The Theoretical Minimum

Classical Mechanics - Solutions

M. Bivert

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Abstract

Below are solution proposals to the exercises of *The Theoretical Minimum - Classical Mechanics*, written by Leonard Susskind and George Hrabovsky. An effort has been so as to recall from the book all the referenced equations, and to be rather verbose regarding mathematical details, rather in line with the general tone of the serie.

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Aristotle's Law of Motion

Exercise 1/4

Exercise 1. *Given a force that varies with time according to $F = 2t^2$, and with the initial condition at time zero, $x(0) = \pi$, use Aristotle's law to find $x(t)$ at all times.*

Let us recall that Aristotle's law of motion is defined, for a one-dimensional particle (otherwise, $F(t)$ and $x(t)$ would be vector-values functions $\boldsymbol{F}(t)$ and $\boldsymbol{x}(t)$) earlier in the book as:

$$\frac{d}{dt}x(t) = \frac{F(t)}{m}$$

And that by integrating both sides, thanks to the fundamental theorem of calculus[1], assuming the mass is constant over time, we obtain:

$$x(t) = \frac{1}{m}\int F(t)\,dt$$

Which is our case, for $F(t) = 2t^2$, develops in:

$$\begin{aligned} x(t) &= \frac{1}{m}\int 2t^2\,dt \\ &= \frac{2}{3m}t^3 + c,\ c \in \mathbb{R} \end{aligned}$$

The initial condition $x(0) = \pi$ implies that $c = \pi$, hence the position at all time would be:

$$\boxed{x(t) = \frac{2}{3m}t^3 + \pi} \quad \square$$

[1] https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

Mass, Acceleration and Force

Some Simple Examples of Solving Newton's Equations

Exercise 2/4

Exercise 2. *Integrate this equation. Hint: Use definite integrals.*

The equation in question resulting from Newton's second law in the case of a constant force F_z being applied to an object of mass m following the z-axis:

$$\dot{v_z} = \dot{v_z(t)} = \frac{F_z}{m}$$

By integrating both sides, thanks to the fundamental theorem of calculus, assuming the mass is constant over time, we obtain:

$$\begin{aligned} v_z(t) &= \int \frac{F_z}{m}\,dt \\ &= \frac{F_z}{m} \int dt \\ &= \frac{F_z}{m} t + c,\ c \in \mathbb{R} \end{aligned}$$

Generally, c would be determine from an initial condition $v_z(0)$, which is our case, would precisely be c, hence:

$$\boxed{v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

Which is exactly the solution proposed in the book.

Exercise 3/4

Exercise 3. *Show by differentiation that this satisfies the equation of motion.*

Contrary to the previous exercice, instead of integrating to find the solution, we start from the solution and climb back to our original equation of motion, which are, in the case of a constant force F_z applied to a mass m following the z-axis:

$$v_z(t) = \dot{z}(t) = v_z(0) - \frac{F_z}{m} t$$

The proposed solution is:

$$z(t) = z_0 + v_z(0)t + \frac{F_z}{2m} t^2$$

Immediately, by derivation, constants goes to 0, t becomes 1 and t^2 becomes $2t$, we indeed obtain:

$$\boxed{\frac{d}{dt} z(t) = \dot{z}(t) = v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

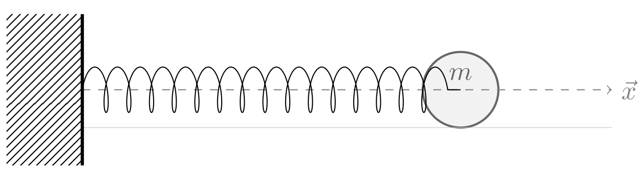
Exercise 4/4

Exercise 4. *Show by differentiation that the general solution to Eq.* (6) *is given in terms of two constants A and B by*

$$x(t) = A\cos\omega t + B\sin\omega t$$

Determine the initial position and velocity at time $t = 0$ *in terms of* A *and* B.

We're in the case of a 1-dimensional harmonic oscillator, depicted below as a mass m set on a frictionless ground attached to a fixed element to the left by a horizontally positioned spring of constant k.



Remark 1. *Almost equivalently, we could have worked in a vertical setting, which, on one hand, would have avoided us the need for a frictionless ground, but on the other hand, would make the equation slightly more complicated.*

There are interesting aspects to both cases, so because the difference between both is so small in the end, we'll discuss both systems.

Harmonic oscillators play a major role in physics, as explained by Feynman in Chapter 21 of the first volume of his *Lectures on Physics*:

> The harmonic oscillator, which we are about to study, has close analogs in many other fields; although we start with a mechanical example of a weight on a spring, or a pendulum with a small swing, or certain other mechanical devices, we are really studying a certain differential equation. This equation appears again and again in physics and in other sciences, and in fact it is a part of so many phenomena that its close study is well worth our while.
>
> Some of the phenomena involving this equation are the oscillations of a mass on a spring; the oscillations of charge flowing back and forth in an electrical circuit; the vibrations of a tuning fork which is generating sound waves; the analogous vibrations of the electrons in an atom, which generate light waves; the equations for the operation of a servosystem, such as a thermostat trying to adjust a temperature; complicated interactions in chemical reactions; the growth of a colony of bacteria in interaction with the food supply and the poisons the bacteria produce; foxes eating rabbits eating grass, and so on; all these phenomena follow equations which are very similar to one another, and this is the reason why we study the mechanical oscillator in such detail.

Thus, we'll take the time to analyze it more precisely than what is expected here: this is the occasion to present some aspects not covered, or not covered as explicitely in the book.

Despite everything happening in one-dimension, we'll use a vector position $\boldsymbol{x}$, which is a function of time. Hence, the vectors for speed and acceleration will respectively be $\boldsymbol{v} = \dot{\boldsymbol{x}} = \dfrac{d}{dt}\boldsymbol{x}(t)$ and $\boldsymbol{a} = \ddot{\boldsymbol{x}} = \dfrac{d^2}{dt^2}\boldsymbol{x}(t)$. Note that we're representing vectors with bold font instead of arrows; bold font is rather common in physics. This should be the only additional thing needed to follow through for someone having read the book up to this stage.

Contact force, normal force, friction force
The contact force is the force resulting from the contact of two objects. It is generally decomposed into vertical and horizontal components. The horizontal component perhaps is the most intuitive: it corresponds to friction forces. For instance, after having given a gentle push to an object an on ordinary table, it will only move up to a certain point: it is progressively stopped by the friction between the object and the table. This friction is caused in part because at a microscopic scale, both the object and the surface are far, far from being perfectly plane, even more so at an atomic scale.

The normal force is the vertical component of the contact force. An object on a table is affected by the Earth gravity, but naturally resists going through the table, unless of course, the object is really massive and/or the table very weak. There's a bunch of complicated interactions at the atomic level from which this situation occurs: at a macroscopic scale, we simplify things and wrap this complexity by abstracting it as a (macroscopic) force.

