Chapter 13. Symmetry Groups

Groups

Definitions:
A **group** is a set G with an operation $\circ$ that is closed and associative, has an identity e, and every element g has an inverse g^{-1} such that $g \circ g^{-1} = e = g^{-1} \circ g$.

A group G is **Abelian** if it is commutative: $g \circ h = h \circ g$ for all g, h in G.

A **subgroup** is a subset of G that is a group under $\circ$.

Let H be a subgroup of G. A **coset of H** is a set $H \circ g = \{h \circ g : h \in H\}$, where $g \in G$. The only coset of H that is a group is the set H itself: $H = H \circ e$ where e is the identity element. The cosets of H form a partition of G.

A **normal subgroup** is a subgroup H that satisfies $g \circ H = H \circ g$ for all g in G, or equivalently $H = g^{-1} \circ H \circ g$.

A group is **simple** if it contains no non-trivial normal subgroup. The simple groups are the fundamental "building blocks" of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

- Classical Families: A_m, B_m, C_m, D_m having dimensions $m(m+2)$, $m(2m+1)$, $m(2m+1)$, and $m(2m-1)$, respectively where $m \in \mathbb{Z}^+$.
- Exceptional Groups: E_6, E_7, E_8, F_4, G_2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has $\approx 10^{60}$ elements and is known as **the monster.**

Definition. The **Product Group** of groups G and H is **GxH** = $\{(g,h) : g \in G, h \in H\}$ with group operation $(g_1,h_1) \circ (g_2,h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.

Definition. Let N be a subgroup of G. The **Factor Space G/N** is the collection of cosets $N \circ g$ along with the operation $(N \circ g_1) \circ (N \circ g_2) = N \circ (g_1 \circ g_2)$.

Theorem. If N is normal then G/N is a group, called the **Factor Group.**

Theorem. [13.10] $H \cong (G \times H) / G$.

The group operation is a function, $\circ : G \times G \to G$. If G is also a topological space, then $\circ$ can either be continuous or not.

Definition. A **group (G,$\circ$) is continuous** if $\circ : G \times G \to G$ is a continuous function when G is considered as a topological space. A **Lie group** is a continuous group where the inversion operation is also continuous.

The **dimension of a group** is its dimension as a topological space, which we now define. Intuitively the dimension of a space is 1 + dim (boundary of space). For example, a disk has dimension 2 and its boundary, a circle, has dimension 1. A line segment has dimension 1 and its boundary, 2 points, has dimension zero. If a point has dimension 0, its boundary, the empty set $\varnothing$, is defined to have dimension -1. It is standard to start with $\varnothing$ and define topological dimension inductively.

There are several definitions of dimension. We give the **Small Inductive Dimension** and briefly two others.

Definition. A collection $\mathcal{U} = \{U_\alpha\}$ is an **open basis for a topological space** X if every open set is a union of sets from $\mathcal{U}$.

Examples. In $\mathbb{R}$, the collection of open intervals forms a basis. In $\mathbb{R}^3$, the collection of open balls forms a basis.

Definition. The **closure** $\overline{A}$ of a set A is the smallest closed set that contains A. The **boundary** of A is $\partial A = \overline{A} - A$.

Definition. Let X be a topological space and $\mathcal{U}$ an open basis for X.

(7) **dim X = -1** if X = $\varnothing$

(8) dim X ≤ n if for all points x and open sets W such that $x \in W$ there exists $U \in \mathcal{U}$ such that $x \in U \subseteq \overline{U} \subseteq W$ and dim $\partial U \leq n-1$

(9) **dim X = *n*** if (2) is true for n but false for $n-1$

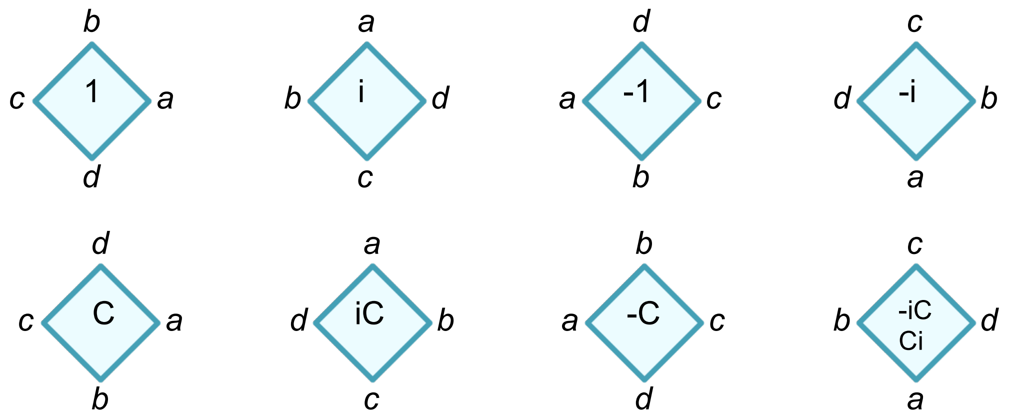
(10) **dim X = ∞** if for every n, dim X ≤ n is false

The **Large Inductive Dimension** is similar but replaces points with closed sets.

The **Lebesgue Covering Dimension** defines dim X = n if every open cover of X has an open refinement in which no point belongs to more that n + 1 sets in the refinement.

All of these definitions agree on spaces like $\mathbb{R}^n$ that are separable and metrizable.

Symmetries of a Square



Definitions:
Non-reflecting Group: < i > = {1, i, -1, -i}
Reflecting Group: < i, C > = {1, i, -1, -i, C, iC, -C, -iC = Ci}
C is complex conjugation: $a+bi \mapsto a-bi$. **1** is the null rotation, which is the group identity element. ***i*** is the 90° counter-clockwise rotation of the square

Convention: ab means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

Examples:

Normal subgroups of < i, C > :

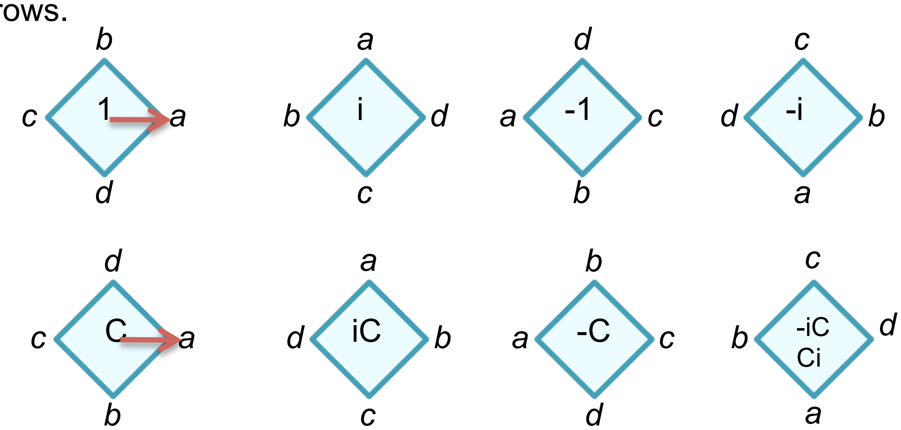
{1, -1, C, -C} , {1, -1}, {1, -1}

Non-normal subgroups of < i, C > :

