The Theoretical Minimum
Quantum Mechanics - Solutions

M. Bivert

April 2, 2023

Abstract

Below are solution proposals to the exercises of *The Theoretical Minimum - Quantum Mechanics*, written by Leonard Susskind and Art Friedman. An effort has been so as to recall from the book all the referenced equations, and to be rather verbose regarding mathematical details, rather in line with the general tone of the serie.

Contents

1 Systems and Experiments **1**
1.1 Inner Products . . . 1

2 Quantum States **3**
2.1 Along the x Axis . . . 3
2.2 Along the y Axis . . . 4

3 Principles of Quantum Mechanics **10**
3.1 Mathematical Interlude: Linear Operators . . . 10
3.1.1 Hermitian Operators and Orthonormal Bases . . . 10
3.1.2 The Gram-Schmidt Procedure . . . 10
3.2 The Principles . . . 10
3.3 An Example: Spin Operators . . . 10
3.4 Constructing Spin Operators . . . 10
3.5 A Common Misconception . . . 10
3.6 3-Vector Operators Revisited . . . 10
3.7 Reaping the Results . . . 10

1 Systems and Experiments

1.1 Inner Products

Exercise 1. *a) Using the axioms for inner products, prove*

$$\Big(\langle A| + \langle B|\Big)|C\rangle = \langle A|C\rangle + \langle B|C\rangle$$

b) Prove $\langle A|A\rangle$ is a real number.

---

a) Let us recall the two axioms in question:

Axiom 1.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 2.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

Let us recall also that if

- $\langle A|$ is the bra of $|A\rangle$
- $\langle B|$ is the bra of $|B\rangle$

Then $\langle A| + \langle B|$ is the bra of $|A\rangle + |B\rangle$.

Let us also observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$\begin{aligned}(a+b)^* &= (x_a + iy_a + x_b + iy_b)^* \\ &= x_a - iy_a + x_b - iy_b \\ &= a^* + b^*\end{aligned}$$

We thus have:

$$\begin{aligned}\Big(\langle A| + \langle B|\Big)|C\rangle &= \langle C|\Big(|A\rangle + |B\rangle\Big)^* \\ &= \Big(\ \langle C|A\rangle + \langle C|B\rangle\ \Big)^* \\ &= \langle C|A\rangle^* + \langle C|B\rangle^* \\ &= \langle A|C\rangle + \langle B|C\rangle \quad \square\end{aligned}$$

b) Mainly from the second axiom:

$$\begin{aligned}x + iy &= \langle A|A\rangle \\ &= \langle A|A\rangle^* \\ &= x - iy \\ &\Rightarrow 2iy = 0 \\ &\Rightarrow y = 0 \\ &\Rightarrow \langle A|A\rangle = x \in \mathbb{R} \quad \square\end{aligned}$$

Exercise 2. *Show that the inner product defined by Eq.* 1.2 *satisfies all the axioms of inner products.*

Let us recall the two axioms in question:

Axiom 3.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 4.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

And let us recall Eq. 1.2 of the book:

$$\begin{aligned}\langle B|A\rangle &= \begin{pmatrix}\beta_1^* & \beta_2^* & \beta_3^* & \beta_4^* & \beta_5^*\end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5\end{pmatrix} \\ &= \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3 + \beta_4^*\alpha_4 + \beta_5^*\alpha_5\end{aligned}$$

For the first axiom, considering $\langle C| = (\gamma_i^*)$:

$$
\begin{aligned}
\langle C|\Big(|A\rangle+|B\rangle\Big) &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3+\beta_3\\ \alpha_4+\beta_4\\ \alpha_5+\beta_5\end{pmatrix}\\
&= \gamma_1^*(\alpha_1+\beta_1)+\gamma_2^*(\alpha_2+\beta_2)+\gamma_3^*(\alpha_3+\beta_3)+\gamma_4^*(\alpha_4+\beta_4)+\gamma_5^*(\alpha_5+\beta_5)\\
&= \left(\gamma_1^*\alpha_1+\gamma_2^*\alpha_2+\gamma_3^*\alpha_3+\gamma_4^*\alpha_4+\gamma_5^*\alpha_5\right)+\left(\gamma_1^*\beta_1+\gamma_2^*\beta_2+\gamma_3^*\beta_3+\gamma_4^*\beta_4+\gamma_5^*\beta_5\right)\\
&= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}+\begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}\\
&= \langle C|A\rangle+\langle C|B\rangle \quad \square
\end{aligned}
$$

