

The Theoretical Minimum

Classical Mechanics - Solutions

L05E02

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Exercise 1. Consider a particle in two dimensions, x and y . The particle has mass m . The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitly that the total energy is conserved.

Equations of motion: For this system, the potential energy V is:

$$V = \frac{1}{2}k(x^2 + y^2) \quad (1)$$

By Newton's second law of motion, given $\mathbf{r} = (x, y)$, we have:

$$\mathbf{F} = m\mathbf{a} = m\dot{\mathbf{v}} = m\ddot{\mathbf{r}} \quad (2)$$

Or,

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \end{aligned} \quad (3)$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i , derived from the potential energy V :

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \quad (4)$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x} V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y} V(x, y) = -ky \end{aligned} \quad (5)$$

Combining (3) and (5):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \quad (6)$$

Which are known by L03E04 to be differential equations associated to harmonic motion, and solved by a slightly more general solution than the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \quad (7)$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$\begin{aligned}
v(t) &= \dot{x}(t) = \omega(-\alpha \sin(\omega t - \theta) + \beta \cos(\omega t - \theta)) \\
a(t) &= \ddot{x}(t) = -\omega^2(\alpha \cos(\omega t - \theta) + \beta \sin(\omega t - \theta)) \\
&= -\omega^2 x(t)
\end{aligned} \tag{8}$$

Where, to differentiate e.g. $\alpha \cos(\omega t - \theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$\frac{d}{dx} f(g(x)) = g'(x) f'(g(x)) \tag{9}$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t=0)$, $\dot{x}(t=0)$, $y(t=0)$, $\dot{y}(t=0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$\begin{aligned}
x(0) &= \alpha_x \cos(0) + \beta_x \sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha \sin(0) + \beta \cos(0)) \\
&= \omega \beta \\
&= \sqrt{\frac{k}{m}} \beta
\end{aligned} \tag{10}$$

Circular orbits: The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r , centered on point (a, b) , with $(a, b, r) \in \mathbb{R}^3$ if:

$$(\forall t \geq 0), (x(t) - a)^2 + (y(t) - b)^2 = r^2 \tag{11}$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$\sin(\theta \pm \varphi) = \sin \theta \cos \varphi \pm \cos \theta \sin \varphi$$

Then, let's introduce two angles φ_x and φ_y such as:

$$\begin{aligned}
\sin \varphi_x &= \alpha_x & \sin \varphi_y &= \alpha_y \\
\cos \varphi_x &= \beta_x & \cos \varphi_y &= \beta_y
\end{aligned}$$

From, there, we can use the previous identity to rewrite our equations of motions (7) as:

$$\begin{aligned}
x(t) &= \sin \varphi_x \cos(\omega t - \theta_x) + \cos \varphi_x \sin(\omega t - \theta_x) = \sin(\omega t + \varphi_x - \theta_x) = \sin \Omega_x \\
y(t) &= \sin \varphi_y \cos(\omega t - \theta_y) + \cos \varphi_y \sin(\omega t - \theta_y) = \sin(\omega t + \varphi_y - \theta_y) = \sin \Omega_y
\end{aligned} \tag{12}$$

Obvoiusly with $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$ and $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$.

Let us now develop (11) with those two versions of $x(t)$ and $y(t)$:

$$\begin{aligned}
(x(t) - a)^2 + (y(t) - b)^2 &= r^2 \\
\Leftrightarrow (\sin \Omega_x - a)^2 + (\sin \Omega_y - b)^2 &= r^2 \\
\Leftrightarrow \sin^2 \Omega_x + \sin^2 \Omega_y - 2(a \sin \Omega_x + b \sin \Omega_y) + a^2 + b^2 &= r^2
\end{aligned}$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a, b) = (0, 0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$\sin^2 \Omega_x + \sin^2 \Omega_y = r^2$$

Which we can rewrite a little bit using the fact that $\sin \varphi = \cos(\varphi - \pi/2)$:

$$\sin^2 \Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2$$

But we know the pythagorean identity $\sin^2 \varphi + \cos^2 \varphi = 1$, hence we know there will be circular orbits when:

$$\begin{cases} r &= 1 \\ \Omega_x &= \Omega_y - \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r &= 1 \\ \omega t + \varphi_x - \theta_x &= \omega t + \varphi_y - \theta_y - \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r &= 1 \\ \varphi_x - \theta_x &= \varphi_y - \theta_y - \frac{\pi}{2} \end{cases}$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (12) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

Remark 1. For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.

Remark 2. The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.

Energy conservation: Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 \quad (13)$$

Which gives us for this system, expliciting the time-dependancies:

$$T(t) = \frac{1}{2} m \dot{x}(t)^2 + \frac{1}{2} m \dot{y}(t)^2 = \frac{1}{2} m (\dot{x}(t)^2 + \dot{y}(t)^2) \quad (14)$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt} T(t) &= \frac{1}{2} m (2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \quad (15)$$

On the other hand, we can compute the variation of potential energy over time from (1), again using the chain rule:

$$\begin{aligned} \frac{d}{dt} V(t) &= \frac{1}{2} k \left(\frac{d}{dt} x(t)^2 + \frac{d}{dt} y(t)^2 \right) \\ &= \frac{1}{2} k (2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= k(x(t)\dot{x}(t) + y(t)\dot{y}(t)) \end{aligned} \quad (16)$$

From (6), we have:

$$\begin{aligned} x(t) &= -\frac{m}{k} \ddot{x}(t) \\ y(t) &= -\frac{m}{k} \ddot{y}(t) \end{aligned} \quad (17)$$

Injecting in (16):

$$\begin{aligned} \frac{d}{dt} V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \quad (18)$$

Thus from (15) and (18):

$$\frac{d}{dt} E(t) = \frac{d}{dt} T(t) + \frac{d}{dt} V(t) = 0 \quad \square \quad (19)$$

That is, total energy E over time doesn't change.