[13.7] SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections. **(A)** Show that SO(3) is a normal subgroup of O(3) and **(B)** show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of O(3). In Lemma 9.2 I show that if 1 is the identity element (null rotation) of SO(3) and R is the reflection operation, then $\mathcal{I} = \{ 1, \mathrm{R} \}$ is a normal subgroup of O(3). Then I prove (Theorem 9) that there are no others.

(A) Show SO(3) is a normal subgroup of O(3)

Lemma 1.1: A subgroup H of a group G is normal iff $g^{-1}\mathrm{H}g = \mathrm{H} \;\; \forall\, g \in \mathrm{G}$

Proof. By Penrose's definition, H is normal iff $g\mathrm{H} = \mathrm{H}g \;\; \forall\, g \in \mathrm{G}$. Left multiplying by g^{-1} yields the lemma. ■

We use the symbol ○ for the O(3) group operation. In particular, if $f, g \in \mathrm{SO(3)}$, we adopt the convention that $f \circ g$ is the composite rotation of f followed by g. Thus, if $w = (x, y, z)$, we write $w\,(f \circ g) = w\,f \circ g$. This is *the opposite of Penrose's convention* but, as will be seen, is necessitated by the approach taken to solve (B).

Also note that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. That is, if w is rotated by f and then g, the inverse operation is to rotate in the reverse directions by g and then f.

Definition: We use **R** to denote the reflection operator on the unit sphere of $\mathbb{R}^3$:

$$(x, y, z)\,\mathrm{R} = (-x, -y, -z).$$

Note that R is its own inverse: $\mathrm{R}^{-1} = \mathrm{R}$. If w is a point then $w\,\mathrm{R}$ is its reflection.

We write the composition of a reflection followed by a rotation as $\mathrm{R} \circ f$, and a rotation followed by a reflection as $f \circ \mathrm{R}$. An expression involving an odd number of reflections yields a reflective rotation and an expression involving an even number of reflections yields a non-reflective rotation. In particular, we have the following lemma.

Lemma 1.2: If $h \in \mathrm{SO(3)}$, then $\mathrm{R} \circ h \circ \mathrm{R} \in \mathrm{SO(3)}$. ■

Definition: $\mathbf{T} \equiv \mathrm{O(3)} - \mathrm{SO(3)}$ is the coset of reflective rotations of the sphere. That is, $\mathrm{T} = \mathrm{SO(3)} \circ \mathrm{R}$, and O(3) is the disjoint union of SO(3) and T:

$$\mathrm{O(3)} = \mathrm{T} \cup \mathrm{SO(3)}$$

The next theorem, Theorem 1, tells us that if $h \in \mathrm{SO(3)}$ then the normality operation $g^{-1} \circ h \circ g$ produces another member of SO(3), even if $g \in \mathrm{T}$.

Theorem 1: SO(3) is a normal subgroup of O(3)

Proof: Clearly SO(3) has an identity, is closed under $\circ$, and each non-reflective rotation in SO(3) has an inverse (non-reflective) rotation. So it remains only to show that if $h \in$ SO(3) and $g \in$ O(3) that $g^{-1} \circ h \circ g \in$ SO(3). Since the composition of 3 non-reflective rotations is a non-reflective rotation, this holds if $g \in$ SO(3). So suppose $g \in$ T. Then $\exists f \in$ SO(3) such that $g = f \circ$ R. Let $h' = f^{-1} \circ h \circ f \in$ SO(3). So, $g^{-1} \circ h \circ g = \mathrm{R} \circ f^{-1} \circ h \circ f \circ \mathrm{R}$
$= \mathrm{R} \circ h' \circ \mathrm{R} \in$ SO(3) by Lemma 1.2. ■

(B) Show SO(3) and $\mathcal{I}$ are the only proper normal subgroups of O(3)

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball $\mathcal{R}$ of radius π in which antipodal points on the surface of $\mathcal{R}$ are identified. The problem was solved by letting elements of $\mathcal{R}$ be written as $f = \theta\,(a, b, c) = (\theta a, \theta b, \theta c)$, where $a^2 + b^2 + c^2 = 1$, θ is the (counter-clockwise) angle of rotation, $0 \le \theta \le \pi$, and $\{\, t\,(a, b, c) : t \ge 0 \,\}$ is the (positive) axis of rotation.

Conventions:

1. **1** will be used to denote the identity element of SO(3). It is the degenerate 0° rotation.
2. An axis of rotation under discussion will be referred to as the **positive axis of rotation**. Its opposite will be called the **negative axis of rotation**.
3. Absolute value **| *f* |** $= \theta$ will be used to denote the (positive) rotation angle of the rotation *f*.

The representations for points (i.e., rotations) $f = \theta\,(a, b, c)$ are unique except:

- When $\theta = 0$, there are infinitely many representations for the identity element since any unit vector (a, b, c) can be used.
- When $\theta = \pi$, there are 2 representations for each π rotation $f = \pi\,(a, b, c) = \pi\,(-a, -b, -c)$ because antipodal points are identified.

As an intermediate step to proving (B), we prove the following theorem, labeled Theorem 8 because there are several steps that are taken to get there.

Theorem 8: SO(3) does not contain a proper normal subgroup.

Outline of Steps to Prove Theorem 8:

Let H be a proper normal subgroup of SO(3). $\exists\, 1 \ne h \in$ H. Let $\theta = |h| > 0$ be the rotation angle of h. So, $0 < \theta \le \pi$. Let S_θ be the sphere in $\mathcal{R}$ of radius θ. S_θ consists of all rotations of angle θ about all axes.

- Let $g \in$ SO(3). We show that $k = g^{-1} \circ h \circ g$ also has rotation angle θ. That is, $k \in S_\theta$. Thus, starting with $h \in$ H and using only the normality operation $g^{-1} \circ h \circ g$, we cannot build up to all of SO(3). By using just the normality operation, we can only generate a subset of S_θ.

- We next show that in fact $\{ g^{-1} \circ h \circ g : g \in \mathrm{SO}(3) \}$ generates all of S_θ.
- Next, suppose (a) $0 \le \phi \le \pi$, (b) $f_1, g_1 \in S_\theta$, and (c) $|f_1 \circ g_1| = \phi$. We show $\{ f \circ g : f, g \in S_\theta \} \supseteq S_\phi$. Thus $H \supseteq S_\phi$.
- Clearly, if (c) holds, the maximum possible size for ϕ is 2θ (namely $f \circ f$ for any $f \in S_\theta$). The minimum possible size is 0 (namely, $f \circ f^{-1}$). In fact, we will show that $\{f \circ g : f, g \in S_\theta\}$ contains every sphere S_ϕ of radius ϕ between $0 \le \phi \le 2\theta$, and so every such sphere belongs to H. To generate all of SO(3), we just need to generate larger and larger spheres in H until all of SO(3) is included. That is, starting with $S_\theta \subseteq H$ we can generate spheres of size up to $S_{2\theta} \subseteq H$. From $S_{2\theta}$ we can generate spheres of size up to $S_{4\theta} \subseteq H$. Then up to $S_{8\theta}$. When $2^n\theta \ge \pi$, then $\mathrm{SO}(3) \subseteq S_{2^n\theta} \subseteq H$ and we are done.

The reader who wishes to first get an overview of the rest of the proof of (B) can skip ahead to Theorem 9 and read its overview.

GA concepts and notation used in this proof

Non-reflective rotations (henceforth just called "rotations") are commonly represented by rotation matrices, and the composition $f \circ g$ can be computed as a matrix product. Instead, in this proof we learn how to use Geometric Algebra, aka Clifford Algebra, for the computations. In GA, every rotation corresponds to a rotor (described shortly), and geometric product (rather than matrix product) is used to compute the composition of rotations.

Let **$\{ e_1, e_2, e_3 \}$** be an orthonormal basis for 3-space. **Rotors** are objects having the form

$$\boldsymbol{r} = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right]$$

where $a^2 + b^2 + c^2 = 1$, $a, b, c \in \mathbb{R}$, and $-\infty \le \theta \le \infty$.

While e_1 is a **vector**, $e_1 e_2$ is a **bivector**. $e_1 e_2 e_3$ is an example of a **trivector**. **Scalars**, vectors, bivectors, trivectors, etc., belong to the class of objects called **multivectors**.

A rotor is the sum of a scalar, $\mathrm{Cos}\left(\frac{\theta}{2}\right)$, and a bivector (the rest of the expression). This is no stranger than a complex number that is the sum of a real and an imaginary number. The two parts cannot be combined but nonetheless the sum has meaning. In general a multivector can be the sum of multivectors of different **grades** that cannot be combined.

A rotor r represents a rotation of angle θ about the axis $\{\, t\,(a,\, b,\, c) : t \geq 0 \,\}$.

$\frac{\theta}{2}$ is called the **rotor angle** and θ is the **rotation angle**. As this can be confusing, in this proof only the term "rotation angle" will be used.

The scalar $\mathbf{Cos}\,\frac{\theta}{2}$ is referred to as the "**constant term**" of r because it does not contain any basis elements. The constant term determines the rotation angle θ. That is, if the constant term of a rotor r is known to be K, then for some θ,

$K = \mathbf{Cos}\,\frac{\theta}{2}$, and so $\theta = 2\,\text{Arc}\mathbf{Cos}\,(K)$.

The magnitude of a rotor is $\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) = 1$, which is why $a^2 + b^2 + c^2 = 1$.

