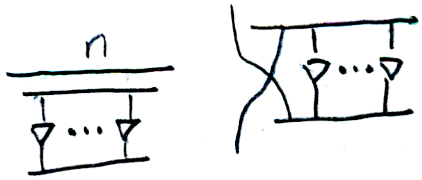
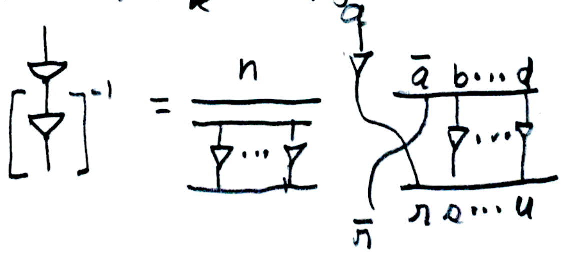
[13.19] Show that $T^{-1} =$ 

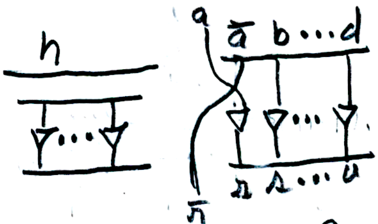
(This annotates Beckmann's solutions)

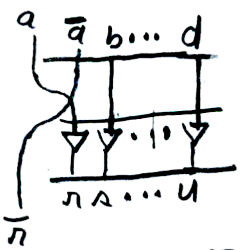
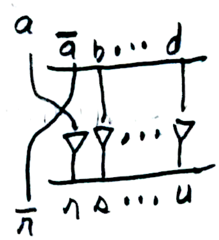
Pf: Suffices to show $TT^{-1} = T^{-1}T = I$. We do the former and leave the latter (but, actually, not) as similar. Note that $T^{\bar a}_{\bar k}$ is simply a number. So it commutes with ϵ and ε. Also, we use the Einstein summation convention.

(A) $TT^{-1} =$ 

$$= \frac{n}{n!\det(T)} T^{a}_{r}\, \delta^{\bar a}_{\bar r}\, \epsilon^{r\cdots u}\, T^{b}_{s}\cdots T^{d}_{u}\, \varepsilon_{\bar a b\cdots d}$$

$$= \frac{n}{n!\det(T)} \delta^{\bar a}_{\bar r}\, \epsilon^{r\cdots u}\, T^{a}_{r} T^{b}_{s}\cdots T^{d}_{u}\, \varepsilon_{\bar a b\cdots d}$$

(B) $=$ 

Claim 1.  $= n!$  :

$$LHS = n!\, \epsilon^{rs\cdots u}\, \delta^{\bar a}_{\bar r}\, T^{[a}{}_{r} T^{b}{}_{s}\cdots T^{d]}{}_{u}\, \varepsilon_{\bar a b\cdots d}$$

$$RHS = n!\, \epsilon^{r\cdots u}\, \delta^{\bar a}_{\bar r}\, T^{a}{}_{r} T^{b}{}_{s}\cdots T^{d}{}_{u}\, \varepsilon_{\bar a b\cdots d}$$

$$LHS = n!\left(\frac{1}{n!}\right) \sum_{\pi} \operatorname{sign}(\pi)\, \epsilon^{r\cdots u}\, \delta^{\bar a}_{\bar r}\, T^{\pi(a)}{}_{r} T^{\pi(b)}{}_{s}\cdots T^{\pi(d)}{}_{u}\, \varepsilon_{\bar a b\cdots d}$$

where π is a permutation of $(a, b, \ldots, d)$ and $\operatorname{sign}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is an even permutation} \\ -1 & \text{o.w.} \end{cases}$

[13.19 con't] Freeze the Einstein summations for a moment. Then LHS has $n!$ terms. We will show

$$\text{sign}(\pi)\, \epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{\pi(a)}{}_{r} \cdots T^{\pi(d)}{}_{u}\, \varepsilon_{\bar a b \ldots d} \tag{1}$$

$$= \epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{a}{}_{r} \cdots T^{d}{}_{u}\, \varepsilon_{\bar a b\ldots d}$$

for each π. That will show that LHS = RHS.

Since every permutation π can be generated from a series of pairwise swaps, it suffices to show that (1) holds for every pairwise swap between members of $(a, \ldots, d)$. There are 2 kinds of swaps. External swaps are ones involving the external superscript "a". Internal swaps are those not involving a.