Before diving into the second axiom, let us observe that for $(a,b)=(x_a+iy_a, x_b+iy_b)\in\mathbb{C}^2$:

$$
\begin{aligned}
(ab)^* &= \Big((x_a+iy_a)\times(x_b+iy_b)\Big)^*\\
&= \Big(x_ax_b-y_ay_b+i(x_by_a+x_ay_b)\Big)^*\\
&= x_ax_b-y_ay_b-i(x_by_a+x_ay_b)\\
&= (x_a-iy_a)\times(x_b-iy_b)\\
&= a^*b^*
\end{aligned}
$$

Or, perhaps more simply using complex numbers' exponential's form:

$$
\begin{aligned}
(ab)^* &= \Big(r_ar_be^{i(\theta_a+\theta_b)}\Big)^*\\
&= r_ar_be^{-i(\theta_a+\theta_b)}\\
&= a^*b^*
\end{aligned}
$$

Hence, regarding the second axiom:

$$
\begin{aligned}
\langle B|A\rangle &= \Big(\left(\,\langle B|A\rangle\,\right)^*\Big)^*\\
&= \Big(\left(\beta_1^*\alpha_1+\beta_2^*\alpha_2+\beta_3^*\alpha_3+\beta_4^*\alpha_4+\beta_5^*\alpha_5\right)^*\Big)^*\\
&= \Big(\beta_1\alpha_1^*+\beta_2\alpha_2^*+\beta_3\alpha_3^*+\beta_4\alpha_4^*+\beta_5\alpha_5^*\Big)*\\
&= \Big(\alpha_1^*\beta_1+\alpha_2^*\beta_2+\alpha_3^*\beta_3+\alpha_4^*\beta_4+\alpha_5^*\beta_5\Big)*\\
&= \left(\begin{pmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* & \alpha_5^*\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}\right)^*\\
&= \langle A|B\rangle^* \quad \square
\end{aligned}
$$

2 Quantum States

2.1 Along the x Axis

Exercise 3. *Prove that the vector $|r\rangle$ in Eq. 2.5 is orthogonal to vector $|l\rangle$ in Eq. 2.6.*

Let us recall respectively Eq. 2.5 and Eq. 2.6:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \qquad\qquad |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

Orthogonality can be detected with the inner-product: $|l\rangle$ and $|r\rangle$ are orthogonals $\Leftrightarrow \langle r|l\rangle = \langle l|r\rangle = 0$.

Remark 1.
The nullity of either inner-product is sufficient, because of the $\langle A|B\rangle = \langle B|A\rangle^*$ *axiom.*

For instance:

$$\begin{aligned}
\langle l|r\rangle &= \begin{pmatrix}\lambda_u^* & \lambda_d^*\end{pmatrix}\begin{pmatrix}\rho_u \\ \rho_d\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\
&= 0 \quad \square
\end{aligned}$$

Or, similarly:

$$\begin{aligned}
\langle r|l\rangle &= \begin{pmatrix}\rho_u^* & \rho_d^*\end{pmatrix}\begin{pmatrix}\lambda_u \\ \lambda_d\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\
&= 0 \quad \square
\end{aligned}$$

2.2 Along the y Axis

Exercise 4. *Prove that* $|i\rangle$ *and* $|o\rangle$ *satisfy all of the conditions in Eqs.* 2.7, 2.8 *and* 2.9. *Are they unique in that respect?*

Let us recall, in order, Eqs. 2.7, 2.8, 2.9, 2.10, which defines $|i\rangle$ and $|o\rangle$, and both 2.5 and 2.6 which defines $|r\rangle$ and $|l\rangle$:

$$\langle i|o\rangle = 0$$

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{2} \qquad\qquad \langle o|d\rangle\langle d|o\rangle = \frac{1}{2}$$
$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{2} \qquad\qquad \langle i|d\rangle\langle d|i\rangle = \frac{1}{2}$$

$$\langle o|r\rangle\langle r|o\rangle = \frac{1}{2} \qquad\qquad \langle o|l\rangle\langle l|o\rangle = \frac{1}{2}$$
$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2} \qquad\qquad \langle i|l\rangle\langle l|i\rangle = \frac{1}{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \qquad |o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \qquad |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

