

[13.7]  $\text{SO}(3)$  is the group of rotations of the unit sphere in 3-space.  $\text{O}(3)$  extends  $\text{SO}(3)$  by including reflections. **(A)** Show that  $\text{SO}(3)$  is a normal subgroup of  $\text{O}(3)$  and **(B)** show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of  $\text{O}(3)$ . In Lemma 9.2 I show that if  $1$  is the identity element (null rotation) of  $\text{SO}(3)$  and  $R$  is the reflection operation, then  $\mathcal{I} = \{1, R\}$  is a normal subgroup of  $\text{O}(3)$ . Then I prove (Theorem 9) that there are no others.

**(A)** Show  $\text{SO}(3)$  is a normal subgroup of  $\text{O}(3)$

**Lemma 1.1:** A subgroup  $H$  of a group  $G$  is normal iff  $g^{-1}Hg = H \quad \forall g \in G$

**Proof.** By Penrose's definition,  $H$  is normal iff  $gH = Hg \quad \forall g \in G$ . Left multiplying by  $g^{-1}$  yields the lemma. ■

We use the symbol  $\circ$  for the  $\text{O}(3)$  group operation. In particular, if  $f, g \in \text{SO}(3)$ , we adopt the convention that  $f \circ g$  is the composite rotation of  $f$  followed by  $g$ .

Thus, if  $w = (x, y, z)$ , we write  $w(f \circ g) = wf \circ g$ . This is *the opposite of Penrose's convention* but, as will be seen, is necessitated by the approach taken to solve (B).

Also note that  $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ . That is, if  $w$  is rotated by  $f$  and then  $g$ , the inverse operation is to rotate in the reverse directions by  $g$  and then  $f$ .

**Definition:** We use  $R$  to denote the reflection operator on the unit sphere of  $\mathbb{R}^3$ :

$$(x, y, z) R = (-x, -y, -z).$$

Note that  $R$  is its own inverse:  $R^{-1} = R$ . If  $w$  is a point then  $w R$  is its reflection.

We write the composition of a reflection followed by a rotation as  $R \circ f$ , and a rotation followed by a reflection as  $f \circ R$ . An expression involving an odd number of reflections yields a reflective rotation and an expression involving an even number of reflections yields a non-reflective rotation. In particular, we have the following lemma.

**Lemma 1.2:** If  $h \in \text{SO}(3)$ , then  $R \circ h \circ R \in \text{SO}(3)$ . ■

**Definition:**  $T \equiv \text{O}(3) - \text{SO}(3)$  is the coset of reflective rotations of the sphere. That is,  $T = \text{SO}(3) \circ R$ , and  $\text{O}(3)$  is the disjoint union of  $\text{SO}(3)$  and  $T$ :

$$\text{O}(3) = T \cup \text{SO}(3)$$

The next theorem, Theorem 1, tells us that if  $h \in \text{SO}(3)$  then the normality operation  $g^{-1} \circ h \circ g$  produces another member of  $\text{SO}(3)$ , even if  $g \in T$ .

**Theorem 1:** SO(3) is a normal subgroup of O(3)

**Proof:** Clearly SO(3) has an identity, is closed under  $\circ$ , and each non-reflective rotation in SO(3) has an inverse (non-reflective) rotation. So it remains only to show that if  $h \in \text{SO}(3)$  and  $g \in \text{O}(3)$  that  $g^{-1} \circ h \circ g \in \text{SO}(3)$ . Since the composition of 3 non-reflective rotations is a non-reflective rotation, this holds if  $g \in \text{SO}(3)$ . So suppose  $g \in \text{T}$ . Then  $\exists f \in \text{SO}(3)$  such that  $g = f \circ R$ . Let  $h' = f^{-1} \circ h \circ f \in \text{SO}(3)$ . So,  $g^{-1} \circ h \circ g = R \circ f^{-1} \circ h \circ f \circ R = R \circ h' \circ R \in \text{SO}(3)$  by Lemma 1.2. ■

**(B)** Show SO(3) and  $\mathcal{I}$  are the only proper normal subgroups of O(3)

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball  $\mathcal{R}$  of radius  $\pi$  in which antipodal points on the surface of  $\mathcal{R}$  are identified. The problem was solved by letting elements of  $\mathcal{R}$  be written as

$f = \theta(a, b, c) = (\theta a, \theta b, \theta c)$ , where  $a^2 + b^2 + c^2 = 1$ ,  $\theta$  is the (counter-clockwise) angle of rotation,  $0 \leq \theta \leq \pi$ , and  $\{t(a, b, c) : t \geq 0\}$  is the (positive) axis of rotation.

#### Conventions:

1. **1** will be used to denote the identity element of SO(3). It is the degenerate  $0^\circ$  rotation.
2. An axis of rotation under discussion will be referred to as the **positive axis of rotation**. Its opposite will be called the **negative axis of rotation**.
3. Absolute value  $|f| = \theta$  will be used to denote the rotation angle of the rotation  $f$ .

The representations for points (i.e., rotations)  $f = \theta(a, b, c)$  are unique except:

- When  $\theta = 0$ , there are infinitely many representations for the identity element since any unit vector  $(a, b, c)$  can be used.
- When  $\theta = \pi$ , there are 2 representations for each  $\pi$  rotation  
 $f = \pi(a, b, c) = \pi(-a, -b, -c)$  because antipodal points are identified.

As an intermediate step to proving (B), we prove the following theorem, labeled Theorem 8 because there are several steps that are taken to get there.

**Theorem 8:** SO(3) does not contain a proper normal subgroup.

#### Outline of Steps to Prove Theorem 8:

Let  $H$  be a proper normal subgroup of SO(3).  $\exists 1 \neq h \in H$ . Let  $\theta = |h| > 0$  be the rotation angle of  $h$ . So,  $0 < \theta \leq \pi$ . Let  $S_\theta$  be the sphere in  $\mathcal{R}$  of radius  $\theta$ .  $S_\theta$  consists of all rotations of angle  $\theta$  about all axes.

- Let  $g \in \text{SO}(3)$ . We show that  $k = g^{-1} \circ h \circ g$  also has rotation angle  $\theta$ . That is,  $k \in S_\theta$ . Thus, starting with  $h \in H$  and using only the normality operation  $g^{-1} \circ h \circ g$ , we cannot build up to all of SO(3). By using just the normality operation, we can only generate a subset of  $S_\theta$ .

- We next show that in fact  $\{ g^{-1} \circ h \circ g : g \in SO(3) \}$  generates all of  $S_\theta$ .
- Next, suppose (a)  $0 \leq \phi \leq \pi$ , (b)  $f_1, g_1 \in S_\theta$ , and (c)  $|f_1 \circ g_1| = \phi$ . We show  $\{ f \circ g : f, g \in S_\theta \} \supseteq S_\phi$ . Thus  $H \supseteq S_\phi$ .
- Clearly, if (c) holds, the maximum possible size for  $\phi$  is  $2\theta$  (namely  $f \circ f$  for any  $f \in S_\theta$ ). The minimum possible size is 0 (namely,  $f \circ f^{-1}$ ). In fact, we will show that  $\{ f \circ g : f, g \in S_\theta \}$  contains every sphere  $S_\phi$  of radius  $\phi$  between  $0 \leq \phi \leq 2\theta$ , and so every such sphere belongs to  $H$ . To generate all of  $SO(3)$ , we just need to generate larger and larger spheres in  $H$  until all of  $SO(3)$  is included. That is, starting with  $S_\theta \subseteq H$  we can generate spheres of size up to  $S_{2\theta} \subseteq H$ . From  $S_{2\theta}$  we can generate spheres of size up to  $S_{4\theta} \subseteq H$ . Then up to  $S_{8\theta}$ . When  $2^n\theta \geq \pi$ , then  $SO(3) \subseteq S_{2^n\theta} \subseteq H$  and we are done.

The reader who wishes to first get an overview of the rest of the proof of (B) can skip ahead to Theorem 9 and read its overview.

### GA concepts and notation used in this proof

Non-reflective rotations (henceforth just called “rotations”) are commonly represented by rotation matrices, and the composition  $f \circ g$  can be computed as a matrix product. Instead, in this proof we learn how to use Geometric Algebra, aka Clifford Algebra, for the computations. In GA, every rotation corresponds to a rotor (described shortly), and geometric product (rather than matrix product) is used to compute the composition of rotations.

Let  $\{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \}$  be an orthonormal basis for 3-space. **Rotors** are objects having the form

$$\mathbf{r} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2]$$

where  $a^2 + b^2 + c^2 = 1$ ,  $a, b, c \in \mathbb{R}$ , and  $-\infty \leq \theta \leq \infty$ .

While  $\mathbf{e}_1$  is a **vector**,  $\mathbf{e}_1 \mathbf{e}_2$  is a **bivector**.  $\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$  is an example of a **trivector**. **Scalars**, vectors, bivectors, trivectors, etc., belong to the class of objects called **multivectors**.

A rotor is the sum of a scalar,  $\cos\left(\frac{\theta}{2}\right)$ , and a bivector (the rest of the expression). This is no stranger than a complex number that is the sum of a real and an imaginary number. The two parts cannot be combined but nonetheless the sum has meaning. In general a multivector can be the sum of multivectors of different **grades** that cannot be combined.

A rotor  $r$  represents a rotation of angle  $\theta$  about the axis  $\{ t(a, b, c) : t \geq 0 \}$ .

$\frac{\theta}{2}$  is called the **rotor angle** and  $\theta$  is the **rotation angle**. As this can be confusing, in this proof only the term “rotation angle” will be used.

