

The Theoretical Minimum

Classical Mechanics - Solutions

L06E03

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Exercise 1. Use the Euler-Lagrange equations to derive the equations of motions from the Lagrangian in Eq. (12).

Again, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: ($\forall i \in \llbracket 1, n \rrbracket$),

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}_i} L \right) = \frac{\partial}{\partial x_i} L \quad (1)$$

In the case of this exercise, the Lagrangian L is defined in Eq. (12) as:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X)$$

Let's compute the partial derivatives of L on \dot{X} , X , \dot{Y} and Y :

$$\begin{aligned} \frac{\partial}{\partial \dot{X}} L &= \frac{\partial}{\partial \dot{X}} \left(\frac{m}{2} \dot{X}^2 + m\omega \dot{X}Y \right) & \frac{\partial}{\partial X} L &= \frac{\partial}{\partial X} \left(\frac{m\omega^2}{2} X^2 - m\omega \dot{Y}X \right) \\ &= m\dot{X} + m\omega Y & &= m\omega^2 X - m\omega \dot{Y} \\ \\ \frac{\partial}{\partial \dot{Y}} L &= \frac{\partial}{\partial \dot{Y}} \left(\frac{m}{2} \dot{Y}^2 - m\omega \dot{Y}X \right) & \frac{\partial}{\partial Y} L &= \frac{\partial}{\partial Y} \left(\frac{m\omega^2}{2} Y^2 + m\omega \dot{X}Y \right) \\ &= m\dot{Y} - m\omega X & &= m\omega^2 Y + m\omega \dot{X} \end{aligned} \quad (2)$$

Finally, by plugging (2) into (1), we obtain:

$$\begin{aligned} \frac{d}{dt} (m\dot{X} + m\omega Y) &= m\omega^2 X - m\omega \dot{Y} & \frac{d}{dt} (m\dot{Y} - m\omega X) &= m\omega^2 Y + m\omega \dot{X} \\ \Leftrightarrow \boxed{m\ddot{X} = m\omega^2 X - 2m\omega \dot{Y}} & & \Leftrightarrow \boxed{m\ddot{Y} = m\omega^2 Y + 2m\omega \dot{X}} & \quad \square \end{aligned}$$

Remark 1. Those results indeed matches the equations proposed in the book just slightly before this exercise.