The Theoretical Minimum
Quantum Mechanics - Solutions
L06E05

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Exercise 1. *Prove the following theorem:*

When any of Alice's or Bob's spin operators acts on a product state, the result is still a product state.

Show that in a product state, the expectation value of any component of $\boldsymbol{\sigma}$ or $\boldsymbol{\tau}$ is exactly the same as it would be in the individual single-spin states.

Remark 1. *This is a bit long, but fairly straightforward.*

As usual, let's recall the context. We have two state spaces, one for Alice, and one for Bob, each sufficient to describe a spin.

Spin states for Alice's and Bob's spaces are respectively denoted:

$$\alpha_u|u\} + \alpha_d|d\}, \quad (\alpha_u, \alpha_d) \in \mathbb{C}^2; \qquad \beta_u|u\rangle + \beta_d|d\rangle, \quad (\beta_u, \beta_d) \in \mathbb{C}^2$$

Such states are normalized:

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

We use a tensor product to join the two spaces. Among all the possible linear combination from the resulting product space, which is a vector space, product states are those of the form (where the αs and βs are constrained by the previous normalization conditions):

$$|\Psi >= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

Now, we want to act on such a product state with an operator from either Alice's state space ($\boldsymbol{\sigma}$) or Bob's ($\boldsymbol{\tau}$), which, as we've saw earlier, can naturally be extended from the individual spaces to the product spaces. Recall that the operators's definition in their own respective state spaces are identical

$$\tau_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \tau_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \tau_z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

However, when acting on a product state (and more generally, on a vector from the product space), each will respectively only act on the corresponding part of the tensor product gluing basis vectors, for instance:

$$\sigma_x(\gamma|ab\rangle) = \gamma\sigma_x(|a\} \otimes |b\rangle) = \gamma|(\sigma_x(a))b\rangle$$
$$\tau_x(\gamma|ab\rangle) = \gamma\tau_x(|a\} \otimes |b\rangle) = \gamma|a(\tau_x(b))\rangle$$

Because the computation will be exactly symmetric, we're only going to do the work for Alice's operators.

Remark 2. *It would be interesting to see under which circumstances the result generalizes to arbitrary observables (Hermitian operators). It seems we would need for such an operator σ to transform the basis vectors $|u\rangle$ and $|d\rangle$ in such a way that the induced rotation and scaling to reach $\sigma|u\rangle$ and $\sigma|d\rangle$, would somehow balance, so as to preserve the product state constraint. In particular, $\sigma|u\rangle$ and $\sigma|d\rangle$ should be orthogonal.*

This is exactly what happens below, for the spin operators.

Note that:

$$\sigma_x|u\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |d\}; \qquad \sigma_x|d\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\}$$

Then:

$$\begin{aligned}
\sigma_x|\Psi\rangle = \; & \alpha_u\beta_u\Big(\underbrace{(\sigma_x|u\})}_{|d\}}\otimes|u\rangle\Big) + \alpha_u\beta_d\Big(\underbrace{(\sigma_x|u\})}_{|d\}}\otimes|d\rangle\Big) + \alpha_d\beta_u\Big(\underbrace{(\sigma_x|d\})}_{|u\}}\otimes|u\rangle\Big) + \alpha_d\beta_d\Big(\underbrace{(\sigma_x|d\})}_{|u\}}\otimes|d\rangle\Big) \\
= \; & \alpha_u\beta_u|du\rangle + \alpha_u\beta_d|dd\rangle + \alpha_d\beta_u|uu\rangle + \alpha_d\beta_d|ud\rangle \\
= \; & \alpha_d\beta_u|uu\rangle + \alpha_d\beta_d|ud\rangle + \alpha_u\beta_u|du\rangle + \alpha_u\beta_d|dd\rangle \\
= \; & \gamma_u\delta_u|uu\rangle + \gamma_u\delta_d|ud\rangle + \gamma_d\delta_u|du\rangle + \gamma_d\delta_d|dd\rangle
\end{aligned}$$

Where, for the last step, we've just introduced some renaming (it'll be made explicit in a moment). Such a state will be a product state if the following hold:

$$\gamma_u^*\gamma_u + \gamma_d^*\gamma_d = 1; \quad \delta_u^*\delta_u + \delta_d^*\delta_d = 1$$

Let's transcribe this in terms of αs and βs:

$$\alpha_d^*\alpha_d + \alpha_u^*\alpha_u = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

Which are but the normalization conditions underlying $|\Psi\rangle$:

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

Hence, $\sigma_x|\Psi\rangle$ is a state product. $\square$

We'll now do similar computations, but for σ_y and σ_z. Starting with σ_y, note that:

$$\sigma_y|u\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|d\}; \qquad \sigma_y|d\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|u\}$$