Remark 2. *In the case of the vertical setup, with a mass attached to a vertical spring, the mass is not in contact with a ground surface, so there would have been no need to discuss contact forces.*

Remark 3. *A static object on a flat surface corresponds to a special case where the horizontal component of the contact force (friction) is null. Hence, the contact force and the normal force are, in such a special case, one and the same. The exact same thing would happen in the case of a frictionless surface.*

Newton's laws
Let us start by recalling Newton's laws of motion:

1. **Principle of inertia**: Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it;
2. "$\boldsymbol{F} = m\boldsymbol{a}$": The change of motion [*momentum*] of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed;
3. "action $\Rightarrow$ reaction": To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Remark 4. *A force always is applied from an object, to another object; with the restriction that an object cannot apply a force to itself.*

Remark 5. *The second law is a actually a statement about momentum, from which we can derive* $\boldsymbol{F} = m\boldsymbol{a}$. *More precisely, the momentum, called "motion" by Newton, is defined as* $\boldsymbol{p} = m\boldsymbol{v}$, *which literally captures the idea of the speed* $\boldsymbol{v}$ *of a certain amount of matter* m. *Then, it follows that the general case,* $\boldsymbol{p}$ *is as* $\boldsymbol{v}$ *a function of time.*
The "change of motion" then refers to an infinitesimal change of the momentum over time, mathematically captured by the time derivative $\frac{d}{dt}\boldsymbol{p}$. *The law can then be progressively written, assuming the mass* m *is constant over time:*

$$\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p} = \frac{d}{dt}m\boldsymbol{v} = m\frac{d}{dt}\boldsymbol{v} = m\boldsymbol{a} \quad \square$$

Remark 6. *It may seem weird to mention in the previous Remark 5 that we assume the mass* m *to be constant over time. In the current context, this will be the case, but in general, likely contrary to what any reasonable human would expect, they are exceptions. More precisely, in special relativity, the mass is a function of the velocity* v*:*

$$m_{rel} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We won't delve into the details here, but note that if $v = \|\boldsymbol{v}\|$ *is much smaller that* c*, the speed of light, then the relative mass* m_{rel} *and* m *are very close.*

To say it otherwise, this assumption doesn't hold at high velocities.

Remark 7. *In the second law, the force impressed ("*$\boldsymbol{F}$*") is actually the force resulting from all the other forced applied to an object,* $\sum_i \boldsymbol{F}_i$*, which is sometimes called* $\boldsymbol{F}_{net}$.

For instance, if you push a cart, $\boldsymbol{F}_{net}$ *should contain at least the force you are exerting, the gravity exerted by Earth, and the contact force generated by the ground on the cart's wheel. The sum of all the forces applied to the cart that will ultimately influence its motion.*

As a result, in practice, when analyzing a situation so as to establish the equation of motion of an object, one should start by identifying all the external forces applied to this object.

Remark 8. *A common special case of the second law is to consider objects that, within a certain frame of reference, do not move. Hence, by definition, their speed* $\boldsymbol{v}$ *would be* 0*, and so would be their acceleration. That is,* $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.

Another special case for the second law is when things move at constant speed. Again, the acceleration would then be $\boldsymbol{0}$ *and thus still* $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.

Remark 9. *It is interesting to spend a moment to ponder what the mass* m *of an object really "is". Intuitively, for a non-physicist mass is another word for weight, but not to a physicist.*

Instead, in the context of the second law, physicists will observe that the more mass there is, the more force will be needed to accelerate an object. Conversely, the less mass there is, the less force will be needed to alter the motion of an object. Thus, they would conceptualize the mass as a ***measure of resistance to acceleration*** *(change of movement).*

This is the second time we meet a subtlety regarding masses, the first one being Remark 6 about how mass is a function of the velocity. We'll come back later once again in a moment at some more subtlety regarding mass.

Remark 10. *In the case of a static object laying on a flat surface, one may think that the gravitational force exerted by the Earth on the object and the normal force/contact force arising from the contact of the object with the surface are two opposite forces, in the sense of Newton's third law: after all, they are indeed of equal intensity and opposite directions.*

But this is incorrect: in such a situation, we actually have two pairs of opposite forces, in the sense of Newton's third law:

1. $\boldsymbol{F}_{o,e}$ *and* $\boldsymbol{F}_{e,o}$*, respectively the gravitational force exerted by the object on Earth, and the gravitational force exerted by Earth on the object;*
2. $\boldsymbol{F}_{o,t}$ *and* $\boldsymbol{F}_{t,o}$*, respectively the contact force exerted by the object on the table, and the contact force exerted by the table on the object.*

Now to be more complete, there's even a third pair of forces, which would be in almost all practical situations negligible, namely, the gravitational forces exerted by the surface on the object, and by the object on the surface.

A good exercise to check one understanding of this topic would be to try to identify forces in the case of a hand pressed on a (static) wall, or in the case of two hands pressed against each other.

Remark 11. *Newton's third law is actually a consequence of the principle of conservation of momentum: essentially, in a closed system, the total momentum is a constant over time.*

This is a key idea in modern physics, and will be key later in the book.

Hooke's law

Hooke's law, as Newton's laws, is an *empirical law*, that is, which has been found by repeated experimentation. Essentially, it states that the force needed to extend or compress a spring by a given distance varies linearly with the distance during which the spring is extended/compressed. That linear factor k *is* the spring's constant. This force is thus a function of that distance/displacement $\boldsymbol{\delta}$:

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = k\boldsymbol{\delta}$$

The equation could be read as something like:

- If I don't compress/stretch the spring ($\boldsymbol{\delta} = \boldsymbol{0}$), no force is exerted;
- The more I compress/stretch the spring, the greater the force, where the relation between both varies linearly.

Remark 12. *In most practical cases,* $\boldsymbol{\delta}$ *will be a function of time* $\boldsymbol{\delta}(t)$*; thus, the force itself will be a function of time:* $\boldsymbol{F}_s(\boldsymbol{\delta}(t))$*.*

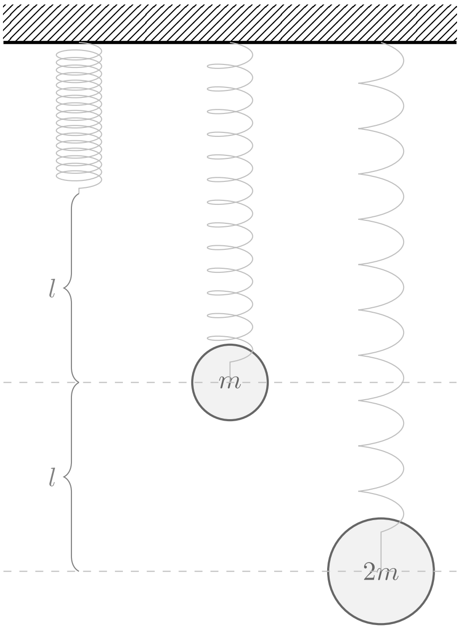
Remark 13. *By Newton's third law, if there's an object exerting a force on the spring, then the spring also exerts a force on the object, of equal intensity, but in the opposite direction. Hence, equivalently, Hooke's law could be reformulated in the form presented in the book:*