The scalar  $\mathbf{Cos} \frac{\theta}{2}$  is referred to as the “**constant term**” of  $r$  because it does not contain any basis elements. The constant term determines the rotation angle  $\theta$ . That is, if the constant term of a rotor  $r$  is known to be  $K$ , then for some  $\theta$ ,

$$K = \mathbf{Cos} \frac{\theta}{2}, \text{ and so } \theta = 2 \operatorname{ArcCos}(K).$$

The magnitude of a rotor is  $\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) = 1$ , which is why  $a^2 + b^2 + c^2 = 1$ .

But even though rotors have magnitude 1, in this proof  $|r|$  will continue to be used to denote the rotation angle  $\theta$  and not 1 (and also not  $\frac{\theta}{2}$ ) in order to be consistent with the definition already given for the magnitude  $|f|$  of a point (rotation) in  $\mathcal{R}$ .

The **geometric product** of 2 multivectors is just the regular polynomial product with a couple of modifications. First, it is non-commutative, so the order of multiplication matters. Second, the basis elements are combined according to the rules

$$\begin{aligned} \mathbf{e}_i^2 &= 1, \quad i = 1, 2, 3 \quad \text{and} \\ \mathbf{e}_i \mathbf{e}_j &= -\mathbf{e}_j \mathbf{e}_i \quad \text{if } i \neq j \quad (\text{antisymmetry}) \end{aligned}$$

While a rotor can have any angle,  $\mathcal{S}$  is defined in this proof as the subset of rotors having (counter-clockwise) rotation angle  $0 \leq \theta \leq \pi$  under the operation of geometric product. If the rotor  $r \in \mathcal{S}$  (i.e., if  $0 \leq \theta \leq \pi$ ), it corresponds to the point  $(\theta a, \theta b, \theta c)$  of  $\mathcal{R}$ .

The symbol  $\circ$  will again be used to denote the geometric product, as in  $f \circ g$ . That is, whether  $f$  and  $g$  belong to  $\mathcal{R}$  or  $\mathcal{S}$ ,  $f \circ g$  will represent their composition.

To compute the sometimes algebra-intensive geometric products I wrote a software package in Mathematica that calculates geometric products and also performs other GA operations such as wedge product, multivector inverse, pseudoscalar, etc. The package can be downloaded for free at <https://github.com/matrixbud/Geometric-Algebra>. I have also saved the Mathematica file having the GA calculations used in this proof in the same directory as this file.

Note: Mathematica always displays the bivector  $\mathbf{e}_3 \mathbf{e}_1$  as  $\mathbf{e}_1 \mathbf{e}_3$ . Thus, when using Mathematica, the negative of the  $b$  term must be used, as in

$$r = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} [a e_2 e_3 - b e_1 e_3 + c e_1 e_2].$$

Since the normality operation  $g^{-1} \circ f \circ g$  involves inverses, this is a good time to provide the GA formula for the inverse of a rotor.

**Theorem 2:** The GA **inverse of the rotor  $r$**  has the formula

$$r^{-1} = \mathbf{Cos} \frac{\theta}{2} - \mathbf{Sin} \frac{\theta}{2} [a e_2 e_3 + b e_3 e_1 + c e_1 e_2].$$

Moreover,

$$r^{-1} \in \mathcal{S} \text{ if } r \in \mathcal{S}.$$

**Proof.** By computing the geometric product we find that

$r^{-1} \circ r = \mathbf{Cos}^2 \frac{\theta}{2} + \mathbf{Sin}^2 \frac{\theta}{2} (a^2 + b^2 + c^2) = 1$  and similarly  $r \circ r^{-1} = 1$ . Thus,  $r^{-1}$  is the inverse of  $r$ . ✓

Another way of seeing that  $r^{-1} \circ r = 1$  is that  $r^{-1}$  can be written

$$r^{-1} = \mathbf{Cos} \left( -\frac{\theta}{2} \right) + \mathbf{Sin} \left( -\frac{\theta}{2} \right) [a e_2 e_3 + b e_3 e_1 + c e_1 e_2],$$

representing a clockwise

rotation about the positive axis that cancels the counter-clockwise rotation of  $r$ , yielding the identity rotation, 1. ✓

Additionally, since  $r^{-1} = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} [-a e_2 e_3 - b e_3 e_1 - c e_1 e_2]$ ,  $r^{-1}$  can also be

regarded as a (counter-clockwise)  $\theta$  rotation about the negative axis, showing that  $r^{-1} \in \mathcal{S}$  (since  $\mathcal{S}$  is the set of all [counter-clockwise] rotations  $0 \leq \theta \leq \pi$  about all axes). ■

In rotor theory the formula for a rotor  $r$  to rotate a *point*  $w = x e_1 + y e_2 + z e_3$  in 3-space to some other point  $v$  is

$$v = r^{-1} \circ w \circ r.$$

Effectively,  $r^{-1}$  performs half the  $\theta$  rotation (i.e.,  $\frac{\theta}{2}$ ) and  $r$  performs the rest. The reader is encouraged to confirm this for a  $90^\circ$  rotation of the point  $(1,0,0)$  in the  $xy$ -plane. To do this, let  $\theta = \frac{\pi}{2}$ ,  $c = 1$ , and  $a = b = 0$  in the rotor  $r$ .

To write the expression to rotate a point  $w$  by  $f$  followed by  $g$ :

- Start with  $w$
- Wrap with  $f$ :  $f^{-1} \circ w \circ f$
- Wrap with  $g$ :  $g^{-1} \circ f^{-1} \circ w \circ f \circ g$

Thus  $f \circ g$  in this expression means to rotate first by  $f$ , then by  $g$ . This is the reason for the left-right convention adopted on page 1.

Every rotation can be represented by a rotor in  $\mathcal{S}$ ; i.e., by a rotor of the form

$$r = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \text{ where } 0 \leq \theta \leq \pi. \text{ Every rotation has}$$

an additional representation by a rotor with  $\pi < \theta < 2\pi$  that is not in  $\mathcal{S}$ . The geometric product of 2 rotors in  $\mathcal{S}$  can be outside of  $\mathcal{S}$ . To make  $\mathcal{S}$  a group under  $\circ$  we need to be able to recognize the formulas of rotors not in  $\mathcal{S}$  and be able to identify them with equivalent rotors in  $\mathcal{S}$ . Thus, we will actually be using  $\circ$  to represent the geometric product mod  $\mathcal{S}$ . However, we will only explicitly write mod  $\mathcal{S}$  when it is necessary for the sake of clarity.

**Definition:** We say that 2 rotors  $r_1$  and  $r_2$  are **equivalent** if they generate the same rotation.

**Example 1:** Does the geometric product of two  $\frac{2}{3}\pi$  rotations about the  $x$ -axis belong to  $\mathcal{S}$ ?

Answer: The result is a  $\frac{4}{3}\pi$  rotation about the  $x$ -axis, not in  $\mathcal{S}$  since  $\frac{4}{3}\pi > \pi$ .

Theorem 3, below, shows how to recognize a rotor not in  $\mathcal{S}$  and how to generate its equivalent rotor in  $\mathcal{S}$ .

**Theorem 3:** Let  $\pi < \theta < 2\pi$ . Then

$$s = \mathbf{Cos} \frac{\theta}{2} + \mathbf{Sin} \frac{\theta}{2} (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \notin \mathcal{S}, \text{ and}$$

$$r = \mathbf{Cos} \left( \pi - \frac{\theta}{2} \right) + \mathbf{Sin} \left( \pi - \frac{\theta}{2} \right) (a e_2 e_3 + b e_3 e_1 + c e_1 e_2) \in \mathcal{S}.$$

Also, the constant term of  $s$  is negative; the constant term of  $r$  is positive,  $-s = r^{-1} \in \mathcal{S}$ , and  $s$  is equivalent to  $-s$ .

**Proof:**  $s \notin \mathcal{S}$  because  $\theta \notin [0, \pi]$ . ✓

Since  $(2\pi - \theta) \in [0, \pi]$ ,  $r \in \mathcal{S}$ . ✓

The constant term of  $s$  is negative:  $\mathbf{Cos}\frac{\theta}{2} < 0$  because  $\frac{\pi}{2} < \frac{\theta}{2} < \pi$ . ✓

The constant term of  $r$  is positive:  $\mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) > 0$  since  $0 < \pi - \frac{\theta}{2} < \frac{\pi}{2}$ .

$r^{-1} \in \mathcal{S}$  from Theorem 2. ✓

Claim  $r^{-1} = -s$ :

Without loss of generality, by a suitable rotation of 3-space, we can assume  $s$ , and hence  $r$ , are rotations about the  $z$ -axis.

$$s = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \quad \text{and} \quad r = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2.$$

These formulas for  $s$  and  $r$  are rotors because they satisfy the rotor definition with  $c = 1$  and  $a = b = 0$ . They rotate about the  $z$ -axis because  $e_1 e_2$  rotates the  $xy$ -plane.

$$r^{-1} = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) - \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 = -\mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) = -s \quad \checkmark$$

To show  $s$  is equivalent to  $r^{-1}$ , we must show that if  $w = (x, y, z) \in \mathbb{R}^3$  then  $s^{-1} \circ w \circ s = r \circ w \circ r^{-1}$ , i.e., that the rotations of  $w$  generated by  $s$  and  $r^{-1}$  are the same for every  $w \in \mathbb{R}^3$ :

$$\begin{aligned} w &= xe_1 + ye_2 + ze_3. \\ s^{-1} \circ w \circ s &= \left[ x \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - 2y \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - x \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_1 \\ &\quad + \left[ y \mathbf{Cos}^2\left(\frac{\theta}{2}\right) + 2x \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - y \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_2 + \\ &\quad + z \left[ \mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right] e_3 \\ &= [x \mathbf{Cos}(\theta) - y \mathbf{Sin}(\theta)] e_1 + [y \mathbf{Cos}(\theta) + x \mathbf{Sin}(\theta)] e_2 + ze_3 \\ &= r \circ w \circ r^{-1} \end{aligned} \quad \blacksquare$$

**Aside:** Note the change from  $\frac{\theta}{2}$  to  $\theta$  in the equation above, evidence that  $\theta$  is indeed the rotation angle generated by the expression  $s^{-1} \circ w \circ s$ .

