

The Theoretical Minimum

Quantum Mechanics - Solutions

L03E04

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Exercise 1. Let $n_z = \cos \theta$, $n_x = \sin \theta \cos \phi$ and $n_y = \sin \theta \sin \phi$. Angles θ and ϕ are defined according to the usual conventions for spherical coordinates (Fig. 3.2). Compute the eigenvalues and eigenvectors for the matrix of Eq. 3.23.

Let's recall Eq. 3.23, which is general form of the spin 3-vector operator:

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix} = \begin{pmatrix} \cos \theta & (\sin \theta \cos \phi - i(\sin \theta \sin \phi)) \\ (\sin \theta \cos \phi + i(\sin \theta \sin \phi)) & -\cos \theta \end{pmatrix}$$

Observe (e.g. from the trigonometric circle) that:

$$\cos \theta = \cos(-\theta); \quad \sin \theta = -\sin(-\theta)$$

Hence:

$$\exp(-i\theta) := \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$$

And we can simplify our previous expression of σ_n to:

$$\sigma_n = \begin{pmatrix} \cos \theta & \exp(-i\phi) \sin \theta \\ \exp(i\phi) \sin \theta & -\cos \theta \end{pmatrix}$$

Note that as we're now in the general case, we indeed have two degrees of freedom, encoded by the two angles θ and ϕ ; the *why* has been explicited in subsection 2.5.

We're still confronted to a spin operator: we expect the eigenvalues to be $+1$ and -1 ¹. But let's check this first: an eigenvector $|\lambda\rangle$ associated to an eigenvalue λ must obey:

$$\sigma_n |\lambda\rangle = \lambda |\lambda\rangle$$

$$\Leftrightarrow \sigma_n |\lambda\rangle - \lambda |\lambda\rangle = 0 \Leftrightarrow (\sigma_n - I_2 \lambda) |\lambda\rangle = 0$$

But eigenvectors are non-zero, hence, again with 0_2 being the 2×2 zero matrix:

$$\Leftrightarrow \sigma_n - I_2 \lambda = 0_2$$

¹Remember from the real spectral theorem, or as the authors call it, the *fundamental theorem*, that because we have a Hermitian matrix, we know it's diagonalizable, that its eigenvalues are real, and that the corresponding eigenvectors form a orthogonal basis

And so this matrix $\sigma_n - I_2\lambda$ cannot be invertible². This translates to a condition on the determinant:

$$\begin{aligned}
\det(\sigma_n - I_2\lambda) = 0 &\Leftrightarrow \begin{vmatrix} \cos\theta - \lambda & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0 \\
&\Leftrightarrow -(\cos\theta - \lambda)(\cos\theta + \lambda) - \underbrace{\exp(i\phi)\exp(-i\phi)}_{=1}\sin^2\theta = 0 \\
&\Leftrightarrow -(\cos^2\theta - \lambda^2) - \sin^2\theta = 0 \\
&\Leftrightarrow \lambda^2 - \underbrace{(\sin^2\theta + \cos^2\theta)}_{=1} = 0 \\
&\Leftrightarrow \lambda^2 = 1 \\
&\Leftrightarrow \lambda = \begin{cases} +1 \\ -1 \end{cases}
\end{aligned}$$

The remaining difficulty is then in finding the eigenvectors. We can use the following argument³.

Consider as a first guess an eigenvector of the form:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \quad (z_1, z_2) \in \mathbb{C}^2$$

We can put both complex numbers in exponential form:

$$\begin{pmatrix} r_1 \exp(i\phi_1) \\ r_2 \exp(i\phi_2) \end{pmatrix} = \exp(i\phi_1) \begin{pmatrix} r_1 \\ r_2 \exp(i(\phi_2 - \phi_1)) \end{pmatrix}; \quad (r_1, r_2, \phi_1, \phi_2) \in \mathbb{R}^4$$

We can then ignore the general phase factor $\exp(i\phi_1)$, e.g. set $\phi_1 = 0$. Furthermore, we want the vector to be normalized (this is an eigenvector associated to the eigenvalue of a Hermitian operator: it must be normalized per the real spectral theorem), i.e.

$$|r_1|^2 + |r_2 \exp(i\phi_2)|^2 = 1 \Leftrightarrow |r_1|^2 + |r_2|^2 = 1$$

But we're then losing a degree of freedom, meaning, r_1 and r_2 are not independent from each other: we can express them both in term of a single parameter, as long as the previous equation is satisfied. We can choose, as it'll make computation easier, $r_1 = \cos\alpha$, $r_2 = \sin\alpha$, with $\alpha \in \mathbb{R}$. Finally, let's rename $\phi_2 = \phi_\alpha$ ⁴, which brings us to consider eigenvectors of the form:

$$\begin{pmatrix} \cos\alpha \\ \exp(i\phi_\alpha)\sin\alpha \end{pmatrix}$$

As for the previous exercise, we can use two different parameter α and β for each eigenvector. Again, because of the diagonalization process, we have the following relation

$$\sigma_n = PDP^{-1} \Leftrightarrow \sigma_n P = PD \underbrace{(PP^{-1})}_{:=I_2} = PD = P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