But even though rotors have magnitude 1, in this proof **$|\, r \,|$** will continue to be used to denote the rotation angle θ and not 1 (and also not $\frac{\theta}{2}$) in order to be consistent with the definition already given for the magnitude $|\, f \,|$ of a point (rotation) in $\mathcal{R}$.

The **geometric product** of 2 multivectors is just the regular polynomial product with a couple of modifications. First, it is non-commutative, so the order of multiplication matters. Second, the basis elements are combined according to the rules

$$e_i^{\,2} = 1, \quad i = 1, 2, 3 \quad \text{and}$$
$$e_i\, e_j = -\, e_j\, e_i \ \text{ if } \ i \neq j \quad \text{(antisymmetry)}$$

While a rotor can have any angle, $\mathcal{S}$ is defined in this proof as the subset of rotors having (counter-clockwise) rotation angle $0 \leq \theta \leq \pi$ under the operation of geometric product. If the rotor $r \in \mathcal{S}$ (i.e., if $0 \leq \theta \leq \pi$), it corresponds to the point $(\theta a,\, \theta b,\, \theta c)$ of $\mathcal{R}$.

The symbol $\circ$ will again be used to denote the geometric product, as in $f \circ g$. That is, whether f and g belong to $\mathcal{R}$ or $\mathcal{S}$, $f \circ g$ will represent their composition.

To compute the sometimes algebra-intensive geometric products I wrote a software package in Mathematica that calculates geometric products and also performs other GA operations such as wedge product, multivector inverse, pseudoscalar, etc. The package can be downloaded for free at https://github.com/matrixbud/Geometric-Algebra. I have also saved the Mathematica file having the GA calculations used in this proof in the same directory as this file.

Note: Mathematica always displays the bivector $e_3 e_1$ as $e_1 e_3$. Thus, when using Mathematica, the negative of the b term must be used, as in

$$r = \textbf{Cos}\frac{\theta}{2} + \textbf{Sin}\frac{\theta}{2}\left[a\, e_2\, e_3 - b\, e_1\, e_3 + c\, e_1\, e_2\right].$$

Since the normality operation $g^{-1} \circ f \circ g$ involves inverses, this is a good time to provide the GA formula for the inverse of a rotor.

Theorem 2: The GA **inverse of the rotor *r*** has the formula

$$r^{-1} = \textbf{Cos}\frac{\theta}{2} - \textbf{Sin}\frac{\theta}{2}\left[a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right].$$

Moreover,

$$r^{-1} \in \mathcal{S} \text{ if } r \in \mathcal{S}.$$

Proof. By computing the geometric product we find that

$r^{-1} \circ r = \textbf{Cos}^2\frac{\theta}{2} + \textbf{Sin}^2\frac{\theta}{2}\left(a^2 + b^2 + c^2\right) = 1$ and similarly $r \circ r^{-1} = 1$. Thus, r^{-1} is the inverse of *r*. ✔

Another way of seeing that $r^{-1} \circ r = 1$ is that r^{-1} can be written

$r^{-1} = \textbf{Cos}\left(-\frac{\theta}{2}\right) + \textbf{Sin}\left(-\frac{\theta}{2}\right)\left[a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right]$, representing a clockwise rotation about the positive axis that cancels the counter-clockwise rotation of *r*, yielding the identity rotation, 1. ✔

Additionally, since $r^{-1} = \textbf{Cos}\frac{\theta}{2} + \textbf{Sin}\frac{\theta}{2}\left[-a\, e_2\, e_3 - b\, e_3\, e_1 - c\, e_1\, e_2\right]$, r^{-1} can also be regarded as a (counter-clockwise) θ rotation about the negative axis, showing that $r^{-1} \in \mathcal{S}$ (since $\mathcal{S}$ is the set of all [counter-clockwise] rotations $0 \le \theta \le \pi$ about all axes). ■

In rotor theory the formula for a rotor *r* to rotate a *point* $w = x\, e_1 + y\, e_2 + z\, e_3$ in 3-space to some other point *v* is

$$v = r^{-1} \circ w \circ r.$$

Effectively, r^{-1} performs half the θ rotation (i.e., $\frac{\theta}{2}$) and *r* performs the rest. The reader is encouraged to confirm this for a 90° rotation of the point (1,0,0) in the *xy*-plane. To do this, let $\theta = \frac{\pi}{2}$, $c = 1$, and $a = b = 0$ in the rotor *r*.

To write the expression to rotate a point *w* by *f* followed by *g*:

- Start with *w*
- Wrap with *f*: $f^{-1} \circ w \circ f$
- Wrap with *g*: $g^{-1} \circ f^{-1} \circ w \circ f \circ g$

Thus $f \circ g$ in this expression means to rotate first by *f*, then by *g*. This is the reason for the left-right convention adopted on page 1.

Every rotation can be represented by a rotor in $\mathcal{S}$; i.e., by a rotor of the form $r = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left(a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right)$ where $0 \le \theta \le \pi$. Every rotation has an additional representation by a rotor with $\pi < \theta < 2\pi$ that is not in $\mathcal{S}$. The geometric product of 2 rotors in $\mathcal{S}$ can be outside of $\mathcal{S}$. To make $\mathcal{S}$ a group under $\circ$ we need to be able to recognize the formulas of rotors not in $\mathcal{S}$ and be able to identify them with equivalent rotors in $\mathcal{S}$. Thus, we will actually be using $\circ$ to represent the geometric product mod $\mathcal{S}$. However, we will only explicitly write mod $\mathcal{S}$ when it is necessary for the sake of clarity.

Definition: We say that 2 rotors r_1 and r_2 are **equivalent** if they generate the same rotation.

Example 1: Does the geometric product of two $\frac{2}{3}\pi$ rotations about the *x*-axis belong to $\mathcal{S}$?

Answer: The result is a $\frac{4}{3}\pi$ rotation about the *x*-axis, not in $\mathcal{S}$ since $\frac{4}{3}\pi > \pi$.
Theorem 3, below, shows how to recognize a rotor not in $\mathcal{S}$ and how to generate its equivalent rotor in $\mathcal{S}$.

Theorem 3: Let $\pi < \theta < 2\pi$. Then

$$s = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left(a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right) \notin \mathcal{S}, \text{ and}$$

$$r = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right)\left(a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right) \in \mathcal{S}.$$

Also, the constant term of *s* is negative; the constant term of *r* is positive, $-s = r^{-1} \in \mathcal{S}$, and *s* is equivalent to $-s$.

Proof: $s \notin \mathcal{S}$ because $\theta \notin [0, \pi]$. ✔

Since $(2\pi - \theta) \in [0, \pi]$, $r \in \mathcal{S}$. ✔

The constant term of *s* is negative: $\mathbf{Cos}\frac{\theta}{2} < 0$ because $\frac{\pi}{2} < \frac{\theta}{2} < \pi$. ✔

The constant term of *r* is positive: $\mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) > 0$ since $0 < \pi - \frac{\theta}{2} < \frac{\pi}{2}$.

$r^{-1} \in \mathcal{S}$ from Theorem 2. ✔

Claim $r^{-1} = -s$:

Without loss of generality, by a suitable rotation of 3-space, we can assume *s,* and hence *r*, are rotations about the *z*-axis.

$$s = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \quad \text{and} \quad r = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 .$$

These formulas for *s* and *r* are rotors because they satisfy the rotor definition with *c* = 1 and *a* = *b* = 0. They rotate about the *z*-axis because $e_1 e_2$ rotates the *xy*-plane.

$$r^{-1} = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) - \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 = -\mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) = -s \quad ✔$$

To show *s* is equivalent to r^{-1}, we must show that if *w* = (*x*, *y*, *z*) $\in \mathbb{R}^3$ then $s^{-1} \circ w \circ s = r \circ w \circ r^{-1}$, i.e., that the rotations of *w* generated by *s* and r^{-1} are the same for every $w \in \mathbb{R}^3$:

$$w = x e_1 + y e_2 + z e_3.$$

$$\begin{aligned} s^{-1} \circ w \circ s = &\left[x\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - 2y\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\theta}{2}\right) - x\mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_1 \\ &+ \left[y\,\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + 2x\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\theta}{2}\right) - y\,\mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_2 + \\ &+ z\left[\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_3 \\ &= \left[x\mathbf{Cos}(\theta) - y\,\mathbf{Sin}(\theta)\right] e_1 + \left[y\,\mathbf{Cos}(\theta) + x\mathbf{Sin}(\theta)\right] e_2 + z e_3 \\ &= r \circ w \circ r^{-1} \qquad \blacksquare \end{aligned}$$

Aside: Note the change from $\frac{\theta}{2}$ to θ in the equation above, evidence that θ is indeed the rotation angle generated by the expression $s^{-1} \circ w \circ s$.

Example 2: Find a rotor in $\mathcal{S}$ that is equivalent to the rotor $s = -\frac{\sqrt{3}}{2} + \frac{1}{2} e_1 e_2$

Solution: First, by Theorem 3, $s \notin \mathcal{S}$ because the constant term is negative. Also by Theorem 3, *s* is equivalent to $-s = \frac{\sqrt{3}}{2} - \frac{1}{2} e_1 e_2$ and $-s \in \mathcal{S}$.