**Example 2:** Find a rotor in  $\mathcal{S}$  that is equivalent to the rotor  $s = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{e}_1\mathbf{e}_2$

Solution: First, by Theorem 3,  $s \notin \mathcal{S}$  because the constant term is negative. Also by Theorem 3,  $s$  is equivalent to  $-s = \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{e}_1\mathbf{e}_2$  and  $-s \in \mathcal{S}$ .

**Example 3:** If a rotor  $s$  is a  $\pi$  rotation about any axis, then  $s^2$  should be the identity, a zero rotation. But if, for example,  $s = \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_2 \mathbf{e}_3$ , then  $s^2 = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_3 = -1$ . How can that be?

Answer: By Theorem 3,  $s^2 \notin \mathcal{S}$  because the constant term is negative. Also by Theorem 3,  $s^2$  is equivalent to  $-s^2 = 1$ , the identity rotor. The fact that  $-1 \equiv 1 \pmod{\mathcal{S}}$  is important in Theorem 9.

**Aside:** If we consider  $-\infty < \theta < \infty$ , then there are infinitely many rotors equivalent to  $r$  since sine and cosine have period  $2\pi$ . But, for  $0 < \theta < 2\pi$  there are only two representations for each rotor, and for  $0 < \theta < \pi$  there is only one.

The next theorem is the analog to Penrose's problem [12.17].

**Theorem 4:**  $\mathcal{S}$  is group isomorphic to  $\mathcal{R}$  [which is group isomorphic to  $\text{SO}(3)$ ]

**Proof:** Let  $s = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2] \in \mathcal{S}$ .

Then  $0 \leq \theta \leq \pi$  and  $a^2 + b^2 + c^2 = 1$ .

$s$  can be represented as  $s = (\theta, a, b, c) \in \mathcal{S}$ . This representation in  $\mathcal{S}$  is unique for  $0 \leq \theta \leq \pi$  except when  $\theta = 0$ :

- Like  $\mathcal{R}$ , when  $\theta = 0$ , there are infinitely many unit points  $(a, b, c)$  that can be used to denote the identity rotation.
- Unlike  $\mathcal{R}$ , when  $\theta = \pi$  there is only 1 representation for each point  $s$  because while the antipodal point represents the same rotation, it is not in  $\mathcal{S}$ . That is, if  $s = (\pi, a, b, c) \in \mathcal{S}$ , the antipodal point is  $(-\pi, -a, -b, -c)$ , not in  $\mathcal{S}$  because  $\theta = -\pi$ .

Define

$$\begin{aligned} T: \mathcal{S} &\rightarrow \mathcal{R}: T(s) = r \text{ where} \\ s &= (\theta, a, b, c) \text{ and} \\ r &= \theta(a, b, c) = (\theta a, \theta b, \theta c). \end{aligned}$$

That is,  $T: (\theta, a, b, c) \mapsto \theta(a, b, c)$

$T$  is well-defined:

To show that  $T$  is well-defined we must show that (a)  $T(s) \in \mathcal{R}$  and (b) if  $s$  has 2 or more representations, then  $T$  assigns the same element of  $\mathcal{S}$  in all cases.

(a) Since  $0 \leq \theta \leq \pi$  and  $a^2 + b^2 + c^2 = 1$ ,  $r \in \mathcal{R}$ .

(b) If  $\theta = 0$ , then  $T(0, a, b, c) = 0$  ( $a, b, c$ ) = (0, 0, 0) for any unit vector ( $a, b, c$ ).

$T$  is obviously 1–1 and onto since  $T: (\theta, a, b, c) \rightarrow \theta(a, b, c)$ .

$T$  is a homomorphism:

We need to show that  $T(s_1 \circ s_2) = T(s_1) \circ T(s_2)$

$s_1 \circ s_2$  is the composition of 2 rotations in  $\mathcal{S}$ , and  $T(s_1) \circ T(s_2)$  is the composition of 2 rotations in  $\mathcal{R}$ . So,

$$\begin{aligned} \text{if } s_1 \circ s_2 &= \text{Composition of } (\theta_1, a_1, b_1, c_1) \& (\theta_2, a_2, b_2, c_2) = (\theta_3, a_3, b_3, c_3) \\ \text{then } T(s_1) \circ T(s_2) &= \text{Composition of } \theta_1(a_1, b_1, c_1) \& \theta_2(a_2, b_2, c_2) \\ &= \theta_3(a_3, b_3, c_3) \end{aligned}$$

$$\text{So, } T(s_1 \circ s_2) = T(\theta_3, a_3, b_3, c_3) = \theta_3(a_3, b_3, c_3) = T(s_1) \circ T(s_2) \quad \blacksquare$$

### Steps leading to Theorem 8

As we move towards Theorem 8, we will be positing a normal group  $H$  and an element  $1 \neq h_1 \in H$ . We will expand the singleton set  $\{h_1\}$  to a sphere of elements in  $H$  using just the normality operation  $g^{-1} \circ h_1 \circ g$  where  $g \in SO(3)$ . Then we will switch to the product operation between elements of the sphere to expand  $H$  to all of  $SO(3)$ .

**WLOG**, the axes of the unit 3-sphere can be rotated so that  $h_1$  is a rotation about the  $x$ -axis. The following lemma provides geometric insight into how to use the normality operation  $g^{-1} \circ f \circ g$  to generate a rotor whose axis is perpendicular to that of a given rotor. This is accomplished by letting  $g$  be a  $90^\circ$  rotation about an axis perpendicular to the axis of  $f$ . For example, if  $f$  rotates by  $\theta$  about the  $x$ -axis and  $g$  is a  $90^\circ$  rotation about the  $y$ -axis, then  $g^{-1} \circ f \circ g$  is a rotation by  $\theta$  about the  $z$ -axis.

**Lemma 5.1:** Let

$$h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e_2e_3 \quad (\theta \text{ rotation about positive } x\text{-axis})$$

$$h_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e_3e_1 \quad (\theta \text{ rotation about positive } y\text{-axis}), \text{ and}$$

$$\mathbf{h}_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \quad (\theta \text{ rotation about positive z-axis})$$

If  $h_1$  belongs to a normal group  $H$ , then  $h_2, h_3, h_1^{-1}, h_2^{-1}, h_3^{-1} \in H$ .

**Proof:** Define rotors

$$g_3 = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \mathbf{e}_1 \mathbf{e}_2 = \frac{1 + \mathbf{e}_1 \mathbf{e}_2}{\sqrt{2}} \text{ and}$$

$$g_2 = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \mathbf{e}_3 \mathbf{e}_1 = \frac{1 - \mathbf{e}_3 \mathbf{e}_1}{\sqrt{2}}.$$

Then

$$g_3^{-1} = \frac{1 - \mathbf{e}_1 \mathbf{e}_2}{\sqrt{2}},$$

$$g_3^{-1} \circ h_1 = \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_2 \mathbf{e}_3 + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_3 \mathbf{e}_1 - \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_1 \mathbf{e}_2$$

$$g_3^{-1} \circ h_1 \circ g_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_3 \mathbf{e}_1 = h_2$$

and

$$g_2^{-1} = \frac{1 + \mathbf{e}_3 \mathbf{e}_1}{\sqrt{2}},$$

$$g_2^{-1} \circ h_1 = \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_2 \mathbf{e}_3 + \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_3 \mathbf{e}_1 + \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}} \mathbf{e}_1 \mathbf{e}_2$$

$$g_2^{-1} \circ h_1 \circ g_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 = h_3$$

Since  $H$  is normal,  $h_2, h_3 \in H$ . Since groups are closed under inverses,  $h_1^{-1}, h_2^{-1}, h_3^{-1} \in H$ . ■

**Definition.** In  $\mathcal{R}$ , let  $S_\theta$  denote the **sphere of radius  $\theta$** . It consists of all rotations of amount  $\theta$  about all axes in 3-space. The formula for  $S_\theta$  in terms of rotors is

$$S_\theta = \left\{ \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) (a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2) : a^2 + b^2 + c^2 = 1 \right\}.$$

Given  $h, r \in \mathcal{R}$  it is easy to find  $g \in \mathcal{R}$  such that  $r = h \circ g$ . Namely,  $g = h^{-1} \circ r$ . It is much more difficult to find  $g \in \mathcal{R}$  such that  $r = g^{-1} \circ h \circ g$ . In fact, we will prove in

Lemma 6.1 that it is not possible at all unless  $h$  and  $r$  have the same magnitude. The next lemma (5.2) proves that it is in fact always possible to find such a point  $g$  if the magnitudes of  $h$  and  $r$  are the same, and the lemma provides an explicit formula for  $g$  (actually 2 formulas, depending on whether  $r$  has a  $x$ -component). This lemma (or the alternate proof to Theorem 5, discussed below) is the key step in achieving Theorem 8.

Lemma 5.2 gives formulas for  $r$  that work but does not explain where they come from. In Appendix 2, I provide what insight I can into how I discovered the formula for rotor  $g_{23}$ . Discovery of the rotor  $g$  in part (b) is slightly more transparent because  $a$  and  $b$  can be viewed as the sides of a right triangle and hence there is an angle  $\alpha$  such that  $b = \text{Sin } \alpha$ .

My inability to provide clear insight for the equations in Lemma 5.2 nudged me to develop another proof for Theorem 5 that is intuitive and does not depend on either Lemma 5.1 or 5.2. It does use an additional lemma (5.3) that is a generalization of Lemma 5.1. I provide both of the Theorem 5 proofs, though only one is needed in order to proceed. I include the second proof because it is short, clever, geometric and provides an insightful process to find, for a given rotation  $r$ , a rotation  $g$  such that  $g^{-1} \circ h \circ g = r$ . I include the original proof because it is short and Lemma 5.2 includes explicit, compact formulas for generating  $r$ .