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = -k\boldsymbol{\delta}$$

Remark 14. *You may sometimes find that last version of the law, as is the case in the book, written as* $\boldsymbol{F}_s(\boldsymbol{x}) = -k\boldsymbol{x}$*. This is a convenient choice that physicists can make, where, by identifying the origin of the coordinate system as the rest position of the mass, the measure of displacement* $\boldsymbol{\delta}$ *will indeed always match the position vector* $\boldsymbol{x}$*.*

This can be summarized by saying that bmF_s *is the the force pulling the mass back to its rest position.*

Remark 15. *If you have a spring or two around (architect lamps are a good source) and some regular weight (small tableware, such as iron tea cups would do), you should be able to easily verify this law experimentally. In this case, the objects' weight will be the source of a gravitational force (we'll come* *back to the gravitational force in a moment): doubling the mass (attaching two cups) should double the deformation of the spring.*



Remark 16. *Were you at least to imagine the previous experiment, you would realize that there must so some limits to this law: the spring could break or be irremediably deformed. Hooke's law is actually a first-order linear approximation, that works reasonably well in plenty of practical cases. There are more advanced models who generalize Hooke's law, but they are of no use to us in this context.*

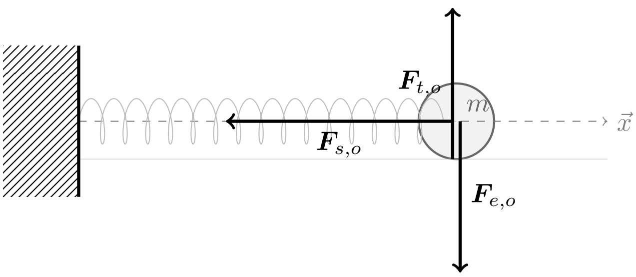
Forces diagram

The goal of a force diagram, or a free body diagram, is to identify the forces acting on an object, with the intent of applying Newton's second law, so as to later determine an equation of movement for that object: we want to identify the sum of all the forces **acting on the object**, $\boldsymbol{F}_{\text{net}} = \sum_i \boldsymbol{F}_i$, who are likely to contribute to the shape of its trajectory. Let's complete the diagram of our horizontal spring setup with:

$\boldsymbol{F}_{e,o}$ the gravitational force exerted by the earth on our object. Note that we haven't talked about gravitation yet, and as you'll see in a moment, we don't need to, but this will be necessary for the vertical spring setup;

$\boldsymbol{F}_{t,o}$ the normal force exerted by the table on our object. Because the surface is frictionless, the normal force *is* the contact force;

$\boldsymbol{F}_{s,o}$ the force exerted by the spring on our object, which can be described by Hooke's law.



Placement of the origin of the $\vec{x}$-axis As we've already observed before, it can be interesting to identify the origin of our coordinate system to the position where the system will be at equilibrium/rest.

Intensities and signs Thanks the previous convention, as discussed in Remark 14, Hooke's law can be used to determine the force exerted by the spring on the object, and can be written as $\boldsymbol{F}_{s,o} = -k\boldsymbol{x}$.

We know, from Newton's second law, that the contact/normal force exerted by the table on the object is equal to $-\boldsymbol{F}_{o,t}$, where $\boldsymbol{F}_{o,t}$ is the force exerted by the object on the table, which itself results from the gravitational force $\boldsymbol{F}_{e,o}$, i.e.

$$\boldsymbol{F}_{t,o} = -\boldsymbol{F}_{o,t} = -\boldsymbol{F}_{e,o}$$

As we'll see quickly, we don't actually need to compute anything more, in this situation. We'll nevertheless show how to compute the gravitational force in the next section, when considering the similar case of a mass attached to a vertical spring.

Applying Newton's second law We can now use Newton's second law, $\sum_i \boldsymbol{F}_i = m\boldsymbol{a} = m\ddot{\boldsymbol{x}}$ to deduce an equation of motion:

$$\begin{aligned} & \boldsymbol{F}_{e,o} + \boldsymbol{F}_{t,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \boldsymbol{F}_{e,o} - \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & -k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \quad \square \end{aligned}$$

Which is a second order, linear differential equation, homogeneous, with constant coefficients. It's often rewritten, as is the case in the book, by defining $\omega^2 = \dfrac{k}{m}$:

$$\boxed{\ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x}}$$

Gravitational force
To handle the case of a vertical spring, we won't be able to brush the gravitational force away as we did previously, and we'll need to compute it.

There are a few different ways to reach a formula for the gravitational force, but perhaps the simplest one, in the context of studying a mass attached to a vertical spring, would be to empirically study the fall of an object of mass m in Earth's gravitational field.

One could proceed for instance by video-taping a vertically falling object of known mass m. The video can then be discretized into a series of images, on which we could measure the evolution of the position of the mass over time. From there, we could compute the speed, as the variation of position, and the acceleration, as the variation of speed.

Then, neglecting the air resistance (well, we *could* perform the experiment in a vacuum), we can pretend that there's a single force acting on the mass, which *is* our gravitational force.

The experiment could obviously be performed multiple times to improve the accuracy. In the end, physicists have found that Earth's gravity is an acceleration vector $\boldsymbol{g}$ of intensity $9.81\,m.s^{-2}$, directed towards the center of the Earth, which experimentally gives a gravitational force of:

$$\boldsymbol{F}_{e,o} = m\boldsymbol{g}$$

Remark 17. *A home-made reenactment of the experiment would be difficult, especially if there's a need for strong precision, which would involve the presence of a vacuum. At least, more difficult than experimentally validating Hooke's law on a single spring.*

Historically, a simpler and more primitive setup was employed, for instance by Galileo, which would involved measuring the speed of a ball rolling on an inclined plane. A great way to lower the speed of the "fall" and thus to get more accurate time measurements.

Remark 18. *1) Note that* $\boldsymbol{g}$ *captures not only the gravitational force exerted by the Earth, but also captures the contribution of the Earth's rotation (centrifugal force).*

2) Because the Earth is not a perfect sphere, the value of $\|\boldsymbol{g}\|$ *is not uniform everywhere on Earth; there are noticeable differences from city to city for instance.*

Remark 19. *In Remark 9, we conceptualized the mass as a measure of resistance to acceleration. Physicists call this the inertial mass. And they conceptually distinguish it from what they call the gravitational mass, which is the property of matter that qualify how an object will interact with a gravitational field.*

We tacitly assumed that both are the same: this is the equivalence principle, a cornerstone of modern theories of gravities. So far, physicists haven't been able to distinguish the two masses experimentally either. But conceptually, they refer to two distinct notions.

Mathematically, this means that in the case of a falling object subject to a single force exerted by gravitation, by applying Newton's second law, we have:

$$\begin{aligned} & \boldsymbol{F}_{net} = \boldsymbol{F}_{e,o} = m_i \boldsymbol{a} \\ \Leftrightarrow \quad & m_g \boldsymbol{g} = m_i \boldsymbol{a} \\ \Leftrightarrow \quad & \boldsymbol{g} = \boldsymbol{a} \end{aligned}$$

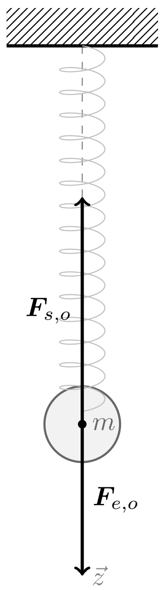
Remark 20. *1) Another approach could have been to rely on Newton's law of universal gravitation.*

$$\boldsymbol{F}_{e,o} = -G \frac{m_e m_o}{\|\boldsymbol{r}_{e,o}\|^2} \hat{\boldsymbol{r}}_{e,o}$$

But not only is this law also empirical, but it would involve knowing the mass of the Earth m_e, the gravitational constant G, etc.

2) Similarly we could delve into Einstein's theory of general relativity, but the prerequisites would again have been more sophisticated than exposing a basic experiment.

Vertical spring
We know have all we need to study a variant of our previous setup, slightly more complicated mathematically, where a mass m is attached to the end side of a vertical spring of constant k, itself firmly attached to a fixed element at its top:



We'll choose the same convention regarding the origin of our coordinate system (see Remark 14). Contrary to the previous case, there's no other force to counterbalance the gravitational force $\boldsymbol{F}_{e,o}$; applying Newton's second law gives:

$$\begin{aligned} & \sum_i \boldsymbol{F}_i = \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & m\boldsymbol{g} - k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x} + \boldsymbol{g} \end{aligned}$$

Again, with $\omega^2 = \frac{k}{m}$; so the only difference is that we get an additional constant $\boldsymbol{g}$, but this still is a second-order linear differential equation with constant coefficients.

Remark 21. *However, this differential equation is said to be non-homogeneous because of the non-zero constant* $\boldsymbol{g}$*, by opposition with the previous homogeneous equation obtained from the horizontal spring setup.*

Verifying the solution
Solving differential equations, mathematically, can be rather involved. For such simple equations, mathematicians themselves may find it sufficient to make an educated guess by tweaking an exponential-based function, despite having explored the subject with much more depth.

We won't discuss the details here, and will pragmatically satisfy ourselves with verifying that the proposed solution actually solves the equation:

$$x(t) = A\cos\omega t + B\sin\omega t$$

Remark 22. *We've been before expressing all our equations with vectors so far, but because we're working in one-dimension, we can divide everything by* $\hat{\boldsymbol{x}}$ *and work from the resulting scalar equations.*

And indeed, the proposed solution does solve the equation for the horizontal spring setup.

$$\begin{aligned}\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\\ &= -\omega^2(A\cos\omega t + B\sin\omega t)\\ &= -\omega^2 x(t) \quad \square\end{aligned}$$

But it doesn't solve the vertical setup; we'll get there quickly, this isn't much more complicated. You may want to try by yourself to see if you can tweak the solution of the homogeneous equation to solve the non-homogeneous one.

Initial conditions
Let's start by computing the expression of the velocity:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Then, at $t = 0$, we have:

$$\begin{aligned}x(0) &= \quad A\cos(0) + B\sin(0) = \boxed{B}\\ \dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square\end{aligned}$$

Equations and initial conditions for the vertical setup
There are general mathematical methods for reaching a solution to a non-homogeneous linear second-order differential equation with constant coefficient from a homogeneous, but again, suffice for us to verify that the following would work:

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}$$

Indeed:

$$\begin{aligned}\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\\ &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t - \frac{\omega^2}{\omega^2}g + g\\ &= -\omega^2(A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}) + g\\ &= -\omega^2 x(t) + g \quad \square\end{aligned}$$

From there, we can again express the initial position and speed as functions of A and B, first by computing the expression of the speed:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

And then by looking at what's happening at $t = 0$:

$$\begin{aligned}x(0) &= \quad A\cos(0) + B\sin(0) + \frac{g}{\omega^2} = \boxed{B + \frac{g}{\omega^2}}\\ \dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square\end{aligned}$$

Interlude 3: Partial Differentiation

Lecture 4: Systems of More Than One Particle

Lecture 5: Energy

More than one dimension

Exercise 2/3

Exercise 5. *Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2+y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitely that the total energy is conserved.*

Equations of motion: For this system, the potential energy V is:

$$V = \frac{1}{2}k(x^2+y^2) \tag{1}$$

By Newton's second law of motion, given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{2}$$

Or,

$$\begin{aligned} F_x &= \quad m\ddot{x} \\ F_y &= \quad m\ddot{y} \end{aligned} \tag{3}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{4}$$

Which in our case, translates to:

$$\begin{aligned} F_x(x,y) &= -\frac{\partial}{\partial x}V(x,y) = -kx \\ F_y(x,y) &= -\frac{\partial}{\partial y}V(x,y) = -ky \end{aligned} \tag{5}$$

Combining (3) and (5):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \tag{6}$$

Which are known by *L03E04* to be differential equations associated to harmonic motion, and solved by a slightly more general solution that the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{7}$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$\begin{aligned} v(t) = \quad \dot{x}(t) &= \quad \omega(-\alpha\sin(\omega t - \theta) + \beta\cos(\omega t - \theta) \\ a(t) = \quad \ddot{x}(t) &= \quad -\omega^2(\alpha\cos(\omega t - \theta) + \beta\sin(\omega t - \theta)) \\ &= \quad -\omega^2 x(t) \end{aligned} \tag{8}$$

Where, to differentiate e.g. $\alpha\cos(\omega t - \theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \tag{9}$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t=0), \dot{x}(t=0), y(t=0), \dot{y}(t=0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$\begin{aligned}
x(0) &= \alpha_x \cos(0) + \beta_x \sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha \sin(0) + \beta \cos(0)) \\
&= \omega\beta \\
&= \sqrt{\frac{k}{m}}\beta
\end{aligned} \tag{10}$$

Circular orbits: The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r, centered on point (a, b), with $(a, b, r) \in \mathbb{R}^3$ if:

$$(\forall t \geq 0),\ (x(t) - a)^2 + (y(t) - b)^2 = r^2 \tag{11}$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$$

Then, let's introduce two angles φ_x and φ_y such as:

$$\begin{array}{ll}
\sin\varphi_x = \alpha_x & \sin\varphi_y = \alpha_y \\
\cos\varphi_x = \beta_x & \cos\varphi_y = \beta_y
\end{array}$$

From, there, we can use the previous identity to rewrite our equations of motions (7) as:

$$\begin{aligned}
x(t) &= \sin\varphi_x \cos(\omega t - \theta_x) + \cos\varphi_x \sin(\omega t - \theta_x) &&= \sin(\omega t + \varphi_x - \theta_x) = \sin\Omega_x \\
y(t) &= \sin\varphi_y \cos(\omega t - \theta_y) + \cos\varphi_y \sin(\omega t - \theta_y) &&= \sin(\omega t + \varphi_y - \theta_y) = \sin\Omega_y
\end{aligned} \tag{12}$$

Obvoiusly with $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$ and $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$.