Example 3: If a rotor *s* is a π rotation about any axis, then s^2 should be the identity, a zero rotation. But if, for example, $s = \mathbf{Cos}\frac{\pi}{2} + \mathbf{Sin}\frac{\pi}{2} e_2 e_3 = e_2 e_3$, then $s^2 = e_2 e_3 e_2 e_3 = -1$. How can that be?

Answer: By Theorem 3, $s^2 \notin \mathcal{S}$ because the constant term is negative. Also by Theorem 3, s^2 is equivalent to $-s^2 = 1$, the identity rotor. The fact that $-1 = 1 \bmod \mathcal{S}$ is important in Theorem 9.

Aside: If we consider $-\infty < \theta < \infty$, then there are infinitely many rotors equivalent to r since sine and cosine have period 2π. But, for $0 < \theta < 2\pi$ there are only two representations for each rotor, and for $0 < \theta < \pi$ there is only one.

The next theorem is the analog to Penrose's problem [12.17].

Theorem 4: $\mathcal{S}$ is group isomorphic to $\mathcal{R}$ [which is group isomorphic to SO(3)]

Proof: Let $\mathbf{s} = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\,e_2\,e_3 + b\,e_3\,e_1 + c\,e_1\,e_2\right] \in \mathcal{S}$.
Then $0 \le \theta \le \pi$ and $a^2 + b^2 + c^2 = 1$.

s can be represented as $s = (\theta, a, b, c) \in \mathcal{S}$. This representation in $\mathcal{S}$ is unique for $0 \le \theta \le \pi$ except when $\theta = 0$:

- Like $\mathcal{R}$, when $\theta = 0$, there are infinitely many unit points (a, b, c) that can be used to denote the identity rotation.
- Unlike $\mathcal{R}$, when $\theta = \pi$ there is only 1 representation for each point s because while the antipodal point represents the same rotation, it is not in $\mathcal{S}$. That is, if $s = (\pi, a, b, c) \in \mathcal{S}$, the antipodal point is $(-\pi, -a, -b, -c)$, not in $\mathcal{S}$ because $\theta = -\pi$.

Define

T: $\mathcal{S} \to \mathcal{R}$: $T(s) = r$ where
$s = (\theta, a, b, c)$ and
$r = \theta\,(a, b, c) = (\theta\,a, \theta\,b, \theta\,c)$.

That is, T: $(\theta, a, b, c) \longmapsto \theta\,(a, b, c)$

T is well-defined:

To show that T is well-defined we must show that (a) $T(s) \in \mathcal{R}$ and (b) if s has 2 or more representations, then T assigns the same element of $\mathcal{S}$ in all cases.

(a) Since $0 \le \theta \le \pi$ and $a^2 + b^2 + c^2 = 1$, $r \in \mathcal{R}$.

(b) If $\theta = 0$, then $T(0, a, b, c) = 0\,(a, b, c) = (0, 0, 0)$ for any unit vector (a, b, c).

T is obviously 1–1 and onto since T: $(\theta, a, b, c) \to \theta\,(a, b, c)$.

T is a homomorphism:

We need to show that $T(s_1 \circ s_2) = T(s_1) \circ T(s_2)$
$s_1 \circ s_2$ is the composition of 2 rotations in $\mathcal{S}$, and $T(s_1) \circ T(s_2)$ is the

composition of 2 rotations in $\mathcal{R}$. So,

if $s_1 \circ s_2$ = Composition of $(\theta_1, a_1, b_1, c_1)$ & $(\theta_2, a_2, b_2, c_2) = (\theta_3, a_3, b_3, c_3)$
then $T(s_1) \circ T(s_2)$ = Composition of $\theta_1(a_1, b_1, c_1)$ & $\theta_2(a_2, b_2, c_2)$
$= \theta_3(a_3, b_3, c_3)$

So, $T(s_1 \circ s_2) = T(\theta_3, a_3, b_3, c_3) = \theta_3(a_3, b_3, c_3) = T(s_1) \circ T(s_2)$ ■

Steps leading to Theorem 8

As we move towards Theorem 8, we will be positing a normal group H and an element $1 \neq h_1 \in$ H. We will expand the singleton set $\{ h_1 \}$ to a sphere of elements in H using just the normality operation $g^{-1} \circ h_1 \circ g$ where $g \in$ SO(3). Then we will switch to the product operation between elements of the sphere to expand H to all of SO(3).

WLOG, the axes of the unit 3-sphere can be rotated so that h_1 is a rotation about the x-axis. The following lemma provides geometric insight into how to use the normality operation $g^{-1} \circ f \circ g$ to generate a rotor whose axis is perpendicular to that of a given rotor. This is accomplished by letting g be a 90° rotation about an axis perpendicular to the axis of f. For example, if f rotates by θ about the x-axis and g is a 90° rotation about the y-axis, then $g^{-1} \circ f \circ g$ is a rotation by θ about the z-axis.

Lemma 5.1: Let

$\boldsymbol{h_1}$ = $\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$ (θ rotation about positive x-axis)

$\boldsymbol{h_2}$ = $\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_3 e_1$ (θ rotation about positive y-axis), and

$\boldsymbol{h_3}$ = $\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2$ (θ rotation about positive z-axis)

If h_1 belongs to a normal group H, then $h_2, h_3, h_1^{-1}, h_2^{-1}, h_3^{-1} \in$ H.

Proof: Define rotors

$$g_3 = \mathbf{Cos}\left(\frac{\pi}{4}\right) + \mathbf{Sin}\left(\frac{\pi}{4}\right) e_1 e_2 = \frac{1 + e_1 e_2}{\sqrt{2}} \text{ and}$$
$$g_2 = \mathbf{Cos}\left(\frac{\pi}{4}\right) - \mathbf{Sin}\left(\frac{\pi}{4}\right) e_3 e_1 = \frac{1 - e_3 e_1}{\sqrt{2}}.$$

Then

$$g_3^{-1} = \frac{1 - e_1 e_2}{\sqrt{2}},$$

$$g_3^{-1} \circ h_1 = \frac{\mathbf{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\mathbf{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_2 e_3 + \frac{\mathbf{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_3 e_1 - \frac{\mathbf{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_2$$

$$g_3^{-1} \circ h_1 \circ g_3 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_3 e_1 = h_2$$

and

$$g_2^{-1} = \frac{1 + e_3 e_1}{\sqrt{2}},$$

$$g_2^{-1} \circ h_1 = \frac{\mathbf{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\mathbf{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_2 e_3 + \frac{\mathbf{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_3 e_1 + \frac{\mathbf{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_2$$

$$g_2^{-1} \circ h_1 \circ g_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 = h_3$$

Since H is normal, h_2, $h_3 \in$ H. Since groups are closed under inverses, h_1^{-1}, h_2^{-1}, $h_3^{-1} \in$ H. ■

Definition. In $\mathcal{R}$, let **S_θ** denote the **sphere of radius *θ***. It consists of all rotations of amount *θ* about all axes in 3-space. The formula for S_θ in terms of rotors is

$$S_\theta = \left\{ \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left(a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right) \mathbf{:}\ a^2 + b^2 + c^2 = 1 \right\}.$$

Given h, $r \in \mathcal{R}$ it is easy to find $g \in \mathcal{R}$ such that $r = h \circ g$. Namely, $g = h^{-1} \circ r$. It is much more difficult to find $g \in \mathcal{R}$ such that $r = g^{-1} \circ h \circ g$. In fact, we will prove in Lemma 6.1 that it is not possible at all unless *h* and *r* have the same magnitude. The next lemma (5.2) proves that it is in fact always possible to find such a point *g* if the magnitudes of *h* and *r* are the same, and the lemma provides an explicit formula for *g* (actually 2 formulas, depending on whether *r* has a *x*-component). This lemma (or the alternate proof to Theorem 5, discussed below) is the key step in achieving Theorem 8.

Lemma 5.2 gives formulas for *r* that work but does not explain where they come from. In Appendix 2, I provide what insight I can into how I discovered the formula for rotor g_{23}. Discovery of the rotor *g* in part (b) is slightly more transparent because *a* and *b* can be viewed as the sides of a right triangle and hence there is an angle *α* such that *b* = ***Sin*** *α*.

My inability to provide clear insight for the equations in Lemma 5.2 nudged me to develop another proof for Theorem 5 that *is* intuitive and does not depend on either Lemma 5.1 or 5.2. It does use an additional lemma (5.3) that is a generalization of Lemma 5.1. I provide both of the Theorem 5 proofs, though only one is needed in order to proceed. I include the second proof because it is short, clever, geometric and provides an insightful process to find, for a given rotation *r,* a rotation *g* such that $g^{-1} \circ h \circ g = r$. I include the original proof because it is short and Lemma 5.2 includes explicit, compact formulas for generating *r*.

Lemma 5.2: Let

$$h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3\,,\ h_3 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2\,, \text{ and}$$

$$r = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right] \in S_\theta\,,$$

where $a^2 + b^2 + c^2 = 1$ and $0 < \theta \le \pi$.

(a) If $a \neq 0$ then

$$\boxed{r = g_{23}^{\ -1} \circ h_1 \circ g_{23}} \quad \text{where}$$

$$g_2 = \mathbf{Cos}\left(\frac{\beta}{2}\right) - \mathbf{Sin}\left(\frac{\beta}{2}\right) e_3\, e_1,$$

$$g_3 = \mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\gamma}{2}\right) e_1\, e_2,$$

$g_{23} = g_2 \circ g_3$, and

$\beta = Arc\,\mathbf{Cos}\,(\sqrt{a^2 + b^2}\,)$ and $\gamma = Arc\,\mathbf{Tan}\,(a, b\,)$.

