Chapter 12. Manifolds of n Dimensions

Configuration Space

Definition. $\mathcal{R}$ = { (ϕ, ψ, θ) = rotation in $\mathbb{R}^3$, ϕ = roll, ψ = pitch, and θ = yaw } is the **rotation space** of $\mathbb{R}^3$. $\mathcal{C} = \mathbb{R}^3 \times \mathcal{R}$ = { (x, y, z, ϕ, ψ, θ) } is the (6-dimensional) **configuration space** of $\mathbb{R}^3$, where (x, y, z) is position.

Note: (1) $-\pi \leq u, v, w \leq \pi$
(2) $\mathcal{R}$ is a ball of radius π
(3) Antipodal points coincide (e.g., roll of - π = roll of π)

Theorem: $\mathcal{C}$ is non-Euclidean.

Proof: In problem 12.17 I show that $\mathcal{R}$ contains a Mobius Strip ■

Manifolds and Scalar Fields

Definitions (from Chapter 10):

- An ***n*-manifold** $\mathcal{M}$ is a locally n-Euclidean paracompact Hausdorff space.
- A **coordinate patch** is an open subset of $\mathbb{R}^n$.
- We denote an axis of a coordinate patch in $\mathbb{R}^n$ by x^k.
- $\mathcal{M}$ is the gluing of a collection of coordinate patches via infinitely smooth transition functions, say F and f, such that $F(X^1, \cdots, X^n) = f(x^1, \cdots, x^n)$ on overlapping patches.
- A **scalar field** Φ on an n-manifold $\mathcal{M}$ is a C^∞-smooth map Φ: $\mathcal{M} \to \mathbb{R}$ or $\mathcal{M} \to \mathbb{C}$.
- A scalar field Φ acting on a point P $\in \mathcal{M}$ is just a common scalar.
- For a point P = (x^1, …, x^n) of $\mathcal{M}$, Φ (P) = f (x^1, …, x^n) where f is a C^∞-smooth **transition function** for Φ on a coordinate patch containing P.

Vector Fields

Definition. Let $\mathcal{S}$ = Set of scalar fields Φ: $\mathcal{M} \to \mathbb{R}$. A **vector field** on $\mathcal{M}$ is a differential operator

$$\boxed{\xi : \mathcal{S} \to \mathcal{S} : \xi = \sum_{k=1}^{n} \xi^k \frac{\partial}{\partial x^k} = < \xi^1, \dots , \xi^n >} \qquad \text{(i)}$$

where < ξ_1, … , ξ_n > are the **components of** ξ. The components ξ_k are scalar fields on $\mathcal{M}$ (*not generally just constants*), and

$$\left[\xi(\Phi)\right](\mathrm{P}) = \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x^k}\bigg|_{\mathrm{P}} \xi^k(\mathrm{P})$$

Geometric Interpretation of Vector Field ξ:

- ξ is a family of arrows at points P = (x^1, …, x^n) in $\mathcal{M}$ with direction $\sum_{k=1}^{n} \xi^k x^k$ and magnitude $\sqrt{\sum_{k=1}^{n} \left(\xi^k\right)^2}$
- ξ is a differentiation operator, with $\xi(\Phi)$ representing the rate of change of Φ in the direction of the arrows
- $\partial / \partial x^k$ represents rate of change in the direction of the k^{th} coordinate axis
- When a vector field ξ acts on a point P$\in \mathcal{M}$, we refer to ξ as a **vector**
- The collection of vector fields constitutes a vector space.
  - This is true whether we are referring to entire fields or merely vector fields defined at a point
  - In the next theorem we prove this for fields defined at a point in order to clarify the meaning of a field defined at a point

Definition. The **tangent space** $\mathcal{T}_{\mathbf{P}}$ to an *n*-manifold $\mathcal{M}$ at a point P $\in \mathcal{M}$ is the collection of all vectors fields ξ evaluated at P.

Theorem. $\mathcal{T}_{\text{P}}$ is a vector space of dimension *n*.

Proof: We assume the vectors are defined over a scalar field Φ. For other symbols we use Ψ and Ξ for scalars, and ξ, η , and ζ for vectors over $\mathcal{M}$. To show that {ξ} is a vector space we must define addition and scalar multiplication and identity the zero vector, the additive inverses, and the unit scalar.

In order to define the zero vector, we first define the zero scalar. It is the scalar field that sends all points in the manifold to 0:

$\mathbf{0_S}: \mathcal{M} \rightarrow \mathbb{R} : \mathbf{0_S}(\text{P}) = 0 \in \mathbb{R}$

Then the zero vector can be defined in terms of the zero scalar:

$\mathbf{0_V}: \mathcal{S} \rightarrow \mathcal{S}: \mathbf{0_V}(\Phi) = \mathbf{0_S}$

Define the unit scalar to send all points to 1.

$\mathbf{1_S}: \mathcal{M} \rightarrow \mathbb{R} : \mathbf{1_S}(\text{P}) = 1 \in \mathbb{R}$

Define addition, $\xi + \eta$: $\left[(\xi+\eta)(\Phi)\right](\text{P}) \equiv \left[\xi\Phi\right](P) + \left[\eta(\Phi)\right](P)$, sum of 2 reals

Define scalar multiplication, $\Psi\,\xi$: $\left[\Psi\xi(\Phi)\right](\text{P}) = \Psi(\text{P})\left[\xi(\Phi)\right](\text{P})$, product of 2 reals

Define -ξ: $\left[(-\xi)(\Phi)\right](\text{P}) = -\left[\xi(\Phi)\right](\text{P})$

With these definitions it is straight-forward to show that $\mathcal{T}_{\text{P}}$ is a vector space:

Abelian group under +:

(a) $\xi + \eta \in \mathcal{T}_{\text{P}}$ (i.e., is $\mathcal{T}_{\text{P}}$ closed under addition)

(b) $\mathbf{0_V} + \xi = \xi = \xi + \mathbf{0_V}$ (Additive identity)

(c) $\xi + -\xi = \mathbf{0_V}$ (Additive inverse)

(d) $\xi + \eta = \eta + \xi$ (Commutative)

(e) $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$ (Associative)

Vector space scalar multiplication relations:

(f) $\boldsymbol{\Psi}\left(\xi+\eta\right)=\boldsymbol{\Psi}\xi+\boldsymbol{\Psi}\eta$

(g) $\left(\boldsymbol{\Psi}+\boldsymbol{\Xi}\right)\xi=\boldsymbol{\Psi}\xi+\boldsymbol{\Xi}\xi$

(h) $\boldsymbol{\Psi}\left(\boldsymbol{\Xi}\xi\right)=\left(\boldsymbol{\Psi}\boldsymbol{\Xi}\right)\xi$

(i) $\mathbf{1}_s\xi=\xi$: $\quad \left[\left(\mathbf{1}_s\xi\right)\left(\Phi\right)\right](\mathrm{P})=\left[\mathbf{1}_s(\mathrm{P})\right]\left\{\left[\xi\left(\left(\Phi\right)\right)\right](\mathrm{P})\right\}=(1)\left[\xi\left(\left(\Phi\right)\right)\right](\mathrm{P})$

$$=\left[\xi\left(\Phi\right)\right](\mathrm{P})$$

Since $\left\{\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\cdots,\frac{\partial}{\partial x^n}\right\}$ is a basis for $\mathcal{T}_{\mathrm{P}}$, dim ($\mathcal{T}_{\mathrm{P}}$) = n. ■

Theorem. $\xi: S\to S$ is a vector field iff for all scalar fields Φ and Ψ

(a) $\xi(\Phi+\Psi)=\xi(\Phi)+\xi(\Psi)$

(b) $\xi(\Phi\Psi)=\Phi\ \xi(\Psi)+\Psi\ \xi(\Phi)$ (1st x derivative of 2nd + 2nd x derivative of 1st)

(c) $\xi(k)=0$ if k is constant

Proof: If ξ is a vector field, clearly (a – c) are satisfied. We attempt the converse. Suppose ξ satisfies (a-c). Let Φ be a scalar field. We must show that

$$\xi=\sum_{k=1}^{n}\xi^k\frac{\partial}{\partial x^k}\ \text{ or }\ \xi\left(\Phi\right)=\sum_{k=1}^{n}\xi^k\frac{\partial\Phi}{\partial x^k}.$$

$\xi(\Phi^2)=\xi(\Phi\Phi)\overset{(b)}{=}\Phi\,\xi\left(\Phi\right)+\Phi\,\xi\left(\Phi\right)=2\Phi\,\xi\Phi.$

$\xi(\Phi^3)=\xi(\Phi\Phi^2)\overset{(b)}{=}2\Phi^2\,\xi\left(\Phi\right)+\Phi^2\,\xi\left(\Phi\right)=3\Phi^2\,\xi\left(\Phi\right).$

$\vdots$

$\xi(\Phi^k)=k\,\Phi^{k-1}\,\xi(\Phi)=\xi\left(\Phi\right)\frac{d\Phi^k}{d\Phi}.$

Let $a\in\mathbb{R}$ be constant. Then $\xi(a\Phi^k)\overset{(b)}{=}a\,\xi(\Phi^k)+\Phi^k\,\xi(a)\overset{(c)}{=}a\,\xi(\Phi^k)$.

Let $p_t(x)=\sum_{k=0}^{t}a_kx^k$ be a polynomial. Then

$$\xi\left[p_t\left(\Phi\right)\right]=\xi\left[\sum_{k=0}^{t}a_k\Phi^k\right]\overset{(a)}{=}\sum_{k=0}^{t}\xi\left[a_k\Phi^k\right]=\left(\sum_{k=1}^{t}a_k\frac{\mathrm{d}}{\mathrm{d}\Phi}\Phi^k\right)\xi\left(\Phi\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\sum_{k=1}^{t}a_k\Phi^k\right)\xi\Phi=\frac{\mathrm{d}p_t\left(\Phi\right)}{\mathrm{d}\Phi}\xi\left(\Phi\right)$$

If $f\in C^\infty$, then $f(x)=\lim_{k\to\infty}p_k\left(x\right)$ for some sequence of polynomials. So,

$$\xi\left[f\left(\Phi\right)\right]=\lim_{t\to\infty}\xi\left[p_t\left(\Phi\right)\right]=\lim_{t\to\infty}\frac{\mathrm{d}p_t\Phi}{\mathrm{d}\Phi}\xi\left(\Phi\right)=\frac{\mathrm{d}f\left(\Phi\right)}{\mathrm{d}\Phi}\xi\left(\Phi\right).$$

Thus a vector field is a differential operator, namely the rate of change of $f(\Phi)$ in the direction of the arrows ([i.e., of $\xi(\Phi)$.]

I don't see how this implies $\xi=\sum_{k=1}^{n}\xi^k\frac{\partial}{\partial x^k}$ ■

1-Forms and Covector Fields

Definition. Recall the definitions of **components:**

1-form: $\boldsymbol{\alpha} = \sum_{k=1}^{n} \alpha_k dx^k = \left\langle \alpha_1, \cdots, \alpha_n \right\rangle$ where $\alpha_k : \mathcal{S} \to \mathbb{R}$ is C^∞-smooth (ii)

vector field: $\boldsymbol{\xi} = \sum_{k=1}^{n} \xi^k \frac{\partial}{\partial x^k} = \left\langle \xi^1, \cdots, \xi^n \right\rangle$ where $\xi^k : \mathcal{M} \to \mathbb{R}$ is C^∞-smooth

Thus the **dot product** between a 1-form $\boldsymbol{\alpha}$ and a vector field ξ is defined as

$$\boldsymbol{\alpha} \bullet \boldsymbol{\xi} = \left\langle \alpha_1, \cdots, \alpha_n \right\rangle \bullet \left\langle \xi^1, \cdots, \xi^n \right\rangle = \sum_{k=1}^{n} \alpha_k \xi^k$$

Note the meaning of the product of the components:

$$\alpha_k \xi^k : \mathcal{S} \to \mathcal{S} : \alpha_k \xi^k \left(\boldsymbol{\Phi}\right) = \alpha_k \left(\boldsymbol{\Phi}\right) \xi^k \left(\boldsymbol{\Phi}\right),$$

the product of a real number $\alpha_k \left(\boldsymbol{\Phi}\right)$ with a scalar $\xi^k \left(\boldsymbol{\Phi}\right)$.

Definition. Let $\mathcal{V}$ be the space of vector fields on $\mathcal{M}$. A **covector field** is a 1-form

$$\boxed{\boldsymbol{\alpha} = \sum_{k=1}^{n} \alpha_k dx^k : \mathcal{V} \to \mathcal{S} : \boldsymbol{\alpha}(\boldsymbol{\xi}) = \boldsymbol{\alpha} \bullet \boldsymbol{\xi}} \qquad \text{(iii)}$$

Thus any 1-form $\boldsymbol{\alpha}$ can be viewed as a covector by defining $\boldsymbol{\alpha}(\boldsymbol{\xi}) = \boldsymbol{\alpha} \bullet \boldsymbol{\xi}$.
Keep in mind that covectors are forms, not vectors.

Theorem. Covectors are linear. That is, for any 2 vector fields ξ and $\boldsymbol{\eta}$, covector field $\boldsymbol{\alpha}$, and scalar field $\boldsymbol{\Phi}$, we have

Proof: $\boldsymbol{\alpha}\left(\xi + \boldsymbol{\eta}\right) = \boldsymbol{\alpha} \bullet \left(\xi + \boldsymbol{\eta}\right) = \boldsymbol{\alpha} \bullet \xi + \boldsymbol{\alpha} \bullet \boldsymbol{\eta} = \boldsymbol{\alpha}\left(\xi\right) + \boldsymbol{\alpha}\left(\boldsymbol{\eta}\right)$

$$\boldsymbol{\alpha}\left(\boldsymbol{\Phi}\xi\right) = \boldsymbol{\alpha} \bullet \left(\boldsymbol{\Phi}\xi\right) + \boldsymbol{\Phi}\left(\boldsymbol{\alpha} \bullet \xi\right) = \boldsymbol{\Phi}\,\boldsymbol{\alpha}\left(\xi\right) \quad ■$$

Note: By $\boldsymbol{\Phi}\xi$ we simply mean they are multiplied. That is, if $\boldsymbol{\Psi} \in \mathcal{S}$ and $P \in \mathcal{M}$, then [($\boldsymbol{\Phi}$ ξ) ($\boldsymbol{\Psi}$)] (P) = $\boldsymbol{\Phi}$(P) [(ξ ($\boldsymbol{\Psi}$)] (P)

Definition. The **Exterior Derivative** is the 1-form $d = \frac{\partial}{\partial x^1} dx^1 + \cdots + \frac{\partial}{\partial x^n} dx^n$ (iv)

So

$$d\boldsymbol{\Phi} = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n = \left\langle \frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n} \right\rangle \qquad \text{(v)}$$

on a coordinate patch $\left(x^1, \cdots, x^n\right)$. d$\boldsymbol{\Phi}$ is called the **gradient 1-form**.

Theorem. ξ ($\boldsymbol{\Phi}$) = $d\boldsymbol{\Phi}$ • ξ = d $\boldsymbol{\Phi}$(ξ) (vi)

Proof: $d\boldsymbol{\Phi}$ is a 1-form so it can be regarded as a covector:

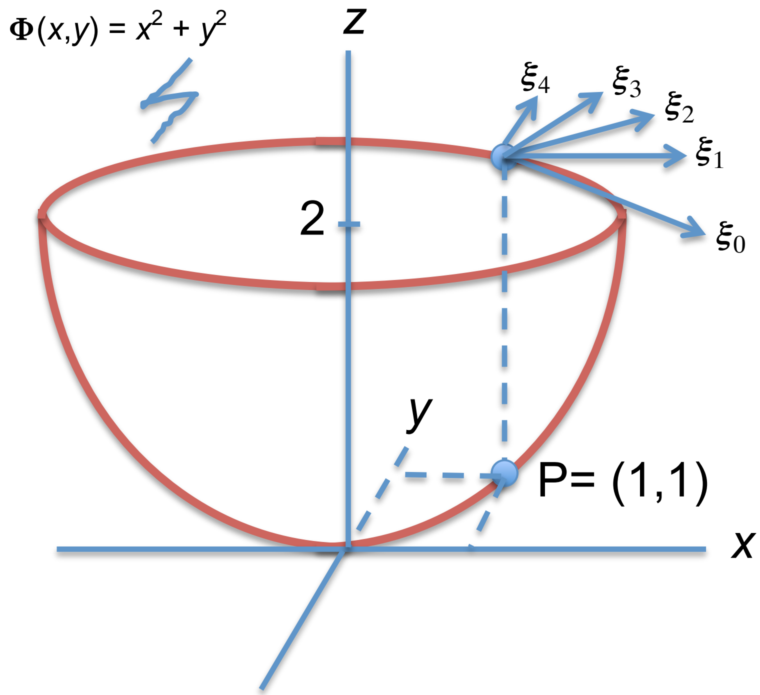
$$d\,\boldsymbol{\Phi}(\xi) = d\boldsymbol{\Phi} \bullet \xi = \left\langle \frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n} \right\rangle \bullet \left\langle \xi^1, \cdots, \xi^n \right\rangle = \sum_{k=1}^{n} \xi^k \frac{\partial f}{\partial x^k}$$

$$= \left(\sum_{k=1}^{n} \xi^k \frac{\partial}{\partial x^k} \right) \left(\boldsymbol{\Phi}\right) = \xi\,(\boldsymbol{\Phi}) \quad ■$$

Example: A manifold, scalar field, vector field, and exterior derivative

Let manifold $\mathcal{M}$ be the xy-plane and scalar $\Phi : \mathbb{R}^2 \to \mathbb{R} : \Phi(x, y) = x^2 + y^2$

We develop vectors, from ξ_0 tangential to the surface of constant $\Phi = 2$ to ξ_4 pointing in the direction of maximum rate of increase of Φ.



A vector has the form $\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$.

$\xi(\Phi)$ represents the rate of change of Φ in the direction of ξ. Define vectors

$\xi_0 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$	$\xi_0(\Phi) = 2x - 2y$	$[\xi_0(\Phi)](\mathrm{P}) = 0$
$\xi_1 \approx 1.3\frac{\partial}{\partial x} - .54\frac{\partial}{\partial y}$	$\xi_1(\Phi) = 2.6x - 1.1y$	$[\xi_1(\Phi)](\mathrm{P}) = 1.5$
$\xi_2 \approx 1.4\frac{\partial}{\partial x}$	$\xi_2(\Phi) = 2.8x$	$[\xi_2\Phi](\mathrm{P}) = 2.8$
$\xi_3 \approx 1.3\frac{\partial}{\partial x} + .54\frac{\partial}{\partial y}$	$\xi_3(\Phi) = 2.6x + 1.1y$	$[\xi_3\Phi](\mathrm{P}) = 3.7$
$\xi_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$	$\xi_4(\Phi) = 2x + 2y$	$[\xi_4\Phi](\mathrm{P}) = 4$

The exterior derivative is $d = \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy$ so $d\Phi = 2x\,dx + 2y\,dy = \langle 2x, 2y\rangle$ and $d\Phi \bullet \xi_k = \langle 2x, 2y\rangle \bullet \langle a_k, b_k\rangle = 2a_k x + 2b_k y$ where $\xi_k = \langle a_k, b_k\rangle$. But $\xi_k(\Phi) = 2a_k x + 2b_k y$ also, so we have that $\boxed{\xi_k(\Phi) = d(\Phi) \bullet \xi_k}$.

Finally, $d\Phi : \mathcal{V} \to \mathcal{S} : d\Phi(\xi) \overset{(iii)}{=} d\Phi \bullet \xi$ so $d\Phi(\xi_k) = d\Phi \bullet \xi_k = \xi_k(\Phi)$.

Note that the gradient $d\Phi$ is an $n - 1 = 2 - 1 = 1$ dimensional *form* and points in the same direction as *vector* ξ_4.

Geometric Interpretation of a Covector Field

- dx^k refers to the (n–1)-plane element spanned by all the coordinate axes except for the x^k-axis
- $\boldsymbol{\alpha} = \sum_{k=1}^{n} \alpha_k \, dx^k = \langle \alpha_1, \cdots, \alpha_n \rangle$ determines the (n–1)-dimensional plane element composed of the directions of vectors ξ that satisfy
  $\boldsymbol{\alpha} \bullet \boldsymbol{\xi} = \langle \alpha_1, \cdots, \alpha_n \rangle \bullet \langle \xi^1, \cdots, \xi^n \rangle = 0$
- By comparison, recall $\partial / \partial x^k$ represents a vector field, the rate of change in the direction of the k^{th} coordinate axis.
- When a covector field $\boldsymbol{\alpha}$ acts on a point $P \in \mathcal{M}$, $\boldsymbol{\alpha}$ is called a **covector**

Theorem. For any 2 covector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, covector field ξ, and scalar field $\boldsymbol{\Phi}$, we have linearity:

(a) $(\boldsymbol{\alpha} + \boldsymbol{\beta}) \bullet \boldsymbol{\xi} = \boldsymbol{\alpha} \bullet \boldsymbol{\xi} + \boldsymbol{\beta} \bullet \boldsymbol{\xi}$
(b) $(\boldsymbol{\Phi}\ \boldsymbol{\alpha}) \bullet \boldsymbol{\xi} = \boldsymbol{\Phi}\ (\boldsymbol{\alpha} \bullet \xi)$

Note: Like $\boldsymbol{\Phi}\ \xi$, above, $\boldsymbol{\Phi} \boldsymbol{\alpha}$ simply means multiplication of $\boldsymbol{\Phi}$ and $\boldsymbol{\alpha}$:

$$[\ (\boldsymbol{\Phi}\ \boldsymbol{\alpha})\ (\xi)\ ]\ (P) = \boldsymbol{\Phi}(P)\ \ [\ (\boldsymbol{\alpha}(\xi)\ ]\ (P)$$

The symmetry of the two previous theorems sets the groundwork for the next theorem, that covectors are the vector space dual to the space of vectors

Since the action of $\boldsymbol{\alpha}$ on ξ is $\boldsymbol{\alpha}(\xi) \overset{(iii)}{=} \boldsymbol{\alpha} \bullet \xi$, the theorem above allows us to define addition and scalar multiplication of covectors:

Definition: $(\boldsymbol{\alpha} + \boldsymbol{\beta})\ (\xi) = \boldsymbol{\alpha}(\ \xi\ ) + \boldsymbol{\beta}(\ \xi\ )$ (Addition of covectors)
$(\boldsymbol{\Phi} \boldsymbol{\alpha})\ (\xi) = \boldsymbol{\Phi} \boldsymbol{\alpha}\ (\xi)$ (Product of scalar with covector)

Theorem. Let $\mathcal{C}$ be the set of covector fields on $\mathcal{M}$. The dual space $\hat{\mathcal{C}}$ of $\mathcal{C}$ is (vector space) isomorphic to $\mathcal{V}$. That is, $\mathcal{C} = \hat{\mathcal{V}}$, and $\hat{\mathcal{C}} = \hat{\hat{\mathcal{V}}} \cong \mathcal{V}$.

Proof. $\hat{\mathcal{C}} = \hat{\hat{\mathcal{V}}} \equiv \left\{ \boldsymbol{\Lambda}_\lambda : \hat{\mathcal{V}} \to \mathcal{S} : \boldsymbol{\Lambda}_\lambda(\boldsymbol{\alpha}) = \boldsymbol{\alpha}\left(\boldsymbol{\Lambda}_\lambda\right) \text{ for } \boldsymbol{\lambda} \in \mathcal{V} \text{ and } \boldsymbol{\alpha} \in \hat{\mathcal{V}} \right\}$.

To make it into a vector space, define +: $\boldsymbol{\Lambda}_\lambda + \boldsymbol{\Lambda}_\nu \equiv \boldsymbol{\Lambda}_{\lambda+\nu}$ (1)

Define scalar multiplications: $\boldsymbol{\Phi}\boldsymbol{\Lambda}_\lambda = \boldsymbol{\Lambda}_{\Phi\lambda}$ (2)

The additive identify is Λ_{0_V}. It is straight-forward to show that $\left(\hat{\hat{\mathcal{V}}}, +\right)$ is an Abelian group and that the required vector space scalar operations are satisfied:

For $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathcal{S}$ and $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathcal{V}$

$$\boldsymbol{\Phi}(\boldsymbol{\Lambda}_\lambda + \boldsymbol{\Lambda}_\nu) = \boldsymbol{\Phi}\boldsymbol{\Lambda}_\lambda + \boldsymbol{\Phi}\boldsymbol{\Lambda}_\nu \quad (3)$$

$$\left(\boldsymbol{\Phi}\boldsymbol{\Psi}\right)(\boldsymbol{\Lambda}_\lambda) = \boldsymbol{\Phi}\left(\boldsymbol{\Psi}\boldsymbol{\Lambda}_\lambda\right) \quad (4)$$

$$\left(\boldsymbol{\Phi} + \boldsymbol{\Psi}\right)\boldsymbol{\Lambda}_\lambda = \boldsymbol{\Phi}\boldsymbol{\Lambda}_\lambda + \boldsymbol{\Psi}\boldsymbol{\Lambda}_\lambda \quad (5)$$

Define

$$f:\hat{\hat{\mathcal{V}}}\rightarrow\mathcal{V}:\ f\left(\Lambda_{\lambda}\right)=\lambda \qquad (6)$$

Clearly *f* is 1 – 1 and onto. *f* is a homomorphism because

$$f\left[\Phi\Lambda_{\lambda}+\Psi\Lambda_{\nu}\right]\overset{(2)}{=}f\left[\Lambda_{\Phi\lambda}+\Lambda_{\Psi\nu}\right]\overset{(1)}{=}f\left(\Lambda_{\Phi\lambda+\Psi\nu}\right)\overset{(6)}{=}\Phi\boldsymbol{\lambda}+\Psi\boldsymbol{\nu}$$

Therefore *f* is a vector space isomorphism. ■

Definition. The **cotangent space** $\mathcal{T}_{\mathbf{P}}^{*}$ to $\mathcal{M}$ at P is the vector space of covectors defined at P.

Geometric Interpretation of Covector at a point P$\in\mathcal{M}$.

- For α at P $\exists\,\Phi$ such that $\boldsymbol{\alpha}=d\Phi$ (viii, stated by Penrose without proof).
- dΦ at P determines an (n-1) dimensional plane element tangential to the (n-1) dimensional surface at P of constant Φ (Problem [12.4])

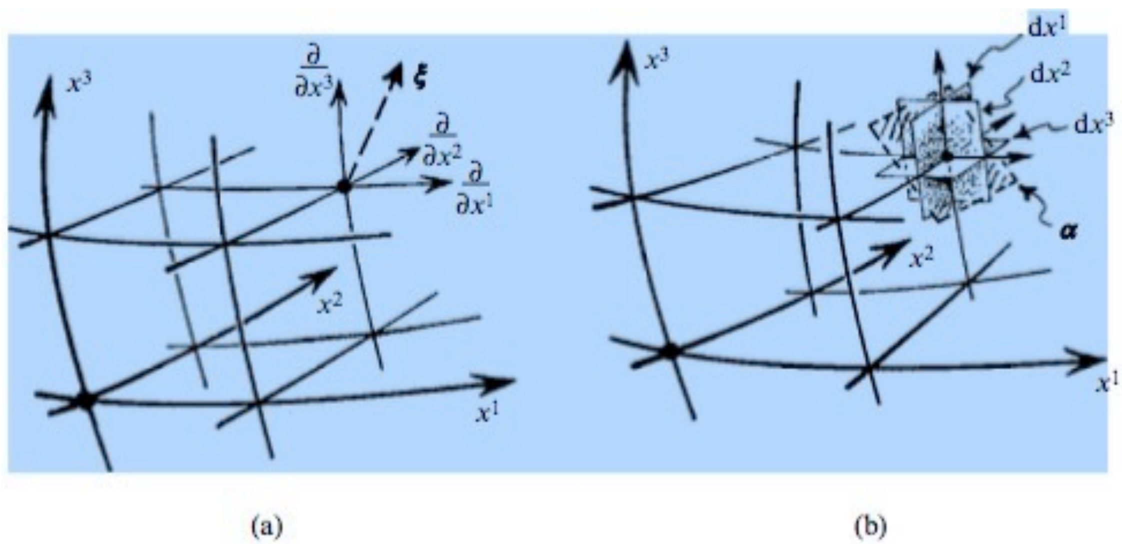


Fig. 12.9 Components in a coordinate patch $(x^1,\ldots,x^n)$ (with $n=3$ here). (a) For a vector (field) $\boldsymbol{\xi}$, these are the coefficients $(\xi^1,\xi^2,\ldots,\xi^n)$ in $\boldsymbol{\xi}=\xi^1\partial/\partial x^1+\xi^2\partial/\partial x^2+\cdots+\xi^n\partial/\partial x^n$, where '$\partial/\partial x^r$' stands for 'rate of change along the rth coordinate axis' (see also Fig. 10.9). (b) For a covector (field) $\boldsymbol{\alpha}$, these are the coefficients $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ in $\boldsymbol{\alpha}=\alpha_1\mathrm{d}x^1+\alpha_2\mathrm{d}x^2+\cdots+\alpha_n\mathrm{d}x^n$, where d$x^r$ stands for 'the gradient of x^r', and refers to the $(n-1)$-plane element spanned by the coordinate axes except for the x^r-axis.

Multivector Fields

We define multivectors in terms of Grassmann algebra wedge products (rather than Clifford algebra) because we need $\left(\frac{\partial}{\partial x^r}\right)^2 = \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^r} = 0$.

Definition. Let $\boldsymbol{\xi} = \sum_{r=1}^{n} \xi^r \frac{\partial}{\partial x^r}$ and $\boldsymbol{\eta} = \sum_{s=1}^{n} \eta^s \frac{\partial}{\partial x^s}$ be vector fields, where ξ^r and η^s are scalar fields on $\mathcal{M}$. Then

$$\begin{aligned}\gamma = \boldsymbol{\xi} \wedge \boldsymbol{\eta} &\equiv \frac{\boldsymbol{\xi}\boldsymbol{\eta} - \boldsymbol{\eta}\boldsymbol{\xi}}{2} = \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{1}{2}\left(\xi^r \eta^s - \xi^s \eta^r\right) \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} \\ &= \sum_{r=1}^{n}\sum_{s=1}^{n} \xi^{[r}\eta^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} \\ &\equiv \sum_{r,s} \gamma^{rs} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}\end{aligned}$$

is a **simple bivector field**, or a **field of 2-plane elements** at points of the Manifold $\mathcal{M}$.

Note 1. The components $\gamma^{rs} = \xi^{[r}\eta^{s]}$ are antisymmetric in r and s ($r \neq s$).

Note 2. $\frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} = -\frac{\partial}{\partial x^s} \wedge \frac{\partial}{\partial x^r}$ ($r \neq s$).

Note 3. γ also equals the un-antisymmetrized sum: $\gamma = \sum_{r=1}^{n}\sum_{s=1}^{n} \xi^r \eta^s \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}$.
See the first theorem in the Forms section, below, for a (parallel) proof for this.

Definition. A **bivector field** is a sum of (q) simple bivector fields:

$\gamma = \sum_{k=1}^{q} \gamma_k = \sum_{k=1}^{q} \boldsymbol{\xi}_k \wedge \boldsymbol{\eta}_k$, where $\boldsymbol{\xi}_k$ and $\boldsymbol{\eta}_k$ are vector fields. We write $\gamma = \left\langle \gamma^{rs} \right\rangle$.

To identify the components γ^{rs}, we expand γ:

$$\gamma = \sum_{k=1}^{q}\left(\sum_{r=1}^{n}\sum_{s=1}^{n} \xi_k^{[r}\eta_k^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}\right) = \sum_{r,s}\left(\sum_{k=1}^{q} \xi_k^{[r}\eta_k^{s]}\right) \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} \equiv \sum_{r,s} \gamma^{rs} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}.$$

So, $\gamma^{rs} = \sum_{k=1}^{q} \xi_k^{[r}\eta_k^{s]} \equiv \sum_{k=1}^{q} \gamma_k^{rs}$, and this implies that γ^{rs} is antisymmetric in r and s: $\gamma^{rs} = -\gamma^{sr}$:

$$\gamma^{rs} = \sum_{k=1}^{q} \xi_k^{[r}\eta_k^{s]} = \sum_{k=1}^{q}\left(\xi_k^r \eta_k^s - \xi_k^s \eta_k^r\right) = -\sum_{k=1}^{q}\left(\xi_k^s \eta_k^r - \xi_k^r \eta_k^s\right) = -\sum_{k=1}^{q} \xi_k^{[s}\eta_k^{r]} = -\gamma^{sr}.$$

Note that a sum of simple bivectors is not necessarily simple because in general $\nexists\ \sigma^r, \tau^s$ such that $\gamma^{rs} = \sum_{k=1}^{q} \xi_k^{[r} \eta_k^{s]} = \sigma^{[r} \tau^{s]}$.

Definition. Let $\zeta = \sum_{t=1}^{n} \zeta^t \frac{\partial}{\partial x^t}$. A **simple trivector field**, or a **field of 3-planes**, is the wedge product of 3 vectors, $\gamma = \xi \wedge \eta \wedge \zeta \equiv \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n} \xi^{[r}\eta^{s}\zeta^{t]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} \wedge \frac{\partial}{\partial x^t}$, with components

$$\gamma^{rst} = \psi^{[r}\eta^{s}\zeta^{t]} = \frac{1}{6}\left(\xi^r\eta^s\zeta^t + \xi^s\eta^t\zeta^r + \cdots + \xi^t\eta^r\zeta^s - \xi^r\eta^t\zeta^s - \xi^t\eta^s\zeta^r - \xi^s\eta^r\zeta^t\right).$$

Definition. A **trivector field** is a sum of (q) simple trivectors fields:

$\gamma = \sum_{k=1}^{q} \xi_k \wedge \eta_k \wedge \zeta_k$. As with bivectors, the components of τ are

$\gamma^{rst} = \sum_{k=1}^{q} \xi_k^{\ [r} \eta_k^{\ s} \zeta_k^{\ t]} = \left(\xi_1^{\ [r}\eta_1^{\ s}\zeta_1^{\ t]} + \cdots + \xi_p^{\ [r}\eta_q^{\ s}\zeta_q^{\ t]}\right)$ and hence { γ^{rst} } is pairwise antisymmetric in r, s & t.

Continuing in this manner we can define ***p*-vector fields**.

Definition. Let $\xi = \sum_{r=1}^{n} \xi^r \frac{\partial}{\partial x^r}$, … , $\eta = \sum_{u=1}^{n} \eta^u \frac{\partial}{\partial x^u}$ be a set of size p of vector fields. A **simple *p*-vector field** is the wedge product of p vector fields:

$$\begin{aligned}
\gamma = \xi \wedge \cdots \wedge \eta &= \sum_{r=1}^{n} \cdots \sum_{u=1}^{n} \xi^{[r} \cdots \eta^{u]} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u} \\
&= \sum_{(r,\cdots,u)\in \mathrm{M}^p} \xi^{[r} \cdots \eta^{u]} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u} \quad \text{(where M} = \{1,2,\cdots,n\}\text{)} \\
&\equiv \sum_{(r,\cdots,u)\in \mathrm{M}^p} \gamma^{r\cdots u} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u}.
\end{aligned}$$

The **components** of γ are $\gamma^{r\cdots u} = \xi^{[r} \cdots \eta^{u]} = \frac{1}{p!} \sum_{\pi \in \mathcal{P}_{r\cdots u}} \mathrm{sign}(\pi) \xi^{\pi(r)} \cdots \eta^{\pi(u)}$ where $\mathcal{P}_{r\cdots u}$ is the set of permutations of (r, … , u). Note that $\gamma^{r\cdots u}$ are antisymmetric in (r, … , u) and that terms having repeating indices may be ignored since $\frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^r} = 0$.

Example 1. We illustrate both the M^p and the permutations notations for a simple 2-vector.

