

[13.7] $\text{SO}(3)$ is the group of rotations of the unit sphere in 3-space. $\text{O}(3)$ extends $\text{SO}(3)$ by including reflections. **(A)** Show that $\text{SO}(3)$ is a normal subgroup of $\text{O}(3)$ and **(B)** show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of $\text{O}(3)$. In Lemma 9.2 I show that if 1 is the identity element (null rotation) of $\text{SO}(3)$ and R is the reflection operation, then $\mathcal{I} = \{1, R\}$ is a normal subgroup of $\text{O}(3)$. Then I prove (Theorem 9) that there are no others.

(A) Show $\text{SO}(3)$ is a normal subgroup of $\text{O}(3)$

Lemma 1.1: A subgroup H of a group G is normal iff $g^{-1}Hg = H \quad \forall g \in G$

Proof. By Penrose's definition, H is normal iff $gH = Hg \quad \forall g \in G$. Left multiplying by g^{-1} yields the lemma. ■

We use the symbol \circ for the $\text{O}(3)$ group operation. In particular, if $f, g \in \text{SO}(3)$, we adopt the convention that $f \circ g$ is the composite rotation of f followed by g .

Thus, if $w = (x, y, z)$, we write $w(f \circ g) = wf \circ g$. This is *the opposite of Penrose's convention* but, as will be seen, is necessitated by the approach taken to solve (B).

Also note that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. That is, if w is rotated by f and then g , the inverse operation is to rotate in the reverse directions by g and then f .

Definition: We use R to denote the reflection operator on the unit sphere of \mathbb{R}^3 :
 $(x, y, z) R = (-x, -y, -z)$.

Note that R is its own inverse: $R^{-1} = R$. If w is a point then $w R$ is its reflection.

We write the composition of a reflection followed by a rotation as $R \circ f$, and a rotation followed by a reflection as $f \circ R$. An expression involving an odd number of reflections yields a reflective rotation and an expression involving an even number of reflections yields a non-reflective rotation. In particular, we have the following lemma.

Lemma 1.2: If $h \in \text{SO}(3)$, then $R \circ h \circ R \in \text{SO}(3)$. ■

Definition: $T = \text{O}(3) - \text{SO}(3)$ is the coset of reflective rotations of the sphere. That is, $T = \text{SO}(3) \circ R$, and $\text{O}(3)$ is the disjoint union of $\text{SO}(3)$ and T :

$$\text{O}(3) = T \cup \text{SO}(3)$$

The next theorem, Theorem 1, tells us that if $h \in \text{SO}(3)$ then the normality operation $g^{-1} \circ h \circ g$ produces another member of $\text{SO}(3)$, even if $g \in T$.

Theorem 1: SO(3) is a normal subgroup of O(3)

Proof: Clearly SO(3) has an identity, is closed under \circ , and each non-reflective rotation in SO(3) has an inverse (non-reflective) rotation. So it remains only to show that if $h \in \text{SO}(3)$ and $g \in \text{O}(3)$ that $g^{-1} \circ h \circ g \in \text{SO}(3)$. Since the composition of 3 non-reflective rotations is a non-reflective rotation, this holds if $g \in \text{SO}(3)$. So suppose $g \in \text{O}$. Then $\exists f \in \text{SO}(3)$ such that $g = f \circ R$. Let $h' = f^{-1} \circ h \circ f \in \text{SO}(3)$. So, $g^{-1} \circ h \circ g = R \circ f^{-1} \circ h \circ f \circ R = R \circ h' \circ R \in \text{SO}(3)$ by Lemma 1.2. ■

(B) Show SO(3) and \mathcal{I} are the only proper normal subgroups of O(3)

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball \mathcal{R} of radius π in which antipodal points on the surface of \mathcal{R} are identified. The problem was solved by letting elements of \mathcal{R} be written as

$f = \theta(a, b, c) = (\theta a, \theta b, \theta c)$, where $a^2 + b^2 + c^2 = 1$, θ is the (counter-clockwise) angle of rotation, $0 \leq \theta \leq \pi$, and $\{t(a, b, c) : t \geq 0\}$ is the (positive) axis of rotation.

Conventions:

1. **1** will be used to denote the identity element of SO(3). It is the degenerate 0° rotation.
2. An axis of rotation under discussion will be referred to as the **positive axis of rotation**. Its opposite will be called the **negative axis of rotation**.
3. Absolute value $|f| = \theta$ will be used to denote the rotation angle of the rotation f .

The representations for points (i.e., rotations) $f = \theta(a, b, c)$ are unique except:

- When $\theta = 0$, there are infinitely many representations for the identity element since any unit vector (a, b, c) can be used.
- When $\theta = \pi$, there are 2 representations for each π rotation
 $f = \pi(a, b, c) = \pi(-a, -b, -c)$ because antipodal points are identified.

As an intermediate step to proving (B), we prove the following theorem, labeled Theorem 8 because there are several steps that are taken to get there.

Theorem 8: SO(3) does not contain a proper normal subgroup.

Outline of Steps to Prove Theorem 8:

Let H be a proper normal subgroup of SO(3). $\exists 1 \neq h \in H$. Let $\theta = |h| > 0$ be the rotation angle of h . So, $0 < \theta \leq \pi$. Let S_θ be the sphere in \mathcal{R} of radius θ . S_θ consists of all rotations of angle θ about all axes.

- Let $g \in \text{SO}(3)$. We show that $k = g^{-1} \circ h \circ g$ also has rotation angle θ . That is, $k \in S_\theta$. Thus, starting with $h \in H$ and using only the normality operation $g^{-1} \circ h \circ g$, we cannot build up to all of SO(3). By using just the normality operation, we can only generate a subset of S_θ .

- We next show that in fact $\{ g^{-1} \circ h \circ g : g \in SO(3) \}$ generates all of S_ϕ .
- Next, suppose (a) $0 \leq \phi \leq \pi$, (b) $f_1, g_1 \in S_\phi$, and (c) $|f_1 \circ g_1| = \phi$. We show $\{ f \circ g : f, g \in S_\phi \} \supseteq S_\phi$. Thus $H \supseteq S_\phi$.
- Clearly, if (c) holds, the maximum possible size for ϕ is 2θ (namely $f \circ f$ for any $f \in S_\phi$). The minimum possible size is 0 (namely, $f \circ f^{-1}$). In fact, we will show that $\{ f \circ g : f, g \in S_\phi \}$ contains every sphere S_ϕ of radius ϕ between $0 \leq \phi \leq 2\theta$, and so every such sphere belongs to H . To generate all of $SO(3)$, we just need to generate larger and larger spheres in H until all of $SO(3)$ is included. That is, starting with $S_\phi \subseteq H$ we can generate spheres of size up to $S_{2\theta} \subseteq H$. From $S_{2\theta}$ we can generate spheres of size up to $S_{4\theta} \subseteq H$. Then up to $S_{8\theta}$. When $2^n\theta \geq \pi$, then $SO(3) \subseteq S_{2^n\theta} \subseteq H$ and we are done.

The reader who wishes to first get an overview of the rest of the proof of (B) can skip ahead to Theorem 9 and read its overview.

GA concepts and notation used in this proof

Non-reflective rotations (henceforth just called “rotations”) are commonly represented by rotation matrices, and the composition $f \circ g$ can be computed as a matrix product. Instead, in this proof we learn how to use Geometric Algebra, aka Clifford Algebra, for the computations. In GA, every rotation corresponds to a rotor (described shortly), and geometric product (rather than matrix product) is used to compute the composition of rotations.

Let $\{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \}$ be an orthonormal basis for 3-space. **Rotors** are objects having the form

$$\mathbf{r} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2]$$

where $a^2 + b^2 + c^2 = 1$, $a, b, c \in \mathbb{R}$, and $-\infty \leq \theta \leq \infty$.

While \mathbf{e}_1 is a **vector**, $\mathbf{e}_1 \mathbf{e}_2$ is a **bivector**. $\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ is an example of a **trivector**. **Scalars**, vectors, bivectors, trivectors, etc., belong to the class of objects called **multivectors**.

A rotor is the sum of a scalar, $\cos\left(\frac{\theta}{2}\right)$, and a bivector (the rest of the expression). This is no stranger than a complex number that is the sum of a real and an imaginary number. The two parts cannot be combined but nonetheless the sum has meaning. In general a multivector can be the sum of multivectors of different **grades** that cannot be combined.

A rotor r represents a rotation of angle θ about the axis $\{ t(a, b, c) : t \geq 0 \}$.

$\frac{\theta}{2}$ is called the **rotor angle** and θ is the **rotation angle**. As this can be confusing, in this proof only the term “rotation angle” will be used.

