The Theoretical Minimum

Classical Mechanics - Solutions

L02E05

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Remark 1. *This is a WIP; some intermediate results still aren't proved.*

Exercise 1. *Prove each of the formulas in Eq.s* (2)*. Hint: Look up trigonometric identities and limit properties in a reference book.*

Let's recall the formulas of Eq.s (2):

$$\begin{aligned}
\frac{d}{dt}(\sin t) &= \cos t \\
\frac{d}{dt}(\cos t) &= \sin t \\
\frac{d}{dt}(e^t) &= e^t \\
\frac{d}{dt}(\ln t) &= \frac{1}{t}
\end{aligned}$$

Remark 2. *Interestingly, there are multiple ways of defining those functions*[1]*. As a result, they are different ways to compute the derivatives, depending on which definitions we choose.*

As the definitions given in the book for sin, cos *and the exponential are rather standard, we'll recall them and use those. The natural logarithm hasn't been clearly defined though, so we'll have to do it.*

Remark 3. *While the book suggest to look up trigonometric identities and common properties in some reference material, we will actually take the time to prove the intermediate results.*

One reason to do so for a physicist would be that this is an interesting mathematical exercise, but also to useful to sharpen one's rigor and reasoning skills.

Saying it otherwise, the treatment of this exercise ***goes far beyond*** *what is expected; the goal being to refresh one's maths skills.*

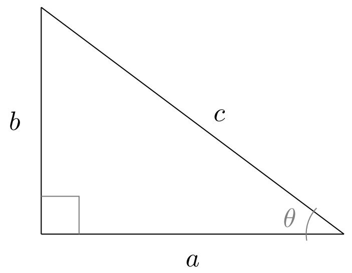
Let's start by recalling that a function $\varphi : E \rightarrow \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon \to 0} \frac{\varphi(e+\epsilon) - \varphi(e)}{\epsilon}}$$

$d\sin t/dt$

Then, let's remind ourselves of the common definitions of cos and sin, in the context of a right triangle:

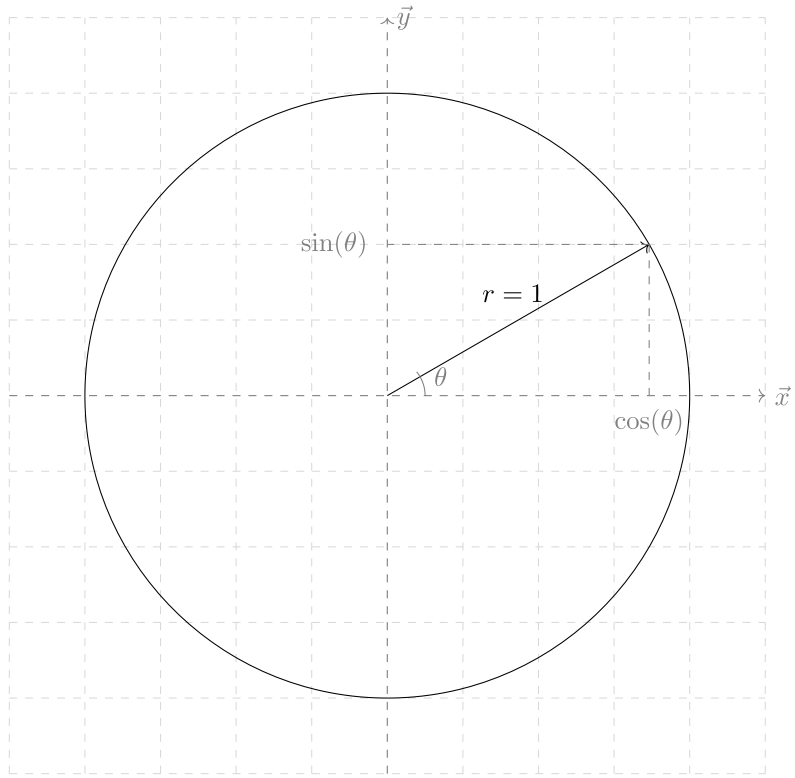
[1] For instance, consider this page containing 6 equivalent definitions of the exponentials: `https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function`



$$\boxed{\cos\theta \triangleq \frac{c}{a}}$$

$$\boxed{\sin\theta \triangleq \frac{b}{c}}$$

In particular, we can identify points on the unit-circle by the angle between the x axis and the radius connecting the center of the circle to such points. Then, each point will then be located in the xy-plane as $(\cos\theta, \sin\theta)$, where θ is the angle previously described, associated to the point.



