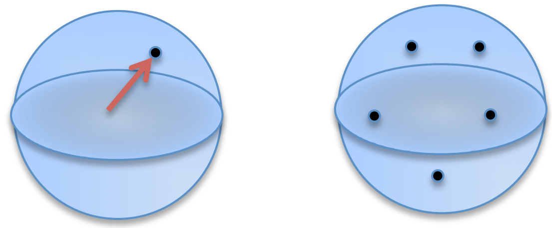
O(3) = SO(3) $\cup$ T, the disjoint union of O(3) with the coset of reflective symmetries

T $= \mathrm{R}\,\mathrm{SO}(3) = \{\mathrm{R}g : g \in \mathrm{SO}(3)\}$ where **R** is the reflection operator on the sphere.

Recall problem [12.7]: SO(3) is group isomorphic to the solid sphere $\mathcal{R}$ of radius π with antipodal points identified.

Theorem. (Problem [13.7]) SO(3) and {1, R} are the only normal subgroups of O(3), where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups



The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.

Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

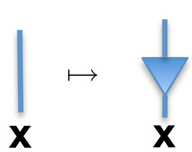
- $f : \mathrm{V} \to \mathrm{W}$ is a **homomorphism** if *it* preserves the vector space structure:
  - $f(au + bv) = af(u) + bf(\mathrm{v})$ for all vectors u and v and scalars a and b.
- **Hom(V,W)** is the set of homomorphisms from V to W.
- $\mathcal{A}$**(V)** = Hom(V,V).
- A **linear transformation** is a member $T \in \mathcal{A}(\mathrm{V})$.
  - That is, a linear transformation is a function $T : \mathrm{V} \to V$ such that $T(au + bv) = aTu + bTv$.

Theorem. [13.12, 13.13] Let V = $\mathbb{R}^3$, using (x^1, x^2, x^2) instead of (x, y, z). Then a linear transformation T takes the form $T : x^r \mapsto T^r{}_s x^s = ax^1 + bx^2 + cx^3$.

Note . Linear transformations are represented by matrices:

$$T : \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \text{or} \qquad \mathrm{x} \mapsto T\mathrm{x}$$

In diagrammatic form this is



Definition. The **transpose** of the matrix $T = T^i{}_j$ is the matrix $T^{\mathrm{T}} = \left(T^{\mathrm{T}}\right)^i{}_j = T^j{}_i$.

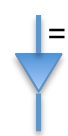
Note. The indices are dummy variables. We can express T as $T = T^r{}_s$ and also $T = T^s{}_r$. This can become subtle when expressing either the transpose or the inverse of T. Do not confuse $T = T^s{}_r$ as an expression for the transpose. The transpose must be expressed as $\left(T^{\mathrm{T}}\right)^r{}_s$ or $\left(T^{\mathrm{T}}\right)^s{}_r$ unless it is part of an expression involving T such as $T^s{}_r = \left(T^r{}_s\right)^{\mathrm{T}}$.

Definition. The **inverse** of the matrix T is the matrix T^{-1} satisfying $TT^{-1} = I = T^{-1}T$. If $S = T^{-1}$, we can also write $S^r{}_s = \left(T^r{}_s\right)^{-1}$ and $S^s{}_r = \left(T^s{}_r\right)^{-1}$ but not $S^s{}_r = \left(T^r{}_s\right)^{-1}$. The indices should match.

Definition. A matrix T is **orthogonal** if $T^{-1} = T^{\mathrm{T}}$.

Theorem. If $R = ST$ then $R^a{}_c = S^a{}_b T^b{}_c$. That is, the composition, R, of 2 linear transformations is the result of matrix multiplication of S and T. In diagrammatic notation:

$R =$  $=$  $= ST$

Example. $TI = T = IT$ is written in diagrammatic form as  =  = .

and, in $\mathbb{R}^3$, $I = \delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if a} \neq \text{b} \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b range over {1, 2, 3}.

Definitions. A linear transformation T is **singular** if Dim(TV) < Dim W; that is, T is not *onto*.

Theorem. [13.17] T is singular iff $\exists$ $v \neq 0$ such that $Tv = 0$.

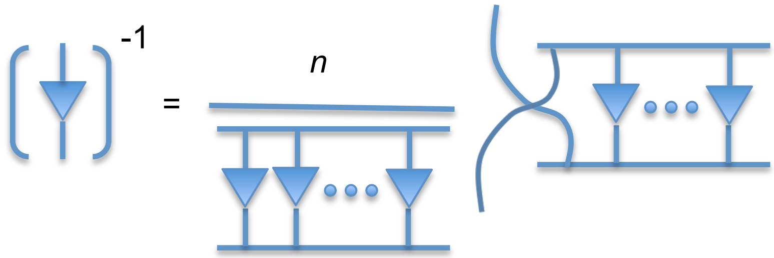
Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1 $\Leftrightarrow \forall v \neq w \;\; T(v - w) = T(v) - T(w) \neq 0 \overset{(*)}{\Leftrightarrow} \forall u \neq 0 \;\; T(u) \neq 0$

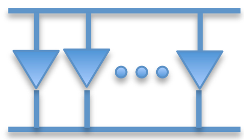
$\overset{[13.17]}{\Leftrightarrow}$ T is non-singular $\Leftrightarrow$ T is onto.

(*) Set $v = 3u$ and $w = 2u$. ∎

Theorem. [13.18] If T is nonsingular, then it has an inverse T^{-1}.

Theorem. [13.19] T^{-1} = 

Determinants and Traces

Definition. Det $T = \frac{1}{n!}$  $= \frac{1}{n!} \in^{ab\cdots d} T^e{}_a T^f{}_b \cdots T^h{}_d \, \varepsilon_{ef\cdots h}$.

Theorem. [Bud] Det $T = \sum_{\pi \in \mathcal{P}_{12\cdots n}} \mathrm{Sign}(\pi)\, T^1{}_{\pi(1)} \cdots T^n{}_{\pi(n)}$ (the normal definition of Det)

Proof. Let $\mathcal{P}_{1\ldots n}$ be the set of permutations of (1, …, n).

$$\text{Det } T = \frac{1}{n!} \varepsilon_{r\cdots s} \in^{t\cdots u} T^r{}_t \cdots T^s{}_u$$

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(1)\cdots\pi(n)} T^{\pi^*(1)}{}_{\pi(1)} \cdots T^{\pi^*(n)}{}_{\pi(n)}$$

(Replace Einstein notation.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(\pi^*(1))\cdots\pi(\pi^*(n))} T^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots T^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Replace π by $\pi \circ \pi^*$ in $\in$ and T. The double sum over π and π^* is unchanged, in both expressions stepping over all permutations of (1, …, n), and the exponents of $\in$ continue to match the subscripts of T.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \mathbf{Sign}(\pi)\, \cancel{\varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi^*(1)\cdots\pi^*(n)}}\, T^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots T^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Re-order superscripts of $\in$ by applying an inverse π permutation.)

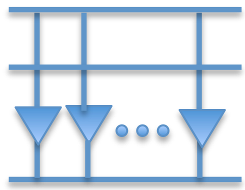
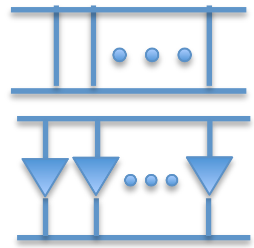
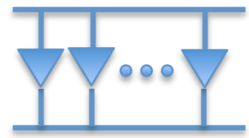
$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi) \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} T^{1}{}_{\pi(1)} \cdots T^{n}{}_{\pi(n)}$$

(This is just a simpler way to label the subscripts and superscripts of T. For example, if $\pi^*(3) = 1$ then $T^{\pi^*(3))}{}_{\pi(\pi^*(3))} = T^1{}_{\pi(1)}$.)

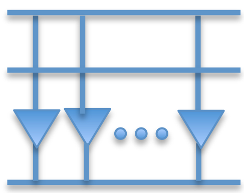
$$= \frac{n!}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi)\, T^{1}{}_{\pi(1)}\cdots T^{n}{}_{\pi(n)}$$

$$= \sum_{\pi\in\mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi)\, T^{1}{}_{\pi(1)}\cdots T^{n}{}_{\pi(n)} \qquad \blacksquare$$

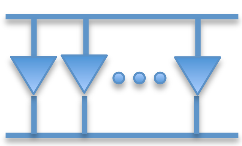
(See my solution to [13.21] for examples of this for n = 2 and 3.)

Theorem. [Bud]  =  = $n!$ 

Proof: Let $P_{a\ldots g}$ be the set of permutations of (a, …, g). Then

 $= \quad n!\,\varepsilon_{a\cdots g} \in^{r\cdots x} T^{[a}{}_{r}\cdots T^{g]}{}_{x}$

$$= \frac{n!}{n!}\varepsilon_{a\cdots g} \in^{r\cdots x} \sum_{\pi\in\mathcal{P}_{ab\cdots g}} \mathrm{Sign}(\pi)\, T^{\pi(a)}{}_{r}\cdots T^{\pi(g)}{}_{x} \overset{(*)}{=} n!\,\varepsilon_{a\cdots g} T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x}$$

= $n!$ 

(*) π is the composition of transmutations (i.e., of pairwise permutations).

