The Theoretical Minimum

Quantum Mechanics - Solutions

L06E09

M. Bivert

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Exercise 1. *Prove that the four vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are eigenvectors of $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$. What are their eigenvalues?*

Recall the definition of those four vectors:

$$\begin{aligned}
|\text{sing}\rangle &= \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right); \qquad & |T_1\rangle &= \frac{1}{\sqrt{2}}\left(|ud\rangle + |du\rangle\right) \\
|T_2\rangle &= \frac{1}{\sqrt{2}}\left(|uu\rangle + |dd\rangle\right); \qquad & |T_3\rangle &= \frac{1}{\sqrt{2}}\left(|uu\rangle - |dd\rangle\right)
\end{aligned}$$

And the definition of $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z$$

Again for this exercise, we won't need to explicitly use the Pauli matrices σ_i/τ_j. But actually, we won't even need the multiplication table either, as we've already done most of the work in earlier exercises. Indeed, if we want to prove that $|\Psi\rangle$ is an eigenvector for $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$, we expect to be able to carry some computation following this pattern:

$$\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})|\Psi\rangle &= (\sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z)|\Psi\rangle \\
&= (\sigma_x\tau_x)|\Psi\rangle + (\sigma_y\tau_y)|\Psi\rangle + (\sigma_z\tau_z)|\Psi\rangle \\
&= \dots \\
&= \lambda_\Psi|\Psi\rangle
\end{aligned}$$

But we know from the book that:

$$\sigma_x\tau_x|\text{sing}\rangle = \sigma_y\tau_y|\text{sing}\rangle = \sigma_z\tau_z|\text{sing}\rangle = -|\text{sing}\rangle$$

From L06E07 that

$$\begin{aligned}
\sigma_x\tau_x|T_1\rangle &= \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) &&=: \quad T_1; \\
\sigma_y\tau_y|T_1\rangle &= \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) &&=: \quad T_1; \\
\sigma_z\tau_z|T_1\rangle &= -\frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) &&=: \quad -T_1;
\end{aligned}$$

And from L06E08 that:

$$\begin{aligned}
\sigma_x\tau_x|T_2\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) &&=: \quad T_2; & \sigma_x\tau_x|T_3\rangle &= \frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) =: \quad -T_3; \\
\sigma_y\tau_y|T_2\rangle &= -\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) &&=: \quad -T_2; & \sigma_y\tau_y|T_3\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) =: \quad T_3; \\
\sigma_z\tau_z|T_2\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) &&=: \quad T_2; & \sigma_z\tau_z|T_3\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) =: \quad T_3.
\end{aligned}$$

It follows that:

$$
\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|\text{sing}\rangle &= \sigma_x\tau_x)|\text{sing}\rangle + (\sigma_y\tau_y)|\text{sing}\rangle + (\sigma_z\tau_z)|\text{sing}\rangle &&= \boxed{-3}|\text{sing}\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_1\rangle &= \sigma_x\tau_x)|T_1\rangle + (\sigma_y\tau_y)|T_1\rangle + (\sigma_z\tau_z)|T_1\rangle &&= \boxed{+1}|T_1\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_2\rangle &= \sigma_x\tau_x)|T_2\rangle + (\sigma_y\tau_y)|T_2\rangle + (\sigma_z\tau_z)|T_2\rangle &&= \boxed{+1}|T_2\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_3\rangle &= \sigma_x\tau_x)|T_3\rangle + (\sigma_y\tau_y)|T_3\rangle + (\sigma_z\tau_z)|T_3\rangle &&= \boxed{+1}|T_3\rangle
\end{aligned}
$$

Hence, as foretold by the authors after this exercise, the triplets share a degenerate eigenvalue $(+1)$, while the singlet is associated to a unique eigenvalue (-3), which justifies *a posteriori* their names.