Note that we have:

Theorem 1.

$$(\forall x \in \mathbb{R}), \quad \boxed{\sin^2 x + \cos^2 x = 1}$$

Proof. This follows immediately from the Pythagorean theorem applied to the right triangle formed by $r = 1$, $\cos\theta$ and $\sin\theta$. □

In order to establish $\sin'$, we will need a few intermediate results that we're going to prove now. First will be to find a formula for $\sin(\alpha+\beta)$. Indeed, if you try to apply the definition of the derivative to sin, you should see a $\sin(x+\epsilon)$: we will need to have it expressed differently to develop the proof.

$$\sin'(x) \triangleq \lim_{\epsilon\to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon}$$

Theorem 2 ($\sin(\alpha+\beta)$)**.**

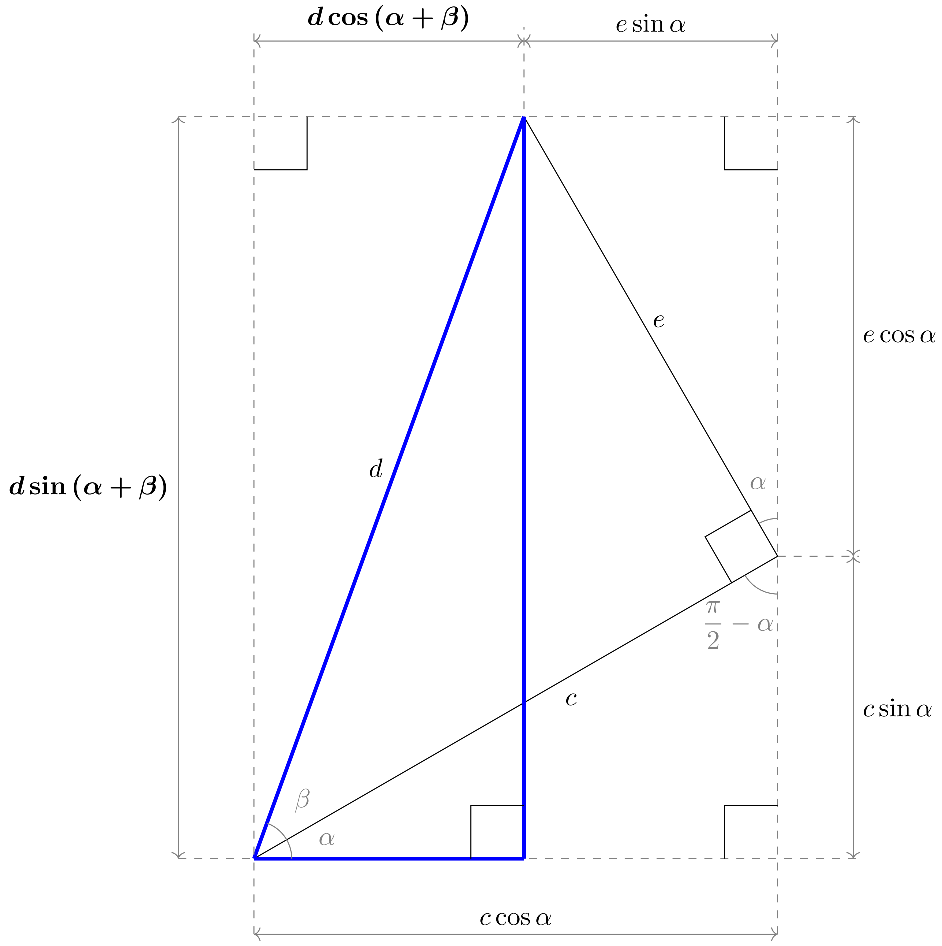
$$(\forall(\alpha,\beta) \in \mathbb{R}^2), \quad \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \tag{1}$$

There's a also a formula for the cosine of a sum of angles, that we will need later, for the derivative of cos, but that will be rather immediate to prove along the one regarding the sine of a sum of angles.

