The Theoretical Minimum

Quantum Mechanics - Solutions

L02E02

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Exercise 1. *Prove that* $|i\rangle$ *and* $|o\rangle$ *satisfy all of the conditions in Eqs.* 2.7, 2.8 *and* 2.9. *Are they unique in that respect?*

Let us recall, in order, Eqs. 2.7, 2.8, 2.9, 2.10, which defines $|i\rangle$ and $|o\rangle$, and both 2.5 and 2.6 which defines $|r\rangle$ and $|l\rangle$:

$$\langle i|o\rangle = 0$$

$$\langle o|u\rangle \langle u|o\rangle = \frac{1}{2} \qquad \langle o|d\rangle \langle d|o\rangle = \frac{1}{2}$$
$$\langle i|u\rangle \langle u|i\rangle = \frac{1}{2} \qquad \langle i|d\rangle \langle d|i\rangle = \frac{1}{2}$$

$$\langle o|r\rangle \langle r|o\rangle = \frac{1}{2} \qquad \langle o|l\rangle \langle l|o\rangle = \frac{1}{2}$$
$$\langle i|r\rangle \langle r|i\rangle = \frac{1}{2} \qquad \langle i|l\rangle \langle l|i\rangle = \frac{1}{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \qquad |o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \qquad |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

For clarity, let us recall that $\langle u|A\rangle$ is the component of $|A\rangle$ on the orthonormal vector $|u\rangle$. This is because in a $(|i\rangle)_{i\in F}$ *orthonormal* basis we have:

$$|A\rangle = \sum_{i\in F} \alpha_i |i\rangle$$
$$\Rightarrow \langle j|A\rangle = \langle j| \sum_{i\in F} \alpha_i |i\rangle = \sum_{i\in F} \alpha_i \langle j|i\rangle = \alpha_j$$

And to make better sense of those equations, let us recall that $\alpha_u^*\alpha_u = \langle A|u\rangle \langle u|A\rangle$ is the probability of a state vector $|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ to be measured in the state $|u\rangle$.

For Eq. 2.7, we have

$$\begin{aligned}
\langle i|o\rangle &= \begin{pmatrix}\iota_u^* & \iota_d^*\end{pmatrix}\begin{pmatrix}o_u\\ o_d\end{pmatrix} \\
&= \iota_u^* o_u + \iota_d^* o_d \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0 \quad \square
\end{aligned}$$

For Eqs. 2.8, we can rely on the projection on an orthonormal vector:

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square \qquad \langle o|d\rangle\langle d|o\rangle = \frac{i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} \quad \square$$

$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square \qquad \langle i|d\rangle\langle d|i\rangle = \frac{-i}{\sqrt{2}}\frac{i}{\sqrt{2}} = \frac{1}{2} \quad \square$$

For Eqs. 2.9, we need to rely on the column form of the inner-product:

$$\begin{aligned}
\langle o|r\rangle\langle r|o\rangle &= \begin{pmatrix}o_u^* & o_d^*\end{pmatrix}\begin{pmatrix}\rho_u\\ \rho_d\end{pmatrix}\begin{pmatrix}\rho_u^* & \rho_d^*\end{pmatrix}\begin{pmatrix}o_u\\ o_d\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle o|l\rangle\langle l|o\rangle &= \begin{pmatrix}o_u^* & o_d^*\end{pmatrix}\begin{pmatrix}\lambda_u\\ \lambda_d\end{pmatrix}\begin{pmatrix}\lambda_u^* & \lambda_d^*\end{pmatrix}\begin{pmatrix}o_u\\ o_d\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1-i+i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle i|r\rangle\langle r|i\rangle &= \begin{pmatrix}\iota_u^* & \iota_d^*\end{pmatrix}\begin{pmatrix}\rho_u\\ \rho_d\end{pmatrix}\begin{pmatrix}\rho_u^* & \rho_d^*\end{pmatrix}\begin{pmatrix}\iota_u\\ \iota_d\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1+i+i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle i|l\rangle\langle l|i\rangle &= \begin{pmatrix}\iota_u^* & \iota_d^*\end{pmatrix}\begin{pmatrix}\lambda_u\\ \lambda_d\end{pmatrix}\begin{pmatrix}\lambda_u^* & \lambda_d^*\end{pmatrix}\begin{pmatrix}\iota_u\\ \iota_d\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

Regarding the unicity of $|i\rangle, |o\rangle$, as for $|r\rangle, |l\rangle$, there definitely is a phase ambiguity, meaning, we can multiply either $|i\rangle$ or $|o\rangle$ by a *phase factor*, say $e^{i\theta}$, without disturbing any of the constraints: orthogonality, probabilities, and the resulting vectors are still unitary.

But as stated by the authors for $|r\rangle, |l\rangle$, measurable quantities are independant of any phase factors. So up to it, they seem to be unique so far.

However, let's try to change the i's place for instance in $|i\rangle$:

$$|i\rangle = \frac{i}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

The vector is still unitary, we still have orthogonality with $|o\rangle$, and if you try to compute $\langle i|u\rangle \langle u|i\rangle$, $\langle i|d\rangle \langle d|i\rangle$, $\langle i|r\rangle \langle r|i\rangle$ or $\langle i|l\rangle \langle l|i\rangle$, you'll still have the same probablities.
Now the question is, is this "swapping" of the i a phase factor? Meaning, can encode this transformation as a multiplication by some $e^{i\theta}$, for some $\theta \in \mathbb{R}$?

Well, the first term of $|i\rangle$ is multiplied by i; recall the definition of the complex exponential:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So this means the first term is multiplied by

$$\exp(i\frac{\pi}{2}) = 0 + i$$

The second term though, is multiplied by $-i$, this means, multiplied by:

$$\exp(-i\frac{\pi}{2}) = 0 + i \times (-1)$$

So we've found a variant of $|i\rangle$, that cannot be obtained by multiplying $|i\rangle$ by a phase factor, and hence:

$$\boxed{\text{The proposed solution is } \textit{not} \text{ unique [up to a phase factor].}}$$

Remark 1. *It may be interesting/possible to classify all such variants, meaning, see how much variety there is / how much structure they share and so forth.*