# The Theoretical Minimum
## Classical Mechanics - Solutions
### L11E05

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**Exercise 1.** *Show that in the $x, y$ plane, the solution to Eq.* (25) *and the solution to Eq.* (26) *are a circular orbit with the center of the orbit being anywhere on the plane. Find the radius of the orbit in terms of the velocity.*

Let's recall Eq. (25) and Eq. (26):

$$a_y = -\frac{eb}{mc}v_x; \qquad a_x = \frac{eb}{mc}v_y$$

Let's rewrite them with dots instead:

$$\ddot{y} = -\frac{eb}{mc}\dot{x}; \qquad \ddot{x} = \frac{eb}{mc}\dot{y}$$

First, note that each equation was obtained from a different gauge. But, as the Hamiltonian is gauge-invariant, the equation of motions aren't affected by a gauge shift. So they do both describe the motion we're interested in and can be "combined".

Then, we've been using similar-looking differential equations (for instance when considering the harmonic oscillator), and we know they were solved by a sine/cosine-like function. So we're going to make some guess and calibrate a sine to have it work. We'll then have an expression for both $x(t)$ and $y(t)$, and if we can find a $r$ such as:

$$(x(t) - a)^2 + (y(t) - b)^2 = r^2$$

We'll know that our coordinate function can draw a circle of radius $r$, centered at $(a, b)$.

Alright so let's say $\dot{x}$ is a sin function; then

- $x$ would be a $-\sin$;
- $\ddot{x}$ would be a sin;
- $\ddot{y}$ would also be a cos;
- $\dot{y}$ would be a $-\sin$;
- and $y$ would be a $-\cos$.

Looks promising, as $x^2 + y^2$ would involve a $\cos^2 + \sin^2 = 1^2$. Now we'd just have to "caliber" the sin properly. Say, if it's a $\sin(\omega t)$, then by repeated differentiation, the $\omega$ would become a multiplicative factor, outside of the sin. So, the following feels reasonable:

$$\omega = \frac{eb}{mc}$$

Let's see what would happen to the first equation, if we start with a $\dot{x} = \sin(\omega t)$:

$$\ddot{y} = -\frac{eb}{mc}\dot{x} = -\omega\sin(\omega t)$$

From which we can derive both:

$$\dot{y} = \cos(\omega t); \qquad \ddot{x} = \omega \cos(\omega t)$$

And those two do validate the second equation:

$$\ddot{x} = \frac{eb}{mc}\dot{y}$$

So we've found a solution; note that we can shift both component by arbitrary constants $a$ and $b$ without affecting their correctness:

$$\boxed{y(t) = \frac{1}{\omega}\sin(\omega t) + a; \qquad x(t) = -\frac{1}{\omega}\cos(\omega t) + b}$$

And $\omega$ was well-named, as it correspond to the angular velocity. Does it describe an orbit around a point $(a, b)$? Let's find out:

$$(x(t) - a)^2 + (y(t) - b)^2 = \frac{1}{\omega^2}\Big(\sin^2(\omega t) + \cos^2(\omega t)\Big) = \Big(\frac{1}{\omega}\Big)^2$$

So they do draw a circle of radius:

$$\boxed{r = \frac{1}{\omega} = \frac{mc}{eb}}$$