The scalar $\mathbf{Cos} \frac{\theta}{2}$ is referred to as the “**constant term**” of r because it does not contain any basis elements. The constant term determines the rotation angle θ . That is, if the constant term of a rotor r is known to be K , then for some θ ,

$$K = \mathbf{Cos} \frac{\theta}{2}, \text{ and so } \theta = 2 \operatorname{ArcCos}(K).$$

The magnitude of a rotor is $\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) = 1$, which is why $a^2 + b^2 + c^2 = 1$.

But even though rotors have magnitude 1, in this proof $|r|$ will continue to be used to denote the rotation angle θ and not 1 (and also not $\frac{\theta}{2}$) in order to be consistent with the definition already given for the magnitude $|f|$ of a point (rotation) in \mathcal{R} .

The **geometric product** of 2 multivectors is just the regular polynomial product with a couple of modifications. First, it is non-commutative, so the order of multiplication matters. Second, the basis elements are combined according to the rules

$$\begin{aligned} \mathbf{e}_i^2 &= 1, \quad i = 1, 2, 3 \quad \text{and} \\ \mathbf{e}_i \mathbf{e}_j &= -\mathbf{e}_j \mathbf{e}_i \quad \text{if } i \neq j \quad (\text{antisymmetry}) \end{aligned}$$

While a rotor can have any angle, \mathcal{S} is defined in this proof as the subset of rotors having (counter-clockwise) rotation angle $0 \leq \theta \leq \pi$ under the operation of geometric product. If the rotor $r \in \mathcal{S}$ (i.e., if $0 \leq \theta \leq \pi$), it corresponds to the point $(\theta a, \theta b, \theta c)$ of \mathcal{R} .

The symbol \circ will again be used to denote the geometric product, as in $f \circ g$. That is, whether f and g belong to \mathcal{R} or \mathcal{S} , $f \circ g$ will represent their composition.

To compute the sometimes algebra-intensive geometric products I wrote a software package in Mathematica that calculates geometric products and also performs other GA operations such as wedge product, multivector inverse, pseudoscalar, etc. The package can be downloaded for free at <https://github.com/matrixbud/Geometric-Algebra>. I have also saved the Mathematica file having the GA calculations used in this proof in the same directory as this file.

Note: Mathematica always displays the bivector $\mathbf{e}_3 \mathbf{e}_1$ as $\mathbf{e}_1 \mathbf{e}_3$. Thus, when using Mathematica, the negative of the b term must be used, as in

$$r = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} [a e_2 e_3 - b e_1 e_3 + c e_1 e_2].$$

Since the normality operation $g^{-1} \circ f \circ g$ involves inverses, this is a good time to provide the GA formula for the inverse of a rotor.

Theorem 2: The GA **inverse of the rotor r** has the formula

$$r^{-1} = \mathbf{Cos} \frac{\theta}{2} - \mathbf{Sin} \frac{\theta}{2} [a e_2 e_3 + b e_3 e_1 + c e_1 e_2].$$

Moreover,

$$r^{-1} \in \mathcal{S} \text{ if } r \in \mathcal{S}.$$

Proof. By computing the geometric product we find that

$r^{-1} \circ r = \mathbf{Cos}^2 \frac{\theta}{2} + \mathbf{Sin}^2 \frac{\theta}{2} (a^2 + b^2 + c^2) = 1$ and similarly $r \circ r^{-1} = 1$. Thus, r^{-1} is the inverse of r . ✓

Another way of seeing that $r^{-1} \circ r = 1$ is that r^{-1} can be written

$$r^{-1} = \mathbf{Cos} \left(-\frac{\theta}{2} \right) + \mathbf{Sin} \left(-\frac{\theta}{2} \right) [a e_2 e_3 + b e_3 e_1 + c e_1 e_2],$$

representing a clockwise

rotation about the positive axis that cancels the counter-clockwise rotation of r , yielding the identity rotation, 1. ✓

Additionally, since $r^{-1} = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} [-a e_2 e_3 - b e_3 e_1 - c e_1 e_2]$, r^{-1} can also

be regarded as a (counter-clockwise) θ rotation about the negative axis, showing that $r^{-1} \in \mathcal{S}$ (since \mathcal{S} is the set of all [counter-clockwise] rotations $0 \leq \theta \leq \pi$ about all axes). ■

In rotor theory the formula for a rotor r to rotate a *point* $w = x e_1 + y e_2 + z e_3$ in 3-space to some other point v is

$$v = r^{-1} \circ w \circ r.$$

Effectively, r^{-1} performs half the θ rotation (i.e., $\frac{\theta}{2}$) and r performs the rest. The reader is encouraged to confirm this for a 90° rotation of the point $(1,0,0)$ in the xy -plane. To do this, let $\theta = \frac{\pi}{2}$, $c = 1$, and $a = b = 0$ in the rotor r .

To write the expression to rotate a point w by f followed by g :

- Start with w
- Wrap with f : $f^{-1} \circ w \circ f$
- Wrap with g : $g^{-1} \circ f^{-1} \circ w \circ f \circ g$

Thus $f \circ g$ in this expression means to rotate first by f , then by g . This is the reason for the left-right convention adopted on page 1.

Every rotation can be represented by a rotor in \mathcal{S} ; i.e., by a rotor of the form

$$r = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \text{ where } 0 \leq \theta \leq \pi. \text{ Every rotation has}$$

an additional representation by a rotor with $\pi < \theta < 2\pi$ that is not in \mathcal{S} . The geometric product of 2 rotors in \mathcal{S} can be outside of \mathcal{S} . To make \mathcal{S} a group under \circ we need to be able to recognize the formulas of rotors not in \mathcal{S} and be able to identify them with equivalent rotors in \mathcal{S} . Thus, we will actually be using \circ to represent the geometric product mod \mathcal{S} . However, we will only explicitly write mod \mathcal{S} when it is necessary for the sake of clarity.

Definition: We say that 2 rotors r_1 and r_2 are **equivalent** if they generate the same rotation.

Example 1: Does the geometric product of two $\frac{2}{3}\pi$ rotations about the x -axis belong to \mathcal{S} ?

Answer: The result is a $\frac{4}{3}\pi$ rotation about the x -axis, not in \mathcal{S} since $\frac{4}{3}\pi > \pi$.

Theorem 3, below, shows how to recognize a rotor not in \mathcal{S} and how to generate its equivalent rotor in \mathcal{S} .

Theorem 3: Let $\pi < \theta < 2\pi$. Then

$$s = R(\theta) \equiv \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \notin \mathcal{S}, \text{ and}$$

$$r = R(2\pi - \theta) = \mathbf{Cos} \left(\pi - \frac{\theta}{2} \right) + \mathbf{Sin} \left(\pi - \frac{\theta}{2} \right) (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \in \mathcal{S}.$$

Also, the constant term of s is negative; the constant term of r is positive, and s is equivalent to $-s = r^{-1} \in \mathcal{S}$.

Proof: $s \notin \mathcal{S}$ because $\theta \notin [0, \pi]$. ✓

Since $(2\pi - \theta) \in [0, \pi]$, $r \in \mathcal{S}$. ✓

The constant term of s is negative: $\mathbf{Cos}\frac{\theta}{2} < 0$ because $\frac{\pi}{2} < \frac{\theta}{2} < \pi$. ✓

The constant term of r is positive: $\mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) > 0$ since $0 < \pi - \frac{\theta}{2} < \frac{\pi}{2}$.

$r^{-1} \in \mathcal{S}$ from Theorem 2. ✓

Claim $r^{-1} = -s$:

Without loss of generality, by a suitable rotation of 3-space, we can assume s , and hence r , are rotations about the z -axis.

$$s = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \quad \text{and} \quad r = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2.$$

These formulas for s and r are rotors because they satisfy the rotor definition with $c = 1$ and $a = b = 0$. They rotate about the z -axis because $e_1 e_2$ rotates the xy -plane.

$$r^{-1} = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) - \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 = -\mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) = -s \quad \checkmark$$

To show s is equivalent to r^{-1} , we must show that if $w = (x, y, z) \in \mathbb{R}^3$ then

$s^{-1} \circ w \circ s = r \circ w \circ r^{-1}$, i.e., that the rotations of w generated by s and r^{-1} are the same for every $w \in \mathbb{R}^3$:

$$w = x e_1 + y e_2 + z e_3.$$

$$\begin{aligned} s^{-1} \circ w \circ s &= \left[x \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - 2y \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - x \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_1 \\ &\quad + \left[y \mathbf{Cos}^2\left(\frac{\theta}{2}\right) + 2x \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - y \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_2 + \\ &\quad + z \left[\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_3 \\ &= [x \mathbf{Cos}(\theta) - y \mathbf{Sin}(\theta)] e_1 + [y \mathbf{Cos}(\theta) + x \mathbf{Sin}(\theta)] e_2 + z e_3 \\ &= r \circ w \circ r^{-1} \end{aligned}$$



Aside: Note the change from $\frac{\theta}{2}$ to θ in the equation above, evidence that θ is indeed the rotation angle generated by the expression $s^{-1} \circ w \circ s$.

