The Theoretical Minimum

Classical Mechanics - Solutions

L06E06

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Exercise 1. *Explain how we derived this.*

Let us recall that "this" refers to the following expression for the kinetic energy:

$$T = m(\dot{x_+}^2 + \dot{x_-}^2)$$

Starting from the following Lagrangian, involving two particles moving on a line with respective position and velocity x_i, $\dot{x}_i$:

$$L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) - V(x_1 - x_2) \tag{1}$$

After having performed the following change of coordinates:

$$x_+ = \frac{x_1 + x_2}{2} \qquad\qquad x_- = \frac{x_1 - x_2}{2} \tag{2}$$

From the Lagrangian, (1) we have the kinetic energy:

$$T = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \tag{3}$$

By first both summing and substracting the two equations of (2), and then by linearity of the derivation, we get:

$$\begin{aligned} x_+ + x_- &= x_1 & \qquad x_+ - x_- &= x_2 \\ \dot{x_+} + \dot{x_-} &= \dot{x_1} & \qquad \dot{x_+} - \dot{x_-} &= \dot{x_2} \end{aligned} \tag{4}$$

It's now simply a matter of injecting (4) into (3):

$$\begin{aligned} T &= \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= \frac{m}{2}((\dot{x_+} + \dot{x_-})^2 + (\dot{x_+} - \dot{x_-})^2) \\ &= \frac{m}{2}(2\dot{x_+}^2 + 2\dot{x_-}^2 + 2\dot{x_+}\dot{x_-} - 2\dot{x_+}\dot{x_-}) \\ &= m(\dot{x_+}^2 + \dot{x_-}^2) \quad \square \end{aligned}$$