**Lemma 5.2:** Let

$$h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3, \quad h_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_1 e_2, \text{ and}$$

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} [a e_2 e_3 + b e_3 e_1 + c e_1 e_2] \in S_\theta,$$

where  $a^2 + b^2 + c^2 = 1$  and  $0 < \theta \leq \pi$ .

(a) If  $a \neq 0$  then

$$r = g_{23}^{-1} \circ h_1 \circ g_{23} \quad \text{where}$$

$$g_2 = \cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\beta}{2}\right) e_3 e_1,$$

$$g_3 = \cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\gamma}{2}\right) e_1 e_2,$$

$$g_{23} = g_2 \circ g_3, \text{ and}$$

$$\beta = \text{Arc Cos} (\sqrt{a^2 + b^2}) \text{ and } \gamma = \text{Arc Tan} (a, b).$$

(b) If  $a = 0$  then

$$r = g^{-1} \circ h_3 \circ g \quad \text{where}$$

$$g = \mathbf{Cos}\left(\frac{\alpha}{2}\right) + \mathbf{Sin}\left(\frac{\alpha}{2}\right)e_2 e_3, \text{ and}$$

$$\alpha = \text{Arc Sin}(-b).$$

**Proof:** ArcTan( $a, b$ ) denotes the arc tangent of  $\frac{b}{a}$  taking into account which quadrant the point ( $a, b$ ) is in.

$$(a) \quad g_{23} = \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_2 e_3$$

$$- \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)e_3 e_1 + \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_1 e_2$$

$$g_{23}^{-1} = \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_2 e_3$$

$$+ \mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right)e_3 e_1 - \mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)e_1 e_2$$

$$g_{23}^{-1} \circ h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right)$$

$$- \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) \right] e_2 e_3$$

$$+ \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) \right] e_3 e_1$$

$$- \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}\left(\frac{\beta}{2}\right)\mathbf{Sin}\left(\frac{\gamma}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\beta}{2}\right)\mathbf{Cos}\left(\frac{\gamma}{2}\right) \right] e_1 e_2$$

$$\begin{aligned}
g_{23}^{-1} \circ h_1 \circ g_{23} &= \cos\left(\frac{\theta}{2}\right) \left[ \cos^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
&\quad \left. + \cos^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] \\
&\quad + \sin\left(\frac{\theta}{2}\right) \left[ \cos^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) - \sin^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
&\quad \left. - \cos^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] e_2 e_3 \\
&\quad + \sin\left(\frac{\theta}{2}\right) \left[ 2\cos^2\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right. \\
&\quad \left. - 2\sin^2\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right] e_3 e_1 \\
&\quad + \sin\left(\frac{\theta}{2}\right) \left[ 2\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\gamma}{2}\right) \right. \\
&\quad \left. + 2\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right] e_1 e_2 \\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) [\cos(\beta) \cos(\gamma) e_2 e_3 + \cos(\beta) \sin(\gamma) e_3 e_1 + \sin(\beta) e_1 e_2] \\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) [ae_2 e_3 + be_3 e_1 + ce_1 e_2] \quad \checkmark \\
&= r
\end{aligned}$$

Note: The interested reader can find intermediate steps for this and subsequent equations in the accompanying Mathematica file.

(b) Since  $a = 0$ , then  $b = \sqrt{1 - c^2}$ , and

$$\begin{aligned}
g^{-1} \circ h_3 &= \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\alpha}{2}\right) e_2 e_3 \\
&\quad - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\alpha}{2}\right) e_3 e_1 + \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\alpha}{2}\right) e_1 e_2 \\
g^{-1} \circ h_3 \circ g &= \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) \sin(\alpha) e_3 e_1 + \sin\left(\frac{\theta}{2}\right) \cos(\alpha) e_1 e_2 \\
&= \cos\left(\frac{\theta}{2}\right) + b \sin\left(\frac{\theta}{2}\right) e_3 e_1 + c \sin\left(\frac{\theta}{2}\right) e_1 e_2 \\
&= r \quad \checkmark \quad \blacksquare
\end{aligned}$$

The next theorem summarizes Lemma 5.2. It shows that for a normal subgroup  $H$ , if  $h \in H$  has rotation angle  $\theta$ , then the sphere  $S_\theta$  of radius  $\theta$  lies in  $H$

**Theorem 5:** If  $H$  is a normal subgroup of  $SO(3)$ ,  $1 \neq h \in H$  and  $\theta = |h|$ , then  $H \supseteq S_\theta$ .

**Proof:** WLOG we can assume  $h$  is a  $\theta$  rotation about the  $x$ -axis. That is,

$$h = h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e_2 e_3.$$

Let  $r \in S_\theta$ . Then for some  $a, b, c \in \mathbb{R}$  such that  $a^2 + b^2 + c^2 = 1$ ,

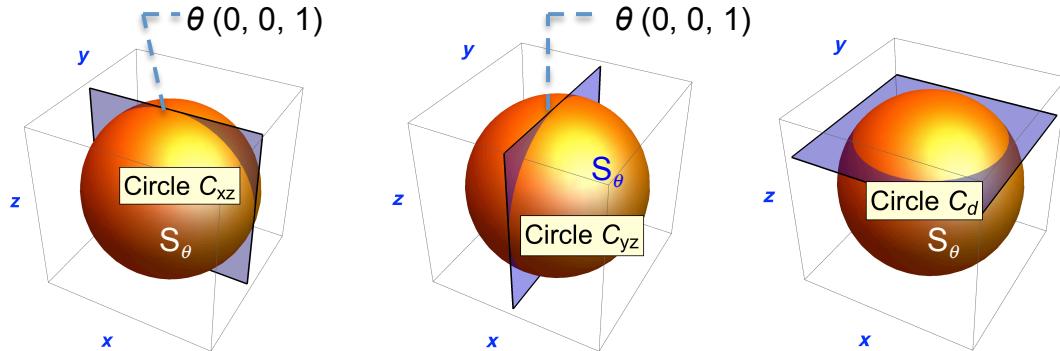
$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}[a e_2 e_3 + b e_3 e_1 + c e_1 e_2].$$

If  $a \neq 0$ , by Lemma 5.2a,  $r \in H$  since  $r = g_{23}^{-1} \circ h_1 \circ g_{23}$  and  $H$  is normal.

If  $a = 0$ , by Lemma 5.2b,  $r \in H$  since  $r = g^{-1} \circ h_3 \circ g$ ,  $H$  is normal, and  $h_3 \in H$  (by Lemma 5.1).  $\blacksquare$

We now present the alternate proof for Theorem 5.

**Outline of Geometric-based Proof of Theorem 5:** As before we begin with  $h_1 = \theta(1, 0, 0) \in H$  and  $f = \theta(a, b, c) \in S_\theta$ . We are done when we show that  $f \in H$ .



In the figure above, circle  $C_{xz}$  of radius  $\theta$  about the origin is formed by slicing the sphere  $S_\theta$  with the  $xz$ -plane of  $\mathcal{R}$ . Circle  $C_{yz}$  is similarly formed by slicing  $S_\theta$  with the  $yz$ -plane. Lemma 5.3 will prove that these circles lie in  $H$ .

The third circle in the figure is formed by taking a horizontal slice of  $S_\theta$  at height  $z = c\theta$ , where  $0 \leq c \leq 1$ . This generates a circle  $C_d$  of radius  $d = \theta\sqrt{1 - c^2}$  that contains  $f$ . Thus, points on  $C_d$  have coordinates  $\theta(d \cos \phi, d \sin \phi, c)$  where  $\phi$  is

the counter-clockwise angle from the  $x$ -axis of an  $xy$ -plane raised to height  $c$ . We show that the formula for  $\phi$  is  $\text{Arc Tan} \frac{b}{a}$ .

Next we show that if  $g_\phi = \phi(0, 0, 1)$  and  $h = \theta(d, 0, c)$ , then  $g_\phi^{-1} \circ h \circ g_\phi \in C_d$  and, in fact,  $\{g_\phi^{-1} \circ h \circ g_\phi : -\pi \leq \phi \leq \pi\}$  generates  $C_d$ . In particular, it generates  $f$ , proving  $f \in H$  since  $h \in C_{xz} \subseteq H$

**Geometric Insight:** We have again used rotations (specifically,  $g_\phi$  and  $h$ ) about axes perpendicular to the axis of  $f$  to generate  $f$  via the normality operation. This time, however,  $g_\phi$  is not required to have a  $90^\circ$  rotation angle but, rather, as  $\phi$  varies from  $-\pi$  to  $\pi$ , the normality operation  $g_\phi^{-1} \circ h \circ g_\phi$  traces out the Circle  $C_d$ .

**Lemma 5.3:** If  $h_1 = (\theta, 0, 0) \in \mathcal{R}$  belongs to a normal group  $H$ , then  $H$  contains  $C_{xy}$ ,  $C_{yz}$ , and  $C_{xz}$ , the circles of radius  $\theta$  centered at the origin in the  $xy$ -,  $yz$ -, and  $xz$ -planes of  $\mathcal{R}$ , respectively.