Example 2: Find a rotor in \mathcal{S} that is equivalent to the rotor $s = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{e}_1\mathbf{e}_2$

Solution: First, by Theorem 3, $s \notin \mathcal{S}$ because the constant term is negative. Also by Theorem 3, s is equivalent to $-s = \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{e}_1\mathbf{e}_2$ and $-s \in \mathcal{S}$.

Example 3: If a rotor s is a π rotation about any axis, then s^2 should be the identity, a zero rotation. But if, for example, $s = \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_2 \mathbf{e}_3$, then $s^2 = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_3 = -1$. How can that be?

Answer: By Theorem 3, $s^2 \notin \mathcal{S}$ because the constant term is negative. Also by Theorem 3, s^2 is equivalent to $-s^2 = 1$, the identity rotor. The fact that $-1 \equiv 1 \pmod{\mathcal{S}}$ is important in Theorem 9.

Aside: If we consider $-\infty < \theta < \infty$, then there are infinitely many rotors equivalent to r since sine and cosine have period 2π . But, for $0 < \theta < 2\pi$ there are only two representations for each rotor, and for $0 < \theta < \pi$ there is only one.

The next theorem is the analog to Penrose's problem [12.17].

Theorem 4: \mathcal{S} is group isomorphic to \mathcal{R} [which is group isomorphic to $\text{SO}(3)$]

Proof: Let $s = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2] \in \mathcal{S}$.

Then $0 \leq \theta \leq \pi$ and $a^2 + b^2 + c^2 = 1$.

s can be represented as $s = (\theta, a, b, c) \in \mathcal{S}$. This representation in \mathcal{S} is unique for $0 \leq \theta \leq \pi$ except when $\theta = 0$:

- Like \mathcal{R} , when $\theta = 0$, there are infinitely many unit points (a, b, c) that can be used to denote the identity rotation.
- Unlike \mathcal{R} , when $\theta = \pi$ there is only 1 representation for each point s because while the antipodal point represents the same rotation, it is not in \mathcal{S} . That is, if $s = (\pi, a, b, c) \in \mathcal{S}$, the antipodal point is $(-\pi, -a, -b, -c)$, not in \mathcal{S} because $\theta = -\pi$.

Define

$$\begin{aligned} T: \mathcal{S} &\rightarrow \mathcal{R}: T(s) = r \text{ where} \\ s &= (\theta, a, b, c) \text{ and} \\ r &= \theta(a, b, c) = (\theta a, \theta b, \theta c). \end{aligned}$$

That is, $T: (\theta, a, b, c) \mapsto \theta(a, b, c)$

T is well-defined:

To show that T is well-defined we must show that (a) $T(s) \in \mathcal{R}$ and (b) if s has 2 or more representations, then T assigns the same element of \mathcal{S} in all cases.

(a) Since $0 \leq \theta \leq \pi$ and $a^2 + b^2 + c^2 = 1$, $r \in \mathcal{R}$.

(b) If $\theta = 0$, then $T(0, a, b, c) = 0$ (a, b, c) = (0, 0, 0) for any unit vector (a, b, c).

T is obviously 1–1 and onto since $T: (\theta, a, b, c) \rightarrow \theta(a, b, c)$.

T is a homomorphism:

We need to show that $T(s_1 \circ s_2) = T(s_1) \circ T(s_2)$

$s_1 \circ s_2$ is the composition of 2 rotations in \mathcal{S} , and $T(s_1) \circ T(s_2)$ is the composition of 2 rotations in \mathcal{R} . So,

if $s_1 \circ s_2$ = Composition of $(\theta_1, a_1, b_1, c_1)$ & $(\theta_2, a_2, b_2, c_2)$ = $(\theta_3, a_3, b_3, c_3)$
 then $T(s_1) \circ T(s_2)$ = Composition of $\theta_1(a_1, b_1, c_1)$ & $\theta_2(a_2, b_2, c_2)$
 $= \theta_3(a_3, b_3, c_3)$

So, $T(s_1 \circ s_2) = T(\theta_3, a_3, b_3, c_3) = \theta_3(a_3, b_3, c_3) = T(s_1) \circ T(s_2)$ ■

Steps leading to Theorem 8

As we move towards Theorem 8, we will be positing a normal group H and an element $1 \neq h_1 \in H$. We will expand the singleton set $\{h_1\}$ to a sphere of elements in H using just the normality operation $g^{-1} \circ h_1 \circ g$ where $g \in SO(3)$. Then we will switch to the product operation between elements of the sphere to expand H to all of $SO(3)$.

WLOG, the axes of the unit 3-sphere can be rotated so that h_1 is a rotation about the x -axis. The following lemma provides geometric insight into how to use the normality operation $g^{-1} \circ f \circ g$ to generate a rotor whose axis is perpendicular to that of a given rotor. This is accomplished by letting g be a 90° rotation about an axis perpendicular to the axis of f . For example, if f rotates by θ about the x -axis and g is a 90° rotation about the y -axis, then $g^{-1} \circ f \circ g$ is a rotation by θ about the z -axis.

Lemma 5.1: Let

$$\mathbf{h}_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_2 \mathbf{e}_3 \quad (\theta \text{ rotation about positive x-axis})$$

$$\mathbf{h}_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_3 \mathbf{e}_1 \quad (\theta \text{ rotation about positive y-axis}), \text{ and}$$

$$\mathbf{h}_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \quad (\theta \text{ rotation about positive z-axis})$$

If h_1 belongs to a normal group H , then $h_2, h_3, h_1^{-1}, h_2^{-1}, h_3^{-1} \in H$.

Proof: Define rotors

$$g_3 = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \mathbf{e}_1 \mathbf{e}_2 = \frac{1 + \mathbf{e}_1 \mathbf{e}_2}{\sqrt{2}} \text{ and}$$

$$g_2 = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \mathbf{e}_3 \mathbf{e}_1 = \frac{1 - \mathbf{e}_3 \mathbf{e}_1}{\sqrt{2}}.$$

Then

$$g_3^{-1} = \frac{1 - \mathbf{e}_1 \mathbf{e}_2}{\sqrt{2}},$$

$$g_3^{-1} \circ h_1 = \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_2 \mathbf{e}_3 + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_3 \mathbf{e}_1 - \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_1 \mathbf{e}_2$$

$$g_3^{-1} \circ h_1 \circ g_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_3 \mathbf{e}_1 = h_2$$

and

$$g_2^{-1} = \frac{1 + \mathbf{e}_3 \mathbf{e}_1}{\sqrt{2}},$$

$$g_2^{-1} \circ h_1 = \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_2 \mathbf{e}_3 + \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_3 \mathbf{e}_1 + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_1 \mathbf{e}_2$$

$$g_2^{-1} \circ h_1 \circ g_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 = h_3$$

Since H is normal, $h_2, h_3 \in H$. Since groups are closed under inverses, $h_1^{-1}, h_2^{-1}, h_3^{-1} \in H$. ■

Definition. In \mathcal{R} , let S_θ denote the **sphere of radius θ** . It consists of all rotations of amount θ about all axes in 3-space. The formula for S_θ in terms of rotors is

$$S_\theta = \left\{ \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)(a e_2 e_3 + b e_3 e_1 + c e_1 e_2) : a^2 + b^2 + c^2 = 1 \right\}.$$

Given $h, r \in \mathcal{R}$ it is easy to find $g \in \mathcal{R}$ such that $r = h \circ g$. Namely, $g = h^{-1} \circ r$. It is much more difficult to find $g \in \mathcal{R}$ such that $r = g^{-1} \circ h \circ g$. In fact, we will prove in Lemma 6.1 that it is not possible at all unless h and r have the same magnitude. The next lemma (5.2) proves that it is in fact always possible to find such a point g if the magnitudes of h and r are the same, and the lemma provides an explicit formula for g (actually 2 formulas, depending on whether r has a x -component). This lemma (or the alternate proof to Theorem 5, discussed below) is the key step in achieving Theorem 8.

Lemma 5.2 gives formulas for r that work but does not explain where they come from. In Appendix 2, I provide what insight I can into how I discovered the formula for rotor g_{23} . Discovery of the rotor g in part (b) is slightly more transparent because a and b can be viewed as the sides of a right triangle and hence there is an angle α such that $b = \mathbf{Sin} \alpha$.

My inability to provide clear insight for the equations in Lemma 5.2 nudged me to develop another proof for Theorem 5 that is intuitive and does not depend on either Lemma 5.1 or 5.2. It does use an additional lemma (5.3) that is a generalization of Lemma 5.1. I provide both of the Theorem 5 proofs, though only one is needed in order to proceed. I include the second proof because it is short, clever, geometric and provides an insightful process to find, for a given rotation r , a rotation g such that $g^{-1} \circ h \circ g = r$. I include the original proof because it is short and Lemma 5.2 includes explicit, compact formulas for generating r .