Theorem 3 $(\cos(\alpha+\beta))$**.**

$$(\forall(\alpha,\beta)\in\mathbb{R}^2), \qquad \boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta} \tag{2}$$

Proof. This will be a "visual proof". Besides the aforementioned definitions of sin and cos, we will also use the "fact" that sum of angles in a right triangle is π, which is actually the *triangle postulate*, an axiom of Euclidean geometry, equivalent to the parallel postulate.



In the previous picture, considering the right triangle formed by c, d and e, we have:

$$c = d\cos\beta; \qquad e = d\sin\beta$$

If we look at the blue/thick triangle right triangle (of hypotenuse d, with an angle of $\beta+\alpha$, and whose other sides are created by projecting the point formed by d and e down to the bottom), we find a new relation, to which we can inject our previous results for c and e:

$$\begin{aligned}
& d\sin(\alpha+\beta) &= \;& c\sin\alpha + e\cos\alpha \\
\Leftrightarrow & &= \;& (d\cos\beta)\sin\alpha + (d\sin\beta)\cos\alpha \\
\Leftrightarrow & \sin(\alpha+\beta) &= \;& \boxed{\cos\beta\sin\alpha+\sin\beta\cos\alpha}
\end{aligned}$$

□

In the same blue/thick triangle, we can also establish a relation for $\cos(\alpha+\beta)$, using the same definition of c and e as before to conclude:

$$\begin{aligned}
& d\cos(\alpha+\beta) &= \;& c\cos\alpha - e\sin\alpha \\
\Leftrightarrow & &= \;& (d\cos\beta)\cos\alpha - (d\sin\beta)\sin\alpha \\
\Leftrightarrow & \cos(\alpha+\beta) &= \;& \boxed{\cos\beta\cos\alpha-\sin\beta\sin\alpha}
\end{aligned}$$

□

Now if you try to write down $\sin'$ as previously suggested, and if you decompose $\cos(x+\epsilon)$ with that formula, you see that this will give birth to two limits:

$$\begin{aligned}
\sin'(x) &\triangleq \lim_{\epsilon\to 0} \frac{\sin(x+\epsilon)-\sin x}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\sin x\cos\epsilon + \cos x \sin\epsilon - \sin x}{\epsilon} \\
&= \lim_{\epsilon\to 0} \left(\frac{\sin x\cos\epsilon - 1}{\epsilon} + \frac{\cos x\sin\epsilon}{\epsilon} \right)
\end{aligned}$$

As we've already explained in L02E04, we can recursively split the previous limits, assuming each individual limit exists:

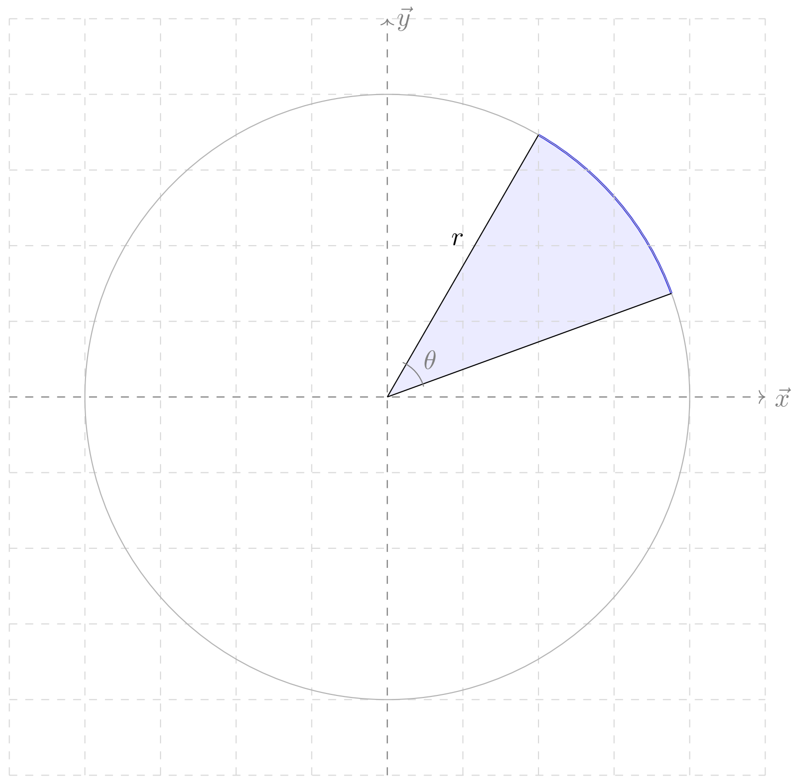
$$\begin{aligned}
\lim_{x\to a} (\varphi(x)+\psi(x)) &= \lim_{x\to a}\varphi(x) + \lim_{x\to a}\psi(x) \\
\lim_{x\to a} (\varphi(x)\psi(x)) &= \lim_{x\to a}\varphi(x) \times \lim_{x\to a}\psi(x)
\end{aligned}$$