Case 1. External Swaps

We show (1) holds for a swap of a with d but the argument holds for any other internal superscript besides d. To swap a and d, first a undergoes $(n-1)$ swaps to the right; then d undergoes $(n-2)$ swaps to the left. The total is $2n-3$, an odd number. If π is the swap of a with d, $\text{sign}(\pi) = -1$. So

$$\text{LHS} = -\epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{d}{}_{r} T^{b}{}_{s} \cdots T^{a}{}_{u}\, \varepsilon_{\bar a\ldots d}$$

$$= -\epsilon^{n s\ldots m}\, \delta^{\bar a}_{r}\, T^{\bar a}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar a\ldots d} \qquad \text{[Rename } r \text{ as } n \text{ and } u \text{ as } m\text{]}$$

$$= -(-1)^{2n-3}\, \epsilon^{m s\ldots n}\, \delta^{\bar a}_{r}\, T^{\bar a}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar a\ldots d} \qquad \text{[Swap } n \text{ and } m \text{ in } \epsilon\text{, which is antisymmetric]}$$

$$= \epsilon^{m s\ldots n}\, \delta^{\bar a}_{r}\, T^{d}{}_{n} T^{b}{}_{s} \cdots T^{a}{}_{m}\, \varepsilon_{\bar a\ldots d}$$

$$= \epsilon^{r s\ldots u}\, \delta^{\bar a}_{r}\, T^{d}{}_{u} T^{b}{}_{s} \cdots T^{a}{}_{r}\, \varepsilon_{\bar a\ldots d} \qquad \text{[Rename } m \text{ as } r \text{ and } n \text{ as } u\text{]}$$

$$= \epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{a}{}_{r} T^{b}{}_{s} \cdots T^{d}{}_{u}\, \varepsilon_{\bar a\ldots d} \qquad \text{[Rearrange the numbers } T^{a}{}_{r}, \ldots, T^{d}{}_{u}\text{]}$$

$$= \text{RHS}$$

Case 2. Internal Swaps

We show for a swap of b and d but the argument holds for any 2 internal superscripts. The swap involves $(n-2)$ pairwise swaps to the right followed by $(n-3)$ swaps to the left, for a total of $2n-5$, an odd number of swaps. Thus

$$\text{LHS} = -\epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{a}{}_{r} T^{d}{}_{s} \cdots T^{b}{}_{u}\, \varepsilon_{\bar a b\ldots d}$$

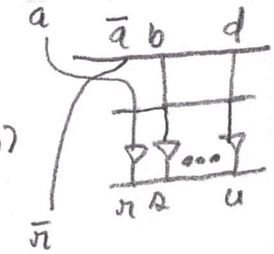
$$= +\epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{a}{}_{r} T^{d}{}_{s} \cdots T^{b}{}_{u}\, \varepsilon_{\bar a d\ldots b} \qquad \text{[Swap b and d in } \varepsilon\text{]}$$

$$= \epsilon^{r\ldots u}\, \delta^{\bar a}_{r}\, T^{a}{}_{r} T^{b}{}_{s} \cdots T^{d}{}_{u}\, \varepsilon_{\bar a b\ldots d} \qquad \text{[Rename b as d and d as b]}$$

$$= \text{RHS}$$

Bud's simpler proof of cases 1 & 2

Note 1. (Juergan Beckmann)



$= n!\, \delta^{\bar a}_{\bar r}\, \varepsilon_{\bar a b \ldots d}\, \epsilon^{rs\ldots u}\, T^{[a}{}_r T^{b}{}_s \cdots T^{d]}{}_u$

$= \delta^{\bar a}_{\bar r}\, \epsilon^{rs\ldots u} \sum_{\pi \in P_{ab\ldots d}} \mathrm{sign}(\pi)\, \varepsilon_{\bar a b \ldots d}\, T^{\pi(a)}{}_r\, T^{\pi(b)}{}_s \cdots T^{\pi(d)}{}_u$

a permutation can be built up from a sequence of pairwise index-swaps.

Case 1: Swaps that involve a

Let $\pi : \begin{matrix} a \to d \\ d \to \bar a \end{matrix}$ and let π fix all other exponents of T

$\epsilon^{rs\ldots u}\, \mathrm{sign}(\pi)\, \varepsilon_{\bar a b \ldots d}\, T^{\pi(a)}{}_r\, T^{\pi(b)}{}_s \cdots T^{\pi(d)}{}_u = \epsilon^{rs\ldots u}\, \mathrm{sign}(\pi)\, \varepsilon_{\bar a b\ldots d}\, T^{d}{}_r T^{b}{}_s \cdots T^{a}{}_u$

$= \epsilon^{us\ldots r}\, \mathrm{sign}(\pi)^2\, \varepsilon_{\bar a b \ldots d}\, T^{a}{}_u T^{b}{}_s \cdots T^{d}{}_r$ swap u and r in ϵ; Swap order of $T^{d}{}_r$ and $T^{a}{}_u$