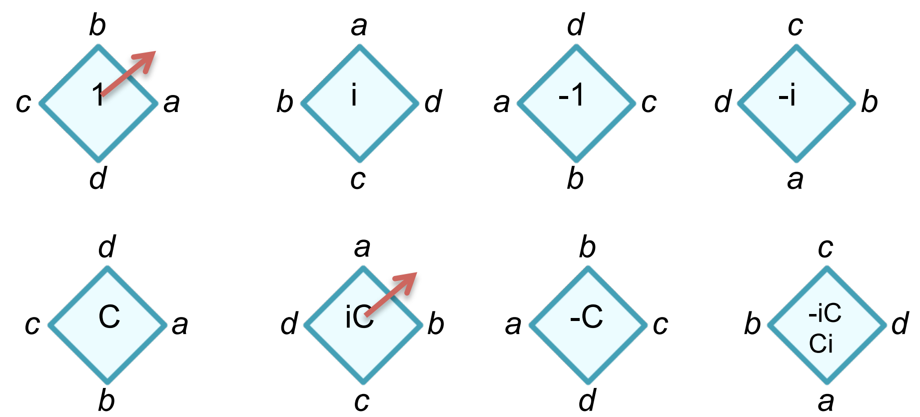
{1, -C}, {1, iC}, {1, C}

For example, {1, C} i = {i, Ci} ≠ {i, -Ci} = i {1, C}

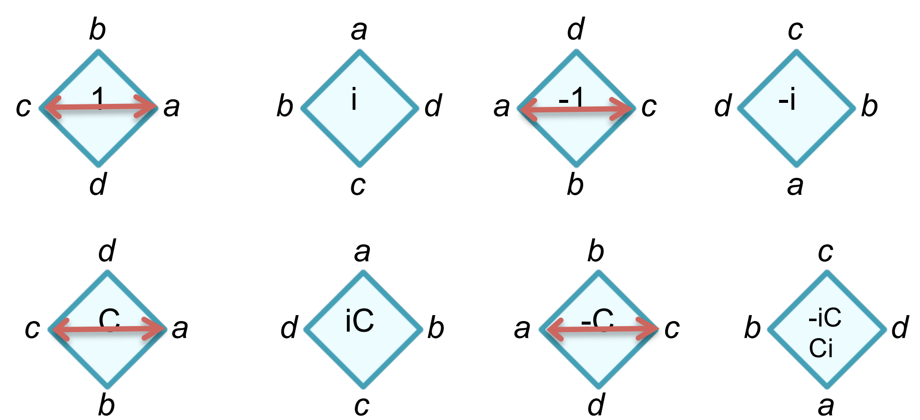
Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.



{1, C} is a reduced symmetry group



{1, iC} is a reduced symmetry group



{1, -1, C, -C} is a reduced symmetry group

Symmetries of a Sphere

Definitions:
A group G whose underlying set is continuous is called a **Lie Group.**
SO(3) is the group of non-reflective symmetries of a 3-sphere
O(3) is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

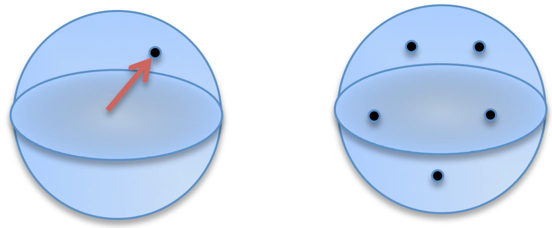
O(3) = SO(3) $\cup$ T, the disjoint union of O(3) with the coset of reflective symmetries

T $= \mathrm{R\,SO}(3) = \{\mathrm{R}g : g \in \mathrm{SO}(3)\}$ where **R** is the reflection operator on the sphere.

Recall problem [12.7]: SO(3) is group isomorphic to the solid sphere $\mathcal{R}$ of radius π with antipodal points identified.

Theorem. (Problem [13.7]) SO(3) and {1, R} are the only normal subgroups of O(3), where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups



The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.

Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

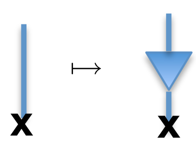
- $f : V \to W$ is a **homomorphism** if *it* preserves the vector space structure:
  - $f(au + bv) = af(u) + bf(v)$ for all vectors u and v and scalars a and b.
- **Hom(V,W)** is the set of homomorphisms from V to W.
- $\mathcal{A}$**(V)** = Hom(V,V).
- A **linear transformation** is a member $T \in \mathcal{A}(V)$.
  - That is, a linear transformation is a function $T : V \to V$ such that $T(au + bv) = aTu + bTv$.

Theorem. [13.12, 13.13] Let V = $\mathbb{R}^3$, using (x^1, x^2, x^2) instead of (x, y, z). Then a linear transformation T takes the form $T : x^r \mapsto T^r{}_s x^s = ax^1 + bx^2 + cx^3$.

Note. Linear transformations are represented by matrices:

$$T : \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{or} \quad x \mapsto Tx$$

In diagrammatic form this is



Theorem. If $R = ST$ then $R^a{}_c = S^a{}_b T^b{}_c$. That is, the composition, R, of 2 linear transformations is the result of matrix multiplication of S and T. In diagrammatic notation:

$R =$  =  = ST

Example. $TI = T = IT$ is written in diagrammatic form as  =  =  .

and, in $\mathbb{R}^3$, $I = \delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b range over {1, 2, 3}.

Definitions. A linear transformation T is **singular** if Dim(TV) < Dim W; that is, T is not *onto*.

Theorem. [13.17] T is singular iff $\exists\ v \neq 0$ such that $Tv = 0$.

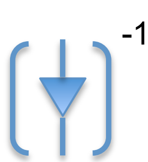
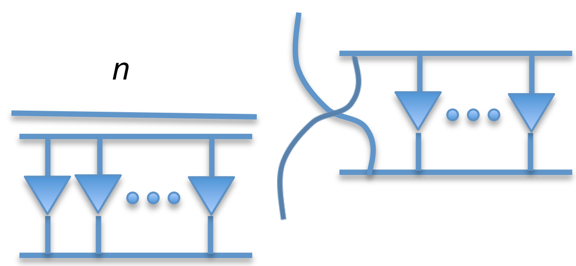
Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1 $\Leftrightarrow \forall v \neq w\ \ T(v - w) = T(v) - T(w) \neq 0 \overset{(*)}{\Leftrightarrow} \forall u \neq 0\ T(u) \neq 0$

$\overset{[13.17]}{\Leftrightarrow}$ T is non-singular $\Leftrightarrow$ T is onto.

(*) Set $v = 3u$ and $w = 2u$. ■

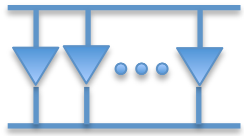
Theorem. [13.18] If T is nonsingular, then it has an inverse T^{-1}.

Theorem. [13.19] T^{-1} =  = 

Definition. The **transpose** of the matrix $T = (T^i{}_j)$ is the matrix $\boldsymbol{T}^{\mathbf{T}} = (T^j{}_i)$.

Definition. A matrix T is **orthogonal** if $T^{-1} = T^{\mathrm{T}}$.

Determinants and Traces

Definition. $\text{Det}\, T = \frac{1}{n!}$  $= \frac{1}{n!} \in^{ab\cdots d} T^e{}_a T^f{}_b \cdots T^h{}_d \,\varepsilon_{ef\cdots h}$.

Theorem. [Bud] $\text{Det}\, T = \sum_{\pi \in \mathcal{P}_{12\cdots n}} \text{Sign}(\pi)\, T^1{}_{\pi(1)} \cdots T^n{}_{\pi(n)}$ (the normal definition of Det)

Proof. Let $\mathcal{P}_{1\ldots n}$ be the set of permutations of (1, …, n).

$$\text{Det}\, T = \frac{1}{n!} \varepsilon_{r\cdots s} \in^{t\cdots u} T^r{}_t \cdots T^s{}_u$$

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(1)\cdots\pi(n)} T^{\pi^*(1)}{}_{\pi(1)} \cdots T^{\pi^*(n)}{}_{\pi(n)}$$

(Replace Einstein notation.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(\pi^*(1))\cdots\pi(\pi^*(n))} T^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots T^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Replace π by $\pi \circ \pi^*$ in $\in$ and T. The double sum over π and π^* is unchanged, in both expressions stepping over all permutations of (1, …, n), and the exponents of $\in$ continue to match the subscripts of T.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \mathbf{Sign}(\pi)\, \cancel{\varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi^*(1)\cdots\pi^*(n)}}\, T^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots T^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Re-order superscripts of $\in$ by applying an inverse π permutation.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \text{Sign}(\pi) \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} T^{\mathbf{1}}{}_{\pi(\mathbf{1})} \cdots T^{\mathbf{n}}{}_{\pi(\mathbf{n})}$$

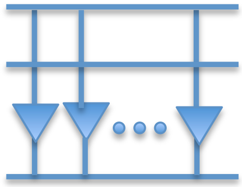
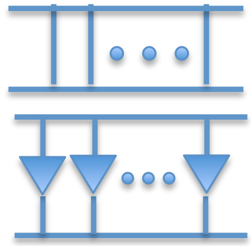
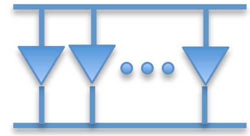
(This is just a simpler way to label the subscripts and superscripts of T. For example, if $\pi^*(3) = 1$ then $T^{\pi^*(3))}{}_{\pi(\pi^*(3))} = T^1{}_{\pi(1)}$.)

$$= \frac{n!}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \text{Sign}(\pi)\, T^1{}_{\pi(1)} \cdots T^n{}_{\pi(n)}$$

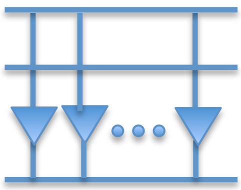
$$= \sum_{\pi \in \mathcal{P}_{1\cdots n}} \text{Sign}(\pi)\, T^1{}_{\pi(1)} \cdots T^n{}_{\pi(n)}$$

■

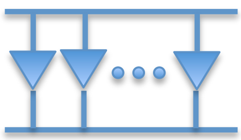
(See my solution to [13.21] for examples of this for n = 2 and 3.)