Let us now develop (11) with those two versions of $x(t)$ and $y(t)$:

$$\begin{aligned}
&(x(t) - a)^2 + (y(t) - b)^2 &&= r^2 \\
\Leftrightarrow &(\sin\Omega_x - a)^2 + (\sin\Omega_y - b)^2 &&= r^2 \\
\Leftrightarrow &\sin^2\Omega_x + \sin^2\Omega_y - 2(a\sin\Omega_x + b\sin\Omega_y)) + a^2 + b^2 &&= r^2
\end{aligned}$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a, b) = (0, 0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$\sin^2\Omega_x + \sin^2\Omega_y = r^2$$

Which we can rewrite a little bit using the fact that $\sin\varphi = \cos(\varphi - \pi/2)$:

$$\sin^2\Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2$$

But we know the pythagorean identity $\sin^2\varphi + \cos^2\varphi = 1$, hence we know there will be circular orbits when:

$$\left\{\begin{array}{lcl} r &=& 1 \\ \Omega_x &=& \Omega_y - \dfrac{\pi}{2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} r &=& 1 \\ \omega t + \varphi_x - \theta_x &=& \omega t + \varphi_y - \theta_y - \dfrac{\pi}{2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} r &=& 1 \\ \varphi_x - \theta_x &=& \varphi_y - \theta_y - \dfrac{\pi}{2} \end{array}\right.$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (12) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

Remark 23. *For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.*

Remark 24. *The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.*

Energy conservation: Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2}\sum_i m_i \dot{x_i}^2 \tag{13}$$

Which gives us for this system, expliciting the time-dependancies:

$$T(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}m\dot{y}(t)^2 = \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2) \tag{14}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt}T(t) &= \frac{1}{2}m(2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{15}$$

On the other hand, we can compute the variation of potential energy over time from (1), again using the chain rule:

$$\begin{aligned} \frac{d}{dt}V(t) &= \frac{1}{2}k(\frac{d}{dt}x(t)^2 + \frac{d}{dt}y(t)^2) \\ &= \frac{1}{2}k(2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= k(x(t)\dot{x}(t) + y(t)\dot{y}(t)) \end{aligned} \tag{16}$$

From (6), we have:

$$\begin{aligned} x(t) &= -\frac{m}{k}\ddot{x}(t) \\ y(t) &= -\frac{m}{k}\ddot{y}(t) \end{aligned} \tag{17}$$

Injecting in (16):

$$\begin{aligned} \frac{d}{dt}V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{18}$$

Thus from (15) and (18):

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square \tag{19}$$

That is, total energy E over time doesn't change.

Exercise 3/3

Lecture 6: The Principle of Least Action

The Transition to Advanced Mechanics

Action and the Lagrangian

Derivation of the Euler-Lagrange Equation

Exercise 1/6

Exercise 6. *Show that Eq.* (4) *is just another form of Newton's equation of motion* $F = ma$.

Where Eq. (4) are the freshly derived Euler-Lagrange equations of motions:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \tag{20}$$

In the context of a single particle moving in one dimension, with kinetic and potential energy given by:

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ V &= V(x) \end{aligned}$$

From which results the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}^2 - V(x) \end{aligned} \tag{21}$$

Let us recall that we also have the *potential energy principle*, stated in one-dimension as Eq. (1) of the previous chapter, *Lecture 5: Energy*:

$$F(x) = -\frac{d}{dx}V(x) \tag{22}$$

Which is also stated more generally in that same chapter, for an abstract configuration space $\{x\} = \{x_i\}$, as Eq. (5):

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\})$$

Thus, deriving each part of (20) with our Lagrangian (21), and considering the *definition* of a potential energy $V(x)$ (22) yields:

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{x}}L &= \frac{d}{dt}m\dot{x} \\ &= m\ddot{x} \end{aligned} \qquad\qquad \begin{aligned} \frac{\partial}{\partial x}L &= \frac{\partial}{\partial x}V(x) \\ &= -F \end{aligned}$$

Then indeed, Euler-Lagrange equations become equivalent to Newton's law of motion:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \\ \Leftrightarrow &m\ddot{x} - (-F) = 0 \\ \Leftrightarrow &\boxed{F = m\ddot{x} = ma} \quad \square \end{aligned}$$

More Particles and More dimensions

Exercise 2/6

Exercise 7. *Show that Eq.* (6) *is just another form of Newton's equation of motion* $F_i = m_i\ddot{x}_i$.

Where Eq. (6) are the following set of equation, defined for all $i \in [\![1, n]\!]$:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}L\right) = \frac{\partial}{\partial x_i}L \tag{23}$$

Remark 25. *This exercice is simply a generalization of the previous exercice (L06E01) to a configuration space of size $n \in \mathbb{N}$.*

Then again, let us recall the Lagrangian defined slightly earlier in the related section of the book:

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} m_i \dot{x_i}^2 \right) - V(\{x\}) \tag{24}$$

Hence, ($\forall i \in [\![1, n]\!]$):

$$\begin{aligned} \frac{\partial}{\partial \dot{x_i}} L &= \frac{\partial}{\partial \dot{x_i}} \sum_{j=1}^{n} \frac{1}{2} m_j \dot{x_j}^2 \\ &= \sum_{j=1}^{n} m_j \dot{x_j} \delta_{ij} \\ &= m_i \dot{x_i} \end{aligned} \qquad \frac{\partial}{\partial x_i} L = -\frac{\partial}{\partial x_i} V(\{x\}) \tag{25}$$

Again, we need the *potential energy principle*, stated as Eq. (5) of the previous chapter *Lecture 5: Energy*, for abstract configuration space $\{x\} = \{x_i\}$, as:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \tag{26}$$

From which we can conclude, by injecting (26) in the second half of (25), and connecting each side with Euler-Lagrange's equations (23), ($\forall i \in [\![1, n]\!]$):

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x_i}} L \right) = \frac{\partial}{\partial x_i} L \\ \Leftrightarrow & \frac{d}{dt} m_i \dot{x_i} = F_i(\{x\}) \\ \Leftrightarrow & \boxed{F_i = m_i \ddot{x_i}} \quad \square \end{aligned}$$

What's Good about Least Action?