M^p Notation: Let $\gamma = \boldsymbol{\xi} \wedge \boldsymbol{\eta}$ be a simple 2-vector in $\mathbb{R}^2$. Thus p = 2 and n = 2.

Since ξ and $\boldsymbol{\eta}$ are vectors in $\mathbb{R}^2$, $\boldsymbol{\xi} = \xi^1 \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}} + \xi^2 \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^2}}$ and $\boldsymbol{\eta} = \eta^1 \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}} + \eta^2 \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^2}}$.

M = {1,2} and $M^p = M^2 = \{(1,1), (1,2), (2,1), (2,2)\}$. We can ignore (1,1) and (2,2) since $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}} \wedge \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^2}} = 0 = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^2}} \wedge \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}}$. Thus,

$$\gamma = \boldsymbol{\xi} \wedge \boldsymbol{\eta} = \sum_{(r,s)\in M^2} \xi^{[r}\eta^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s} = \xi^{[1}\eta^{2]} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \xi^{[2}\eta^{1]} \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1}. \quad ✔$$

Permutation Notation: $\mathcal{P} = \{\pi_1, \pi_2\}$ where $\pi_1 : \left\{ \begin{array}{c} 1 \to 1 \\ 2 \to 2 \end{array} \right\}$, $\pi_2 : \left\{ \begin{array}{c} 1 \to 2 \\ 2 \to 1 \end{array} \right\}$, and

$sign(\pi_1) = +1$, and $sign(\pi_2) = -1$. So

$$\gamma^{12} = \xi^{[1}\eta^{2]} = \frac{1}{2!}\sum_{\pi\in\mathcal{P}} sign(\pi)\xi^{\pi(1)}\eta^{\pi(2)} = \frac{1}{2!}\left(sign(\pi_1)\xi^{\pi_1(1)}\eta^{\pi_1(2)} + sign(\pi_2)\xi^{\pi_2(1)}\eta^{\pi_2(2)}\right)$$

$$= \tfrac{1}{2}\left(\xi^1\eta^2 - \xi^2\eta^1\right) \text{ and}$$

$$\gamma_{21} = \tfrac{1}{2}\left(\xi^2\eta^1 - \xi^1\eta^2\right) = -\gamma_{12}. \quad ✔$$

Moreover, $\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} = -\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1}$. So

$$\gamma = 2\xi^{[1}\eta^{2]} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} = \left(\xi^1\eta^2 - \xi^2\eta^1\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} = \xi^1\eta^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \xi^2\eta^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1}$$

as expected.

Definition. (p-vector field)

Let {ξ_k, …, η_k} be sets of p vectors, k = 1, …, q. That is, each set has p elements, and each element is a vector field:

$$\boldsymbol{\xi}_k = \sum_{r=1}^{n} \xi_k^r \frac{\partial}{\partial x^r}, \ \dots, \ \boldsymbol{\eta}_k = \sum_{u=1}^{n} \eta_k^u \frac{\partial}{\partial x^u}.$$

For k = 1, 2, …, n let $\boldsymbol{\gamma_k}$ be simple p-vector fields:

$$\gamma_k = \boldsymbol{\xi}_k \wedge \cdots \wedge \boldsymbol{\eta}_k = \sum_{(r,\cdots,u)\in M^p} \xi^{[r} \cdots \eta^{u]} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u}$$

$$\equiv \sum_{(r,\cdots,u)\in M^p} \gamma_k^{r\cdots u} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u} \qquad \text{(where M = } \{1,2,\cdots,n\}\text{)}$$

The components of $\boldsymbol{\gamma_k}$ are

$$\gamma_k^{r\cdots u} = \xi_k^{[r} \cdots \eta_k^{u]} = \frac{1}{p!} \sum_{\pi\in\mathcal{P}_{r\cdots u}} \text{sign}(\pi)\xi_k^{\pi(r)} \cdots \eta_k^{\pi(u)}.$$

$\gamma_k^{r\cdots u}$ are antisymmetric in (r, … , u).

A ***p*-vector field** is a sum of (q) simple p-vectors fields:

$$\boldsymbol{\gamma} = \sum_{k=1}^{q} \boldsymbol{\gamma}_k = \sum_{k=1}^{q} \boldsymbol{\xi}_k \wedge \cdots \wedge \boldsymbol{\eta}_k = \sum_{k=1}^{q} \sum_{(r\cdots u)\in M^p} \boldsymbol{\xi}_k^{[r} \wedge \cdots \wedge \boldsymbol{\eta}_k^{u]} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u}$$

$$= \sum_{(r\cdots u)\in M^p} \left(\sum_{k=1}^{q} \boldsymbol{\xi}_k^{[r} \wedge \cdots \wedge \boldsymbol{\eta}_k^{u]} \right) \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u}$$

$$\equiv \sum_{(r\cdots u)\in M^p} \boldsymbol{\gamma}^{r\cdots u} \frac{\partial}{\partial x^r} \wedge \cdots \wedge \frac{\partial}{\partial x^u}.$$

Thus **the components of γ** are

$$\gamma^{r\cdots u} = \sum_{k=1}^{q} \xi_k^{[r} \cdots \eta_k^{u]} = \frac{1}{p!} \sum_{k=1}^{q} \sum_{\pi\in\mathcal{P}_{r\cdots u}} \operatorname{sign}(\pi)\, \xi_k^{\pi(r)} \cdots \eta_k^{\pi(u)} = \sum_{k=1}^{q} \gamma_k^{r\cdots u},$$

antisymmetric in (r, … , u).

Theorem. Let $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ be p-vector fields. Then $\boldsymbol{\psi} + \boldsymbol{\omega}$ is a p-vector field.

Proof. We use bivectors as an illustration. We need to show that $\boldsymbol{\psi} + \boldsymbol{\omega}$ can be expressed as a sum of simple bivector fields. Let

$$\boldsymbol{\psi} = \sum_{k=1}^{t} \boldsymbol{\psi}_k$$

be a sum of t simple bivectors and

$$\boldsymbol{\omega} = \sum_{k=1}^{t} \boldsymbol{\omega}_k$$

be a sum of q simple bivectors. For each k,

$$\boldsymbol{\psi}_k = \boldsymbol{\xi}_k \wedge \boldsymbol{\eta}_k$$

is the wedge product of two vector fields, where

$$\boldsymbol{\xi}_k = \sum_{r=1}^{n} \xi_k^r \frac{\partial}{\partial x^r} \text{ and } \boldsymbol{\eta}_k = \sum_{s=1}^{n} \eta_k^s \frac{\partial}{\partial x^s}.$$

Therefore

$$\boldsymbol{\psi}_k = \left(\sum_{r=1}^{n} \xi_k^r \frac{\partial}{\partial x^r} \right) \wedge \left(\sum_{s=1}^{n} \eta_k^s \frac{\partial}{\partial x^r} \right) = \sum_{r=1}^{n} \sum_{s=1}^{n} \xi_k^{[r} \eta_k^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}$$

and

$$\boldsymbol{\psi} = \sum_{k=1}^{t} \sum_{r=1}^{n} \sum_{s=1}^{n} \xi_k^{[r} \eta_k^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}.$$

This is the bottom-up expression that represents a sum of simple bivector fields. $\boldsymbol{\omega}$ has a similar expression as a sum of simple bivector fields:

$$\boldsymbol{\omega} = \sum_{k=t+1}^{t+q} \sum_{r=1}^{n} \sum_{s=1}^{n} \xi_k^{[r} \eta_k^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}.$$

Therefore

$$\boldsymbol{\psi} + \boldsymbol{\omega} = \sum_{k=1}^{t+q} \sum_{r=1}^{n} \sum_{s=1}^{n} \xi_k^{[r} \eta_k^{s]} \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial x^s}.$$

This bottom-up expression shows that $\boldsymbol{\psi} + \boldsymbol{\omega}$ can be expressed as a sum of simple bivector fields. ■

Definition. Let $\psi = \sum_{(r,\cdots,u)\in M^p} \psi_{r\cdots u} \frac{\partial}{\partial x^r}\wedge\cdots\wedge\frac{\partial}{\partial x^u}$ be a p-vector field and

$\omega = \sum_{(j,\cdots,m)\in M^q} \omega_{j\cdots m} \frac{\partial}{\partial x^j}\wedge\cdots\wedge\frac{\partial}{\partial x^m}$ a q-vector field. We define their **wedge product**

as $\psi\wedge\omega = \sum_{(r,\cdots,u)\in M^p}\sum_{(j,\cdots,m)\in M^q} \psi_{[r\cdots u}\,\omega_{j\cdots m]} \frac{\partial}{\partial x^r}\wedge\cdots\wedge\frac{\partial}{\partial x^u}\wedge\frac{\partial}{\partial x^j}\wedge\cdots\wedge\frac{\partial}{\partial x^m}$.

Theorem. Let $\psi = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_p$ be a p-vector field and $\omega = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_q$ a q-vector field. Then the $(p + q)$-vector field $\psi \wedge \omega = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_p \wedge \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_q$. That is, antisymmetrization of the two antisymmetrizations equals antisymmetrization of the combined set.

Proof: Analogous to Problem [12.8], proven for p and q forms. ■

p–Forms

Similar to multivectors, we define p-forms in terms of Grassmann wedge products of 1-forms.

Definition. Let $\mathcal{M}$ be an n-manifold and let $\alpha = \sum_{r=1}^{n}\alpha_r dx^r$ and $\beta = \sum_{s=1}^{n}\beta_s dx^s$ be independent 1-forms. A **simple 2-form** is the **wedge product of 1-forms:**

$$\alpha\wedge\beta \equiv \frac{\alpha\beta-\beta\alpha}{2} = \sum_{r=1}^{n}\sum_{s=1}^{n}\tfrac{1}{2}\left(\alpha_r\beta_s - \alpha_s\beta_r\right)\mathrm{d}x^r\wedge\mathrm{d}x^s = \sum_{r=1}^{n}\sum_{s=1}^{n}\alpha_{[r}\beta_{s]}\,\mathrm{d}x^r\wedge\mathrm{d}x^s.$$

The next theorem shows that $\alpha \wedge \beta$ could have been defined without the antisymmetric coefficients.

Theorem. $\alpha \wedge \beta = \sum_{r=1}^{n}\sum_{s=1}^{n}\alpha_r\beta_s\, dx^r\wedge dx^s$

Proof: Fix $r \neq s$. Since $\alpha_{[s}\beta_{r]} = \tfrac{1}{2}\left(\alpha_s\beta_r - \alpha_r\beta_s\right) = -\tfrac{1}{2}\left(\alpha_r\beta_s - \alpha_s\beta_r\right) = -\alpha_{[r}\beta_{s]}$,

$$\begin{aligned}
\alpha_{[r}\beta_{s]}\,dx^r\wedge dx^s + \alpha_{[s}\beta_{r]}\,dx^s\wedge dx^r &= \left(\alpha_{[r}\beta_{s]} - \alpha_{[s}\beta_{r]}\right)dx^r\wedge dx^s \\
&= 2\alpha_{[r}\beta_{s]}\,dx^r\wedge dx^s \\
&= 2\left(\frac{1}{2}\right)\left(\alpha_r\beta_s - \alpha_s\beta_r\right)dx^r\wedge dx^s \\
&= \alpha_r\beta_s\,dx^r\wedge dx^s + \alpha_s\beta_r\,dx^s\wedge dx^r
\end{aligned}$$ ■

Remark 1 In the non-antisymmetric definition, the terms $\alpha_r\,\beta_s$ and $\alpha_s\,\beta_r$ are allowed to have any values whereas in the original definition they each have the average $\alpha_{[r}\,\beta_{s]}$ of the two values. For most purposes this is desired.

Remark 2. $dx^r\wedge dx^s = -dx^s\wedge dx^r$.

Definition. For $k = 1, \ldots, q$ let $\boldsymbol{\phi}^k = \boldsymbol{\alpha}^k \wedge \boldsymbol{\beta}^k$ be simple 2-forms. A **2-form** is a sum

$$\boldsymbol{\phi} = \sum_{k=1}^{q} \boldsymbol{\phi}^k = \sum_{k=1}^{q} \boldsymbol{\alpha}^k \wedge \boldsymbol{\beta}^k$$

of (q) simple 2-forms. Analogous to bivectors,

$$\boldsymbol{\phi} = \sum_{k=1}^{q} \left(\sum_{r=1}^{n} \sum_{s=1}^{n} \alpha^k_{[r} \beta^k_{s]}\, dx^r \wedge dx^s \right) = \sum_{r,s} \left(\sum_{k=1}^{q} \alpha^k_{[r} \beta^k_{s]} \right) dx^r \wedge dx^s \equiv \sum_{r,s} \phi_{rs}\, dx^r \wedge dx^s,$$

so that **components** of ϕ are

$$\phi_{rs} = \sum_{k=1}^{q} \alpha^k_{[r} \beta^k_{s]} = \sum_{k=1}^{q} \frac{1}{2} \left(\alpha^k_r \beta^k_s - \alpha^k_s \beta^k_r \right)$$

and, hence, ϕ_{rs} is antisymmetric in r and s.

Definition. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, …, $\boldsymbol{\delta}$ be p independent 1-forms in an n-manifold $\mathcal{M}$.

$$\boldsymbol{\alpha} = \sum_{r=1}^{n} \alpha_r dx^r \;, \;\ldots\;, \; \boldsymbol{\delta} = \sum_{u=1}^{n} \delta_u\, dx^u \;.$$

A **simple *p*-form** in $\mathcal{M}$ is the wedge product of p 1-forms:

$$\begin{aligned} \boldsymbol{\phi} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \cdots \wedge \boldsymbol{\delta} &= \sum_{r=1}^{n} \sum_{s=1}^{n} \cdots \sum_{u=1}^{n} \alpha_{[r} \beta_s \cdots \delta_{u]}\, dx^r \wedge dx^s \wedge \cdots \wedge dx^u \\ &= \sum_{(r,s,\cdots,u) \in M^p} \alpha_{[r} \beta_s \cdots \delta_{u]}\, dx^r \wedge dx^s \wedge \cdots \wedge dx^u \\ &\equiv \sum_{(r,s,\cdots,u) \in M^p} \phi_{r \cdots u}\, dx^r \wedge dx^s \wedge \cdots \wedge dx^u \quad \text{where } M = \{1, 2, \ldots, n\}. \end{aligned}$$

Thus the **components *of*** ϕ are

$$\phi_{r \cdots u} = \alpha_{[r} \beta_s \cdots \delta_{u]} = \frac{1}{p!} \sum_{\pi \in \mathcal{P}_{r \cdots u}} \mathrm{sign}(\pi)\, \alpha_{\pi(r)} \beta_{\pi(s)} \cdots \delta_{\pi(u)},$$

where $\mathcal{P}_{r \cdots u}$ is the set of permutations of (r, s, …, u).

Observation. $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \cdots \wedge \boldsymbol{\delta}$ determines an (n–p)-plane element at each point P of $\mathcal{M}$, namely the intersection of the (n–1)-plane elements determined by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \cdots, \boldsymbol{\delta}$. (See Figure 12.9 on page 8 of these notes.)

Definition. A ***p*-form** $\boldsymbol{\phi}$ is a sum of (q) simple p-forms $\boldsymbol{\phi}^k$:

Consider p symbols $\alpha - \delta$ and an index k ranging from $1 - q$.

For fixed k, $\boldsymbol{\alpha}^k, \cdots, \boldsymbol{\delta}^k$ is a set of (p) 1-forms:

$$\boldsymbol{\alpha}^k = \sum_{r=1}^{n} \alpha^k_r\, dx^r \;, \;\ldots\;, \; \boldsymbol{\delta}^k = \sum_{u=1}^{n} \delta^k_u\, dx^u$$

$\boldsymbol{\phi}^k = \boldsymbol{\alpha}^k \wedge \boldsymbol{\beta}^k \wedge \cdots \wedge \boldsymbol{\delta}^k = \sum_{(r,\cdots,u) \in M^p} \phi^k_{r \cdots u}\, dx^r \wedge \cdots \wedge dx^u$ is a simple p-form.

The components of $\boldsymbol{\phi}^k$ are

$$\phi^k_{r \cdots u} = \alpha^k_{[r} \beta^k_s \cdots \delta^k_{u]} = \frac{1}{p!} \sum_{\pi \in \mathcal{P}_{r \cdots u}} \mathrm{sign}(\pi)\, \alpha^k_{\pi(r)} \cdots \delta^k_{\pi(u)} \;.$$

The components $\phi^k_{r\cdots u}$ are pairwise antisymmetric in (r, s, … , u).

Now we can define the p-form $\boldsymbol{\phi}$.

$$\begin{aligned}\boldsymbol{\phi} &= \sum_{k=1}^{q} \boldsymbol{\phi}^k = \sum_{k=1}^{q} \boldsymbol{\alpha}^k \wedge \boldsymbol{\beta}^k \wedge \cdots \wedge \boldsymbol{\delta}^k \\ &= \sum_{k=1}^{q} \sum_{(r,\cdots,u)\in \mathrm{M}^p} \alpha^k_{[r}\, \beta^k_s \cdots \delta^k_{u]}\, dx^r \wedge dx^s \wedge \cdots \wedge dx^u \\ &= \sum_{(r,\cdots,u)\in \mathrm{M}^p} \left(\sum_{k=1}^{q} \alpha^k_{[r}\, \beta^k_s \cdots \delta^k_{u]} \right) dx^r \wedge dx^s \wedge \cdots \wedge dx^u \\ &\equiv \sum_{(r,\cdots,u)\in \mathrm{M}^p} \phi_{rs\cdots u}\, dx^r \wedge dx^s \wedge \cdots \wedge dx^u .\end{aligned}$$

By definition, the components of $\boldsymbol{\phi}$ are

$$\phi_{rs\cdots u} = \sum_{k=1}^{q} \alpha^k_{[r}\, \beta^k_s \cdots \delta^k_{u]} = \sum_{k=1}^{q} \phi^k_{r\cdots u} .$$

They are pairwise antisymmetric in (r, s, …, u). Also, as with 2-forms,

$$\phi_{[r\cdots u]} = \phi_{r\cdots u} \Leftrightarrow \sum_{k=1}^{q} \alpha^k_{[[r} \cdots \delta^k_{u]]} = \sum_{k=1}^{q} \alpha^k_{[r} \cdots \delta^k_{u]}$$

and

$$\forall k\ \phi^k_{[r\cdots u]} = \phi^k_{r\cdots u} \Leftrightarrow \alpha^k_{[[r} \cdots \delta^k_{u]]} = \alpha^k_{[r} \cdots \delta^k_{u]} .$$

Definition. Let $\boldsymbol{\phi}$ be a p-form and let $\boldsymbol{\chi}$ be a q-form. Penrose defines their wedge product as $\boldsymbol{\phi} \wedge \boldsymbol{\chi} = \sum_{(r,\cdots,s)\in \mathrm{M}^p} \sum_{(j,\cdots,m)\in \mathrm{M}^q} \phi_{[r\cdots u}\chi_{j\cdots m]}\, dx^r \wedge \cdots \wedge dx^u \wedge dx^j \wedge \cdots \wedge dx^m$.

Theorem. (i) $\phi^k_{rs\cdots u} = \phi^k_{[rs\cdots u]}$ (Problem [12.7])

(ii) If $\boldsymbol{\phi} = \boldsymbol{\alpha} \wedge \cdots \wedge \boldsymbol{\delta}$ is a p-form and $\boldsymbol{\chi} = \boldsymbol{\lambda} \wedge \cdots \wedge \boldsymbol{\nu}$ is a q-form, then the (p+q)-form $\boldsymbol{\phi} \wedge \boldsymbol{\chi}$ satisfies

$$\boldsymbol{\phi} \wedge \boldsymbol{\chi} = \boldsymbol{\alpha} \wedge \cdots \wedge \boldsymbol{\delta} \wedge \boldsymbol{\lambda} \wedge \cdots \wedge \boldsymbol{\nu}$$ (Problem [12.8])

Corollary [11.13]. $\boldsymbol{\phi} \wedge \boldsymbol{\chi} = \begin{cases} \boldsymbol{\chi} \wedge \boldsymbol{\phi} & \text{if either } p \text{ or } q \text{ is even} \\ -\boldsymbol{\chi} \wedge \boldsymbol{\phi} & \text{if both } p \text{ and } q \text{ are odd} \end{cases}$

Proof: $\boldsymbol{\phi} \wedge \boldsymbol{\chi} = \boldsymbol{\alpha} \wedge \cdots \wedge \boldsymbol{\gamma} \wedge \boldsymbol{\lambda} \wedge \cdots \wedge \boldsymbol{\nu} = (-1)^{pq}\, \boldsymbol{\lambda} \wedge \cdots \wedge \boldsymbol{\nu} \wedge \boldsymbol{\alpha} \wedge \cdots \wedge \boldsymbol{\gamma}$

$= (-1)^{pq}\, \boldsymbol{\lambda} \wedge \boldsymbol{\phi}$ ■

Integrals of Forms

Observation. An (n–1)-plane element can be represented in 2 different ways.

(1) It can be determined by a 1-form (covector). While change occurs in the direction determined by the covector, the (n–1) perpendicular directions are held constant.

(2) An (n–1)-vector formed by wedging 1-vectors is an (n–1)-plane element

(3) We think of forms as a type of density suitable for integration, but not so $(n–1)$-vectors. For example, forms contain dx^r but vectors contain $\frac{\partial}{\partial x^r}$.

(4) Forms have a built-in method for dealing with change of variables. Thus explicit representation of change of variables, like the Jacobian, is hidden with forms, greatly simplifying the notation.

Let us begin with the concept of integration of 1-forms. The simplest case is integration over a 1-dimensional curve γ in an n-manifold $\mathcal{M}$.

Definitions.

- A **curve** γ in an n-manifold $\mathcal{M}$ is the image of a piecewise C^1-smooth function $T : [0, 1] \to \gamma \subset \mathcal{M}$: $T(t) = (x^1, x^2, \ldots, x^n)$, $0 \le t \le 1$
- A **curve is simple** if $T(s) \ne T(t)$ except possibly for $T(0) = T(1)$
- A **loop** (or a **closed curve**) is a simple curve with the beginning point equal to the endpoint: $T(0) = T(1)$
- A curve is **rectifiable** means it has arc length.
- Let γ_0 and γ_1 be curves with the same starting point $\gamma_0(0) = \gamma_1(0)$ and ending point $\gamma_0(1) = \gamma_1(1)$. γ_0 is **homotopic** to γ_1 if $\exists$ continuous map $f : [0, 1] \times [0, 1] \to \mathcal{M}$ satisfying
  (1) $f(0,t) = \gamma_0(t)\ \ \forall t$ (i.e., $f(0,\bullet)$ is curve γ_0)
  (2) $f(1,t) = \gamma_1(t)\ \ \forall t$ (i.e., $f(1,\bullet)$ is curve γ_1)
  (3) $f(s,0) = \gamma_0(0) = \gamma_1(0)$ and $f(s,1) = \gamma_0(1) = \gamma_1(1)$ (i.e., the family of curves $f(s,\bullet)$ have the same first point and same last point as γ_0 and γ_1)
- Note: A point P is a degenerate loop $\gamma(t) = P\ \ \forall t \in [0, 1]$. So, **a curve γ is homotopic to a point** P if $\exists$ continuous $f : [0, 1] \times [0, 1] \to \mathcal{M}$ such that $f(0,\bullet) = \gamma$, $f(1,\bullet) = P$, and $f(s, 0) = f(s, 1) = P\ \ \forall\ s \in [0, 1]$
- **A manifold is simply-connected** if each loop is homotopic to a point. Otherwise we say that $\mathcal{M}$ **is multiply-connected**.

Example 2. An open disk about the origin with the origin removed is connected but not simply connected.

Definition. Let $\mathcal{M}$ be an n-manifold, $\gamma = T(\,[0,1]\,)$ a curve in $\mathcal{M}$ from $a = T(0)$ to $b = T(1)$, $(x) = (x^1, x^2, \ldots, x^n)$ a coordinate patch, $f : \mathcal{M} \to \mathbb{R}$ a real-valued function, and $\alpha = f(x)\,dx = \sum_{r=1}^{n} f_r(x^1, \cdots, x^n)\,dx^r$ a 1-form. The **line integral of the 1-form α over a curve γ** is

$$\int_\gamma \alpha = \int_{x=a}^{b} f(x)\,dx \equiv \sum_{r=1}^{n} \int_{t=0}^{1} f_r\left(x^1(t), x^2(t), \cdots, x^n(t)\right) dx^r(t)$$

$$= \sum_{r=1}^{n} \int_{t=0}^{1} \left[f_r\left(x^1(t), x^2(t), \cdots, x^n(t)\right) \frac{dx^r(t)}{dt} \right] dt$$

where the last term is a sum of standard Riemann integrals.

For example, if $\alpha = Fdx + Gdy + Hdz$ then

$$\int_\gamma \alpha = \int_\gamma F\,dx + G\,dy + H\,dz$$
$$= \int_{t=0}^{1} \left[F\big(x(t), y(t), z(t)\big)\frac{dx}{dt} + G\big(x(t), y(t), z(t)\big)\frac{dy}{dt} + H\big(x(t), y(t), z(t)\big)\frac{dz}{dt} \right] dt.$$

Theorem. [12.9] Let $(x) = (x^1, x^2, \ldots, x^n)$ and $(X) = (X^1, X^2, \ldots, X^n)$ be overlapping coordinate patches on an n-manifold $\mathcal{M}$ having transition functions $x = x(X)$ [i.e., $x^k = x^k(X^1, X^2, \ldots, X^n)$ for $1 \le k \le n$] and $X = X(x)$, respectively. Then

$$\int_\gamma \alpha = \int_a^b f(x)\,dx = \int_A^B F(X)\,dX$$

where

$A = X(a) = (\, X^1(a^1, a^2, \ldots, a^n), \ldots, X^n(a^1, a^2, \ldots, a^n)\,)$,
$B = X(b) = (\, X^1(b^1, b^2, \ldots, b^n), \ldots, X^n(b^1, b^2, \ldots, b^n)\,)$,
$F(X) = f[x(X)]$, and $dX = d[x(X)]$.

As mentioned in Observation (4), above, the coordinate-free integral generates a far simpler expression that leads to greater insights.

Example 3. Integration of a 1-form over a 1-manifold

Let $\mathcal{M} = \mathbb{R}$ be the 1-manifold, $f(x) = x^2$, α $f(x)\,dx = x^2\,dx$ the 1-form, $\gamma = T([0,1]) = [2, 4]$, and $X = X(x) = 2x$. Find a, b, A, B, $F(X)$, and dX such that

$$\int_\gamma \alpha = \int_a^b f(x)\,dx = \int_A^B F(X)\,dX.$$

$a = T(0) = x(0) = 2$, $b = T(1) = x(1) = 4$, $A = X(a) = 2a = 4$, $B = X(b) = 2b = 8$,
$F(X) = f\big(x(X)\big) = x(X)^2 = (\tfrac{1}{2}X)^2 = \tfrac{1}{4}X^2$, and $dx = d[x(X)] = d[\tfrac{1}{2}X] = \tfrac{1}{2}dX$.

$$\int_\gamma \alpha = \int_{x=2}^{4} x^2\,dx = \int_{X=4}^{8} (\tfrac{1}{4}X^2)\,(\tfrac{1}{2}dX) = \int_{X=4}^{8} \tfrac{1}{8}X^2\,dX.$$

Check: $\displaystyle\int_{x=2}^{4} x^2\,dx = \frac{56}{3}$ and $\displaystyle\int_{X=4}^{8} \tfrac{1}{8}X^2\,dX = \frac{56}{3}$ ✔

Example 4. Integration of a 1-form over a 2-manifold

Given:

Let $\mathcal{M} = \mathbb{R}^2$ be the 2-manifold, $f(x,y) = x^2$, $g(x,y) = xy$, and
$\alpha = f(x,y)\,dx + g(x,y)\,dy = x^2\,dx + xy\,dy$ the 1-form,
$\gamma = T\big([0,1]\big) = \overline{\mathrm{PQ}}$ is a curve in $\mathcal{M}$, where $\mathrm{P} = (2,1)$ and $\mathrm{Q} = (8,4)$, and

$$\begin{cases} X = X(x,y) = 2x \\ Y = Y(x,y) = x + y. \end{cases}$$

Perform the following steps:

(a) Compute $\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy$.

(b) Find $F(X,Y)\,dX$ and $G(X,Y)\,dY$

(c) Show directly that $\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy = \int_\gamma F\,dX + G dY$.

(a) From the table at the right we can compute the parametric equations of the line from P to Q:

$x = 6t+2$ and $y = 3t+1$. So $\frac{dx}{dt} = 6$ and $\frac{dy}{dt} = 3$.

t	x	y
0	2	1
1	8	4

$$\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy = \int_{t=0}^{1}\left[f(x(t),y(t))\frac{dx(t)}{dt} + g(x(t),y(t))\frac{dy(t)}{dt}\right]dt$$

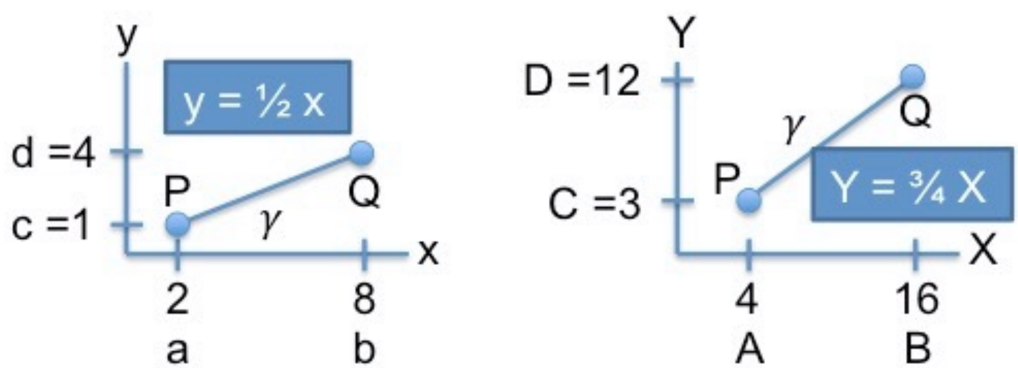
$$= \int_{t=0}^{1}\left[x(t)^2(6) + x(t)\,y(t))(3)\right]dt$$

$$= \int_{t=0}^{1}\left[[2(3t+1)]^2(6) + [2(3t+1)](3t+1)(3)\right]dt$$

$$= \int_{t=0}^{1}\left[24(3t+1)^2 + 6(3t+1)^2\right]dt = \int_{t=0}^{1} 30(3t+1)^2\,dt = 210$$

(b) We use P and Q to represent the endpoints of γ in coordinate-free notation. P = (a, c) = (x(0), y(0)) = (2,1) and Q = (b, d) = (x(1), y(1)) = (8,4) in (x,y)-coordinates.

P = (A, C) = (2a, a + c) = (4, 3) and
Q = (B, D) = (2b, b + d) = (16, 12) in (X,Y)-coordinates.
The (x,y)- and (X,Y)-plots are below, along with their equations.



We next solve for the inverse transition function and its exterior derivative.

$$\begin{cases} x = x(X,Y) = \frac{1}{2}X \\ y = y(X,Y) = Y - x = -\frac{1}{2}X + Y \end{cases} \quad \text{and} \quad \begin{cases} dx = \frac{1}{2}dX \\ dy = -\frac{1}{2}dX + dY \end{cases}.$$

Note that *dy* involves *dX* as well as *dY*, causing the need to group by *dX* terms and *dY* terms in solving for *F* and *G*, below:

$$F(X,Y)\,dX + G(X,Y)\,dY = \alpha$$
$$= f(x,y)\,dx + g(x,y)\,dy = x^2\,dx + xy\,dy$$
$$= x(X,Y)^2\,d[x(X,Y)] + x(X,Y)\,y(X,Y)\,d[y(X,Y)]$$
$$= \tfrac{1}{4}X^2\left(\tfrac{1}{2}dX\right) + \left(\tfrac{1}{2}X\right)\left(-\tfrac{1}{2}X + Y\right)\left(-\tfrac{1}{2}dX + dY\right)$$
$$= \tfrac{1}{8}X^2\,dX + \tfrac{1}{2}X\left[\left(\tfrac{1}{4}X - \tfrac{1}{2}Y\right)dX + \left(-\tfrac{1}{2}X + Y\right)dY\right]$$
$$= \tfrac{1}{8}X^2\,dX + \left[\left(\tfrac{1}{8}X^2 - \tfrac{1}{4}XY\right)dX + \left(-\tfrac{1}{4}X^2 + \tfrac{1}{2}XY\right)dY\right]$$
$$= \left(\tfrac{1}{4}X^2 - \tfrac{1}{4}XY\right)dX + \left(-\tfrac{1}{4}X^2 + \tfrac{1}{2}XY\right)dY.$$

<u>This results in *dX* terms involving *Y* and *dY* terms involving *X*:</u>

$$F(X,Y)\,dX \;=\; \tfrac{1}{4}\left(X^2 - \;\;XY\right)dX, \text{ and}$$
$$G(X,Y)\,dY \;= -\tfrac{1}{4}\left(X^2 - 2XY\right)dY.$$

From the plots, $Y = \frac{3}{4}X$ and $X = \frac{4}{3}Y$. Substituting yields:

$$F(X,Y) = \frac{1}{4}\left(X^2 - \frac{3}{4}X^2\right) = \frac{1}{16}X^2$$
$$G(X,Y) = -\frac{1}{4}\left[\frac{16}{9}Y^2 - 2\left(\frac{4}{3}Y\right)Y\right] = \frac{2}{9}Y^2$$

(c) $$\int_\gamma f\,dx + g\,dy = \int_{x=2}^{8} x^2\,dx + \int_{y=1}^{4} xy\,dy = \int_{x=2}^{8} x^2\,dx + \int_{y=1}^{4} 2y^2\,dy$$
$$= \frac{504}{3} + \frac{2}{3}(63) = 210$$

$$\int_\gamma F\,dX + G\,dY = \tfrac{1}{16}\int_{X=4}^{16} X^2\,dX + \tfrac{2}{9}\int_{Y=3}^{12} Y^2\,dY = \tfrac{1}{48}X^3\Big|_4^{16} + \tfrac{2}{27}Y^3\Big|_3^{12}$$
$$= 84 + 126 = 210. \quad \blacksquare$$

Note. In solving for *F* and *G* we set $FdX + GdY = fdx + gdy$. We did not actually set $F(X,Y) = f\left(x(X,Y), y(X,Y)\right)$ and $G(X,Y) = g\left(x(X,Y), y(X,Y)\right)$ though *in theory* we could have. *In practice* we grouped by *dX* and *dY* and set *F* equal to the factors of *dX* and G equal to the factors of *dY*.

The next simplest integral to consider is integration of a 2-form over a 2-dimensinal surface in $\mathbb{R}^n$.

Definition. Let D be a bounded open subset of $\mathbb{R}^2$ whose boundary is a rectifiable simple closed curve. A **surface** $\mathcal{S}$ in an n-manifold $\mathcal{M}$ is the image of a C^1-smooth 1-1 function: $\mathcal{S} = T\left(\overline{D}\right)$ where

$T:\overline{D} \to \mathcal{M} : T(u,v) = (x^1,\cdots,x^n)$, and where $x^r = x^r(u, v)$ for all r.