**Proof:** The lemma is just a slight generalization of Lemma 5.1. Recall

$$h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3. \text{ Similarly to Lemma 5.1, let}$$

$$g_2 = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) e_3 e_1 \text{ and } g_3 = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) e_1 e_2 \text{ where } 0 \leq \phi \leq \frac{\pi}{2}.$$

$$\text{Also set } g_1 = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) e_2 e_3. \text{ Then}$$

$$g_3^{-1} \circ h_1 \circ g_3 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) [\cos(\phi) e_2 e_3 + \sin(\phi) e_3 e_1].$$

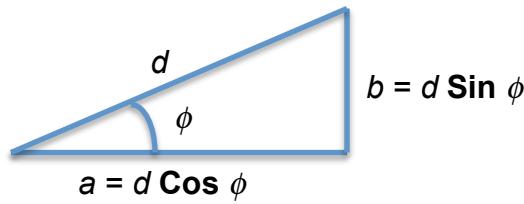
Recall from page 2 of this proof that the axis of rotation of  $g_3^{-1} \circ h_1 \circ g_3$  is  $\{t(\cos \phi, \sin \phi, 0) : t \geq 0\}$ . Thus rotor  $g_3^{-1} \circ h_1 \circ g_3$  is represented in  $\mathcal{R}$  as the point  $\theta(\cos \phi, \sin \phi, 0)$ . As  $\phi$  varies from  $-\pi$  to  $\pi$ , these points sweep out the circle  $C_{xy}$ . Similarly, the set of points  $\{g_2^{-1} \circ h_1 \circ g_2\}$  forms the circle  $C_{xz}$  and

$\{g_1^{-1} \circ h_2 \circ g_1\}$  forms the circle  $C_{yz}$ , where  $h_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_3 e_1$ . These

circles are contained in  $H$  because  $H$  is normal and, in the case of the 3<sup>rd</sup> circle, also because  $h_2 \in H$  (because  $h_2 \in C_{xz} \subseteq H$ ). ■

**Theorem 5 Proof #2:** Let  $H$  be a non-trivial normal subgroup of  $\text{SO}(3)$ .

WLOG let  $(0, 0, 0) \neq h_1 = (\theta, 0, 0) \in H$  be the element of  $\mathcal{R}$  representing a  $\theta$  rotation about the  $x$ -axis. Let  $f = \theta(a, b, c) \in S_\theta$ . It suffices to show that  $f \in H$ .



If  $a = 0$ , then  $f \in C_{yz} \subseteq H$  by Lemma 5.3.

So we can assume  $a \neq 0$ .

Let  $C_d$  be the circle formed by taking a horizontal slice of  $S_\theta$  at height  $z = c\theta$ .

The radius of  $C_d$  is  $d = \sqrt{1 - c^2}$ .

We wish to represent  $f$  also as  $f = \theta(d \cos \phi, d \sin \phi, c)$  since it is a point on the circle  $C_d$ . That is, we desire an angle  $\phi$  such that  $a = d \cos \phi$  and  $b = d \sin \phi$ . To

do this, we define  $\phi = \text{ArcTan}(a, b)$ , which is  $\text{ArcTan} \frac{b}{a}$  but adjusts  $\phi$  for the

quadrant of  $(a, b)$ .  $\phi$  is well-defined because  $a \neq 0$ . In order to make  $\sin \phi = \frac{b}{d}$

and  $\cos \phi = \frac{a}{d}$  well-defined, we also must have  $d \neq 0$ . But, if  $d = 0$  then

$f = \theta(0, 0, \pm 1) \in C_{yz} \subseteq H$  (see Circle  $C_{yz}$  figure), and we are done. So we can assume  $d \neq 0$ .

Let

$$g_\phi = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) e_1 e_2 = \phi(0, 0, 1)$$

be the rotation of angle  $\phi$  about the  $z$ -axis.

Let

$$h = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) (d e_2 e_3 + c e_1 e_2) = \theta(d, 0, c),$$

a point of the circle  $C_{xz}$ .  $h \in H$  by Lemma 5.3.

Finally,

$$\begin{aligned} g_\phi^{-1} \circ h \circ g_\phi &= \cos\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\phi}{2}\right) + \cos\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\phi}{2}\right) \\ &\quad + \left( d \sin\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\phi}{2}\right) - d \sin\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\phi}{2}\right) \right) e_2 e_3 \\ &\quad + 2d \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi}{2}\right) \sin\left(\frac{\phi}{2}\right) e_3 e_1 \\ &\quad + \left( c \sin\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\phi}{2}\right) + c \sin\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\phi}{2}\right) \right) e_1 e_2 \end{aligned}$$

$$\begin{aligned}
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) [d \mathbf{Cos}(\phi) \mathbf{e}_2 \mathbf{e}_3 + d \mathbf{Sin}(\phi) \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2] \\
&= \theta(d \mathbf{Cos}(\phi), d \mathbf{Sin}(\phi), c) \\
&= f.
\end{aligned}$$

Therefore  $f \in H$  and  $S_\theta \subseteq H$ . ■

The next lemma shows that for any  $g \in SO(3)$ , the normality operation  $g^{-1} \circ h \circ g$  results in a rotation having the same rotation angle as  $h$ . Thus, for a given  $h \in SO(3)$ ,  $\{g^{-1} \circ h \circ g : g \in SO(3)\}$  cannot generate all of  $SO(3)$ . The prior Lemma 5.2 (as well as the alternate proof of Theorem 5) tells us that it at least generates *all* of  $S_\theta$ .

**Lemma 6.1:** Let  $0 \leq \theta \leq \pi$ ,  $h \in S_\theta$ , and  $g \in SO(3)$ . Then  $g^{-1} \circ h \circ g \in S_\theta$ .

**Proof:** WLOG  $h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{e}_2 \mathbf{e}_3$ . Since 2 vectors determine a plane, WLOG we can also assume that  $g$  lies in the  $xy$ -plane. That is, if the magnitude of  $g$  is  $0 \leq \phi \leq \pi$  then  $g = \mathbf{Cos}\frac{\phi}{2} + \mathbf{Sin}\frac{\phi}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1]$  where  $a^2 + b^2 = 1$ .

Then

$$\begin{aligned}
g^{-1} \circ h \circ g &= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) + b^2 \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&\quad + \left[ \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \\
&\quad + 2ab \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \\
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&\quad - \mathbf{Sin}\left(\frac{\theta}{2}\right) \left\{ \begin{aligned}
&+ \left[ \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \\
&+ 2ab \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2
\end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{Cos}\left(\frac{\theta}{2}\right) \\
&\quad + \mathbf{Sin}\left(\frac{\theta}{2}\right) \left\{ + \left[ \mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2 \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \right] \mathbf{e}_2 \mathbf{e}_3 \right. \\
&\quad \left. + 2ab \mathbf{Sin}^2\left(\frac{\phi}{2}\right) \mathbf{e}_3 \mathbf{e}_1 - 2b \mathbf{Sin}\left(\frac{\phi}{2}\right) \mathbf{Cos}\left(\frac{\phi}{2}\right) \mathbf{e}_1 \mathbf{e}_2 \right\}
\end{aligned}$$

Since the constant term of  $g^{-1} \circ h \circ g$  is  $\mathbf{Cos}\frac{\theta}{2}$ ,  $g^{-1} \circ h \circ g \in S_\theta$ . ■

Now we switch tactics, using the product operation rather than the normality operation to generate additional elements of H. We will show that geometric products of pairs of elements in  $S_\theta$  generate every sphere  $S_\omega$  and, hence, all of SO(3).

The next theorem starts the process. It shows the ability to generate one sphere in  $\mathcal{R}$  from another by taking products of pairs of elements from the first sphere.

**Theorem 6:** Let  $k_1, k_2 \in S_\theta$  for some  $0 < \theta \leq \pi$ , and let  $\omega = |k_1 \circ k_2|$ . Then  $S_\omega$  can be generated from geometric products of pairs of elements from  $S_\theta$ .

**Proof.** The theorem is trivially true if  $\omega = 0$ , so assume  $\omega > 0$ . Let  $r \in S_\omega$ . The claim is that we can find  $f_1, f_2 \in S_\theta$  such that  $r = f_1 \circ f_2$ . WLOG let  $k_1 \circ k_2$  be the  $\omega$  rotation about the positive x-axis. That is  $h_1 \equiv k_1 \circ k_2 = \mathbf{Cos}\left(\frac{\omega}{2}\right) + \mathbf{Sin}\left(\frac{\omega}{2}\right) \mathbf{e}_2 \mathbf{e}_3$ .

Since  $r \in S_\omega$ ,  $\exists a, b, c \in \mathbb{R}$  such that  $r = \mathbf{Cos}\frac{\omega}{2} + \mathbf{Sin}\frac{\omega}{2} [a \mathbf{e}_2 \mathbf{e}_3 + b \mathbf{e}_3 \mathbf{e}_1 + c \mathbf{e}_1 \mathbf{e}_2]$ , where  $a^2 + b^2 + c^2 = 1$ .

Lemma 5.2 yields that either (a)  $r = g_{23}^{-1} \circ h_1 \circ g_{23}$  or (b)  $r = g^{-1} \circ h_3 \circ g$ , where  $h_3 \in S_\omega$  is the  $\omega$  rotation about the z-axis and  $g_{23}$  and  $g$  are as defined in the lemma (using  $\omega$  rather than  $\theta$ ).

(a) Define  $f_1 = g_{23}^{-1} \circ k_1 \circ g_{23}$  and  $f_2 = g_{23}^{-1} \circ k_2 \circ g_{23}$ . By Lemma 6.1,  $f_1, f_2 \in S_\theta$ .