Exercise 3/6

Exercise 8. *Use the Euler-Lagrange equations to derive the equations of motions from the Lagrangian in Eq.* (12)*.*

Again, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: ($\forall i \in [\![1, n]\!]$),

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{x_i}} L \right) = \frac{\partial}{\partial x_i} L \tag{27}$$

In the case of this exercice, the Lagrangian L is defined in Eq. (12) as:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X)$$

Let's compute the partial derivatives of L on $\dot{X}$, X, $\dot{Y}$ and Y:

$$\begin{aligned} \frac{\partial}{\partial \dot{X}} L &= \frac{\partial}{\partial \dot{X}} \left(\frac{m}{2} \dot{X}^2 + m\omega \dot{X} Y \right) \\ &= m\dot{X} + m\omega Y \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial X} L &= \frac{\partial}{\partial X} \left(\frac{m\omega^2}{2} X^2 - m\omega \dot{Y} X \right) \\ &= m\omega^2 X - m\omega \dot{Y} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \dot{Y}} L &= \frac{\partial}{\partial \dot{Y}} \left(\frac{m}{2} \dot{Y}^2 - m\omega \dot{Y} X \right) \\ &= m\dot{Y} - m\omega X \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial Y} L &= \frac{\partial}{\partial Y} \left(\frac{m\omega^2}{2} Y^2 + m\omega \dot{X} Y \right) \\ &= m\omega^2 Y + m\omega \dot{X} \end{aligned} \tag{28}$$

Finally, by plugging (28) into (27), we obtain:

$$\frac{d}{dt}\left(m\dot{X} + m\omega Y\right) = m\omega^2 X - m\omega\dot{Y} \qquad \frac{d}{dt}\left(m\dot{Y} - m\omega X\right) = m\omega^2 Y + m\omega\dot{X}$$
$$\Leftrightarrow \boxed{m\ddot{X} = m\omega^2 X - 2m\omega\dot{Y}} \qquad \Leftrightarrow \boxed{m\ddot{Y} = m\omega^2 Y + 2m\omega\dot{X}} \quad \square$$

Remark 26. *Those results indeed matches the equations proposed in the book just slightly before this exercice.*

Exercise 4/6

Exercise 9. *Work out George's Lagrangian and Euler-Lagrange equations in polar coordinates.*

As always, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: $(\forall i \in [\![1, n]\!])$,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \tag{29}$$

The original Lagrangian L in our case is defined by the Eq. (10) of this chapter as:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{30}$$

After the following coordinate shift (Eq. (9) of the book):

$$x = X\cos(\omega t) + Y\sin(\omega t) \qquad y = -X\sin(\omega t) + Y\cos(\omega t) \tag{31}$$

We obtained this Lagrangian (Eq. (12) of the book):

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X) \tag{32}$$

For the current exercice, the coordinate shift to polar equations is:

$$X = R\cos\theta \qquad Y = R\sin\theta \tag{33}$$

Where, implicitely, both R and θ are, as X and Y, functions of time.

Now, we have at least two ways of solving this exercice:

1. Either perform the coordinate shit (33) in (32): this will be a tedious but very similar developement as the one performed in the book to obtain (32) from (30) and (31);
2. or perform this new coordinate shift (33) directly in the first coordinate shift (31), and work from the first Lagrangian (30) instead: some trigonometric identities are likely to ease at least the beginning of the work here.

We will try both aproaches, and expect to find the exact same solutions in the end.

First approach

Let use start by computing the time derivative of X and Y as defined by (33), using the both the product [2] and the chain rule [3]:

$$\dot{X} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta \qquad \dot{Y} = \dot{R}\sin\theta + R\dot{\theta}\cos\theta \tag{34}$$

Remark 27. *For clarity, as a similar developement will happen a few times, let's go into details for the first one: the product rule for two functions u and v of a single variable, of respective derivatives u' and v' is*

$$(uv)' = u'v + uv'$$

[2]`https://en.wikipedia.org/wiki/Product_rule`
[3]`https://en.wikipedia.org/wiki/Chain_rule`

Now the chain rule is, again for the same kind of functions:

$$(u(v(x))' = v'(x)u'(v(x))$$

In the present case, we have a $X(t)$ *defined as the product of two functions:* $X(t) = R(t)\cos(\omega(t))$*, where the second one is itself a composition of two functions* $\cos(\omega(t))$*. Hence, by applying first the product rule, we obtain:*

$$X'(t) = R'(t)\cos(\omega(t)) + R(t)\left(\cos(\omega(t))\right)'$$

While the chain rule gives us:

$$(\cos(\omega(t)))' = -\omega'(t)\sin(\omega(t))$$

Hence,

$$X'(t) = R'(t)\cos(\omega(t)) - R(t)\omega'(t)\sin(\omega(t))$$

Now, our goal will be to plug (33) and (34) into the Lagrangian (32) obtained after the first coordinate shift, but doing that transformation at once will gives a difficult to read equation. Instead, we'll work in smaller steps, simplying our results using trigonometric identities along the way.

Let us start with $X^2 + Y^2$, using the fact that $\sin^2\theta + \cos^2\theta = 1$:

$$\begin{aligned} X^2 + Y^2 &= R^2\cos^2\theta + R^2\sin^2\theta \\ &= R^2(\cos^2\theta + \sin^2\theta) \\ &= R^2 \end{aligned} \tag{35}$$

Now for $\dot{X}^2 + \dot{Y}^2$, using the same trigonometric identity:

$$\begin{aligned} \dot{X}^2 &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)^2 \\ &= \dot{R}^2\cos^2\theta - 2R\dot{R}\dot{\theta}\cos\theta\sin\theta + R^2\dot{\theta}^2\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}^2 &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)^2 \\ &= \dot{R}^2\sin^2\theta + 2R\dot{R}\dot{\theta}\sin\theta\cos\theta + R^2\dot{\theta}^2\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}^2 + \dot{Y}^2 &= \dot{R}^2(\cos^2\theta + \sin^2\theta) + R^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \\ &= \dot{R}^2 + R^2\dot{\theta}^2 \end{aligned} \tag{36}$$

Finally, for $\dot{X}Y - \dot{Y}X$:

$$\begin{aligned} \dot{X}Y &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)R\sin\theta \\ &= R\dot{R}\cos\theta\sin\theta - R^2\dot{\theta}\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}X &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)R\cos\theta \\ &= R\dot{R}\cos\theta\sin\theta + R^2\dot{\theta}\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}Y - \dot{Y}X &= -R^2\dot{\theta}(\sin^2\theta + \cos^2\theta) \\ &= -R^2\dot{\theta} \end{aligned} \tag{37}$$

Now we're ready to plug (35), (36) and (37) into (32):

$$L = \boxed{\frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}} \tag{38}$$

