L07E03

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Exercise 1. *Show that the combination* aq_1+bq_2*, along with the Lagrangian, is invariant under Equations* (7)*.*

Let us first recall the equations for the potential (Equations (3)):

$$V(q_1, q_2) = V(aq_1 - bq_2)$$

Which is meant to be considered in the case of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{1}$$

Finally, the "Equations (7)" relate to the following change of coordinates:

$$\begin{aligned} q_1 &\to q_1 - b\delta \\ q_2 &\to q_2 + a\delta \end{aligned} \tag{2}$$

Remark 1. *There are typos around here in the book. In my printed version, it is as previously described, but in an online version, it is given by (mind the signs):*

$$\begin{aligned} q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 - a\delta \end{aligned}$$

yet in that same online version, the potential is said to depend on $aq_1 + bq_2$ *in accordance to Equations* (3)*, but said Equations* (3) *actually make it depend on* $aq_1 - bq_2$*!*

To summarize, with a $V(aq_1+bq_2)$*, the two previous transformations will keep the Lagrangian unchanged. But with a* $V(aq_1 - bq_2)$*, none of the previous transformations will keep the Lagrangian; those two will:*

$$\begin{aligned} q_1 &\to q_1 - b\delta & \qquad q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 - a\delta & \qquad q_2 &\to q_2 + b\delta \end{aligned} \tag{3}$$

In what follows, we will arbitrarily assume a $V(aq_1 - bq_2)$*, and, say, the first transformation of* (3)*.*

Assuming a, b and δ are time-invariant, it follows that $\dot{q_1}$ and $\dot{q_2}$ are unchanged by this transformation, hence

$$\begin{aligned} \dot{q_i} &\to \dot{q_i} \\ \dot{q_1}^2 + \dot{q_2}^2 &\to \dot{q_1}^2 + \dot{q_2}^2 \end{aligned}$$

Injecting (2) into (1) gives us the following Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(a(q_1 - b\delta) - b(q_2 - a\delta)) \\ &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \end{aligned}$$

We can see that indeed, the Lagrangian is unchanged; because the $\dot{q_i}$ are also unchanged, we would derive the exact same equation of motions as we did for the previous exercise.