Theorem. [Bud]  =  = $n!$ 

Proof: Let $P_{a\ldots g}$ be the set of permutations of (a, …, g). Then

 = $n!\,\varepsilon_{a\cdots g} \in^{r\cdots x} T^{[a}{}_r \cdots T^{g]}{}_x$

$$= \frac{n!}{n!}\varepsilon_{a\cdots g} \in^{r\cdots x} \sum_{\pi\in\mathcal{P}_{ab\cdots g}} \mathrm{Sign}(\pi) T^{\pi(a)}{}_r \cdots T^{\pi(g)}{}_x \overset{(*)}{=} n!\,\varepsilon_{a\cdots g} T^a{}_r \cdots T^g{}_x \in^{r\cdots x}$$

= $n!$ 

(*) π is the composition of transmutations (i.e., of pairwise permutations).

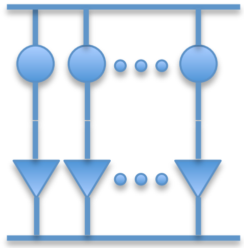
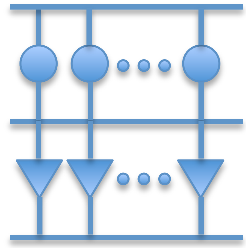
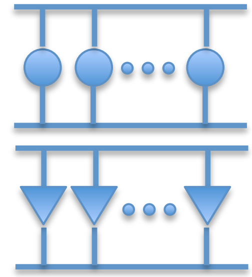
Let π^*: $\begin{matrix} c \mapsto e \\ e \mapsto c \end{matrix}$ be a transmutation. Then

$$\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*) T^{\pi(a)}{}_r \cdots T^{\pi(c)}{}_t \cdots T^{\pi(e)}{}_v \cdots T^{\pi(g)}{}_x$$
$$= \varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*) T^a{}_r \cdots T^e{}_t \cdots T^c{}_v \cdots T^g{}_x$$
$$= \varepsilon_{a\cdots e\cdots c\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*) T^a{}_r \cdots T^c{}_t \cdots T^e{}_v \cdots T^g{}_x \text{ (Rename } c \mapsto e \text{ \& } e \mapsto c\text{)}$$
$$= \mathrm{Sign}(\pi^*)\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*) T^a{}_r \cdots T^c{}_t \cdots T^e{}_v \cdots T^g{}_x$$
$$= \varepsilon_{a\cdots g} T^a{}_r \cdots T^g{}_x \in^{r\cdots x}.$$

So, for any permutation π, we have

$$\varepsilon_{a\cdots g} \in^{r\cdots x} \mathrm{Sign}(\pi) T^{\pi(a)}{}_r \cdots T^{\pi(g)}{}_x = \varepsilon_{a\cdots g} T^a{}_r \cdots T^g{}_x \in^{r\cdots x} \quad \blacksquare$$

Theorem. [13.22]

Det AB = $\frac{1}{n!}$  $= \left(\frac{1}{n!}\right)^2$  $= \left(\frac{1}{n!}\right)^2$ 

= DetA DetB

Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A ≠ 0. ■

Definition. Vectors v and w are **orthogonal** if $v \bullet w = 0$. That is, the angle between them is 90°.

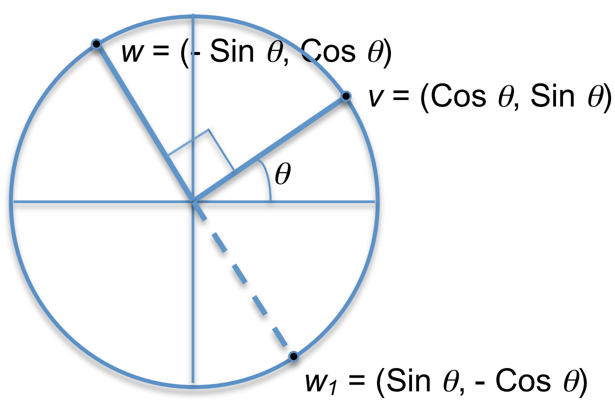
Theorem. A matrix is orthogonal (i.e., $T^{\mathrm{T}} = T^{-1}$) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B

Let $A = \begin{pmatrix} v & w \end{pmatrix}$

$= \begin{pmatrix} \text{Cos } \theta & \text{- Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix}.$

$A^{\mathrm{T}} = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta \\ \text{- Sin } \theta & \text{Cos } \theta \end{pmatrix}.$



$A^{\mathrm{T}} = A^{-1}$:

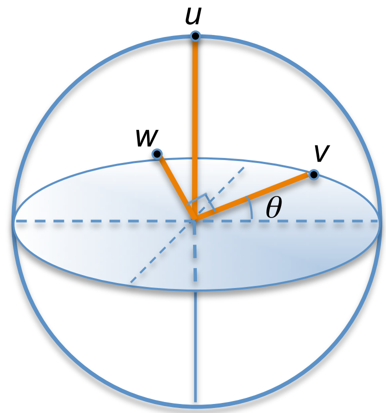
$$A\,A^{\mathrm{T}} = \begin{pmatrix} \text{Sin}^2\theta + \text{Cos}^2\theta & 0 \\ 0 & \text{Sin}^2\theta + \text{Cos}^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I} \quad ✔$$

Similarly $A^{\mathrm{T}} A = \mathrm{I}$ ✔

So A is an orthogonal matrix ✔
Det A = Det A^{T} = $\text{Cos}^2\theta + \text{Sin}^2\theta = 1$ ✔
The column vectors of A are orthogonal: $v \perp w$ ✔

Let $B = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta \\ \text{Sin } \theta & -\text{Cos } \theta \end{pmatrix} = B^T$. Then $B\,B^T = \mathrm{I}$, Det B = Det B^T = -1, and its column vectors v and w_1 are orthogonal.

Examples. Orthogonal 3 x 3 Matrices: A, B, and C



Let $v = \begin{pmatrix} \text{Cos } \theta \\ \text{Sin } \theta \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} -\text{Sin } \theta \\ \text{Cos } \theta \\ 0 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let $A = \begin{pmatrix} v & w & u \end{pmatrix} = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta & 0 \\ \text{Sin } \theta & \text{Cos } \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$A^T = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta & 0 \\ -\text{Sin } \theta & \text{Cos } \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A is orthogonal, its columns are orthogonal vectors, and its determinant is +1. ✔

Let $B = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta & 0 \\ \text{Sin } \theta & \text{Cos } \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$. B is orthogonal and its determinant is -1. ✔