Let $\pi^{*}\colon \begin{matrix} c \mapsto e \\ e \mapsto c \end{matrix}$ be a transmutation. Then

$$\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^{*})\, T^{\pi(a)}{}_{r}\cdots T^{\pi(c)}{}_{t}\cdots T^{\pi(e)}{}_{v}\cdots T^{\pi(g)}{}_{x}$$

$$= \varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^{*})\, T^{a}{}_{r}\cdots T^{e}{}_{t}\cdots T^{c}{}_{v}\cdots T^{g}{}_{x}$$

$$= \varepsilon_{a\cdots e\cdots c\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^{*})\, T^{a}{}_{r}\cdots T^{c}{}_{t}\cdots T^{e}{}_{v}\cdots T^{g}{}_{x} \text{ (Rename } c \mapsto e \text{ \& } e \mapsto c\text{)}$$

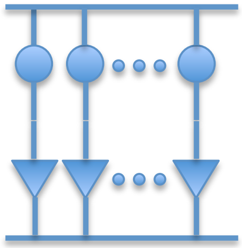
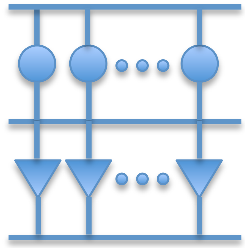
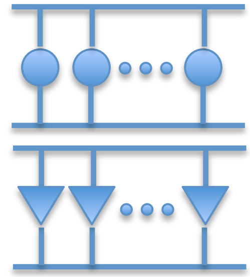
$$= \mathrm{Sign}(\pi^{*})\,\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^{*})\, T^{a}{}_{r}\cdots T^{c}{}_{t}\cdots T^{e}{}_{v}\cdots T^{g}{}_{x}$$

$$= \varepsilon_{a\cdots g} T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x}.$$

So, for any permutation π, we have

$$\varepsilon_{a\cdots g} \in^{r\cdots x} \mathrm{Sign}(\pi)\, T^{\pi(a)}{}_{r}\cdots T^{\pi(g)}{}_{x} = \varepsilon_{a\cdots g} T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x} \qquad \blacksquare$$

Theorem. [13.22]

Det AB = $\frac{1}{n!}$  $= \left(\frac{1}{n!}\right)^2$  $= \left(\frac{1}{n!}\right)^2$ 

= DetA DetB

Corollary. Det $T^{-1} = \dfrac{1}{\text{Det } T}$.

Proof. $1 = \text{Det } I = \text{Det}\left[T\,T^{-1}\right] = \text{Det } T \ \text{Det } T^{-1}$ ■

Corollary. If T is orthogonal, then Det $T = \pm 1$.

Proof. Clearly Det T^{T} = Det T. But $T^{T} = T^{-1}$. So

$$\text{Det } T = \frac{1}{\text{Det }\left(T^{-1}\right)} = \frac{1}{\text{Det }\left(T^{T}\right)} = \frac{1}{\text{Det } T}$$ ■

Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A ≠ 0. ■

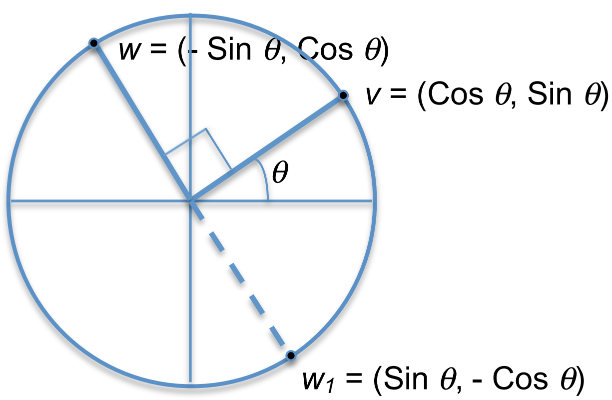
Definition. Vectors v and w are **orthogonal** if $v \bullet w = 0$. That is, the angle between them is 90°.

Theorem. A matrix is orthogonal (i.e., $T^{T} = T^{-1}$) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B

$$\text{Let A} = \begin{pmatrix} v & w \end{pmatrix} = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix}.$$

$$\text{A}^{T} = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta \\ -\text{Sin } \theta & \text{Cos } \theta \end{pmatrix}.$$



$A^T = A^{-1}$:

$$A\,A^T = \begin{pmatrix} \text{Sin}^2\theta + \text{Cos}^2\theta & 0 \\ 0 & \text{Sin}^2\theta + \text{Cos}^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I} \quad ✔$$

Similarly $A^T A = \mathrm{I}$ ✔

So A is an orthogonal matrix ✔

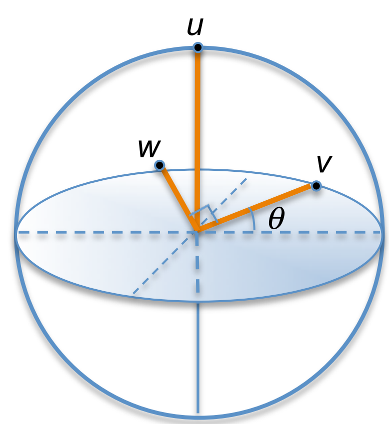
Det A = Det A^T = $\text{Cos}^2\theta + \text{Sin}^2\theta$ = 1 ✔

The column vectors of A are orthogonal: $v \perp w$ ✔

Let $B = \begin{pmatrix} \text{Cos}\,\theta & \text{Sin}\,\theta \\ \text{Sin}\,\theta & -\text{Cos}\,\theta \end{pmatrix} = B^T$. Then $B\,B^T = \mathrm{I}$, Det B = Det B^T = -1, and its column vectors v and w_1 are orthogonal.

Examples. Orthogonal 3 x 3 Matrices: A, B, and C

Let $v = \begin{pmatrix} \text{Cos}\,\theta \\ \text{Sin}\,\theta \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} -\text{Sin}\,\theta \\ \text{Cos}\,\theta \\ 0 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.



Let $A = \begin{pmatrix} v & w & u \end{pmatrix} = \begin{pmatrix} \text{Cos}\,\theta & -\text{Sin}\,\theta & 0 \\ \text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$A^T = \begin{pmatrix} \text{Cos}\,\theta & \text{Sin}\,\theta & 0 \\ -\text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A is orthogonal, its columns are orthogonal vectors, and its determinant is +1. ✔

Let $B = \begin{pmatrix} \text{Cos}\,\theta & -\text{Sin}\,\theta & 0 \\ \text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$. B is orthogonal and its determinant is -1. ✔

Let C be a θ-rotation of A about an axis $\left\{t(a,b,c):\ 0 < t < \infty,\ a^2 + b^2 + c^2 = 1\right\}$:

$$C = \begin{pmatrix} \frac{1}{2}\left[1 + a^2 - b^2 - c^2 + \left(1 - a^2 + b^2 + c^2\right)\text{Cos}\,\theta\right] & 2\text{Sin}\,\frac{\theta}{2}\left(-c\,\text{Cos}\,\frac{\theta}{2} + ab\,\text{Sin}\,\frac{\theta}{2}\right) & 2\text{Sin}\,\frac{\theta}{2}\left(b\,\text{Cos}\,\frac{\theta}{2} + ac\,\text{Sin}\,\frac{\theta}{2}\right) \\ 2\text{Sin}\,\frac{\theta}{2}\left(c\,\text{Cos}\,\frac{\theta}{2} + ab\,\text{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1 - a^2 + b^2 - c^2 + \left(1 + a^2 - b^2 + c^2\right)\text{Cos}\,\theta\right] & 2\text{Sin}\,\frac{\theta}{2}\left(-a\,\text{Cos}\,\frac{\theta}{2} + bc\,\text{Sin}\,\frac{\theta}{2}\right) \\ 2\text{Sin}\,\frac{\theta}{2}\left(-b\,\text{Cos}\,\frac{\theta}{2} + ac\,\text{Sin}\,\frac{\theta}{2}\right) & 2\text{Sin}\,\frac{\theta}{2}\left(a\,\text{Cos}\,\frac{\theta}{2} + bc\,\text{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1 - a^2 - b^2 + c^2 + \left(1 + a^2 + b^2 - c^2\right)\text{Cos}\,\theta\right] \end{pmatrix}$$

It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✔

Definition. A **symmetry** of a vector space (V,+) is a transformation $T : V \mapsto V$ that is 1-1 and onto that preserves the vector space structure:

$$T(a\,v + b\,w) = a\,Tv + b\,Tw$$

Definition. The **General Linear Group GL(*n*)** is the group of symmetries of an *n*-dimensional vector space.

Theorem. GL(n) is the group of non-singular (n x n) matrices.

Proof. Let $T \in$ GL(n). Since $T(a\,v + b\,w) = a\,Tv + b\,Tw$, T is a linear transformation. Were T singular, then by [13.17] Dim $TV < n \;\Rightarrow\; T$ is not onto, a contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. ■

Definition. The **Special Linear Group SL(*n*)** is the subset of GL(n) having determinant = 1.

Theorem. SL(n) is a normal subgroup of GL(n).

Proof. First, SL(n) is a group:

Closed: If $S_1, S_2 \in$ SL(n), then $\mathrm{Det}(S_1 S_2) = \mathrm{Det}(S_1)\,\mathrm{Det}(S_2) = 1$
$\Rightarrow S_1\,S_2 \in$ SL(n).

Identity: $\mathrm{Det}(I) = 1 \Rightarrow I \in$ SL(n)

Inverse: $1 = \mathrm{Det}(I) = \mathrm{Det}(S_1 S_1^{-1}) = \mathrm{Det}(S_1)\,\mathrm{Det}(S_1^{-1}) = \mathrm{Det}(S_1^{-1})$
$\Rightarrow S_1^{-1} \in$ SL(n)

Also, SL(n) is normal:

Let $S \in$ SL(n) and $G \in$ GL(n). Then

$$\mathrm{Det}(G^{-1} S\, G) = \mathrm{Det}(G^{-1})\,\mathrm{Det}(S)\,\mathrm{Det}(G) = \mathrm{Det}(G^{-1})\,\mathrm{Det}(G)$$
$$= \mathrm{Det}(G\,G^{-1}) = \mathrm{Det}(I) = 1$$
$$\Rightarrow G^{-1} S\, G \in \mathrm{SL}(n) \;\Rightarrow\; G^{-1}\,\mathrm{SL}(n)\,G = \mathrm{SL}(n)$$ ■

The groundwork has now been laid to introduce the table, below, that shows the relationships between SO(3), O(3), SL(3), GL(3), general linear transformations, orthogonality, determinants, and symmetries. The table shows that $SO(3) \subset O(3) \subset GL(3) \subset \mathcal{A}(\mathbb{R}^3)$ and $SO(3) \subset SL(3) \subset GL(3)$. It show that GL(3) is both the set of symmetries of $\mathbb{R}^3$ and the set of non-singular matrices.

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe.

They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant ≠ ±1 then orthogonal matrices also uniformly expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than 90°, squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90°, stretching the sphere along that plane.

