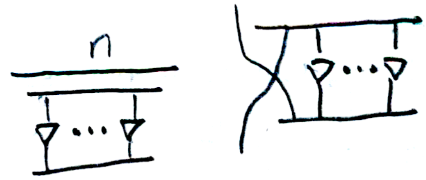
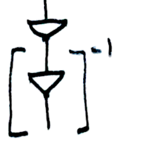
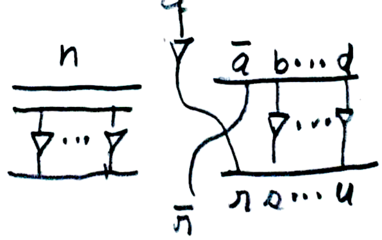
[13.19] Show that $T^{-1} =$ 

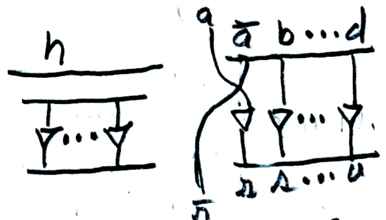
(This annotates Beckmann's solutions)

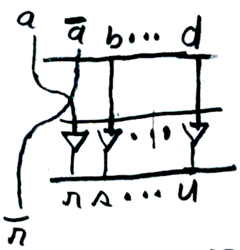
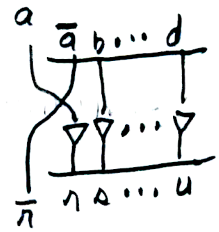
Pf: Suffices to show $TT^{-1} = T^{-1}T = I$. We do the former and leave the latter as similar (but, actually, not). Note that $T^{\bar{a}}_{\bar{r}}$ is simply a number. So it commutes with $\in$ and ε. Also, we use the Einstein summation convention.

(A) $TT^{-1} =$  $=$ 

$$= \frac{n}{n!\,\det(T)} T^{a}_{\ r}\, \delta^{\bar{a}}_{\bar{r}}\, \in^{r \cdots u}\, T^{b}_{\ s} \cdots T^{d}_{\ u}\, \varepsilon_{\bar{a} b \cdots d}$$

$$= \frac{n}{n!\,\det(T)}\, \delta^{\bar{a}}_{\bar{r}}\, \in^{r \cdots u}\, T^{a}_{\ r} T^{b}_{\ s} \cdots T^{d}_{\ u}\, \varepsilon_{\bar{a} b \cdots d}$$

(B) $=$ 

Claim 1.  $= n!$  :

$$LHS = n!\, \in^{rs \cdots u}\, \delta^{\bar{a}}_{\bar{r}}\, T^{[a}_{\ \ r} T^{b}_{\ s} \cdots T^{d]}_{\ \ u}\, \varepsilon_{\bar{a} b \cdots d}$$

$$RHS = n!\, \in^{r \cdots u}\, \delta^{\bar{a}}_{\bar{r}}\, T^{a}_{\ r} T^{b}_{\ s} \cdots T^{d}_{\ u}\, \varepsilon_{\bar{a} b \cdots d}$$

$$LHS = n!\left(\frac{1}{n!}\right) \sum_{\pi} \mathrm{sign}(\pi)\, \in^{r \cdots u}\, \delta^{\bar{a}}_{\bar{r}}\, T^{\pi(a)}_{\ \ r} T^{\pi(b)}_{\ \ s} \cdots T^{\pi(d)}_{\ \ u}\, \varepsilon_{\bar{a} b \cdots d}$$

where π is a permutation of $(a, b, \ldots, d)$ and $\mathrm{sign}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is an even permutation} \\ -1 & \text{o.w.} \end{cases}$

[13.19 con't] Freeze the Einstein summations for a moment. Then LHS has $n!$ terms. We will show $\overset{(1)}{}$ $\mathrm{sign}(\pi)\, \epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u}\, \varepsilon_{\bar{a}b\ldots d}$

$= \epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{a}{}_{r} \cdots T^{d}{}_{u}\, \varepsilon_{\bar{a}b\ldots d}$ for each π. That will show that LHS = RHS.

Since every permutation π can be generated from a series of pairwise swaps, it suffices to show that (1) holds for every pairwise swap between members of $(a, \ldots, d)$. There are 2 kinds of swaps. External swaps are ones involving the external superscript "a". Internal swaps are those not involving a.