(b) If $a = 0$ then

$$\boxed{r = g^{-1} \circ h_3 \circ g} \quad \text{where}$$

$$g = \mathbf{Cos}\left(\frac{\alpha}{2}\right) + \mathbf{Sin}\left(\frac{\alpha}{2}\right) e_2\, e_3, \text{ and}$$

$\alpha = Arc\,\mathbf{Sin}\,(-\,b)$.

Proof: ArcTan(a, b) denotes the arc tangent of $\frac{b}{a}$ taking into account which quadrant the point (a, b) is in.

(a) $$g_{23} = \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_2\,e_3 - \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)e_3\,e_1 + \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_1\,e_2$$

$$g_{23}^{-1} = \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_2\,e_3 + \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)e_3\,e_1 - \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_1\,e2$$

$$\begin{aligned} g_{23}^{-1}\circ h_1 = {} & \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) \\ & -\left[\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)\right]e_2\,e_3 \\ & +\left[\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)\right]e_3\,e_1 \\ & -\left[\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)\right]e_1\,e_2 \end{aligned}$$

$$\begin{aligned} g_{23}{}^{-1}\circ h_1\circ g_{23} = {} & \mathbf{Cos}\left(\frac{\theta}{2}\right)\left[\begin{array}{l} \mathbf{Cos}^2\left(\frac{\beta}{2}\right)\mathbf{Cos}^2\left(\frac{\gamma}{2}\right) + \mathbf{Sin}^2\left(\frac{\beta}{2}\right)\mathbf{Cos}^2\left(\frac{\gamma}{2}\right) \\ + \mathbf{Cos}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}^2\left(\frac{\gamma}{2}\right) + \mathbf{Sin}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}^2\left(\frac{\gamma}{2}\right) \end{array}\right] \\ & + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[\begin{array}{l} \mathbf{Cos}^2\left(\frac{\beta}{2}\right)\mathbf{Cos}^2\left(\frac{\gamma}{2}\right) - \mathbf{Sin}^2\left(\frac{\beta}{2}\right)\mathbf{Cos}^2\left(\frac{\gamma}{2}\right) \\ - \mathbf{Cos}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}^2\left(\frac{\gamma}{2}\right) + \mathbf{Sin}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}^2\left(\frac{\gamma}{2}\right) \end{array}\right]e_2\,e_3 \\ & + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[\begin{array}{l} 2\,\mathbf{Cos}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) \\ -2\,\mathbf{Sin}^2\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) \end{array}\right]e_3\,e_1 \\ & + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[\begin{array}{l} 2\,\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}^2\left(\frac{\gamma}{2}\right) \\ +2\,\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}^2\left(\frac{\gamma}{2}\right) \end{array}\right]e_1\,e_2 \end{aligned}$$

$$= \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[\mathbf{Cos}(\beta)\mathbf{Cos}(\gamma)e_2 e_3 + \mathbf{Cos}(\beta)\mathbf{Sin}(\gamma)e_3 e_1 + \mathbf{Sin}(\beta)e_1 e_2\right]$$

$$= \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[a e_2 e_3 + b e_3 e_1 + c e_1 e_2\right] \qquad ✔$$

$$= r$$

Note: The interested reader can find intermediate steps for this and subsequent equations in the accompanying Mathematica file.

(b) Since $a = 0$, then $b = \sqrt{1 - c^2}$, and

$$g^{-1} \circ h_3 = \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\alpha}{2}\right) - \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\alpha}{2}\right)e_2 e_3$$

$$- \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\alpha}{2}\right)e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\alpha}{2}\right)e_1 e_2$$

$$g^{-1} \circ h_3 \circ g = \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}(\alpha)\, e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}(\alpha)\, e_1 e_2$$

$$= \mathbf{Cos}\left(\frac{\theta}{2}\right) + b\, \mathbf{Sin}\left(\frac{\theta}{2}\right)e_3 e_1 + c\, \mathbf{Sin}\left(\frac{\theta}{2}\right)e_1 e_2$$

$= r$ ✔ ■

The next theorem summarizes Lemma 5.2. It shows that for a normal subgroup H, if $h \in$ H has rotation angle θ, then the sphere S_θ of radius θ lies in H

Theorem 5: If H is a normal subgroup of SO(3), $1 \neq h \in$ H and $\theta = |h|$, then H $\supseteq S_\theta$.

Proof: WLOG we can assume h is a θ rotation about the x-axis. That is,

$$h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_2 e_3 .$$

Let $r \in S_\theta$. Then for some a, b, $c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 = 1$,

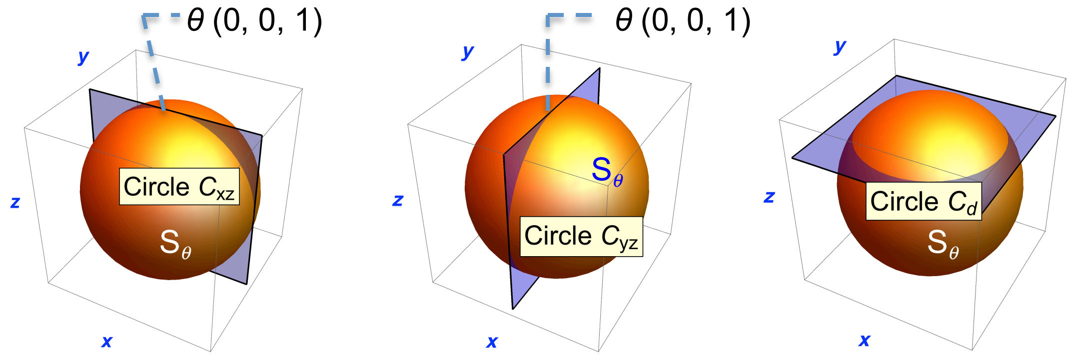
$$r = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right] .$$

If $a \neq 0$, by Lemma 5.2a, $r \in$ H since $r = g_{23}^{-1} \circ h_1 \circ g_{23}$ and H is normal.

If $a = 0$, by Lemma 5.2b, $r \in$ H since $r = g^{-1} \circ h_3 \circ g$, H is normal, and $h_3 \in$ H (by Lemma 5.1). ■

We now present the alternate proof for Theorem 5.

Outline of Geometric-based Proof of Theorem 5: As before we begin with $h_1 = \theta\,(1, 0, 0) \in$ H and $f = \theta\,(a, b, c) \in S_\theta$. We are done when we show that $f \in$ H.



In the figure above, circle C_{xz} of radius θ about the origin is formed by slicing the sphere S_θ with the *xz*-plane of $\mathcal{R}$. Circle C_{yz} is similarly formed by slicing S_θ with the *yz*-plane. Lemma 5.3 will prove that these circles lie in H.

The third circle in the figure is formed by taking a horizontal slice of S_θ at height $z = c\theta$, where $0 \le c \le 1$. This generates a circle C_d of radius $d = \theta\sqrt{1-c^2}$ that contains *f*. Thus, points on C_d have coordinates $\theta(d\,\mathbf{Cos}\,\phi,\ d\,\mathbf{Sin}\,\phi,\ c)$ where ϕ is the counter-clockwise angle from the *x*-axis of an *xy*-plane raised to height *c*. We show that the formula for ϕ is $Arc\,\mathbf{Tan}\frac{b}{a}$.

Next we show that if $g_\phi = \phi\,(0, 0, 1)$ and $h = \theta\,(d, 0, c)$, then $g_\phi^{-1} \circ h \circ g_\phi \in C_d$ and, in fact, $\{\, g_\phi^{-1} \circ h \circ g_\phi : -\pi \le \phi \le \pi \,\}$ generates C_d. In particular, it generates *f*, proving $f \in$ H since $h \in C_{xz} \subseteq$ H

Geometric Insight: We have again used rotations (specifically, g_ϕ and *h*) about axes perpendicular to the axis of *f* to generate *f* via the normality operation. This time, however, g_ϕ is not required to have a 90° rotation angle but, rather, as ϕ varies from $-\pi$ to π, the normality operation $g_\phi^{-1} \circ h \circ g_\phi$ traces out the Circle C_d.

Lemma 5.3: If $h_1 = (\theta, 0, 0) \in \mathcal{R}$ belongs to a normal group H, then H contains C_{xy}, C_{yz}, and C_{xz}, the circles of radius θ centered at the origin in the *xy*-, *yz*-, and *xz*-planes of $\mathcal{R}$, respectively.

Proof: The lemma is just a slight generalization of Lemma 5.1 Recall

$h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$. Similarly to Lemma 5.1, let

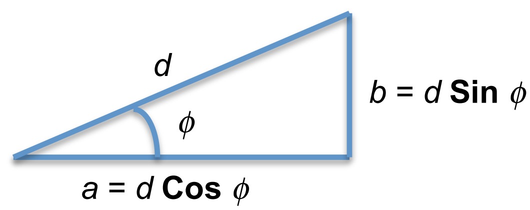
$g_2 = \mathbf{Cos}\left(\frac{\phi}{2}\right) + \mathbf{Sin}\left(\frac{\phi}{2}\right) e_3 e_1$ and $g_3 = \mathbf{Cos}\left(\frac{\phi}{2}\right) + \mathbf{Sin}\left(\frac{\phi}{2}\right) e_1 e_2$ where $0 \le \phi \le \frac{\pi}{2}$.

Also set $g_1 = \textbf{Cos}\left(\frac{\phi}{2}\right) + \textbf{Sin}\left(\frac{\phi}{2}\right) e_2 e_3$. Then

$g_3^{-1} \circ h_1 \circ g_3 = \textbf{Cos}\left(\frac{\theta}{2}\right) + \textbf{Sin}\left(\frac{\theta}{2}\right)\left[\textbf{Cos}(\phi) e_2 e_3 + \textbf{Sin}(\phi) e_3 e_1\right]$.

Recall from page 2 of this proof that the axis of rotation of $g_3^{-1} \circ h_1 \circ g_3$ is $\{ t\,(\textbf{Cos}\ \phi, \textbf{Sin}\ \phi, 0) : t \geq 0 \}$. Thus rotor $g_3^{-1} \circ h_1 \circ g_3$ is represented in $\mathcal{R}$ as the point $\theta\,(\textbf{Cos}\ \phi, \textbf{Sin}\ \phi, 0)$. As ϕ varies from $-\pi$ to π, these points sweep out the circle C_{xy}. Similarly, the set of points $\left\{g_2^{-1} \circ h_1 \circ g_2\right\}$ forms the circle C_{xz} and $\left\{g_1^{-1} \circ h_2 \circ g_1\right\}$ forms the circle C_{yz}, where $h_2 = \textbf{Cos}\left(\frac{\theta}{2}\right) + \textbf{Sin}\left(\frac{\theta}{2}\right) e_3 e_1$. These circles are contained in H because H is normal and, in the case of the 3rd circle, also because $h_2 \in$ H (because $h_2 \in C_{xz} \subseteq$ H).■

Theorem 5 Proof #2: Let H be a non-trivial normal subgroup of SO(3). WLOG let $(0, 0, 0) \neq h_1 = (\theta, 0, 0) \in$ H be the element of $\mathcal{R}$ representing a θ rotation about the *x*-axis. Let $f = \theta\,(a, b, c) \in S_\theta$. It suffices to show that $f \in$ H. If $a = 0$, then $f \in C_{yz} \subseteq$ H by Lemma 5.3. So we can assume $a \neq 0$.



Let C_d be the circle formed by taking a horizontal slice of S_θ at height $z = c\,\theta$. The radius of C_d is $d = \theta\sqrt{1 - c^2}$.

We wish to represent *f* also as $f = \theta\,(d\,\textbf{Cos}\ \phi, d\,\textbf{Sin}\ \phi, c)$ since it is a point on the circle C_d.That is, we desire an angle ϕ such that $a = d\,\textbf{Cos}\ \phi$ and $b = \textbf{Sin}\ \phi$. To do this, we define $\phi = \textbf{ArcTan}\ (a, b)$, which is **ArcTan** $\frac{b}{a}$ but adjusts ϕ for the quadrant of (a, b). ϕ is well-defined because $a \neq 0$. In order to make $\textbf{Sin}\ \phi = \frac{b}{d}$ and $\textbf{Cos}\ \phi = \frac{a}{d}$ well-defined, we also must have $d \neq 0$. But, if $d = 0$ then $f = \theta\,(0, 0, \pm 1) \in C_{yz} \subseteq$ H (see Circle C_{yz} figure), and we are done. So we can assume $d \neq 0$.

Let

$$g_\phi = \textbf{Cos}\left(\frac{\phi}{2}\right) + \textbf{Sin}\left(\frac{\phi}{2}\right) e_1 e_2 = \phi\,(0, 0, 1)$$

be the rotation of angle ϕ about the z-axis.

Let

$$h = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left(d\, e_2\, e_3 + c\, e_1\, e_2\right) = \theta\,(d,\, 0,\, c),$$

a point of the circle C_{xz}. $h \in$ H by Lemma 5.3.

Finally,

$$\begin{aligned} g_\phi^{-1} \circ h \circ g_\phi = {} & \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\ & + \left(d\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) - d\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right) e_2\, e_3 \\ & + \;2d\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)\mathbf{Sin}\left(\frac{\phi}{2}\right)\; e_3\, e_1 \\ & + \left(c\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + c\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right) e_1\, e_2 \\ & = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left[d\,\mathbf{Cos}(\phi)e_2\, e_3 + d\,\mathbf{Sin}(\phi)e_3\, e_1 + c\, e_1\, e_2\right] \\ & = \theta\left(d\,\mathbf{Cos}(\phi),\, d\,\mathbf{Sin}(\phi),\, c\right) \\ & = f\,. \end{aligned}$$

Therefore $f \in$ H and $S_\theta \subseteq$ H. ■

The next lemma shows that for any $g \in$ SO(3), the normality operation $g^{-1} \circ h \circ g$ results in a rotation having the same rotation angle as h. Thus, for a given $h \in$ SO(3), $\left\{g^{-1} \circ h \circ g : g \in \text{SO(3)}\right\}$ cannot generate all of SO(3). The prior Lemma 5.2 (as well as the alternate proof of Theorem 5) tells us that it at least generates *all* of S_θ.

Lemma 6.1: Let $0 \le \theta \le \pi$, $h \in S_\theta$, and $g \in$ SO(3). Then $g^{-1} \circ h \circ g \in S_\theta$.

Proof: WLOG $h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2\, e_3$. Since 2 vectors determine a plane, WLOG we can also assume that g lies in the xy-plane. That is, if the magnitude of g is $0 \le \phi \le \pi$ then $g = \mathbf{Cos}\,\frac{\phi}{2} + \mathbf{Sin}\,\frac{\phi}{2}\left[a\, e_2\, e_3 + b\, e_3\, e_1\right]$ where $a^2 + b^2 = 1$.

Then

$$g^{-1}\circ h\circ g= \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right)+a^2\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)+b^2\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)$$
$$+\left[\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right)+a^2\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)-b^2\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3$$
$$+2ab\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_3e_1-2b\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2$$
$$=\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right)+\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)$$
$$-\mathbf{Sin}\left(\frac{\theta}{2}\right)\left\{\begin{array}{l}+\left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right)+a^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)-b^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3\\+2ab\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_3e_1-2b\,\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2\end{array}\right\}$$
$$=\mathbf{Cos}\left(\frac{\theta}{2}\right)$$
$$+\mathbf{Sin}\left(\frac{\theta}{2}\right)\left\{\begin{array}{l}+\left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right)+a^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)-b^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3\\+2ab\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_3e_1-2b\,\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2\end{array}\right\}$$

Since the constant term of $g^{-1}\circ h\circ g$ is $\mathbf{Cos}\frac{\theta}{2}$, $g^{-1}\circ h\circ g\in S_\theta$.■

Now we switch tactics, using the product operation rather than the normality operation to generate additional elements of H. We will show that geometric products of pairs of elements in S_θ generate every sphere S_ω and, hence, all of SO(3).

The next theorem starts the process. It shows the ability to generate one sphere in $\mathcal{R}$ from another by taking products of pairs of elements from the first sphere.

Theorem 6: Let k_1, $k_2\in S_\theta$ for some $0<\theta\le\pi$, and let $\omega=|k_1\circ k_2|$. Then S_ω can be generated from geometric products of pairs of elements from S_θ.

Proof. The theorem is trivially true if ω = 0, so assume ω > 0. Let $r\in S_\omega$. The claim is that we can find f_1, $f_2\in S_\theta$ such that $r=f_1\circ f_2$. WLOG let $k_1\circ k_2$ be the ω rotation about the positive *x*-axis. That is $h_1\equiv k_1\circ k_2=\mathbf{Cos}\left(\frac{\omega}{2}\right)+\mathbf{Sin}\left(\frac{\omega}{2}\right)e_2e_3$.

Since $r \in S_\omega$, $\exists\ a, b, c \in \mathbb{R}$ such that $r = \textbf{Cos}\,\frac{\omega}{2} + \textbf{Sin}\,\frac{\omega}{2}\left[a\,e_2\,e_3 + b\,e_3\,e_1 + c\,e_1\,e_2\right]$, where $a^2 + b^2 + c^2 = 1$.

Lemma 5.2 yields that either (a) $r = g_{23}{}^{-1} \circ h_1 \circ g_{23}$ or (b) $r = g^{-1} \circ h_3 \circ g$, where $h_3 \in S_\omega$ is the ω rotation about the *z*-axis and g_{23} and g are as defined in the lemma (using ω rather than θ).

(a) Define $f_1 = g_{23}{}^{-1} \circ k_1 \circ g_{23}$ and $f_2 = g_{23}{}^{-1} \circ k_2 \circ g_{23}$. By Lemma 6.1, $f_1, f_2 \in S_\theta$.

Thus,

$$\begin{aligned} r &= g_{23}{}^{-1} \circ h_1 \circ g_{23} = g_{23}{}^{-1} \circ k_1 \circ k_2 \circ g_{23} \\ &= \left(g_{23}{}^{-1} \circ k_1 \circ g_{23}\right) \circ \left(g_{23}{}^{-1} \circ k_2 \circ g_{23}\right) \\ &= f_1 \circ f_2 \end{aligned} \quad ✔$$

(b) As was shown in Lemma 5.1, h_3 can be obtained from h_1 by a normality operation using the 90° rotation $g_2 = \frac{1 - e_3\,e_1}{\sqrt{2}}$. The rotation formula is $h_3 = g_2{}^{-1} \circ h_1 \circ g_2$. Apply the same 90° rotation to k_1 and k_2 :

$$k_3 = g_2{}^{-1} \circ k_1 \circ g_2 \quad \text{and} \quad k_4 = g_2{}^{-1} \circ k_2 \circ g_2 .$$

By Lemma 6.1, $k_3, k_4 \in S_\theta$ and

$$\begin{aligned} h_3 &= g_2{}^{-1} \circ\ h_1 \circ\ g_2 = g_2{}^{-1} \circ\ k_1 \circ k_2 \circ g_2 \\ &= \left(g_2{}^{-1} \circ\ k_1 \circ\ g_2\right) \circ \left(g_2{}^{-1} \circ\ k_2 \circ\ g_2\right) . \\ &= k_3 \circ k_4 \end{aligned}$$

Define $f_1 = g^{-1} \circ k_3 \circ g$ and $f_2 = g^{-1} \circ k_4 \circ g$. By Lemma 6.1, $f_1, f_2 \in S_\theta$. Thus

$$\begin{aligned} r &= g^{-1} \circ h_3 \circ g = g^{-1} \circ k_3 \circ k_4 \circ g = \left(g^{-1} \circ k_3 \circ g\right) \circ \left(g^{-1} \circ k_4 \circ g\right) \\ &= f_1 \circ f_2 \end{aligned} \quad \blacksquare$$

The next theorem and its corollary show how to generate all of SO(3) from geometric products of pairs of elements of S_θ.

Theorem 7: Suppose $\theta \in \left[\frac{\pi}{2}, \pi\right]$ and $0 < \omega \leq \pi$. Then there is a pair of elements in S_θ whose geometric product lies in S_ω.

Proof: Let h_1 be the rotor in S_θ that rotates by θ about the x-axis:

$$h_1 = \textbf{Cos}\left(\frac{\theta}{2}\right) + \textbf{Sin}\left(\frac{\theta}{2}\right)\, e_2\, e_3.$$

h_1 is the first desired element of S_θ. We will find a second rotor, $h \in S_\theta$, such that $h \circ h_1 \in S_\omega$. To define h, we need a, b, $c \in \mathbb{R}$ such that

$$a^2 + b^2 + c^2 = 1,$$

$$\boldsymbol{h} = \textbf{Cos}\left(\frac{\theta}{2}\right) + \textbf{Sin}\left(\frac{\theta}{2}\right)\left(a\, e_2\, e_3 + b\, e_3\, e_1 + c\, e_1\, e_2\right), \text{ and}$$

$$|h \circ h_1| = \omega.$$

Recall that $|h \circ h_1| = \omega \Leftrightarrow$ Constant term of $h \circ h_1$ = $\textbf{Cos}\left(\frac{\omega}{2}\right)$.

By performing the geometric product calculation we find that

$$\begin{aligned} h \circ h_1 = \; & \textbf{Cos}^2\left(\frac{\theta}{2}\right) - a\, \textbf{Sin}^2\left(\frac{\theta}{2}\right) \\ & + \left(\textbf{Cos}\left(\frac{\theta}{2}\right) \textbf{Sin}(\theta) + a\, \textbf{Cos}(\theta)\, \textbf{Sin}\left(\frac{\theta}{2}\right) \right) e_2\, e_3 \\ & + \left(b\, \textbf{Cos}\left(\frac{\theta}{2}\right) \textbf{Sin}\left(\frac{\theta}{2}\right) - c\, \textbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_3\, e_1 \\ & + \left(c\, \textbf{Cos}\left(\frac{\theta}{2}\right) \textbf{Sin}\left(\frac{\theta}{2}\right) + b\, \textbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_1\, e_2 \end{aligned}$$

The constant term of rotor $h \circ h_1$ = $\textbf{Cos}^2\left(\frac{\theta}{2}\right) - a\, \textbf{Sin}^2\left(\frac{\theta}{2}\right)$.

Setting $\textbf{Cos}\left(\frac{\omega}{2}\right) = \textbf{Cos}^2\left(\frac{\theta}{2}\right) - a\, \textbf{Sin}^2\left(\frac{\theta}{2}\right)$ and solving for the coefficient a yields

$$\boldsymbol{a} = \frac{\textbf{Cos}^2\left(\frac{\theta}{2}\right) - \textbf{Cos}\left(\frac{\omega}{2}\right)}{\textbf{Sin}^2\left(\frac{\theta}{2}\right)}.$$

This is well defined except when $\textbf{Sin}^2\left(\frac{\theta}{2}\right) = 0$, which occurs only when θ = 0 mod 2π. Since 0 is outside the domain $\left[\frac{\pi}{2}, \pi\right]$ of θ, the definition of a is well defined. "b" and "c" can be any numbers so long as $a^2 + b^2 + c^2 = 1$. In fact it is convenient to set b = 0 and have $c = \sqrt{1 - a^2}$.

Finally, we must show that a, b, c $\in \mathbb{R}$ which will conclude the proof of the theorem. To do this we must show that $a^2 \leq 1$.

Since $0 < \omega \leq \pi$,

$$0 < \frac{\omega}{2} \leq \frac{\pi}{2},$$

$$\mathbf{Cos}\frac{\pi}{2} \leq \mathbf{Cos}\frac{\omega}{2} < \mathbf{Cos}(0),$$

$$0 \leq \mathbf{Cos}\left(\frac{\omega}{2}\right) < 1,$$

$$-1 < -\mathbf{Cos}\left(\frac{\omega}{2}\right) \leq 0, \tag{iv}$$

Since $\frac{\pi}{2} \leq \theta \leq \pi$,

$$\frac{\pi}{4} \leq \frac{\theta}{2} \leq \frac{\pi}{2},$$

$$0 \leq \mathbf{Cos}\left(\frac{\theta}{2}\right) \leq \frac{1}{\sqrt{2}},$$

$$0 \leq \mathbf{Cos}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2}, \tag{v}$$

$$0 \leq 1 - \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2},$$

$$-\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) - 1 \leq 0,$$

$$\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq 1. \tag{vi}$$

Thus,

$$-1 = \frac{0-1}{1} \overset{\text{(iv, v, vi)}}{\leq} \frac{\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\theta}{2}\right)} = a \overset{\text{(iv, v, vi)}}{\leq} \frac{\frac{1}{2}+0}{\frac{1}{2}} = 1$$

That is, $|\, a \,| \leq 1$. ■

Corollary 7.1: Suppose $\frac{\pi}{2} \leq \theta \leq \pi$, $0 < \omega \leq \pi$, and $S_\theta \subseteq H$, where H is a group, not necessarily normal. Then S_ω is generated by geometric products of pairs of elements from S_θ, and so $S_\omega \subseteq H$.

Proof: By Theorem 7, $\exists\, k_1, k_2 \in S_\theta \subseteq H$ such that $k_1 \circ k_2 \in S_\omega$. By Theorem 6, $S_\omega = \{h \circ k : h, k \in S_\theta\} \subseteq H$. ■

Theorem 8: SO(3) contains no proper normal subgroup.

Proof: Let H be a non-trivial normal subgroup of SO(3). H contains a non-identity element $1 \neq h_0 \in H$ where h_0 is a rotation by some angle $\phi \leq \pi$ about some axis.

a. Since $h_0 \neq 1$, $\phi > 0$.

b. Since $\phi > 0$, there is a positive integer n such that $\frac{\pi}{2} \leq n\phi \leq \pi$.

Define $h = h_0^n = h_0 \circ h_0 \circ \cdots \circ h_0$ and $\theta = n\,\phi$.

c. h Is a rotation of angle θ, and $\frac{\pi}{2} \leq \theta \leq \pi$. Thus $h \neq 1$.

d. Since $1 \neq h$, by Theorem 5, $S_\theta \subseteq H$.

We wish to show that H = SO(3). It suffices to show that H contains every sphere S_ω in $\mathcal{R}$. For $\omega = 0$, $S_\omega \subseteq H$ since $1 \in H$. So let $0 < \omega \leq \pi$ be an arbitrary rotation angle. By Corollary 7.1, $S_\omega \subseteq H$. ■

Geometric Algebra Treatment of Reflections and Reflective Rotations:

Recall that O(3) is the disjoint union **O(3) = SO(3) ∪ T** of the non-reflective and reflective rotations, respectively. In GA terminology, we can write this as **O(3) =** $\mathcal{S} \cup \mathcal{T}$. We have already defined $\mathcal{S}$. Its elements are rotors, multivectors having the form

$$f = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a_{23}\,e_2\,e_3 + a_{31}\,e_3\,e_1 + a_{12}\,e_1\,e_2\right],$$

where $a_{23}^2 + a_{31}^2 + a_{12}^2 = 1$, $a_{23}, a_{31}, a_{12} \in \mathbb{R}$, and $0 \leq \theta \leq \pi$.

We wish to define $\mathcal{T}$. That is, we wish to describe a reflective rotation, h, in GA terms. We cannot represent h as a rotor because all rotors represent non-reflective rotations. Instead, we seek to represent h as a 3-dimensional multivector. Moreover, just as $f^{-1} \circ v \circ f$ (non-reflectively) rotates the vector v into some vector w, we require the multivector h *to* have the property that $h^{-1} \circ v \circ h$ reflectively rotates v into $-w$. That is, we require

$$-w = h^{-1} \circ v \circ h \qquad \text{(vii)}$$

Because $h = f \circ R$, a rotation followed by a reflection, we also wish to find a multivector representation for the reflection operator R.