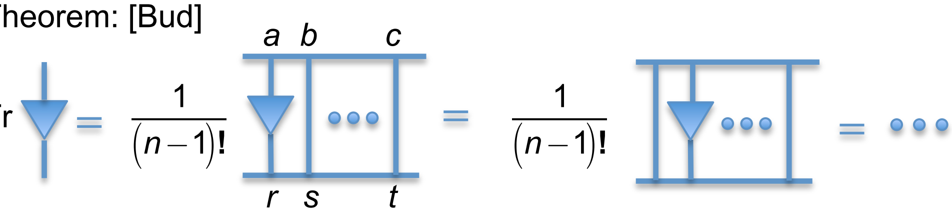
$\mathcal{A}(\mathbb{R}^3)$ = 3 x 3 Matrices

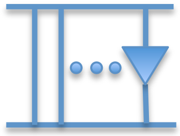
Determinant	Orthogonal	Sphere maps to a …	Matrix Type
0	No	Circle, Ellipse, line segment or point	Singular
Between -1 and 0	No	Contracted reflected sphere or ellipsoid	
Between 0 and +1	No	Contracted sphere or ellipsoid	**GL(3)**
-1	Yes	Reflected sphere	
	No	Reflected ellipsoid O(3)	Non-singular
+1	Yes	SO(3) = sphere SL(3)	
	No	Ellipsoid	Symmetries of $\mathbb{R}^3$
< -1	No	Expanded reflected sphere or ellipsoid	
> 1	No	Expanded sphere or ellipsoid	

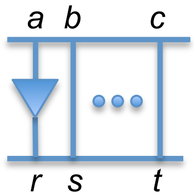
Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

Definition. The **Trace** of A is $T_r(A)$ = Tr  $= T^k{}_k = T^1{}_1 + \cdots T^n{}_n$.

Theorem: [Bud]

Tr 

$= \dfrac{1}{(n-1)!}$ 

Proof: Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{rs\ldots t}$ be the sets of permutations of $(a,b,\ldots,c)$ and $(r,s,\ldots,t)$, respectively. Let B = 

$$= \in^{rs\cdots t} \varepsilon_{ab\cdots c} T^a{}_r \delta^b_s \cdots \delta^c_t = \sum_{\pi\in\mathcal{P}_{ab\cdots c}} \sum_{\pi'\in\mathcal{P}_{rs\cdots t}} \in^{\pi'(r)\pi'(s)\cdots\pi'(t)} \varepsilon_{\pi(a)\pi(b)\cdots\pi(c)} T^{\pi(a)}{}_{\pi'(r)} \delta^{\pi(b)}_{\pi'(s)} \cdots \delta^{\pi(c)}_{\pi'(t)}.$$

Fix π. The only non-zero term in the sum is

$$\in^{\pi(a)\pi(b)\cdots\pi(c)} \varepsilon_{\pi(a)\pi(b)\cdots\pi(c)} T^{\pi(a)}{}_{\pi(a)} \delta^{\pi(b)}_{\pi(b)} \cdots \delta^{\pi(c)}_{\pi(c)} = T^{\pi(a)}{}_{\pi(a)}.$$

I showed in Problem [13.22] that $\in^{xy\cdots z} \varepsilon_{xy\cdots z} = 1$ for any fixed $(x,y,\ldots,z)$.

Thus, B = $\sum_{\pi\in\mathcal{P}_{ab\cdots c}} T^{\pi(a)}{}_{\pi(a)}$. This sum has n! terms composed of $(n-1)$! terms equal to $T^a{}_a$, $(n-1)$! terms equal to $T^b{}_b$, …, and $(n-1)$! terms equal to $T^c{}_c$. So,

B = $(n-1)!\,(T^a{}_a + T^b{}_b + \ldots + T^c{}_c) = (n-1)!\ \mathrm{Tr}\,(\mathrm{A}) = (n-1)!\ \mathrm{Tr}$ 

Similarly for the other figures. ■

Theorem. [13.24] $\mathrm{Det}(\mathrm{I} + \in \mathrm{A}) = 1 + \in \mathrm{Tr}(\mathrm{A})$ if we ignore 2nd order and higher $\in$ terms.

Theorem. [13.25] Det $e^{\mathrm{A}} = e^{\mathrm{Tr}(\mathrm{A})}$.

Definition. An **Eigenvector** is a non-zero vector v for which $\exists\lambda\in\mathbb{C}$ such that $Tv = \lambda v \;\Leftrightarrow\; (T - \lambda I)v = 0$. λ is called an **Eigenvalue.**

Note: $\mathrm{Det}(T - \lambda I) = 0$ and so $(T - \lambda I)$ is singular

Theorem. [13.26] $\mathrm{Det}(T\text{-}\lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) = 0$ is a polynomial equation of degree n.

Definition. λ **has multiplicity** $\boldsymbol{r}$ means that λ appears r times in the equation above. Eigenvalue multiplicities are called **degeneracies** in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to λ is a linear space called an **Eigenspace**.

Theorem. If d is the dimension of the Eigenspace of λ and r is the multiplicity of λ then $1 \le d \le r$.

Theorem. [13.27] Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then $\sum r_i = n$.

Corollary. A linear transformation T has at least 1 Eigenvector.
Theorem. [13.30] Suppose $\{e_k\}$ and $\{f_k\}$ are bases for a vector space V, and $f_k = Te_k$. Then

$$f_j = \begin{pmatrix} T^1{}_j \\ \vdots \\ T^n{}_j \end{pmatrix}.$$

That is, the components of f_j in basis $\{e_k\}$ are $\left(T^1{}_j, \cdots, T^n{}_j\right)$.

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

$$T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then there is a basis for V such that the matrix of T in this basis is

$$T = \left(\begin{array}{ccccc|cccc|cccc}
\lambda_1 & 1 & & & & & & & & & & \cdots & \mathbf{0} \\
 & \lambda_1 & 1 & & & & & & & & & \iddots & \vdots \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & \ddots & 1 & & & & & & & & \\
 & & & & \lambda_1 & 0 & & & & & & & \\
\hline
 & & & & & \lambda_2 & 1 & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & \ddots & 1 & & & & \\
 & & & & & & & & \lambda_{n-1} & 0 & & & \\
\hline
 & & & & & & & & & \lambda_n & 1 & & \\
 & & & & & & & & & & \lambda_n & \ddots & \\
\vdots & \iddots & & & & & & & & & & \ddots & 1 \\
\mathbf{0} & \cdots & & & & & & & & & & & \lambda_n
\end{array}\right).$$

Representations and Lie Algebras

Definition. Let $T : \mathrm{G} \to \mathcal{G}$ be a homomorphism of a group G to some well-known standard group $\mathcal{G}$. The image T(G) is called a **Group Representation of G**. In this section we take $\mathcal{G}$ to be GL(n), the multiplicative group of non-singular $n \times n$ matrices. T is **faithful** if it is 1-1.

Theorem. [13.32] Every finite group has a faithful representation. Every finite dimensional Lie group has a locally faithful (?) representation.

We will see in Chapter 14 that the theory of representations of continuous groups by linear transformations can be converted to the study of representations of Lie algebras, which we define next.

Definition. A nonempty set (R, +, •) is a **ring** if for all a, b, c in R:

(1) (R, +) is an Abelian group

(2) R is closed under multiplication •

(3) $a \bullet (b + c) = a \bullet b + a \bullet c$ and $(b + c) \bullet a = b \bullet a + c \bullet a$ (left and right distributive)

A ring R is an **associative ring** if it is associative under multiplication:

(4) $r \bullet (s \bullet t) = (r \bullet s) \bullet t$ for $r, s, t \in \mathrm{R}$

There are rings that have no multiplicative identity (i.e., no element 1). Rings that do have a multiplicative identity are said to be **rings with unit element**.

Definition. An **algebra** is ring that is also a vector space (that is, it has scalar multiplication in addition to addition and regular multiplication) and that for all *a*, *b* in R and scalar α we have $\alpha(a\,b) = (\alpha a)b = a(\alpha b)$. If the underlying ring is associative, then it is an **associative algebra**.

Example. If V is a vector space then the set of linear transformations, $\mathcal{A}$(V), is an algebra. E.g., $\mathcal{A}(\mathbb{R}^3)$, the set of 3 × 3 matrices, is an algebra: you can add and multiply matrices as well as multiply them by scalars.

Example. GL(3) is not an algebra. It is just a group. It is not even a ring because it is not closed under matrix addition. For example, addition of 2 non-singular matrices can yield the zero matrix, which is singular and, hence, not in GL(3).

I give below the standard definition of a Lie algebra. Penrose does not give this definition so I will shortly prove that his definition (of a special case) is indeed a Lie algebra.

Definition. (Standard definition) ($\mathfrak{g}, [\bullet,\bullet]$) is a **Lie algebra** if $\mathfrak{g}$ is a vector space over a field F with the **Lie bracket**, a binary operator that satisfies

- **Bilinearity:**
  $[ax+by,z]=a[x,z]+b[y,z]$ and $[z,ax+by]=a[z,x]+b[z,y]$
  $\forall a,b\in F$ and $x,y,z\in\mathfrak{g}$
- **Alternativity:** $[x,x]=0\ \forall x\in\mathfrak{g}$
- **Jacobi Identity:** $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0\ \forall x,y,z\in\mathfrak{g}$

Note that blinearity and alternativity imply

- **Anticommutativity**: [*x,y*] = - [*y,x*] :

$$0=[x+y,x+y]=\cancel{[x,x]}+[x,y]+[y,x]+\cancel{[y,y]} \qquad ✔$$

Theorem. [13.34] Let A, B $\in\mathfrak{g}$. Then

(a) $(I+\in A)(I+\in B)=I+\in\ A+B$ if we ignore terms $o(\in)^2$

(b) $(I+\in A)(I+\in B)(I+\in A)^{-1}(I+\in B)^{-1}=I+\in^2[A,B]$ if we ignore $o(\in)^3$

If “infinitesimals” *a* and *b* are represented by $(I+\in A)$ and $(I+\in B)$, then we see that the product $aba^{-1}b^{-1}$ is represented by Lie brackets. Thus we make the following definition.

