[13.39] Let V be an n-dimensional vector space. Let $\mathcal{V} = V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$ be the vector space of $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors on V, where V* is the dual vector space of V. Show that a linear transformation T on V induces a linear transformation $\mathcal{T} : \mathcal{V} \to \mathcal{V} : Q^{f \cdots h}_{a \cdots c} \mapsto S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, Q^{f' \cdots h'}_{a' \cdots c'}$ on $\mathcal{V}$ where $S = \left(T^{-1}\right)^{\mathsf{T}}$, the transpose of the inverse of T.

Solution. We must first show that $\mathcal{T}\left(Q\right) \in \mathcal{V}$:

Let $Q^{f \cdots h}_{a \cdots c} = y_a \otimes \cdots \otimes y_c \otimes x^f \otimes \cdots \otimes x^h$ where $y_a, \cdots, y_c \in V^{\mathsf{T}}$ and $x^f, \cdots, x^h \in V$.

Recall that $S^{a'}{}_{a} \cdots S^{c'}{}_{c} T^{f}{}_{f'} \cdots T^{h}{}_{h'} = S^{a'}{}_{a} \otimes \cdots \otimes S^{c'}{}_{c} \otimes T^{f}{}_{f'} \otimes \cdots \otimes T^{h}{}_{h'}$. Therefore

$$\begin{aligned} \mathcal{T}\left(Q\right) &= S^{a'}{}_{a} \otimes \cdots \otimes S^{c'}{}_{c} \otimes T^{f}{}_{f'} \otimes \cdots \otimes T^{h}{}_{h'} \left(y_{a'} \otimes \cdots \otimes y_{c'} \otimes x^{f'} \otimes \cdots \otimes x^{h'}\right) \\ &= S^{a'}{}_{a} y_{a'} \otimes \cdots \otimes S^{c'}{}_{c} y_{c'} \otimes T^{f}{}_{f'} x^{f'} \otimes \cdots \otimes T^{h}{}_{h'} x^{h'}. \end{aligned}$$

Also, $S : V^{\mathsf{T}} \to V^{\mathsf{T}}$ and $T : V \to V$. So, for example, $S^{a'}{}_{a} y_{a'}$ is a sum of vectors in V^{T}. Therefore

$$S^{a'}{}_{a} y_{a'} \in V^{\mathsf{T}} \cong V^*,$$

and thus

$$\mathcal{T}\left(Q\right) \in V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V = \mathcal{V} \qquad ✔$$

To show that $\mathcal{T}$ is linear, let P and Q be $\begin{bmatrix} p \\ q \end{bmatrix}$-valent tensors, α a scalar, and $R = P + Q$. Because the tensor product is multilinear,

$$\begin{aligned} \mathcal{T}\left(P^{f \cdots h}_{a \cdots c} + Q^{f \cdots h}_{a \cdots c}\right) &= \mathcal{T}\left(R^{f \cdots h}_{a \cdots c}\right) = S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, R^{f' \cdots h'}_{a' \cdots c'} \\ &= S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \left(P^{f' \cdots h'}_{a' \cdots c'} + Q^{f' \cdots h'}_{a' \cdots c'}\right) \\ &= S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, P^{f' \cdots h'}_{a' \cdots c'} \;+\; S^{a'}{}_{a} \cdots S^{c'}{}_{c}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'}\, Q^{f' \cdots h'}_{a' \cdots c'} \\ &= \mathcal{T}\left(P^{f \cdots h}_{a \cdots c}\right) + \mathcal{T}\left(Q^{f \cdots h}_{a \cdots c}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}\left(\alpha\, Q^{f \cdots h}_{a \cdots c}\right) &= \mathcal{T}\left(\left(\alpha\, Q\right)^{f \cdots h}_{a \cdots c}\right) = S^{a}{}_{a'} \cdots S^{c}{}_{c'}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'} \left(\alpha\, Q\right)^{f' \cdots h'}_{a' \cdots c'} \\ &= \alpha\, S^{a}{}_{a'} \cdots S^{c'}{}_{c'}\, T^{f}{}_{f'} \cdots T^{h}{}_{h'} Q^{f' \cdots h'}_{a' \cdots c'} \\ &= \alpha \mathcal{T}\left(Q^{f \cdots h}_{a \cdots c}\right) \qquad ✔ \end{aligned}$$