The Theoretical Minimum

Classical Mechanics - Solutions

L11E04

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Exercise 1. *Using the Hamiltonian, Eq. (24), work out Hamilton's equations of motion and show that you just get back to the Newton-Lorentz equation of motion.*

The Hamiltonian is:

$$H = \frac{1}{2m}\sum_i\Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)^2$$

Hamilton's equation of motion are given by the pair:

$$\dot{q_j} = \frac{\partial H}{\partial p_j}; \qquad \dot{p_j} = -\frac{\partial H}{\partial q_j}$$

Let's get started with the first one:

$$\begin{aligned}
\dot{q_j} &= \frac{\partial H}{\partial p_j} \\
&= \frac{1}{2m}\frac{\partial}{\partial p_j}\sum_i\Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)^2 \\
&= \frac{1}{2m}\frac{\partial}{\partial p_j}\sum_i\Big(p_i^2 - 2\frac{e}{c}p_iA_i(\boldsymbol{q}) + (\frac{e}{c})^2A_i(\boldsymbol{q})^2\Big) \\
&= \frac{1}{2m}\sum_i\Big(\frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c}(\frac{\partial p_i}{\partial p_j}A_i(\boldsymbol{q}) + p_i\underbrace{\frac{\partial A_i(\boldsymbol{q})}{\partial p_j}}_{=0}) + (\frac{e}{c})^2\underbrace{\frac{\partial A_i(\boldsymbol{q})^2}{\partial p_j}}_{=0}\Big) \\
&= \frac{1}{2m}\sum_i\Big(\frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c}\frac{\partial p_i}{\partial p_j}A_i(\boldsymbol{q})\Big) \\
&= \frac{1}{2m}\Big(2p_j - 2\frac{e}{c}A_j(\boldsymbol{q})\Big) \\
&= \boxed{\frac{1}{m}(p_j - \frac{e}{c}A_j(\boldsymbol{q}))}
\end{aligned}$$

We've found back the expression of the moment, tweaked a little. Let's now take the time derivative of the previous equation to get:

$$\begin{aligned}
\ddot{q_j} &= \frac{1}{m}\Big(\dot{p_j} - \frac{e}{c}\dot{A}_j(\boldsymbol{q})\Big) \\
\Leftrightarrow \quad m\ddot{q_j} &= \dot{p_j} - \frac{e}{c}\dot{A}_j(\boldsymbol{q})
\end{aligned}$$

Which starts to look like an equation of motion. We'll develop $\dot{A}_j(\boldsymbol{q})$ using the chain rule later. For now, let's compute $\dot{p_j}$, but this time, let's make our life a bit simpler by using the chain rule:

$$\begin{aligned}
\dot{p_j} &= -\frac{\partial H}{\partial q_j} \\
&= -\frac{1}{2m}\frac{\partial}{\partial q_j}\sum_i \Big(\underbrace{p_i - \frac{e}{c}A_i(\boldsymbol{q})}_{=\phi}\Big)^2 \\
&= -\frac{1}{2m}\sum_i 2\phi\frac{\partial \phi}{\partial q_j} \\
&= \sum_i \underbrace{\frac{1}{m}\Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)}_{=\dot{q}_i}\frac{e}{c}\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} \\
&= \boxed{\frac{e}{c}\sum_i \dot{q}_i\frac{\partial A_i(\boldsymbol{q})}{\partial q_i}}
\end{aligned}$$

Now let's use the multi-dimensional chain rule to develop $\dot{A}_j(\boldsymbol{q})$:

$$\dot{A}_j(\boldsymbol{q}) = \dot{A}_j(q_x(t), q_y(t), q_z(t)) = \sum_i \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\dot{q}_i$$

Finally, let's use this and the expression of $\dot{p_j}$ to rewrite our embryo of motion equation:

$$\begin{aligned}
m\ddot{q}_j &= \dot{p_j} - \frac{e}{c}\dot{A}_j(\boldsymbol{q}) \\
&= \sum_i \dot{q}_i\frac{e}{c}\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} - \frac{e}{c}\sum_i \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\dot{q}_i \\
&= \frac{e}{c}\sum_i \dot{q}_i\Big(\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\Big) \\
&= \frac{e}{c}\Big(\dot{q}_j\underbrace{(\frac{\partial A_j(\boldsymbol{q})}{\partial q_j} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_j})}_{=0} + \dot{q}_k\underbrace{(\frac{\partial A_k(\boldsymbol{q})}{\partial q_k} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_k})}_{=B_l} + \dot{q}_l\underbrace{(\frac{\partial A_l(\boldsymbol{q})}{\partial q_l} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_l})}_{=-B_k}\Big) \\
&= \frac{e}{c}\Big(B_l\dot{q}_k - B_k\dot{q}_l\Big)
\end{aligned}$$

And so for j, k, l three distinct elements of $\{x, y, z\}$, we indeed have found our equations of Newton-Lorentz:

$$\boxed{m\ddot{q}_j = \frac{e}{c}\Big(B_l\dot{q}_k - B_k\dot{q}_l\Big)}$$