Definition. Let G be a group. $\{aba^{-1}b^{-1}: a,b \in G\}$ is the **set of group commutators.**

Definition. (Penrose's definition) Let $\mathcal{G}$ be a subgroup of GL(n). Let $\mathcal{G}^*$ be the vector space generated from $\mathcal{G}$ by the addition of scalar multiplication. For A, B $\in \mathcal{G}^*$, a **Lie bracket** is **[A,B] ≡ AB – BA** (commutator operation). Note that [A,B] is not necessarily in $\mathcal{G}$ because $\mathcal{G}$ is not required to be closed under subtraction. The **Lie algebra generated by** $\mathcal{G}$ is the algebra $\mathfrak{g}$ generated from $\mathcal{G}^*$ by the addition of +, –, and Lie bracket operations.

We wish to show that $\mathfrak{g}$ is a Lie algebra. The next theorem shows that $\mathfrak{g}$ is more than simply a vector space as required in the standard definition of Lie algebra.

Theorem. $\mathfrak{g}$ is an associative algebra

Proof: We begin with a (multiplicative) subgroup $\mathcal{G}$ of GL(n) and its expansion to a vector space $\mathcal{G}^*$. The set $\mathfrak{g}$ is obtained from $\mathcal{G}^*$ by repeated application of the operations +, –, and $[\bullet,\bullet]$.

($\mathfrak{g}$, +) is an Abelian group:
- Closed under + since + is a repeated operation until nothing new is obtained
- The zero matrix is the additive identity and belongs to $\mathfrak{g}$
- \- A is the additive inverse of A and is in $\mathfrak{g}$ since – is a repeated operation
- A+B = B+A since these are matrices

$\mathfrak{g}$ is closed under group multiplication : We show this by induction, constructing $\mathfrak{g}$ by "levels". Let

$$\mathfrak{g}_0 = \mathcal{G}^* \qquad \text{(level 0)}$$
$$\mathfrak{g}_1 = \{A \pm B : A,B \in \mathcal{G}^*\} \cup \{[A,B] : A,B \in \mathcal{G}^*\} \qquad \text{(level 1)}$$
$$\mathfrak{g}_2 = \{A \pm B : A,B \in \mathfrak{g}_1\} \cup \{[A,B] : A,B \in \mathfrak{g}_1\} \qquad \text{(level 2)}$$
$$\mathfrak{g}_3 = \{A \pm B : A,B \in \mathfrak{g}_2\} \cup \{[A,B] : A,B \in \mathfrak{g}_2\} \qquad \text{(level 3)}$$
$$\vdots$$

Level 0 is closed under multiplication since it is a multiplicative group. Supposed level n – 1 is closed under multiplication and let E and F belong to level n. Then E = A±B or E = [A,B] where A, B $\in \mathfrak{g}_{n-1}$ and F = C±D or F = [C,D] where C, D$\in \mathfrak{g}_{n-1}$. If EF = (A ± B) (C ±D) then EF = AC ± AD ± BC ± BD $\in \mathfrak{g}_n$ because AC, AD, BC, and BD $\in \mathfrak{g}_{n-1}$ because $\mathfrak{g}_{n-1}$ is closed under multiplication. If EF = (A±B) [C,D], then EF = ACD – ADC ± BCD $\mp$ BDC $\in \mathfrak{g}_n$ because ACD, ADC, BCD, and BDC $\in \mathfrak{g}_{n-1}$ because

$\mathfrak{g}_{n-1}$ is closed under multiplication. Similarly, if EF = [A,B] [C,D], then EF = ABCD – ABDC + … + BADC $\in \mathfrak{g}_n$.

We also have to show that the construction process stops at (the first) infinity, and it does because if A, B$\in \mathfrak{g}_\infty$ then $\exists\, n$ such that A, B $\in \mathfrak{g}_n$ and so A±B and [A,B] $\in \mathfrak{g}_{n+1} \subseteq \mathfrak{g}_\infty$.

The distributive property holds:

If A, B, C $\in \mathfrak{g}$ then A(B+C) = AB + AC since A, B, and C are matrices.

Similarly, the associative property for multiplication holds because we are dealing with matrices.

Finally, $\mathfrak{g}$ is a vector space:

The only property that is not obvious is closure of scalar multiplication. If α is a scalar and A $\in \mathfrak{g}$, then A is constructed via the induction process from a finite number of elements B, …, D belonging to $\mathcal{G}^*$. Since $\mathcal{G}^*$ is a vector space then αB, …, αD also belong to $\mathcal{G}^*$, and αA is constructed from them in a parallel process. So αA $\in \mathfrak{g}$, proving $\mathfrak{g}$ is a vector space. ■

Theorem. [13.35] Let A, B $\in \mathfrak{g}$ and $\lambda \in \mathbb{C}$. Then

(a) [A+B,C] = [A,C] + [B,C] and [λA,B] = λ [A,B] (Lie bracket left distributivity)

(b) [B,A] = - [A,B] (Lie bracket antisymmetry, also called anticommutativity)

(c) [A, [B,C]] + [B, [C,A]] + [C, [A,B]] = 0 (Jacobi identity)

(d) dim $\mathcal{G}^*$ ≤ dim G . If T is faithful, then dim $\mathcal{G}^*$ = dim G

Corollary. The Lie algebra $\mathfrak{g}$ generated by $\mathcal{G}$ is a Lie algebra.

Proof.

Right distributivity:

[C,A+B] = [C,A]+[C,B] and [A,λB] = λ [A,B] follows from (a) and (b).

Bilinearity:

This follows from (a) and right distributivity.

Alternativity:

Let A $\in \mathfrak{g}$. By (b), B = [A,A] = - [A,A] = - B. Since a vector space has a unique zero, B= 0. ■

Convention. Henceforth we assume T is faithful. Thus, dim $\mathcal{G}^*$ = n = dim $\mathcal{G}$.

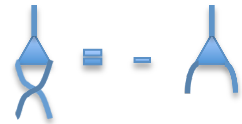
Definition. Let n be the dimension of the vector space $\mathcal{G}^*$ and (E_1, E_2, …, E_n) a basis for $\mathcal{G}^*$. Then

$$\exists \gamma_{\alpha\beta}{}^{\chi} \text{ where } \alpha,\beta,\chi \in \{1,\ 2,\ \cdots,\ n\} \text{ such that } \left[E_\alpha, E_\beta\right] = \gamma_{\alpha\beta}{}^{\chi} E_\chi .$$

The n^3 components $\gamma_{\alpha\beta}{}^{\chi}$ are called the **structural constants for G** and can be expressed in diagrammatic form as shown.

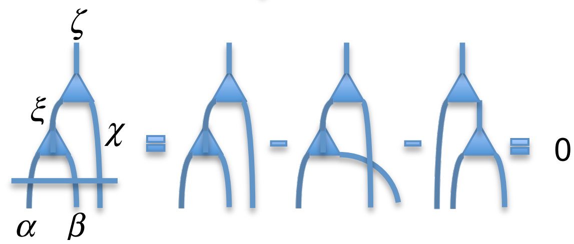
The $\gamma_{\alpha\beta}{}^{\chi}$ are not all independent because they satisfy relations in the next theorem.

Theorem. [13.36] $\gamma_{\beta\alpha}{}^{\chi} = -\gamma_{\alpha\beta}{}^{\chi}$ and $\gamma_{[\alpha\beta}{}^{\xi}\gamma_{\chi]\xi}{}^{\zeta} = 0$



Proof. This follows from Lie bracket antisymmetry and the Jacobi identity. ■

This theorem can be expressed in diagrammatic form as shown.



Definition. Let V be a vector space. The **dual space V*** is defined to be $V^* = \left\{ f : V \to \mathbb{R} \text{ or } \mathbb{C} : f \text{ is a linear map} \right\}$. By convention the vector space consists of column vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. The next theorem says that the dual space consists of row vectors $y = \left[y_1 \cdots y_n \right]$.

Theorem. Let V be a vector space (of column vectors) and $V^{\mathrm{T}} = \left[y : y \text{ is a row vector and } y^{\mathrm{T}} \in V \right]$ where the superscript T means transpose. Then

$$(1)\ V^* = \left\{ f_y : V \to \mathbb{R} \text{ or } \mathbb{C} : \ f_y : x \mapsto y\,x \text{ for } x \in V \text{ and } y \in V^{\mathrm{T}} \right\}$$

and

$$(2)\ V^* \cong V^{\mathrm{T}}.$$

Proof. Let $x \in$ V denote a column vector: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Consider the basis for V composed of {e_1, e_2, …, e_n} *where* $e_k = \begin{bmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{bmatrix}$.

Let $f \in \mathrm{V}^*$. Define $y_k = f(e_k)$ for all k and set $y = [y_1, \cdots, y_n]$. Then

$$f(x) = f\left(\sum_{k=1}^{n} x_k\, e_k\right) = \sum_{k=1}^{n} f(x_k\, e_k) = \sum_{k=1}^{n} x_k\, f(\, e_k) = \sum_{k=1}^{n} x_k\, y_k$$

$$= [y_1, \cdots, y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = y\, x$$

$$= f_y(x)$$

Thus every linear map $f : \mathrm{V} \to \mathbb{R}$ or $\mathbb{C}$ equals f_y for some $y \in \mathrm{V}$, and this shows also that every map f_y is linear and thus belongs to V*. This proves (1) ✔

The mapping $f : \mathrm{V}^{\mathrm{T}} \to \mathrm{V}^* : f(y) = f_y$ is an isomorphism because

$$[f(y+z)](x) = f_{y+z}(x) = (y+z)\, x = y\, x + z\, x = f_y(x) + f_z(x) = [f(y) + f(z)](x)$$

for y, $z \in \mathrm{V}^{\mathrm{T}}$, and

$[f(\alpha\, y)](x) = f_{\alpha y}(x) = \alpha\, y\, x = \alpha f_y(x) = [\alpha f(y)](x)$ for a scalar α. ■

Note: Is V^{T} isomorphic to V? Consider the natural map $\mu : \mathrm{V}^{\mathrm{T}} \to \mathrm{V} : y \mapsto y^{\mathrm{T}}$ where y is a row vector in V^{T} and y^{T} is a column vector in V. Set $y_3 = y_1 + y_2$. Then $\mu(y_1 + y_2) = \mu(y_3) = {y_3}^{\mathrm{T}} = {y_1}^{\mathrm{T}} + {y_2}^{\mathrm{T}} = \mu(y_1) + \mu(y_2)$ and $\mu(\alpha\, y) = (\alpha\, y)^{\mathrm{T}} = \alpha\, y^{\mathrm{T}} = \alpha\, \mu(y)$. Thus μ is a homomorphism and it is clearly 1-1 and onto.

