The Theoretical Minimum

Classical Mechanics - Solutions

L06E04

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Exercise 1. *Work out George's Lagrangian and Euler-Lagrange equations in polar coordinates.*

As always, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: $(\forall i \in [\![1, n]\!])$,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \tag{1}$$

The original Lagrangian L in our case is defined by the Eq. (10) of this chapter as:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{2}$$

After the following coordinate shift (Eq. (9) of the book):

$$x = X\cos(\omega t) + Y\sin(\omega t) \qquad y = -X\sin(\omega t) + Y\cos(\omega t) \tag{3}$$

We obtained this Lagrangian (Eq. (12) of the book):

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X) \tag{4}$$

For the current exercise, the coordinate shift to polar equations is:

$$X = R\cos\theta \qquad Y = R\sin\theta \tag{5}$$

Where, implicitly, both R and θ are, as X and Y, functions of time.

Now, we have at least two ways of solving this exercise:

1. Either perform the coordinate shift (5) in (4): this will be a tedious but very similar development as the one performed in the book to obtain (4) from (2) and (3);
2. or perform this new coordinate shift (5) directly in the first coordinate shift (3), and work from the first Lagrangian (2) instead: some trigonometric identities are likely to ease at least the beginning of the work here.

We will try both approaches, and expect to find the exact same solutions in the end.

First approach

Let use start by computing the time derivative of X and Y as defined by (5), using the both the product [1] and the chain rule [2]:

$$\dot{X} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta \qquad \dot{Y} = \dot{R}\sin\theta + R\dot{\theta}\cos\theta \tag{6}$$

[1] `https://en.wikipedia.org/wiki/Product_rule`
[2] `https://en.wikipedia.org/wiki/Chain_rule`

Remark 1. *For clarity, as a similar development will happen a few times, let's go into details for the first one: the product rule for two functions u and v of a single variable, of respective derivatives u' and v' is*

$$(uv)' = u'v + uv'$$

Now the chain rule is, again for the same kind of functions:

$$(u(v(x))' = v'(x)u'(v(x))$$

In the present case, we have a $X(t)$ defined as the product of two functions: $X(t) = R(t)\cos(\omega(t))$, where the second one is itself a composition of two functions $\cos(\omega(t))$. Hence, by applying first the product rule, we obtain:

$$X'(t) = R'(t)\cos(\omega(t)) + R(t)\left(\cos(\omega(t))\right)'$$

While the chain rule gives us:

$$(\cos(\omega(t)))' = -\omega'(t)\sin(\omega(t))$$

Hence,

$$X'(t) = R'(t)\cos(\omega(t)) - R(t)\omega'(t)\sin(\omega(t))$$

Now, our goal will be to plug (5) and (6) into the Lagrangian (4) obtained after the first coordinate shift, but doing that transformation at once will gives a difficult to read equation. Instead, we'll work in smaller steps, simplifying our results using trigonometric identities along the way.

Let us start with $X^2 + Y^2$, using the fact that $\sin^2\theta + \cos^2\theta = 1$:

$$\begin{aligned} X^2 + Y^2 &= R^2\cos^2\theta + R^2\sin^2\theta \\ &= R^2(\cos^2\theta + \sin^2\theta) \\ &= R^2 \end{aligned} \tag{7}$$

Now for $\dot{X}^2 + \dot{Y}^2$, using the same trigonometric identity:

$$\begin{aligned} \dot{X}^2 &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)^2 \\ &= \dot{R}^2\cos^2\theta - 2R\dot{R}\dot{\theta}\cos\theta\sin\theta + R^2\dot{\theta}^2\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}^2 &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)^2 \\ &= \dot{R}^2\sin^2\theta + 2R\dot{R}\dot{\theta}\sin\theta\cos\theta + R^2\dot{\theta}^2\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}^2 + \dot{Y}^2 &= \dot{R}^2(\cos^2\theta + \sin^2\theta) + R^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \\ &= \dot{R}^2 + R^2\dot{\theta}^2 \end{aligned} \tag{8}$$

Finally, for $\dot{X}Y - \dot{Y}X$:

$$\begin{aligned} \dot{X}Y &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)R\sin\theta \\ &= R\dot{R}\cos\theta\sin\theta - R^2\dot{\theta}\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}X &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)R\cos\theta \\ &= R\dot{R}\cos\theta\sin\theta + R^2\dot{\theta}\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}Y - \dot{Y}X &= -R^2\dot{\theta}(\sin^2\theta + \cos^2\theta) \\ &= -R^2\dot{\theta} \end{aligned} \tag{9}$$

Now we're ready to plug (7), (8) and (9) into (4):

$$L = \boxed{\frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}} \tag{10}$$