Lemma 5.2: Let

$$h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_2 e_3, \quad h_3 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_1 e_2, \text{ and}$$

$$r = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}[a e_2 e_3 + b e_3 e_1 + c e_1 e_2] \in S_\theta,$$

where $a^2 + b^2 + c^2 = 1$ and $0 < \theta \leq \pi$.

(a) If $a \neq 0$ then

$$r = g_{23}^{-1} \circ h_1 \circ g_{23} \quad \text{where}$$

$$g_2 = \cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\beta}{2}\right) e_3 e_1,$$

$$g_3 = \cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\gamma}{2}\right) e_1 e_2,$$

$$g_{23} = g_2 \circ g_3, \text{ and}$$

$$\beta = \text{Arc Cos } (\sqrt{a^2 + b^2}) \text{ and } \gamma = \text{Arc Tan } (a, b).$$

(b) If $a = 0$ then

$$r = g^{-1} \circ h_3 \circ g \quad \text{where}$$

$$g = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) e_2 e_3, \text{ and}$$

$$\alpha = \text{Arc Sin } (-b).$$

Proof: $\text{ArcTan}(a, b)$ denotes the arc tangent of $\frac{b}{a}$ taking into account which quadrant the point (a, b) is in.

$$(a) \quad g_{23} = \cos\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) e_2 e_3 \\ - \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) e_3 e_1 + \cos\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) e_1 e_2$$

$$g_{23}^{-1} = \cos\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) e_2 e_3 \\ + \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) e_3 e_1 - \cos\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) e_1 e_2$$

$$\begin{aligned}
g_{23}^{-1} \circ h_1 = & \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \\
& - \left[\cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right] e_2 e_3 \\
& + \left[\cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \right] e_3 e_1 \\
& - \left[\cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right] e_1 e_2
\end{aligned}$$

$$\begin{aligned}
g_{23}^{-1} \circ h_1 \circ g_{23} = & \cos\left(\frac{\theta}{2}\right) \left[\cos^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
& \quad \left. + \cos^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] \\
& + \sin\left(\frac{\theta}{2}\right) \left[\cos^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) - \sin^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
& \quad \left. - \cos^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] e_2 e_3 \\
& + \sin\left(\frac{\theta}{2}\right) \left[2 \cos^2\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right. \\
& \quad \left. - 2 \sin^2\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right] e_3 e_1 \\
& + \sin\left(\frac{\theta}{2}\right) \left[2 \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
& \quad \left. + 2 \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] e_1 e_2 \\
= & \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) [\cos(\beta) \cos(\gamma) e_2 e_3 + \cos(\beta) \sin(\gamma) e_3 e_1 + \sin(\beta) e_1 e_2] \\
= & \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) [ae_2 e_3 + be_3 e_1 + ce_1 e_2] \quad \checkmark \\
= & r
\end{aligned}$$

Note: The interested reader can find intermediate steps for this and subsequent equations in the accompanying Mathematica file.

(b) Since $a = 0$, then $b = \sqrt{1 - c^2}$, and

$$\begin{aligned}
g^{-1} \circ h_3 &= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}\left(\frac{\alpha}{2}\right) - \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\alpha}{2}\right) e_2 e_3 \\
&\quad - \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\alpha}{2}\right) e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}\left(\frac{\alpha}{2}\right) e_1 e_2 \\
g^{-1} \circ h_3 \circ g &= \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}(\alpha) e_3 e_1 + \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}(\alpha) e_1 e_2 \\
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) + b \mathbf{Sin}\left(\frac{\theta}{2}\right) e_3 e_1 + c \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \\
&= r \quad \checkmark
\end{aligned}$$

■

The next theorem summarizes Lemma 5.2. It shows that for a normal subgroup H , if $h \in H$ has rotation angle θ , then the sphere S_θ of radius θ lies in H

Theorem 5: If H is a normal subgroup of $\text{SO}(3)$, $1 \neq h \in H$ and $\theta = |h|$, then $H \supseteq S_\theta$.

Proof: WLOG we can assume h is a θ rotation about the x -axis. That is,

$$h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3.$$

Let $r \in S_\theta$. Then for some $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 = 1$,

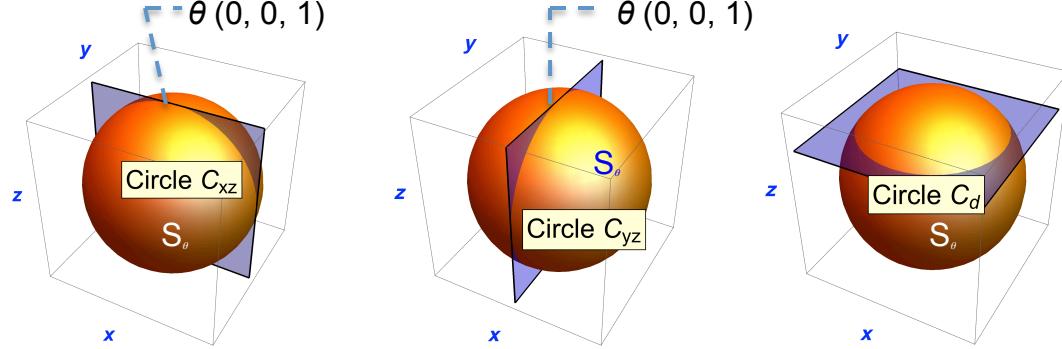
$$r = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2} [a e_2 e_3 + b e_3 e_1 + c e_1 e_2].$$

If $a \neq 0$, by Lemma 5.2a, $r \in H$ since $r = g_{23}^{-1} \circ h_1 \circ g_{23}$ and H is normal.

If $a = 0$, by Lemma 5.2b, $r \in H$ since $r = g^{-1} \circ h_3 \circ g$, H is normal, and $h_3 \in H$ (by Lemma 5.1). ■

We now present the alternate proof for Theorem 5.

Outline of Geometric-based Proof of Theorem 5: As before we begin with $h_1 = \theta(1, 0, 0) \in H$ and $f = \theta(a, b, c) \in S_\theta$. We are done when we show that $f \in H$.



In the figure above, circle C_{xz} of radius θ about the origin is formed by slicing the sphere S_θ with the xz -plane of \mathcal{R} . Circle C_{yz} is similarly formed by slicing S_θ with the yz -plane. Lemma 5.3 will prove that these circles lie in H .

The third circle in the figure is formed by taking a horizontal slice of S_θ at height $z = c\theta$, where $0 \leq c \leq 1$. This generates a circle C_d of radius $d = \theta\sqrt{1 - c^2}$ that contains f . Thus, points on C_d have coordinates $\theta(d \cos \phi, d \sin \phi, c)$ where ϕ is the counter-clockwise angle from the x -axis of an xy -plane raised to height c . We show that the formula for ϕ is $\text{Arc Tan} \frac{b}{a}$.

Next we show that if $g_\circ = \phi(0, 0, 1)$ and $h = \theta(d, 0, c)$, then $g_\circ^{-1} \circ h \circ g_\circ \in C_d$ and, in fact, $\{g_\circ^{-1} \circ h \circ g_\circ : -\pi \leq \phi \leq \pi\}$ generates C_d . In particular, it generates f , proving $f \in H$ since $h \in C_{xz} \subseteq H$

Geometric Insight: We have again used rotations (specifically, g_\circ and h) about axes perpendicular to the axis of f to generate f via the normality operation. This time, however, g_\circ is not required to have a 90° rotation angle but, rather, as ϕ varies from $-\pi$ to π , the normality operation $g_\circ^{-1} \circ h \circ g_\circ$ traces out the Circle C_d .

Lemma 5.3: If $h_1 = (\theta, 0, 0) \in \mathcal{R}$ belongs to a normal group H , then H contains C_{xy} , C_{yz} , and C_{xz} , the circles of radius θ centered at the origin in the xy -, yz -, and xz -planes of \mathcal{R} , respectively.

