The Theoretical Minimum

Classical Mechanics - Solutions

L05E03

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Exercise 1. *Rework Exercise 2 for the potential* $V = \frac{k}{2(x^2+y^2)}$. *Are there circular orbits? If so, do they all have the same period? Is the total energy conserved?*

Equations of motion
The approach is similar to what has been done for the previous exercise: for this system, the potential energy V is:

$$V = \frac{k}{2(x^2+y^2)} \tag{1}$$

By Newton's second law of motion[1], given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{2}$$

Or,

$$\begin{aligned} F_x &= \quad m\ddot{x} \\ F_y &= \quad m\ddot{y} \end{aligned} \tag{3}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \tag{4}$$

As for the previous exercise, we make heavy use of the chain rule[2] for derivation:

$$\frac{d}{dx} f(g(x)) = g'(x) f'(g(x)) \tag{5}$$

To compute e.g. $F_x(x, y)$, we define $\phi(x) = x^2 + y^2$:

$$\begin{aligned} F_x(x,y) &= -\frac{\partial}{\partial x} V(x,y) \\ &= \frac{k}{2} \frac{d}{dx} \frac{1}{\phi(x)} \\ &= \frac{k}{2} \phi'(x) \frac{-1}{\sqrt{\phi(x)}} \\ &= \frac{kx}{(x^2+y^2)^2} \end{aligned} \tag{6}$$

Thus finally:

[1] https://en.wikipedia.org/wiki/Newton%27s_laws_of_motion#Second
[2] https://en.wikipedia.org/wiki/Chain_rule

$$\begin{aligned} F_x(x,y) &= \frac{kx}{(x^2+y^2)^2} \\ F_y(x,y) &= \frac{ky}{(x^2+y^2)^2} \end{aligned} \tag{7}$$

Hence combining (7) and (3):

$$\begin{aligned} F_x(x,y) &= \boxed{m\ddot{x}(t) = k\frac{x(t)}{(x(t)^2+y(t)^2)^2}} \\ F_y(x,y) &= \boxed{m\ddot{y}(t) = k\frac{y(t)}{(x(t)^2+y(t)^2)^2}} \end{aligned} \tag{8}$$

Circular orbits

Let's make a guess, and see what would happen were we to plug the simplest circular motion, that we've already studied in the book at the end of Chapter 2 (Motion), given by:

$$x(t) = R\cos(\omega t); \qquad y(t) = R\sin(\omega t)$$

Which is very convenient for us, because if we try this solution in (8), the (common) denominator simplifies:

$$\left(x(t)^2+y(t)^2\right)^2 = \left((R\cos(\omega t))^2 + (R\sin(\omega t))^2\right)^2 = R^4\underbrace{(\cos^2(\omega t)+\sin^2(\omega t))^2}_{=1} = R^4$$

Let's now consider the velocities and accelerations we would obtain by differentiating our guess for $x(t)$ and $y(t)$:

$$\begin{aligned} \dot{x}(t) &= -R\omega\sin(\omega t); & \dot{y}(t) &= R\omega\cos(\omega t) \\ \ddot{x}(t) &= -R\omega^2\cos(\omega t); & \ddot{y}(t) &= -R\omega^2\sin(\omega t) \end{aligned}$$

There are two ways for this guess to actually work:

1. Either we set $\omega^2 = -k/mR^4$, which implies either:
   - $\boxed{k \text{ to be zero}}$ (trivial solution then);
   - or that mR to be close to infinite (unrealistic);
   - or $\boxed{\text{that } k \text{ is (strictly) negative;}}$
   - or that either m or R are negative (unrealistic);
   - or, mathematically, that ω is an imaginary (complex) number, which would be difficult to interpret, physically;
2. The other option would be for R to be negative, which again doesn't make a lot of sense, physically-wise.

Remark 1. *Note that our guess would have worked for a negated V:*

$$V = -\frac{k}{2(x^2+y^2)}$$

Remark 2. *What is commonly referred to as "the trivial solution", especially in the context of differential equations, is the solution $x(t) = 0$, which is of little interest, mathematically and physically.*

We can conclude that, at least physically speaking, $\boxed{\textbf{there are no circular orbits}\text{, unless } k \text{ is negative}}$. This is because, if there were circular orbits, then they would be a coordinate change away from being in the form of our guess.

The only remaining issue is that k hasn't been clearly defined, physically speaking, so we can't really know for sure if assuming k to be negative (with a reminder that $k = 0$ leads to the trivial solution).

Remark 3. *Another approach, used for instance in the official solutions*[3]*, relies on the polar coordinate* (r, θ)*: the existence of a circular orbit then translate to* r *being a constant, or equivalently,* $\dot{r} = 0$.

We'll dive deeper into polar coordinates in a later exercise, alongside a bunch of other elements related to circular motion (L06E05, which involves a pendulum).

Energy conservation

Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2}\sum_i m_i \dot{x_i}^2 \tag{9}$$

Which gives us for this system, expliciting the time-dependencies:

$$T(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}m\dot{y}(t)^2 = \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2) \tag{10}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt}T(t) &= \frac{1}{2}m(2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{11}$$

On the other hand, we can compute the variation of potential energy over time from (1). We'll use the chain rule again, with $\phi(t) = x(t)^2 + y(t)^2$ and thus:

$$\begin{aligned} \phi'(t) &= 2x'(t)x(t) + 2y'(t)y(t) \\ &= 2\dot{x}x + 2\dot{y}y \end{aligned}$$

It follows that:

$$\begin{aligned} \frac{d}{dt}V(t) &= \frac{d}{dt}\frac{k}{2(x(t)^2 + y(t)^2)} \\ &= \frac{k}{2}\frac{d}{dt}\phi(t)^{-1} \\ &= -\frac{k}{2}\phi'(t)\phi(t)^{-2} \\ &= -\frac{k}{2}\frac{2\dot{x}x + 2\dot{y}y}{(x(t)^2 + y(t)^2} \\ &= -k\frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^2} \\ &= -k\frac{\dot{x}x + \dot{y}y}{\phi(t)^2} \end{aligned} \tag{12}$$

Then, from (8), we can extract

$$x(t) = \frac{m}{k}\ddot{x}\phi(t)^2; \qquad y(t) = \frac{m}{k}\ddot{y}\phi(t)^2$$

Injecting in (12) gives:

$$\begin{aligned} \frac{d}{dt}V(t) &= -\frac{k}{\phi(t)^2}\left(\dot{x}\frac{m}{k}\ddot{x}\phi(t)^2 + \dot{y}\frac{m}{k}\ddot{y}\phi(t)^2\right) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{13}$$

And so by combining (13) and (11) we can indeed see that the energy is conserved:

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square$$

[3] `http://www.madscitech.org/tm/slns/l5e3.pdf`