Definition. Let $\mathcal{M}$ be an n-manifold, $\mathcal{S}$ a surface in $\mathcal{M}$, (x) a coordinate patch, and

$$\alpha = \sum_{r=1}^{n}\sum_{s=1}^{n} f_{rs}(x^1,\cdots,x^n)\, dx^r \wedge dx^s$$

a 2-form. The **surface integral of the 2-form α over a surface $\mathcal{S}$** is

$$\int_{\mathcal{S}} \alpha \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} \iint_{\overline{D}} f_{rs}\left(x^1(u,v),\cdots,x^n(u,v)\right) dx^r(u,v) \wedge dx^s(u,v)$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} \iint_{D} f_{rs}\left(x^1(u,v),\cdots,x^n(u,v)\right) \frac{\partial\left(x^r, x^s\right)}{\partial\left(u,v\right)}\, du\, dv.$$

Note 1. $dx^r = \dfrac{\partial x^r}{\partial u}du + \dfrac{\partial x^r}{\partial v}dv$ and $dx^s = \dfrac{\partial x^s}{\partial u}du + \dfrac{\partial x^s}{\partial v}dv$. So,

$$dx^r \wedge dy^s = \left(\frac{\partial x^r}{\partial u}du + \frac{\partial x^r}{\partial v}dv\right) \wedge \left(\frac{\partial x^s}{\partial u}du + \frac{\partial x^s}{\partial v}dv\right)$$

$$= \left(\frac{\partial x^r}{\partial u}\frac{\partial x^s}{\partial v} - \frac{\partial x^r}{\partial v}\frac{\partial x^s}{\partial u}\right) du \wedge dv = \frac{\partial(x^r, x^s)}{\partial(u,v)}\, du \wedge dv.$$

Note 2. The RHS is simply a sum of standard Riemann double integrals.

Note 3. $\dfrac{\partial(x^r, x^s)}{\partial(u,v)}$ is a **Jacobian**:

$$J_{rs} = \frac{\partial(x^r, x^s)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x^r}{\partial u} & \dfrac{\partial x^r}{\partial v} \\ \dfrac{\partial x^s}{\partial u} & \dfrac{\partial x^s}{\partial v} \end{vmatrix} = \frac{\partial x^r}{\partial u}\frac{\partial x^s}{\partial v} - \frac{\partial x^r}{\partial v}\frac{\partial x^s}{\partial u}.$$

As before, if (X) is an overlapping coordinate patch, then

$$\alpha = \sum_{r=1}^{n}\sum_{s=1}^{n} f_{rs}(x^1,\cdots,x^n)\, dx^r \wedge dx^s = \sum_{r=1}^{n}\sum_{s=1}^{n} F_{rs}(X^1,\cdots,X^n)\, dX^r \wedge dX^s,$$

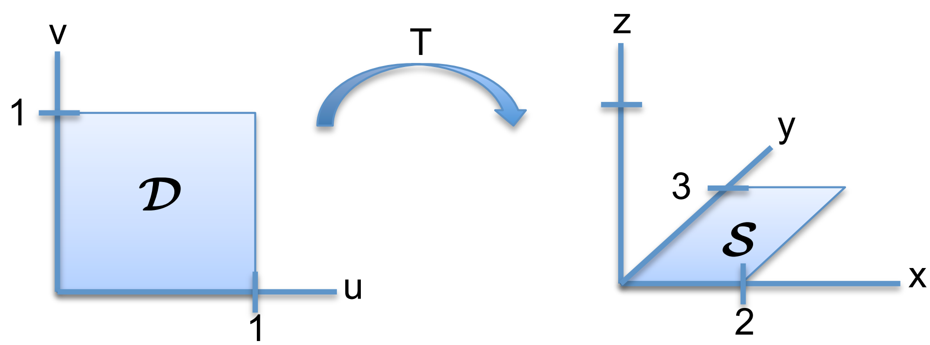
where $\sum_{r=1}^{n}\sum_{s=1}^{n} F_{rs}(X^1,\cdots,X^n)\, dX^r \wedge dX^s$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} f_{rs}\left(x^1(X^1,\cdots,X^n),\cdots,x^n(X^1,\cdots,X^n)\right) dx^r(X^1,\cdots,X^n) \wedge dx^s(X^1,\cdots,X^n).$$

Note 4. Even though $\sum_{r=1}^{n}\sum_{s=1}^{n} F_{rs}\, dx^r \wedge dx^s = \sum_{r=1}^{n}\sum_{s=1}^{n} f_{rs}\, dx^r \wedge dx^s$, it is *not* true that each $F_{rs}(X^1,\cdots,X^n)$ must equal each $f_{rs}\left(x^1(X^1,\cdots,X^n),\cdots,x^n(X^1,\cdots,X^n)\right)$. For example $dx^r \wedge dx^s$ may contain dX^t terms where $t \neq r$ or s.

Example 5. Integration of a 2-form over a 3-manifold:

Let $\alpha = f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx$ be the 2-form where $f(x,y,z) = x^2$, $g(x,y,z) = x + y$, and $h(x,y,z) = xz$.



Let $\mathcal{D}$ be the open unit square as shown above in the *uv*-plane, and let $\bar{\mathcal{D}}$ represent the closed square. Let $\mathcal{M} = \mathbb{R}^3$ be the 3-manifold and let $\mathcal{S}$ be the surface (closed parallelogram) shown above in the xy-plane of $\mathcal{M}$. Find $\int_{\mathcal{S}} \alpha$

T: $\mathcal{D} \to \mathbb{R}^3$: $T(u,v) = (x,y,z) = (2u,3v,0)$ and $\mathcal{S} \equiv T\left(\overline{\mathcal{D}}\right)$. Then

$$\begin{cases} x(u,v) = 2u \\ y(u,v) = 3v \\ z(u,v) = 0 \end{cases} \quad \text{and} \quad \begin{cases} dx(u,v) = 2\,du \\ dy(u,v) = 3\,dv \\ dz(u,v) = 0 \end{cases}$$

$$\int_{\mathcal{S}} \alpha = \int_{\mathcal{S}} f(x,y,z)\,dx\wedge dy + g(x,y,z)\,dy\wedge dz + h(x,y,z)\,dz\wedge dx$$

$$= \iint_{\mathcal{D}} f\big(x(u,v),y(u,v),z(u,v)\big)dx(u,v)\wedge dy(u,v)$$

$$+ g\big(x(u,v),y(u,v),z(u,v)\big)dy(u,v)\wedge dz(u,v)$$

$$+ \ h\big(x(u,v),y(u,v),z(u,v)\big)dz(u,v)\wedge dx(u,v)$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} x(u,v)^2\,dx(u,v)\wedge dy(u,v) + [x(u,v)+y(u,v)]\,dy(u,v)\wedge dz(u,v)$$

$$+ \quad x(u,v)\,z(u,v)\,dz(u,v)\wedge dx(u,v)$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} (2u)^2(2\,du)\wedge(3\,dv) + (2u+3v)(3\,dv)\wedge 0 + (2u)(0)\wedge(2\,du)$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} (4u^2)(6\,du\wedge dv)$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} 24u^2\,du\,dv = 8. \quad \blacksquare$$

Observe the Jacobian: From Note 3, above,

$$dx\wedge dy = \frac{\partial(x,y)}{\partial(u,v)}du\wedge dv = \big((2)(3)-(0)(0)\big)du\wedge dv = 6\,du\wedge dv \qquad ✔$$

Similarly to 2-forms, we can define surface integrals for 3-forms, 4-forms, etc.

Definition. Let $\mathcal{D}$ be a bounded open subset of $\mathbb{R}^p$ whose boundary is a rectifiable simple closed (p-1)-hyperspace. A **(p–1)-surface $\mathcal{S}$** in an n-manifold $\mathcal{M}$, $p \le n$, is the image of a C^1-smooth 1-1 function

$$T : \overline{\mathcal{D}} \to \mathcal{S} : T(u^1,\cdots,u^p) = (x^1,\cdots,x^n).$$

Definition. Let $\mathcal{M}$ be an n-manifold, $\mathcal{S}$ a (p–1)-surface in $\mathcal{M}$, (x) a coordinate patch, and $\alpha = \sum_{r=1}^{n}\cdots\sum_{t=1}^{n} f_{r\cdots t}(x^1,\cdots,x^n)\,dx^r\wedge\cdots\wedge dx^t$ a p-form (i.e., $r \ldots t$ has p terms). The **surface integral of α over the (p–1)-surface $\mathcal{S}$** is $\int_{\mathcal{S}} \alpha$

$$= \sum_{r=1}^{n}\cdots\sum_{t=1}^{n}\int\cdots\int_{\mathcal{D}} f_{r\cdots t}\Big[x^1(u^1,\cdots,u^p),\cdots,x^n(u^1,\cdots,u^p)\Big]\,dx^r(u^1,\cdots,u^p)\wedge\cdots\wedge dx^t(u^1,\cdots,u^p)$$

$$= \sum_{r=1}^{n}\cdots\sum_{t=1}^{n}\int\cdots\int_{\mathcal{D}} f_{r\cdots t}\Big[x^1(u^1,\cdots,u^p),\cdots,x^n(u^1,\cdots,u^p)\Big]\frac{\partial\big(x^r,\cdots,x^t\big)}{\partial\big(u^1,\cdots,u^p\big)}du^1\cdots du^p$$

Note: As in Example 5, an integral of a form over a surface $\mathcal{S}$ in a manifold is computed as a standard Riemann integral over a compact subset $\overline{\mathcal{D}}$ of $\mathbb{R}^n$.

Theorem. Let $\alpha = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} f_{r \cdots t}(x^1, \cdots, x^n)\, dx^r \wedge \cdots \wedge dx^t$ and let (X) be an overlapping coordinate patch with transition functions $X = X(x^1, \cdots, x^n)$ and $x = x(X^1, \cdots, X^n)$. Then there exist functions $F_{r \cdots t}(X^1, \cdots, X^n)$ such that

$$\int_S \alpha = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} \int_S f_{r \cdots t}\left(x^1, \cdots, x^n\right) dx^r \wedge \cdots \wedge dx^t$$

$$\sum_{r=1}^{n} \cdots \sum_{t=1}^{n} \int_S F_{r \cdots t}\left(X^1, \cdots, X^n\right) dX^r \wedge \cdots \wedge dX^t.$$

Proof: Define $F_{r \cdots t}(X^1, \cdots, X^n) = f_{r \cdots t}\left(x^1(X^1, \cdots, X^n), \cdots, x^n(X^1, \cdots, X^n)\right)$.

Then $\alpha = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} F_{r \cdots t}\left(X^1, \cdots, X^n\right) dX^r \wedge \cdots \wedge dX^t$ and

$$\int_S \alpha = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} \int_S F_{r \cdots t}\left(X^1, \cdots, X^n\right) dX^r \wedge \cdots \wedge dX^t. \qquad \blacksquare$$

Note. Refer again to the Note to Example 2. It points out that the above definition of $F_{r \cdots t}$ is correct *in theory* though *in practice* we generate $F_{r \cdots t}$ by equating

$$\sum_{r=1}^{n} \cdots \sum_{t=1}^{n} F_{r \cdots t}\, dX^r \wedge \cdots \wedge dX^t = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} f_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t,$$

grouping the factors of $dX^r \wedge \cdots \wedge dX^t$ on RHS (i.e., $x^s = x^s(X^1, \cdots, X^n)\ \forall s$), and setting $F_{r \cdots t}$ to be the factor of $dX^r \wedge \cdots \wedge dX^t$.

Exterior Derivative

We desire a coordinate-free definition for the exterior derivative, d. In Chapter 10, d was defined in $\mathbb{R}^3$ as $d = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz$ for a coordinate patch (x, y, z). A coordinate-free approach can use properties (a-c) in the next theorem. Penrose gives no proof nor does he mention whether or not the converse to this theorem holds.

Theorem. Let $\mathcal{M}$ be an n-manifold, $0 \le p, q < n$, α and β p-forms, and γ a q-form. The exterior derivative d is the unique operator taking p-forms to (p+1)-forms that satisfies:

(a) $d(\alpha + \beta) = d\alpha + d\beta$

(b) $d(\alpha \wedge \gamma) = (d\alpha) \wedge \gamma + (-1)^p \alpha \wedge d\gamma$

(c) $d^2\alpha \equiv d(d\alpha) = 0$

Note 1. Property (c) implies Grassmann algebra and equality of mixed partials:

In $\mathbb{R}^2$, $0 = d^2 = \frac{\partial^2}{\partial x^2} dx \wedge dx + \frac{\partial^2}{\partial y^2} dy \wedge dy + \left(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} \right) dx \wedge dy$.

$dx \wedge dx$ = 0 = $dy \wedge dy$ because $\frac{\partial^2}{\partial x^2} \neq 0$. For example, $\frac{\partial^2 x^3}{\partial x^2} = 6x \neq 0$. This is the signature of a Grassmann algebra. Also, since $dx \wedge dy \neq 0$,

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$$

Note 2. Some geometric examples of Property (c) are:

- ✓ The boundary of a closed disk α in $\mathbb{R}^2$ is a circle $\partial\alpha$, which has no boundary ($\partial^2\alpha$)
- ✓ The boundary of a solid sphere α in $\mathbb{R}^3$ is its surface $\partial\alpha$, which has no boundary ($\partial^2\alpha$)

Since Penrose gives no proof of the above theorem, it is helpful to develop a few examples. Property (a) is intuitively satisfied. Property (b) is less intuitive, even though Property (b) is roughly the way a derivative should work ("first times the derivative of the 2nd plus second times the derivative of the first") except for the alternating minus sign to account for moving *d* through α. It is also insightful to have algebraic examples of Property (c) in terms of operator *d*. This is provided in Examples 6 – 7

Example 6: Forms in $\mathbb{R}^2$

Let $\alpha = f(x,y)dx + g(x,y)dy$ (1-form) and $\gamma = h(x,y)$ (0-form). Since α is a 1-form, p = 1 so that $(-1)^p = -1$.

Property (b): Show $d(\alpha \wedge \gamma) = (d\alpha) \wedge \gamma + (-1)^1 \alpha \wedge d\gamma$:

$$\alpha \wedge \gamma = (f\,dx + g\,dy) \wedge h \underset{\text{prdt defn}}{\overset{\text{wedge}}{=}} f\,h\,dx + g\,h\,dy \text{ ,}$$

$$d\gamma = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right) h = h_x dx + h_y dy \text{ , and}$$

$$d\alpha = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right) (f\,dx + g\,dy) = (g_x - f_y)\, dx \wedge dy. \text{ Thus}$$

$$\begin{aligned} d(\alpha \wedge \gamma) &= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right) (f\,h\,dx + g\,h\,dy) \\ &= \left[(g\,h_x + g_x h) - (f\,h_y + f_y h) \right] dx \wedge dy \\ &= \left[(g_x - f_y) h - (f\,h_y - g\,h_x) \right] dx \wedge dy \\ &= \left[(g_x - f_y) h\, dx \wedge dy \right] - \left[(f\,dx + g\,dy) \wedge (h_x\,dx + h_y\,dy) \right] \\ &= (d\alpha) \wedge \gamma - \alpha \wedge d\gamma \qquad ✔ \end{aligned}$$

Property (c): Show $d^2 = 0$:

$$d(d\alpha) = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)\left(g_x - f_y\right) dx \wedge dy = 0 \qquad \blacksquare$$

Example 7: Forms in $\mathbb{R}^4$

Let $\alpha = f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx$, $\gamma = q\,dw$, and

$d = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw\right).$

Since α is a 2-form, $p = 2$ so that $(-1)^p = (-1)^2$.

Property (b): Show $d(\alpha \wedge \gamma) = (d\alpha) \wedge \gamma + (-1)^2 \alpha \wedge d\gamma$:

$$d\alpha = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw\right)\left(f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx\right)$$

$$= \left(g_x + h_y + f_z\right) dx \wedge dy \wedge dz + f_w\, dx \wedge dy \wedge dw + g_w\, dy \wedge dz \wedge dw$$

$$+ h_w\, dz \wedge dx \wedge dw$$

$$d\gamma = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw\right)\left(q\,dw\right)$$

$$= q_x\, dx \wedge dw + q_y\, dy \wedge dw + q_z\, dz \wedge dw$$

$$\alpha \wedge \gamma = \left(f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx\right) \wedge q\,dw$$

$$= f\,q\, dx \wedge dy \wedge dw + g\,q\, dy \wedge dz \wedge dw + hq\, dz \wedge dx \wedge dw$$

So,

$$d(\alpha \wedge \gamma) = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw\right)$$

$$\left(f\,q\, dx \wedge dy \wedge dw + g\,q\, dy \wedge dz \wedge dw + hq\, dz \wedge dx \wedge dw\right)$$

$$= dx\frac{\partial}{\partial x}\left(g\,q\, dy \wedge dz \wedge dw\right) + dy\frac{\partial}{\partial y}\left(hq\, dz \wedge dx \wedge dw\right)$$

$$+ dz\frac{\partial}{\partial z}\left(f\,q\, dx \wedge dy \wedge dw\right)$$

$$= \frac{\partial}{\partial x}\left(g\,q\, dx \wedge dy \wedge dz \wedge dw\right) + \frac{\partial}{\partial y}\left(hq\, dy \wedge dz \wedge dx \wedge dw\right)$$

$$+ \frac{\partial}{\partial z}\left(f\,q\, dz \wedge dx \wedge dy \wedge dw\right)$$

$$= \left(g_x q + g q_x + h_y q + h q_y + f_z q + f q_z\right) dx \wedge dy \wedge dz \wedge dw$$

$$= \left[\left(g_x q + h_y q + f_z q\right) + \left(f q_z + g q_x + h q_y\right)\right] dx \wedge dy \wedge dz \wedge dw$$

$$= \left\{ \left[\left(g_x\, q + h_y\, q + f_z\, q \right) + 0 \right] + \left[f\, q_z + g\, q_x + h\, q_y \right] \right\} dx \wedge dy \wedge dz \wedge dw$$

$$= \left(\left(g_x + h_y + f_z \right) dx \wedge dy \wedge dz + \begin{bmatrix} f_w\; dx \wedge dy \wedge dw + g_w\; dy \wedge dz \wedge dw \\ +h_w\; dz \wedge dx \wedge dw \end{bmatrix} \right) \wedge q\, dw$$

$$+ \left(f\, dx \wedge dy + g\, dy \wedge dz + h\, dz \wedge dx \right) \wedge \left(q_x\, dx \wedge dw + q_y\, dy \wedge dw + q_z\, dz \wedge dw \right)$$

$$= (d\alpha) \wedge \gamma + (-1)^2\, \alpha \wedge d\gamma$$ ✔

Property (b): Show the reverse expression, $d\left(\gamma \wedge \alpha\right) = \left(d\gamma\right) \wedge \alpha + \left(-1\right)^1 \gamma \wedge d\alpha$:
Note p = 1 because γ is a 1-form. Also, we already have expressions, above, for α, γ, and $d\gamma$.