Let C be a θ-rotation of A about an axis $\left\{t(a,b,c)\mathbf{:}\ 0 < t < \infty,\ a^2+b^2+c^2=1\right\}$:

$$C = \begin{pmatrix} \frac{1}{2}\left[1+a^2-b^2-c^2+\left(1-a^2+b^2+c^2\right)\text{Cos } \theta\right] & 2\text{Sin } \frac{\theta}{2}\left(-c\,\text{Cos } \frac{\theta}{2}+ab\,\text{Sin } \frac{\theta}{2}\right) & 2\text{Sin } \frac{\theta}{2}\left(b\,\text{Cos } \frac{\theta}{2}+ac\,\text{Sin } \frac{\theta}{2}\right) \\ 2\text{Sin } \frac{\theta}{2}\left(c\,\text{Cos } \frac{\theta}{2}+ab\,\text{Sin } \frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2+b^2-c^2+\left(1+a^2-b^2+c^2\right)\text{Cos } \theta\right] & 2\text{Sin } \frac{\theta}{2}\left(-a\,\text{Cos } \frac{\theta}{2}+bc\,\text{Sin } \frac{\theta}{2}\right) \\ 2\text{Sin } \frac{\theta}{2}\left(-b\,\text{Cos } \frac{\theta}{2}+ac\,\text{Sin } \frac{\theta}{2}\right) & 2\text{Sin } \frac{\theta}{2}\left(a\,\text{Cos } \frac{\theta}{2}+bc\,\text{Sin } \frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2-b^2+c^2+\left(1+a^2+b^2-c^2\right)\text{Cos } \theta\right] \end{pmatrix}$$

It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✔

Definition. A **symmetry** of a vector space (V,+) is a transformation $T: \mathrm{V} \mapsto \mathrm{V}$ that is 1-1 and onto that preserves the vector space structure:

T(*a v* + *b w*) = *a Tv* + *b Tw*

Definition. The **General Linear Group GL(*n*)** is the group of symmetries of an *n*-dimensional vector space.

Theorem. GL(n) is the group of non-singular (n x n) matrices.

Proof. Let $T \in$ GL(n). Since $T(a\, v + b\, w) = a\, Tv + b\, Tw$, T is a linear transformation. Were T singular, then by [13.17] Dim $TV < n \Rightarrow$ T is not onto, a contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. ■

Definition. The **Special Linear Group SL(*n*)** is the subset of GL(n) having determinant = 1.

Theorem. SL(n) is a normal subgroup of GL(n).

Proof. First, SL(n) is a group:

Closed: If $S_1, S_2 \in SL(n)$, then $\mathrm{Det}(S_1 S_2) = \mathrm{Det}(S_1)\,\mathrm{Det}(S_2) = 1$
$\Rightarrow S_1 S_2 \in SL(n)$.

Identity: $\mathrm{Det}(I) = 1 \Rightarrow I \in SL(n)$

Inverse: $1 = \mathrm{Det}(I) = \mathrm{Det}(S_1 S_1^{-1}) = \mathrm{Det}(S_1)\,\mathrm{Det}(S_1^{-1}) = \mathrm{Det}(S_1^{-1})$
$\Rightarrow S_1^{-1} \in SL(n)$

Also, SL(n) is normal:

Let $S \in SL(n)$ and $G \in GL(n)$. Then

$$\mathrm{Det}(G^{-1} S\, G) = \mathrm{Det}(G^{-1})\,\mathrm{Det}(S)\,\mathrm{Det}(G) = \mathrm{Det}(G^{-1})\,\mathrm{Det}(G)$$
$$= \mathrm{Det}(G\, G^{-1}) = \mathrm{Det}(I) = 1$$
$$\Rightarrow G^{-1} S\, G \in SL(n) \Rightarrow G^{-1}\, SL(n)\, G = SL(n)$$ ■

The groundwork has now been laid to introduce the table, below, that shows the relationships between SO(3), O(3), SL(3), GL(3), general linear transformations, orthogonality, determinants, and symmetries. The table shows that $SO(3) \subset O(3) \subset GL(3) \subset \mathcal{A}(\mathbb{R}^3)$ and $SO(3) \subset SL(3) \subset GL(3)$. It show that GL(3) is both the set of symmetries of $\mathbb{R}^3$ and the set of non-singular matrices. It also shows that the orthogonal group O(3) is a proper subgroup of the group of orthogonal matrices (union of the shaded-blue entries).

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe. They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant ≠ ±1 then orthogonal matrices also uniformly expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than 90°, squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90°, stretching the sphere along that plane.

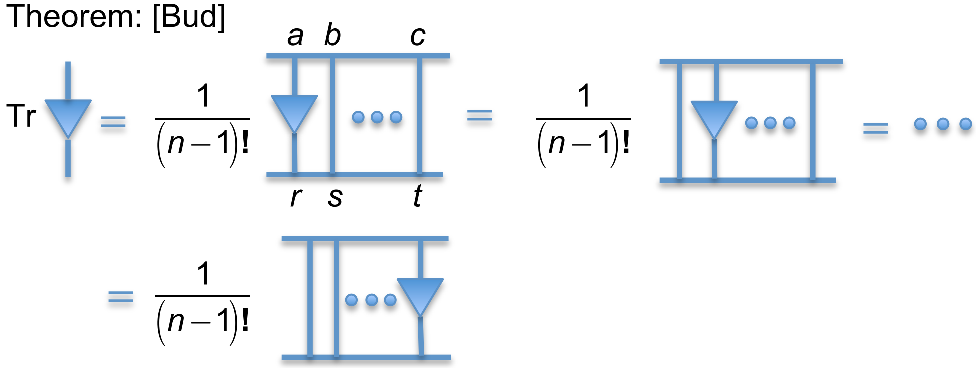
$\mathcal{A}(\mathbb{R}^3)$ = 3 x 3 Matrices

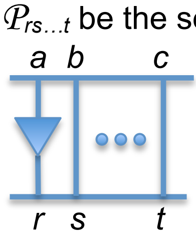
Determinant	Orthogonal	Sphere maps to a …	Matrix Type
0	Yes	Circle or line or point	Singular
	No	Ellipse or line or point	
Between -1 and 0	Yes	Contracted reflected sphere	**GL(3)** Non-singular Symmetries of $\mathbb{R}^3$
	No	Contracted reflected ellipsoid	
Between 0 and +1	Yes	Contracted sphere	
	No	Contracted ellipsoid	
-1	Yes	Reflected sphere	
	No	Reflected ellipsoid O(3)	
+1	Yes	SO(3) = sphere SL(3)	
	No	Ellipsoid	
< -1	Yes	Expanded reflected sphere	
	No	Expanded reflected ellipsoid	
> 1	Yes	Expanded sphere	
	No	Expanded ellipsoid	

Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

Definition. The **Trace** of A is T_r(A) = Tr  $= T^k{}_k = T^1{}_1 + \cdots T^n{}_n$.

Theorem: [Bud]



Proof: Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{rs\ldots t}$ be the sets of permutations of ($a,b,\ldots,c$) and ($r,s,\ldots,t$), respectively. Let B = 

$= \in^{rs\cdots t} \varepsilon_{ab\cdots c} T^{a}{}_{r} \delta^{b}_{s} \cdots \delta^{c}_{t} = \sum_{\pi \in \mathcal{P}_{ab\cdots c}} \sum_{\pi' \in \mathcal{P}_{rs\cdots t}} \in^{\pi'(r)\pi'(s)\cdots\pi'(t)} \varepsilon_{\pi(a)\pi(b)\cdots\pi(c)} T^{\pi(a)}{}_{\pi'(r)} \delta^{\pi(b)}_{\pi'(s)} \cdots \delta^{\pi(c)}_{\pi'(t)}$.

Fix π. The only non-zero term in the sum is

$\in^{\pi(a)\pi(b)\cdots\pi(c)} \varepsilon_{\pi(a)\pi(b)\cdots\pi(c)} T^{\pi(a)}{}_{\pi(a)} \delta^{\pi(b)}_{\pi(b)} \cdots \delta^{\pi(c)}_{\pi(c)} = T^{\pi(a)}{}_{\pi(a)}$.

I showed in Problem [13.22] that $\in^{xy\cdots z} \varepsilon_{xy\cdots z} = 1$ for any fixed $(x,y,\ldots,z)$.

