The Theoretical Minimum

Classical Mechanics - Solutions

L07E02

M. Bivert

September 15, 2022

Exercise 1. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{1}$$

Because the question is a unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

As for the previous exercice, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}} L\right) = \frac{\partial}{\partial q_i} L$$

Which was in the book followed by the definition of the conjugate momentum p_i:

$$p_i = \frac{\partial}{\partial \dot{q_1}} L$$

For our Lagrangian (1), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}} L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}} L = \dot{q_2} \tag{2}$$

$$\frac{d}{dt} p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt} p_2 = \dot{p_2} = \ddot{q_2} \tag{3}$$

Using the chain rule[1] for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\begin{aligned}
\frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\
&= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\
&= -(a\frac{\partial}{\partial q_1} V)(aq_1 - bq_2) & &= +(b\frac{\partial}{\partial q_2} V)(aq_1 - bq_2)
\end{aligned} \tag{4}$$

As for the previous exercice, it seems that there a tacit assumption of a symmetry within the potential V so that we can write $V' = \dfrac{\partial}{\partial q_i} V$; then, combining (2), (3) and (4):

[1] https://en.wikipedia.org/wiki/Chain_rule

$$\dot{p_1} = -aV'(aq_1 - bq_2) \qquad \dot{p_2} = +bV'(aq_1 - bq_2)$$

As suggested, let's multiply the first equation by b, the second by a, and sum the result:

$$b\dot{p_1} + a\dot{p_2} = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is a indeed a constant over time, i.e. that it is conserved (over time).

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particules respectively by a and b, and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance