[13.32] Show that every finite group **G** has a faithful representation in GL(*n*) where *n* is the order of **G.**

Solution

Part A. Show *T*(*G)* is a group representation

Proof of Part A is just an elaboration of Robin's method, which is very slick.

Let **G** = {g_1, …, g_n}. A representation *T*(*G)* is the image of a group homomorphism $T\,\mathbf{:}\ \boldsymbol{G} \to \mathrm{GL}(n)$. A homomorphism *T* is a function that preserves the group structure:

For all $g_i, g_j \in \boldsymbol{G},\ T(g_i)\,T(g_j) = T(g_i\, g_j)$.

$T(g_i)$ is an invertible *n* x *n* matrix. I use Penrose's hint to label the rows and columns of matrix $T(g_i)$ to indicate that the matrix takes g_r to g_s :

$$T(g_i) = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & \cdots & s & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ r \\ \vdots \\ n \end{array} & \begin{pmatrix} T(g_i)^1{}_1 & T(g_i)^1{}_2 & \cdots & T(g_i)^1{}_s & \cdots & T(g_i)^1{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_i)^r{}_1 & T(g_i)^r{}_2 & \cdots & T(g_i)^r{}_s & \cdots & T(g_i)^r{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_i)^n{}_1 & T(g_i)^n{}_2 & \cdots & T(g_i)^n{}_s & \cdots & T(g_i)^n{}_n \end{pmatrix} \end{array}.$$

Matrix $T(g_i)$ can be written

$$T(g_j) = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & \cdots & t & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ s \\ \vdots \\ n \end{array} & \begin{pmatrix} T(g_j)^1{}_1 & T(g_j)^1{}_2 & \cdots & T(g_j)^1{}_t & \cdots & T(g_j)^1{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_j)^s{}_1 & T(g_j)^s{}_2 & \cdots & T(g_j)^s{}_t & \cdots & T(g_j)^s{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_j)^n{}_1 & T(g_j)^n{}_2 & \cdots & T(g_j)^n{}_t & \cdots & T(g_j)^n{}_n \end{pmatrix} \end{array},$$

matrix $T(g_i\, g_j)$ can be written

$$\mathrm{T}(g_i g_j) = \begin{array}{c} \\ r \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & t & \end{array} \\ \begin{pmatrix} & \vdots & \\ \cdots & \mathrm{T}(g_i g_j)^r_t & \cdots \\ & \vdots & \end{pmatrix} \end{array},$$

and the product matrix $T(g_i)T(g_j)$ is

$$\mathrm{T}(g_i)\mathrm{T}(g_j) = \begin{array}{c} \\ r \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & t & \end{array} \\ \begin{pmatrix} & \vdots & \\ \cdots & \sum_{s=1}^{n} \mathrm{T}(g_i)^r_s \mathrm{T}(g_j)^s_t & \cdots \\ & \vdots & \end{pmatrix} \end{array}.$$

A strategy to define T such that $T(g_i g_j) = T(g_i)T(g_j)$ is to put as many zeros as possible into the matrix so that the calculation becomes simpler. To that end, define

$$T(g_i)^r_s \equiv \delta_{g_r\, g_i g_s} = \begin{cases} 1 & \text{if } g_r = g_i\, g_s \\ 0 & \text{Otherwise} \end{cases}.$$

So,

$$T(g_j)^s_t \equiv \delta_{g_s\, g_j g_t} = \begin{cases} 1 & \text{if } g_s = g_j\, g_t \\ 0 & \text{Otherwise} \end{cases}$$

and

$$T(g_i g_j)^r_t \equiv \delta_{g_r\, g_i g_j g_t} = \begin{cases} 1 & \text{if } g_r = g_i\, g_j\, g_t \\ 0 & \text{Otherwise} \end{cases}.$$

Hence

$$T(g_i) = \begin{array}{c} \\ r \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & s & \end{array} \\ \begin{pmatrix} & \vdots & \\ \cdots & \delta_{g_r\, g_i g_s} & \cdots \\ & \vdots & \end{pmatrix} \end{array}, \quad T(g_j) = \begin{array}{c} \\ s \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & t & \end{array} \\ \begin{pmatrix} & \vdots & \\ \cdots & \delta_{g_s\, g_j g_t} & \cdots \\ & \vdots & \end{pmatrix} \end{array},$$

$$T\left(g_ig_j\right) = \begin{array}{cc} & t \\ r & \begin{pmatrix} & \vdots & \\ \cdots & \delta_{g_r\, g_ig_jg_t} & \cdots \\ & \vdots & \end{pmatrix} \end{array}, \text{ and}$$

$$\mathrm{T}\left(g_i\right)\mathrm{T}\left(g_j\right) = \begin{array}{cc} & t \\ r & \begin{pmatrix} & \vdots & \\ \cdots & \sum_{s=1}^{n} \delta_{g_r\, g_ig_s}\, \delta_{g_s\, g_jg_t} & \cdots \\ & \vdots & \end{pmatrix} \end{array}.$$

The matrices $T\left(g_i\right)$, $T\left(g_j\right)$, and $T\left(g_ig_j\right)$ have precisely one 1 in every row and every column. The element $\sum_{s=1}^{n} \delta_{g_r\, g_ig_s}\, \delta_{g_s\, g_jg_t}$ of the matrix $T\left(g_i\right)T\left(g_j\right)$ then becomes

$$\sum_{s=1}^{n} \delta_{g_r\, g_ig_s}\, \delta_{g_s\, g_jg_t} = \left\{ \begin{array}{ll} & \text{if } g_r = g_i\, g_s \text{ and } g_s = g_j\, g_t \text{ for some } s \\ 1 & \Leftrightarrow \text{if } g_i^{-1}g_r = g_s = g_j\, g_t \text{ for some } s \\ & \Leftrightarrow \text{if } \left(g_i\, g_j\right)g_t = g_r \\ 0 & \text{Otherwise} \end{array} \right\} = \delta_{g_r\, g_ig_jg_t}.$$

That is, $T\left(g_i\right)T\left(g_j\right) = T\left(g_i\, g_j\right)$. ✔

Part B Show T is faithful

T is faithful if it is one-to-one; i.e., if $T\left(g_i\right) = T\left(g_j\right) \Rightarrow g_i = g_j$. So, suppose

$$\begin{aligned} T\left(g_i\right) = T\left(g_j\right) &\Leftrightarrow \forall\, a, b \;\; T\left(g_i\right)^a{}_b = T\left(g_j\right)^a{}_b \\ &\Rightarrow \forall\, a,b \;\; T\left(g_i\right)^a{}_b = 1 \;\text{ if and only if }\; T\left(g_j\right)^a{}_b = 1 \\ &\Leftrightarrow \forall\, a,b \;\; g_i\, g_b = g_a = g_j\, g_b \\ &\Rightarrow g_i = g_j. \quad ✔ \end{aligned}$$