

L07E02

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September 8, 2022

Exercise 1. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(aq_1 - bq_2) \quad (1)$$

Because the question is unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

As for the previous exercise, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_i} L \right) = \frac{\partial}{\partial q_i} L$$

Which was in the book followed by the definition of the conjugate momentum p_i :

$$p_i = \frac{\partial}{\partial \dot{q}_i} L$$

For our Lagrangian (1), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q}_1} L = \dot{q}_1 \quad p_2 \equiv \frac{\partial}{\partial \dot{q}_2} L = \dot{q}_2 \quad (2)$$

$$\frac{d}{dt} p_1 = \dot{p}_1 = \ddot{q}_1 \quad \frac{d}{dt} p_2 = \dot{p}_2 = \ddot{q}_2 \quad (3)$$

Using the chain rule¹ for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\begin{aligned} \frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -\frac{\partial}{\partial q_1} \varphi(q_1) \frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2} \varphi(q_2) \frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -(a \frac{\partial}{\partial q_1} V)(aq_1 - bq_2) & &= +(b \frac{\partial}{\partial q_2} V)(aq_1 - bq_2) \end{aligned} \quad (4)$$

As for the previous exercise, it seems that there is a tacit assumption of a symmetry within the potential V so that we can write $V' = \frac{\partial}{\partial q_i} V$; then, combining (2), (3) and (4):

$$\dot{p}_1 = -aV'(aq_1 - bq_2) \quad \dot{p}_2 = +bV'(aq_1 - bq_2)$$

¹https://en.wikipedia.org/wiki/Chain_rule

As suggested, let's multiply the first equation by b , the second by a , and sum the result:

$$b\dot{p}_1 + a\dot{p}_2 = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is indeed a constant over time, i.e. that it is conserved (over time).

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particles respectively by a and b , and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance