[13.40] Let V be an n-dimensional vector space, $\mathcal{V} = \mathrm{V} \otimes \mathrm{V}$ the tensor product of V with itself, and $Q^{ab} \in \mathcal{V}$, a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$-tensor over V. Let $Q^{(ab)} = \frac{1}{2}\left(Q^{ab} + Q^{ba}\right)$ be the symmetric part and $Q^{[ab]} = \frac{1}{2}\left(Q^{ab} - Q^{ba}\right)$ be the antisymmetric part. Define $\mathcal{V}_+ = \left\{Q^{(ab)} : Q^{ab} \in \mathcal{V}\right\}$ and $\mathcal{V}_- = \left\{Q^{[ab]} : Q^{ab} \in \mathcal{V}\right\}$. Then

(1) $\mathcal{V}_+$ and $\mathcal{V}_-$ are vector spaces,

(2) $\mathcal{V}_+ \cap \mathcal{V}_- = \{0\}$,

(3) $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$ (i.e., $\mathcal{V} = \left\{v_+ + v_- : v_+ \in \mathcal{V}_+ \text{ and } v_- \in \mathcal{V}_-\right\}$),

(4) dim $\mathcal{V}_+ = \dfrac{n}{2}(n+1)$ and dim $\mathcal{V}_- = \dfrac{n}{2}(n-1)$.

Proof. Penrose only asks us to show (4) and the reader can safely skip to that step now. However, I found (1-3) useful for enhancing my understanding of this topic since Penrose stated these without proof.

Notation. $Q^{ab} = v^a \otimes w^b = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \otimes \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}$ and $Q^{ba} = w^b \otimes v^a$ where $v, w \in \mathrm{V}$.

(1) Let $P^{ab}, Q^{ab} \in \mathcal{V}$, α a scalar, and $R^{ab} = P^{ab} + Q^{ab}$. Then[1]

$$P^{(ab)} + Q^{(ab)} = \frac{1}{2}\left(P^{ab} + P^{ba} + Q^{ab} + Q^{ba}\right) \overset{?}{=} \frac{1}{2}\left(R^{ab} + R^{ba}\right) = R^{(ab)} \in \mathcal{V}_+.$$

So $\mathcal{V}_+$ is closed under +. It is also closed under scalar multiplication:

$$\alpha Q^{(ab)} = \frac{\alpha}{2}\left(Q^{ab} + Q^{ba}\right) = \frac{1}{2}\left[(\alpha Q)^{ab} + (\alpha Q)^{ba}\right] = (\alpha Q)^{ab} \in \mathcal{V}_+.$$

Also,

$$0 = \frac{1}{2}\left(0^{ab} + 0^{ba}\right) = 0^{(ab)} \in \mathcal{V}_+$$

and

$$-\left(Q^{(ab)}\right) = -\frac{1}{2}\left[Q^{ab} - Q^{ba}\right] = \frac{1}{2}\left(Q^{ba} - Q^{ab}\right) = Q^{(ba)} \in \mathcal{V}_+. \; .$$

Since it is a subset of $\mathcal{V}$, the remaining vector space axioms hold. ✔

[1] The proof of the line below is actually quite tricky

(2) Let $Q \in \mathcal{V}_+ \cap \mathcal{V}_-$. Since $Q \in \mathcal{V}_+$, $Q = Q^{(ab)} = \frac{1}{2}\left(Q^{ab} + Q^{ba}\right)$. Since

$$Q \in \mathcal{V}_-,\ Q = Q^{[ab]} = \frac{1}{2}\left(Q^{ab} - Q^{ba}\right) \quad \Rightarrow \quad Q = 0.\ ✔$$

(3) Let $Q \in \mathcal{V}$. Then $Q^{(ab)} \in \mathcal{V}_+$, $Q^{[ab]} \in \mathcal{V}_-$, and

$$Q = \frac{1}{2}\left(Q^{ab} + Q^{ba}\right) + \frac{1}{2}\left(Q^{ab} - Q^{ba}\right) = Q^{(ab)} + Q^{[ab]} \in \mathcal{V}_+ \oplus \mathcal{V}_-.\ ✔$$

(4) Let $\mathscr{B} = \left\{e^1, \cdots, e^n\right\}$ be the basis for V where $e^k = \begin{bmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{bmatrix}$. Set

$$e^{ab} = e^a \otimes e^b = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1_{ab} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

By definition, $\mathscr{B} = \left\{e^{ab}\right\}$ is a basis for $\mathcal{V}$, and it has n^2 terms.

Define $\mathscr{B}_+ = \left\{e^{(ab)} : a \le b\right\}$ and $\mathscr{B}_- = \left\{e^{[ab]} : a \le b\right\}$.

Observe that $e^{(aa)} = e^{aa}$ and $e^{[aa]} = 0$. So, $e^{11}, \cdots, e^{nn} \in \mathscr{B}_+$ and thus $\mathscr{B}_+$ has $\frac{n}{2}\left(n+1\right)$ terms with $a \le b$ and $\mathscr{B}_-$ has $\frac{n}{2}\left(n-1\right)$ terms with $a < b$.

Note: The reason for defining $\mathscr{B}_+$ and $\mathscr{B}_-$ with $a \le b$ is that $e^{(ab)} = e^{(ba)}$ and $e^{[ba]} = -e^{[ab]}$, so terms with $b > a$ are not independent from the others.

Each set $\mathscr{B}_+$ and $\mathscr{B}_-$ is clearly consists of linearly independent vectors, so each constitutes a basis for a subspace of $\mathcal{V}$. Moreover, $\mathscr{B}_+$ clearly is a basis for $\mathcal{V}_+$ and $\mathscr{B}_-$ is a basis for $\mathcal{V}_-$. ✔

An example with n = 2 may clarify part (4). Let $Q^{ab} = v^a \otimes w^b$ where $v^a = \begin{bmatrix} r \\ s \end{bmatrix}$ and $w^b = \begin{bmatrix} t \\ u \end{bmatrix}$. Then $e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathscr{B} = \left\{ e^1, e^2 \right\}$ is a basis for V. Next, $e^{11} = e^1 \otimes e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e^{12} = e^1 \otimes e^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $e^{21} = e^2 \otimes e^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $e^{22} = e^2 \otimes e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\mathscr{B} = \left\{ e^{11}, e^{12}, e^{21}, e^{22} \right\}$ is a basis for $\mathcal{V} = \mathrm{V} \otimes \mathrm{V}$. Observe that $e^{(11)} = \frac{1}{2}\left(e^{11} + e^{11} \right) = e^{11}$ and $e^{(22)} = e^{22}$. These are 2 elements of $\mathscr{B}_+$. Note that $e^{[11]} = 0 = e^{[22]}$ so they do not contribute to $\mathscr{B}_-$. The other term in $\mathscr{B}_+$ is $e^{(12)}$ $\left[\text{which equals } \frac{1}{2}\left(e^{12} + e^{21} \right) = e^{(21)} \right]$. The only term in $\mathscr{B}_-$ is $e^{[12]}$ $\left[\text{which equals } -e^{[21]} \right]$. Thus dim $\mathcal{V}_+ = \frac{n}{2}\left(n+1 \right) =$ 3 and dim $\mathcal{V}_- = \frac{n}{2}\left(n-1 \right) = 1$.