# The Theoretical Minimum
## Quantum Mechanics - Solutions
### L07E03

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**Exercise 1.** *a) Rewrite Eq. 7.10 in component form, replacing the symbols A, B, a, and b with the matrices and column vectors from Eqs. 7.7 and 7.8.*

*b) Perform the matrix multiplications Aa and Bb on the right-hand side. Verify that each result is a $4 \times 1$ matrix.*

*c) Expand all three Kronecker products.*

*d) Verify the row and column sizes of each Kronecker product:*

- $A \otimes B : 4 \times 4$

- $a \otimes b : 4 \times 1$

- $Aa \otimes Bb : 4 \times 1$

*e) Perform the matrix multiplication on the left-hand side, resulting in a $4 \times 1$ column vector. Each row should be the sum of four separate terms.*

*f) Finally, verify that the resulting column vectors on the left and right sides are identical.*

Recall Eq. 7.10

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

And Eq. 7.7 and 7.8:

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}; \qquad \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}$$

Our goal is to prove Eq. 7.10 by following all the recommended steps. It's a bit tedious, but otherwise presents no major difficulties.

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a) Let's rewrite the equation (that's still to be proved) in component form:

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

$$\Leftrightarrow \left( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \left( \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) = \left( \left( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \right) \otimes \left( \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) \right)$$

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b) Let's expand $Aa$ and $Bb$:

$$Aa = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix}; \quad Bb = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix};$$

From Eqs. 7.7 and 7.8, we can see that all Kronecker products indeed expand to $4 \times 1$ matrices. Equation 7.10 is then equivalent to:

$$\left( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \left( \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) = \left( \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix} \otimes \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix} \right)$$

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c), d), e), f) I'll be mixing all those steps together, because this is fairly trivial. First, $A \otimes B$ and $a \otimes b$ are respectively Eqs. 7.7 and 7.8. This gives us already:

$$\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} = \left( \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix} \otimes \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix} \right)$$

It remains to expand the last Kronecker product, for which we can use 7.8:

$$\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} = \begin{pmatrix} (A_{11}a_{11} + A_{12}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{11}a_{11} + A_{12}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix}$$

And it's now trivial to verify that this holds, as expected.  $\square$