The Theoretical Minimum

Quantum Mechanics - Solutions

L05E02

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Exercise 1. *1) Show that* $\Delta A^2 = \langle \bar{A}^2 \rangle$ *and* $\Delta B^2 = \langle \bar{B}^2 \rangle$

2) Show that $[\bar{A}, \bar{B}] = [A, B]$

3) Using these relations, show that

$$\Delta A\ \Delta B \geq \frac{1}{2} \langle \Psi | [A, B] | \Psi \rangle$$

OK, let's as usual recall the context: A and B are two observables. We defined the expectation value of an observable C with eigenvalues labelled as c to be:

$$\langle C \rangle := \langle \Psi | C | \Psi \rangle = \sum_c c P(c)$$

We construct from C a new observable $\bar{C}$:

$$\bar{C} := C - \langle C \rangle I$$

Where the identity I is sometimes implicit. The eigenvalues of $\bar{C}$ are denoted $\bar{c}$ and can be expressed in terms of C's eigenvalues, denoted c:

$$\bar{c} = c - \langle C \rangle$$

From there, we defined the *standard deviation*, or the square of the uncertainty of C, assuming a "well-behaved" probability distribution P, by:

$$(\Delta C)^2 := \sum_c \bar{c}^2 P(c)$$

Let's first quickly prove that $\bar{c} = c - \langle C \rangle$ are indeed the eigenvalues of $\bar{C} = C - \langle C \rangle I$. Consider an eigenvalue c of C, with associated eigenvector $|c\rangle$. It follows that:

$$\begin{aligned} & C|c\rangle = && c|c\rangle \\ \Leftrightarrow \quad & C|c\rangle - \langle C \rangle |c\rangle = && c|c\rangle - \langle C \rangle |c\rangle \\ \Leftrightarrow \quad & (C - \langle C \rangle I)|c\rangle = && (c - \langle C \rangle)|c\rangle \\ \Leftrightarrow \quad & \bar{C}|c\rangle = && (c - \langle C \rangle)|c\rangle \end{aligned}$$

Meaning, $|c\rangle$ is still an eigenvector of $\bar{C}$, but now associated to the eigenvalue $c - \langle C \rangle$. The $|c\rangle$ still make an orthonormal basis of the state space, so there are no other eigenvectors (there can't be more eigenvectors than the dimension of the surrounding state-space). $\square$

Similarly, we can prove that c^2 are the eigenvalues associated to C^2, for an observable C: again start from an eigenvalue c of C, associated to an eigenvector $|C\rangle$:

$$C|c\rangle = c|c\rangle \Leftrightarrow C(C|c\rangle) = C(c|c\rangle) \Leftrightarrow C^2|c\rangle = c(\underbrace{C|c\rangle}_{c|c\rangle}) \Leftrightarrow C^2|c\rangle = c^2|c\rangle) \quad \square$$

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1) We'll prove the fact for an arbitrary observable C: it'll naturally hold for both A and B.

$$\begin{aligned}
(\Delta C)^2 := & \sum_c \bar{c}^2 P(c) \\
= & \sum_c (c - \langle c \rangle)^2 P(c) \quad \text{(definition of } \bar{c}) \\
= & \langle \Psi | \bar{C}^2 | \Psi \rangle =: \langle \bar{C}^2 \rangle \quad \text{(two previous properties)} \quad \square
\end{aligned}$$

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2) This is an elementary calculation:

$$\begin{aligned}
[\bar{A}, \bar{B}] := & \bar{A}\bar{B} - \bar{B}\bar{A} && \text{(commutator's definition)} \\
= & (A - \langle A \rangle I)(B - \langle B \rangle I) - (B - \langle B \rangle I)(A - \langle A \rangle I) && \text{(definition of } \bar{C}) \\
= & \Big(AB - \langle A \rangle B - \langle B \rangle A + \langle A \rangle \langle B \rangle I \Big) - \Big(BA - \langle B \rangle A - \langle A \rangle B + \langle B \rangle \langle A \rangle I \Big) \\
= & AB - BA \\
=: & [A, B] && \text{(commutator's definition)} \quad \square
\end{aligned}$$

Remember, $\langle A \rangle$ and $\langle B \rangle$ are real numbers (their multiplication is then commutative).

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3) This is now just about following the reasoning preceding the exercise in the book, as suggested by the authors, by replacing A and B with $\bar{A}$ and $\bar{B}$.

So let:

$$|X\rangle = \bar{A}|\Psi\rangle = (A - \langle A \rangle I)|\Psi\rangle; \qquad |Y\rangle = i\bar{B}|\Psi\rangle = i(B - \langle B \rangle I)|\Psi\rangle$$

Recall the general form of Cauchy-Schwarz for a complex vector space[1]:

$$2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$$

Where the norm is defined from the inner-product:

$$|X| = \sqrt{\langle X|X\rangle}$$

Injecting our two vectors in such a Cauchy-Schwarz equation yields:

$$\begin{aligned}
2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} \geq & |i(\langle \Psi|\bar{A}\bar{B}|\Psi\rangle - \langle \Psi|\bar{B}\bar{A}|\Psi\rangle)| \\
\geq & |\langle \Psi|[\bar{A}, \bar{B}]|\Psi\rangle| && \text{(commutator definition)} \\
\geq & |\langle \Psi|[A, B]|\Psi\rangle| && \text{(from 2), } [\bar{A}, \bar{B}] = [A, B])
\end{aligned}$$

But from 1), we know that

$$2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} = 2\sqrt{(\Delta A)^2 (\Delta B)^2} = 2\Delta A \Delta B$$

Note that the $\sqrt{\cdot}$ can be removed "safely" as the ΔC^2 are defined as a sum of positive terms (no absolute values necessary).

Putting the two together yields the expected, *general uncertainty principle*:

$$\boxed{\Delta A \Delta B \geq |\langle \Psi|[A, B]|\Psi\rangle|} \quad \square$$

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[1] I'm sticking to the authors' terminology and notations.