Case 1. External Swaps

We show (1) holds for a swap of a with d but the argument holds for any other internal superscript besides d. To swap a and d, first a undergoes (n-1) swaps to the right; then d undergoes (n-2) swaps to the left. The total is 2n-3, an odd number. If π is the swap of a with d, $\mathrm{sign}(\pi) = -1$. So

$LHS = -\epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{d}{}_{r} T^{b}{}_{s} \cdots T^{a}{}_{u}\, \varepsilon_{\bar{a}\ldots d}$

$= -\epsilon^{ns\ldots m}\, \delta^{\bar{a}}_{r}\, T^{\bar{a}}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar{a}\ldots d}$ [Rename r as n and u as m]

$= -(-1)^{2n-3}\, \epsilon^{ms\ldots n}\, \delta^{\bar{a}}_{r}\, T^{\bar{a}}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar{a}\ldots d}$ [Swap n and m in ϵ, which is antisymmetric]

$= \epsilon^{ms\ldots n}\, \delta^{\bar{a}}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar{a}\ldots d}$

$= \epsilon^{rs\ldots u}\, \delta^{\bar{a}}_{r}\, T^{d}{}_{u} T^{b}{}_{s} \cdots T^{a}{}_{r}\, \varepsilon_{\bar{a}\ldots d}$ [Rename m as r and n as u]

$= \epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{a}{}_{r} T^{b}{}_{s} \cdots T^{d}{}_{u}\, \varepsilon_{\bar{a}\ldots d}$ [Rearrange the numbers $T^{a}{}_{r}, \ldots, T^{d}{}_{u}$]

$= RHS$

Case 2. Internal Swaps

We show for a swap of b and d but the argument holds for any 2 internal superscripts. The swap involves (n-2) pairwise swaps to the right followed by (n-3) swaps to the left, for a total of 2n-5, an odd number of swaps. Thus

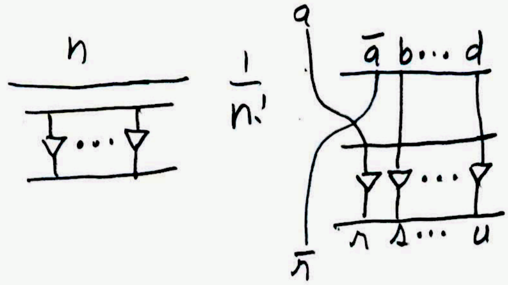
$LHS = -\epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{a}{}_{r} T^{d}{}_{s} \cdots T^{b}{}_{u}\, \varepsilon_{\bar{a}b\ldots d}$

$= +\epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{a}{}_{r} T^{d}{}_{s} \cdots T^{b}{}_{u}\, \varepsilon_{\bar{a}d\ldots b}$ [Swap b and d in ϵ]

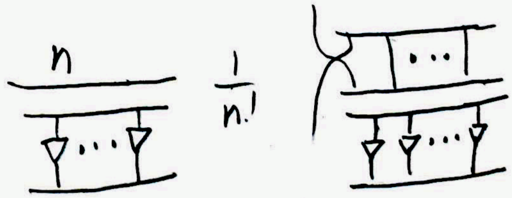
$= \epsilon^{r\ldots u}\, \delta^{\bar{a}}_{r}\, T^{a}{}_{r} T^{b}{}_{s} \cdots T^{d}{}_{u}\, \varepsilon_{\bar{a}b\ldots d}$ [Rename b as d and d as b]

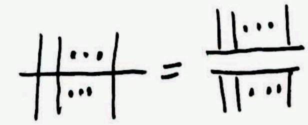
$= RHS$

[13.19 con't]. We have shown that (B.) can be rewritten as

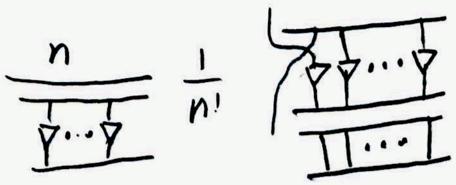
(C) 

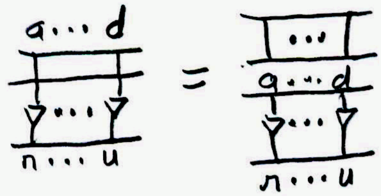
Beckmann claims that by splitting RHS according to Fig 12.18 yields

(D) 

Recall Fig 12.18:  This split lacks both ▽ and externals.