Definition. Let G be a group. A vector space V is called a **representation space for G** if G is represented by a group $\mathcal{G}$ of linear transformations on V.

Let $T \in \mathcal{G}$. For x $\in$ V, $T: \mathrm{V} \to \mathrm{V}$ can be written as $x \mapsto Tx$, or in matrix form as $x^a \mapsto T^a{}_b x^b$ where $T^a{}_b$ is an n x n matrix and x^a and x^b are column n-vectors.

Set

$$S = T^{-1}, \text{ or } S^a{}_b = \left(T^a{}_b\right)^{-1}.$$

$$\begin{tikzcd} \mathrm{V} \arrow[r, "S"] \arrow[dr, dashed, "yS"'] & \mathrm{V} \arrow[d, "y"] \\ & \mathbb{R} \text{ or } \mathbb{C} \end{tikzcd}$$

Then

$$ST = I, \text{ or } S^a{}_b T^b{}_c = \delta^a_c .$$

Let $y \in$ V* (actually, y $\in \mathrm{V}^{\mathrm{T}} \cong$ V*). Then

$$y: \mathrm{V} \to \mathbb{R} \text{ or } \mathbb{C}: x \mapsto y\,x$$

for $x \in$ V. Consider the identity map

$$y\,x \mapsto y\,x = y(ST)x = (yS)(Tx).$$

Since

$$x \mapsto Tx,$$

if this map is a homomorphism preserving yx, then it must also map

$$y \mapsto yS, \text{ or } y_a \mapsto y_b S^b{}_a$$

where $S^b{}_a = \left(T^b{}_a\right)^{-1}$ is a matrix and y_a and y_b are row vectors. We prefer to use column vectors so we write

$$y^{\mathrm{T}} \mapsto S^{\mathrm{T}} y^{\mathrm{T}}, \text{ or } y^a \mapsto \left(S^{\mathrm{T}}\right)^a{}_b\, y^b$$

Observe from the figure that $yS \in \mathrm{V}^*$. Also observe that the inverse operation reverses the order of multiplication and the transpose reverses it back to the same order as for T.

The mapping $y^{\mathrm{T}} \mapsto S^{\mathrm{T}} y^{\mathrm{T}}$ plays the central role in the representation-space theorem for tensors (in a few pages). In the representation-space theorem for the dual space, next, the mapping $(y \mapsto yS): \mathrm{V}^* \to \mathrm{V}^*$ plays the central role.

Theorem. If V is a representation space for a group G, then so is V*.

Proof. By definition of representation, there is a subgroup $\mathcal{G} \subset \mathrm{GL}(n)$ and an isomorphism $T: G \to \mathcal{G}$ such that $\forall g \in G$ $T(g): \mathrm{V} \to \mathrm{V}$ is a linear transformation on V. We seek another subgroup $\mathcal{G}^* \subset \mathrm{GL}(n)$ and an isomorphism $\mathcal{T}: G \to \mathcal{G}^*$ such that $\forall g \in G$ $\mathcal{T}(g): \mathrm{V}^* \to \mathrm{V}^*$ is a linear transformation on V*.

Define

(1) $T_g = T(g)$ for $g \in$ G.

Since T is an isomorphism,

(2) $T_{g_1 g_2} \overset{(1)}{=} T(g_1 g_2) = T(g_1)\, T(g_2) \overset{(1)}{=} T_{g_1}\, T_{g_2}$ for $g_1, g_2 \in G$.

Let

(3) $\quad S_g = T_g^{-1}$.

Define a mapping

(4) $\quad \mathcal{T}_g : \mathrm{V}^* \to \mathrm{V}^* : \mathcal{T}_g(y) = y\,S_g$ for $y \in \mathrm{V}^*$.

Recall that y is a row vector as is $y\,S$. Define

$$\mathcal{G}^* = \left\{\mathcal{T}_g : g \in G\right\} \text{ and}$$

(5) $\quad \mathcal{T} : G \to \mathcal{G}^* : \mathcal{T}(g) = \mathcal{T}_g$ for $g \in$ G.

$\mathcal{G}^*$ Is clearly a subgroup of GL(n). We need to show that $\mathcal{T}$ is an isomorphism.

Let $g_1, g_2 \in$ G and set

(6) $\quad g_3 = g_1\, g_2$.

Then

(7) $\quad T_{g_3} \overset{(6)}{=} T_{g_1 g_2} \overset{(2)}{=} T_{g_1}\, T_{g_2}$

which implies

(8) $\quad S_{g_3} \overset{(3)}{=} T_{g_3}^{\;-1} \overset{(7)}{=} T_{g_2}^{\;-1}\, T_{g_1}^{\;-1} \overset{(3)}{=} S_{g_2}\, S_{g_1}$.

Let $y \in$ V*. Then

$$\begin{aligned}\mathcal{T}(g_1 g_2)(y) &\overset{(6)}{=} \mathcal{T}(g_3)(y) \overset{(5)}{=} \mathcal{T}_{g_3}(y) \overset{(4)}{=} y\,S_{g_3} \overset{(8)}{=} y\,S_{g_2}\, S_{g_1} \overset{(4)}{=} \left[\mathcal{T}_{g_2}(y)\right] S_{g_1} \\ &\overset{(4)}{=} \mathcal{T}_{g_1}\left[\mathcal{T}_{g_2}(y)\right] \overset{(5)}{=} \mathcal{T}(g_1)\left[\mathcal{T}(g_2)(y)\right] = \left[\mathcal{T}(g_1)\mathcal{T}(g_2)\right](y),\end{aligned}$$

or

$$\mathcal{T}(g_1 g_2) = \mathcal{T}(g_1)\mathcal{T}(g_2).$$

That is, $\mathcal{T}$ is a homomorphism ✔

To show that $\mathcal{T}$ is an isomorphism, we must show that it is 1-1.

$$\begin{aligned}&\mathcal{T}(g_1) = \mathcal{T}(g_2) \quad \overset{(5)}{\Leftrightarrow} \quad \mathcal{T}_{g_1} = \mathcal{T}_{g_2} \quad \overset{(4)}{\Leftrightarrow} \quad y\,S_{g_1} = y\,S_{g_2} \quad \forall\, \mathrm{y} \in \mathrm{V}^* \\ &\Leftrightarrow \quad S_{g_1} = S_{g_2} \quad \overset{(3)}{\Leftrightarrow} \quad T_{g_1} = T_{g_2} \quad \overset{(1)}{\Leftrightarrow} \quad T(g_1) = T(g_2) \\ &\Rightarrow \quad g_1 = g_2 \text{ since } T \text{ is an isomorphism.}\end{aligned}$$

So $\mathcal{T}$ is 1-1. ✔ ■

Tensors

Definitions. Let V and W be vector spaces (over the same field) having bases

$\left\{e^i\right\}_{i=1}^m$ and $\left\{f^j\right\}_{j=1}^n$, respectively, where $e^i = \begin{bmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{bmatrix}$ and $f^j = \begin{bmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{bmatrix}$ are vectors.

Let $v = v^i = v_i e^i = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, $w = w^j = w_j e^j = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, and $T: \mathrm{V} \times \mathrm{W} \to \mathrm{V} \times \mathrm{W}$.

A map *T* is an outer product if $T(v,w) = \sum_{i,j} v_i * w_j$ for some operation.

If $m = n$, **a map *T* is an inner product** if $T(v,w) = \sum_{i=1}^{n} v_i * w_i$

***T* is bilinear** if it is linear in v and w independently; i.e., T satisfies

(1) $\alpha T(v,w) = T(\alpha v,w) = T(v,\alpha w)$

(2) $T(u+v,w) = T(u,w) + T(v,w)$

(3) $T(v,w+x) = T(v,w) + T(v,x)$

The **tensor product of vector spaces V and W** is the vector space

$$\mathbf{V} \otimes \mathbf{W} : \left\{\alpha_{ij}\, e^i \otimes f^j : \alpha \text{ is a scalar}\right\}$$

with addition defined component-wise

$$P^{ij} + Q^{ij} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix} = \begin{bmatrix} \alpha_{11}+\beta_{11} & \cdots & \alpha_{1n}+\beta_{1n} \\ \vdots & & \vdots \\ \alpha_{m1}+\beta_{m1} & \cdots & \alpha_{mn}+\beta_{mn} \end{bmatrix}$$

Note. Dim ($\mathrm{V} \otimes \mathrm{W}$) = (Dim V) (Dim W):

$\left\{e_i \otimes f_j\right\}$ has nm elements and is a basis for $\mathrm{V} \otimes \mathrm{W}$.

The **tensor product** $\otimes : \mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes \mathrm{W}$ is the outer product

$$\boldsymbol{v}\otimes\boldsymbol{w}=\begin{bmatrix} v_1\\ \vdots\\ v_m\end{bmatrix}\otimes\begin{bmatrix} w_1\\ \vdots\\ w_n\end{bmatrix}=\begin{bmatrix} v_1\otimes\begin{bmatrix} w_1\\ \vdots\\ w_n\end{bmatrix}\\ \vdots\\ v_m\otimes\begin{bmatrix} w_1\\ \vdots\\ w_n\end{bmatrix}\end{bmatrix}=\begin{bmatrix} v_1\otimes w_1 & \cdots & v_1\otimes w_n\\ \vdots & & \vdots\\ v_m\otimes w_1 & \cdots & v_m\otimes w_n\end{bmatrix}$$

$$=\begin{bmatrix} v_1 w_1 e^1\otimes f^1 & \cdots & v_1 w_n e^1\otimes f^n\\ \vdots & & \vdots\\ v_m w_1 e^m\otimes f^1 & \cdots & v_m w_n e^m\otimes f^n\end{bmatrix} = \mathrm{v}_i\,\mathrm{w}_j\, e^i\otimes e^j = v^i\otimes w^j$$

or $\begin{bmatrix} v_1 w_1 & \cdots & v_1 w_n\\ \vdots & & \vdots\\ v_m w_1 & \cdots & v_m w_n\end{bmatrix}$ for short,

with the following rules (that make $\otimes$ bilinear):

(1) $\alpha(v\otimes w)=(\alpha v)\otimes w=v\otimes(\alpha w)$, where α is a scalar.