$= + \epsilon^{rs\ldots u}\, \varepsilon_{\bar a b \ldots d}\, T^{a}{}_r T^{b}{}_s \cdots T^{d}{}_u$ Rename $r \mapsto u$ and $u \mapsto r$

Case 2 Swaps that don't involve a

Let $\pi : \begin{matrix} b \to d \\ d \to b \end{matrix}$ and let π fix all other exponents of T

$\epsilon^{rs\ldots u}\, \mathrm{sign}(\pi)\, \varepsilon_{\bar a b \ldots d}\, T^{\pi(a)}{}_r\, T^{\pi(b)}{}_s \cdots T^{\pi(d)}{}_u = \epsilon^{rs\ldots u}\, \mathrm{sign}(\pi)\, \varepsilon_{\bar a b \ldots d}\, T^{a}{}_r T^{d}{}_s \cdots T^{b}{}_u$

$= \epsilon^{rs\ldots u}\, \mathrm{sign}(\pi)^2\, \varepsilon_{\bar a d \ldots b}\, T^{a}{}_r T^{d}{}_s \cdots T^{b}{}_u$ Swap b & d in ε;

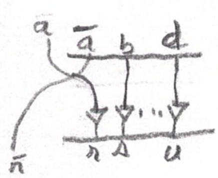
$= + \epsilon^{rs\ldots u}\, \varepsilon_{\bar a b \ldots d}\, T^{a}{}_r T^{b}{}_s \cdots T^{d}{}_u$ Rename $b \mapsto d$ and $d \mapsto b$

Thus $\delta^{\bar a}_{\bar r}\, \epsilon^{rs\ldots u} \sum_{\pi} \mathrm{sign}(\pi)\, \varepsilon_{\bar a b \ldots d}\, T^{\pi(a)}{}_r\, T^{\pi(b)}{}_s \cdots T^{\pi(d)}{}_u$

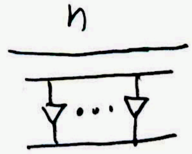
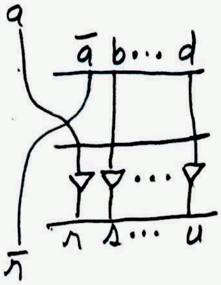
$= \delta^{\bar a}_{\bar r}\, \epsilon^{rs\ldots u} \sum_{\pi} \varepsilon_{\bar a b \ldots d}\, T^{a}{}_r T^{b}{}_s \cdots T^{d}{}_u$

$= n!\, \delta^{\bar a}_{\bar r}\, \epsilon^{rs\ldots u}\, \varepsilon_{\bar a b \ldots d}\, T^{a}{}_r T^{b}{}_s \cdots T^{d}{}_u$

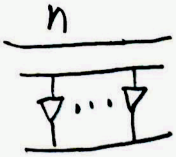
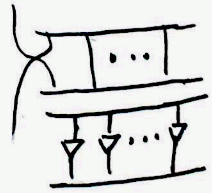
$= n!$

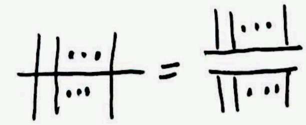


[13.19 con't]. We have shown that (B.) can be rewritten as

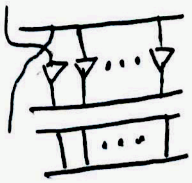
(C)  $\frac{1}{n!}$ 

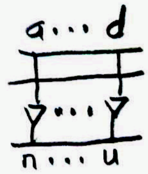
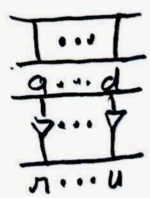
Beckmann claims that by splitting RHS according to Fig 12.18 yields

(D)  $\frac{1}{n!}$ 

Recall Fig 12.18:  This split lacks both $\triangledown$ and externals.

I believe the split from (C) to (D) requires proof. I cannot prove it. I can.
My "claim" on p.1 results in
prove 2 things that generalize Fig 12.18 but not (D).