Then:

$$\begin{aligned}
\sigma_y|\Psi\rangle = \; & \alpha_u\beta_u\Big(\underbrace{(\sigma_y|u\})}_{i|d\}}\otimes|u\rangle\Big) + \alpha_u\beta_d\Big(\underbrace{(\sigma_y|u\})}_{i|d\}}\otimes|d\rangle\Big) + \alpha_d\beta_u\Big(\underbrace{(\sigma_y|d\})}_{-i|u\}}\otimes|u\rangle\Big) + \alpha_d\beta_d\Big(\underbrace{(\sigma_y|d\})}_{-i|u\}}\otimes|d\rangle\Big) \\
= \; & i\alpha_u\beta_u|du\rangle + i\alpha_u\beta_d|dd\rangle - i\alpha_d\beta_u|uu\rangle - i\alpha_d\beta_d|ud\rangle \\
= \; & -i\alpha_d\beta_u|uu\rangle - i\alpha_d\beta_d|ud\rangle + i\alpha_u\beta_u|du\rangle + i\alpha_u\beta_d|dd\rangle \\
= \; & \gamma_u\delta_u|uu\rangle + \gamma_u\delta_d|ud\rangle + \gamma_d\delta_u|du\rangle + \gamma_d\delta_d|dd\rangle
\end{aligned}$$

Where again, for the last step, we've performed some renaming (again, made explicit in a few lines). For this to be a product state, the following must hold:

$$\gamma_u^*\gamma_u + \gamma_d^*\gamma_d = 1; \quad \delta_u^*\delta_u + \delta_d^*\delta_d = 1$$

Again, transcribed in terms of αs and βs this yields:

$$(-i\alpha_d)^*(-i\alpha_d) + (i\alpha_u)^*(i\alpha_u) = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

$$\Leftrightarrow ((i\alpha_d^*)(-i\alpha_d) + (-i\alpha_u^*)(i\alpha_u) = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1)$$

$$\Leftrightarrow (\alpha_d^*\alpha_d + \alpha_u^*\alpha_u = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1)$$

Which again, is the normalization conditions for $|\Psi\rangle$. Hence, $\sigma_y|\Psi\rangle$ is a product state. $\square$

One last time for σ_z, start by observing:

$$\sigma_y|u\} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |u\}; \qquad \sigma_y|d\} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -|d\}$$

Then:

$$\begin{aligned}
\sigma_z|\Psi\rangle = \quad & \alpha_u\beta_u\Big(\underbrace{(\sigma_z|u\})}_{|u\}}\otimes|u\rangle\Big) + \alpha_u\beta_d\Big(\underbrace{(\sigma_z|u\})}_{|u\}}\otimes|d\rangle\Big) + \alpha_d\beta_u\Big(\underbrace{(\sigma_z|d\})}_{-|d\}}\otimes|u\rangle\Big) + \alpha_d\beta_d\Big(\underbrace{(\sigma_z|d\})}_{-|d\}}\otimes|d\rangle\Big) \\
= \quad & \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle \\
= \quad & \gamma_u\delta_u|uu\rangle + \gamma_u\delta_d|ud\rangle + \gamma_d\delta_u|du\rangle + \gamma_d\delta_d|dd\rangle
\end{aligned}$$

The renaming is much simpler this time. Let's recall one last time the product state condition:

$$\gamma_u^*\gamma_u + \gamma_d^*\gamma_d = 1; \quad \delta_u^*\delta_u + \delta_d^*\delta_d = 1$$

Or, transcribed in terms of αs and βs:

$$\alpha_u^*\alpha_u + (-\alpha_d)^*(-\alpha_d) = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

$$\Leftrightarrow (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1)$$

Which again, is but the condition for $|\Psi\rangle$ to be a state product. Hence, $\sigma_z|\Psi\rangle$ is a state product. $\square$

It remains to establish the last part of the exercise, namely, that the expectation is unchanged. Recall that for an observable A, given a state $|\Psi\rangle$, the expected value is defined as:

$$\langle A\rangle := \langle\Psi|A|\Psi\rangle$$

Now, we've been computing $A|\Psi\rangle$ in the previous section for all "component" of Alice's spin; so we just have to take a product with $\langle\Psi|$ to get the expected value.

Now remember, we consider an ordered basis $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$ to create column/row vectors, for instance:

$$|\Psi >= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle = \begin{pmatrix}\alpha_u\beta_u\\ \alpha_u\beta_d\\ \alpha_d\beta_u\\ \alpha_d\beta_d\end{pmatrix}$$

We previously established that:

$$\sigma_x|\Psi\rangle = \alpha_d\beta_u|uu\rangle + \alpha_d\beta_d|ud\rangle + \alpha_u\beta_u|du\rangle + \alpha_u\beta_d|dd\rangle$$

