The Theoretical Minimum

Classical Mechanics - Solutions

L11E03

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Exercise 1. *Show that the vector potentials in Equations (8) and Equations (9) both give the same uniform magnetic field. This means that the two differ by a gradient. Find the scalar whose gradient, when added to Equations (8), gives Equations (9).*

We're in the context of exploring how a magnetic field $\boldsymbol{B}$ *must* "derive" from vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

That is:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \\ \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{pmatrix}$$

Now the authors gave us two vector potential $\boldsymbol{A}$, $\boldsymbol{A'}$ in the referenced Equations (8) and (9):

$$\boldsymbol{A} = \begin{pmatrix} 0 \\ bx \\ 0 \end{pmatrix}; \qquad \boldsymbol{A'} = \begin{pmatrix} -by \\ 0 \\ 0 \end{pmatrix}$$

And we must prove that they correspond to an uniform magnetic field pointing in the z axis with intensity b (i.e $\boldsymbol{B} = (0, 0, b)$)
We just have to compute the curl of $\boldsymbol{A}$ and $\boldsymbol{A'}$:

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \\ \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ b-0 \end{pmatrix} = \boldsymbol{B} \quad \square$$

$$\boldsymbol{\nabla} \times \boldsymbol{A'} = \begin{pmatrix} \dfrac{\partial A'_z}{\partial y} - \dfrac{\partial A'_y}{\partial z} \\ \dfrac{\partial A'_x}{\partial z} - \dfrac{\partial A'_z}{\partial x} \\ \dfrac{\partial A'_y}{\partial x} - \dfrac{\partial A'_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ 0-(-b) \end{pmatrix} = \boldsymbol{B} \quad \square$$

Now the two vector fields must differ by gradient field generate from some scalar field $s(x, y, z)$:

$$\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla} s$$

Which means

$$\boldsymbol{\nabla} s = \boldsymbol{A}' - \boldsymbol{A} = \begin{pmatrix} -by \\ -bx \\ 0 \end{pmatrix} = -b \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial s}{\partial x} \\ \dfrac{\partial s}{\partial y} \\ \dfrac{\partial s}{\partial z} \end{pmatrix}$$

We can "see" that $s(x, y, z) = -bxy$ fits:

$$\frac{\partial s}{\partial x} = -by; \qquad \frac{\partial s}{\partial y} = -bx$$