Proof: The lemma is just a slight generalization of Lemma 5.1. Recall

$$h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e_2e_3. \text{ Similarly to Lemma 5.1, let}$$

$$g_2 = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)e_3e_1 \text{ and } g_3 = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)e_1e_2 \text{ where } 0 \leq \phi \leq \frac{\pi}{2}.$$

Also set $g_1 = \mathbf{Cos}\left(\frac{\phi}{2}\right) + \mathbf{Sin}\left(\frac{\phi}{2}\right)e_2 e_3$. Then

$$g_3^{-1} \circ h_1 \circ g_3 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)[\mathbf{Cos}(\phi)e_2 e_3 + \mathbf{Sin}(\phi)e_3 e_1].$$

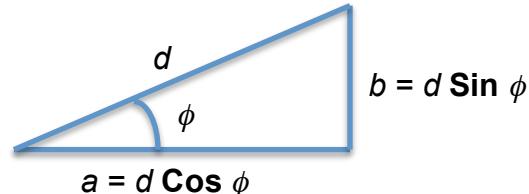
Recall from page 2 of this proof that the axis of rotation of $g_3^{-1} \circ h_1 \circ g_3$ is $\{t(\mathbf{Cos} \phi, \mathbf{Sin} \phi, 0) : t \geq 0\}$. Thus rotor $g_3^{-1} \circ h_1 \circ g_3$ is represented in \mathcal{R} as the point $\theta(\mathbf{Cos} \phi, \mathbf{Sin} \phi, 0)$. As ϕ varies from $-\pi$ to π , these points sweep out the circle C_{xy} . Similarly, the set of points $\{g_2^{-1} \circ h_1 \circ g_2\}$ forms the circle C_{xz} and $\{g_1^{-1} \circ h_2 \circ g_1\}$ forms the circle C_{yz} , where $h_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_3 e_1$. These circles are contained in H because H is normal and, in the case of the 3rd circle, also because $h_2 \in H$ (because $h_2 \in C_{xz} \subseteq H$). ■

Theorem 5 Proof #2: Let H be a non-trivial normal subgroup of $\text{SO}(3)$. WLOG let $(0, 0, 0) \neq h_1 = (\theta, 0, 0) \in H$ be the element of \mathcal{R} representing a θ rotation about the x -axis. Let $f = \theta(a, b, c) \in S_\circ$. It suffices to show that $f \in H$.

If $a = 0$, then $f \in C_{yz} \subseteq H$ by Lemma 5.3.
So we can assume $a \neq 0$.

Let C_d be the circle formed by taking a horizontal slice of S_\circ at height $z = c\theta$.

The radius of C_d is $d = \theta\sqrt{1 - c^2}$.



We wish to represent f also as $f = \theta(d \mathbf{Cos} \phi, d \mathbf{Sin} \phi, c)$ since it is a point on the circle C_d . That is, we desire an angle ϕ such that $a = d \mathbf{Cos} \phi$ and $b = d \mathbf{Sin} \phi$. To

do this, we define $\phi = \text{ArcTan}(a, b)$, which is $\text{ArcTan} \frac{b}{a}$ but adjusts ϕ for the

quadrant of (a, b) . ϕ is well-defined because $a \neq 0$. In order to make $\mathbf{Sin} \phi = \frac{b}{d}$

and $\mathbf{Cos} \phi = \frac{a}{d}$ well-defined, we also must have $d \neq 0$. But, if $d = 0$ then

$f = \theta(0, 0, \pm 1) \in C_{yz} \subseteq H$ (see Circle C_{yz} figure), and we are done. So we can assume $d \neq 0$.

Let

$$g_\phi = \mathbf{Cos}\left(\frac{\phi}{2}\right) + \mathbf{Sin}\left(\frac{\phi}{2}\right)e_1 e_2 = \phi(0, 0, 1)$$

be the rotation of angle ϕ about the z -axis.

Let

$$h = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)(d e_2 e_3 + c e_1 e_2) = \theta(d, 0, c),$$

a point of the circle C_{xz} . $h \in H$ by Lemma 5.3.

Finally,

$$\begin{aligned} g_{\phi}^{-1} \circ h \circ g_{\phi} &= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\ &\quad + \left(d \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) - d \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right) e_2 e_3 \\ &\quad + 2d \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{Sin}\left(\frac{\phi}{2}\right) e_3 e_1 \\ &\quad + \left(c \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + c \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right) e_1 e_2 \\ &= \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) [d \mathbf{Cos}(\phi) e_2 e_3 + d \mathbf{Sin}(\phi) e_3 e_1 + c e_1 e_2] \\ &= \theta(d \mathbf{Cos}(\phi), d \mathbf{Sin}(\phi), c) \\ &= f. \end{aligned}$$

Therefore $f \in H$ and $S_o \subseteq H$. ■

The next lemma shows that for any $g \in SO(3)$, the normality operation $g^{-1} \circ h \circ g$ results in a rotation having the same rotation angle as h . Thus, for a given $h \in SO(3)$, $\{g^{-1} \circ h \circ g : g \in SO(3)\}$ cannot generate all of $SO(3)$. The prior Lemma 5.2 (as well as the alternate proof of Theorem 5) tells us that it at least generates all of S_o .

Lemma 6.1: Let $0 \leq \theta \leq \pi$, $h \in S_o$, and $g \in SO(3)$. Then $g^{-1} \circ h \circ g \in S_o$.

Proof: WLOG $h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_2 e_3$. Since 2 vectors determine a plane,

WLOG we can also assume that g lies in the xy -plane. That is, if the magnitude of g is $0 \leq \phi \leq \pi$ then $g = \mathbf{Cos}\frac{\phi}{2} + \mathbf{Sin}\frac{\phi}{2}[a e_2 e_3 + b e_3 e_1]$ where $a^2 + b^2 = 1$.

Then

$$\begin{aligned}
g^{-1} \circ h \circ g &= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) + b^2 \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&\quad + \left[\mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \\
&\quad + 2ab \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \\
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&\quad - \mathbf{Sin}\left(\frac{\theta}{2}\right) \left\{ \begin{array}{l} + \left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \\ + 2ab \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \end{array} \right\} \\
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) \\
&\quad + \mathbf{Sin}\left(\frac{\theta}{2}\right) \left\{ \begin{array}{l} + \left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \\ + 2ab \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \end{array} \right\}
\end{aligned}$$

Since the constant term of $g^{-1} \circ h \circ g$ is $\mathbf{Cos}\frac{\theta}{2}$, $g^{-1} \circ h \circ g \in S_\circ$. ■

Now we switch tactics, using the product operation rather than the normality operation to generate additional elements of H. We will show that geometric products of pairs of elements in S_\circ generate every sphere S_ω and, hence, all of SO(3).

The next theorem starts the process. It shows the ability to generate one sphere in \mathcal{R} from another by taking products of pairs of elements from the first sphere.

Theorem 6: Let $k_1, k_2 \in S_\circ$ for some $0 < \theta \leq \pi$, and let $\omega = |k_1 \circ k_2|$. Then S_ω can be generated from geometric products of pairs of elements from S_\circ .

Proof. The theorem is trivially true if $\omega = 0$, so assume $\omega > 0$. Let $r \in S_\omega$. The claim is that we can find $f_1, f_2 \in S_\circ$ such that $r = f_1 \circ f_2$. WLOG let $k_1 \circ k_2$ be the ω rotation about the positive x -axis. That is $h_1 \equiv k_1 \circ k_2 = \mathbf{Cos}\left(\frac{\omega}{2}\right) + \mathbf{Sin}\left(\frac{\omega}{2}\right) \mathbf{e}_2 \mathbf{e}_3$.

Since $r \in S_\omega$, $\exists a, b, c \in \mathbb{R}$ such that $r = \cos \frac{\omega}{2} + \sin \frac{\omega}{2} [a e_2 e_3 + b e_3 e_1 + c e_1 e_2]$, where $a^2 + b^2 + c^2 = 1$.

Lemma 5.2 yields that either (a) $r = g_{23}^{-1} \circ h_1 \circ g_{23}$ or (b) $r = g^{-1} \circ h_3 \circ g$, where $h_3 \in S_\omega$ is the ω rotation about the z-axis and g_{23} and g are as defined in the lemma (using ω rather than θ).

(a) Define $f_1 = g_{23}^{-1} \circ k_1 \circ g_{23}$ and $f_2 = g_{23}^{-1} \circ k_2 \circ g_{23}$. By Lemma 6.1, $f_1, f_2 \in S_\omega$.

Thus,

$$\begin{aligned} r &= g_{23}^{-1} \circ h_1 \circ g_{23} = g_{23}^{-1} \circ k_1 \circ k_2 \circ g_{23} \\ &= (g_{23}^{-1} \circ k_1 \circ g_{23}) \circ (g_{23}^{-1} \circ k_2 \circ g_{23}) \\ &= f_1 \circ f_2 \quad \checkmark \end{aligned}$$

(b) As was shown in Lemma 5.1, h_3 can be obtained from h_1 by a normality operation using the 90° rotation $g_2 = \frac{1 - e_3 e_1}{\sqrt{2}}$. The rotation formula is

$h_3 = g_2^{-1} \circ h_1 \circ g_2$. Apply the same 90° rotation to k_1 and k_2 :

$$k_3 = g_2^{-1} \circ k_1 \circ g_2 \quad \text{and} \quad k_4 = g_2^{-1} \circ k_2 \circ g_2.$$

By Lemma 6.1, $k_3, k_4 \in S_\omega$ and

$$\begin{aligned} h_3 &= g_2^{-1} \circ h_1 \circ g_2 = g_2^{-1} \circ k_1 \circ k_2 \circ g_2 \\ &= (g_2^{-1} \circ k_1 \circ g_2) \circ (g_2^{-1} \circ k_2 \circ g_2) \\ &= k_3 \circ k_4 \end{aligned}$$

Define $f_1 = g^{-1} \circ k_3 \circ g$ and $f_2 = g^{-1} \circ k_4 \circ g$. By Lemma 6.1, $f_1, f_2 \in S_\omega$. Thus

$$\begin{aligned} r &= g^{-1} \circ h_3 \circ g = g^{-1} \circ k_3 \circ k_4 \circ g = (g^{-1} \circ k_3 \circ g) \circ (g^{-1} \circ k_4 \circ g) \\ &= f_1 \circ f_2 \quad \blacksquare \end{aligned}$$

The next theorem and its corollary show how to generate all of $SO(3)$ from geometric products of pairs of elements of S_ω .