But the columns of P must contain our eigenvectors, so this is equivalent to:

$$\begin{aligned}
\underbrace{\begin{pmatrix} \cos\theta & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta \end{pmatrix}}_{=\sigma_n} \underbrace{\begin{pmatrix} \cos\alpha & \cos\beta \\ \exp(i\phi_\alpha)\sin\alpha & \exp(i\phi_\beta)\sin\beta \end{pmatrix}}_{=P} &= \begin{pmatrix} \cos\alpha & \cos\beta \\ \exp(i\phi_\alpha)\sin\alpha & \exp(i\phi_\beta)\sin\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha & -\cos\beta \\ \exp(i\phi_\alpha)\cos\alpha & -\exp(i\phi_\beta)\sin\beta \end{pmatrix}
\end{aligned}$$

²Again for otherwise, as recalled in L03E03, multiply both sides of the equation by its inverse, get an identity on the left-hand-side and still the zero matrix on the right-hand-side

³<https://physics.stackexchange.com/a/720025>

⁴Note that I'm not yet identifying ϕ_α with ϕ ; this will come naturally later on

Let's perform the matrix multiplication on the left:

$$\begin{pmatrix} \cos \theta \cos \alpha + \exp(i(\phi_\alpha - \phi)) \sin \theta \sin \alpha & \cos \theta \cos \beta + \exp(i(\phi_\beta - \phi)) \sin \theta \sin \beta \\ \exp(i\phi) \sin \theta \cos \alpha - \exp(i\phi_\alpha) \cos \theta \sin \alpha & \exp(i\phi) \sin \theta \cos \beta - \exp(i\phi_\beta) \cos \theta \sin \beta \end{pmatrix} \\ = \begin{pmatrix} \cos \alpha & -\cos \beta \\ \exp(i\phi_\alpha) \cos \alpha & -\exp(i\phi_\beta) \sin \beta \end{pmatrix}$$

From which we can extract the following system of equations:

$$\begin{cases} \cos \theta \cos \alpha + \exp(i(\phi_\alpha - \phi)) \sin \theta \sin \alpha & = \cos \alpha \\ \cos \theta \cos \beta + \exp(i(\phi_\beta - \phi)) \sin \theta \sin \beta & = -\cos \beta \end{cases}$$

Remark 1. As for the previous exercise, I leave it to you to check that the solution we'll find for this system also solve the two other omitted equations.

It's tempting to set $\phi = \phi_\alpha = \phi_\beta$, but can we do so? Well, we know the two eigenvectors will have to be orthogonal: this adds an additional constraint, which decrease our degrees of freedom by one, meaning there's one superfluous variable in $\{\alpha, \beta, \phi_\alpha, \phi_\beta\}$. We can *choose* to implement this constraint by setting $\phi_\alpha = \phi_\beta$.

From there, we can indeed set $\phi_\alpha = \phi_\beta = \phi$, as this allows us to solve the equation for α and β more easily:

$$\Leftrightarrow \begin{cases} \cos \theta \cos \alpha + \sin \theta \sin \alpha & = \cos \alpha \\ \sin \theta \cos \beta - \cos \theta \sin \beta & = -\cos \beta \end{cases}$$

Which is exactly the same system we had for the previous exercise, which was solved by:

$$\begin{cases} \alpha & = \theta/2 \\ \beta & = \frac{1}{2}(\theta + \pi) \end{cases}$$

With the same trigonometric identities as for the previous exercise:

$$\cos(\alpha + \frac{\pi}{2}) = -\sin \alpha; \quad \sin(\alpha + \frac{\pi}{2}) = \cos \alpha$$

We reach the following eigenvectors

$$\boxed{\begin{cases} | +1 \rangle & = \begin{pmatrix} \cos \alpha \\ \exp(i\phi) \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \exp(i\phi) \sin(\theta/2) \end{pmatrix} \\ | -1 \rangle & = \begin{pmatrix} \cos \beta \\ \exp(i\phi) \sin \beta \end{pmatrix} = \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi) \cos(\theta/2) \end{pmatrix} \end{cases}}$$

Alright, let's make the same verifications the authors did in the book after the previous exercise. First, we get the expected eigenvalues $+1, -1$, which are the only two eigenvalues we have for a spin operator.

Then the two eigenvectors must be orthogonal, indeed (I only do it one way; the other is trivially similar):

$$\begin{aligned} \langle +1 | -1 \rangle &= (\cos(\theta/2) \quad \exp(-i\phi) \sin(\theta/2)) \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi) \cos(\theta/2) \end{pmatrix} \\ &= -\cos(\theta/2) \sin(\theta/2) + \exp(-i\phi + i\phi) \cos(\theta/2) \sin(\theta/2) = 0 \end{aligned}$$

Finally, if we prepare a spin along the z -axis in the up state $|u\rangle$, then rotate our apparatus to lie along the \hat{n} axis, which *is not* restricted to the xz -plane anymore, we have according to the fourth principle⁵:

$$\begin{aligned} P(+1) &= |\langle u | +1 \rangle|^2 = \cos^2(\theta/2) \\ P(-1) &= |\langle u | -1 \rangle|^2 = \sin^2(\theta/2) \end{aligned}$$

Which then lead to the exact same computation regarding the expected value for the measurement:

$$\langle \sigma_n \rangle = \sum_i \lambda_i P(\lambda_i) = (+1) \cos^2(\theta/2) + (-1) \sin^2(\theta/2) = \boxed{\cos \theta}$$

Note also that $P(+1) + P(-1) = 1$.

⁵Don't hesitate to get back to the definition of $|u\rangle$ and that of the inner-product if this isn't clear enough.