Again, you may want to refer to *Paul's Online Notes* [2] for a proof. So we need to see if those limits actually exists, and because we want to compute $\sin'$ in the end, we also want to know their value.

But, to prove those two limits (the proof of one can depend on the other), we first need to define what a *circular sector* is, and how to express its *area*, from which, you'll see, computing the area of a circle is but a special case. We will also need to establish the *squeeze theorem*: we shied away from proving results on limits so far, but we'll do this one. So we will also need to understand *limits* a little deeper.

Let's start with circular sector:

Definition 1 (circular arc, circular sector)**.** *A* <u>*circular arc*</u> *is a portion of the circle between two points of that circle A* <u>*circular sector*</u> *is a portion of disk enclosed between two (usually distinct) radii and a circular arc.*



Remark 4. *A circle then, is but a circular sector of angle 2π.*

Unfortunately, if we want to compute the area of a circular sector, which can be obtained from the area of a circle by that of a circle, the process being similar), we need some integral calculus, which will only be discussed in the book in the next Interlude. Nevertheless:

[2] `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

Theorem 4 (circle area)**.** *The total area of a circle of radius r is given by:*

$$\boxed{\pi r^2}$$

Proof. TODO □

Theorem 5 (circular sector area)**.** *The area of a circular sector, of angle θ, in a circle of radius r is given by:*

$$\boxed{\frac{1}{2}r^2\theta}$$

Proof. Because the area is evenly distributed on a circle, this is a simple cross-multiplication[3] involving the area of a circle:

$$\begin{aligned} 2\pi &\to \pi r^2 \\ \theta &\to A_\theta = \frac{\theta\pi r^2}{2\pi} = \boxed{\frac{1}{2}r^2\theta} \end{aligned}$$

□

Remark 5. *We could also have proved it directly with an integral, as we did for the circle.*

Definition 2 ((ϵ, δ)-definition of a limit)**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$. Saying that $\lim_{x\to e} \varphi(x) = L$ is the same as saying:*

$$(\forall\epsilon > 0), (\exists\delta \in \mathbb{R}), (\forall x \in \mathbb{R})\, (|x - e| < \delta) \Rightarrow (|\varphi(x) - L| < \epsilon)$$

The definition is rather clumsy looking, so let's unpack it a little. Let's fix an ϵ to some value close to zero, say 0.1*, or* 0.0001*.*

Now the idea is that, for this ϵ, we will always be able to find a distance δ such that if we pick an x between $e - \delta$ and $e + \delta$ (i.e. $|x - e| < \delta$), then $\varphi(x)$ will be between $L - \epsilon$ and $L + \epsilon$ (i.e. $|\varphi(x) - L| < \epsilon$).

But, this is true for all strictly positive ϵ. So in particular, this is true for an ever so smaller ϵ. In other words, regardless of how close we want $\varphi(x)$ and L to be, we will always be able to achieve it if we bring x and e close enough.