Theorem 7: Suppose $\theta \in \left[\frac{\pi}{2}, \pi \right]$ and $0 < \omega \leq \pi$. Then there is a pair of elements in S_ω whose geometric product lies in S_ω .

Proof: Let h_1 be the rotor in S_θ that rotates by θ about the x -axis:

$$h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3.$$

h_1 is the first desired element of S_ω . We will find a second rotor, $h \in S_\omega$, such that $h \circ h_1 \in S_\omega$. To define h , we need $a, b, c \in \mathbb{R}$ such that

$$a^2 + b^2 + c^2 = 1,$$

$$\mathbf{h} = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)(a e_2 e_3 + b e_3 e_1 + c e_1 e_2), \text{ and}$$

$$|h \circ h_1| = \omega.$$

Recall that $|h \circ h_1| = \omega \Leftrightarrow \text{Constant term of } h \circ h_1 = \mathbf{Cos}\left(\frac{\omega}{2}\right)$.

By performing the geometric product calculation we find that

$$\begin{aligned} h \circ h_1 &= \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \\ &\quad + \left(\mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}(\theta) + a \mathbf{Cos}(\theta) \mathbf{Sin}\left(\frac{\theta}{2}\right) \right) e_2 e_3 \\ &\quad + \left(b \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - c \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_3 e_1 \\ &\quad + \left(c \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) + b \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_1 e_2 \end{aligned}$$

The constant term of rotor $h \circ h_1 = \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right)$.

Setting $\mathbf{Cos}\left(\frac{\omega}{2}\right) = \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right)$ and solving for the coefficient a yields

$$a = \frac{\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\theta}{2}\right)}.$$

This is well defined except when $\mathbf{Sin}^2\left(\frac{\theta}{2}\right) = 0$, which occurs only when

$\theta = 0 \bmod 2\pi$. Since 0 is outside the domain $\left[\frac{\pi}{2}, \pi\right]$ of θ , the definition of a is well defined. “ b ” and “ c ” can be any numbers so long as $a^2 + b^2 + c^2 = 1$. In fact

it is convenient to set $b = 0$ and have $c = \sqrt{1-a^2}$.

Finally, we must show that $a, b, c \in \mathbb{R}$ which will conclude the proof of the theorem. To do this we must show that $a^2 \leq 1$.

Since $0 < \omega \leq \pi$,

$$0 < \frac{\omega}{2} \leq \frac{\pi}{2},$$

$$\mathbf{Cos}\frac{\pi}{2} \leq \mathbf{Cos}\frac{\omega}{2} < \mathbf{Cos}(0),$$

$$0 \leq \mathbf{Cos}\left(\frac{\omega}{2}\right) < 1,$$

$$-1 < -\mathbf{Cos}\left(\frac{\omega}{2}\right) \leq 0, \quad (\text{iv})$$

Since $\frac{\pi}{2} \leq \theta \leq \pi$,

$$\frac{\pi}{4} \leq \frac{\theta}{2} \leq \frac{\pi}{2},$$

$$0 \leq \mathbf{Cos}\left(\frac{\theta}{2}\right) \leq \frac{1}{\sqrt{2}},$$

$$0 \leq \mathbf{Cos}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2}, \quad (\text{v})$$

$$0 \leq 1 - \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2},$$

$$-\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) - 1 \leq 0,$$

$$\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq 1. \quad (\text{vi})$$

Thus,

$$-1 = \frac{0-1}{1} \stackrel{(\text{iv, v, vi})}{\leq} \frac{\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\theta}{2}\right)} = a \stackrel{(\text{iv, v, vi})}{\leq} \frac{\frac{1}{2} + 0}{\frac{1}{2}} = 1$$

That is, $|a| \leq 1$. ■

Corollary 7.1: Suppose $\frac{\pi}{2} \leq \theta \leq \pi$, $0 < \omega \leq \pi$, and $S_\omega \subseteq H$, where H is a group,

not necessarily normal. Then S_ω is generated by geometric products of pairs of elements from S_ω , and so $S_\omega \subseteq H$.

Proof: By Theorem 7, $\exists k_1, k_2 \in S_\omega \subseteq H$ such that $k_1 \circ k_2 \in S_\omega$. By Theorem 6, $S_\omega = \{h \circ k : h, k \in S_\theta\} \subseteq H$. ■

Theorem 8: $SO(3)$ contains no proper normal subgroup.

Proof: Let H be a non-trivial normal subgroup of $SO(3)$. H contains a non-identity element $1 \neq h_0 \in H$ where h_0 is a rotation by some angle $\phi \leq \pi$ about some axis.

a. Since $h_0 \neq 1$, $\phi > 0$.

b. Since $\phi > 0$, there is a positive integer n such that $\frac{\pi}{2} \leq n\phi \leq \pi$.

Define $h = h_0^n = h_0 \circ h_0 \circ \dots \circ h_0$ and $\theta = n\phi$.

c. h is a rotation of angle θ , and $\frac{\pi}{2} \leq \theta \leq \pi$. Thus $h \neq 1$.

d. Since $1 \neq h$, by Theorem 5, $S_\omega \subseteq H$.

We wish to show that $H = SO(3)$. It suffices to show that H contains every sphere S_ω in \mathcal{R} . For $\omega = 0$, $S_\omega \subseteq H$ since $1 \in H$. So let $0 < \omega \leq \pi$ be an arbitrary rotation angle. By Corollary 7.1, $S_\omega \subseteq H$. ■

Geometric Algebra Treatment of Reflections and Reflective Rotations:

Recall that $O(3)$ is the disjoint union $O(3) = SO(3) \cup T$ of the non-reflective and reflective rotations, respectively. In GA terminology, we can write this as

$O(3) = S \cup T$. We have already defined S . Its elements are rotors, multivectors having the form

$$f = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a_{23} e_2 e_3 + a_{31} e_3 e_1 + a_{12} e_1 e_2],$$

where $a_{23}^2 + a_{31}^2 + a_{12}^2 = 1$, $a_{23}, a_{31}, a_{12} \in \mathbb{R}$, and $0 \leq \theta \leq \pi$.

We wish to define T . That is, we wish to describe a reflective rotation, h , in terms of a multivector. We cannot represent h as a rotor because all rotors represent non-reflective rotations. Instead, we seek to represent h as a 3-dimensional multivector. Moreover, just as $f^{-1} \circ v \circ f$ (non-reflectively) rotates the vector v into some vector w , we require the multivector h to have the property that $h^{-1} \circ v \circ h$ reflectively rotates v into $-w$. That is, we require

$$-w = h^{-1} \circ v \circ h \quad (\text{vii})$$

Because $h = f \circ R$, a rotation followed by a reflection, we also wish to find a multivector representation for the reflection operator R .

We quickly discover that the expression for the multivector R to reflect a vector w into $-w$ must be $-w = R \circ w \circ R$. To see this, let

- $v \in \mathbb{R}^3$: a vector in \mathbb{R}^3
- $-v \in \mathbb{R}^3$: the vector obtained by reflecting v
- $f \in \mathcal{S}$: a rotation
- $w = f^{-1} \circ v \circ f \in \mathbb{R}^3$: the vector obtained by rotating v by f
- $-w \in \mathbb{R}^3$: the vector obtained by reflecting w
- $h = f \circ R \in \mathcal{T}$: the reflective rotation generated by f

Since a reflection followed by a reflection is the identity, R is its own inverse, $R = R^{-1}$. So,

$$\begin{aligned} -w &= h^{-1} \circ v \circ h = (f \circ R)^{-1} \circ v \circ (f \circ R) = (R^{-1} \circ f^{-1}) \circ v \circ (f \circ R) \\ &= R \circ (f^{-1} \circ v \circ f) \circ R \\ &= R \circ w \circ R \end{aligned} \quad (\text{viii})$$

Defining $R = i = \sqrt{-1}$ satisfies equation (viii). i is a scalar, and a scalar is a multivector, so this satisfies our goal of defining R as a multivector. In Appendix 1 it is shown that while there are other multivector expressions for R that satisfy equation (viii), all such expressions involve imaginary terms. In Definition (1), below, we define $R = i$, the simplest such expression.