For clarity, let us recall that $\langle u|A\rangle$ is the component of $|A\rangle$ on the orthonormal vector $|u\rangle$. This is because in a $(|i\rangle)_{i\in F}$ *orthonormal* basis we have:

$$\begin{aligned} |A\rangle &= \sum_{i\in F}\alpha_i|i\rangle \\ \Rightarrow \langle j|A\rangle &= \langle j|\sum_{i\in F}\alpha_i|i\rangle = \sum_{i\in F}\alpha_i\,\langle j|i\rangle = \alpha_j \end{aligned}$$

And to make better sense of those equations, let us recall that $\alpha_u^*\alpha_u = \langle A|u\rangle\,\langle u|A\rangle$ is the probability of a state vector $|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ to be measured in the state $|u\rangle$.
For Eq. 2.7, we have

$$\begin{aligned} \langle i|o\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\ &= \iota_u^* o_u + \iota_d^* o_d \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0 \quad \square \end{aligned}$$

For Eqs. 2.8, we can rely on the projection on an orthonormal vector:

$$\begin{aligned} \langle o|u\rangle\,\langle u|o\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle o|d\rangle\,\langle d|o\rangle &= \frac{i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} \quad \square \\ \langle i|u\rangle\,\langle u|i\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle i|d\rangle\,\langle d|i\rangle &= \frac{-i}{\sqrt{2}}\frac{i}{\sqrt{2}} = \frac{1}{2} \quad \square \end{aligned}$$

For Eqs. 2.9, we need to rely on the column form of the inner-product:

$$
\begin{aligned}
\langle o|r\rangle \langle r|o\rangle &= \begin{pmatrix} o_u^* & o_d^* \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} \begin{pmatrix} \rho_u^* & \rho_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\langle o|l\rangle \langle l|o\rangle &= \begin{pmatrix} o_u^* & o_d^* \end{pmatrix} \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} \begin{pmatrix} \lambda_u^* & \lambda_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1-i+i+1) = \frac{1}{2} \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\langle i|r\rangle \langle r|i\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} \begin{pmatrix} \rho_u^* & \rho_d^* \end{pmatrix} \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1+i+i+1) = \frac{1}{2} \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\langle i|l\rangle \langle l|i\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} \begin{pmatrix} \lambda_u^* & \lambda_d^* \end{pmatrix} \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}
$$

Regarding the unicity of $|i\rangle, |o\rangle$, as for $|r\rangle, |l\rangle$, there definitely is a phase ambiguity, meaning, we can multiply either $|i\rangle$ or $|o\rangle$ by a *phase factor*, say $e^{i\theta}$, without disturbing any of the constraints: orthogonality, probabilities, and the resulting vectors are still unitary.

But as stated by the authors for $|r\rangle, |l\rangle$, measurable quantities are independant of any phase factors. So up to it, they seem to be unique so far.

However, let's try to change the i's place for instance in $|i\rangle$:

$$
|i\rangle = \frac{i}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle
$$

The vector is still unitary, we still have orthogonality with $|o\rangle$, and if you try to compute $\langle i|u\rangle \langle u|i\rangle$, $\langle i|d\rangle \langle d|i\rangle$, $\langle i|r\rangle \langle r|i\rangle$ or $\langle i|l\rangle \langle l|i\rangle$, you'll still have the same probablities.
Now the question is, is this "swapping" of the i a phase factor? Meaning, can encode this transformation as a multiplication by some $e^{i\theta}$, for some $\theta \in \mathbb{R}$?

Well, the first term of $|i\rangle$ is multiplied by i; recall the definition of the complex exponential:

$$
e^{i\theta} = \cos\theta + i\sin\theta
$$

So this means the first term is multiplied by

$$
\exp(i\frac{\pi}{2}) = 0 + i
$$

The second term though, is multiplied by $-i$, this means, multiplied by:

$$
\exp(-i\frac{\pi}{2}) = 0 + i \times (-1)
$$

So we've found a variant of $|i\rangle$, that cannot be obtained by multiplying $|i\rangle$ by a phase factor, and hence:

$$
\boxed{\text{The proposed solution is \textit{not} unique [up to a phase factor].}}
$$

Exercise 5. *For the moment, forget that Eqs.* 2.10 *give us working definitions for* $|i\rangle$ *and* $|o\rangle$ *in terms of* $|u\rangle$ *and* $|d\rangle$*, and assume that the components* α, β, γ *and* δ *are unknown:*

$$|o\rangle = \alpha|u\rangle + \beta|d\rangle \qquad\qquad |i\rangle = \gamma|u\rangle + \delta|d\rangle$$

a) Use Eqs. 2.8 *to show that*

$$\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$$

b) Use the above results and Eqs. 2.9 *to show that*

$$\alpha^*\beta + \alpha\beta^* = \gamma^*\delta + \gamma\delta^* = 0$$

c) Show that $\alpha^*\beta$ *and* $\gamma^*\delta$ *must each be pure imaginary.*