$$\left(d\gamma\right) \wedge \alpha = \left(q_x dx \wedge dw + q_y dy \wedge dw + q_z dz \wedge dw\right) \wedge \left(f\, dx \wedge dy + g\, dy \wedge dz + h\, dz \wedge dx\right)$$

$$= \left(q_x g + q_y h + q_z f\right) dx \wedge dy \wedge dz \wedge dw$$

$$\gamma \wedge d\alpha = \left(q\, dw\right) \wedge \begin{bmatrix} \left(g_x + h_y + f_z\right) dx \wedge dy \wedge dz \\ +f_w\, dx \wedge dy \wedge dw + g_w\, dy \wedge dz \wedge dw + h_w dz \wedge dx \wedge dw \end{bmatrix}$$

$$= \left(-1\right)^1 q \left(g_x + h_y + f_z\right) dx \wedge dy \wedge dz \wedge dw$$

$$\left(-1\right)^1 \gamma \wedge d\alpha = q\left(g_x + h_y + f_z\right) dx \wedge dy \wedge dz \wedge dw$$

$$\gamma \wedge \alpha = q\, dw \wedge \left(f\, dx \wedge dy + g\, dy \wedge dz + h\, dz \wedge dx\right)$$

$$= q\, f\, dw \wedge dx \wedge dy + q\, g\, dw \wedge dy \wedge dz + q\, h\, dw \wedge dz \wedge dx$$

$$d\left(\gamma \wedge \alpha\right) = \left(q_z f + q\, f_z\right) dz \wedge dw \wedge dx \wedge dy + \left(q_x g + q\, g_x\right) dx \wedge dw \wedge dy \wedge dz$$

$$+ \left(q_y h + q\, h_y\right) dy \wedge dw \wedge dz \wedge dx$$

$$= \left[\left(q_x g + q_y h + q_z f\right) + q\left(g_x + h_y + f_z\right)\right] dx \wedge dy \wedge dz \wedge dw$$

$$= \left(d\gamma\right) \wedge \alpha + \left(-1\right)^1 \gamma \wedge d\alpha$$ ✔

Property (c): Show d^2 = 0:

$$d(d\alpha) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz + \frac{\partial}{\partial w} dw \right) \begin{pmatrix} \left(g_x + h_y + f_z\right) dx \wedge dy \wedge dz \\ +f_w\; dx \wedge dy \wedge dw + g_w\; dy \wedge dz \wedge dw \\ +h_w\; dz \wedge dx \wedge dw \end{pmatrix}$$

$$= \left(g_{xw} + h_{yw} + f_{zw} - g_{wx} - h_{wy} - f_{wz}\right) dx \wedge dy \wedge dz \wedge dw$$

$$= \left[\left(g_{xw} - g_{wx}\right) + \left(h_{yw} - h_{wy}\right) + \left(f_{zw} - f_{wz}\right)\right]$$

$$= 0$$ ■

Note. We have reinforced that d^2 = 0 is due to the mixed partials being equal.

Theorem (Problem 12.11]. Given $A(x, y)$ and $B(x, y)$, using only the coordinate-free properties show that $d(Adx+Bdy)=\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)dx\wedge dy$ (a - c).

Proof.

$$d(A\,dx+B\,dy)\overset{(a)}{=}d(A\,dx)+d(B\,dy)\underset{\text{definition}}{\overset{\text{wedge prdt}}{=}}d(A\wedge dx)+d(B\wedge dy)$$

$$\overset{(b)}{=}dA\wedge dx-\cancel{A\wedge d(dy)}+dB\wedge dy-\cancel{B\wedge d(dy)}$$

$$\overset{(c)}{=}dA\wedge dx+dB\wedge dy$$

$$\underset{\text{Rule}}{\overset{\text{Chain}}{=}}\left(\frac{\partial A}{\partial x}dx+\frac{\partial A}{\partial y}dy\right)\wedge dx+\left(\frac{\partial B}{\partial x}dx+\frac{\partial B}{\partial y}dy\right)\wedge dx$$

$$=\left(\frac{\partial B}{dx}-\frac{\partial A}{\partial y}\right)dx\wedge dy \qquad \blacksquare$$

Theorem (Problem [12.13] Poincare Lemma for $p = 1$ in $\mathbb{R}$). Let $\beta = A(x, y)\,dx + B(x, y)\,dy$ be a 1-form such that $d\beta = 0$. Show there exists a scalar field $\Phi:\mathbb{R}^2\to\mathbb{R}$ such that locally $\beta = d\Phi$.

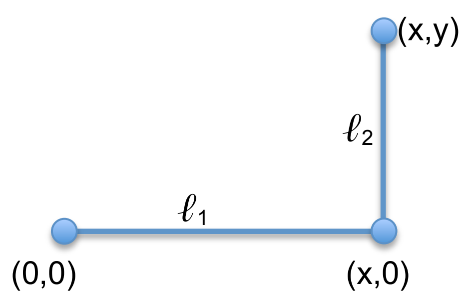
Proof.

$$0=d\beta=d(A\,dx+B\,dy)\overset{(a)}{=}d(A\,dx)+d(B\,dy)=d(A\wedge dx)+d(B\wedge dy)$$

$$\overset{(b)}{=}dA\wedge dx-A\wedge\cancel{d(dx)}+dB\wedge dy-B\wedge\cancel{d(dy)}\overset{(c)}{=}dA\wedge dx+dB\wedge dy$$

$$\underset{\text{Rule}}{\overset{\text{Chain}}{=}}\left(\frac{\partial A}{\partial x}dx+\frac{\partial A}{\partial y}dy\right)\wedge dx+\left(\frac{\partial B}{\partial x}dx+\frac{\partial B}{\partial y}dy\right)\wedge dy=\left(\frac{\partial B}{dx}-\frac{\partial A}{\partial y}\right)dx\wedge dy$$

$$\Rightarrow (i)\quad \frac{\partial B(x,y)}{\partial x}=\frac{\partial A(x,y)}{\partial y}$$



Without loss of generality, let's choose our local point to be (0,0), and assume the point (x,y) is in an open connected neighborhood of (0,0) so that we can join them with the lines ℓ_1 and ℓ_2 as shown.

Define $\Phi(x,y)\equiv\int_{l_1\cup l_2}A(x,y)+B(x,y)=\int_0^x A(t,0)\,dt+\int_0^y B(x,t)\,dt.$ That is, we integrate from (0,0) to (x,y) using A along ℓ_1 and B along ℓ_2.

Restricted to ℓ_2, $B_x(y)\equiv B(x,y)$ is a function of just y. Let $b(y)$ be the antiderivative of $B_x(y)$. That is, $\int_0^y B_x(t)\,dt=b(y)-b(0).$ So,

(*ii*) $\dfrac{\partial\Phi}{\partial y}=\dfrac{\partial}{\partial y}\displaystyle\int_0^x A(t,0)\,dt+\dfrac{\partial}{\partial y}\int_0^y B(x,t)\,dt=\dfrac{\partial}{\partial y}\int_0^y B_x(t)\,dt=\dfrac{\partial}{\partial y}\big[b(y)-b(0)\big]=B_x(y)$

$= B(x,y)$ since $A(t,0)$ is a function of t $\Rightarrow \displaystyle\int_0^x A(t,0)\,dt$ is a function of (just) x.

Restricted to ℓ_1, $A_0(x)\equiv A(x,0)$ is a function of just x. Let $a(x)$ be the antiderivative of $A_0(x)$. That is, $\int_0^x A(t,0)\,dt=\int_0^x A_0(t)\,dt=a(x)-a(0)$. Similarly, restricted to ℓ_2, $A_x(t)\equiv A(x,t)$ is a function of just t. So,

(*iii*) $$\frac{\partial\Phi}{\partial x}=\frac{\partial}{\partial x}\int_0^x A(t,0)\,dt+\frac{\partial}{\partial x}\int_0^y B(x,t)\,dt=\frac{\partial}{\partial x}\big[a(x)-a(0)\big]+\frac{\partial}{\partial x}\int_0^y B(x,t)\,dt$$
$$\underset{\text{of Calculus}}{\overset{\text{Fund Th}}{=}} A_0(x)+\int_0^y \frac{\partial}{\partial x}B(x,t)\,dt\overset{(i)}{=}A(x,0)+\int_0^y\frac{\partial}{\partial t}A(x,t)\,dt=A(x,0)+\int_0^y\frac{\partial}{\partial t}A_x(t)\,dt$$
$$\underset{\text{of Calculus}}{\overset{\text{Fund Th}}{=}} A(x,0)+\big[A_x(y)-A_x(0)\big]=A(x,0)+\big[A(x,y)-A(x,0)\big]=A(x,y).$$

Finally, we have

$$d\Phi=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy\overset{\text{(ii \& iii)}}{=}A(x,y)dx+B(x,y)dy$$
$$=\beta.$$

■

Definition. Let $\boldsymbol{\alpha}$ be a p-form in $\mathbb{R}^n$. Then in a coordinate patch (x^1, …, x^n),

$$\boldsymbol{\alpha}=\sum_{(r,\cdots,t)\in M^P}^{\infty}\alpha_{r\cdots t}\,dx^r\wedge\cdots\wedge dx^t=\sum_{(r,\cdots,t)\in M^P}^{\infty}\alpha_{[r\cdots t]}\,dx^r\wedge\cdots\wedge dx^t$$ where M = {1, 2, …, n}

and $\alpha_{[r\cdots t]}=\dfrac{1}{p!}\displaystyle\sum_{\pi\in\mathcal{P}_{r\cdots t}} sign(\pi)\,\alpha_{\pi(r)\cdots\pi(t)}$, where $\mathcal{P}_{r\,\ldots\,t}$ = Set of permutations of (r, …, t).

The **exterior derivative of the *p*-form** $\boldsymbol{\alpha}$ is the (p+1)-form

$$d\boldsymbol{\alpha}=\sum_{(q,r,\cdots,t)\in M^{P+1}}^{\infty}\frac{\partial}{\partial x^{[q}}\alpha_{r\cdots t]}\,dx^q\wedge dx^r\wedge\cdots\wedge dx^t$$, where

$$\frac{\partial}{\partial x^{[q}}\alpha_{r\cdots t]}=\frac{1}{(p+1)!}\sum_{\pi\in\mathcal{P}_{qr\cdots t}} sign(\pi)\frac{\partial}{\partial^{\pi(q)}}\alpha_{\pi(r)\cdots\pi(t)},$$ and $\mathcal{P}_{qr\ldots t}$ = Set of permutations of (q, r, …, t).

Examples of the summation and permutation notation were given for simple p-vector fields in $\mathbb{R}^2$. We now provide examples to illustrate this notation for forms in $\mathbb{R}^3$. We also provide examples of the exterior derivative notation in $\mathbb{R}^3$.

Example 8.

Let $\boldsymbol{\alpha}$ be a 2-form in $\mathbb{R}^3$. That is, p = 2 and

$$\begin{aligned}
\boldsymbol{\alpha} &= \alpha_{12}\, dx^1 \wedge dx^2 + \alpha_{13}\, dx^1 \wedge dx^3 + \alpha_{23}\, dx^2 \wedge dx^3 \\
&\quad + \alpha_{21}\, dx^2 \wedge dx^1 + \alpha_{31}\, dx^3 \wedge dx^1 + \alpha_{32}\, dx^3 \wedge dx^2 \\
&= \tfrac{1}{2}\left(\alpha_{12} - \alpha_{21}\right) dx^1 \wedge dx^2 + \tfrac{1}{2}\left(\alpha_{13} - \alpha_{31}\right) dx^1 \wedge dx^3 + \tfrac{1}{2}\left(\alpha_{23} - \alpha_{32}\right) dx^2 \wedge dx^3 \\
&\quad + \tfrac{1}{2}\left(\alpha_{21} - \alpha_{12}\right) dx^2 \wedge dx^1 + \tfrac{1}{2}\left(\alpha_{31} - \alpha_{13}\right) dx^3 \wedge dx^1 + \tfrac{1}{2}\left(\alpha_{32} - \alpha_{23}\right) dx^3 \wedge dx^2 \\
&= \alpha_{[12]}\, dx^1 \wedge dx^2 + \alpha_{[13]}\, dx^1 \wedge dx^3 + \alpha_{[23]}\, dx^2 \wedge dx^3 \\
&\quad + \alpha_{[21]}\, dx^2 \wedge dx^1 + \alpha_{[31]}\, dx^3 \wedge dx^1 + \alpha_{[32]}\, dx^3 \wedge dx^2.
\end{aligned}$$

$$\mathcal{P}_{12} = \left\{\pi_1, \pi_2\right\} = \mathcal{P}_{21} \text{ where } \pi_1 : \left\{\begin{array}{c} 1 \to 1 \\ 2 \to 2 \end{array}\right\},\ \pi_2 : \left\{\begin{array}{c} 1 \to 2 \\ 2 \to 1 \end{array}\right\},$$

$sign(\pi_1) = +1$, and $sign(\pi_2) = -1$.

$$\begin{aligned}
\alpha_{[12]} &= \tfrac{1}{2}\left(\alpha_{12} - \alpha_{21}\right) = \frac{1}{2!}\left(sign(\pi_1)\,\alpha_{\pi_1(1)\,\pi_1(2)} + sign(\pi_2)\,\alpha_{\pi_2(1)\,\pi_2(2)}\right) \\
&= \sum_{\pi \in \mathcal{P}_{12}} sign(\pi)\,\alpha_{\pi(1)\,\pi(2)},
\end{aligned}$$

and

$$\alpha_{[21]} = \tfrac{1}{2}\left(\alpha_{21} - \alpha_{12}\right) = \frac{1}{2!}\left(sign(\pi_1)\,\alpha_{\pi_1(2)\,\pi_1(1)} + sign(\pi_2)\,\alpha_{\pi_2(2)\,\pi_2(1)}\right) = \sum_{\pi \in \mathcal{P}_{21}} sign(\pi)\,\alpha_{\pi(1)\,\pi(2)}.$$

Note that $\mathcal{P}_{21} = \{\pi_1, \pi_2\}$ uses the exact same π_1 and π_2 as $\mathcal{P}_{12}$. Similar equations hold for (1,3) and (2,3).

M = {1, 2, …, n} = {1, 2, 3} since we are working with $\mathbb{R}^3$. So,

$\mathrm{M}^p = \mathrm{M}^2 = \left\{1,2,3\right\} \mathrm{x} \left\{1,2,3\right\} = \left\{(1,1),(1,2)\cdots,(3,3)\right\}$. Therefore

$$\begin{aligned}
\sum_{(r,s)\in M^2} \alpha_{rs}\, dx^r \wedge dx^s &= \cancel{\alpha_{11}\, dx^1 \wedge dx^1} + \alpha_{12}\, dx^1 \wedge dx^2 + \alpha_{13}\, dx^1 \wedge dx^3 \\
&\quad + \alpha_{21}\, dx^2 \wedge dx^1 + \cancel{\alpha_{22}\, dx^2 \wedge dx^2} + \alpha_{23}\, dx^2 \wedge dx^3 \\
&\quad + \alpha_{31}\, dx^3 \wedge dx^1 + \alpha_{32}\, dx^3 \wedge dx^2 + \cancel{\alpha_{33}\, dx^3 \wedge dx^3} \\
&= \boldsymbol{\alpha}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\sum_{(r,s)\in M^2} \alpha_{[rs]}\, dx^r \wedge dx^s &= \cancel{\alpha_{[11]}\, dx^1 \wedge dx^1} + \alpha_{[12]}\, dx^1 \wedge dx^2 + \alpha_{[13]}\, dx^1 \wedge dx^3 \\
&+ \alpha_{[21]}\, dx^2 \wedge dx^1 + \cancel{\alpha_{[22]}\, dx^2 \wedge dx^2} + \alpha_{[23]}\, dx^2 \wedge dx^3 \\
&+ \alpha_{[31]}\, dx^3 \wedge dx^1 + \alpha_{[32]}\, dx^3 \wedge dx^2 + \cancel{\alpha_{[33]}\, dx^3 \wedge dx^3} \\
&= \tfrac{1}{2}\left(\alpha_{12} - \alpha_{21}\right) dx^1 \wedge dx^2 + \tfrac{1}{2}\left(\alpha_{13} - \alpha_{31}\right) dx^1 \wedge dx^3 + \tfrac{1}{2}\left(\alpha_{23} - \alpha_{33}\right) dx^2 \wedge dx^3 \\
&+ \tfrac{1}{2}\left(\alpha_{21} - \alpha_{12}\right) dx^2 \wedge dx^1 + \tfrac{1}{2}\left(\alpha_{31} - \alpha_{13}\right) dx^3 \wedge dx^1 + \tfrac{1}{2}\left(\alpha_{32} - \alpha_{23}\right) dx^3 \wedge dx^2 \\
&= \alpha_{12}\, dx^1 \wedge dx^2 + \alpha_{13}\, dx^1 \wedge dx^3 + \alpha_{23}\, dx^2 \wedge dx^3 \\
&+ \alpha_{21}\, dx^2 \wedge dx^1 + \alpha_{31}\, dx^3 \wedge dx^1 + \alpha_{32}\, dx^3 \wedge dx^2 \\
&= \boldsymbol{\alpha}.
\end{aligned}
$$

We next illustrate the formula for $\dfrac{\partial}{\partial x^{[q}} \alpha_{r\cdots t]}$.

$\mathcal{P}_{123} = \{\pi_1, \ldots, \pi_6\}$ where

$$
\pi_1 : \left\{ \begin{array}{c} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{array} \right\},\ \
\pi_2 : \left\{ \begin{array}{c} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{array} \right\},\ \
\pi_3 : \left\{ \begin{array}{c} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{array} \right\},
$$

$$
\pi_4 : \left\{ \begin{array}{c} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{array} \right\},\ \
\pi_5 : \left\{ \begin{array}{c} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{array} \right\},\ \
\pi_6 : \left\{ \begin{array}{c} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{array} \right\} \text{ and}
$$

$$
\begin{aligned}
&sign(\pi_1) = sign(\pi_3) = sign(\pi_5) = +1 \text{ and} \\
&sign(\pi_2) = sign(\pi_4) = sign(\pi_6) = -1.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x^{[1}} \alpha_{23]} &= \frac{1}{3!} \sum_{\pi \in \mathcal{P}_{123}} sign(\pi) \frac{\partial}{\partial x^{\pi(1)}} \alpha_{\pi(2)\pi(3)} \\
&= \frac{1}{6} \left(\begin{array}{l}
sign(\pi_1) \dfrac{\partial}{\partial x^{\pi_1(1)}} \alpha_{\pi_1(2)\pi_1(3)} + sign(\pi_2) \dfrac{\partial}{\partial x^{\pi_2(1)}} \alpha_{\pi_2(2)\pi_2(3)} \\
+ sign(\pi_3) \dfrac{\partial}{\partial x^{\pi_3(1)}} \alpha_{\pi_3(2)\pi_3(3)} + sign(\pi_4) \dfrac{\partial}{\partial x^{\pi_4(1)}} \alpha_{\pi_4(2)\pi_4(3)} \\
+ sign(\pi_5) \dfrac{\partial}{\partial x^{\pi_5(1)}} \alpha_{\pi_5(2)\pi_5(3)} + sign(\pi_6) \dfrac{\partial}{\partial x^{\pi_6(1)}} \alpha_{\pi_6(2)\pi_6(3)}
\end{array} \right) \\
&= \frac{1}{6} \left(\frac{\partial}{\partial x^1} \alpha_{23} - \frac{\partial}{\partial x^1} \alpha_{32} + \frac{\partial}{\partial x^2} \alpha_{31} - \frac{\partial}{\partial x^2} \alpha_{13} + \frac{\partial}{\partial x^3} \alpha_{12} - \frac{\partial}{\partial x^3} \alpha_{21} \right).
\end{aligned}
$$

Observe that $\mathcal{P}_{123} = \mathcal{P}_{132} = \mathcal{P}_{231} = \mathcal{P}_{213} = \mathcal{P}_{312} = \mathcal{P}_{321}$. So, for example,

$$
\begin{aligned}
\frac{\partial}{\partial x^{[1}}\alpha_{32]} &= \frac{1}{3!}\sum_{\pi\in\mathcal{P}_{132}} sign(\pi)\frac{\partial}{\partial^{\pi(1)}}\alpha_{\pi(3)\pi(2)} \\
&= \frac{1}{6}\begin{pmatrix} sign(\pi_1)\frac{\partial}{\partial^{\pi_1(1)}}\alpha_{\pi_1(3)\pi_1(2)} + sign(\pi_2)\frac{\partial}{\partial^{\pi_2(1)}}\alpha_{\pi_2(3)\pi_2(2)} \\ + sign(\pi_3)\frac{\partial}{\partial^{\pi_3(1)}}\alpha_{\pi_3(3)\pi_3(2)} + sign(\pi_4)\frac{\partial}{\partial^{\pi_4(1)}}\alpha_{\pi_4(3)\pi_4(2)} \\ + sign(\pi_5)\frac{\partial}{\partial^{\pi_5(1)}}\alpha_{\pi_5(3)\pi_5(2)} + sign(\pi_6)\frac{\partial}{\partial^{\pi_6(1)}}\alpha_{\pi_6(3)\pi_6(2)} \end{pmatrix} \\
&= \frac{1}{6}\left(\frac{\partial}{\partial x^1}\alpha_{32} - \frac{\partial}{\partial x^1}\alpha_{23} + \frac{\partial}{\partial x^2}\alpha_{13} - \frac{\partial}{\partial x^2}\alpha_{31} + \frac{\partial}{\partial x^3}\alpha_{21} - \frac{\partial}{\partial x^3}\alpha_{12}\right) \\
&= -\frac{\partial}{\partial x^{[1}}\alpha_{23]}.
\end{aligned}
$$

We find that

$$\frac{\partial}{\partial x^{[1}}\alpha_{23]} = \frac{\partial}{\partial x^{[2}}\alpha_{31]} = \frac{\partial}{\partial x^{[3}}\alpha_{12]} = -\frac{\partial}{\partial x^{[1}}\alpha_{32]} = -\frac{\partial}{\partial x^{[2}}\alpha_{13]} = -\frac{\partial}{\partial x^{[3}}\alpha_{21]}.$$

Next, $M^{p+1} = \{1,2,3\}^3 = \{ (1,1,1), (1,1,2), \ldots, (3,3,3) \}$. Thus

$$
\begin{aligned}
&\sum_{(q,r,s)\in M^3}\frac{\partial}{\partial x^{[q}}\alpha_{rs]}\, dx^q \wedge dx^r \wedge dx^s \\
&= \cancel{\frac{\partial}{\partial x^{[1}}\alpha_{11]}\, dx^1\wedge dx^1\wedge dx^1} + \cancel{\frac{\partial}{\partial x^{[1}}\alpha_{12]}\, dx^1\wedge dx^1\wedge dx^2} + \cancel{\frac{\partial}{\partial x^{[1}}\alpha_{13]}\, dx^1\wedge dx^1\wedge dx^3} \\
&+ \cancel{\frac{\partial}{\partial x^{[1}}\alpha_{21]}\, dx^1\wedge dx^2\wedge dx^1} + \cancel{\frac{\partial}{\partial x^{[1}}\alpha_{22]}\, dx^1\wedge dx^2\wedge dx^2} + \frac{\partial}{\partial x^{[1}}\alpha_{23]}\, dx^1\wedge dx^2\wedge dx^3 + \cdots \\
&= \frac{\partial}{\partial x^{[1}}\alpha_{23]}\, dx^1\wedge dx^2\wedge dx^3 + \frac{\partial}{\partial x^{[2}}\alpha_{31]}\, dx^2\wedge dx^3\wedge dx^1 + \frac{\partial}{\partial x^{[3}}\alpha_{12]}\, dx^3\wedge dx^1\wedge dx^2 \\
&+ \frac{\partial}{\partial x^{[1}}\alpha_{32]}\, dx^1\wedge dx^3\wedge dx^2 + \frac{\partial}{\partial x^{[2}}\alpha_{13]}\, dx^2\wedge dx^1\wedge dx^3 + \frac{\partial}{\partial x^{[3}}\alpha_{21]}\, dx^3\wedge dx^2\wedge dx^1 \\
&= \left(\frac{\partial}{\partial x^1}dx^1 + \frac{\partial}{\partial x^2}dx^2 + \frac{\partial}{\partial x^3}dx^3\right)\begin{pmatrix} \alpha_{[12]}\, dx^1\wedge dx^2 + \alpha_{[13]}\, dx^1\wedge dx^3 \\ + \alpha_{[23]}\, dx^2\wedge dx^3 + \alpha_{[21]}\, dx^2\wedge dx^1 \\ + \alpha_{[31]}\, dx^3\wedge dx^1 + \alpha_{[32]}\, dx^3\wedge dx^2 \end{pmatrix} \\
&= (d)(\boldsymbol{\alpha}) = d\boldsymbol{\alpha}. \qquad \blacksquare
\end{aligned}
$$

Definition. Let $\boldsymbol{R}$ be a compact, oriented (p+1)-dimensional region in $\mathbb{R}^n$. Then $\partial\boldsymbol{R}$ denotes the **boundary of *R*** (also compact and oriented).

Theorem (Fundamental Theorem of Exterior Calculus). Let $\boldsymbol{\Phi}$ be a p-form on a compact, oriented (p+1)-dimensional region R in $\mathbb{R}^n$. Then $\int_R d\boldsymbol{\Phi} = \int_{\partial R}\boldsymbol{\Phi}.$

Example 9 (Fundamental Theorem of Calculus). Let $R = [a, b]$. Then $\partial R = b - a$. Let $f(x)\, dx$ be a 1-form on $\mathbb{R}$. Then $\int_R df = \int_a^b f'(x)\,dx = f(b) - f(a) = \int_{\partial R} f.$

Example 10 (Green's Theorem for a rectangle in xy-plane). Let D be a closed oriented rectangle in $\mathbb{R}^2$ with boundary C. Let $\Phi = P\,dx + Q\,dy$ be a 1-form. Then $\int_D d\Phi = \int_C \Phi$.

In vector calculus this was $\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P\,dx + Q\,dy$. Note that

$$d\Phi = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)\wedge\left(P\,dx + Q\,dy\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

Example 11 (Stoke's Theorem). Let S be a smooth oriented surface in $\mathbb{R}^3$ with smooth oriented boundary C. Let $\Phi = P\,dx + Q\,dy + R\,dz$ be a C^1-smooth 1-form on an open set in $\mathbb{R}^3$ that contains S. Then $\int_S d\Phi = \int_C \Phi$.

In vector calculus this was $\iint_S \left(\vec{\nabla} x \vec{F}\right)\bullet d\vec{S} = \oint_C \vec{F}\bullet d\vec{r}$ where $\vec{F} = P\vec{\boldsymbol{i}} + Q\vec{\boldsymbol{j}} + R\vec{\boldsymbol{k}}$,

$d\vec{S} = dy\,dz\,\vec{i} + dx\,dz\,\vec{j} + dx\,dy\,\vec{k}$, and $d\vec{r} = dx\vec{i} + dy\,\vec{j} + dz\,\vec{k}$,. Observe that

$$\begin{aligned} d\Phi &= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\wedge\left(P\,dx + Q\,dy + R\,dz\right) \\ &= \left(R_y - Q_z\right)dy\,dz + \left(P_z - R_x\right)dz\,dx + \left(Q_x - P_y\right)dx\,dy\,, \end{aligned}$$

$$\vec{\nabla}\times\vec{F} = \left(R_y - Q_z\right)\vec{i} + \left(P_z - R_x\right)\vec{j} + \left(Q_x - P_y\right)\vec{k}\,,$$

$$\begin{aligned} \left(\vec{\nabla}\times\vec{F}\right)\bullet d\vec{S} &= \left[\left(R_y - Q_z\right)\vec{i} + \left(P_z - R_x\right)\vec{j} + \left(Q_x - P_y\right)\vec{k}\right]\bullet\left[dy\,dz\;\vec{i} + dx\,dz\;\vec{j} + dx\,dy\;\vec{k}\right] \\ &= d\Phi \end{aligned}$$

and

$$\vec{F}\bullet d\vec{r} = \left(P\vec{\boldsymbol{i}} + Q\vec{\boldsymbol{j}} + R\vec{\boldsymbol{k}}\right)\bullet\left(dx\vec{\boldsymbol{i}} + dy\,\vec{\boldsymbol{j}} + dz\,\vec{\boldsymbol{k}}\right) = \Phi\,.$$

Example 12 (Divergence Theorem for a rectangular parallelepiped). Let V be a solid parallelepiped in $\mathbb{R}^3$ with surface S. Let

$\Phi = R\,dx\wedge dy + P\,dy\wedge dz + Q\,dz\wedge dx$ be a 2-form on an open set in $\mathbb{R}^3$ containing V. Then $\int_V d\Phi = \int_S \Phi$.

In vector calculus this was $\iiint_V \left(\vec{\nabla}\bullet\vec{F}\right)dV = \iint_S \vec{F}\bullet d\vec{S}$ where

$\vec{F} = R\vec{\boldsymbol{i}} + P\,\vec{\boldsymbol{j}} + Q\vec{\boldsymbol{k}}$.

Einstein Summation Notation and Levi-Civita Antisymmetization Quantities

Definition. **Einstein Summation Convention** calls to sum over all pairs of alike upper and lower index variables (but not indexed numbers). The remainder of the notes in this chapter will adhere to this convention.

Example. $\alpha_k \xi^k$ replresents the summation in the RHS: $\boldsymbol{\alpha} \bullet \boldsymbol{\xi} = \sum_{k=1}^{n} \alpha_k \xi^k$.

Example. We can write an n-form in Einstein notation as

$$\boldsymbol{\alpha} = \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t .$$

In non-Einstein notation this is written

$$\boldsymbol{\alpha} = \sum_{r=1}^{n} \cdots \sum_{t=1}^{n} \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t = \sum_{(r,\cdots,t) \in \mathrm{M}^p} \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t$$

Moreover, since any tuple (r, …, t) with a repeating entry yields a zero term, we need only consider non-repeating p=tuples (r, …, t):

$$\boldsymbol{\alpha} = \sum_{\{r,\cdots,t\} \subset \mathrm{M}} \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t .$$

Definition.Let $\mathcal{M}$ be an n-manifold, $n = p + q$, $1 \le p$, $q < n$. Let M = {1, 2, …, n}. Let $\boldsymbol{\psi} = \psi^{u \cdots w} \frac{\partial}{\partial x^u} \wedge \cdots \wedge \frac{\partial}{\partial x^w}$ (Einstein notation) be a q-vector. Recall that $\psi^{u \cdots v} : S \rightarrow \mathrm{S}$ is a differentiation operator . Let $\boldsymbol{\varepsilon} = \varepsilon_{1 \cdots n}\, dx^1 \wedge \cdots \wedge dx^n$ (not Einstein summation) be an n-form. Keep in mind that $\varepsilon_{1 \ldots n}$ is not necessarily a constant. Rather it is a function from $\mathcal{S}$ to $\mathbb{R}$. $\boldsymbol{\varepsilon}$ is called a **volume *n*-form**. Geometrically $\boldsymbol{\varepsilon}$ represents a point P$\in\mathcal{M}$ that is the intersection of n (n – 1)-plane elements as illustrated in Fig 12.9 on page 8.

Definition. Given a point P in a scalar field $\boldsymbol{\Phi}$ over $\mathcal{M}$, $[\varepsilon(\Phi)](\mathrm{P})$ is is called a **volume element for $\mathcal{M}$.** It is the tangent plane to $\boldsymbol{\varepsilon}$ at P.

Definition. Let {r, …, t} = M – {u, …, w}. Let K be any proportionality constant. The **Hodge contraction operator *** converts $\boldsymbol{\psi}$ to a p-form $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = \boldsymbol{\varepsilon} * \boldsymbol{\psi} \equiv K \varepsilon_{r \cdots t u \cdots w} \psi^{u \cdots w}\, dx^r \wedge \cdots \wedge dx^t$$

$$= \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t \quad \text{(uses Einstein Summation)}$$

Note that $\alpha_{r \cdots t} = K \varepsilon_{r \cdots t u \cdots w} \psi^{u \cdots w}$ is antisymmetric since $\varepsilon_{r \cdots t u \cdots w}$ is.

We say that the **contraction** $\alpha_{r \cdots t} = K\, \varepsilon_{r \cdots t u \cdots w} \psi^{u \cdots w}$ converts the q-vector ψ into the p-form $\boldsymbol{\alpha}$. We have "glued" the q upper indices of $\boldsymbol{\psi}$ to q of the lower indices of $\varepsilon_{r \ldots w}$ leaving the p unattached lower indices of $\alpha_{r \ldots t}$.

We say that α and ψ are **Hodge-like duals**. (True Hodge duals are generated in an analogous fashion in Clifford algebras between multivectors and pseudo-multivectors by multiplying by the pseudo-scalar.)

Definition. The **reciprocal volume element of $\boldsymbol{\varepsilon}$** is the vector

$$\in\; = \;\in^{1 \cdots n} \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \text{ where } \in^{1 \cdots n} = \frac{1}{\varepsilon_{1 \cdots n}} .$$

$\boldsymbol{\varepsilon}$ and $\in$ are called the **Levi-Civita Antisymmetrization Quantities**.

Definition. Let L be any proportionality constant. The **inverse Hodge contraction operator **** converts $\boldsymbol{\alpha}$ back to $\boldsymbol{\psi}$:

$$\begin{aligned}\boldsymbol{\psi} = \boldsymbol{\alpha} ** \boldsymbol{\varepsilon} &\equiv L\,\alpha_{r\cdots t} \in^{r\cdots tu\cdots w} \frac{\partial}{\partial x^u} \wedge \cdots \wedge \frac{\partial}{\partial x^w} \\ &= \psi^{u\cdots w} \frac{\partial}{\partial x^u} \wedge \cdots \wedge \frac{\partial}{\partial x^w} \quad \text{(Einstein Summation)}\end{aligned}$$

We say that the contraction $\psi^{u\cdots w} = L\,\alpha_{r\cdots t} \in^{r\cdots tu\cdots w}$ converts the p-form $\boldsymbol{\alpha} = \alpha_{r\cdots t}\, dx^r \wedge \cdots \wedge dx^t$ into the q-vector $\boldsymbol{\psi} = \psi^{u\cdots w} \frac{\partial}{\partial x^u} \wedge \cdots \wedge \frac{\partial}{\partial x^w}$.

Also,

$$\begin{aligned}&\alpha_{r\cdots t} = K\,\varepsilon_{r\cdots tu\cdots w}\,\psi^{u\cdots w} = K\,\varepsilon_{r\cdots tu\cdots w}\,L\,\alpha_{r\cdots t} \in^{r\cdots w} = n!\,K\,L\,\alpha_{r\cdots t} \\ &\Rightarrow \quad K\,L = \frac{1}{n!}\end{aligned}$$

The usual choice is to set $K = L = \frac{1}{\sqrt{n!}}$. So the Hodge-like dual definitions become

$$\begin{aligned}\alpha_{r\cdots t} &= \frac{1}{\sqrt{n!}}\varepsilon_{r\cdots tu\cdots w}\psi^{u\cdots w} \\ \psi^{u\cdots w} &= \frac{1}{\sqrt{n!}}\alpha_{r\cdots t} \in^{r\cdots tu\cdots w}\end{aligned}$$

Theorem. $\boldsymbol{\varepsilon} \bullet \in = n!$

Proof:

$\boldsymbol{\varepsilon} \bullet \in = \varepsilon_{r\cdots w} \in^{r\cdots w} = \sum_{\pi \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi(r)\cdots\pi(w)} \in^{\pi(r)\cdots\pi(w)} = n!(1) = n!$ because there are n! permutations of (1, …, n)

Definition: **Symmetrization** (round brackets):

$$\begin{aligned}&\psi^{(ab)} = \tfrac{1}{2}\left(\psi^{ab} + \psi^{ba}\right) \\ &\psi^{(abc)} = \frac{1}{6}\left(\psi^{abc} + \psi^{bca} + \psi^{cab} + \psi^{bac} + \psi^{cba} + \psi^{acb}\right)\end{aligned}$$

Etc.

