[13.39] Let V be an n-dimensional vector space. Let $\mathcal{V} = V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$ be the vector space of $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors on V, where V* is the dual vector space of V. Show that a linear transformation T on V induces a linear transformation $\mathcal{T} : \mathcal{V} \to \mathcal{V} : Q^{f \cdots h}_{a \cdots c} \mapsto S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \, Q^{f' \cdots h'}_{a' \cdots c'}$ on $\mathcal{V}$ where $S = \left(T^{-1}\right)^{\mathrm{T}}$, the transpose of the inverse of T.

Proof. To show that $\mathcal{T}$ is linear, let P and Q be $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors, α a scalar, and $R = P + Q$. Then

$$\begin{aligned}
\mathcal{T}\left(P^{f \cdots h}_{a \cdots c} + Q^{f \cdots h}_{a \cdots c}\right) = \mathcal{T}\left(R^{f \cdots h}_{a \cdots c}\right) &= S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \, R^{f' \cdots h'}_{a' \cdots c'} \\
&= S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \left(P^{f' \cdots h'}_{a' \cdots c'} + Q^{f' \cdots h'}_{a' \cdots c'}\right) \\
&= S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \, P^{f' \cdots h'}_{a' \cdots c'} \; + \; S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \, Q^{f' \cdots h'}_{a' \cdots c'} \\
&= \mathcal{T}\left(P^{f \cdots h}_{a \cdots c}\right) + \mathcal{T}\left(Q^{f \cdots h}_{a \cdots c}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}\left(\alpha \, Q^{f \cdots h}_{a \cdots c}\right) = \mathcal{T}\left(\left(\alpha \, Q\right)^{f \cdots h}_{a \cdots c}\right) &= S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \left(\alpha \, Q\right)^{f' \cdots h'}_{a' \cdots c'} \\
&= \alpha \, S^{a'}{}_{a} \cdots S^{c'}{}_{c} \, T^{f}{}_{f'} \cdots T^{h}{}_{h'} Q^{f' \cdots h'}_{a' \cdots c'} \\
&= \alpha \, \mathcal{T}\left(Q^{f \cdots h}_{a \cdots c}\right)
\end{aligned}$$