

# The Theoretical Minimum

## Classical Mechanics - Solutions

L05E02

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**Exercise 1.** Consider a particle in two dimensions,  $x$  and  $y$ . The particle has mass  $m$ . The potential energy is  $V = \frac{1}{2}k(x^2 + y^2)$ . Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitly that the total energy is conserved.

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### Equations of motion

For this system, the potential energy  $V$  is:

$$V = \frac{1}{2}k(x^2 + y^2) \quad (1)$$

By Newton's second law of motion, given  $\mathbf{r} = (x, y)$ , we have:

$$\mathbf{F} = m\mathbf{a} = m\dot{\mathbf{v}} = m\ddot{\mathbf{r}} \quad (2)$$

Or,

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \end{aligned} \quad (3)$$

We know by equation (5) of this lecture that to each coordinate  $x_i$  of the configuration space  $\{x\}$ , there is a force  $F_i$ , derived from the potential energy  $V$ :

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \quad (4)$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x} V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y} V(x, y) = -ky \end{aligned} \quad (5)$$

Combining (3) and (5):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \quad (6)$$

Which are known by L03E04 to be differential equations associated to harmonic motion, and solved by a slightly more general solution than the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \quad (7)$$

Indeed, considering e.g.  $x(t)$ , with simplified variable names:

$$\begin{aligned}
v(t) = \dot{x}(t) &= \omega(-\alpha \sin(\omega t - \theta) + \beta \cos(\omega t - \theta)) \\
a(t) = \ddot{x}(t) &= -\omega^2(\alpha \cos(\omega t - \theta) + \beta \sin(\omega t - \theta)) \\
&= -\omega^2 x(t)
\end{aligned} \tag{8}$$

Where, to differentiate e.g.  $\alpha \cos(\omega t - \theta)$ , we define  $\phi(\omega) = \omega t - \theta$ , so as to use the chain rule for derivation:

$$\frac{d}{dx} f(g(x)) = g'(x) f'(g(x)) \tag{9}$$

Note that, in this case, as already suggested on L03E04  $\alpha_{x,y}$  and  $\beta_{x,y}$  can be determined from the initial position and velocity, that is, from  $x(t=0)$ ,  $\dot{x}(t=0)$ ,  $y(t=0)$ ,  $\dot{y}(t=0)$ . For instance, assuming  $\theta_{x,y} = 0$  to simplify the calculus:

$$\begin{aligned}
x(0) &= \alpha_x \cos(0) + \beta_x \sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha \sin(0) + \beta \cos(0)) \\
&= \omega \beta \\
&= \sqrt{\frac{k}{m}} \beta
\end{aligned} \tag{10}$$

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## Circular orbits

**Remark 1.** *This exercise was completed a few months before writing the present remark. The approach taken is mathematically interesting, but clumsy, from a physicist's point of view. Given the question, and given the fact that the simplest circular orbit has already been studied at the end of Chapter 2 (Motion), it would have been wiser to start by verifying this motion, rather than working from the most general solutions of the differential equation.*

*In the absolute, the quest for the general solution to the differential equation itself should have been polarized by a wiser choice of referencial, to reduce the complexity of the solution.*

*In short, the solution presented here was executed with a mathematician/computer scientist mindset (both being professionally conditioned to explicitly handle all special cases), while the present remark is written with a better understanding of physicists' approach.*

The existence of a (potential) circular orbit is determined an additional constraint tying the equation of  $x(t)$  and  $y(t)$ . Namely, the equation of motion will describe a circle of radius  $r$ , centered on point  $(a, b)$ , with  $(a, b, r) \in \mathbb{R}^3$  if:

$$(\forall t \geq 0), (x(t) - a)^2 + (y(t) - b)^2 = r^2 \tag{11}$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$\sin(\theta \pm \varphi) = \sin \theta \cos \varphi \pm \cos \theta \sin \varphi$$