(2) $(x+v)\otimes w=x\otimes w+v\otimes w$

$v\otimes(y+w)=v\otimes y+v\otimes w$.

Rules (1) and (2) appear more natural in matrix form:

(1) $$\alpha\begin{bmatrix} v_1 w_1 & \cdots & v_1 w_n\\ \vdots & & \vdots\\ v_m w_1 & \cdots & v_m w_n\end{bmatrix}=\begin{bmatrix} (\alpha v_1)w_1 & \cdots & (\alpha v_1)w_n\\ \vdots & & \vdots\\ (\alpha v_m)w_1 & \cdots & (\alpha v_m)w_n\end{bmatrix}$$

$$=\begin{bmatrix} v_1(\alpha w_1) & \cdots & v_1(\alpha w_n)\\ \vdots & & \vdots\\ v_m(\alpha w_1) & \cdots & v_m(\alpha w_n)\end{bmatrix}$$

(2) $\begin{bmatrix} (x_1+v_1)w_1 & \cdots & (x_1+v_1)w_n \\ \vdots & & \vdots \\ (x_m+v_m)w_1 & \cdots & (x_m+v_m)w_n \end{bmatrix}$

$$= \begin{bmatrix} x_1w_1 & \cdots & x_1w_n \\ \vdots & & \vdots \\ x_mw_1 & \cdots & x_mw_n \end{bmatrix} + \begin{bmatrix} v_1w_1 & \cdots & v_1w_n \\ \vdots & & \vdots \\ v_mw_1 & \cdots & v_mw_n \end{bmatrix}$$

Note 1. The components $v_i \otimes w_j = v_i w_j$ may not be multiplied out, reverse ordered, or combined in any way because they belong to different vector spaces. Also, in Quantum Mechanics, *v* is the state of one system and *w* the state of another system. So combining them in any way doesn't make sense.

Note 2. $v \otimes w \in \mathrm{V} \otimes \mathrm{V}$ as well as sums of such products.

Note 3. 0 = $0 \otimes w = v \otimes 0 = 0 \otimes 0$ and $-(v \otimes w) = (-v) \otimes w = v \otimes (-w)$:

$0 = 0 \otimes w = v \otimes 0$ follows from (1) by setting α = 0. Of course this includes $0 = 0 \otimes 0$ since it holds for *v* = 0 = *w*. Also,

$$0 + v \otimes w = 0 \otimes w + v \otimes w \overset{(2)}{=} (0+v) \otimes w = v \otimes w \quad ✔$$

$$v \otimes w + (-v) \otimes w \overset{(2)}{=} (v-v) \otimes w = 0 \quad ✔$$

Example. $Q = \begin{bmatrix} 0 & 1\otimes 1 \\ -1\otimes 1 & 0 \end{bmatrix}$ is an example of an element of $\mathrm{V} \otimes \mathrm{W}$ that cannot be expressed as $v \otimes w$ for any $v \in \mathrm{V}$ and $w \in \mathrm{W}$:

Suppose $v = \begin{bmatrix} a \\ b \end{bmatrix}$ and $w = \begin{bmatrix} r \\ s \end{bmatrix}$. Then $v \otimes w = \begin{bmatrix} a\otimes r & a\otimes s \\ b\otimes r & b\otimes s \end{bmatrix}$. So *a* = 0 or *r* = 0. If *a* = 0 then $v \otimes w = \begin{bmatrix} 0 & 0 \\ b\otimes r & b\otimes s \end{bmatrix} \Rightarrow 1\otimes 1 = 0$, a contradiction. If *r* = 0 then $v \otimes w = \begin{bmatrix} 0 & a\otimes s \\ 0 & b\otimes s \end{bmatrix} \Rightarrow -1\otimes 1 = 0$, a contraction.

Theorem. Suppose $x \otimes y = v \otimes w$ where $x = x_i e^i$, $y = y_j e^j$, $v = v_i e^i$, and $w = w_j e^j$. Then $\forall\, i, j \;\; x_i y_j = v_i w_j$.

Proof. Since $\otimes$ is a bilinear operation,

$$0 = x \otimes y - v \otimes w = x_i e^i \otimes y_j f^j - v_i e^i \otimes w_j f^j$$

$$= x_i y_j e^i \otimes f^j - v_i w_j e^i \otimes f^j = (x_i y_j - v_i w_j) e^i \otimes f^j \qquad ■$$

Note. If $x=\alpha v$ and $y=\frac{1}{\alpha}w$, then $x_i y_j=\alpha v_i \frac{1}{\alpha}w_j = v_i w_j$ and so $x\otimes y = v\otimes w$. However, it does not have to be so tidy. There can be a different $\alpha=\alpha_{ij}$ for each *i j*-pair.

Notation. $Q^{ab}=x^a\otimes w^b$ denotes the *a*-*b* component of the tensor $Q=x\otimes w$. $Q^{ba}=x^b\otimes w^a$ denotes the *b*-*a* component. When there is no confusion we often refer to the tensor as Q^{ab}.

Definition. Let S be a linear transformation on V and *T* a linear transformation on W. The **tensor product of S and T** is the bilinear transformation

$$S\otimes T : \mathrm{V}\otimes\mathrm{W}\to\mathrm{V}\otimes\mathrm{W} : S\otimes T\left(v\otimes w\right)=Sv\otimes Tw\,.$$

Note 1. **Bilinear** means linear in each of V and W separately, a reminder that there is no mixing of V and W.

Note 2. **Multilinear** means linear in each of several vector spaces separately.

Note 3. In matrix notation, $S=S^a{}_b$ and $T=T^c{}_d$ are 2-dimensional arrays, i.e., *n* x *n* matrices. Since we don't mix *S* and *T*, $S\otimes T=R^{ac}_{bd}$, a 4-dimensional array, or an *n* x *n* x *n* x *n* matrix. Similarly, $v\otimes w=u^{ij}$ is a 2-dimensional array.

Definition. Let V be an *n*-dimensional vector space and let V* be its dual space. Recall that if a vector space is composed of column vectors $x^f,\cdots,x^h$ then the dual space is composed of row vectors $y_a,\cdots,y_c$.

Let *p* and *q* be positive integers and let V_a, …, V_c, be *q* copies of V and let V_e , …, V_f be *p* copies of V. The **tensor product space of V** is

$$\mathcal{V}=\mathrm{V}_a{}^*\otimes\cdots\otimes\mathrm{V}_c{}^*\otimes\mathrm{V}_f\otimes\cdots\otimes\mathrm{V}_h\,.$$

Theorem. [13.38] $\mathcal{V}$ is an n^{p+q}-dimensional vector space.

Proof. This follows from the prior theorem since Dim $\mathcal{V}=\underbrace{n\times\cdots\times n}_{q\ terms}\times\underbrace{n\times\cdots\times n}_{p\ terms}$. ■

Definition. **An element of** $\mathcal{V}$ can be denoted

$$\boldsymbol{Q}=\boldsymbol{Q}^{f\cdots h}_{a\cdots c}=y_a\otimes\cdots\otimes y_c\otimes x^f\otimes\cdots\otimes x^h\,.$$

Recall from Chapter 12 that ***Q* is a** $\begin{bmatrix} p \\ q \end{bmatrix}$**-valent tensor over V**, an **abstract quantity** with *p* upper and *q* lower indices. Recall that "abstract" means that *Q* is not tied to a particular basis for V.

Q can be expressed as a generalized $n \times n \times \cdots \times n$ matrix, a **(p+q)-dimensional array**. For example, Q^a_b is an n x n matrix, a 2-dimensional array. Q^{ab}_c is an n x n x n matrix, a 3-dimensional array.

Q^{T} can be considered to be an element of $\mathcal{V}$* , a multilinear function, as follows. Consider q row vectors $A_a, \cdots, C_c$ and p column vectors $F^f, \cdots, H^h$. Then $Q^{\mathrm{T}} : A_a \cdots C_c\, F^f \cdots H^h \mapsto A_a \cdots C_c\, Q^{a \cdots c}_{f \cdots h}\, F^f \cdots H^h$. We sometimes write this as $Q^{\mathrm{T}} : \mathcal{V} \to \mathbb{R}$ or $\mathbb{C} : Q^{\mathrm{T}}\left(y, \cdots, y, x, \cdots, x\right) = y_a \cdots y_c\, Q^{a \cdots c}_{f \cdots h}\, x^f \cdots x^h$.

Array Dimension	Tensor	# of Entries
0	Scalar	1
1	Vector	n
2	Matrix	n^2
3	3-Tensor (cube of numbers)	n^3
n	n-Tensor (n-dimensional hypercube of numbers)	n^k

Example. Let V = $\mathbb{R}^2$, Let S and T be linear transformations on V. Given a basis for V, S and T can be represented by matrices A and B, respectively:

$$A^a{}_{a'} = \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & b^2{}_2 \end{pmatrix} \text{ and } B^b{}_{b'} = \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix}.$$

Then

$$A^a{}_{a'} \otimes B^b{}_{b'} = \begin{bmatrix} a^1{}_1 \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix} & a^1{}_2 \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix} \\ a^2{}_1 \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix} & a^2{}_2 \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} a^1{}_1 b^1{}_1 & a^1{}_1 b^1{}_2 \\ a^1{}_1 b^2{}_1 & a^1{}_1 b^2{}_2 \end{pmatrix} & \begin{pmatrix} a^1{}_2 b^1{}_1 & a^1{}_2 b^1{}_2 \\ a^1{}_2 b^2{}_1 & a^1{}_2 b^2{}_2 \end{pmatrix} \\ \begin{pmatrix} a^2{}_1 b^1{}_1 & a^2{}_1 b^1{}_2 \\ a^2{}_1 b^2{}_1 & a^2{}_1 b^2{}_2 \end{pmatrix} & \begin{pmatrix} a^2{}_2 b^1{}_1 & a^2{}_2 b^1{}_2 \\ a^2{}_2 b^2{}_1 & a^2{}_2 b^2{}_2 \end{pmatrix} \end{bmatrix},$$

a 4-dimensional hypercube. Note that a 2x2 array is a matrix with 4 entries, a 2x2x2 array is a cube with 8 entries, and C is a 2x2x2x2 4-dimensional array with 16 entries.