Thus,

$$\begin{aligned}
r &= g_{23}^{-1} \circ h_1 \circ g_{23} = g_{23}^{-1} \circ k_1 \circ k_2 \circ g_{23} \\
&= (g_{23}^{-1} \circ k_1 \circ g_{23}) \circ (g_{23}^{-1} \circ k_2 \circ g_{23}) \\
&= f_1 \circ f_2
\end{aligned}$$

✓

(b) As was shown in Lemma 5.1,  $h_3$  can be obtained from  $h_1$  by a normality operation using the  $90^\circ$  rotation  $g_2 = \frac{1 - e_3 e_1}{\sqrt{2}}$ . The rotation formula is

$$h_3 = g_2^{-1} \circ h_1 \circ g_2. \text{ Apply the same } 90^\circ \text{ rotation to } k_1 \text{ and } k_2 :$$

$$k_3 = g_2^{-1} \circ k_1 \circ g_2 \quad \text{and} \quad k_4 = g_2^{-1} \circ k_2 \circ g_2.$$

By Lemma 6.1,  $k_3, k_4 \in S_\theta$  and

$$\begin{aligned} h_3 &= g_2^{-1} \circ h_1 \circ g_2 = g_2^{-1} \circ k_1 \circ k_2 \circ g_2 \\ &= (g_2^{-1} \circ k_1 \circ g_2) \circ (g_2^{-1} \circ k_2 \circ g_2) \\ &= k_3 \circ k_4 \end{aligned}$$

Define  $f_1 = g^{-1} \circ k_3 \circ g$  and  $f_2 = g^{-1} \circ k_4 \circ g$ . By Lemma 6.1,  $f_1, f_2 \in S_\theta$ . Thus

$$\begin{aligned} r &= g^{-1} \circ h_3 \circ g = g^{-1} \circ k_3 \circ k_4 \circ g = (g^{-1} \circ k_3 \circ g) \circ (g^{-1} \circ k_4 \circ g) \\ &= f_1 \circ f_2 \end{aligned}$$

■

The next theorem and its corollary show how to generate all of  $\text{SO}(3)$  from geometric products of pairs of elements of  $S_\theta$ .

**Theorem 7:** Suppose  $\theta \in \left[ \frac{\pi}{2}, \pi \right]$  and  $0 < \omega \leq \pi$ . Then there is a pair of elements in  $S_\theta$  whose geometric product lies in  $S_\omega$ .

**Proof:** Let  $h_1$  be the rotor in  $S_\theta$  that rotates by  $\theta$  about the  $x$ -axis:

$$h_1 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) e_2 e_3.$$

$h_1$  is the first desired element of  $S_\theta$ . We will find a second rotor,  $h \in S_\theta$ , such that  $h \circ h_1 \in S_\omega$ . To define  $h$ , we need  $a, b, c \in \mathbb{R}$  such that

$$a^2 + b^2 + c^2 = 1,$$

$$h = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) (a e_2 e_3 + b e_3 e_1 + c e_1 e_2), \text{ and}$$

$$|h \circ h_1| = \omega.$$

Recall that  $|h \circ h_1| = \omega \Leftrightarrow \text{Constant term of } h \circ h_1 = \cos\left(\frac{\omega}{2}\right)$ .

By performing the geometric product calculation we find that

$$\begin{aligned}
h \circ h_1 &= \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \\
&\quad + \left( \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}(\theta) + a \mathbf{Cos}(\theta) \mathbf{Sin}\left(\frac{\theta}{2}\right) \right) e_2 e_3 \\
&\quad + \left( b \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) - c \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_3 e_1 \\
&\quad + \left( c \mathbf{Cos}\left(\frac{\theta}{2}\right) \mathbf{Sin}\left(\frac{\theta}{2}\right) + b \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \right) e_1 e_2
\end{aligned}$$

The constant term of rotor  $h \circ h_1 = \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right)$ .

Setting  $\mathbf{Cos}\left(\frac{\omega}{2}\right) = \mathbf{Cos}^2\left(\frac{\theta}{2}\right) - a \mathbf{Sin}^2\left(\frac{\theta}{2}\right)$  and solving for the coefficient  $a$  yields

$$a = \frac{\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\theta}{2}\right)}.$$

This is well defined except when  $\mathbf{Sin}^2\left(\frac{\theta}{2}\right) = 0$ , which occurs only when

$\theta = 0 \bmod 2\pi$ . Since 0 is outside the domain  $\left[\frac{\pi}{2}, \pi\right]$  of  $\theta$ , the definition of  $a$  is well defined. “ $b$ ” and “ $c$ ” can be any numbers so long as  $a^2 + b^2 + c^2 = 1$ . In fact it is convenient to set  $b = 0$  and have  $c = \sqrt{1-a^2}$ .

Finally, we must show that  $a, b, c \in \mathbb{R}$  which will conclude the proof of the theorem. To do this we must show that  $a^2 \leq 1$ .

Since  $0 < \omega \leq \pi$ ,

$$0 < \frac{\omega}{2} \leq \frac{\pi}{2},$$

$$\mathbf{Cos}\frac{\pi}{2} \leq \mathbf{Cos}\frac{\omega}{2} < \mathbf{Cos}(0),$$

$$0 \leq \mathbf{Cos}\left(\frac{\omega}{2}\right) < 1,$$

$$-1 < -\mathbf{Cos}\left(\frac{\omega}{2}\right) \leq 0, \tag{iv}$$

Since  $\frac{\pi}{2} \leq \theta \leq \pi$ ,

$$\frac{\pi}{4} \leq \frac{\theta}{2} \leq \frac{\pi}{2},$$

$$0 \leq \mathbf{Cos}\left(\frac{\theta}{2}\right) \leq \frac{1}{\sqrt{2}},$$

$$0 \leq \mathbf{Cos}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2}, \quad (\text{v})$$

$$0 \leq 1 - \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq \frac{1}{2},$$

$$-\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) - 1 \leq 0,$$

$$\frac{1}{2} \leq \mathbf{Sin}^2\left(\frac{\theta}{2}\right) \leq 1. \quad (\text{vi})$$

Thus,

$$-1 = \frac{0-1}{1} \stackrel{(\text{iv, v, vi})}{\leq} \frac{\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\theta}{2}\right)} = a \stackrel{(\text{iv, v, vi})}{\leq} \frac{\frac{1}{2} + 0}{\frac{1}{2}} = 1$$

That is,  $|a| \leq 1$ . ■

**Corollary 7.1:** Suppose  $\frac{\pi}{2} \leq \theta \leq \pi$ ,  $0 < \omega \leq \pi$ , and  $S_\theta \subseteq H$ , where  $H$  is a group, not necessarily normal. Then  $S_\omega$  is generated by geometric products of pairs of elements from  $S_\theta$ , and so  $S_\omega \subseteq H$ .

**Proof:** By Theorem 7,  $\exists k_1, k_2 \in S_\theta \subseteq H$  such that  $k_1 \circ k_2 \in S_\omega$ . By Theorem 6,  $S_\omega = \{h \circ k : h, k \in S_\theta\} \subseteq H$ . ■

**Theorem 8:**  $SO(3)$  contains no proper normal subgroup.

**Proof:** Let  $H$  be a non-trivial normal subgroup of  $SO(3)$ .  $H$  contains a non-identity element  $1 \neq h_0 \in H$  where  $h_0$  is a rotation by some angle  $\phi \leq \pi$  about some axis.

a. Since  $h_0 \neq 1$ ,  $\phi > 0$ .

b. Since  $\phi > 0$ , there is a positive integer  $n$  such that  $\frac{\pi}{2} \leq n\phi \leq \pi$ .

Define  $h = h_0^n = h_0 \circ h_0 \circ \dots \circ h_0$  and  $\theta = n\phi$ .

c.  $h$  is a rotation of angle  $\theta$ , and  $\frac{\pi}{2} \leq \theta \leq \pi$ . Thus  $h \neq 1$ .

d. Since  $1 \neq h$ , by Theorem 5,  $S_\theta \subseteq H$ .

We wish to show that  $H = SO(3)$ . It suffices to show that  $H$  contains every sphere  $S_\omega$  in  $\mathcal{R}$ . For  $\omega = 0$ ,  $S_\omega \subseteq H$  since  $1 \in H$ . So let  $0 < \omega \leq \pi$  be an arbitrary rotation angle. By Corollary 7.1,  $S_\omega \subseteq H$ . ■

### Geometric Algebra Treatment of Reflections and Reflective Rotations:

Recall that  $O(3)$  is the disjoint union  $O(3) = SO(3) \cup T$  of the non-reflective and reflective rotations, respectively. In GA terminology, we can write this as

$O(3) = S \cup T$ . We have already defined  $S$ . Its elements are rotors, multivectors having the form

$$f = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [a_{23} e_2 e_3 + a_{31} e_3 e_1 + a_{12} e_1 e_2],$$

where  $a_{23}^2 + a_{31}^2 + a_{12}^2 = 1$ ,  $a_{23}, a_{31}, a_{12} \in \mathbb{R}$ , and  $0 \leq \theta \leq \pi$ .

We wish to define  $T$ . That is, we wish to describe a reflective rotation,  $h$ , in GA terms. We cannot represent  $h$  as a rotor because all rotors represent non-reflective rotations. Instead, we seek to represent  $h$  as a 3-dimensional multivector. Moreover, just as  $f^{-1} \circ v \circ f$  (non-reflectively) rotates the vector  $v$  into some vector  $w$ , we require the multivector  $h$  to have the property that  $h^{-1} \circ v \circ h$  reflectively rotates  $v$  into  $-w$ . That is, we require

$$-w = h^{-1} \circ v \circ h \tag{vii}$$

Because  $h = f \circ R$ , a rotation followed by a reflection, we also wish to find a multivector representation for the reflection operator  $R$ .