Now, let's compute the partial derivatives of our new Lagrangian:

$$\begin{aligned} \frac{\partial}{\partial \dot{R}}L &= \frac{\partial}{\partial \dot{R}}\left(\frac{m}{2}\dot{R}^2\right) \\ &= m\dot{R} \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial R}L &= \frac{\partial}{\partial R}\left(\frac{m}{2}R^2\dot{\theta}^2 + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}\right) \\ &= (\dot{\theta}^2 + \omega^2 - 2\omega\dot{\theta})mR \\ &= (\dot{\theta} - \omega)^2 mR \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \dot{\theta}}L &= \frac{\partial}{\partial \dot{\theta}}\left(\frac{m}{2}R^2\dot{\theta}^2 - m\omega R^2\dot{\theta}\right) \\ &= mR^2(\dot{\theta} - \omega) \end{aligned} \qquad \frac{\partial}{\partial \theta}L = 0 \tag{39}$$

And from there, plug (39) in Euler-Lagrange (29) to derive the equations of motion (again for the second one, we use a combination of the product and chain rules for derivatives):

$$\frac{d}{dt}\left(m\dot{R}\right) = (\dot{\theta}-\omega)^2 mR \qquad\qquad \frac{d}{dt}\left(mR^2(\dot{\theta}-\omega)\right) = 0$$

$$\Leftrightarrow \boxed{\ddot{R} = (\dot{\theta}-\omega)^2 R} \qquad\qquad \Leftrightarrow m\left((\dot{\theta}-\omega)2\dot{R}R + R^2\ddot{\theta}\right) = 0$$

$$\Leftrightarrow \boxed{R\ddot{\theta} = (\omega-\dot{\theta})2\dot{R}} \quad \square$$

Second approach
We'll now try to see if we can get a cleaner derivation, hopefully with the same results, by combining the two coordinate shifts (31) and (33) first, and then rely on the original Lagrangian (30).

The combined coordinate shift is:

$$\begin{aligned} x &= R\cos\theta\cos(\omega t) + R\sin\theta\sin(\omega t) \\ y &= -R\cos\theta\sin(\omega t) + R\sin\theta\cos(\omega t) \end{aligned}$$

We have the four following trigonometric identities[4]:

$$cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+cos(\theta+\varphi)}{2} \qquad sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-cos(\theta+\varphi)}{2}$$

$$cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-sin(\theta-\varphi)}{2} \qquad sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+sin(\theta-\varphi)}{2}$$

Hence the coordinate shift can be rewritten:

$$\begin{aligned} x &= R\cos(\theta-\omega t) \\ y &= R\sin(\theta-\omega t) \end{aligned} \tag{40}$$

To inject it in the original Lagrangian (30), we need to compute $\dot{x}^2+\dot{y}^2$. For the derivation, as previously, we'll rely on a combination of the product/chain rule; we'll note $\varphi = \theta - \omega t$:

$$\begin{aligned} \dot{x} &= \dot{R}\cos\varphi - R(\dot{\theta}-\omega)\sin\varphi \\ \dot{y} &= \dot{R}\sin\varphi + R(\dot{\theta}-\omega)\cos\varphi \end{aligned}$$

$$\begin{aligned} \dot{x}^2 &= \dot{R}^2\cos^2\varphi - 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\sin^2\varphi \\ \dot{y}^2 &= \dot{R}^2\sin^2\varphi + 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\cos^2\varphi \end{aligned}$$

Hence the Lagrangian becomes, again using the pythagorean trigonometric identity $\cos^2\theta+\sin^2\theta = 1$:

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) \\ &= \boxed{\frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}-\omega)^2\right)} \\ &= \frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}^2 - 2\dot{\theta}\omega + \omega^2)\right) \\ &= \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m}{2}R^2\omega^2 - m\omega R^2\dot{\theta} \end{aligned}$$

Which is the same Lagrangian we had before in (38), from which we would obviously derive the exact same equation of motion. $\square$.

Remark 28. *As expected, the derivation is overall less tedious, but only because the complexity is now hidden behind the trigonometric identities.*

[4] `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities`

Remark 29. *A little later in the book, a solution to this exercice is proposed: it starts with this Lagrangian:*

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Which is exactly our Lagrangian, however assuming for some reason that $\omega = 0$. From which follows the same equation of motions, again with the same assumption regarding ω:

$$\ddot{r} = r\dot{\theta}^2$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

Generalized Coordinates and Momenta

Exercise 5/6

Cyclic coordinates

Exercise 6/6

Exercise 10. *Explain how we derived this.*

Let us recall that "this" refers to the following expression for the kinetic energy:

$$T = m(\dot{x_+}^2 + \dot{x_-}^2)$$

Starting from the following Lagrangian, involving two particles moving on a line with respective position and velocity x_i, $\dot{x_i}$:

$$L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) - V(x_1 - x_2) \tag{41}$$

After having performed the following change of coordinates:

$$x_+ = \frac{x_1 + x_2}{2} \qquad\qquad x_- = \frac{x_1 - x_2}{2} \tag{42}$$

From the Lagrangian, (41) we have the kinetic energy:

$$T = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \tag{43}$$

By first both summing and substracting the two equations of (42), and then by linearity of the derivation, we get:

$$\begin{aligned} x_+ + x_- &= x_1 & \qquad x_+ - x_- &= x_2 \\ \dot{x_+} + \dot{x_-} &= \dot{x_1} & \qquad \dot{x_+} - \dot{x_-} &= \dot{x_2} \end{aligned} \tag{44}$$

It's now simply a matter of injecting (44) into (43):

$$\begin{aligned} T &= \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= \frac{m}{2}((\dot{x_+} + \dot{x_-})^2 + (\dot{x_+} - \dot{x_-})^2) \\ &= \frac{m}{2}(2\dot{x_+}^2 + 2\dot{x_-}^2 + 2\dot{x_+}\dot{x_-} - 2\dot{x_+}\dot{x_-}) \\ &= m(\dot{x_+}^2 + \dot{x_-}^2) \quad \square \end{aligned}$$

Lecture 7: Symmetries and Conservation Laws

Preliminaries

Exercise 1/7

Exercise 11. *Derive Equations* (2) *and explain the sign difference.*

Let us recall Equations (2):

$$\dot{p_1} = -V'(q_1 - q_2) \qquad \dot{p_2} = +V'(q_1 - q_2)$$

We have to derive them from the Lagrangian given in Equation (1), which represents a system of two generalized coordinates q_1 and q_2:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1 - q_2) \tag{45}$$

To retrieve the equations of motions from a Lagrangian, we need to use Euler-Lagrange's equations, for instance recalled as Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Let us also recall, again from previous chapter, right after Equation (13), that the conjugate momentum is defined by

$$p_i = \frac{\partial}{\partial \dot{q_1}}L$$

For our Lagrangian (45), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}}L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}}L = \dot{q_2} \tag{46}$$

$$\frac{d}{dt}p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt}p_2 = \dot{p_2} = \ddot{q_2} \tag{47}$$

Using the chain rule[5] for the other half, with $\varphi(q_i) = q_1 - q_2$, we get:

$$\begin{aligned}
\frac{\partial}{\partial q_1}L &= -\frac{\partial}{\partial q_1}V(\varphi(q_1)) & \frac{\partial}{\partial q_2}L &= -\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\
&= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1}V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\
&= -(\frac{\partial}{\partial q_1}V)(q_1 - q_2) & &= +(\frac{\partial}{\partial q_2}V)(q_1 - q_2)
\end{aligned} \tag{48}$$

By noting $V' = \frac{\partial}{\partial q_i}V$, and combining equations (46), (47) and (48), we indeed obtain the expected equations of motion $\square$.

