The Theoretical Minimum

Quantum Mechanics - Solutions

L02E03

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Exercise 1. *For the moment, forget that Eqs.* 2.10 *give us working definitions for* $|i\rangle$ *and* $|o\rangle$ *in terms of* $|u\rangle$ *and* $|d\rangle$*, and assume that the components* α, β, γ *and* δ *are unknown:*

$$|o\rangle = \alpha|u\rangle + \beta|d\rangle \qquad\qquad |i\rangle = \gamma|u\rangle + \delta|d\rangle$$

a) Use Eqs. 2.8 *to show that*

$$\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$$

b) Use the above results and Eqs. 2.9 *to show that*

$$\alpha^*\beta + \alpha\beta^* = \gamma^*\delta + \gamma\delta^* = 0$$

c) Show that $\alpha^*\beta$ *and* $\gamma^*\delta$ *must each be pure imaginary.*

If $\alpha^*\beta$ *is pure imaginary, then* α *and* β *cannot both be real. The same reasoning applies to* $\gamma^*\delta$*.*

Let's start by recalling Eqs. 2.8, 2.9 and 2.10, which are respectively:

$$\begin{aligned} \langle o|u\rangle \langle u|o\rangle = \frac{1}{2} \qquad \langle o|d\rangle \langle d|o\rangle = \frac{1}{2} \\ \langle i|u\rangle \langle u|i\rangle = \frac{1}{2} \qquad \langle i|d\rangle \langle d|i\rangle = \frac{1}{2} \end{aligned} \tag{1}$$

$$\begin{aligned} \langle o|r\rangle \langle r|o\rangle = \frac{1}{2} \qquad \langle o|l\rangle \langle l|o\rangle = \frac{1}{2} \\ \langle i|r\rangle \langle r|i\rangle = \frac{1}{2} \qquad \langle i|l\rangle \langle l|i\rangle = \frac{1}{2} \end{aligned} \tag{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \qquad |o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \tag{3}$$

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a) Let's start by recalling that the inner-product in a Hilbert space is defined between a bra and a ket, and that it should satisfy the following two axioms:

$$\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle \text{ (linearity)}$$

$$\langle B|A\rangle = \langle A|B\rangle^* \text{ (complex conjugation)}$$

Furthermore, the scalar-multiplication of a ket is linear:

$$z \in \mathbb{C}, \qquad |zA\rangle = z|A\rangle$$

Then we can multiply $|o\rangle = \alpha|u\rangle + \beta|d\rangle$ to the left by $\langle u|$ to compute $\langle u|o\rangle$, using the linearity of the inner-product/scalar multiplication, and the fact that $|u\rangle$ and $|d\rangle$ are, by definition, orthogonal vectors (meaning, $\langle u|d\rangle = 0$ and $\langle u|u\rangle = \langle d|d\rangle = 1$)

$$\langle u|o\rangle = \alpha \langle u|u\rangle + \beta \langle u|d\rangle = \alpha$$

Because of the complex conjugation rule, we have

$$\langle o|u\rangle = \langle u|o\rangle^* = \alpha^*$$

And so by Eqs. 2.8 and the previous computation we have

$$\frac{1}{2} = \underbrace{\langle o|u\rangle}_{\alpha} \underbrace{\langle u|o\rangle}_{\alpha^*} = \alpha\alpha^* \quad \square$$

The process is very similar to prove $\beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$:

$$\begin{aligned}
\frac{1}{2} &= \langle o|d\rangle \langle d|o\rangle \\
&= (\langle d|o\rangle)^* \langle d|o\rangle \\
&= \Big(\langle d|\{\alpha|u\rangle + \beta|d\rangle\}\Big)^* \Big(\langle d|\{\alpha|u\rangle + \beta|d\rangle\}\Big) \\
&= \Big(\alpha \underbrace{\langle d|u\rangle}_{=0} + \beta \underbrace{\langle d|d\rangle}_{=1}\Big)^* \Big(\alpha \underbrace{\langle d|u\rangle}_{=0} + \beta \underbrace{\langle d|d\rangle}_{=1}\Big) \\
&= \beta^*\beta \quad \square
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \langle i|u\rangle \langle u|i\rangle \\
&= (\langle u|i\rangle)^* \langle u|i\rangle \\
&= \Big(\langle u|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^* \Big(\langle u|\{\gamma|u\rangle + \delta|d\rangle\}\Big) \\
&= \Big(\gamma \underbrace{\langle u|u\rangle}_{=1} + \delta \underbrace{\langle u|d\rangle}_{=0}\Big)^* \Big(\gamma \underbrace{\langle u|u\rangle}_{=1} + \delta \underbrace{\langle u|d\rangle}_{=0}\Big) \\
&= \gamma^*\gamma \quad \square
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \langle i|d\rangle \langle d|i\rangle \\
&= (\langle d|i\rangle)^* \langle d|i\rangle \\
&= \Big(\langle d|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^* \Big(\langle d|\{\gamma|u\rangle + \delta|d\rangle\}\Big) \\
&= \Big(\gamma \underbrace{\langle d|u\rangle}_{=0} + \delta \underbrace{\langle d|d\rangle}_{=1}\Big)^* \Big(\gamma \underbrace{\langle d|u\rangle}_{=0} + \delta \underbrace{\langle d|d\rangle}_{=1}\Big) \\
&= \delta^*\delta \quad \square
\end{aligned}$$

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b) I don't think we can conclude here without recalling the definition of $|r\rangle$:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

Let's start with a piece from Eqs. 2.9, arbitrarily (we could use $\langle i|l\rangle \langle l|i\rangle = \frac{1}{2}$, but I think we'd still need the previous definition of $|r\rangle$):

$$\langle i|r\rangle \langle r|i\rangle = \frac{1}{2}$$

But:

$$\langle r|i\rangle = \langle r|\{\alpha + |u\rangle + \beta|d\rangle\} = \alpha \langle r|u\rangle + \beta \langle r|d\rangle$$

And:

$$\langle i|r\rangle = (\langle r|i\rangle)^* = (\alpha\langle r|u\rangle + \beta\langle r|d\rangle)^* = \alpha^*\langle u|r\rangle + \beta^*\langle d|r\rangle$$

So

$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}$$
$$\Leftrightarrow \Big(\alpha^*\langle u|r\rangle + \beta^*\langle d|r\rangle\Big)\Big(\alpha\langle r|u\rangle + \beta\langle r|d\rangle\Big) = \frac{1}{2}$$
$$\Leftrightarrow \underbrace{\alpha^*\alpha}_{=1/2}\langle u|r\rangle\langle r|u\rangle + \alpha^*\beta\langle u|r\rangle\langle r|d\rangle + \beta^*\alpha\langle d|r\rangle\langle r|u\rangle + \underbrace{\beta^*\beta}_{=1/2}\langle d|r\rangle\langle r|d\rangle = \frac{1}{2}$$
$$\Leftrightarrow \frac{1}{2}\Big(\langle u|r\rangle\langle r|u\rangle + \langle d|r\rangle\langle r|d\rangle\Big) + \alpha^*\beta\langle u|r\rangle\langle r|d\rangle + \beta^*\alpha\langle d|r\rangle\langle r|u\rangle = \frac{1}{2}$$

Now if $|r\rangle = \rho_u|u\rangle + \rho_d|d\rangle$, then

$$\langle u|r\rangle\langle r|u\rangle + \langle d|r\rangle\langle r|d\rangle = \rho_u\rho_u^* + \rho_d\rho_d^* = 1$$

As $\rho_u\rho_u^*$ would be the probability of $|r\rangle$ to be up, and $\rho_d\rho_d^*$ would the probability of $|r\rangle$ to be down, which are two orthogonal states in a two-states setting, and so the sum of their probability must be 1.

