[13.40] Let V be an n-dimensional vector space, $\mathcal{V} = \mathrm{V} \otimes \mathrm{V}$ the tensor product of V with itself, and let $Q^{ab} \in \mathcal{V}$ be a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-tensor. Let

$$Q^{(ab)} = \tfrac{1}{2}\left(Q^{ab} + Q^{ba}\right) \text{ be the symmetric part}$$

and

$$Q^{[ab]} = \tfrac{1}{2}\left(Q^{ab} - Q^{ba}\right) \text{ be the antisymmetric part.}$$

Define

$$\mathcal{V}_{+} = \left\{Q^{(ab)} : Q^{ab} \in \mathcal{V}\right\} \text{ and } \mathcal{V}_{-} = \left\{Q^{[ab]} : Q^{ab} \in \mathcal{V}\right\}.$$

Then

$$\dim \mathcal{V}_{+} = \frac{n}{2}\left(n+1\right) \text{ and } \dim \mathcal{V}_{-} = \frac{n}{2}\left(n-1\right).$$

Solution.

Let $\mathscr{B} = \left\{e^{1}, \cdots, e^{n}\right\}$ be the basis for V where $e^{k} = \begin{pmatrix} 0 \\ \vdots \\ 1_{k} \\ \vdots \\ 0 \end{pmatrix}$. Set

$$e^{ab} = e^{a} \otimes e^{b} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1_{ab} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}.$$

By definition, $\mathscr{B} = \left\{e^{ab}\right\}$ is a basis for $\mathcal{V}$, and it has n^2 terms. Observe that

$$e^{(aa)} = e^{aa} \text{ and } e^{[aa]} = 0.$$

So, we define

$$\mathscr{B}_{+} = \left\{e^{(ab)} : a \leq b\right\} \text{ and } \mathscr{B}_{-} = \left\{e^{[ab]} : a < b\right\}.$$

$\mathscr{B}_{+}$ has $\frac{n}{2}\left(n+1\right)$ terms with $a \leq b$ and $\mathscr{B}_{-}$ has $\frac{n}{2}\left(n-1\right)$ terms with $a < b$.

Note: The reason for defining $\mathscr{B}_{+}$ and $\mathscr{B}_{-}$ with $a \leq b$ and $a < b$ is that

$$e^{(ab)} = e^{(ba)}$$

and

$$e^{[ba]} = -e^{[ab]}.$$

So, terms with $b > a$ are not independent from the others.

Set $\mathscr{B} = \mathscr{B}_+ \cup \mathscr{B}_-$. Claim $\mathscr{B}$ is a basis for $\mathcal{V}$:

Consider two typical elements of $\mathscr{B}$,

$$e^{(ab)} = \begin{array}{c} \\ 1 \\ a \\ b \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & a & b & n \end{array} \\ \begin{pmatrix} & & & \\ & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \\ & & & \end{pmatrix} \end{array}$$

and

$$e^{[ab]} = \begin{array}{c} \\ 1 \\ a \\ b \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & a & b & n \end{array} \\ \begin{pmatrix} & & & \\ & 0 & \frac{1}{2} & \\ & -\frac{1}{2} & 0 & \\ & & & \end{pmatrix} \end{array}.$$

$e^{(ab)}$ and $e^{[ab]}$ are linearly independent because there is no scalar *a* such that $e^{(ab)} = \alpha e^{[ab]}$. Moreover, $\mathscr{B}$ is a linearly independent set because all other elements of $\mathscr{B}$ have 0's in the *a*-*b* and *b*-*a* positions. $\mathcal{V}$ has dimension n^2, and since $\mathscr{B}$ has n^2 independent elements, it is a basis for $\mathcal{V}$. ✔

Observe that dim span $\left(\mathscr{B}_+\right) = \dfrac{n}{2}(n+1)$ and dim span $\left(\mathscr{B}_-\right) = \dfrac{n}{2}(n-1)$.

We proceed to show that $\mathcal{V}_+ =$ span $\left(\mathscr{B}_+\right)$ and $\mathcal{V}_- =$ span $\left(\mathscr{B}_-\right)$, which will complete the problem.

Claim: $\mathscr{B}_+ \subseteq \mathcal{V}_+$:

Let $E^{ab} = e^{(ab)}$ for $a,\ b \le n$. When $a \le b$, $E^{ab} \in \mathscr{B}_+$ and

$$E^{ab} = e^{(ab)} = \tfrac{1}{2}\left(e^{(ab)} + e^{(ab)}\right) = \tfrac{1}{2}\left(e^{(ab)} + e^{(ba)}\right) = \tfrac{1}{2}\left(E^{ab} + E^{ba}\right) = E^{(ab)} \in \mathcal{V}_+$$

$\Rightarrow \quad \mathscr{B}_+ \subseteq \mathcal{V}_+$ ✔

Claim: $\mathscr{B}_- \subseteq \mathcal{V}_-$:

Let $F^{ab} = e^{[ab]}$ for a, $b \le n$. When $a < b$, $F^{ab} \in \mathscr{B}_-$ and

$$F^{ab} = e^{[ab]} = \tfrac{1}{2}\left(e^{[ab]} + e^{[ab]}\right) = \tfrac{1}{2}\left(e^{[ab]} - e^{[ba]}\right) = \tfrac{1}{2}\left(F^{ab} - F^{ba}\right) = F^{[ab]} \in \mathcal{V}_-$$

$\Rightarrow \quad \mathscr{B}_- \subseteq \mathcal{V}_- \quad$ ✔

Thus, Span $\left(\mathscr{B}_+\right) \subseteq \mathcal{V}_+$ and Span $\left(\mathscr{B}_-\right) \subseteq \mathcal{V}_-$.

Consider $Q^{(ab)} \in \mathcal{V}_+$.

Denote $Q^{ab} = \begin{pmatrix} & \vdots & \\ \cdots & q^{ab} & \cdots \\ & \vdots & \end{pmatrix}$. Then

$$Q^{(ab)} = \tfrac{1}{2}\begin{pmatrix} & \vdots & \\ \cdots & q^{ab} + q^{ba} & \cdots \\ & \vdots & \end{pmatrix} = \sum_{ab} \tfrac{1}{2}\left(q^{ab} + q^{ba}\right) e^{ab}.$$

Fix a < b:

$$\tfrac{1}{2}\left(q^{ab} + q^{ba}\right)e^{ab} + \tfrac{1}{2}\left(q^{ba} + q^{ab}\right)e^{ba} = \left(q^{ab} + q^{ba}\right)\left[\tfrac{1}{2}\left(e^{ab} + e^{ba}\right)\right]$$
$$= \left(q^{ab} + q^{ba}\right)e^{(ab)}$$

Since $e^{aa} = e^{(aa)}$, $Q^{(ab)} = \sum_a q^{aa} e^{(aa)} + \sum_{a<b}\left(q^{ab} + q^{ba}\right)e^{(ab)} \in \text{Span}\left(\mathscr{B}_+\right)$

Therefore

$\mathcal{V}_+ \subseteq \text{Span}\left(\mathscr{B}_+\right) \quad \Rightarrow \quad \mathcal{V}_+ = \text{Span}\left(\mathscr{B}_+\right)$

$\Rightarrow \quad \dim \mathcal{V}_+ = \dim \text{Span}\left(\mathscr{B}_+\right) = \tfrac{1}{2}n\left(n+1\right) \quad$ ✔

Similarly, for $a < b$,

$$Q^{[ab]} = \sum_{ab} \tfrac{1}{2}\left(q^{ab} - q^{ba}\right)e^{ab} = \sum_{a<b}\left(q^{ab} - q^{ba}\right)e^{[ab]} \in \text{Span}\left(\mathscr{B}_-\right)$$

$\Rightarrow \quad \mathcal{V}_- \subseteq \text{Span}\left(\mathscr{B}_-\right) \quad \Rightarrow \quad \mathcal{V}_- = \text{Span}\left(\mathscr{B}_-\right)$

$\Rightarrow \quad \dim \mathcal{V}_- = \dim \text{Span}\left(\mathscr{B}_-\right) = \tfrac{1}{2}n\left(n-1\right) \quad$ ✔ $\blacksquare$

Note: Since $\mathcal{V}_+ \cap \mathcal{V}_- = \text{Span}\left(\mathscr{B}_+\right) \cap \text{Span}\left(\mathscr{B}_-\right) = \left[0\right]$, the set containing the zero matrix, then $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$, the sum of disjoint subspaces.

Example with n = 2:

Let

$$e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then $\mathscr{B} = \{e^1, e^2\}$ is a basis for V.

Let $e^{11} = e^1 \otimes e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e^{12} = e^1 \otimes e^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$e^{21} = e^2 \otimes e^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $e^{22} = e^2 \otimes e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then $\mathscr{B} = \{e^{11}, e^{12}, e^{21}, e^{22}\}$ is a basis for $\mathcal{V} = \mathrm{V} \otimes \mathrm{V}$.

Observe that

$$e^{(11)} = \frac{1}{2}\left(e^{11} + e^{11}\right) = e^{11} \text{ and } e^{(22)} = e^{22}.$$

These are 2 elements of $\mathscr{B}_+$.

Note that $e^{[11]} = 0 = e^{[22]}$ so they do not contribute to $\mathscr{B}_-$.

The other term in $\mathscr{B}_+$ is $e^{(12)}$ [which equals $e^{(21)} = \frac{1}{2}\left(e^{12} + e^{21}\right)$].

The only term in $\mathscr{B}_-$ is $e^{[12]}$ (which equals $-e^{[21]}$).

Thus dim $\mathcal{V}_+ = \frac{n}{2}(n+1) =$ 3 and dim $\mathcal{V}_- = \frac{n}{2}(n-1) = 1$. ✔