(D')  $\frac{1}{n!}$  . I can also show

Claim 2:  =  (Note: No externals)

$$LHS = n!\,\epsilon^{r\cdots u}\, T^{[a}{}_{r}\cdots T^{d]}{}_{u}\,\varepsilon_{a\cdots d} = n!\left(\tfrac{1}{n!}\right)\sum_{\pi} \mathrm{sign}(\pi)\,\epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r}\cdots T^{\pi(d)}{}_{u}\,\varepsilon_{a\cdots d}$$

$$= \sum_{\pi} \mathrm{sign}(\pi)\,\epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r}\cdots T^{\pi(d)}{}_{u}\,[\mathrm{sign}(\pi)\,\varepsilon_{\pi(a)\cdots\pi(d)}]$$

$$= \sum_{\pi} \epsilon^{r\cdots u}\, T^{\pi(a)}{}_{r}\cdots T^{\pi(d)}{}_{u}\,\varepsilon_{\pi(a)\cdots\pi(d)} \overset{(*)}{=} \sum_{\pi} \epsilon^{r\cdots u}\, T^{a}{}_{r}\cdots T^{d}{}_{u}\,\varepsilon_{a\cdots u}$$

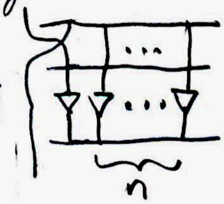
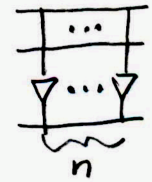
$$= n!\,\epsilon^{r\cdots u}\, T^{a}{}_{r}\cdots T^{d}{}_{u}\,\varepsilon_{a\cdots d}$$

$$= RHS$$

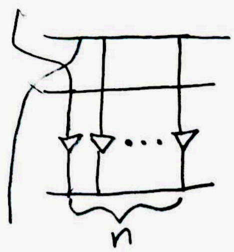
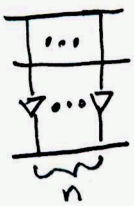
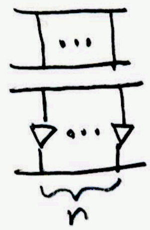
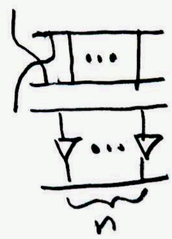
(*) Rename $\pi(a)$ as a, $\pi(b)$ as b, $\ldots$, $\pi(d)$ as d

Proving (D) is much more difficult I believe

[13.19 cont] I can use Fig 12.8 and Claim 2 to reduce (c) $\Rightarrow$ (d) to a different problem,

$(c') \Rightarrow (D')$: Let $(c') = \frac{1}{n!}$  and $(D') =$ 

If $(c') \Rightarrow (D')$, then

$\frac{1}{n!}$  $\overset{c' \Rightarrow D'}{=}$  $\overset{\text{Claim 2}}{=}$  $\overset{\text{Fig 12.8}}{=} \frac{1}{n!}$ 

$\Rightarrow \left[(C) = (D)\right]$

$$(c') = \frac{1}{n!}\,\epsilon^{r\cdots u}\,\delta^{\bar a}_{\bar r}\,T^{[a}{}_{r}\,T^{b}{}_{s}\cdots T^{d]}{}_{u}\,\varepsilon_{\bar a b\cdots d}$$

$$= \frac{1}{n!}\sum_{\pi}\delta^{\bar a}_{\bar r}\,\epsilon^{r\cdots u}\,T^{\pi(a)}{}_{r}\,T^{\pi(b)}{}_{s}\cdots T^{\pi(d)}{}_{u}\,\epsilon_{\bar a b\cdots d}$$

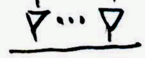
$$= \frac{1}{n!}\sum_{\pi}\delta^{\bar r}_{\bar r}\,\epsilon^{r\cdots u}\,T^{\pi(a)}{}_{r}\,T^{\pi(b)}{}_{s}\cdots T^{\pi(d)}{}_{u}\,\varepsilon_{\bar r b\cdots d}$$

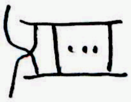
$$= \frac{1}{n!}\,\epsilon^{r\cdots u}\,T^{\pi(a)}{}_{r}\,T^{\pi(b)}{}_{s}\cdots T^{\pi(d)}{}_{u}\,\varepsilon_{\bar a b\cdots d} \qquad \text{(rename } \bar r \text{ to } \bar a\text{)}$$

$$(D')\quad n!\,\epsilon^{r\cdots u}\,T^{[a}{}_{r}\cdots T^{d]}{}_{u}\,\varepsilon_{a\cdots d}$$

$$= n!\,\cancel{\left(\tfrac{1}{n!}\right)}\sum_{\pi}\epsilon^{r\cdots u}\,T^{\pi(a)}{}_{r}\cdots T^{\pi(d)}{}_{u}\,\varepsilon_{a\cdots d}$$

Why are they equal?

Finishing this problem ... the two  cancel, leaving

$(E) = \frac{n}{n!}$  $\overset{\text{Fig 12.8}}{=} \frac{n}{n!}(n-1)!\,\vert = \frac{n!}{n!}\,\vert = I$ $\square$