Thus, B = $\sum_{\pi \in \mathcal{P}_{ab\cdots c}} T^{\pi(a)}{}_{\pi(a)}$. This sum has $n!$ terms composed of $(n-1)!$ terms equal to $T^{a}{}_{a}$, $(n-1)!$ terms equal to $T^{b}{}_{b}$, …, and $(n-1)!$ terms equal to $T^{c}{}_{c}$. So,

B = $(n-1)!\,(T^{a}{}_{a} + T^{b}{}_{b} + \ldots + T^{c}{}_{c}) = (n-1)!\,\mathrm{Tr}\,(A) = (n-1)!\,\mathrm{Tr}$

Similarly for the other figures. ■

Theorem. [13.24] $\mathrm{Det}\left(\mathrm{I} + \in \mathrm{A}\right) = 1 + \in \mathrm{Tr}\left(\mathrm{A}\right)$ if we ignore 2nd order and higher $\in$ terms.

Theorem. [13.25] Det e^{A} = $e^{\mathrm{Tr}(\mathrm{A})}$.

Definition. An **Eigenvector** is a non-zero vector v for which $\exists \lambda \in \mathbb{C}$ such that $Tv = \lambda v \;\Leftrightarrow\; \left(T - \lambda I\right)v = 0$. λ is called an **Eigenvalue.**

Note: $\mathrm{Det}\left(T - \lambda I\right) = 0$ and so $\left(T - \lambda I\right)$ is singular

Theorem. [13.26] $\mathrm{Det}\left(T\text{-}\lambda I\right) = \left(\lambda_1 - \lambda\right)\cdots\left(\lambda_n - \lambda\right) = 0$ is a polynomial equation of degree n.

Definition. **λ has multiplicity r** means that λ appears r times in the equation above. Eigenvalue multiplicities are called **degeneracies** in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to λ is a linear space called an **Eigenspace**.

Theorem. If d is the dimension of the Eigenspace of λ and r is the multiplicity of λ then $1 \le d \le r$.

Theorem. [13.27] Let $\left\{\lambda_i\right\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then $\sum r_i = n$.

Corollary. A linear transformation T has at least 1 Eigenvector.
Theorem. [13.30] Suppose $\{e_k\}$ and $\{f_k\}$ are bases for a vector space V, and $f_k = Te_k$. Then

$$f_j = \begin{pmatrix} T^1{}_j \\ \vdots \\ T^n{}_j \end{pmatrix}.$$

That is, the components of f_j in basis $\{e_k\}$ are $\left(T^1{}_j, \cdots, T^n{}_j\right)$.

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

$$T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then there is a basis for V such that the matrix of T in this basis is

$$T = \left(\begin{array}{ccccc|cccc|cccc}
\lambda_1 & 1 & & & & & & & & & & \cdots & 0 \\
 & \lambda_1 & 1 & & & & & & & & & \iddots & \vdots \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & \ddots & 1 & & & & & & & & \\
 & & & & \lambda_1 & 0 & & & & & & & \\
\hline
 & & & & & \lambda_2 & 1 & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & \ddots & 1 & & & & \\
 & & & & & & & & \lambda_{n-1} & 0 & & & \\
\hline
 & & & & & & & & & \lambda_n & 1 & & \\
 & & & & & & & & & & \lambda_n & \ddots & \\
\vdots & \iddots & & & & & & & & & & \ddots & 1 \\
0 & \cdots & & & & & & & & & & & \lambda_n
\end{array}\right).$$

Representations

Definition. Let L : G $\rightarrow \mathcal{G}$ be a homomorphism of a group G to some well-known standard group $\mathcal{G}$. The image L(G) is clled a **Group Representation of G**. In this section we take $\mathcal{G}$ to be GL(n), the multiplicative group of non-singular $n \times n$ matrices. L is **faithful** if it is 1-1.

Theorem. [13.32] Every finite group has a faithful representation. Every finite dimensional Lie group has a locally faithful (?) representation.

We will see in Chapter 14 that the theory of representations of continuous groups by linear transformations can be converted to the study of representations of Lie algebras, which we define next.

Definition. A nonempty set (R, +, •) is a **ring** if for all a, b, c in R:

(1) (R, +) is an Abelian group
(2) R is closed under multiplication •
(3) $a \bullet (b + c) = a \bullet b + a \bullet c$ and $(b + c) \bullet a = b \bullet a + c \bullet a$ (left and right distributive)

A ring R is an **associative ring** if it is associative under multiplication:

(4) $r \bullet (s \bullet t) = (r \bullet s) \bullet t$ for $r, s, t \in \mathrm{R}$

There are rings that have no multiplicative identity (i.e., no element 1). Rings that do have a multiplicative identity are said to be **rings with unit element**.

Definition. An **algebra** is ring that is also a vector space (that is, it has scalar multiplication in addition to addition and regular multiplication) and that for all a, b in R and scalar α we have $\alpha(a\,b) = (\alpha a)b = a(\alpha\,b)$. If the underlying ring is associative, then it is an **associative algebra**.

Example. If V is a vector space then the set of linear transformations, $\mathcal{A}$(V), is an algebra. E.g., $\mathcal{A}(\mathbb{R}^3)$, the set of 3 × 3 matrices, is an algebra: you can add and multiply matrices as well as multiply them by scalars.

Example. GL(3) is not an algebra. It is just a group. It is not even a ring because it is not closed under matrix addition. For example, addition of 2 non-singular matrices can yield the zero matrix, which is singular and, hence, not in GL(3).

I give below the standard definition of a Lie algebra. Penrose does not give this definition so I will shortly prove that his definition (of a special case) is indeed a Lie algebra.

Definition. (Standard definition) ($\mathfrak{g}, [\bullet,\bullet]$) is a **Lie algebra** if $\mathfrak{g}$ is a vector space over a field F with the **Lie bracket**, a binary operator that satisfies

- Bilinearity: $[ax+by,z]=a[x,z]+b[y,z]$ and $[z,ax+by]=a[z,x]+b[z,y]$ $\forall a,b \in F$ and $x,y,z \in \mathfrak{g}$
- Alternativity: $[x,x]=0 \ \forall x \in \mathfrak{g}$
- Jacobi Identity: $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0 \ \forall x,y,z \in \mathfrak{g}$

Note that blinearity and alternativity imply

- Anticommutativity: $[x,y] = - [y,x]$:

$$0=[x+y,\, x+y]=\cancel{[x,x]}+[x,y]+[y,x]+\cancel{[y,y]} \quad ✔$$

Theorem. [13.34] Let A, B $\in \mathfrak{g}$. Then

(a) $(\mathrm{I}+\in \mathrm{A})(\mathrm{I}+\in \mathrm{B})=\mathrm{I}+\in$ A+B if we ignore terms $o(\in)^2$

(b) $(\mathrm{I}+\in \mathrm{A})(\mathrm{I}+\in \mathrm{B})(\mathrm{I}+\in \mathrm{A})^{-1}(\mathrm{I}+\in \mathrm{B})^{-1}=\mathrm{I}+\in^2[\mathrm{A},\mathrm{B}]$ if we ignore $o(\in)^3$

If "infinitesimals" *a* and *b* are represented by $(\mathrm{I}+\in \mathrm{A})$ and $(\mathrm{I}+\in \mathrm{B})$, then we see that the products $aba^{-1}b^{-1}$ are represented by Lie brackets. Thus we make the following definition.

Definition. Let G be a group. $\{aba^{-1}b^{-1} : a,b \in \mathrm{G}\}$ is the **set of group commutators.**

Definition. (Penrose's definition) Let $\mathcal{G}$ be a subgroup of GL(*n*). Let $\mathcal{G}^*$ be the vector space generated from $\mathcal{G}$ by the addition of scalar multiplication. For A, B $\in \mathcal{G}^*$, a **Lie bracket** is **[A,B] ≡ AB – BA** (commutator operation). Note that [A,B] is not necessarily in $\mathcal{G}$ because $\mathcal{G}$ is not required to be closed under subtraction. The **Lie algebra generated by** $\mathcal{G}$ is the algebra $\mathfrak{g}$ generated from $\mathcal{G}^*$ by the addition of +, –, and Lie bracket operations.

We wish to show that $\mathfrak{g}$ is a Lie algebra. The next theorem shows that $\mathfrak{g}$ is more than simply a vector space as required in the standard definition of Lie algebra.

Theorem. $\mathfrak{g}$ is an associative algebra

Proof: We begin with a (multiplicative) subgroup $\mathcal{G}$ of GL(*n*) and its expansion to a vector space $\mathcal{G}^*$. The set $\mathfrak{g}$ is obtained from $\mathcal{G}^*$ by repeated application of the operations +, –, and $[\bullet,\bullet]$.