Since i is a scalar, it commutes with the geometric product operator:

$$f \circ g \circ i = i(f \circ g) = (if) \circ g = f \circ (ig) \text{ for all multivectors } f \text{ and } g.$$

Definition: The **Reflection Operator** is actually 3 different operators, depending on its domain. The original operator R with domain \mathbb{R}^3 induces operators on \mathcal{S} and \mathcal{T} . Let $i = \sqrt{-1}$, $v \in \mathbb{R}^3$, $f \in \mathcal{S}$, and $h \in \mathcal{T}$. Define

- (1) $\mathbf{R} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : R(v) = R^{-1} \circ v \circ R = i \circ v \circ i = -v$; i.e., $\mathbf{R} = i = R^{-1}$.
- (2) $\mathbf{R}_S : \mathcal{S} \rightarrow \mathcal{T} : R_S(f) = f \circ R_S = f \circ i = if$; i.e., $\mathbf{R}_S = i = R_S^{-1}$.
- (3) $\mathbf{R}_T : \mathcal{T} \rightarrow \mathcal{S} : R_T(h) = h \circ R_T = h \circ (-i) = -ih$; i.e., $\mathbf{R}_T = -i = R_T$.

These definitions require clarification. We begin by observing we are now able to identify the multivector representation of a reflective rotation:

$$(4) \quad \mathbf{h} = f \circ R_S = if = i \left\{ \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} [b_{23} e_2 e_3 + b_{31} e_3 e_1 + b_{12} e_1 e_2] \right\},$$

where $f \in \mathcal{S}$, $b_{23}^2 + b_{31}^2 + b_{12}^2 = 1$, $b_{23}, b_{31}, b_{12} \in \mathbb{R}$, and $0 \leq \theta \leq \pi$.

Lemma 9.1 provides the equation for the inverse of a reflective rotation.

Lemma 9.1: Let $h = f \circ i \in \mathcal{T}$, where $f \in \mathcal{S}$. Then $h^{-1} = f^{-1} \circ i$.

Proof: Let $g = f^{-1} \circ i$. By definition (2), $g \in \mathcal{T}$ since $f^{-1} \in \mathcal{S}$. Because the expression for the geometric product of 2 members of \mathcal{T} involves 2 reflections, $g \circ h \in \mathcal{S}$. Thus,

$$g \circ h = (f^{-1} \circ i) \circ (f \circ i) = i^2 (f^{-1} \circ f) = -1.$$

Recall from Theorem 3 that $-1 \notin \mathcal{S}$ but $-1 \approx 1 \in \mathcal{S}$. Therefore $g \circ h = 1 \text{ mod } \mathcal{S}$. Similarly, $h \circ g = 1$, and we have that $h^{-1} = g = f^{-1} \circ i$. ■

Lemma 9.1 is somewhat surprising because one might expect that $h^{-1} = i^{-1} \circ f^{-1} = -f^{-1} \circ i$. The subtlety here is that when $i \in \mathbb{C}$ then $i^{-1} = -i$, but $i^{-1} = i \text{ mod } \mathcal{T}$ when i is a reflective rotation¹. This is similar to (and follows from) $1 \neq -1$ when $1 \in \mathbb{R}$ but $1 = -1 \text{ mod } \mathcal{S}$ (according to Theorem 3) when 1 is a rotation.

In the reflection operator definition, what does it mean when we say that R induces R_s ? Observe that both $w \circ R$ and $f \circ R$ appear in equation (viii). In $w \circ R$, R is acting on a vector, but in $f \circ R$, R is acting on a rotation. So the R in $f \circ R$ becomes what I am calling the induced reflection R_s . The domain of the reflection operator has changed but the multivector expression hasn't changed, so $R_s = R = i$, which is Definition (2).

Why is $R_T = -i$ (rather than just i) in Definition (3)? Because $1 = R_s \circ R_T = i R_T$.

Outline of Proof of Theorem 9:

Observe that what was called the reflection operation “ R ” in the 2nd paragraph of page 1 is now defined to simply be “ i ”. That is, $i = R_s \in \mathcal{T}$. Using this notation, we have that the subgroup $\mathcal{I} = \{1, i\}$.

Let H be a proper normal subgroup of $O(3)$ such that $H \neq SO(3)$ and $H \neq \mathcal{I}$. From Theorem 8, $\exists i \neq h \in H \cap T$. Then $h^2 \in SO(3) \cap H$. If $1 \neq h^2$, by Theorem 8, $SO(3) \subseteq H$. Also, $\exists f \in SO(3)$ such that $h = f \circ R_s = i f$. To see that $T \subseteq H$, let $t \in T$. $\exists f_1 \in SO(3)$ such that $t = f_1 \circ R_s = i f_1$. Let $f_2 = f_1 \circ f^{-1} \in SO(3)$. Then $f_1 = f_2 \circ f$. So,

$$t = i f_1 = i (f_2 \circ f) = f_2 \circ (i f) = f_2 \circ h.$$

$t \in H$ since $f_2 \in SO(3) \subset H$. Thus $T \subseteq H$. Since, also, $SO(3) \subseteq H$, then $H = O(3)$.

¹ $i^{-1} = R^{-1} = i$.

Unfortunately, h^2 does equal 1 when either $h = i$ (ruled out) or h is the reflection of any π rotation (allowed). Even so, everything in the prior paragraph still holds except we will not have proved that $\text{SO}(3) \subset H$. To prove this, we will generate $h_1 \in H \cap T$ such that $1 \neq h \circ h_1 \in \text{SO}(3)$. Then we will again have, by Theorem 8, that $\text{SO}(3) \subseteq H$.

Notice that $h = f \circ R_S \Leftrightarrow h = i f \Leftrightarrow f = -i h \Leftrightarrow f = h \circ R_T$. That is,

$$h = f \circ R_S \Leftrightarrow f = h \circ R_T \quad (\text{ix})$$

Lemma 9.2: $\mathcal{I} = \{1, i\}$ is a normal subgroup of $O(3)$.

Proof: $g^{-1} \circ 1 \circ g = 1$ and $g^{-1} \circ i \circ g = i$ for every $g \in O(3)$. ■

Theorem 9: $\text{SO}(3)$ and \mathcal{I} are the only proper normal subgroups of $O(3)$.

Proof: To help keep track of variables, we will use s, s_0, s_1, s_{21}, s_3 and s_4 for elements of $\text{SO}(3)$ and t, t_0, t_1 , and t_2 for elements of T .

Suppose H is a non-trivial normal subgroup of $O(3)$ and $H \neq \text{SO}(3)$ and $H \neq \mathcal{I}$.

Claim: There is an element $t_1 \in H \cap T$ such that $t_1 \neq i$:

By Theorem 8, $\exists t_0 \in H \cap T$. If $t_0 \neq i$, the claim is true. If $t_0 = i$, since $H \neq \mathcal{I}$, H contains another element. If that element is in T , the claim is true. If the other element is $s_0 \in \text{SO}(3)$, we can assume $s_0 \neq 1$ since $H \neq \mathcal{I}$. Set $t_1 = s_0 i$. Then $t_1 \in H \cap T$ and $t_1 \neq i$.

As an intermediate step we wish to find an element $t_2 \in H \cap T$ such that $t_1 \circ t_2 \neq 1$. Set

$$s_1 \equiv t_1 \circ R_T = -i t_1 \in \text{SO}(3).$$

WLOG, we can assume s_1 is a rotation about the **x-axis**. Thus,

$$s_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3.$$

By (ix),

$$t_1 = s_1 \circ R_S = i s_1 = i \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3 \right].$$

and

$$it_1 = - \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3 \right].$$

Let $\theta = |s_1|$. $\theta \neq 0$ because $t_1 \neq i$ and, consequently, $s_1 \neq 1$. Define

$$g_2 = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)e_1e_2 = \frac{1+e_3e_1}{\sqrt{2}}$$

a 90° rotation about the **y-axis**. $g_2 \in SO(3)$ since it is a rotation. Hence,

$$s_2 \equiv g_2^{-1} \circ s_1 \circ g_2 = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)e_1e_2.$$

$s_2 \in SO(3)$ is the rotation of angle θ about the negative **z-axis**. Define

$$t_2 \equiv g_2^{-1} \circ t_1 \circ g_2 = g_2^{-1} \circ i s_1 \circ g_2 = i s_2 = i \left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)e_1e_2 \right].$$

So $t_2 \in T \cap H$ and

$$t_1 \circ t_2 = -\cos^2\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1e_2 + \sin^2\left(\frac{\theta}{2}\right)e_3e_1 - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2e_3.$$

Notice that the constant term of $t_1 \circ t_2 < 0$. So, by Theorem 3, we replace the expression for $t_1 \circ t_2$ by its negative:

$$t_1 \circ t_2 = \cos^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1e_2 - \sin^2\left(\frac{\theta}{2}\right)e_3e_1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2e_3.$$

Since $\theta \neq 0$, $t_1 \circ t_2 \neq 1$. This concludes the intermediate step.

