[13.40] Let V be an n-dimensional vector space, $\mathcal{V} = \mathrm{V} \otimes \mathrm{V}$ the tensor product of V with itself, and $Q^{ab} \in \mathcal{V}$ a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-tensor. Let

$$Q^{(ab)} = \tfrac{1}{2}\left(Q^{ab} + Q^{ba}\right) \text{ be the symmetric part}$$

and

$$Q^{[ab]} = \tfrac{1}{2}\left(Q^{ab} - Q^{ba}\right) \text{ be the antisymmetric part.}$$

Define

$$\mathcal{V}_+ = \left\{Q^{(ab)} : Q^{ab} \in \mathcal{V}\right\} \text{ and } \mathcal{V}_- = \left\{Q^{[ab]} : Q^{ab} \in \mathcal{V}\right\}.$$

Then

$$\dim \mathcal{V}_+ = \frac{n}{2}(n+1) \text{ and } \dim \mathcal{V}_- = \frac{n}{2}(n-1).$$

Solution.

Let $\mathscr{B} = \left\{e^1, \cdots, e^n\right\}$ be the basis for V where $e^k = \begin{bmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{bmatrix}$. Set

$$e^{ab} = e^a \otimes e^b = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1_{ab} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

By definition, $\mathscr{B} = \left\{e^{ab}\right\}$ is a basis for $\mathcal{V}$, and it has n^2 terms. Observe that $e^{(aa)} = e^{aa}$ and $e^{[aa]} = 0$. So, we define

$$\mathscr{B}_+ = \left\{e^{(ab)} : a \le b\right\} \text{ and } \mathscr{B}_- = \left\{e^{[ab]} : a < b\right\}.$$

$\mathscr{B}_+$ has $\frac{n}{2}(n+1)$ terms with $a \le b$ and $\mathscr{B}_-$ has $\frac{n}{2}(n-1)$ terms with $a < b$.

Note: The reason for defining $\mathscr{B}_+$ and $\mathscr{B}_-$ with $a \le b$ is that $e^{(ab)} = e^{(ba)}$ and $e^{[ba]} = -e^{[ab]}$, so terms with $b > a$ are not independent from the others. As defined, each of $\mathscr{B}_+$ and $\mathscr{B}_-$ consists of linearly independent vectors, so dim span $\left(\mathscr{B}_+\right)=\frac{n}{2}(n+1)$ and dim span $\left(\mathscr{B}_-\right)=\frac{n}{2}(n-1)$.

We proceed to show that $\mathscr{V}_+ =$ span $\left(\mathscr{B}_+\right)$ and $\mathscr{V}_- =$ span $\left(\mathscr{B}_-\right)$, which completes the problem.

First, since dim span$\left(\mathscr{B}_+\right)+$dim span $\left(\mathscr{B}_-\right)=\frac{n}{2}(n+1)+\frac{n}{2}(n-1)=n^2=$ dim $\mathscr{V}$, we have that $\mathscr{V}=$ span $\left(\mathscr{B}_+\right)+$span $\left(\mathscr{B}_-\right)$.

Next, we claim that span $\left(\mathscr{B}_+\right)\subseteq\mathscr{V}_+$ and span $\left(\mathscr{B}_-\right)\subseteq\mathscr{V}_-$:

$Q\in$ span $\left(\mathscr{B}_+\right)\Rightarrow\exists$ scalars α_i and basis elements $E_i\in\mathscr{B}_+$ such that Q=$\sum_i\alpha_i E_i$.

Since each E_i is symmetric, Q is symmetric. That is, $Q\in\mathscr{V}_+$.

Similarly, if $Q\in$ span $\left(\mathscr{B}_-\right)$ then Q is antisymmetric, or $Q\in\mathscr{V}_-$.

Finally, $\mathscr{V}=$ span $\left(\mathscr{B}_+\right)+$span $\left(\mathscr{B}_-\right)\subseteq\mathscr{V}_++\mathscr{V}_-\subseteq\mathscr{V}$

$\Rightarrow\mathscr{V}_+=$span $\left(\mathscr{B}_+\right)$ and $\mathscr{V}_-=$span $\left(\mathscr{B}_-\right)$ ✔

Example with n = 2:

Let $e^1=\begin{bmatrix}1\\0\end{bmatrix}$ and $e^2=\begin{bmatrix}0\\1\end{bmatrix}$. Then $\mathscr{B}=\left\{e^1,e^2\right\}$ is a basis for V. Let

$e^{11}=e^1\otimes e^1=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$, $e^{12}=e^1\otimes e^2=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$,

$e^{21}=e^2\otimes e^1=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$, and $e^{22}=e^2\otimes e^2=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$. Then

$\mathscr{B}=\left\{e^{11},e^{12},e^{21},e^{22}\right\}$ is a basis for $\mathscr{V}=\mathrm{V}\otimes\mathrm{V}$. Observe that

$e^{(11)}=\frac{1}{2}\left(e^{11}+e^{11}\right)=e^{11}$ and $e^{(22)}=e^{22}$. These are 2 elements of $\mathscr{B}_+$. Note that $e^{[11]}=0=e^{[22]}$ so they do not contribute to $\mathscr{B}_-$. The other term in $\mathscr{B}_+$ is $e^{(12)}$ [which equals $e^{(21)}=\frac{1}{2}\left(e^{12}+e^{21}\right)$]. The only term in $\mathscr{B}_-$ is $e^{[12]}$ (which equals $-e^{[21]}$). Thus dim $\mathscr{V}_+=\frac{n}{2}(n+1)=$ 3 and dim $\mathscr{V}_-=\frac{n}{2}(n-1)=1$.