We quickly discover that the expression for the multivector R to reflect a vector w into $-w$ must be $-w = R \circ w \circ R$. To see this, let

- $v \in \mathbb{R}^3$: a vector in $\mathbb{R}^3$
- $-v \in \mathbb{R}^3$: the vector obtained by reflecting v
- $f \in \mathcal{S}$: a rotation
- $w = f^{-1} \circ v \circ f \in \mathbb{R}^3$: the vector obtained by rotating v by f
- $-w \in \mathbb{R}^3$: the vector obtained by reflecting w
- $h = f \circ R \in \mathcal{T}$: the reflective rotation generated by f

Since a reflection followed by a reflection is the identity, R is its own inverse, $R = R^{-1}$. So,

$$\begin{aligned} -w &\overset{\text{(vii)}}{=} h^{-1} \circ v \circ h = (f \circ R)^{-1} \circ v \circ (f \circ R) = (R^{-1} \circ f^{-1}) \circ v \circ (f \circ R) \\ &= R \circ (f^{-1} \circ v \circ f) \circ R \\ &= R \circ w \circ R \qquad \text{(viii)} \end{aligned}$$

Defining R = $i = \sqrt{-1}$ satisfies equation (viii). i is a scalar, and the scalar i is a multivector, so this satisfies our goal of defining R as a multivector. In Appendix 1 it is shown that while there are other multivector expressions for R that satisfy equation (viii), all such expressions involve imaginary terms. In Definition (1), below, we define R = i, the simplest such expression.

Since i is a scalar, it commutes with the geometric product operator:

$f \circ g \circ i = i(f \circ g) = (if) \circ g = f \circ (ig)$ for all multivectors f and g.

Definition: The **Reflection Operator** is actually 3 different operators, depending on its domain. The original operator R with domain $\mathbb{R}^3$ induces operators on $\mathcal{S}$ and $\mathcal{T}$. Let $i = \sqrt{-1}$, $v \in \mathbb{R}^3$, $f \in \mathcal{S}$, and $h \in \mathcal{T}$. Define

(1) $\mathbf{R} : \mathbb{R}^3 \to \mathbb{R}^3 : R(v) = R^{-1} \circ v \circ R = i \circ v \circ i = \mathbf{-v}$; i.e., $\mathbf{R = \mathit{i} = R^{-1}}$.
(2) $\mathbf{R_S} : \mathcal{S} \to \mathcal{T}: R_S(f) = f \circ R_S = f \circ i = \mathbf{\mathit{i} f}$; i.e., $\mathbf{R_S = \mathit{i} = R_S^{-1}}$.
(3) $\mathbf{R_T} : \mathcal{T} \to \mathcal{S}: R_T(h) = h \circ R_T = h \circ (-i) = \mathbf{-\mathit{i}\, h}$; i.e., $\mathbf{R_T = -\mathit{i} = R_T}$.

These definitions require clarification. We begin by observing we are now able to identify the multivector representation of a reflective rotation:

(4) $$h = f \circ R_S = if = i\left\{\mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[b_{23}\,e_2\,e_3 + b_{31}\,e_3\,e_1 + b_{12}\,e_1\,e_2\right]\right\},$$

where $f \in \mathcal{S}$, $b_{23}^2 + b_{31}^2 + b_{12}^2 = 1$, $b_{23}, b_{31}, b_{12} \in \mathbb{R}$, and $0 \le \theta \le \pi$.

Lemma 9.1 provides the equation for the inverse of a reflective rotation.

Lemma 9.1: Let $h = f \circ i \in \mathcal{T}$, where $f \in \mathcal{S}$. Then $h^{-1} = f^{-1} \circ i$.

Proof: Let $g = f^{-1} \circ i$. By definition (2), $g \in \mathcal{T}$ since $f^{-1} \in \mathcal{S}$. Because the expression for the geometric product of 2 members of $\mathcal{T}$ involves 2 reflections, $g \circ h \in \mathcal{S}$. Thus,

$$g \circ h = (f^{-1} \circ i) \circ (f \circ i) = i^2 (f^{-1} \circ f) = -1.$$

Recall from Theorem 3 that $-1 \notin \mathcal{S}$ but $-1 \approx 1 \in \mathcal{S}$. Therefore $g \circ h = 1 \bmod \mathcal{S}$. Similarly, $h \circ g = 1$, and we have that $h^{-1} = g = f^{-1} \circ i$. ■

Lemma 9.1 is somewhat surprising because one might expect that $h^{-1} = i^{-1} \circ f^{-1} = - f^{-1} \circ i$. The subtlety here is that when $i \in \mathbb{C}$ then $i^{-1} = -i$, but $i^{-1} = i \bmod \mathcal{T}$ when i is a reflective rotation[1]. This is similar to (and follows from) $1 \neq -1$ when $1 \in \mathbb{R}$ but $1 = -1 \bmod \mathcal{S}$ (according to Theorem 3) when 1 is a rotation.

In the reflection operator definition, what does it mean when we say that R *induces* $R_{\mathcal{S}}$? Observe that both $w \circ R$ and $f \circ R$ appear in equation (viii). In $w \circ R$, R is acting on a vector, but in $f \circ R$, R is acting on a rotation. So the R in $f \circ R$ becomes what I am calling the induced reflection $R_{\mathcal{S}}$. The domain of the reflection operator has changed but the multivector expression hasn't changed, so $R_S = R = i$, which is Definition (2).

Why is $R_T = -i$ (rather than just i) in Definition (3)? Because $1 = R_S \circ R_T = i\, R_T$.

Outline of Proof of Theorem 9:
Observe that what was called the reflection operation "R" in the 2nd paragraph of page 1 is now defined to simply be "i". That is, $i = R_S \in T$. Using this notation, we have that the subgroup $\mathcal{I} = \{1, i\}$.

Let H be a proper normal subgroup of O(3) such that $H \neq SO(3)$ and $H \neq \mathcal{I}$. From Theorem 8, $\exists\, i \neq h \in H \cap T$. Then $h^2 \in SO(3) \cap H$. If $1 \neq h^2$, by Theorem 8, $SO(3) \subseteq H$. Also, $\exists\, f \in SO(3)$ such that $h = f \circ R_S = i\, f$. To see that $T \subseteq H$, let $t \in T$. $\exists\, f_1 \in SO(3)$ such that $t = f_1 \circ R_S = i\, f_1$. Let $f_2 = f_1 \circ f^{-1} \in SO(3)$. Then $f_1 = f_2 \circ f$. So,

$$t = i\, f_1 = i\,(f_2 \circ f) = f_2 \circ (i\, f) = f_2 \circ h.$$

$t \in H$ since $f_2 \in SO(3) \subset H$. Thus $T \subseteq H$. Since, also, $SO(3) \subseteq H$, then $H = O(3)$.

[1] $i^{-1} = R^{-1} = i$.

Unfortunately, h^2 does equal 1 when either $h = i$ (ruled out) or h is the reflection of any π rotation (allowed). Even so, everything in the prior paragraph still holds except we will not have proved that SO(3) ⊂ H. To prove this, we will generate $h_1 \in H \cap T$ such that $1 \neq h \circ h_1 \in SO(3)$. Then we will again have, by Theorem 8, that $SO(3) \subseteq H$.

Notice that $h = f \circ R_S \Leftrightarrow h = if \Leftrightarrow f = -i\,h \Leftrightarrow f = h \circ R_T$. That is,

$$h = f \circ R_S \Leftrightarrow f = h \circ R_T \tag{ix}$$

Lemma 9.2: $\mathcal{I} = \{1, i\}$ is a normal subgroup of O(3).

Proof: $g^{-1} \circ 1 \circ g = 1$ and $g^{-1} \circ i \circ g = i$ for every $g \in O(3)$. ■

Theorem 9: SO(3) and $\mathcal{I}$ are the only proper normal subgroups of O(3).

Proof: To help keep track of variables, we will use s, s_0, s_1, s_2, s_3, and s_4 for elements of SO(3) and t, t_0, t_1, and t_2 for elements of T.

Suppose H is a non-trivial normal subgroup of O(3) and H ≠ SO(3) and $H \neq \mathcal{I}$.

Claim: There is an element $t_1 \in H \cap T$ such that $t_1 \neq i$:

By Theorem 8, $\exists\, t_0 \in H \cap T$. If $t_0 \neq i$, the claim is true. If $t_0 = i$, since $H \neq \mathcal{I}$, H contains another element. If that element is in T, the claim is true. If the other element is $s_0 \in SO(3)$, we can assume $s_0 \neq 1$ since $H \neq \mathcal{I}$. Set $t_1 = s_0\, i$. Then $t_1 \in H \cap T$ and $t_1 \neq i$.

As an intermediate step we wish to find an element $t_2 \in H \cap T$ such that $t_1 \circ t_2 \neq 1$. Set

$s_1 \equiv t_1 \circ R_T = -i\, t_1 \in SO(3)$.