Theorem 6 (squeeze theorem)**.** *Let φ, ψ, ϕ be three real-valued functions defined on an interval $I \subset \mathbb{R}$, and a be a point of I. If, $(\forall x \in I/\{a\})$ ($x \in I$ but $x \neq a$), we have:*

$$\varphi(x) \leq \psi(x) \leq \phi(x)$$

With:

$$\lim_{x\to a} \varphi(x) = \lambda = \lim_{x\to a} \phi(x)$$

Then:

$$\boxed{\lim_{x\to a} \psi(x) = \lambda}$$

Proof. TODO □

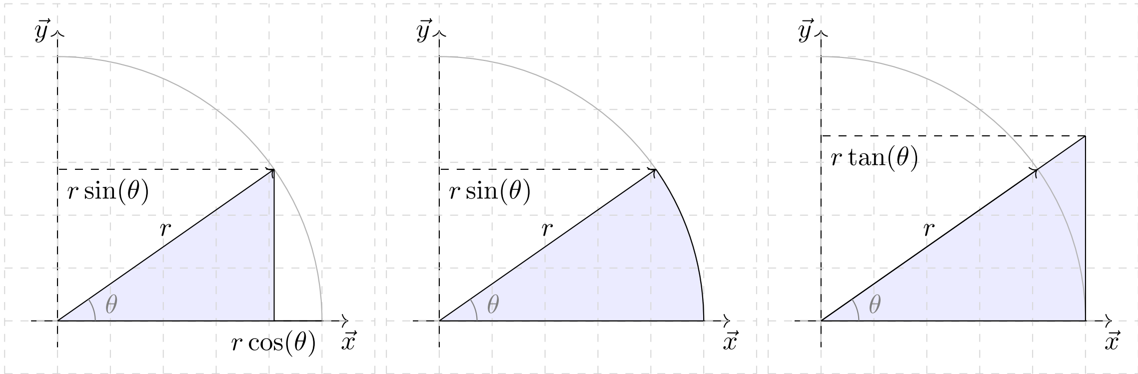
We now have everything we need to start computing the limits involved in the differentiation of sine:

Theorem 7.

$$\boxed{\lim_{\epsilon\to 0} \frac{\sin\epsilon}{\epsilon} = 1}$$

[3] `https://en.wikipedia.org/wiki/Cross-multiplication`

Proof. Consider the three following blueish areas:



The three areas are definitely ordered from smaller to bigger (left to right), and we can also determine them: the middle one is that of a sector, while the two side ones are right triangles (so their areas is half of the corresponding rectangle). We then have the following inequalities:

$$\begin{array}{rrrr} & \frac{1}{2}r^2\cos\theta\sin\theta \leq & \frac{1}{2}r^2\theta \leq & \frac{1}{2}r^2\tan\theta \\ \Leftrightarrow & \cos\theta\sin\theta \leq & \theta \leq & \frac{\sin\theta}{\cos\theta} \\ \Leftrightarrow & \cos\theta \leq & \frac{\theta}{\sin\theta} \leq & \frac{1}{\cos\theta} \\ \Leftrightarrow & \frac{1}{\cos\theta} \geq & \frac{\sin\theta}{\theta} \geq & \cos\theta \end{array}$$

But, as θ goes to zero, the two extremes of this inequalities become:

$$\lim_{\theta\to 0}\frac{1}{\cos\theta} = 1; \qquad \lim_{\theta\to 0}\cos\theta = 1$$

Hence by the *squeeze theorem*, it follows that we *must* have:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$$

□

As for the other limit:

Theorem 8.

$$\boxed{\lim_{\epsilon\to 0}\frac{\cos\epsilon - 1}{\epsilon} = 0}$$

Proof. We will rely on the previous limit; this will be a "proof with a trick" (multiplying by $a/a = 1$; in the present context, a will always be non-zero). Note also at the end that we can apply the product rules for limits given the existence of both sublimits.

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{\cos\epsilon - 1}{\epsilon} &= \lim_{\epsilon\to 0}\left(\frac{\cos\epsilon - 1}{\epsilon} \times \frac{\cos\epsilon + 1)}{\cos\epsilon + 1)}\right) \\ &= \lim_{\epsilon\to 0}\left(\frac{\cos^2\epsilon - 1}{\epsilon(\cos\epsilon + 1))}\right) \\ &= -\lim_{\epsilon\to 0}\left(\frac{\sin^2\epsilon}{\epsilon(\cos\epsilon + 1))}\right) \\ &= -\lim_{\epsilon\to 0}\left(\frac{\sin\epsilon}{\epsilon} \times \frac{\sin\epsilon}{\cos\epsilon + 1}\right) \\ &= -\underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\epsilon})}_{\to 1} \times \underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\cos\epsilon + 1})}_{\to 0/2=0} \\ &= \boxed{0} \end{aligned}$$

