

The Theoretical Minimum

Quantum Mechanics - Solutions

L07E04

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May 6, 2023

Exercise 1. Calculate the density matrix for:

$$|\Psi\rangle = \alpha|u\rangle + \beta|d\rangle$$

Answer:

$$\begin{aligned}\psi(u) &= \alpha; & \psi^*(u) &= \alpha^* \\ \psi(d) &= \beta; & \psi^*(d) &= \beta^*\end{aligned}$$

$$\rho_{a'a} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{pmatrix}$$

Now try plugging in some numbers for α and β . Make sure they are normalized to 1. For example, $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$.

Start by recalling the definition of the density matrix for a single spin in a known state:

$$\rho_{aa'} = \psi^*(a')\psi(a)$$

Now we have no wave function ψ in the exercise statement (the answer set aside), but we can find it by identification with general form of $|\Psi\rangle$:

$$|\Psi\rangle = \sum_{a,b,c,\dots} \psi(a,b,c,\dots)|a,b,c,\dots\rangle$$

Hence, $\psi(u)$ is the component of $|\Psi\rangle$ following the $|u\rangle$ axis, and $\psi(d)$ the one on the $|d\rangle$ axis:

$$\psi(u) = \langle u|\Psi\rangle = \alpha; \quad \psi(d) = \langle d|\Psi\rangle = \beta;$$

Immediately:

$$\psi^*(u) = \alpha^*; \quad \psi^*(d) = \beta^*;$$

Then it's just about packaging all the $\rho_{aa'}$ in a matrix: the basis is ordered ($\{|u\rangle, |d\rangle\}$) hence:

$$\rho = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \begin{pmatrix} \psi^*(u)\psi(u) & \psi^*(d)\psi(u) \\ \psi^*(u)\psi(d) & \psi^*(d)\psi(d) \end{pmatrix} = \boxed{\begin{pmatrix} \alpha^*\alpha & \beta^*\alpha \\ \alpha^*\beta & \beta^*\beta \end{pmatrix}}$$

Remark 1. We could also use the fact that the density operator is defined as a linear combination of projectors corresponding to the potential states of the system, each scaled by a probability, and so that the sum of those probabilities is 1, e.g.:

$$\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|; \quad \text{where: } \sum_i P_i = 1$$

As we're in the case of a single spin in a known state $|\Psi\rangle$, this reduces to

$$\rho = 1|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi|$$

Assuming again the ordered basis $\{|u\rangle, |d\rangle\}$, we can write $\langle\Psi|$ and $|\Psi\rangle$ in column form, and perform the outer-product:

$$\rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

This allows us to double-check our previous result: it seems there's a typo in the exercise statement.

Let's compute a few density matrices for well-known states:

$$\begin{aligned}
|u\rangle &= 1|u\rangle + 0|d\rangle \quad \Rightarrow \quad \rho_{|u\rangle} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
|d\rangle &= 0|u\rangle + 1|d\rangle \quad \Rightarrow \quad \rho_{|d\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
|r\rangle &= \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|r\rangle} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \\
|l\rangle &= \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|l\rangle} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \\
|i\rangle &= \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|i\rangle} = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix} \\
|o\rangle &= \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|o\rangle} = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}
\end{aligned}$$

The French version of this exercise¹ is a bit more interesting, there are a few additional questions. We can for instance check that ρ is Hermitian:

$$\rho^\dagger = (\rho^*)^T = \begin{pmatrix} (\alpha^*\alpha)^* & (\beta^*\alpha)^* \\ (\alpha^*\beta)^* & (\beta^*\beta)^* \end{pmatrix}^T = \begin{pmatrix} \alpha\alpha^* & \beta\alpha^* \\ \alpha\beta^* & \beta\beta^* \end{pmatrix}^T = \begin{pmatrix} \alpha^*\alpha & \beta^*\alpha \\ \alpha^*\beta & \beta^*\beta \end{pmatrix} =: \rho \quad \square$$

Or that its trace is 1, because of the normalization condition on $|\Psi\rangle$:

$$\text{Tr}(\rho) = \alpha^*\alpha + \beta^*\beta = 1 \quad \square$$

Finally, we can check that ρ projects to $|\Psi\rangle$. Consider a vector which has a component perpendicular to $|\Psi\rangle$, that is, in the direction of $|\Psi^\perp\rangle$, and a component in the direction of $|\Psi\rangle$

$$|\Phi\rangle = \gamma|\Psi^\perp\rangle + \delta|\Psi\rangle$$

By linearity:

$$\rho|\Phi\rangle = \gamma\rho|\Psi^\perp\rangle + \delta\rho|\Psi\rangle$$

Using the fact that $\rho = |\Psi\rangle\langle\Psi|$, we see, by associativity on the products, and by the orthogonality condition between $|\Psi\rangle$ and $|\Psi^\perp\rangle$:

$$\rho|\Psi^\perp\rangle = (|\Psi\rangle\langle\Psi|)|\Psi^\perp\rangle = |\Psi\rangle(\underbrace{\langle\Psi|\Psi^\perp\rangle}_{=0}) = 0$$

On the other hand, by the normalization condition on $|\Psi\rangle$:

$$\rho|\Psi\rangle = (|\Psi\rangle\langle\Psi|)|\Psi\rangle = |\Psi\rangle(\underbrace{\langle\Psi|\Psi\rangle}_{=1}) = |\Psi\rangle$$

By injecting the two previous results in the one before, it follows that indeed that ρ projects a vector on the $|\Psi\rangle$ direction:

$$\rho|\Phi\rangle = \delta|\Psi\rangle$$

¹See <https://leminimumtheorique.jimdo.free.com/le%C3%A7on-7/exercice-7-4/> for a relevant excerpt, which by the way seems to confirm the typo hypothesis.