I believe the split from (C) to (D) requires proof. I cannot prove it. I can prove 2 things that generalize Fig 12.18 but not (D). My "claim" on p.1 results in

(D')  . I can also show

Claim 2:  (Note: No externals)

$$LHS = n!\, \epsilon^{r\cdots u}\, T^{[a}{}_{r} \cdots T^{d]}{}_{u}\, \varepsilon_{a\cdots d} = n!\left(\tfrac{1}{n!}\right) \sum_{\pi} \mathrm{sign}(\pi)\, \epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u}\, \varepsilon_{a\cdots d}$$

$$= \sum_{\pi} \mathrm{sign}(\pi)\, \epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u}\, [\mathrm{sign}(\pi)\, \varepsilon_{\pi(a)\cdots\pi(d)}]$$

$$= \sum_{\pi} \epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u}\, \varepsilon_{\pi(a)\cdots\pi(d)} \overset{(*)}{=} \sum_{\pi} \epsilon^{r\cdots u}\, T^{a}{}_{r} \cdots T^{d}{}_{u}\, \varepsilon_{a\cdots u}$$

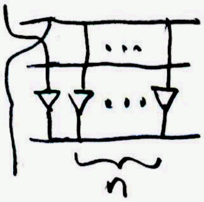
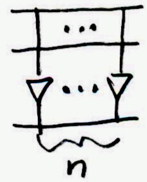
$$= n!\, \epsilon^{r\cdots u}\, T^{a}{}_{r} \cdots T^{d}{}_{u}\, \varepsilon_{a\cdots d}$$

$$= RHS$$

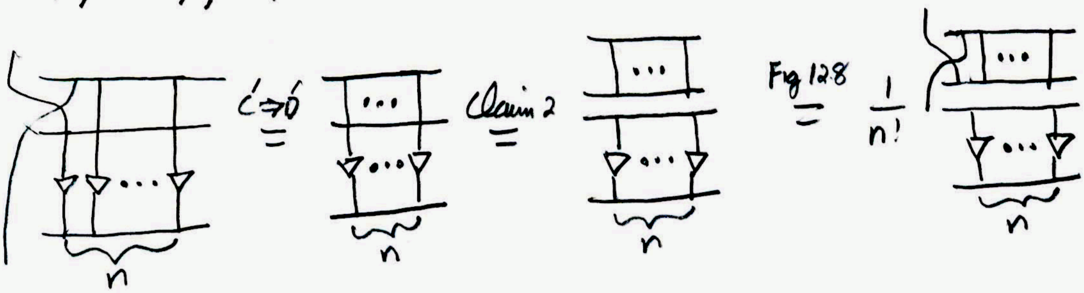
(*) Rename $\pi(a)$ as a, $\pi(b)$ as b, ..., $\pi(d)$ as d

Proving (D) is much more difficult I believe

[13.19 cont] I can use Fig 12.8 and Claim 2 to reduce (C) ⇒ (d) to a different problem,

(C') ⇒ (D'): Let (C') = $\frac{1}{n!}$  and (D') = 

If (C') ⇒ (D'), then

$\frac{1}{n!}$ 

$\Rightarrow \left[(C) = (D) \right]$

$$(c') = \frac{1}{n!} \epsilon^{r\cdots u} \delta^{\bar{a}}_{\bar{r}} T^{[a}{}_{r} T^{b}{}_{s} \cdots T^{d]}{}_{u} \varepsilon_{\bar{a}b\cdots d}$$

$$= \frac{1}{n!} \sum_{\pi} \delta^{\bar{a}}_{\bar{r}} \epsilon^{r\cdots u} T^{\pi(a)}{}_{r} T^{\pi(b)}{}_{s} \cdots T^{\pi(d)}{}_{u} \epsilon_{\bar{a}b\cdots d}$$

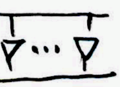
$$= \frac{1}{n!} \sum_{\pi} \delta^{\bar{r}}_{\bar{r}} \epsilon^{r\cdots u} T^{\pi(a)}{}_{r} T^{\pi(b)}{}_{s} \cdots T^{\pi(d)}{}_{u} \varepsilon_{\bar{r}b\cdots d}$$

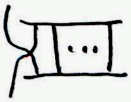
$$= \frac{1}{n!} \epsilon^{r\cdots u} T^{\pi(a)}{}_{r} T^{\pi(b)}{}_{s} \cdots T^{\pi(d)}{}_{u} \varepsilon_{\bar{a}b\cdots d} \qquad \text{(rename } \bar{r} \text{ to } \bar{a}\text{)}$$

$$(D') \quad n!\, \epsilon^{r\cdots u} T^{[a}{}_{r} \cdots T^{d]}{}_{u} \varepsilon_{a\cdots d}$$

$$= n! \cancel{\left(\frac{1}{n!}\right)} \sum_{\pi} \epsilon^{r\cdots u} T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u} \varepsilon_{a\cdots d}$$

Why are they equal?

Finishing this problem ... the two  cancel, leaving

(E) = $\frac{n}{n!}$  $\overset{\text{Fig 128}}{=} \frac{n}{n!} (n-1)!$ | $= \frac{n!}{n!}$ | $= I$ □