We quickly discover that the expression for the multivector  $R$  to reflect a vector  $w$  into  $-w$  must be  $-w = R \circ w \circ R$ . To see this, let

- $v \in \mathbb{R}^3$ : a vector in  $\mathbb{R}^3$
- $-v \in \mathbb{R}^3$ : the vector obtained by reflecting  $v$
- $f \in S$ : a rotation
- $w = f^{-1} \circ v \circ f \in \mathbb{R}^3$ : the vector obtained by rotating  $v$  by  $f$
- $-w \in \mathbb{R}^3$ : the vector obtained by reflecting  $w$
- $h = f \circ R \in T$ : the reflective rotation generated by  $f$

Since a reflection followed by a reflection is the identity,  $R$  is its own inverse,  $R = R^{-1}$ . So,

$$-w = h^{-1} \circ v \circ h = (f \circ R)^{-1} \circ v \circ (f \circ R) = (R^{-1} \circ f^{-1}) \circ v \circ (f \circ R) \stackrel{(vii)}{=} -w$$

$$\begin{aligned}
&= R \circ (f^{-1} \circ v \circ f) \circ R \\
&= R \circ w \circ R
\end{aligned} \tag{viii}$$

Defining  $R = i = \sqrt{-1}$  satisfies equation (viii).  $i$  is a scalar, and the scalar  $i$  is a multivector, so this satisfies our goal of defining  $R$  as a multivector. In Appendix 1 it is shown that while there are other multivector expressions for  $R$  that satisfy equation (viii), all such expressions involve imaginary terms. In Definition (1), below, we define  $R = i$ , the simplest such expression.

Since  $i$  is a scalar, it commutes with the geometric product operator:

$$f \circ g \circ i = i(f \circ g) = (if) \circ g = f \circ (ig) \text{ for all multivectors } f \text{ and } g.$$

**Definition:** The **Reflection Operator** is actually 3 different operators, depending on its domain. The original operator  $R$  with domain  $\mathbb{R}^3$  induces operators on  $\mathcal{S}$  and  $\mathcal{T}$ . Let  $i = \sqrt{-1}$ ,  $v \in \mathbb{R}^3$ ,  $f \in \mathcal{S}$ , and  $h \in \mathcal{T}$ . Define

- (1)  $\mathbf{R} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : R(v) = R^{-1} \circ v \circ R = i \circ v \circ i = -v$ ; i.e.,  $\mathbf{R} = i = \mathbf{R}^{-1}$ .
- (2)  $\mathbf{R}_S : \mathcal{S} \rightarrow \mathcal{T} : R_S(f) = f \circ R_S = f \circ i = if$ , i.e.,  $\mathbf{R}_S = i = \mathbf{R}_S^{-1}$ .
- (3)  $\mathbf{R}_T : \mathcal{T} \rightarrow \mathcal{S} : R_T(h) = h \circ R_T = h \circ (-i) = -ih$ ; i.e.,  $\mathbf{R}_T = -i = \mathbf{R}_T$ .

These definitions require clarification. We begin by observing we are now able to identify the multivector representation of a reflective rotation:

$$(4) \quad h = f \circ R_S = if = i \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [b_{23} e_2 e_3 + b_{31} e_3 e_1 + b_{12} e_1 e_2] \right\},$$

where  $f \in \mathcal{S}$ ,  $b_{23}^2 + b_{31}^2 + b_{12}^2 = 1$ ,  $b_{23}, b_{31}, b_{12} \in \mathbb{R}$ , and  $0 \leq \theta \leq \pi$ .

Lemma 9.1 provides the equation for the inverse of a reflective rotation.

**Lemma 9.1:** Let  $h = f \circ i \in \mathcal{T}$ , where  $f \in \mathcal{S}$ . Then  $h^{-1} = f^{-1} \circ i$ .

**Proof:** Let  $g = f^{-1} \circ i$ . By definition (2),  $g \in \mathcal{T}$  since  $f^{-1} \in \mathcal{S}$ . Because the expression for the geometric product of 2 members of  $\mathcal{T}$  involves 2 reflections,  $g \circ h \in \mathcal{S}$ . Thus,

$$g \circ h = (f^{-1} \circ i) \circ (f \circ i) = i^2 (f^{-1} \circ f) = -1.$$

Recall from Theorem 3 that  $-1 \notin \mathcal{S}$  but  $-1 \approx 1 \in \mathcal{S}$ . Therefore  $g \circ h = 1 \bmod \mathcal{S}$ . Similarly,  $h \circ g = 1$ , and we have that  $h^{-1} = g = f^{-1} \circ i$ . ■

Lemma 9.1 is somewhat surprising because one might expect that  $h^{-1} = i^{-1} \circ f^{-1} = -f^{-1} \circ i$ . The subtlety here is that when  $i \in \mathbb{C}$  then  $i^{-1} = -i$ , but  $i^{-1} = i \bmod \mathcal{T}$  when  $i$  is a reflective rotation<sup>1</sup>. This is similar to (and follows from)

---

<sup>1</sup>  $i^{-1} = R^{-1} = i$ .

$1 \neq -1$  when  $1 \in \mathbb{R}$  but  $1 = -1 \bmod \mathcal{S}$  (according to Theorem 3) when 1 is a rotation.

In the reflection operator definition, what does it mean when we say that  $R$  induces  $R_s$ ? Observe that both  $w \circ R$  and  $f \circ R$  appear in equation (viii). In  $w \circ R$ ,  $R$  is acting on a vector, but in  $f \circ R$ ,  $R$  is acting on a rotation. So the  $R$  in  $f \circ R$  becomes what I am calling the induced reflection  $R_s$ . The domain of the reflection operator has changed but the multivector expression hasn't changed, so  $R_s = R = i$ , which is Definition (2).

Why is  $R_T = -i$  (rather than just  $i$ ) in Definition (3)? Because  $1 = R_S \circ R_T = i R_T$ .

### Outline of Proof of Theorem 9:

Observe that what was called the reflection operation “ $R$ ” in the 2<sup>nd</sup> paragraph of page 1 is now defined to simply be “ $i$ ”. That is,  $i = R_S \in T$ . Using this notation, we have that the subgroup  $\mathcal{I} = \{1, i\}$ .

Let  $H$  be a proper normal subgroup of  $O(3)$  such that  $H \neq SO(3)$  and  $H \neq \mathcal{I}$ . From Theorem 8,  $\exists i \neq h \in H \cap T$ . Then  $h^2 \in SO(3) \cap H$ . If  $1 \neq h^2$ , by Theorem 8,  $SO(3) \subseteq H$ . Also,  $\exists f \in SO(3)$  such that  $h = f \circ R_S = i f$ . To see that  $T \subseteq H$ , let  $t \in T$ .  $\exists f_1 \in SO(3)$  such that  $t = f_1 \circ R_S = i f_1$ . Let  $f_2 = f_1 \circ f^{-1} \in SO(3)$ . Then  $f_1 = f_2 \circ f$ . So,

$$t = i f_1 = i (f_2 \circ f) = f_2 \circ (i f) = f_2 \circ h.$$

$t \in H$  since  $f_2 \in SO(3) \subset H$ . Thus  $T \subseteq H$ . Since, also,  $SO(3) \subseteq H$ , then  $H = O(3)$ .

Unfortunately,  $h^2$  does equal 1 when either  $h = i$  (ruled out) or  $h$  is the reflection of any  $\pi$  rotation (allowed). Even so, everything in the prior paragraph still holds except we will not have proved that  $SO(3) \subset H$ . To prove this, we will generate  $h_1 \in H \cap T$  such that  $1 \neq h \circ h_1 \in SO(3)$ . Then we will again have, by Theorem 8, that  $SO(3) \subseteq H$ .

Notice that  $h = f \circ R_S \Leftrightarrow h = i f \Leftrightarrow f = -i h \Leftrightarrow f = h \circ R_T$ . That is,

$$h = f \circ R_S \Leftrightarrow f = h \circ R_T \tag{ix}$$

**Lemma 9.2:**  $\mathcal{I} = \{1, i\}$  is a normal subgroup of  $O(3)$ .

**Proof:**  $g^{-1} \circ 1 \circ g = 1$  and  $g^{-1} \circ i \circ g = i$  for every  $g \in O(3)$ . ■

**Theorem 9:**  $SO(3)$  and  $\mathcal{I}$  are the only proper normal subgroups of  $O(3)$ .

**Proof:** To help keep track of variables, we will use  $s, s_0, s_1, s_2, s_3$ , and  $s_4$  for elements of  $SO(3)$  and  $t, t_0, t_1$ , and  $t_2$  for elements of  $T$ .

Suppose  $H$  is a non-trivial normal subgroup of  $O(3)$  and  $H \neq SO(3)$  and  $H \neq I$ .

Claim: There is an element  $t_1 \in H \cap T$  such that  $t_1 \neq i$ :

By Theorem 8,  $\exists t_0 \in H \cap T$ . If  $t_0 \neq i$ , the claim is true. If  $t_0 = i$ , since  $H \neq I$ ,  $H$  contains another element. If that element is in  $T$ , the claim is true. If the other element is  $s_0 \in SO(3)$ , we can assume  $s_0 \neq 1$  since  $H \neq I$ . Set  $t_1 = s_0 i$ . Then  $t_1 \in H \cap T$  and  $t_1 \neq i$ .

