The Theoretical Minimum

Classical Mechanics - Solutions

L10E01

M. Bivert

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Exercise 1. *Prove Eq.* (14)

Eq. (14) of the book refers to:

$$\{F(q,p), p_i\} = \frac{\partial F(q,p)}{\partial q_i}$$

Where the brackets $\{.,.\}$ are the Poisson Brackets: for A and B each two functions of $2N$ variables $\{p_i\}_{1\leq i\leq N}$ and $\{q_i\}_{1\leq i\leq N}$, $(N \in \mathbb{N})$:

$$\{A, B\} = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

And $F(p,q)$ a function of q and p. There's a bit of ambiguity regarding what p and q are, which actually doesn't affect the derivation, but let's make things clear anyway. In the previous example, we proved that $\{q^n, p\} = nq^{n-1}$: in this case, $N = 1$ and we had a single q and a single p.

But now we're asked to prove a result involving $F(p,q)$ partially derived according to q_i, which implies, for the result not to be trivial, that F is a function of q_i, and thus that p and q are actually tuples of N p_i and N q_i.

So, let's expand the Poisson brackets to be evaluated, using the definition of the Poisson brackets:

$$\{F(q,p), p_i\} = \sum_{k=1}^{N} \frac{\partial}{\partial q_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial p_k} - \frac{\partial}{\partial p_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial q_k}$$

Now because p_i will never depends on q_k, as those are two distinct variables, $\dfrac{\partial p_i}{\partial q_k} = 0$, and the previous expression shrinks to:

$$\{F(q,p), p_i\} = \sum_{k=1}^{N} \frac{\partial}{\partial q_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial p_k}$$

For similar reasons, $\dfrac{\partial p_i}{\partial p_k} = \delta_i^k$, and the previous expression shrinks again to:

$$\{F(q,p), p_i\} = \frac{\partial}{\partial q_i}F(q_1,\cdots,q_N,p_1,\cdots,p_N) = \boxed{\frac{\partial F(p,q)}{\partial q_i}}$$

Remark 1. *Eq.* (15) *of the book is to be proven as we did for Eq.* (14).

Remark 2. *Earlier in this section, the authors informally invited us to verify the properties of the Poisson brackets (anti-symmetry, linearity, product rule). I won't be doing it, because I think at this stage of the book, this should be elementary: you just have to replace the brackets by their definition, and re-arrange the terms, often using linearity of the differentiation/partial differentiation, and then switch back to expressions involving (the expected) Poisson brackets again.*