($\mathfrak{g}$, +) is an Abelian group:

- Closed under + since + is a repeated operation until nothing new is obtained
- The zero matrix is the additive identity and belongs to $\mathfrak{g}$
- - A is the additive inverse of A and is in $\mathfrak{g}$ since – is a repeated operation
- A+B = B+A since these are matrices

$\mathfrak{g}$ is closed under group multiplication : We show this by induction, constructing $\mathfrak{g}$ by “levels”. Let

$$\mathfrak{g}_0 = \mathcal{G}^* \qquad \text{(level 0)}$$

$$\mathfrak{g}_1 = \{A \pm B : A,B \in \mathcal{G}^*\} \cup \{[A,B] : A,B \in \mathcal{G}^*\} \qquad \text{(level 1)}$$

$$\mathfrak{g}_2 = \{A \pm B : A,B \in \mathfrak{g}_1\} \cup \{[A,B] : A,B \in \mathfrak{g}_1\} \qquad \text{(level 2)}$$

$$\mathfrak{g}_3 = \{A \pm B : A,B \in \mathfrak{g}_2\} \cup \{[A,B] : A,B \in \mathfrak{g}_2\} \qquad \text{(level 3)}$$

$$\vdots$$

Level 0 is closed under multiplication since it is a multiplicative group. Supposed level $n – 1$ is closed under multiplication and let E and F belong to level n. Then E = A±B or E = [A,B] where A, B $\in \mathfrak{g}_{n-1}$ and F = C±D or F = [C,D] where C, D$\in \mathfrak{g}_{n-1}$. If EF = (A ± B) (C ±D) then EF = AC ± AD ± BC ± BD $\in \mathfrak{g}_n$ because AC, AD, BC, and BD $\in \mathfrak{g}_{n-1}$ because $\mathfrak{g}_{n-1}$ is closed under multiplication. If EF = (A±B) [C,D], then EF = ACD – ADC ± BCD $\mp$ BDC $\in \mathfrak{g}_n$ because ACD, ADC, BCD, and BDC $\in \mathfrak{g}_{n-1}$ because $\mathfrak{g}_{n-1}$ is closed under multiplication. Similarly, if EF = [A,B] [C,D], then EF = ABCD – ABDC + … + BADC $\in \mathfrak{g}_n$.

We also have to show that the construction process stops at (the first) infinity, and it does because if A, B$\in \mathfrak{g}_\infty$ then $\exists\, n$ such that A, B $\in \mathfrak{g}_n$ and so A±B and [A,B] $\in \mathfrak{g}_{n+1} \subseteq \mathfrak{g}_\infty$.

The distributive property holds:

If A, B, C $\in \mathfrak{g}$ then A(B+C) = AB + AC since A, B, and C are matrices.

Similarly, the associative property for multiplication holds because we are dealing with matrices.

Finally, $\mathfrak{g}$ is a vector space:

The only property that is not obvious is closure of scalar multiplication. If α is a scalar and A $\in \mathfrak{g}$, then A is constructed via the induction process from a finite number of elements B, …, D belonging to $\mathcal{G}^*$. Since $\mathcal{G}^*$ is a vector space then αB, …, αD also belong to $\mathcal{G}^*$, and αA is constructed from them in a parallel process. So αA $\in \mathfrak{g}$, proving $\mathfrak{g}$ is a vector space. ■

Theorem. [13.35] Let A, B $\in \mathfrak{g}$ and $\lambda \in \mathbb{C}$. Then

(a) [A+B,C] = [A,C] + [B,C] and [λA,B] = λ [A,B] (Lie bracket left distributivity)
(b) [B,A] = - [A,B] (Lie bracket antisymmetry, also called anticommutativity)
(c) [A, [B,C]] + [B, [C,A]] + [C, [A,B]] = 0 (Jacobi identity)
(d) dim $\mathcal{G}^*$ ≤ dim G . If T is faithful, then dim $\mathcal{G}^*$ = dim G

Corollary. The Lie algebra $\mathfrak{g}$ generated by $\mathcal{G}$ is a Lie algebra.

Proof.

Right distributivity:
[C,A+B] = [C,A]+[C,B] and [A,λB] = λ [A,B] follows from (a) and (b).

Bilinearity:
This follows from (a) and right distributivity.

Alternativity:
Let A $\in \mathfrak{g}$. By (b), B = [A,A] = - [A,A] = - B. Since a vector space has a unique zero, B= 0. ■

Convention. Henceforth we assume T is faithful. Thus, dim $\mathcal{G}^*$ = n = dim $\mathcal{G}$.

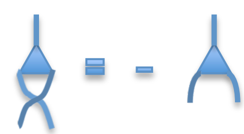
Definition. Let n be the dimension of the vector space $\mathcal{G}^*$ and (E_1, E_2, …, E_n) a basis for $\mathcal{G}^*$. Then

$$\exists \gamma_{\alpha\beta}{}^{\chi} \text{ where } \alpha,\beta,\chi \in \{1,\ 2,\ \cdots,\ n\} \text{ such that } \left[E_\alpha, E_\beta\right] = \gamma_{\alpha\beta}{}^{\chi} E_\chi .$$

The n^3 components $\gamma_{\alpha\beta}{}^{\chi}$ are called the **structural constants for G** and can be expressed in diagrammatic form as shown.

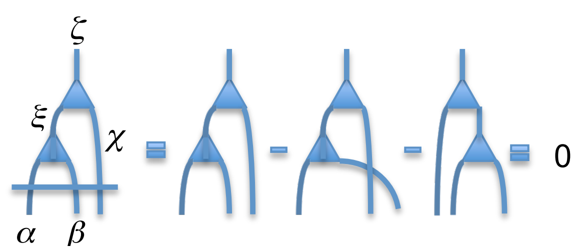
The $\gamma_{\alpha\beta}{}^{\chi}$ are not all independent because they satisfy relations in the next theorem.

Theorem. [13.36] $\gamma_{\beta\alpha}{}^{\chi} = -\gamma_{\alpha\beta}{}^{\chi}$ and $\gamma_{[\alpha\beta}{}^{\xi}\gamma_{\chi]\xi}{}^{\zeta} = 0$



Proof. This follows from Lie bracket antisymmetry and the Jacobi identity. ■

This theorem can be expressed in diagrammatic form as shown.



Tensor Representations

Definition. Let V be a vector space. The **dual space V*** is defined to be $V^* = \{ f : V \to \mathbb{R} \text{ or } \mathbb{C} : f \text{ is a linear map} \}$.

Theorem. V* = $\{y : \mathrm{V} \to \mathbb{R} \text{ or } \mathbb{C} : y \in \mathrm{V} \text{ is a row vector and } y : x \mapsto yx \text{ for } x \in \mathrm{V}\}$.

Proof.

Let $x \in$ V denote a column vector: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Consider the basis for V composed of {e_1, e_2, …, e_n} *where* $e_k = \begin{bmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{bmatrix}$. Let $f \in \mathrm{V}^*$. Define $y_k = f(e_k)$ and set $y = [y_1, \cdots, y_n]$. Then

$$\begin{aligned} f(x) &= f\left(\sum_{k=1}^{n} x_k\, e_k\right) = \sum_{k=1}^{n} f(x_k\, e_k) = \sum_{k=1}^{n} x_k\, f(\,e_k) = \sum_{k=1}^{n} x_k\, y_k \\ &= [y_1, \cdots, y_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= yx \end{aligned}$$

■

Definition. Let G be a group. A vector space V is called a **representation space for G** if G is represented by a group $\mathcal{G}$ of linear transformations on V.