Remark 30. *That is, assuming,* $\frac{\partial}{\partial q_1}V(q_1) = \frac{\partial}{\partial q_2}V(q_2)$*: for all the energy potential presented earlier in the book, there's indeed such a symmetry, e.g.*

$$\begin{aligned}
V &= \frac{1}{2}k(x^2 + y^2), & &p103 \\
V &= \frac{1}{2}\frac{k}{x^2 + y^2}, & &p103 \\
V &= -m\omega^2(X^2 + Y^2), & &p120
\end{aligned}$$

A similar tacit assumption seems to exists in Herbert Goldstein's Classical Mechanics[6]*.*

[5] https://en.wikipedia.org/wiki/Chain_rule
[6] https://physics.stackexchange.com/a/107141

Mathematically, the sign difference comes from the fact that the potential depends on one side from q_1 and on the other from $-q_2$, which will persist when differentiating the potential V. Physically, it reflects that there's an order relation between the two "positions" q_1 and q_2: one will come before the other, and our potential V depends on this ordering.

Exercise 2/7

Exercise 12. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{49}$$

Because the question is a unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

As for the previous exercice, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}} L\right) = \frac{\partial}{\partial q_i} L$$

Which was in the book followed by the definition of the conjugate momentum p_i:

$$p_i = \frac{\partial}{\partial \dot{q_1}} L$$

For our Lagrangian (49), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}} L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}} L = \dot{q_2} \tag{50}$$

$$\frac{d}{dt} p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt} p_2 = \dot{p_2} = \ddot{q_2} \tag{51}$$

Using the chain rule[7] for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\begin{aligned} \frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -(a\frac{\partial}{\partial q_1} V)(aq_1 - bq_2) & &= +(b\frac{\partial}{\partial q_2} V)(aq_1 - bq_2) \end{aligned} \tag{52}$$

As for the previous exercice, it seems that there a tacit assumption of a symmetry within the potential V so that we can write $V' = \frac{\partial}{\partial q_i} V$; then, combining (50), (51) and (52):

$$\dot{p_1} = -aV'(aq_1 - bq_2) \qquad \dot{p_2} = +bV'(aq_1 - bq_2)$$

As suggested, let's multiply the first equation by b, the second by a, and sum the result:

$$b\dot{p_1} + a\dot{p_2} = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is a indeed a constant over time, i.e. that it is conserved (over time).

[7] https://en.wikipedia.org/wiki/Chain_rule

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particules respectively by a and b, and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance

Examples of symmetries

Exercise 3/7

Exercise 13. *Show that the combination $aq_1 + bq_2$, along with the Lagrangian, is invariant under Equations* (7)*.*

Let us first recall the equations for the potential (Equations (3)):

$$V(q_1, q_2) = V(aq_1 - bq_2)$$

Which is meant to be considered in the case of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{53}$$

Finally, the "Equations (7)" relate to the following change of coordinates:

$$\begin{aligned} q_1 &\to q_1 - b\delta \\ q_2 &\to q_2 + a\delta \end{aligned} \tag{54}$$

Remark 31. *There are typos around here in the book. In my printed version, it is as previously described, but in an online version, it is given by (mind the signs):*

$$\begin{aligned} q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 - a\delta \end{aligned}$$

yet in that same online version, the potential is said to depend on $aq_1 + bq_2$ in accordance to Equations (3)*, but said Equations* (3) *actually make it depend on $aq_1 - bq_2$!*

To summarize, with a $V(aq_1+bq_2)$, the two previous transformations will keep the Lagrangian unchanged. But with a $V(aq_1 - bq_2)$, none of the previous transformations will keep the Lagrangian; those two will:

$$\begin{aligned} q_1 &\to q_1 - b\delta \\ q_2 &\to q_2 - a\delta \end{aligned} \qquad\qquad \begin{aligned} q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 + b\delta \end{aligned} \tag{55}$$

In what follows, we will arbitrarily assume a $V(aq_1 - bq_2)$, and, say, the first transformation of (55)*.*

Assuming a, b and δ are time-invariant, it follows that $\dot{q_1}$ and $\dot{q_2}$ are unchanged by this transformation, hence

$$\begin{aligned} \dot{q_i} &\to \dot{q_i} \\ \dot{q_1}^2 + \dot{q_2}^2 &\to \dot{q_1}^2 + \dot{q_2}^2 \end{aligned}$$

Injecting (54) into (53) gives us the following Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(a(q_1 - b\delta) - b(q_2 - a\delta)) \\ &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \end{aligned}$$

We can see that indeed, the Lagrangian is unchanged; because the $\dot{q_i}$ are also unchanged, we would derive the exact same equation of motions as we did for the previous exercise.

Exercise 4/7

Exercise 14. *Show this to be true.*

Where "this" refers to the fact that this Lagrangian (Equation (8)):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \tag{56}$$

does not change to first order in δ, for the infinitesimal transformation described by e.g. Equations (12):

$$\begin{aligned} \delta_v x &= y\delta \\ \delta_v y &= -x\delta \end{aligned} \tag{57}$$

The transformation to the derivatives over time of x and y has already been established in Equations (11):

$$\begin{aligned} \dot{x} &\to \dot{x} + \dot{y}\delta \\ \dot{y} &\to \dot{y} - \dot{x}\delta \end{aligned} \tag{58}$$

Let's then perform the substitution described by (57) and (58) in the Lagrangian (56):

$$\begin{aligned} L &= \frac{m}{2}((\dot{x} + \dot{y}\delta)^2 + (\dot{y} - \dot{x}\delta)^2) - V((x + y\delta)^2 + (y - x\delta)^2) \\ &= \frac{m}{2}\left(\left(\dot{x}^2 + 2\dot{x}\dot{y}\delta + (\dot{y}\delta)^2\right) + \left(\dot{y}^2 - 2\dot{y}\dot{x}\delta + (\dot{x}\delta)^2\right)\right) \\ &\quad - V\left(\left(x^2 + 2xy\delta + (y\delta)^2\right) + \left(y^2 - 2yx\delta + (x\delta)^2\right)\right) \\ &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \delta^2(\dot{x}^2 + \dot{y}^2)\right) - V\left(x^2 + y^2 + \delta^2(x^2 + y^2)\right) \end{aligned}$$

Now, as we care about first-order changes in δ only, changes proportional to $\delta^n|_{n\geq 2}$ will be negligible; it follows that the Lagrangian is indeed unchanged:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \quad \square$$

Back to examples

Exercise 5/7