If $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ are vectors, then

$$A^a{}_{a'} \otimes B^b{}_{b'}\left(v^{a'} \otimes w^{b'}\right) = A^a{}_{a'}\left(v^{a'}\right) \otimes B^b{}_{b'}\left(w^{b'}\right)$$

$$= \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{pmatrix} b^1{}_1 & b^1{}_2 \\ b^2{}_1 & b^2{}_2 \end{pmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{pmatrix} a^1{}_1 v_1 + a^1{}_2 v_2 \\ a^2{}_1 v_1 + a^2{}_2 v_2 \end{pmatrix} \otimes \begin{pmatrix} b^1{}_1 w_1 + b^1{}_2 w_2 \\ b^2{}_1 w_1 + b^2{}_2 w_2 \end{pmatrix}.$$

Remember, we don't multiply this out because we don't mix *v* and *w.*

Theorem. [13.39] The linear transformation $x \mapsto Tx$ (or $x^a \mapsto T^a{}_b x^b$) on V induces a linear transformation $\mathcal{T}:\ Q^{f\cdots h}_{a\cdots c} \mapsto \left(S^{\mathrm{T}}\right)^{a'}{}_a \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_c\, T^f{}_{f'} \cdots T^h{}_{h'}\, Q^{f'\cdots h'}_{a'\cdots c'}$ on $\mathcal{V}$ where *S* = T^{-1} and S^{T} is the transpose of *S*. More precisely, if *Q* is the tensor product of *q* covectors and *p* vectors, then

$$Q^{f\cdots h}_{a\cdots c} = y_a \otimes \cdots \otimes y_c \otimes x^f \otimes \cdots \otimes x^h,$$

and

$$\mathcal{T}\left(Q\right) = \left(S^{\mathrm{T}}\right)^{a'}{}_a \otimes \cdots \otimes \left(S^{\mathrm{T}}\right)^{c'}{}_c \otimes T^f{}_{f'} \otimes \cdots \otimes T^h{}_{h'}\left(y_{a'} \otimes \cdots \otimes y_{c'} \otimes x^{f'} \otimes \cdots \otimes x^{h'}\right)$$

$$= \left(S^{\mathrm{T}}\right)^{a'}{}_a\left(y_{a'}\right) \otimes \cdots \otimes \left(S^{\mathrm{T}}\right)^{c'}{}_c\left(y_{c'}\right) \otimes T^f{}_{f'}\left(x^{f'}\right) \otimes \cdots \otimes T^h{}_{h'}\left(x^{h'}\right).$$

Proof. To show that $\mathcal{T}$ is linear, let *P* and *Q* be $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors, α a scalar, and *R* = *P* + *Q*. Then

$$\mathcal{T}\left(P^{f\cdots h}_{a\cdots c} + Q^{f\cdots h}_{a\cdots c}\right) = \mathcal{T}\left(R^{f\cdots h}_{a\cdots c}\right) = \left(S^{\mathrm{T}}\right)^{a'}{}_a \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_c\, T^f{}_{f'} \cdots T^h{}_{h'}\, R^{f'\cdots h'}_{a'\cdots c'}$$

$$= \left(S^{\mathrm{T}}\right)^{a'}{}_a \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_c\, T^f{}_{f'} \cdots T^h{}_{h'}\left(P^{f'\cdots h'}_{a'\cdots c'} + Q^{f'\cdots h'}_{a'\cdots c'}\right)$$

$$= \left(S^{\mathrm{T}}\right)^{a'}{}_a \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_c\, T^f{}_{f'} \cdots T^h{}_{h'}\, P^{f'\cdots h'}_{a'\cdots c'} + \left(S^{\mathrm{T}}\right)^{a'}{}_a \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_c\, T^f{}_{f'} \cdots T^h{}_{h'}\, Q^{f'\cdots h'}_{a'\cdots c'}$$

$$= \mathcal{T}\left(P^{f\cdots h}_{a\cdots c}\right) + \mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right)$$

and

$$\mathcal{T}\left(\alpha\, Q^{f\cdots h}_{a\cdots c}\right) = \mathcal{T}\left(\left(\alpha\, Q\right)^{f\cdots h}_{a\cdots c}\right) = \left(S^{\mathrm{T}}\right)^{a}{}_{a'} \cdots \left(S^{\mathrm{T}}\right)^{c}{}_{c'}\, T^f{}_{f'} \cdots T^h{}_{h'}\left(\alpha\, Q\right)^{f'\cdots h'}_{a'\cdots c'}$$

$$= \alpha\left(S^{\mathrm{T}}\right)^{a}{}_{a'} \cdots \left(S^{\mathrm{T}}\right)^{c'}{}_{c'}\, T^f{}_{f'} \cdots T^h{}_{h'} Q^{f'\cdots h'}_{a'\cdots c'}$$

$$= \alpha\, \mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right) \quad \blacksquare$$

The next lemma shows that the multilinear tensor product definition enables a certain amount of tensor interchanging even though there is no commutativity per se.

Lemma. Let V be a vector space and $T: \mathrm{V} \to \mathrm{V}$ a linear transformation. Let $S = T^{-1}$. Then

$$\begin{aligned} &S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, S^{a''}{}_{a'} \cdots S^{c''}{}_{c'}\, T^{f'}{}_{f''} \cdots T^{h'}{}_{h''} \\ &= S^{a'}{}_{a}\, S^{a''}{}_{a'} \cdots S^{c'}{}_{c}\, S^{c''}{}_{c'}\, T^{f}{}_{f'}\, T^{f'}{}_{f''} \cdots T^{h}{}_{h'}\, T^{h'}{}_{h''}. \end{aligned}$$

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\mathcal{T}_2"] \arrow[dr, dashed, "\mathcal{T}_{12}"'] & \mathcal{V} \arrow[d, "\mathcal{T}_1"] \\ & \mathcal{V} \end{tikzcd}$$

Proof. The lemma is summarized in the figure at the right. $\mathcal{V} = \mathrm{V}^* \otimes \cdots \otimes \mathrm{V}^* \otimes \mathrm{V} \otimes \cdots \otimes \mathrm{V}$, $\mathcal{T}_1 \mathcal{T}_2$ is represented by the first expression, and $\mathcal{T}_{12}$ is represented by the second one. Let $Q^{f \cdots h}_{a \cdots c} = y_a \otimes \cdots \otimes y_c \otimes x^f \otimes \cdots \otimes x^h \in \mathcal{V}$. We wish to show that $\mathcal{T}_1 \mathcal{T}_2 Q = \mathcal{T}_{12} Q$, which we get by applying the definition of the (multilinear) tensor product twice:

$$\begin{aligned} &S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, S^{a''}{}_{a'} \cdots S^{c''}{}_{c'}\, T^{f'}{}_{f''} \cdots T^{h'}{}_{h''}\, Q^{f'' \cdots h''}_{a'' \cdots c''} \\ &\quad = S^{a'}{}_{a} \otimes \cdots \otimes S^{c'}{}_{c} \otimes T^{f}{}_{f'} \otimes \cdots \otimes T^{h}{}_{h'} \otimes S^{a''}{}_{a'} \otimes \cdots \otimes S^{c''}{}_{c'} \otimes T^{f'}{}_{f''} \otimes \cdots \otimes T^{h'}{}_{h''} \\ &\qquad \left(y_{a''} \otimes \cdots \otimes y_{c''} \otimes x^{f''} \otimes \cdots \otimes x^{h''} \right) \\ &\quad = S^{a'}{}_{a} \otimes \cdots \otimes S^{c'}{}_{c} \otimes T^{f}{}_{f'} \otimes \cdots \otimes T^{h}{}_{h'} \\ &\qquad \left(S^{a''}{}_{a'} y_{a''} \otimes \cdots \otimes S^{c''}{}_{c'} y_{c''} \otimes T^{f'}{}_{f''} x^{f''} \otimes \cdots \otimes T^{h'}{}_{h''} x^{h''} \right) \\ &\quad = S^{a'}{}_{a} S^{a''}{}_{a'} y_{a''} \otimes \cdots \otimes S^{c'}{}_{c} S^{c''}{}_{c'} y_{c''} \otimes T^{f}{}_{f'} T^{f'}{}_{f''} x^{f''} \otimes \cdots \otimes T^{h}{}_{h'} T^{h'}{}_{h''} x^{h''} \\ &\quad = S^{a'}{}_{a}\, S^{a''}{}_{a'} \cdots S^{c'}{}_{c}\, S^{c''}{}_{c'}\, T^{f}{}_{f'}\, T^{f'}{}_{f''} \cdots T^{h}{}_{h'}\, T^{h'}{}_{h''}\, Q^{f'' \cdots h''}_{a'' \cdots c''} \qquad \blacksquare \end{aligned}$$

In the next theorem, the mapping $y \mapsto y S^{\mathrm{T}}$ plays the central role, replacing $y \mapsto y S$ that was used to show that V* is a representation space. Since S is a square matrix, S^{T} (as well as S) is a linear transformation on V and $y S^{\mathrm{T}} \in \mathrm{V}^*$ as the figure at the right shows.

$$\begin{tikzcd} \mathrm{V} \arrow[r, "S^{\mathrm{T}}"] \arrow[dr, dashed, "y S^{\mathrm{T}}"'] & \mathrm{V} \arrow[d, "y"] \\ & \mathbb{R} \text{ or } \mathbb{C} \end{tikzcd}$$

Theorem. If V is a representation space for a group G, then so is the tensor product space $\mathcal{V}$.

Proof. By definition of representation, there is a subgroup $\mathcal{G} \subset \mathrm{GL}(n)$ and an isomorphism $T: G \to \mathcal{G}$ such that $\forall g \in G \;\; T(g): \mathrm{V} \to \mathrm{V}$ is a linear transformation on V. We seek another subgroup $\mathcal{G}^* \subset \mathrm{GL}(n)$ and an isomorphism $\mathcal{T}: G \to \mathcal{G}^*$ such that $\forall g \in G \;\; \mathcal{T}(g): \mathcal{V} \to \mathcal{V}$ is a linear transformation on $\mathcal{V}$.

Denote

(1) $T_g = T(g)$ for $g \in$ G.

Since T is an isomorphism,

(2) $T_{g_1 g_2} \overset{(1)}{=} T(g_1\, g_2) = T(g_1)\, T(g_2) \overset{(1)}{=} T_{g_1}\, T_{g_2}$ for $g_1, g_2 \in G$.

Set

(3) $S_g = T_g^{-1}$.

By prior theorem [13.39], T_g induces a linear transformation $\mathcal{T}_g$ on $\mathcal{V}$:

For $Q^{f\cdots h}_{a\cdots c} \in \mathcal{V}$, define

$$\mathcal{T}_g : \mathcal{V} \to \mathcal{V} : \mathcal{T}_g\left(Q^{f\cdots h}_{a\cdots c}\right) = \left(S^{\mathsf{T}}\right)^{a'}{}_{a} \otimes \cdots \otimes \left(S^{\mathsf{T}}\right)^{c'}{}_{c} \otimes T^{f}{}_{f'} \otimes \cdots \otimes T^{h}{}_{h'} \left(Q^{f'\cdots h'}_{a'\cdots c'}\right)$$

or, in abbreviated format,

(4) $\mathcal{T}_g = \left(S^{\mathsf{T}}\right)^{a'}{}_{a} \cdots \left(S^{\mathsf{T}}\right)^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}$.

Define

$\mathcal{G}^* = \left\{\mathcal{T}_g : g \in G\right\}$ and

(5) $\mathcal{T} : G \to \mathcal{G}^* : \mathcal{T}(g) = \mathcal{T}_g$ for $g \in$ G.

$\mathcal{G}^*$ is clearly a subgroup of GL(n). We show that $\mathcal{T}$ is a homomorphism. Let $g_1, g_2 \in$ G and set

(6) $g_3 = g_1\, g_2$.

We need to show that $\mathcal{T}(g_1\, g_2) = \mathcal{T}(g_1)\, \mathcal{T}(g_2)$.

(7) $T_{g_3} \overset{(6)}{=} T_{g_1 g_2} \overset{(2)}{=} T_{g_1}\, T_{g_2}$ or $T_{g_3}{}^{f}{}_{f'} = T_{g_1}{}^{f}{}_{f'}\, T_{g_2}{}^{f'}{}_{f''}$

$\Rightarrow \qquad S_{g_3} \overset{(3)}{=} T_{g_3}{}^{-1} \overset{(7)}{=} T_{g_2}{}^{-1}\, T_{g_1}{}^{-1} \overset{(3)}{=} S_{g_2}\, S_{g_1}$.

Therefore

(8) $S^{\mathsf{T}}_{g_3} = S^{\mathsf{T}}_{g_1}\, S^{\mathsf{T}}_{g_2}$ or $\left(S_{g_3}{}^{\mathsf{T}}\right)^{a'}{}_{a} = \left(S_{g_1}{}^{\mathsf{T}}\right)^{a'}{}_{a} \left(S_{g_2}{}^{\mathsf{T}}\right)^{a''}{}_{a'}$.

Observe that the inverse operation changed the order of g_1 and g_2. Then the transpose operation changed it back to the desired order. So

$$(9)\quad \mathcal{T}(g_1\, g_2) \overset{(6)}{=} \mathcal{T}(g_3) \overset{(5)}{=} \mathcal{T}_{g_3} \overset{(4)}{=} \left(S_{g_3}{}^{\mathsf{T}}\right)^{a'}{}_{a} \cdots \left(S_{g_3}{}^{\mathsf{T}}\right)^{c'}{}_{c}\, T_{g_3}{}^{f}{}_{f'} \cdots T_{g_3}{}^{h}{}_{h'}$$

$$\overset{(7,\,8)}{=} \left(S_{g_1}{}^{\mathsf{T}}\right)^{a'}{}_{a} \left(S_{g_2}{}^{\mathsf{T}}\right)^{a''}{}_{a'} \cdots \left(S_{g_1}{}^{\mathsf{T}}\right)^{c'}{}_{c} \left(S_{g_2}{}^{\mathsf{T}}\right)^{c''}{}_{c'}\, T_{g_1}{}^{f}{}_{f'} T_{g_2}{}^{f'}{}_{f''} \cdots T_{g_1}{}^{h}{}_{h'} T_{g_2}{}^{h'}{}_{h''}$$

and

$$(10)\quad \mathcal{T}(g_1)\mathcal{T}(g_2) \overset{(5)}{=} \mathcal{T}_{g_1}\mathcal{T}_{g_2}$$

$$\overset{(4)}{=} \left(S_{g_1}{}^{\mathsf{T}}\right)^{a}{}_{a'} \cdots \left(S_{g_1}{}^{\mathsf{T}}\right)^{c}{}_{c'}\, T_{g_1}{}^{f}{}_{f'} \cdots T_{g_1}{}^{h}{}_{h'} \left(S_{g_2}{}^{\mathsf{T}}\right)^{a''}{}_{a'} \cdots \left(S_{g_2}{}^{\mathsf{T}}\right)^{c''}{}_{c'}\, T_{g_2}{}^{f'}{}_{f''} \cdots T_{g_2}{}^{h'}{}_{h''}$$

By the lemma, (9) = (10) and hence $\mathcal{T}$ is a homomorphism. ✔

To show that $\mathcal{T}$ is an isomorphism, we must show that it is 1-1. Again, let $g_1, g_2 \in \mathrm{G}$. We must show that $\mathcal{T}(g_1) = \mathcal{T}(g_2) \;\Rightarrow\; g_1 = g_2$. To simplify notation, set

$$T = T(g_1)$$

and

$$N = T(g_2).$$

T and N are linear transformations on V. Set

$$S = T^{-1}$$

and

$$M = N^{-1}.$$

By (4),

$$\mathcal{T}(g_1) \overset{(5)}{=} \mathcal{T}_{g_1} \overset{(4)}{=} \left(S^{\mathsf{T}}\right)^{a'}{}_{a} \cdots \left(S^{\mathsf{T}}\right)^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}$$

and

$$\mathcal{T}(g_2) \overset{(5)}{=} \mathcal{T}_{g_2} \overset{(4)}{=} \left(M^{\mathsf{T}}\right)^{a'}{}_{a} \cdots \left(M^{\mathsf{T}}\right)^{c'}{}_{c}\, N^{f}{}_{f'} \cdots N^{h}{}_{h'}\,.$$

So,

$$\mathcal{T}(g_1) = \mathcal{T}(g_2) \quad\Leftrightarrow\quad S^{a'}{}_{a} = M^{a'}{}_{a}, \cdots, S^{c'}{}_{c} = M^{c'}{}_{c}, T^{f}{}_{f'} = N^{f}{}_{f'}, \cdots, T^{h}{}_{h'} = N^{h}{}_{h'}$$

because we don't mix dissimilar indices. Each of these expressions is equivalent to $T = N$. So,

$$\mathcal{T}(g_1) = \mathcal{T}(g_2) \quad\Leftrightarrow\quad T = N \quad\Leftrightarrow\quad T(g_1) = T(g_2) \quad\Rightarrow\quad g_1 = g_2$$

because T is an isomorphism. ■

Definition. Let V be an n-dimensional vector space and $Q^{f\cdots h} \in \mathcal{V} = \mathrm{V} \otimes \cdots \otimes \mathrm{V}$ a $\begin{bmatrix} p \\ 0 \end{bmatrix}$-valent tensor. The **symmetric part of Q** is $Q^{(f\cdots h)} = \frac{1}{p!}\sum_{\pi} Q^{\pi(f)\cdots\pi(h)}$ and the **antisymmetric part of Q** is $Q^{[f\cdots h]} = \frac{1}{p!}\sum_{\pi} \mathrm{Sign}(\pi)\, Q^{\pi(f)\cdots\pi(h)}$. Notice that $Q = Q^{(f\cdots h)} + Q^{[f\cdots h]}$. The **symmetric space** is $\mathcal{V}_+ = \left\{Q^{(f\cdots h)} : Q \in \mathcal{V}\right\}$ and the **antisymmetric space** is $\mathcal{V}_- = \left\{Q^{[f\cdots h]} : Q \in \mathcal{V}\right\}$.

Theorem. [13.40]

(1) $\mathcal{V}_+$ and $\mathcal{V}_-$ are vector spaces

(2) $\mathcal{V}_+ \cap \mathcal{V}_- = \{0\}$

(3) $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$

(4) Dim $\mathcal{V}_+ = \frac{n}{2}(n+1)$ and Dim $\mathcal{V}_- = \frac{n}{2}(n-1)$

(5) $P \in \mathcal{V}_+ \Rightarrow P^{f\cdots h} = P^{\pi(f)\cdots\pi(h)}$ for any permutation π,

(6) $P \in \mathcal{V}_- \Rightarrow P^{f\cdots h} = \begin{cases} P^{\pi(f)\cdots\pi(h)} & \text{if } \pi \text{ is even} \\ -P^{\pi(f)\cdots\pi(h)} & \text{if } \pi \text{ is odd} \end{cases}$

Note. There is a parallel theory for $\begin{bmatrix} 0 \\ q \end{bmatrix}$-valent tensors $Q_{a\cdots c} \in \mathrm{V}^* \otimes \cdots \otimes \mathrm{V}^*$.

Example. Let $Q \in \mathcal{V} = \mathrm{V} \otimes \mathrm{V}$ and $P^{ab} = Q^{(ab)} \in \mathcal{V}_+$. Then $P^{ba} = P^{ab}$. Let $R^{ab} = Q^{[ab]} \in \mathcal{V}_-$. Then $R^{ba} = -R^{ab}$.