The Theoretical Minimum

Classical Mechanics - Solutions

L05E02

M. Bivert

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Exercise 1. *Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitely that the total energy is conserved.*

Equations of motion: For this system, the potential energy V is:

$$V = \frac{1}{2}k(x^2 + y^2) \tag{1}$$

By Newton's second law of motion, given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{2}$$

Or,

$$\begin{aligned} F_x &= \quad m\ddot{x} \\ F_y &= \quad m\ddot{y} \end{aligned} \tag{3}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{4}$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x}V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y}V(x, y) = -ky \end{aligned} \tag{5}$$

Combining (3) and (5):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \tag{6}$$

Which are known by *L03E04* to be differential equations associated to harmonic motion, and solved by a slightly more general solution that the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{7}$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$
\begin{aligned}
v(t) = \quad \dot{x}(t) = & \quad \omega(-\alpha\sin(\omega t-\theta)+\beta\cos(\omega t-\theta) \\
a(t) = \quad \ddot{x}(t) = & \quad -\omega^2(\alpha\cos(\omega t-\theta)+\beta\sin(\omega t-\theta)) \\
= & \quad -\omega^2 x(t)
\end{aligned} \tag{8}
$$

Where, to differentiate e.g. $\alpha\cos(\omega t-\theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$
\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \tag{9}
$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t=0), \dot{x}(t=0), y(t=0), \dot{y}(t=0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$
\begin{aligned}
x(0) &= \alpha_x\cos(0)+\beta_x\sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha\sin(0)+\beta\cos(0)) \\
&= \omega\beta \\
&= \sqrt{\frac{k}{m}}\beta
\end{aligned} \tag{10}
$$

Circular orbits: The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r, centered on point (a,b), with $(a,b,r)\in\mathbb{R}^3$ if:

$$
(\forall t \geq 0),\ (x(t)-a)^2+(y(t)-b)^2 = r^2 \tag{11}
$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$
\sin(\theta\pm\varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi
$$

Then, let's introduce two angles φ_x and φ_y such as:

$$
\begin{aligned}
\sin\varphi_x &= \alpha_x \quad & \sin\varphi_y &= \alpha_y \\
\cos\varphi_x &= \beta_x \quad & \cos\varphi_y &= \beta_y
\end{aligned}
$$

From, there, we can use the previous identity to rewrite our equations of motions (7) as:

$$
\begin{aligned}
x(t) &= \sin\varphi_x\cos(\omega t-\theta_x)+\cos\varphi_x\sin(\omega t-\theta_x) &= \sin(\omega t+\varphi_x-\theta_x) = \sin\Omega_x \\
y(t) &= \sin\varphi_y\cos(\omega t-\theta_y)+\cos\varphi_y\sin(\omega t-\theta_y) &= \sin(\omega t+\varphi_y-\theta_y) = \sin\Omega_y
\end{aligned} \tag{12}
$$

Obvoiusly with $\Omega_x = \Omega_x(t) = \omega t+\varphi_x-\theta_x$ and $\Omega_y = \Omega_y(t) = \omega t+\varphi_y-\theta_y$.

Let us now develop (11) with those two versions of $x(t)$ and $y(t)$:

$$
\begin{aligned}
& (x(t)-a)^2+(y(t)-b)^2 &&= r^2 \\
\Leftrightarrow & (\sin\Omega_x-a)^2+(\sin\Omega_y-b)^2 &&= r^2 \\
\Leftrightarrow & \sin^2\Omega_x+\sin^2\Omega_y-2(a\sin\Omega_x+b\sin\Omega_y))+a^2+b^2 &&= r^2
\end{aligned}
$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a,b) = (0,0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$
\sin^2\Omega_x+\sin^2\Omega_y = r^2
$$

Which we can rewrite a little bit using the fact that $\sin\varphi = \cos(\varphi-\pi/2)$:

$$
\sin^2\Omega_x+\cos^2(\Omega_y-\frac{\pi}{2}) = r^2
$$

But we know the pythagorean identity $\sin^2\varphi + \cos^2\varphi = 1$, hence we know there will be circular orbits when:

$$\begin{cases} r &= 1 \\ \Omega_x &= \Omega_y - \dfrac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r &= 1 \\ \omega t + \varphi_x - \theta_x &= \omega t + \varphi_y - \theta_y - \dfrac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r &= 1 \\ \varphi_x - \theta_x &= \varphi_y - \theta_y - \dfrac{\pi}{2} \end{cases}$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (12) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

Remark 1. *For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.*

Remark 2. *The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.*

Energy conservation: Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2}\sum_i m_i \dot{x_i}^2 \tag{13}$$

Which gives us for this system, expliciting the time-dependancies:

$$T(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}m\dot{y}(t)^2 = \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2) \tag{14}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt}T(t) &= \frac{1}{2}m(2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{15}$$

On the other hand, we can compute the variation of potential energy over time from (1), again using the chain rule:

$$\begin{aligned} \frac{d}{dt}V(t) &= \frac{1}{2}k(\frac{d}{dt}x(t)^2 + \frac{d}{dt}y(t)^2) \\ &= \frac{1}{2}k(2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= k(x(t)\dot{x}(t) + y(t)\dot{y}(t)) \end{aligned} \tag{16}$$

From (6), we have:

$$\begin{aligned} x(t) &= -\frac{m}{k}\ddot{x}(t) \\ y(t) &= -\frac{m}{k}\ddot{y}(t) \end{aligned} \tag{17}$$

Injecting in (16):

$$\begin{aligned} \frac{d}{dt}V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{18}$$

Thus from (15) and (18):

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square \tag{19}$$

That is, total energy E over time doesn't change.