As an intermediate step we wish to find an element  $t_2 \in H \cap T$  such that  $t_1 \circ t_2 \neq 1$ . Set

$$s_1 \equiv t_1 \circ R_T = -i t_1 \in SO(3).$$

WLOG, we can assume  $s_1$  is a rotation about the **x-axis**. Thus,

$$s_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3.$$

By (ix),

$$t_1 = s_1 \circ R_S = i s_1 = i \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3 \right].$$

and

$$it_1 = - \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3 \right].$$

Let  $\theta = |s_1|$ .  $\theta \neq 0$  because  $t_1 \neq i$  and, consequently,  $s_1 \neq 1$ . Define

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{4}\right) + \mathbf{Sin}\left(\frac{\pi}{4}\right) e_1 e_2 = \frac{1+e_3 e_1}{\sqrt{2}}$$

a  $90^\circ$  rotation about the **y-axis**.  $g_2 \in SO(3)$  since it is a rotation. Hence,

$$s_2 \equiv g_2^{-1} \circ s_1 \circ g_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2.$$

$s_2 \in SO(3)$  is the rotation of angle  $\theta$  about the negative **z-axis**. Define

$$t_2 \equiv g_2^{-1} \circ t_1 \circ g_2 = g_2^{-1} \circ i s_1 \circ g_2 = i s_2 = i \left[ \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \right].$$

So  $t_2 \in T \cap H$  and

$$t_1 \circ t_2 = -\cos^2\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1 e_2 + \sin^2\left(\frac{\theta}{2}\right)e_3 e_1 - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2 e_3.$$

Notice that the constant term of  $t_1 \circ t_2 < 0$ . So, by Theorem 3, we replace the expression for  $t_1 \circ t_2$  by its negative:

$$t_1 \circ t_2 = \cos^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1 e_2 - \sin^2\left(\frac{\theta}{2}\right)e_3 e_1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2 e_3.$$

Since  $\theta \neq 0$ ,  $t_1 \circ t_2 \neq 1$ . This concludes the intermediate step.

We next wish to show that  $H \supset SO(3)$ . First, we observe that

$$\begin{aligned} s_1 \circ s_2 &= \cos^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_1 e_2 - \sin^2\left(\frac{\theta}{2}\right)e_3 e_1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e_2 e_3 \\ &= t_1 \circ t_2. \end{aligned}$$

We define

$$\textcolor{red}{s}_3 \equiv t_1 \circ t_2 = s_1 \circ s_2.$$

$1 \neq s_3 \in H$  because  $s_3 = t_1 \circ t_2$ .  $s_3 \in SO(3)$  because  $s_3 = s_1 \circ s_2$ . So,  
 $1 \neq s_3 \in H \cap SO(3)$ .

$H \cap SO(3)$  is thus a non-trivial normal subgroup of  $SO(3)$ , so by Theorem 8  
 $H \cap SO(3) = SO(3)$ .

Thus,

$$H \supset SO(3).$$

Finally, we are in a position to show as well that  $H \supset T$ , implying that  $H = O(3)$  since  $H \supset SO(3) \cup T = O(3)$ . Let  $t \in T$  be any element of  $T$ . This proof is finished if we can show  $t \in H$ . Let

$$\textcolor{red}{s} \equiv -it = t \circ R_T \in SO(3).$$

$\exists x, y, \text{ and } z$  where  $x^2 + y^2 + z^2 = 1$  and  $\omega$  satisfying  $0 \leq \omega \leq \pi$  such that

$$s = \cos\frac{\omega}{2} + \sin\frac{\omega}{2} [x e_2 e_3 + y e_3 e_1 + z e_1 e_2].$$

By (ix),

$$t = s \circ R_s = is = i \left\{ \cos\frac{\omega}{2} + \sin\frac{\omega}{2} [x e_2 e_3 + y e_3 e_1 + z e_1 e_2] \right\}.$$

Define

$$\mathbf{s}_4 \equiv s_1^{-1} \circ s \in \text{SO}(3) \subset H.$$

$s_4$  is the rotation whose geometric product with  $s_1$  equals  $s$  :  
 $s = (s_1 \circ s_1^{-1}) \circ s = s_1 \circ (s_1^{-1} \circ s) = s_1 \circ s_4$ .

Therefore  $s_4$  is also the rotation whose geometric product with  $t_1$  equals  $t$  :  
 $t = s \circ i = (s_1 \circ s_4) \circ i = (s_1 \circ i) \circ s_4 = t_1 \circ s_4$

So,  $t \in H$  since  $t_1, s_4 \in H$ , and that completes the proof the theorem and part (B).

■

## APPENDIX 1: GA expressions for defining the reflection operator

$$\textcolor{brown}{R}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Let  $w$  be a vector in  $\mathbb{R}^3$ . Without loss of generality,

$$w = e_1$$

We need to solve for the coefficients in the GA multivector expression for  $R$ :

$$R = c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 + c_{12} e_1 e_2 + c_{23} e_2 e_3 + c_{13} e_1 e_3 + c_{123} e_1 e_2 e_3$$

From equation (viii)  $R$  must satisfy

$$R \circ w \circ R = -w = -e_1.$$

By symmetry,

$$c_1 = c_2 = c_3 \text{ and } c_{12} = c_{23} = c_{13}.$$

Thus,

$$R = c_0 + c_1 e_1 + c_1 e_2 + c_1 e_3 + c_{12} e_1 e_2 + c_{12} e_2 e_3 + c_{12} e_1 e_3 + c_{123} e_1 e_2 e_3.$$

Expanding  $R \circ w \circ R$  yields

$$R \circ w \circ R = A + B e_1 + C e_2 + D e_3 = -w = 0 - e_1 + 0 e_2 + 0 e_3,$$

where

$$A = 2(c_0 c_1 + c_{12} c_{123}),$$

$$B = c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2,$$

$$C = 2(c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123}),$$

and

$$D = 2(c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123}).$$

This leads to 4 equations in the 4 unknowns  $c_0, c_1, c_{12}, c_{123}$ :

$$\left\{ \begin{array}{lcl} c_0 c_1 + c_{12} c_{123} & = & 0 \\ c_0^2 + c_{123}^2 - c_1^2 - c_{12}^2 & = & -1 \\ c_1^2 + c_{12}^2 - c_0 c_{12} + c_1 c_{123} & = & 0 \\ c_1^2 + c_{12}^2 + c_0 c_{12} - c_1 c_{123} & = & 0 \end{array} \right\}$$

Adding equations (3) and (4) yields

$$c_1^2 + c_{12}^2 = 0 \Rightarrow c_{12} = \pm c_1 i.$$

Equation (1) then becomes

$$\pm c_0 c_{12} i + c_{12} c_{123} = \pm c_{12} (c_0 i + c_{123}) = 0 \Rightarrow c_{12} = 0 \text{ or } c_{123} = \pm c_0 i.$$

$c_{123} = \pm c_0 i$  is not possible because, along with  $c_{12} = \pm c_1 i$ , it transforms (2) into  
 $c_0^2 - c_0^2 - c_1^2 + c_1^2 = -1$ , or  $0 = -1$ .

Thus

$$c_{12} = 0 \text{ (and } c_{12} = \pm c_1 i)$$

which implies that

$$c_1 = 0$$

and equation (2) becomes

$$c_0^2 + c_{123}^2 = -1 \Rightarrow c_{123} = \pm \sqrt{-1 - c_0^2} = \pm i \sqrt{c_0^2 + 1}$$

From this it is clear that if  $c_0$  is real, then  $c_{123}$  is imaginary. This proves the claim that every multivector representation for  $R$  involves an imaginary term.

I list below the four simplest expressions for  $R$ :

$$R = \pm i$$

$$R = \pm i e_1 e_2 e_3$$

$$R = 1 \pm i\sqrt{2} e_1 e_2 e_3$$

$$R = i\sqrt{2} \pm e_1 e_2 e_3$$

For the record, if instead of starting with  $w = e_1$  we use  $w = a_1 e_1 + a_2 e_2 + a_3 e_3$ , the resulting equations lead to the same solution set.

## APPENDIX 2: The 2 insights that suggested how to define the rotor $g_{23}$ in Lemma 5.2.

The hardest part of this proof was to invent the rotor  $g_{23}$ . I couldn't solve a certain system of 4 non-linear simultaneous equations that would have provided it, but I was able to guess a solution after examining lots of rotations.

The **1st insight**, gained during Lemma 5.1, is that a rotor, say  $h_3$ , about the z-axis, is obtained by  $g_2^{-1} \circ h_1 \circ g_2$  where  $g_2$  is a rotor about the y-axis with a  $90^\circ$  rotation angle and  $h_1$  is a rotor about the x-axis having the same angle of rotation,  $\theta$ , as  $h_2$ . Loosely speaking, I learned that a  $90^\circ$  rotation  $g_2$  yields a rotation about an axis  $90^\circ$  away from  $h_1$ .

**2<sup>nd</sup> Insight** comes from discovering that reducing the  $g_2$  angle of rotation to  $45^\circ$  generates a rotation about the  $45^\circ$  diagonal between the x-axis and z-axis (i.e., about an axis  $45^\circ$  away from  $h_1$ ), seen below:

Let

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{8}\right) - \mathbf{Sin}\left(\frac{\pi}{8}\right) \mathbf{e}_1 \mathbf{e}_3.$$

Then

$$\begin{aligned} g_2^{-1} \circ h_1 \circ g_2 &= \mathbf{Cos}\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}} \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{e}_1 \mathbf{e}_2 + \frac{1}{\sqrt{2}} \mathbf{Sin}\left(\frac{\theta}{2}\right) \mathbf{e}_2 \mathbf{e}_3 \\ &= 45^\circ \text{ diagonal axis in the } zx\text{-plane.} \end{aligned}$$

To generate a  $45^\circ$  diagonal axis not in the  $xy$ -,  $yz$ -, or  $zx$ -planes, I made a guess that it requires products of rotors like  $g_2$  and  $g_3$  where  $g_2$  generates a skew axis between the z and x-axes, and then  $g_3$  generates an out-of-zx-plane skew axis towards the y-axis.

Armed with the guess  $g_{23} = g_2 \circ g_3$ , for a given angle  $\psi$ :

- I assigned  $g_2$  and  $g_3$  rotor angles  $\beta$  and  $\gamma$ , respectively,
- Performed the  $g_{23}^{-1} \circ h \circ g_{23}$  GA multiplication with Mathematica, and
- Solved for a constant term equal to  $\mathbf{Cos}\left(\frac{\psi}{2}\right)$ .

This yielded the **ArcCos** and **ArcTan** values used in Lemma 5.2. The proof in Lemma 5.2 is just the confirmation multiplication that the solution found is correct.