Hence the previous expression becomes:

$$\alpha^*\beta\langle u|r\rangle\langle r|d\rangle + \beta^*\alpha\langle d|r\rangle\langle r|u\rangle = 0$$

Note that so far, we haven't needed the expression of $|r\rangle$, but I think we don't have a choice but to use it to conclude:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

So, as the coefficient are real numbers:

$$\langle u|r\rangle = \frac{1}{\sqrt{2}} = \langle r|u\rangle\,; \qquad \langle d|r\rangle = \frac{1}{\sqrt{2}} = \langle r|d\rangle$$

Replacing in the previous expression we have:

$$\alpha^*\beta\underbrace{\langle u|r\rangle}_{=1/\sqrt{2}}\underbrace{\langle r|d\rangle}_{=1/\sqrt{2}} + \beta^*\alpha\underbrace{\langle d|r\rangle}_{=1/\sqrt{2}}\underbrace{\langle r|u\rangle}_{=1/\sqrt{2}} = 0$$
$$\Leftrightarrow \frac{1}{2}\alpha^*\beta + \frac{1}{2}\beta^*\alpha = 0$$
$$\Leftrightarrow \boxed{\alpha^*\beta + \beta^*\alpha = 0} \quad \square$$

The process is very similar to prove $\gamma^*\delta + \gamma\delta^* = 0$; one has to start again from a Eqs. 2.9, but this time, from another piece involving o, arbitrarily:

$$\langle o|r\rangle\langle r|o\rangle = \frac{1}{2}$$
$$\Leftrightarrow \Big(\langle r|o\rangle\Big)^*\langle r|o\rangle = \frac{1}{2}$$
$$\Leftrightarrow \Big(\langle r|\{\gamma|u\rangle + \delta|d\rangle\}\Big)^*\Big(\langle r|\{\gamma|u\rangle + \delta|d\rangle\}\Big) = \frac{1}{2}$$
$$\Leftrightarrow \Big(\gamma^*\langle u|r\rangle + \delta^*\langle d|r\rangle\Big)\Big(\gamma\langle r|u\rangle + \delta\langle r|d\rangle\Big) = \frac{1}{2}$$
$$\Leftrightarrow \underbrace{\gamma^*\gamma}_{=1/2}\langle u|r\rangle\langle r|u\rangle + \gamma^*\delta\langle u|r\rangle\langle r|d\rangle + \delta^*\gamma\langle d|r\rangle\langle r|u\rangle + \underbrace{\delta^*\delta}_{=1/2}\langle d|r\rangle\langle r|d\rangle = \frac{1}{2}$$
$$\Leftrightarrow \frac{1}{2}\Big(\underbrace{\langle u|r\rangle\langle r|u\rangle + \langle d|r\rangle\langle r|d\rangle}_{=1}\Big) + \gamma^*\delta\langle u|r\rangle\langle r|d\rangle + \delta^*\gamma\langle d|r\rangle\langle r|u\rangle = \frac{1}{2}$$
$$\Leftrightarrow \gamma^*\delta\underbrace{\langle u|r\rangle\langle r|d\rangle}_{=1/2} + \delta^*\gamma\underbrace{\langle d|r\rangle\langle r|u\rangle}_{=1/2} = 0$$
$$\Leftrightarrow \boxed{\gamma^*\delta + \delta^*\gamma = 0} \quad \square$$

c) Let's consider $\alpha\beta^*$ is *not* to be a complex number; hence we can write it as:

$$\alpha\beta^* = a + ib \qquad , (a, b) \in \mathbb{R}^2$$

But then:

$$\left(\alpha\beta^*\right)^* = a - ib = \alpha^*\beta$$

That's because, for two complex numbers $z = a + ib$ and $w = x + iy$, we have:

$$\left(zw\right)^* = z^*w^*$$

Indeed:

$$zw = (a + ib)(x + iy) = (ax - by) + i(bx + ya)$$

Hence:

$$\left(zw\right)^* = (ax - by) - i(bx + ya)$$

But:

$$z^*w^* = (a - ib)(x - iy) = (ax - by) - i(bx + ya)$$

Hence the result. Back to our α and β, we established in b) that:

$$\alpha^*\beta + \alpha\beta^* = 0$$

Which is equivalent from previous little proof to saying that

$$\alpha^*\beta + \left(\alpha^*\beta\right)^* = 0$$

$$\Leftrightarrow (a + ib) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow \boxed{a = 0}$$

Which is the same as saying that the real part of $\alpha^*\beta$ is zero, or that it's a pure imaginary number. The exact same argument applies for $\gamma^*\delta$.