Let $T \in \mathcal{G}$. For x $\in$ V, $T : \mathrm{V} \to \mathrm{V}$ can be written as $x \mapsto Tx$, or in matrix form as $x^a \mapsto T^a{}_b x^b$ where $T^a{}_b$ is a matrix and x^a and x^b are column vectors. Set $S = T^{-1}$.

$$\begin{tikzcd} \mathrm{V} \arrow[r, "S^{\mathrm{T}}"] \arrow[dr, dashed, "S^{\mathrm{T}} y^{\mathrm{T}}"'] & \mathrm{V} \arrow[d, "y^{\mathrm{T}}"] \\ & \mathbb{R} \text{ or } \mathbb{C} \end{tikzcd}$$

Let $y \in$ V*. Then $y : \mathrm{V} \to \mathbb{R}$ or $\mathbb{C}$ and $y : x \mapsto yx$ for $x \in$ V. Consider the identity map $yx \mapsto yx = y(ST)x = (yS)(Tx)$. Since $x \mapsto Tx$, if this map is a homomorphism preserving *yx*, then it must also map $y \mapsto yS$ or $y_a \mapsto y_b S^b{}_a$ where $S^b{}_a$ is a matrix and y_a and y_b are row vectors. We prefer to use column vectors so we write $y^{\mathrm{T}} \mapsto S^{\mathrm{T}} y^{\mathrm{T}}$ or $y^a \mapsto S^a{}_b y^b$ where the T superscript means matrix transpose. Observe that $S^{\mathrm{T}} y^{\mathrm{T}} \in \mathrm{V}^*$. We can also write $ST = I$, or $S^a{}_b T^b{}_c = \delta^a_c$.

The mapping $\left(y^{\mathsf{T}} \mapsto S^{\mathsf{T}} y^{\mathsf{T}}\right): \mathrm{V}^* \to \mathrm{V}^*$ plays the central role in the next theorem.

Theorem. If V is a representation space for a group G, then so is V*.

Proof. By definition of representation, there is a subgroup $\mathcal{G} \subset \mathrm{GL}(n)$ and an isomorphism $L: G \to \mathcal{G}$ such that $\forall g \in G \;\; L(g): \mathrm{V} \to \mathrm{V}$ is a linear transformation of V. We seek another subgroup $\mathcal{G}^* \subset \mathrm{GL}(n)$ and an isomorphism $L^*: G \to \mathcal{G}^*$ such that $\forall g \in G \;\; L^*(g): \mathrm{V}^* \to \mathrm{V}^*$ is a linear transformation of V*.

Define

(1) $T_g = L(g)$ for $g \in$ G.

Since L is an isomorphism,

(2) $L(g\,g') = L(g)\,L(g')$ for $g, g' \in G$.

Let

(3) $S_g = T_g^{-1}$.

Define a matrix

(4) $\alpha_g : \mathrm{V}^* \to \mathrm{V}^* : \alpha_g\, y^{\mathsf{T}} = S_g^{\mathsf{T}} y^{\mathsf{T}}$ for $y \in \mathrm{V}^*$.

Recall that y is a row vector. y^{T} is a column vector as is $S_g^{\mathsf{T}} y^{\mathsf{T}}$. Define

$\mathcal{G}^* = \left\{\alpha_g : g \in G\right\}$ and

(5) $L^*: G \to \mathcal{G}^* : L^*(g) = \alpha_g$ for $g \in$ G.

We proceed to show that L^* is a homomorphism. Let $g, g' \in$ G and set

(6) $g'' = g\,g'$.

It suffices to show that $L^*(g\,g') = L^*(g)\,L^*(g')$. We have

(7) $L^*(g\,g') \overset{(6)}{=} L^*(g'') \overset{(5)}{=} \alpha_{g''}$ and

(8) $L^*(g)\,L^*(g') \overset{(5)}{=} \alpha_g\, \alpha_{g'}$.

(9) $T_{g''} \overset{(1)}{=} L(g'') \overset{(6)}{=} L(g\,g') \overset{(2)}{=} L(g)\,L(g') \overset{(1)}{=} T_g\, T_{g'}$

$$\Rightarrow \quad S_{g''} \overset{(3)}{=} T_{g''}^{-1} \overset{(9)}{=} T_{g'}^{-1}\, T_g^{-1} \overset{(3)}{=} S_{g'}\, S_g ,$$

or

(10) $S_{g''}^{\mathsf{T}} = S_g^{\mathsf{T}}\, S_{g'}^{\mathsf{T}}$.

Observe that the inverse operation changed the order of g and g'. Then the transpose operation changed it back to the desired order. So

$$L^*(g\,g') \overset{(7)}{=} \alpha_{g''} \overset{(4)}{=} S_{g''}^{\mathsf{T}} \overset{(10)}{=} S_g^{\mathsf{T}}\, S_{g'}^{\mathsf{T}} \overset{(4)}{=} \alpha_g\, \alpha_{g'} \overset{(8)}{=} L^*(g)L^*(g').$$

That is, L^* is a homomorphism ✔

To show that L^* is an isomorphism, we must show that it is 1-1.

$$L^*(g) = L^*(g') \overset{(5)}{\Leftrightarrow} \alpha_g = \alpha_{g'} \overset{(4)}{\Leftrightarrow} S_g\, y^{\mathsf{T}} = S_{g'}\, y^{\mathsf{T}}\ \forall\, \mathrm{y} \in \mathrm{V}^*$$

$$\Leftrightarrow\ S_g = S_{g'} \overset{(3)}{\Leftrightarrow} T_g = T_{g'} \overset{(1)}{\Leftrightarrow} L(g) = L(g')$$

$$\Rightarrow\ g = g' \text{ since } L \text{ is an isomorphism.}$$

So L^* is 1-1. ✔ ■

Recall the definition of a tensor from §12.8.

Definitions. Let V be an n-dimensional vector space and let p and q be positive integers. Let V* be the dual space of V. Let $\mathrm{V}_a, \ldots, \mathrm{V}_c, \mathrm{V}_e, \ldots, \mathrm{V}_f$ be copies of V and define the **direct sum space**

$$\mathcal{V} = \mathrm{V}_f^* \oplus \cdots \oplus \mathrm{V}_h^* \oplus \mathrm{V}_a \oplus \cdots \oplus \mathrm{V}_c$$

$$= \left\{ (y_f, \cdots, y_h, x_a, \cdots, x_c) : y_f, \cdots, y_h \in \mathrm{V}^* \text{ and } x_a, \cdots, x_c \in \mathrm{V} \right\}$$

where $y_f, \cdots, y_h$ are row vectors and $x_a, \cdots, x_c$ are column vectors.

Recall from Chapter 12 that Q is a $\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}$-**valent tensor over V** if Q is an abstract quantity $\boldsymbol{Q} = \boldsymbol{Q}_{a\cdots c}^{f\cdots h}$ with p upper and q lower indices. Just as a linear transformation on V can be represented by a matrix, we can represent the multilinear function Q of q column vectors $A^a, \cdots, C^c$ and p row vectors $F_f, \cdots, H_h$ from V by a multi-dimensional matrix. That is,

$$F_f \cdots H_h A^a \cdots C^c \mapsto Q_{a\cdots c}^{f\cdots h} F_f \cdots H_h A^a \cdots, C^c.$$

Define the **tensor product space** $\mathcal{V}^*$ to be the set of $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors $Q_{a\cdots c}^{f\cdots h}$:

$$\mathcal{V}^* = \mathrm{V}_f^* \otimes \cdots \otimes \mathrm{V}_h^* \otimes \mathrm{V}_a \otimes \cdots \otimes \mathrm{V}_c.$$

If $Q \in \mathcal{V}^*$, then Q is a multilinear function

$$Q : \mathcal{V} \to \mathbb{R} \text{ or } \mathbb{C} : (y_f, \cdots, y_h, x_a, \cdots, x_c) \mapsto Q_{a\cdots c}^{f\cdots h} y_f \cdots y_h\, x^a \cdots x^c.$$

Theorem. [13.38] $\mathcal{V}^*$ is an n^{p+q}-dimensional vector space.

Proof. Like the collection of n x n non-singular matrices, $\mathcal{V}^*$ forms an Abelian group under addition and satisfies the three vector space scalar properties. So $\mathcal{V}^*$ is a vector space. Since each of f, …, h, a, …, c range from 1 – n, there are n^{p+q} entries in $Q^{f\cdots h}_{a\cdots c}$. For each f, …, h, a, …, $c \in \{1,\cdots,n\}$ define $E^{f\cdots h}_{a\cdots c}$ to have a 1 in the (f, …, h, a, …, c) position (of the multi-dimensional matrix) and 0 elsewhere. The collection of $\mathcal{E} = E^{f\cdots h}_{a\cdots c}$ forms a basis for $\mathcal{V}^*$ of size n^{p+q}:

> $\mathcal{E}$ is clearly the largest possible set of independent members of $\mathcal{V}^*$. Also, if $Q^{f\cdots h}_{a\cdots c} \in \mathcal{V}^*$, let $\alpha_{f\cdots h\, a\cdots c}$ be the (scalar) entry at the (f, …, h, a, …, c) position. Then $Q^{f\cdots h}_{a\cdots c} = \sum_{\substack{f\cdots h\\ a\cdots c}} \alpha_{f\cdots h\, a\cdots c}\, E^{f\cdots h}_{a\cdots c}$ is a linear combination of the members of $\mathcal{E}$. ■

Theorem. [13.39] The linear transformation $x \mapsto Tx$ (or $x^a \mapsto T^a{}_b x^b$) on V induces a linear transformation $\mathcal{T}$: $Q^{f\cdots h}_{a\cdots c} \mapsto Q^{f'\cdots h'}_{a'\cdots c'}\, S^f{}_{f'} \cdots S^h{}_{h'}\, T^{a'}{}_a \cdots T^{c'}{}_c$ on $\mathcal{V}^*$.

Proof. $x \mapsto Tx$ induces $y^{\mathsf{T}} \mapsto S^{\mathsf{T}} y^{\mathsf{T}}$ (or $y^a \mapsto S^a{}_b y^b$), and these can be applied to each copy of V and V*, respectively.

To show that $\mathcal{T}$ is linear, let P and Q be $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors, α a scalar, and $R = P + Q$. Then

$$\begin{aligned}
\mathcal{T}\left(P^{f\cdots h}_{a\cdots c} + Q^{f\cdots h}_{a\cdots c}\right) = \mathcal{T}\left(R^{f\cdots h}_{a\cdots c}\right) &= R^{f\cdots h}_{a\cdots c}\, S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \\
&= \left(P^{f\cdots h}_{a\cdots c} + Q^{f\cdots h}_{a\cdots c}\right) S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \\
&= P^{f\cdots h}_{a\cdots c}\, S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \; + \; Q^{f\cdots h}_{a\cdots c}\, S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \\
&= \mathcal{T}\left(P^{f\cdots h}_{a\cdots c}\right) + \mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}\left(\alpha\, Q^{f\cdots h}_{a\cdots c}\right) = \mathcal{T}\left(\left(\alpha\, Q\right)^{f\cdots h}_{a\cdots c}\right) &= \left(\alpha\, Q\right)^{f\cdots h}_{a\cdots c} S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \\
&= \alpha\, Q^{f\cdots h}_{a\cdots c}\, S^f{}_{f'} \cdots S^h{}_{h'} T^{a'}{}_a \cdots T^{c'}{}_c \\
&= \alpha\, \mathcal{T}\left(Q^{f\cdots h}_{a\cdots c}\right)
\end{aligned}$$

■

Theorem. If V is a representation space for a group G, then so is $\mathcal{V}^*$.

Proof. V is a representation space for G means there is an isomorphism

$$L : G \to \mathcal{G} : L(g) = T(g)$$

where $T(g)$ is a linear transformation on V. By [13.39], $T(g)$ induces a linear transformation $\mathcal{T}(g)$ on $\mathcal{V}$ *. Define

$$L^* : G \to \mathcal{V}^* : L^*(g) = \mathcal{T}(g).$$

We must show that L^* is an isomorphism. Let $g_1, g_2 \in \mathrm{G}$ and set $g_3 = g_1\, g_2$. Then

$$L^*(g_1\, g_2) = L^*(g_3) = \mathcal{T}(g_3) = \mathcal{T}(g_1\, g_2) = \mathcal{T}(g_1)\mathcal{T}(g_2) = L^*(g_1)L^*(g_2).$$

Therefore L^* is a homomorphism.

We next show that L^* is 1-1. Again, let $g_1, g_2 \in \mathrm{G}$. To simplify notation, set $T = L(g_1)$ and $N = L(g_2)$. T and N are linear transformations on V. Set $S = T^{-1}$ and $M = N^{-1}$. By [13.39], T induces the linear transformation

$$\mathcal{T} = S^{f}{}_{f'} \cdots S^{h}{}_{h'}\, T^{a'}{}_{a} \cdots T^{c'}{}_{c}$$

on $\mathcal{V}$ * and N induces the linear transformation

$$\mathcal{N} = M^{f}{}_{f'} \cdots M^{h}{}_{h'}\, N^{a'}{}_{a} \cdots N^{c'}{}_{c}$$

on $\mathcal{V}$ *. Then $L^*(g_1) = \mathcal{T}$ and $L^*(g_2) = \mathcal{N}$. Thus, $L^*(g_1) = L^*(g_2)$ is equivalent to $\mathcal{T} = \mathcal{N}$. Claim $T = N$:

Denote $T = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, $S^{\mathrm{T}} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$, $N = \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix}$, and $M^{\mathrm{T}} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$. Consider

$\mathcal{T}\left(E^{1\cdots1}_{1\cdots1}\right) = E^{1\cdots1}_{1\cdots1} S^{f}{}_{f'} \cdots S^{h}{}_{h'}\, T^{a'}{}_{a} \cdots T^{c'}{}_{c}$ and $\mathcal{N}\left(E^{1\cdots1}_{1\cdots1}\right) = E^{1\cdots1}_{1\cdots1} M^{f}{}_{f'} \cdots M^{h}{}_{h'}\, N^{a'}{}_{a} \cdots N^{c'}{}_{c}$

where $E^{1\cdots1}_{1\cdots1}$ is the basis tensor having 1 at the (1, …, 1) position and 0's elsewhere. Just as a matrix is multiplied by a column vector by summing products along the matrix rows and the vector column, the tensor $E^{1\cdots1}_{1\cdots1}$ can be multiplied by the column vectors, one-at-a-time, by summing products along the tensor "rows" and the vector columns. Since $E^{1\cdots1}_{1\cdots1}$ has all zeros except a one in the upper left-hand corner, it is easy to do the multiplication in your head to get $\mathcal{T}\left(E^{1\cdots1}_{1\cdots1}\right) = s_1^{\,q}\, t_1^{\,p}$ and $\mathcal{N}\left(E^{1\cdots1}_{1\cdots1}\right) = m_1^{\,q}\, n_1^{\,p}$.

Moreover, since $S = T^{-1}$ and $M = N^{-1}$, then $s_1 = \dfrac{t_1}{\sum_{k=1}^{n} t_k^{\,2}}$ and $m_1 = \dfrac{n_1}{\sum_{k=1}^{n} n_k^{\,2}}$. So

$$\mathcal{T}\left(E_{1\cdots1}^{1\cdots1}\right)=\left(\frac{t_1}{\sum_{k=1}^{n} t_k^{\,2}}\right)^{p+q}$$ and $$\mathcal{N}\left(E_{1\cdots1}^{1\cdots1}\right)=\left(\frac{n_1}{\sum_{k=1}^{n} n_k^{\,2}}\right)^{p+q}$$. If T and N have norm 1, then $\sum_{k=1}^{n} t_k^{\,2}$ and $\sum_{k=1}^{n} n_k^{\,2}$ and so $\mathcal{T}\left(E_{1\cdots1}^{1\cdots1}\right)=t_1^{\,p+q}$ and $\mathcal{N}\left(E_{1\cdots1}^{1\cdots1}\right)=n_1^{\,p+q}$.

Thus, $\mathcal{T}=\mathcal{N} \quad \Rightarrow \quad t_1^{\,p+q}=\mathcal{T}\left(E_{1\cdots1}^{1\cdots1}\right)=\mathcal{N}\left(E_{1\cdots1}^{1\cdots1}\right)=n_1^{\,p+q}$

$\Rightarrow$ $t_1 = n_1$. Similarly, using other basis tensors, we get that $t_2 = n_2$, …, $t_n = n_n$; that is, $T = N$. If the vectors T and N are not normalized, they are the product of a scalar and a normalized vector and the result still holds.

Therefore $L(g_1)=T=N=L(g_2) \quad \Rightarrow \quad g_1=g_2$. ■