WLOG, we can assume s_1 is a rotation about the ***x*-axis**. Thus,

$$s_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3.$$

By (ix),

$$t_1 = s_1 \circ R_S = i\, s_1 = i\left[\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3\right].$$

and

$$i t_1 = -\left[\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3\right].$$

Let $\theta = |\, s_1 \,|$. $\theta \neq 0$ because $t_1 \neq i$ and, consequently, $s_1 \neq 1$. Define

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{4}\right) + \mathbf{Sin}\left(\frac{\pi}{4}\right) e_1 e_2 = \frac{1 + e_3 e_1}{\sqrt{2}}$$

a 90° rotation about the ***y-axis***. $g_2 \in SO(3)$ since it is a rotation. Hence,

$$s_2 \equiv g_2^{-1} \circ s_1 \circ g_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 .$$

$s_2 \in SO(3)$ is the rotation of angle θ about the negative **z-axis**. Define

$$t_2 \equiv g_2^{-1} \circ t_1 \circ g_2 = g_2^{-1} \circ i s_1 \circ g_2 = i s_2 = i\left[\mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2\right].$$

So $t_2 \in T \cap H$ and

$$t_1 \circ t_2 = -\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_1 e_2 + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) e_3 e_1 - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_2 e_3 .$$

Notice that the constant term of $t_1 \circ t_2 < 0$. So, by Theorem 3, we replace the expression for $t_1 \circ t_2$ by its negative:

$$t_1 \circ t_2 = \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_1 e_2 - \mathbf{Sin}^2\left(\frac{\theta}{2}\right) e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_2 e_3 .$$

Since $\theta \neq 0$, $t_1 \circ t_2 \neq 1$. This concludes the intermediate step.

We next wish to show that $H \supset SO(3)$. First, we observe that

$$\begin{aligned} s_1 \circ s_2 &= \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_1 e_2 - \mathbf{Sin}^2\left(\frac{\theta}{2}\right) e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\theta}{2}\right) e_2 e_3 \\ &= t_1 \circ t_2 . \end{aligned}$$

We define

$$s_3 \equiv t_1 \circ t_2 = s_1 \circ s_2 .$$

$1 \neq s_3 \in H$ because $s_3 = t_1 \circ t_2$. $s_3 \in SO(3)$ because $s_3 = s_1 \circ s_2$. So,

$1 \neq s_3 \in H \cap SO(3)$.

$H \cap SO(3)$ is thus a non-trivial normal subgroup of $SO(3)$, so by Theorem 8

$H \cap SO(3) = SO(3)$.

Thus,

$H \supset SO(3)$.

Finally, we are in a position to show as well that $H \supset T$, implying that $H = O(3)$ since $H \supset SO(3) \cup T = O(3)$. Let $\boldsymbol{t} \in \mathbf{T}$ be any element of T. This proof is finished if we can show $t \in H$. Let

$$\boldsymbol{s} \equiv -\,i\,t = t \circ R_T \in SO(3).$$

$\exists$ x, y, and z where $x^2 + y^2 + z^2 = 1$ and ω satisfying $0 \le \omega \le \pi$ such that

$$s = \mathbf{Cos}\,\frac{\omega}{2} + \mathbf{Sin}\,\frac{\omega}{2}\,\left[x\,e_2\,e_3 + y\,e_3\,e_1 + z\,e_1\,e_2\right].$$

By (ix),

$$t = s \circ R_S = i\,s = i\left\{\mathbf{Cos}\,\frac{\omega}{2} + \mathbf{Sin}\,\frac{\omega}{2}\,\left[x\,e_2\,e_3 + y\,e_3\,e_1 + z\,e_1\,e_2\right]\right\}.$$

Define

$$\boldsymbol{s_4} \equiv s_1^{-1} \circ s \in SO(3) \subset H.$$

s_4 is the rotation whose geometric product with s_1 equals s :

$$s = (s_1 \circ s_1^{-1}) \circ s = s_1 \circ (s_1^{-1} \circ s) = s_1 \circ s_4.$$

Therefore s_4 is also the rotation whose geometric product with t_1 equals t :

$$t = s \circ i = (s_1 \circ s_4) \circ i = (s_1 \circ i) \circ s_4 = t_1 \circ s_4$$

So, $t \in H$ since t_1, $s_4 \in H$, and that completes the proof the theorem and part (B).

■

APPENDIX 1: GA expressions for defining the reflection operator R: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

Let w be a vector in $\mathbb{R}^3$. Without loss of generality,

$w = e_1$

We need to solve for the coefficients in the GA multivector expression for R:

$$R = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 + c_{12} e_1 e_2 + c_{23} e_2 e_3 + c_{13} e_1 e_3 + c_{123} e_1 e_2 e_3$$

From equation (viii) R must satisfy

$$R \circ w \circ R = - w = - e_1.$$

By symmetry,

$$c_1 = c_2 = c_3 \text{ and } c_{12} = c_{23} = c_{13}.$$

Thus,

$$R = c_0 + c_1 e_1 + c_1 e_2 + c_1 e_3 + c_{12} e_1 e_2 + c_{12} e_2 e_3 + c_{12} e_1 e_3 + c_{123} e_1 e_2 e_3.$$

Expanding $R \circ w \circ R$ yields

$$R \circ w \circ R = A + Be_1 + Ce_2 + De_3 = - w = 0 - e_1 + 0\, e_2 + 0\, e_3,$$

where

$$A = 2\left(c_0 c_1 + c_{12} c_{123}\right),$$

$$B = c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2,$$

$$C = 2\left(c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123}\right),$$

and

$$D = 2\left(c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123}\right).$$

This leads to 4 equations in the 4 unknowns c_0, c_1, c_{12}, c_{123} :

$$\left\{ \begin{array}{ccc} c_0 c_1 + c_{12} c_{123} & = & 0 \\ c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2 & = & -1 \\ c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123} & = & 0 \\ c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123} & = & 0 \end{array} \right\}$$

Adding equations (3) and (4) yields

$$c_1^2 + c_{12}^2 = 0 \quad \Rightarrow \quad c_{12} = \pm c_1 i.$$

Equation (1) then becomes

$$\pm c_0 c_{12} i + c_{12} c_{123} = \pm c_{12}\left(c_0 i + c_{123}\right) = 0 \quad \Rightarrow \quad c_{12} = 0 \text{ or } c_{123} = \pm c_0 i.$$

$c_{123} = \pm c_0\, i$ is not possible because, along with $c_{12} = \pm c_1\, i$, it transforms (2) into

$$c_0{}^2 - c_0{}^2 - c_1{}^2 + c_1{}^2 = -1, \text{ or } 0 = -1.$$

Thus

$$c_{12} = 0 \text{ (and } c_{12} = \pm c_1\, i)$$

which implies that

$$c_1 = 0.$$

So

R = $c_0 + c_{123}\, e_1\, e_2\, e_3$

and equation (2) becomes

$$c_0^2 + c_{123}^2 = -1 \quad \Rightarrow \qquad c_{123} = \pm\sqrt{-1 - c_0^2} = \pm i\sqrt{c_0{}^2 + 1}\,.$$

From this it is clear that if c_0 is real, then c_{123} is imaginary. This proves the claim that every multivector representation for R involves an imaginary term.

I list below the four simplest expressions for R:

R = $\pm i$
R = $\pm i\; e_1\; e_2\; e_3$
R = $1 \pm i\sqrt{2}\, e_1\, e_2\, e_3$
R = $i\sqrt{2} \pm e_1\, e_2\, e_3$

For the record, if instead of starting with $w = e_1$ we use $w = a_1\, e_1 + a_2\, e_2 + a_3\, e_3$, the resulting equations lead to the same solution set.

APPENDIX 2: The 2 insights that suggested how to define the rotor g_{23} in Lemma 5.2.

The hardest part of this proof was to invent the rotor g_{23}. I couldn't solve a certain system of 4 non-linear simultaneous equations that would have provided it, but I was able to guess a solution after examining lots of rotations.

The **1st insight**, gained during Lemma 5.1, is that a rotor, say h_3, about the z-axis, is obtained by $g_2^{-1} \circ h_1 \circ g_2$ where g_2 is a rotor about the y-axis with a 90° rotation angle and h_1 is a rotor about the x-axis having the same angle of rotation, θ, *as* h_2. Loosely speaking, I learned that a 90° rotation g_2 yields a rotation about an axis 90° away from h_1.

2nd Insight comes from discovering that reducing the g_2 angle of rotation to 45° generates a rotation about the 45° diagonal between the x-axis and z-axis (i.e., about an axis 45° away from h_1), seen below:

Let

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{8}\right) - \mathbf{Sin}\left(\frac{\pi}{8}\right) e_1\, e_3.$$

Then

$$g_2^{-1} \circ h_1 \circ g_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}}\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1\, e_2 + \frac{1}{\sqrt{2}}\mathbf{Sin}\left(\frac{\theta}{2}\right) e_2\, e_3$$

$$= \text{45° diagonal axis in the } zx\text{-plane.}$$

To generate a 45° diagonal axis not in the xy-, yz-, or zx-planes, I made a guess that it requires products of rotors like g_2 and g_3 where g_2 generates a skew axis between the z and x-axes, and then g_3 generates an out-of-zx-plane skew axis towards the y-axis.

Armed with the guess $g_{23} = g_2 \circ g_3$, for a given angle ψ:

- I assigned g_2 and g_3 rotor angles β and γ, respectively,
- Performed the $g_{23}^{-1} \circ h \circ g_{23}$ GA multiplication with Mathematica, and
- Solved for a constant term equal to $\mathbf{Cos}\left(\frac{\psi}{2}\right)$.

This yielded the *Arc***Cos** and *Arc***Tan** values used in Lemma 5.2. The proof in Lemma 5.2 is just the confirmation multiplication that the solution found is correct.