The Theoretical Minimum

Quantum Mechanics - Solutions

L07E03

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Exercise 1. *a) Rewrite Eq.* 7.10 *in component form, replacing the symbols A, B, a, and b with the matrices and column vectors from Eqs.* 7.7 *and* 7.8*.*

b) Perform the matrix multiplications Aa and Bb on the right-hand side. Verify that each result is a 4×1 matrix.

c) Expand all three Kronecker products.

d) Verify the row and column sizes of each Kronecker product:

- $A \otimes B \,:\, 4 \times 4$
- $a \otimes b \,:\, 4 \times 1$
- $Aa \otimes Bb \,:\, 4 \times 1$

e) Perform the matrix multiplication on the left-hand side, resulting in a 4×1 column vector. Each row should be the sum of four separate terms.

f) Finally, verify that the resulting column vectors on the left and right sides are identical.

Recall Eq. 7.10

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

And Eq. 7.7 and 7.8:

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}; \qquad \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}$$

Our goal is to prove Eq. 7.10 by following all the recommended steps. It's a bit tedious, but otherwise presents no major difficulties.

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a) Let's rewrite the equation (that's still to be proved) in component form:

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

$$\Leftrightarrow \left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) = \left(\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \right) \otimes \left(\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) \right)$$

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b) Let's expand Aa and Bb:

$$Aa = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix}; \quad Bb = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix};$$

From Eqs. 7.7 and 7.8, we can see that all Kronecker products indeed expand to 4×1 matrices. Equation 7.10 is then equivalent to:

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right) = \left(\begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix} \otimes \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix} \right)$$

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c), d), e), f) I'll be mixing all those steps together, because this is fairly trivial. First, $A \otimes B$ and $a \otimes b$ are respectively Eqs. 7.7 and 7.8. This gives us already:

$$\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} = \left(\begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix} \otimes \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix} \right)$$

It remains to expand the last Kronecker product, for which we can use 7.8:

$$\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} = \begin{pmatrix} (A_{11}a_{11} + A_{12}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{11}a_{11} + A_{12}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix}$$

And it's now trivial to verify that this holds, as expected. $\square$