If $\alpha^*\beta$ *is pure imaginary, then* α *and* β *cannot both be real. The same reasoning applies to* $\gamma^*\delta$*.*

Let's start by recalling Eqs. 2.8, 2.9 and 2.10, which are respectively:

$$\begin{aligned} \langle o|u\rangle \langle u|o\rangle &= \frac{1}{2} \qquad \langle o|d\rangle \langle d|o\rangle = \frac{1}{2} \\ \langle i|u\rangle \langle u|i\rangle &= \frac{1}{2} \qquad \langle i|d\rangle \langle d|i\rangle = \frac{1}{2} \end{aligned} \tag{1}$$

$$\begin{aligned} \langle o|r\rangle \langle r|o\rangle &= \frac{1}{2} \qquad \langle o|l\rangle \langle l|o\rangle = \frac{1}{2} \\ \langle i|r\rangle \langle r|i\rangle &= \frac{1}{2} \qquad \langle i|l\rangle \langle l|i\rangle = \frac{1}{2} \end{aligned} \tag{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \qquad |o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \tag{3}$$

---

a) Let's start by recalling that the inner-product in a Hilbert space is defined between a bra and a ket, and that it should satisfy the following two axioms:

$$\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle \text{ (linearity)}$$

$$\langle B|A\rangle = \langle A|B\rangle^* \text{ (complex conjugation)}$$

Furthermore, the scalar-multiplication of a ket is linear:

$$z \in \mathbb{C}, \qquad |zA\rangle = z|A\rangle$$

Then we can multiply $|o\rangle = \alpha|u\rangle + \beta|d\rangle$ to the left by $\langle u|$ to compute $\langle u|o\rangle$, using the linearity of the inner-product/scalar multiplication, and the fact that $|u\rangle$ and $|d\rangle$ are, by definition, orthogonal vectors (meaning, $\langle u|d\rangle = 0$ and $\langle u|u\rangle = \langle d|d\rangle = 1$)

$$\langle u|o\rangle = \alpha \langle u|u\rangle + \beta \langle u|d\rangle = \alpha$$

Because of the complex conjugation rule, we have

$$\langle o|u\rangle = \langle u|o\rangle^* = \alpha^*$$

And so by Eqs. 2.8 and the previous computation we have

$$\frac{1}{2} = \underbrace{\langle o|u\rangle}_{\alpha} \underbrace{\langle u|o\rangle}_{\alpha^*} = \alpha\alpha^* \quad \square$$

The process is very similar to prove $\beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$:

$$
\begin{aligned}
\frac{1}{2} &= \langle o|d\rangle \langle d|o\rangle \\
&= (\langle d|o\rangle)^* \langle d|o\rangle \\
&= \Big(\langle d|\{\alpha|u\rangle + \beta|d\rangle\}\Big)^* \Big(\langle d|\{\alpha|u\rangle + \beta|d\rangle\}\Big) \\
&= \Big(\alpha \underbrace{\langle d|u\rangle}_{=0} + \beta \underbrace{\langle d|d\rangle}_{=1}\Big)^* \Big(\alpha \underbrace{\langle d|u\rangle}_{=0} + \beta \underbrace{\langle d|d\rangle}_{=1}\Big) \\
&= \beta^*\beta \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} &= \langle i|u\rangle \langle u|i\rangle \\
&= (\langle u|i\rangle)^* \langle u|i\rangle \\
&= \Big(\langle u|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^* \Big(\langle u|\{\gamma|u\rangle + \delta|d\rangle\}\Big) \\
&= \Big(\gamma \underbrace{\langle u|u\rangle}_{=1} + \delta \underbrace{\langle u|d\rangle}_{=0}\Big)^* \Big(\gamma \underbrace{\langle u|u\rangle}_{=1} + \delta \underbrace{\langle u|d\rangle}_{=0}\Big) \\
&= \gamma^*\gamma \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} &= \langle i|d\rangle \langle d|i\rangle \\
&= (\langle d|i\rangle)^* \langle d|i\rangle \\
&= \Big(\langle d|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^* \Big(\langle d|\{\gamma|u\rangle + \delta|d\rangle\}\Big) \\
&= \Big(\gamma \underbrace{\langle d|u\rangle}_{=0} + \delta \underbrace{\langle d|d\rangle}_{=1}\Big)^* \Big(\gamma \underbrace{\langle d|u\rangle}_{=0} + \delta \underbrace{\langle d|d\rangle}_{=1}\Big) \\
&= \delta^*\delta \quad \square
\end{aligned}
$$

---

b) I don't think we can conclude here without recalling the definition of $|r\rangle$:

$$
|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle
$$

Let's start with a piece from Eqs. 2.9, arbitrarily (we could use $\langle i|l\rangle \langle l|i\rangle = \frac{1}{2}$, but I think we'd still need the previous definition of $|r\rangle$):

$$
\langle i|r\rangle \langle r|i\rangle = \frac{1}{2}
$$

But:

$$
\langle r|i\rangle = \langle r|\{\alpha + |u\rangle + \beta|d\rangle\} = \alpha \langle r|u\rangle + \beta \langle r|d\rangle
$$

And:

$$
\langle i|r\rangle = (\langle r|i\rangle)^* = (\alpha \langle r|u\rangle + \beta \langle r|d\rangle)^* = \alpha^* \langle u|r\rangle + \beta^* \langle d|r\rangle
$$

So

$$
\begin{gathered}
\langle i|r\rangle \langle r|i\rangle = \frac{1}{2} \\
\Leftrightarrow \Big(\alpha^* \langle u|r\rangle + \beta^* \langle d|r\rangle\Big)\Big(\alpha \langle r|u\rangle + \beta \langle r|d\rangle\Big) = \frac{1}{2} \\
\Leftrightarrow \underbrace{\alpha^*\alpha}_{=1/2} \langle u|r\rangle \langle r|u\rangle + \alpha^*\beta \langle u|r\rangle \langle r|d\rangle + \beta^*\alpha \langle d|r\rangle \langle r|u\rangle + \underbrace{\beta^*\beta}_{=1/2} \langle d|r\rangle \langle r|d\rangle = \frac{1}{2} \\
\Leftrightarrow \frac{1}{2}\Big(\langle u|r\rangle \langle r|u\rangle + \langle d|r\rangle \langle r|d\rangle\Big) + \alpha^*\beta \langle u|r\rangle \langle r|d\rangle + \beta^*\alpha \langle d|r\rangle \langle r|u\rangle = \frac{1}{2}
\end{gathered}
$$

Now if $|r\rangle = \rho_u|u\rangle + \rho_d|d\rangle$, then

$$
\langle u|r\rangle \langle r|u\rangle + \langle d|r\rangle \langle r|d\rangle = \rho_u\rho_u^* + \rho_d\rho_d^* = 1
$$

As $\rho_u \rho_u^*$ would be the probability of $|r\rangle$ to be up, and $\rho_d \rho_d^*$ would the probability of $|r\rangle$ to be down, which are two orthogonal states in a two-states setting, and so the sum of their probability must be 1.

Hence the previous expression becomes:

$$\alpha^* \beta \langle u|r\rangle \langle r|d\rangle + \beta^* \alpha \langle d|r\rangle \langle r|u\rangle = 0$$

Note that so far, we haven't needed the expression of $|r\rangle$, but I think we don't have a choice but to use it to conclude:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

So, as the coefficient are real numbers:

$$\langle u|r\rangle = \frac{1}{\sqrt{2}} = \langle r|u\rangle \,; \qquad \langle d|r\rangle = \frac{1}{\sqrt{2}} = \langle r|d\rangle$$

Replacing in the previous expression we have:

$$\alpha^* \beta \underbrace{\langle u|r\rangle}_{=1/\sqrt{2}} \underbrace{\langle r|d\rangle}_{=1/\sqrt{2}} + \beta^* \alpha \underbrace{\langle d|r\rangle}_{=1/\sqrt{2}} \underbrace{\langle r|u\rangle}_{=1/\sqrt{2}} = 0$$

$$\Leftrightarrow \frac{1}{2}\alpha^* \beta + \frac{1}{2}\beta^* \alpha = 0$$

$$\Leftrightarrow \boxed{\alpha^* \beta + \beta^* \alpha = 0} \quad \square$$

The process is very similar to prove $\gamma^* \delta + \gamma \delta^* = 0$; one has to start again from a Eqs. 2.9, but this time, from another piece involving o, arbitrarily:

$$\langle o|r\rangle \langle r|o\rangle = \frac{1}{2}$$

$$\Leftrightarrow \Big(\langle r|o\rangle\Big)^* \langle r|o\rangle = \frac{1}{2}$$

$$\Leftrightarrow \Big(\langle r|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^* \Big(\langle r|\{\gamma|u\rangle + \delta|d\rangle\}\Big) = \frac{1}{2}$$

$$\Leftrightarrow \Big(\gamma^* \langle u|r\rangle + \delta^* \langle d|r\rangle\Big)\Big(\gamma \langle r|u\rangle + \delta \langle r|d\rangle\Big) = \frac{1}{2}$$

$$\Leftrightarrow \underbrace{\gamma^* \gamma}_{=1/2} \langle u|r\rangle \langle r|u\rangle + \gamma^* \delta \langle u|r\rangle \langle r|d\rangle + \delta^* \gamma \langle d|r\rangle \langle r|u\rangle + \underbrace{\delta^* \delta}_{=1/2} \langle d|r\rangle \langle r|d\rangle = \frac{1}{2}$$

$$\Leftrightarrow \frac{1}{2}\Big(\underbrace{\langle u|r\rangle \langle r|u\rangle + \langle d|r\rangle \langle r|d\rangle}_{=1}\Big) + \gamma^* \delta \langle u|r\rangle \langle r|d\rangle + \delta^* \gamma \langle d|r\rangle \langle r|u\rangle = \frac{1}{2}$$

$$\Leftrightarrow \gamma^* \delta \underbrace{\langle u|r\rangle \langle r|d\rangle}_{=1/2} + \delta^* \gamma \underbrace{\langle d|r\rangle \langle r|u\rangle}_{=1/2} = 0$$

$$\Leftrightarrow \boxed{\gamma^* \delta + \delta^* \gamma = 0} \quad \square$$

---

c) Let's consider $\alpha\beta^*$ is *not* to be a complex number; hence we can write it as:

$$\alpha\beta^* = a + ib \qquad , (a, b) \in \mathbb{R}^2$$

But then:

$$\Big(\alpha\beta^*\Big)^* = a - ib = \alpha^* \beta$$

That's because, for two complex numbers $z = a + ib$ and $w = x + iy$, we have:

$$\Big(zw\Big)^* = z^* w^*$$

Indeed:

$$zw = (a+ib)(x+iy) = (ax-by) + i(bx+ya)$$

Hence:

$$\Big(zw\Big)^* = (ax-by) - i(bx+ya)$$

But:

$$z^*w^* = (a-ib)(x-iy) = (ax-by) - i(bx+ya)$$

Hence the result. Back to our α and β, we established in b) that:

$$\alpha^*\beta + \alpha\beta^* = 0$$

Which is equivalent from previous little proof to saying that

$$\alpha^*\beta + \Big(\alpha^*\beta\Big)^* = 0$$

$$\Leftrightarrow (a+ib) + (a-ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow \boxed{a=0}$$

Which is the same as saying that the real part of $\alpha^*\beta$ is zero, or that it's a pure imaginary number. The exact same argument applies for $\gamma^*\delta$.

3 Principles of Quantum Mechanics

3.1 Mathematical Interlude: Linear Operators

3.1.1 Hermitian Operators and Orthonormal Bases

3.1.2 The Gram-Schmidt Procedure

3.2 The Principles

3.3 An Example: Spin Operators

3.4 Constructing Spin Operators

Exercise 6. *Prove that Eq.* 3.16 *is the unique solution to Eqs.* 3.14 *and* 3.15.

Let's recall all the equations, 3.14, 3.15 and 3.16

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

By developing the matrix product and identifying the vectors components, the first two equations make a system of four equations involving four unknowns $(\sigma_z)_{11}$, $(\sigma_z)_{12}$, $(\sigma_z)_{21}$ and $(\sigma_z)_{22}$:

$$\begin{cases} 1(\sigma_z)_{11} + 0(\sigma_z)_{12} & = 1 \\ 1(\sigma_z)_{21} + 0(\sigma_z)_{22} & = 0 \\ 0(\sigma_z)_{11} + 1(\sigma_z)_{12} & = 0 \\ 0(\sigma_z)_{21} + 1(\sigma_z)_{22} & = -1 \end{cases} \Leftrightarrow \begin{cases} (\sigma_z)_{11} & = 1 \\ (\sigma_z)_{21} & = 0 \\ (\sigma_z)_{12} & = 0 \\ (\sigma_z)_{22} & = -1 \end{cases} \Leftrightarrow \boxed{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \quad \square \tag{7}$$

3.5 A Common Misconception

3.6 3-Vector Operators Revisited

3.7 Reaping the Results