The Theoretical Minimum

Classical Mechanics - Solutions

L07E01

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Exercise 1. *Derive Equations* (2) *and explain the sign difference.*

Let us recall Equations (2):

$$\dot{p_1} = -V'(q_1 - q_2) \qquad \dot{p_2} = +V'(q_1 - q_2)$$

We have to derive them from the Lagrangian given in Equation (1), which represents a system of two generalized coordinates q_1 and q_2:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1 - q_2) \tag{1}$$

To retrieve the equations of motions from a Lagrangian, we need to use Euler-Lagrange's equations, for instance recalled as Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Let us also recall, again from previous chapter, right after Equation (13), that the conjugate momentum is defined by

$$p_i = \frac{\partial}{\partial \dot{q_1}}L$$

For our Lagrangian (1), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}}L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}}L = \dot{q_2} \tag{2}$$

$$\frac{d}{dt}p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt}p_2 = \dot{p_2} = \ddot{q_2} \tag{3}$$

Using the chain rule[1] for the other half, with $\varphi(q_i) = q_1 - q_2$, we get:

$$\begin{aligned}
\frac{\partial}{\partial q_1}L &= -\frac{\partial}{\partial q_1}V(\varphi(q_1)) & \frac{\partial}{\partial q_2}L &= -\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\
&= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1}V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\
&= -(\frac{\partial}{\partial q_1}V)(q_1 - q_2) & &= +(\frac{\partial}{\partial q_2}V)(q_1 - q_2)
\end{aligned} \tag{4}$$

By noting $V' = \frac{\partial}{\partial q_i}V$, and combining equations (2), (3) and (4), we indeed obtain the expected equations of motion $\square$.

Remark 1. *That is, assuming,* $\frac{\partial}{\partial q_1}V(q_1) = \frac{\partial}{\partial q_2}V(q_2)$*: for all the energy potential presented earlier in the book, there's indeed such a symmetry, e.g.*

[1] `https://en.wikipedia.org/wiki/Chain_rule`

$$V = \frac{1}{2}k(x^2 + y^2), \qquad p103$$
$$V = \frac{1}{2}\frac{k}{x^2 + y^2}, \qquad p103$$
$$V = -m\omega^2(X^2 + Y^2), \qquad p120$$

A similar tacit assumption seems to exists in Herbert Goldstein's Classical Mechanics[2].

Mathematically, the sign difference comes from the fact that the potential depends on one side from q_1 and on the other from $-q_2$, which will persist when differentiating the potential V. Physically, it reflects that there's an order relation between the two "positions" q_1 and q_2: one will come before the other, and our potential V depends on this ordering.

[2] `https://physics.stackexchange.com/a/107141`