Theorem. (Problem [12.16]) Let $\mathcal{M}$ be an n-manifold, $q \le n$, and M = $\{1, 2, \ldots, n\}$.

Let $\boldsymbol{\alpha} = \alpha_{r\cdots t}\, dx^r \wedge \cdots \wedge dx^t$ be a p-form and let $\boldsymbol{\psi} = \psi^{uv\cdots w} \dfrac{\partial}{\partial x^u} \cdots \dfrac{\partial}{\partial x^w}$ be its q-vector Hodge-like dual.

(i) The following conditions for simplicity are equivalent:

a. $\boldsymbol{\alpha}$ is simple iff $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$ for all p-tuples $(r, \ldots, t)$ and $(r', \ldots, t')$ in M^p

b. $\boldsymbol{\psi}$ is simple iff $\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'} = 0$ for all q-tuples $(u, v, \ldots, w)$ and $(u', v', \ldots, w')$ in M^q

c. $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ are both simple iff $\psi^{u\cdots w}\,\alpha_{ws\cdots t} = 0$ for all p-tuples $(w, s \ldots, t)$ and q-tuples $(u, \ldots, w)$.

Corollary. One member of a Hodge-like Dual is simple iff the other member is.

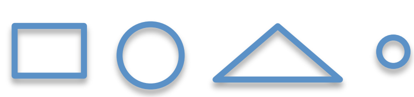
Philosophy Digression. Many problems require the Einstein Summation Convention in order to simplify expressions. Unfortunately this makes such analyses coordinate-dependent. A solution to this dilemma is that no special coordinates are specified, putting the analysis part-way back towards being coordinate-free. To move all the way back, we formalize an abstract index notation.

Definition. **Abstract-Index Notation**: Indices 1, 2, …, n do not refer to dimensions in a coordinate patch but rather are abstract markers for the algebraic equations.

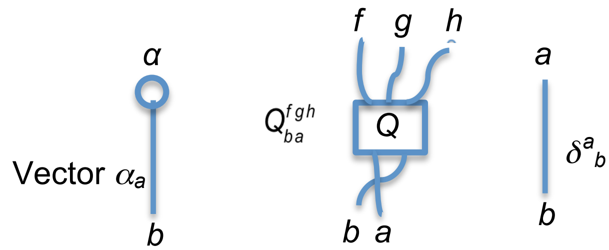
Definition. A **tensor** is an abstract quantity $Q^{f\cdots h}_{a\cdots c}$ with p upper and q lower indices. $\boldsymbol{Q}^{f\cdots h}_{a\cdots c}$ is also called a $\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}$**-valent tensor** or a **tensor of valence** $\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}$.
Q is a multilinear function of p vectors $\boldsymbol{A}, \ldots, \boldsymbol{C}$ and q covectors $\boldsymbol{F}, \ldots, \boldsymbol{H}$ where

$$\boxed{\boldsymbol{Q}(\boldsymbol{A},\cdots,\boldsymbol{C};\boldsymbol{F},\cdots,\boldsymbol{H}) = \boldsymbol{A}^a \cdots \boldsymbol{C}^c\, \boldsymbol{Q}^{f\cdots h}_{a\cdots c}\, \boldsymbol{F}_f \cdots \boldsymbol{H}_h}.$$

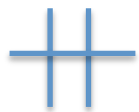
The many tiny subscripts and superscripts make even the combination of Abstract-Index Notation and the Einstein Summation Convention unwieldy. A solution to this is **Diagrammatic Notation:**

- $\boldsymbol{Q}$ is represented by  etc.

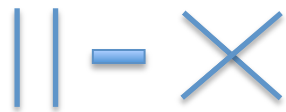
- Vector arms extend upward, Covector arms extend downward

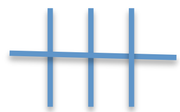


- Lines for contracted variables are vertically connected
- Antisymmetrization is represented by a horizontal bar across index lines



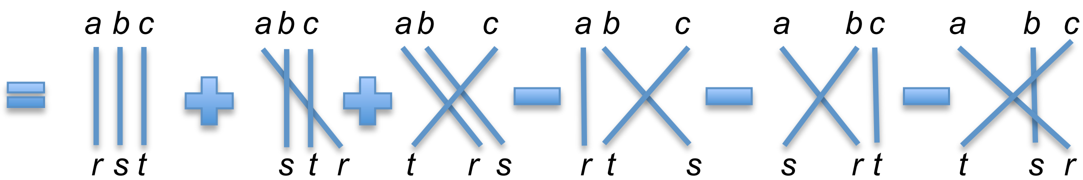
$$= 2!\ \delta^a_{[r}\,\delta^b_{s]} = \left(\delta^a_r\,\delta^b_s - \delta^a_s\,\delta^b_r\right) =$$



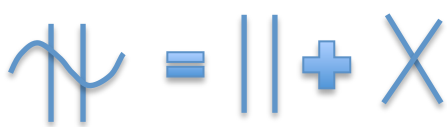


$$= 3!\ \delta^a_{[r}\,\delta^b_s\,\delta^c_{t]}$$

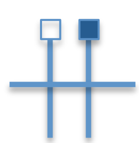
$$= 3!\left(\frac{1}{3!}\right)\left(\delta^a_r\,\delta^b_s\,\delta^c_t + \delta^a_s\,\delta^b_t\,\delta^c_r + \delta^a_t\,\delta^b_r\,\delta^c_s - \delta^a_r\,\delta^b_t\,\delta^c_s - \delta^a_t\,\delta^b_s\,\delta^c_r - \delta^a_s\,\delta^b_r\,\delta^c_t\right)$$



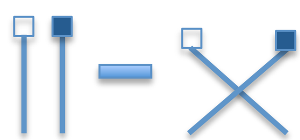
- Symmetrization is represented by a squiggly horizontal bar across index lines

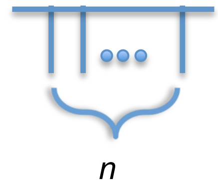


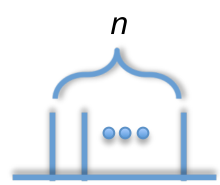
- Example.

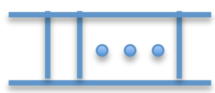


$$= 2\zeta_{[a}\,\xi_{b]} = \left(\zeta_a\,\xi_b - \zeta_b\,\xi_a\right) =$$



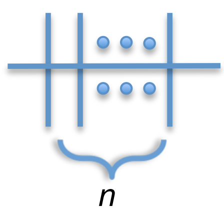
- The volume n-form $\varepsilon_{r\,s\,\ldots\,w}$ is 

- Its Hodge-like dual scalar $\in^{r\,s\,\ldots\,w}$ is 

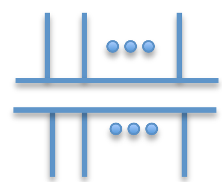
- $n!$ =  $=$ $\in^{r\,s\cdots t}\,\varepsilon_{r\,s\cdots t}$

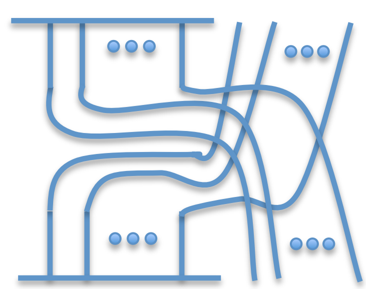
- Multiplication is represented by either horizontal or vertical juxtaposition

Example.

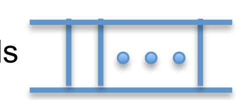


$$= n!\,\delta^{a}_{[r}\cdots\delta^{f}_{w]} \overset{[13.22a]}{=} \in^{a\cdots f}\,\varepsilon_{r\cdots w} =$$

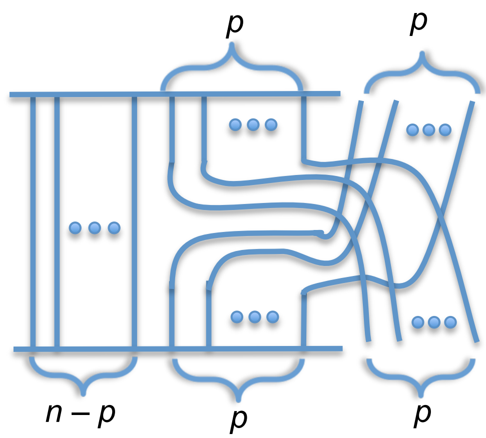


$=$ 

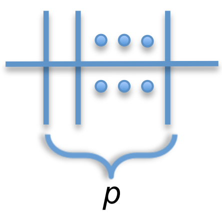
= $n!$ Sign(π) where π is the permutation $\pi:(r,\cdots,w)\mapsto(a,\cdots,f)$

- Note: If (a, …, f) = (r, …, w), then the above equals  $=$ $n!$

- Example.



$$= \varepsilon_{rs\cdots tuv\cdots w} \in^{rs\cdots tde\cdots f} \overset{[13.22b]}{=} (n-p)!\,\varepsilon_{uv\cdots w} \in^{de\cdots f} \equiv (n-p)!\,p!\,\delta^{d}_{[u}\cdots\delta^{f}_{w]}$$

$= (n-p)!$ 

- The book examples and explanations are below.

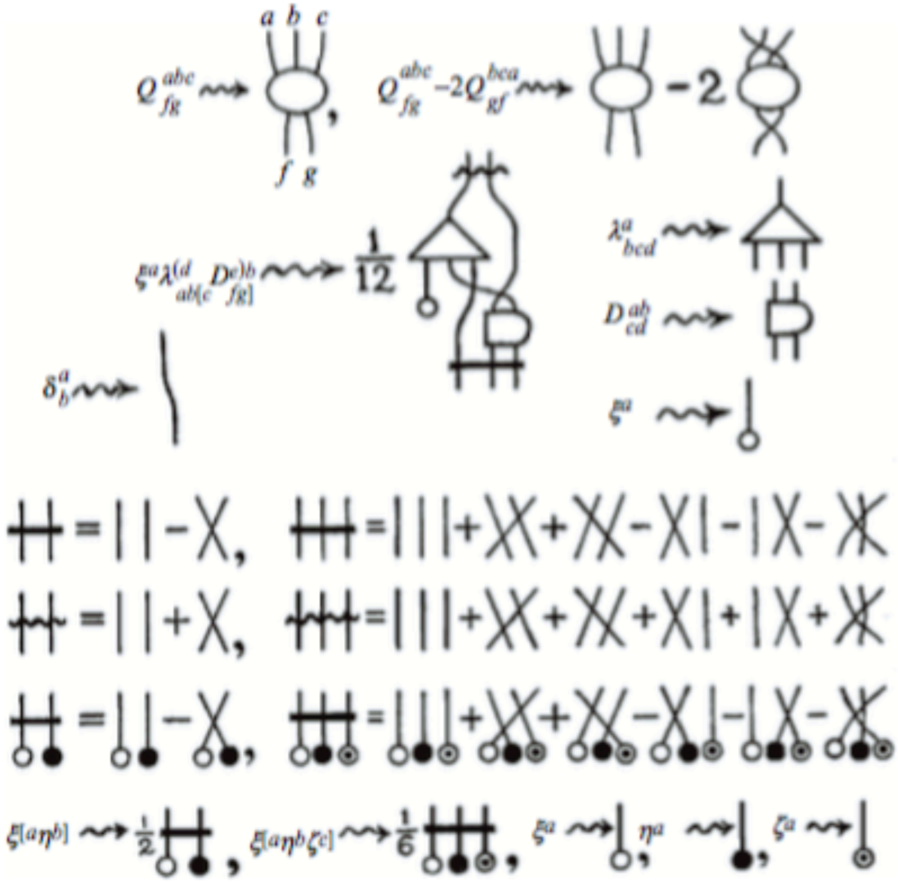


Fig. 12.17 Diagrammatic tensor notation. The $[{}^3_2]$-valent tensor Q is represented by an oval with 3 arms and 2 legs, where the general $[{}^p_q]$-valent tensor picture would have p arms and q legs. In an expression such as $Q^{abc}_{fg} - 2Q^{bca}_{gf}$, the diagrammatic notation uses positioning on the page of the ends of the arms and legs to keep track of which index is which, instead of employing individual index letters. Contractions of tensor indices are represented by the joining of an arm and a leg, as illustrated in the diagram for $\xi^a \lambda^{(d}_{ab[c} D^{e)b}_{fg]}$. This diagram also illustrates the use of a thick bar across index lines to denote antisymmetrization and a wiggly bar to represent symmetrization. The factor $\frac{1}{12}$ in the diagram results from the fact that (to facilitate calculations) the normal factorial denominator for symmetrizers and antisymmetrizers is omitted in the diagrammatic notation (so here we need $\frac{1}{2!} \times \frac{1}{3!} = \frac{1}{12}$). In the lower half of the diagram, antisymmetrizers and symmetrizers are written out as 'disembodied' expressions (by use of the diagrammatic representation of the Kronecker delta δ^a_b that will be introduced in §13.3, Fig. 13.6c). This is then used to express the (multivector) wedge products $\boldsymbol{\xi} \wedge \boldsymbol{\eta}$ and $\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \boldsymbol{\zeta}$.

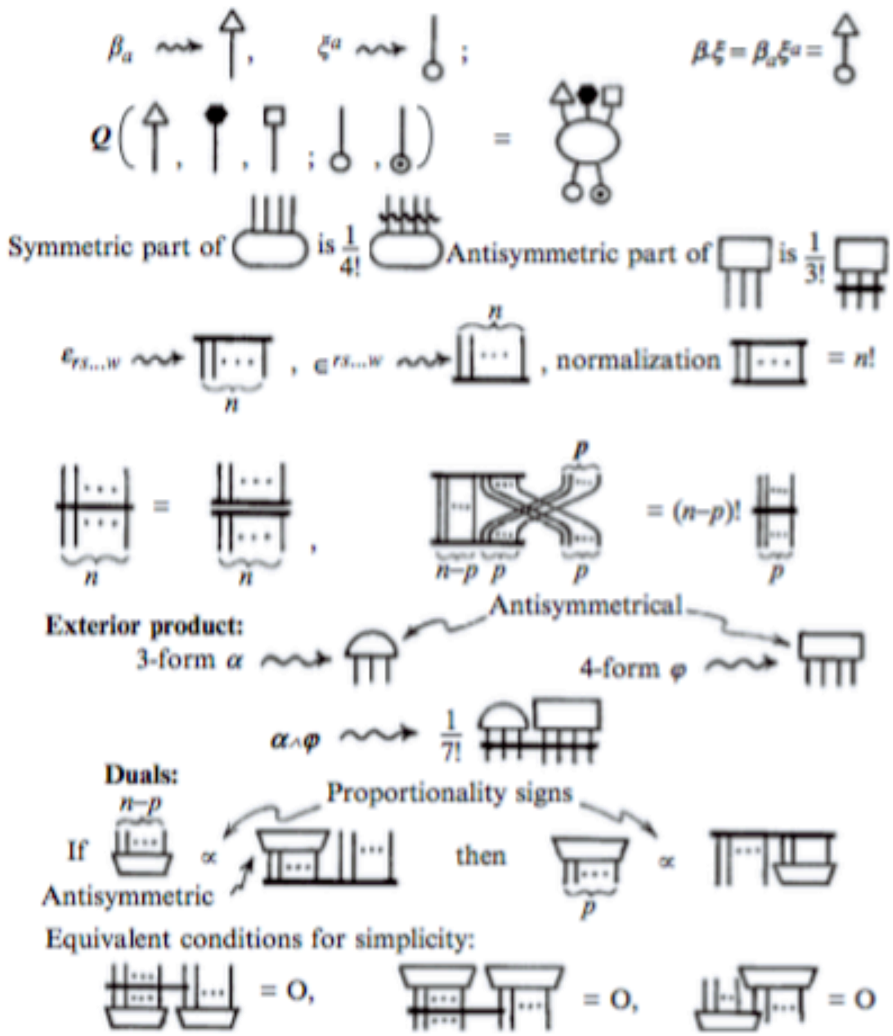


Fig. 12.18 More diagrammatic tensor notation. The diagram for a covector $\boldsymbol{\beta}$ (1-form) has a single leg, which when joined to the single arm of a vector $\boldsymbol{\xi}$ gives their scalar product. More generally, the multilinear form defined by a $[^p_q]$-valent tensor Q is represented by joining the p arms to the legs of p variable covectors and the q legs to the arms of q variable vectors (here $q = 3$ and $p = 2$). Symmetric and antisymmetric parts of general tensors can be expressed using the wiggly lines and thick bars of the operations of Fig. 12.17. Also, the bar notation combines with a related diagrammatic notation for the volume n-form $\varepsilon_{rs...w}$ (for an n-dimensional space) and its dual n-vector $\epsilon^{rs...w}$, normalized according to $\varepsilon_{rs...w}\epsilon^{rs...w} = n!$ Relations equivalent to $n!\delta^a_{[r}\delta^b_s \dots \delta^f_{w]} = \epsilon^{ab...f}\varepsilon_{rs...w}$ (n antisymmetrized indices) and $\varepsilon_{a...cu...w}\,\epsilon^{a...ce...f} = p!(n-p)!\delta^e_{[u}\dots\delta^f_{w]}$ (see § 13.3 and Fig. 13.6c) are also expressed. Exterior products of forms, the 'duality' between p-forms and $(n-p)$-vectors, and the conditions for 'simplicity' are then succinctly represented diagrammatically. (For exterior derivative diagrams, see Fig. 14.18.)

Complex Manifolds

A real 2*n*-manifold that satisfies generalized Cauchy-Riemann equations is equivalent to a complex *n*-manifold. This condition allows us to freely move between real and complex manifold algebra.

The equations are called the Nijenhaus tensor, derived from $J(\eta) = \xi$ and $J(\xi) = -\eta$. Think of *i* (*x* + *yi*) = -*y* + *ix*.