$\square$

We now have everything to conclude: let's recapitulate all the intermediate steps to compute $\sin'$:

Theorem 9 (sine derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\sin'(x) = \cos(x)}$$

Proof.

$$\begin{aligned}
(\forall x \in \mathbb{R}), \quad \sin'(x) &\triangleq \lim_{\epsilon \to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\sin x \cos \epsilon + \cos x \sin \epsilon - \sin x}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left(\frac{\sin x(\cos \epsilon - 1)}{\epsilon} + \frac{\cos x \sin \epsilon}{\epsilon} \right) \\
&= \sin x \underbrace{\lim_{\epsilon \to 0} \frac{\cos \epsilon - 1}{\epsilon}}_{=0} + \cos x \underbrace{\lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon}}_{=1} \\
&= \boxed{\cos x}
\end{aligned}$$

$\square$

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d* cos *t*/*dt

If we apply our previous formulas (1) and (2) regarding respectively the sine and cosine of a sum of two angles, in the case where one angle is $\pi/2$, we have: If one of those angle is $\pi/2$, we have:

$$\begin{aligned}
(\forall x \in \mathbb{R}), \quad \sin(x + \frac{\pi}{2}) &= \sin x \underbrace{\cos \frac{\pi}{2}}_{=0} + \cos x \underbrace{\sin \frac{\pi}{2}}_{=1} \\
&= \boxed{\cos x} \\
(\forall x \in \mathbb{R}), \quad \cos(x + \frac{\pi}{2}) &= \cos x \underbrace{\cos \frac{\pi}{2}}_{=0} - \sin x \underbrace{\sin \frac{\pi}{2}}_{=1} \\
&= \boxed{-\sin x}
\end{aligned}$$

From there:

Theorem 10 (cosine derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\cos'(x) = -\sin(x)}$$

Proof.

$$\begin{aligned}
(\forall x \in \mathbb{R}), \quad \cos'(x) &= \sin'(x + \frac{\pi}{2}) \\
&= (\sin \circ (y \to y + \frac{\pi}{2}))'(x) \\
&= \cos(x + \frac{\pi}{2}) \\
&= \boxed{-\sin(x)}
\end{aligned}$$

$\square$

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de*^{t}/*dt

This one, as mentioned in the book, is "trivial" when we define the exponential function to be precisely the function which is equal to its derivative (and such as $e^0 = 1$).

And this is usually the way the exponential function will be first introduced to students. You may want to have a look at other equivalent characterisation of the function[4]. Trying to compute an exponential defined on a development in infinite series carries a certain aesthetic for instance.

$$\boxed{\frac{d}{dt}e^t \triangleq e^t}$$

$d \ln t / dt$

As for the exponential, there can be some variety here depending on how we *characterize* the ln function [5]. Usually, it will be introduced as the *inverse function* of the exponential:

Definition 3 (natural logarithm)**.** *The natural logarithm function is defined as the function* ln *such that:*

$$(\forall x \in \mathbb{R}), \quad \boxed{e^{\ln(x)} = x}$$

Remark 6. *To rigorously establish this definition, would have needed to prove that the exponential is invertible.*

Theorem 11 (natural logarithm derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\ln'(x) = \frac{1}{x}}$$

Proof. The proof develops from the previous definition of the logarithm by integrating both side and then applying the chain rule:

$$\begin{array}{rrcl} & (\forall x \in \mathbb{R}), \quad e^{\ln(x)} & = & x \\ \Leftrightarrow & \frac{d}{dx}e^{\ln(x)} & = & \frac{d}{dx}x \\ \Leftrightarrow & \ln'(x)\,\underbrace{e^{\ln(x)}}_{=x} & = & 1 \\ \Leftrightarrow & \ln'(x) & = & \boxed{\frac{1}{x}} \end{array}$$

□

Remark 7. *For the sake of completeness, some authors[6], will for instance start by defining the logarithm as an integral, and then define the exponential as the inverse of the logarithm. From which they can prove that the derivative of the exponential is the exponential.*

[4] https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function
[5] https://en.wikipedia.org/wiki/Natural_logarithm#Definitions
[6] https://www.whitman.edu/mathematics/calculus_late_online/section09.02.html