Now, let's compute the partial derivatives of our new Lagrangian:

$$\begin{aligned} \frac{\partial}{\partial\dot{R}}L &= \frac{\partial}{\partial\dot{R}}\left(\frac{m}{2}\dot{R}^2\right) \\ &= m\dot{R} \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial R}L &= \frac{\partial}{\partial R}\left(\frac{m}{2}R^2\dot{\theta}^2 + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}\right) \\ &= (\dot{\theta}^2 + \omega^2 - 2\omega\dot{\theta})mR \\ &= (\dot{\theta} - \omega)^2 mR \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\dot{\theta}}L &= \frac{\partial}{\partial\dot{\theta}}\left(\frac{m}{2}R^2\dot{\theta}^2 - m\omega R^2\dot{\theta}\right) \\ &= mR^2(\dot{\theta} - \omega) \end{aligned} \qquad \frac{\partial}{\partial\theta}L = 0 \tag{11}$$

And from there, plug (11) in Euler-Lagrange (1) to derive the equations of motion (again for the second one, we use a combination of the product and chain rules for derivatives):

$$\begin{aligned} &\frac{d}{dt}\left(m\dot{R}\right) = (\dot{\theta}-\omega)^2 mR \\ \Leftrightarrow &\boxed{\ddot{R} = (\dot{\theta}-\omega)^2 R} \end{aligned} \qquad \begin{aligned} &\frac{d}{dt}\left(mR^2(\dot{\theta}-\omega)\right) = 0 \\ \Leftrightarrow &m\left((\dot{\theta}-\omega)2\dot{R}R + R^2\ddot{\theta}\right) = 0 \\ \Leftrightarrow &\boxed{R\ddot{\theta} = (\omega-\dot{\theta})2\dot{R}} \quad \square \end{aligned}$$

Second approach
We'll now try to see if we can get a cleaner derivation, hopefully with the same results, by combining the two coordinate shifts (3) and (5) first, and then rely on the original Lagrangian (2).

The combined coordinate shift is:

$$\begin{aligned} x &= R\cos\theta\cos(\omega t) + R\sin\theta\sin(\omega t) \\ y &= -R\cos\theta\sin(\omega t) + R\sin\theta\cos(\omega t) \end{aligned}$$

We have the four following trigonometric identities[3]:

$$cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+cos(\theta+\varphi)}{2} \qquad sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-cos(\theta+\varphi)}{2}$$
$$cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-sin(\theta-\varphi)}{2} \qquad sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+sin(\theta-\varphi)}{2}$$

Hence the coordinate shift can be rewritten:

$$\begin{aligned} x &= R\cos(\theta-\omega t) \\ y &= R\sin(\theta-\omega t) \end{aligned} \tag{12}$$

To inject it in the original Lagrangian (2), we need to compute $\dot{x}^2+\dot{y}^2$. For the derivation, as previously, we'll rely on a combination of the product/chain rule; we'll note $\varphi = \theta - \omega t$:

$$\begin{aligned} \dot{x} &= \dot{R}\cos\varphi - R(\dot{\theta}-\omega)\sin\varphi \\ \dot{y} &= \dot{R}\sin\varphi + R(\dot{\theta}-\omega)\cos\varphi \end{aligned}$$

$$\begin{aligned} \dot{x}^2 &= \dot{R}^2\cos^2\varphi - 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\sin^2\varphi \\ \dot{y}^2 &= \dot{R}^2\sin^2\varphi + 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\cos^2\varphi \end{aligned}$$

Hence the Lagrangian becomes, again using the Pythagorean trigonometric identity $\cos^2\theta + \sin^2\theta = 1$:

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) \\ &= \boxed{\frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}-\omega)^2\right)} \\ &= \frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}^2 - 2\dot{\theta}\omega + \omega^2)\right) \\ &= \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m}{2}R^2\omega^2 - m\omega R^2\dot{\theta} \end{aligned}$$

Which is the same Lagrangian we had before in (10), from which we would obviously derive the exact same equation of motion. $\square$.

Remark 2. *As expected, the derivation is overall less tedious, but only because the complexity is now hidden behind the trigonometric identities.*

[3] `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities`

Remark 3. *A little later in the book, a solution to this exercise is proposed: it starts with this Lagrangian:*

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Which is exactly our Lagrangian, however assuming for some reason that $\omega = 0$*. From which follows the same equation of motions, again with the same assumption regarding* ω*:*

$$\ddot{r} = r\dot{\theta}^2$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

Let's remind ourselves that ω *represents the rotation of the polar coordinate system of the present exercise, a rotation which won't exist for a general polar coordinate system, hence the reason we have* $\omega = 0$ *in the general case.*