Hence:

$$\begin{aligned}
\langle\sigma_x\rangle = \quad & \langle\Psi|(\sigma_x|\Psi\rangle) \\
= \quad & \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^*\end{pmatrix}\begin{pmatrix}\alpha_d\beta_u\\ \alpha_d\beta_d\\ \alpha_u\beta_u\\ \alpha_u\beta_d\end{pmatrix} \\
= \quad & \alpha_u^*\beta_u^*\alpha_d\beta_u + \alpha_u^*\beta_d^*\alpha_d\beta_d + \alpha_d^*\beta_u^*\alpha_u\beta_u + \alpha_d^*\beta_d^*\alpha_u\beta_d \\
= \quad & \beta_d^*\beta_d(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) + \beta_u^*\beta_u(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) \\
= \quad & \underbrace{(\beta_d^*\beta_d + \beta_u^*\beta_u)}_{=1}(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) \\
= \quad & \alpha_u^*\alpha_d + \alpha_d^*\alpha_u
\end{aligned}$$

I don't think we've already computed $\langle\Psi|\sigma_x|\Psi\rangle$ in terms of αs and βs before (we did earlier in L03E04 computed it in terms of θ, an angle between two states), so let's do it (I'll use σ_x^A to indicate that we're using σ_x restricted to Alice's space; for clarity, I'll be using the *ordered* basis $\{|u\}, |d\}\}$):

$$\begin{aligned}
\left\langle \sigma_x^A \right\rangle = & \ \{\Psi|\sigma_x^A|\Psi\} \\
= & \ \begin{pmatrix}\alpha_u^* & \alpha_d^*\end{pmatrix} \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \begin{pmatrix}\alpha_u \\ \alpha_d\end{pmatrix} \\
= & \ \begin{pmatrix}\alpha_u^* & \alpha_d^*\end{pmatrix} \begin{pmatrix}\alpha_d \\ \alpha_u\end{pmatrix} \\
= & \ \alpha_u^*\alpha_d + \alpha_d^*\alpha_u \\
= & \ \langle \sigma_x \rangle \quad \square
\end{aligned}$$

Let's do the same thing for $\langle \sigma_y \rangle$; recall that we've computed earlier.

$$\sigma_y|\Psi\rangle = -i\alpha_d\beta_u|uu\rangle - i\alpha_d\beta_d|ud\rangle + i\alpha_u\beta_u|du\rangle + i\alpha_u\beta_d|dd\rangle$$

Hence,

$$\begin{aligned}
\langle \sigma_y \rangle = & \ \langle \Psi|(\sigma_y|\Psi\rangle) \\
= & \ \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^*\end{pmatrix} \begin{pmatrix}-i\alpha_d\beta_u \\ -i\alpha_d\beta_d \\ i\alpha_u\beta_u \\ i\alpha_u\beta_d\end{pmatrix} \\
= & \ i(-\alpha_u^*\beta_u^*\alpha_d\beta_u - \alpha_u^*\beta_d^*\alpha_d\beta_d + \alpha_d^*\beta_u^*\alpha_u\beta_u + \alpha_d^*\beta_d^*\alpha_u\beta_d) \\
= & \ i\Big(\beta_u^*\beta_u(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d) + \beta_d^*\beta_d(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)\Big) \\
= & \ i\underbrace{(\beta_u^*\beta_u + \beta_d^*\beta_d)}_{=1}(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d) \\
= & \ i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\left\langle \sigma_y^A \right\rangle = & \ \{\Psi|\sigma_y^A|\Psi\} \\
= & \ \begin{pmatrix}\alpha_u^* & \alpha_d^*\end{pmatrix} \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \begin{pmatrix}\alpha_u \\ \alpha_d\end{pmatrix} \\
= & \ \begin{pmatrix}\alpha_u^* & \alpha_d^*\end{pmatrix} \begin{pmatrix}-i\alpha_d \\ i\alpha_u\end{pmatrix} \\
= & \ i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d) \\
= & \ \langle \sigma_y \rangle \quad \square
\end{aligned}$$

Finally for $\langle \sigma_z \rangle$, recall:

$$\sigma_z|\Psi\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle$$

Hence,

$$\begin{aligned}
\langle \sigma_z \rangle = & \ \langle \Psi|(\sigma_z|\Psi\rangle) \\
= & \ \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^*\end{pmatrix} \begin{pmatrix}\alpha_u\beta_u \\ \alpha_u\beta_d \\ -\alpha_d\beta_u \\ -\alpha_d\beta_d\end{pmatrix} \\
= & \ \alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d - \alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d \\
= & \ \beta_u^*\beta_u(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d) + \beta_d^*\beta_d(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d) \\
= & \ \underbrace{(\beta_u^*\beta_u + \beta_d^*\beta_d)}_{=1}(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d) \\
= & \ \alpha_u^*\alpha_u - \alpha_d^*\alpha_d
\end{aligned}$$

And on the other hand:

$$
\begin{aligned}
\left\langle \sigma_z^A \right\rangle &= \{\Psi|\sigma_z^A|\Psi\} \\
&= \begin{pmatrix} \alpha_u^* & \alpha_d^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix} \\
&= \begin{pmatrix} \alpha_u^* & \alpha_d^* \end{pmatrix} \begin{pmatrix} \alpha_u \\ -\alpha_d \end{pmatrix} \\
&= \alpha_u^* \alpha_u - \alpha_d^* \alpha_d \\
&= \langle \sigma_y \rangle \quad \square
\end{aligned}
$$