We next wish to show that $H \supset SO(3)$. First, we observe that

$$\begin{aligned} s_1 \circ s_2 &= \cos^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1e_2 - \sin^2\left(\frac{\theta}{2}\right)e_3e_1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2e_3 \\ &= t_1 \circ t_2. \end{aligned}$$

We define

$$s_3 \equiv t_1 \circ t_2 = s_1 \circ s_2.$$

$1 \neq s_3 \in H$ because $s_3 = t_1 \circ t_2$. $s_3 \in SO(3)$ because $s_3 = s_1 \circ s_2$. So, $1 \neq s_3 \in H \cap SO(3)$.

$H \cap SO(3)$ is thus a non-trivial normal subgroup of $SO(3)$, so by Theorem 8 $H \cap SO(3) = SO(3)$.

Thus,

$$H \supset SO(3).$$

Finally, we are in a position to show as well that $H \supset T$, implying that $H = O(3)$ since $H \supset SO(3) \cup T = O(3)$. Let $t \in T$ be any element of T . This proof is finished if we can show $t \in H$. Let

$$\mathbf{s} = -it = t \circ R_T \in SO(3).$$

$\exists x, y, \text{ and } z$ where $x^2 + y^2 + z^2 = 1$ and ω satisfying $0 \leq \omega \leq \pi$ such that

$$s = \cos \frac{\omega}{2} + \sin \frac{\omega}{2} [x e_2 e_3 + y e_3 e_1 + z e_1 e_2].$$

By (ix),

$$t = s \circ R_s = is = i \left\{ \cos \frac{\omega}{2} + \sin \frac{\omega}{2} [x e_2 e_3 + y e_3 e_1 + z e_1 e_2] \right\}.$$

Define

$$\mathbf{s}_4 = s_1^{-1} \circ s \in SO(3) \subset H.$$

s_4 is the rotation whose geometric product with s_1 equals s :

$$s = (s_1 \circ s_1^{-1}) \circ s = s_1 \circ (s_1^{-1} \circ s) = s_1 \circ s_4.$$

Therefore s_4 is also the rotation whose geometric product with t_1 equals t :

$$t = s \circ i = (s_1 \circ s_4) \circ i = (s_1 \circ i) \circ s_4 = t_1 \circ s_4$$

So, $t \in H$ since $t_1, s_4 \in H$, and that completes the proof the theorem and part (B). ■

APPENDIX 1: All possible ways to define the operator $\mathbf{R}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Begin with:

- $w = e_1$: a vector in \mathbb{R}^3 ,
- $\mathbf{R} = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 + c_{12} e_1 e_2 + c_{23} e_2 e_3 + c_{13} e_1 e_3 + c_{123} e_1 e_2 e_3$,
the reflection operator, represented as a multivector

\mathbf{R} must satisfy

$$\mathbf{R}^{-1} \circ w \circ \mathbf{R} = -w = -e_1$$

\mathbf{R} is symmetrical with respect to x, y , and z [i.e., $\mathbf{R}(x, y, z) = (-x, -y, -z)$]. So, $c_1 = c_2 = c_3$ and $c_{12} = c_{23} = c_{13}$. Thus,

$$\begin{aligned} \mathbf{R} &= c_0 + c_1 e_1 + c_1 e_2 + c_1 e_3 + c_{12} e_1 e_2 + c_{12} e_2 e_3 + c_{12} e_1 e_3 + c_{123} e_1 e_2 e_3 \\ \mathbf{R}^{-1} &= c_0 + c_1 e_1 + c_1 e_2 + c_1 e_3 - c_{12} e_1 e_2 - c_{12} e_2 e_3 - c_{12} e_1 e_3 - c_{123} e_1 e_2 e_3, \end{aligned}$$

The inverse \mathbf{R}^{-1} doesn't always exist so that is part of what we must look for.

$$\mathbf{R}^{-1} \circ w \circ \mathbf{R} = A + B e_1 + C e_2 + D e_3 = 0 - e_1 + 0 e_2 + 0 e_3, \text{ where}$$

$$\begin{aligned} A &= 2(c_0 c_1 + c_{12} c_{123}), \quad B = c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2, \quad C = 2(c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123}), \text{ and} \\ D &= 2(c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123}). \end{aligned}$$

This leads to 4 equations in the 4 unknowns $c_0, c_1, c_{12}, c_{123}$:

$$\left\{ \begin{array}{lcl} c_0 c_1 + c_{12} c_{123} & = & 0 \\ c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2 & = & -1 \\ c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123} & = & 0 \\ c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123} & = & 0 \end{array} \right\}$$

Adding equations (3) and (4) yields

$$c_1^2 + c_{12}^2 = 0 \Rightarrow c_1^2 = -c_{12}^2 \leq 0 \Rightarrow c_1 = c_{12} = 0.$$

Equation (2) then becomes

$$c_0^2 + c_{123}^2 = -1 \Rightarrow c_{123} = \pm \sqrt{-1 - c_0^2}.$$

(Equations 1, 3, and 4 all become $0 = 0$, so the solution is consistent.)

From this it is clear that if c_0 is real, then c_{123} is imaginary. This proves the claim that every multivector representation for \mathbf{R} involves an imaginary term.

I list below the four simplest expressions for R , and confirm that R^{-1} exists in each instance.

$$R = i \text{ and } R^{-1} = -i$$

$$R = i e_1 e_2 e_3 \text{ and } R^{-1} = R = i e_1 e_2 e_3$$

$$R = 1 + i\sqrt{2} e_1 e_2 e_3 \text{ and } R^{-1} = -1 + i\sqrt{2} e_1 e_2 e_3$$

$$R = i\sqrt{2} + e_1 e_2 e_3 \text{ and } R^{-1} = -i\sqrt{2} + e_1 e_2 e_3$$

We also need to know whether solutions for the special case of $w = e_1$ are general solutions. It turns out that they are. That is, if $w = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a vector in \mathbb{R}^3 , then the equation $R^{-1} \circ w \circ R = -w$ has the same solution set:

$$c_1 = c_{12} = 0 \text{ and } c_{123} = \pm \sqrt{-1 - c_0^2}.$$

The solutions for R given above are robust.

APPENDIX 2: The 2 insights that suggested how to define the rotor g_{23} in Lemma 5.2.

The hardest part of this proof was to invent the rotor g_{23} . I couldn't solve a certain system of 4 non-linear simultaneous equations that would have provided it, but I was able to guess a solution after examining lots of rotations.

The **1st insight**, gained during Lemma 5.1, is that a rotor, say h_3 , about the z-axis, is obtained by $g_2^{-1} \circ h_1 \circ g_2$ where g_2 is a rotor about the y-axis with a 90° rotation angle and h_1 is a rotor about the x-axis having the same angle of rotation, θ , as h_2 . Loosely speaking, I learned that a 90° rotation g_2 yields a rotation about an axis 90° away from h_1 .

2nd Insight comes from discovering that reducing the g_2 angle of rotation to 45° generates a rotation about the 45° diagonal between the x-axis and z-axis (i.e., about an axis 45° away from h_1), seen below:

Let

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{8}\right) - \mathbf{Sin}\left(\frac{\pi}{8}\right) \mathbf{e}_1 \mathbf{e}_3.$$

Then

$$\begin{aligned} g_2^{-1} \circ h_1 \circ g_2 &= \mathbf{Cos}\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}} \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 + \frac{1}{\sqrt{2}} \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{e}_2 \mathbf{e}_3 \\ &= 45^\circ \text{ diagonal axis in the } zx\text{-plane.} \end{aligned}$$

To generate a 45° diagonal axis not in the xy -, yz -, or zx -planes, I made a guess that it requires products of rotors like g_2 and g_3 where g_2 generates a skew axis between the z and x-axes, and then g_3 generates an out-of-zx-plane skew axis towards the y-axis.

Armed with the guess $g_{23} = g_2 \circ g_3$, for a given angle ψ :

- I assigned g_2 and g_3 rotor angles β and γ , respectively,
- Performed the $g_{23}^{-1} \circ h \circ g_{23}$ GA multiplication with Mathematica, and
- Solved for a constant term equal to $\mathbf{Cos}\left(\frac{\psi}{2}\right)$.

This yielded the **ArcCos** and **ArcTan** values used in Lemma 5.2. The proof in Lemma 5.2 is just the confirmation multiplication that the solution found is correct.