Then, let's introduce two angles  $\varphi_x$  and  $\varphi_y$  such as:

$$\begin{aligned}
\sin \varphi_x &= \alpha_x & \sin \varphi_y &= \alpha_y \\
\cos \varphi_x &= \beta_x & \cos \varphi_y &= \beta_y
\end{aligned}$$

From, there, we can use the previous identity to rewrite our equations of motions (7) as:

$$\begin{aligned}
x(t) &= \sin \varphi_x \cos(\omega t - \theta_x) + \cos \varphi_x \sin(\omega t - \theta_x) = \sin(\omega t + \varphi_x - \theta_x) = \sin \Omega_x \\
y(t) &= \sin \varphi_y \cos(\omega t - \theta_y) + \cos \varphi_y \sin(\omega t - \theta_y) = \sin(\omega t + \varphi_y - \theta_y) = \sin \Omega_y
\end{aligned} \tag{12}$$

Obvoiusly with  $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$  and  $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$ .

Let us now develop (11) with those two versions of  $x(t)$  and  $y(t)$ :

$$\begin{aligned} (x(t) - a)^2 + (y(t) - b)^2 &= r^2 \\ \Leftrightarrow (\sin \Omega_x - a)^2 + (\sin \Omega_y - b)^2 &= r^2 \\ \Leftrightarrow \sin^2 \Omega_x + \sin^2 \Omega_y - 2(a \sin \Omega_x + b \sin \Omega_y) + a^2 + b^2 &= r^2 \end{aligned}$$

For simplicity, we can assume that the circular orbits, if any, will be centered on  $(a, b) = (0, 0)$ ; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$\sin^2 \Omega_x + \sin^2 \Omega_y = r^2$$

Which we can rewrite a little bit using the fact that  $\sin \varphi = \cos(\varphi - \pi/2)$ :

$$\sin^2 \Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2$$

But we know the pythagorean identity  $\sin^2 \varphi + \cos^2 \varphi = 1$ , hence we know there will be circular orbits when:

$$\left\{ \begin{array}{l} r = 1 \\ \Omega_x = \Omega_y - \frac{\pi}{2} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} r = 1 \\ \omega t + \varphi_x - \theta_x = \omega t + \varphi_y - \theta_y - \frac{\pi}{2} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} r = 1 \\ \varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2} \end{array} \right\}$$

Hence we can see that the only condition relating  $x(t)$  and  $y(t)$  is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period  $T$  that we can observed from (12) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

**Remark 2.** For a wave function  $z(t) = \sin(\omega t + \varphi)$ , by definition,  $2\pi/\omega$  is the period, and  $\varphi$  the phase shift.

**Remark 3.** The phase shift condition could be rewritten in terms of  $\alpha_{x,y}$  and  $\beta_{x,y}$ .

### Energy conservation

Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 \quad (13)$$

Which gives us for this system, expliciting the time-dependencies:

$$T(t) = \frac{1}{2} m \dot{x}(t)^2 + \frac{1}{2} m \dot{y}(t)^2 = \frac{1}{2} m (\dot{x}(t)^2 + \dot{y}(t)^2) \quad (14)$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt} T(t) &= \frac{1}{2} m (2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \quad (15)$$

On the other hand, we can compute the variation of potential energy over time from (1), again using the chain rule:

$$\begin{aligned} \frac{d}{dt} V(t) &= \frac{1}{2} k \left( \frac{d}{dt} x(t)^2 + \frac{d}{dt} y(t)^2 \right) \\ &= \frac{1}{2} k (2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= k(x(t)\dot{x}(t) + y(t)\dot{y}(t)) \end{aligned} \quad (16)$$

From (6), we have:

$$\begin{aligned}x(t) &= -\frac{m}{k}\ddot{x}(t) \\y(t) &= -\frac{m}{k}\ddot{y}(t)\end{aligned}\tag{17}$$

Injecting in (16):

$$\begin{aligned}\frac{d}{dt}V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\&= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y})\end{aligned}\tag{18}$$

Thus from (15) and (18):

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square\tag{19}$$

That is, the total energy  $E$  doesn't change over time.