The Theoretical Minimum
Quantum Mechanics - Solutions
L06E03

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Exercise 1. *Prove that the state $|sing\rangle$ cannot be written as a product state.*

Let's recall the definition of the so-called *singlet* state $|\text{sing}\rangle$:

$$|\text{sing}\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right)$$

As for the previous exercise, we're still in the context of combining two state spaces: Alice's and Bob's, each representing the states of a spin, where the general form of Alice's state vectors is:

$$\alpha_u|u\} + \alpha_d|d\}, \quad (\alpha_u, \alpha_d) \in \mathbb{C}^2$$

While spin states for the second space (Bob's) are denoted:

$$\beta_u|u\rangle + \beta_d|d\rangle, \quad (\beta_u, \beta_d) \in \mathbb{C}^2$$

In this context, let's clarify the difference between a product state and a general composite state, with potential entanglement:

Product state obtained by developing a product between two states from Alice and Bob's state spaces, which yield something along the form:

$$\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

Remember from the previous exercise that such a state vector is naturally normalized, as a consequence of the normalization of the underlying vectors from Alice and Bob's space states;

General state for a 2-spins system obtained by linear combination of the vectors from the ordered basis $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$:

$$\Psi = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle$$

And impose a normalization condition on the scalar factors:

$$|\Psi| = 1 \Leftrightarrow \psi_{uu}^*\psi_{uu} + \psi_{ud}^*\psi_{ud} + \psi_{du}^*\psi_{du} + \psi_{dd}^*\psi_{dd} = 1$$

Clearly, $|\text{sing}\rangle$ is normalized: it's at least a general state for a 2-spins system. Assume it is a product state. Then there exists $(\alpha_u, \alpha_d, \beta_u, \beta_d) \in \mathbb{C}^4$ such that:

$$\begin{cases} \alpha_u\beta_d = & \frac{1}{\sqrt{2}} \\ \alpha_d\beta_u = & \frac{1}{\sqrt{2}} \\ \alpha_u\beta_u = & 0 \\ \alpha_d\beta_d = & 0 \end{cases}$$

But now, if $\alpha_u\beta_u = 0$, then at least either $\alpha_u = 0$ or $\beta_u = 0$. Assume $\alpha_u = 0$. But then, we can't have $\alpha_u\beta_d = 1/\sqrt{2}$. Assume then $\beta_u = 0$. Yet in this case, we can't have $\alpha_d\beta_u = 1/\sqrt{2}$.

So the system isn't solvable and our previous assumption can't hold. Hence, there's no such $(\alpha_u, \alpha_d, \beta_u, \beta_d) \in \mathbb{C}^4$, and $|\text{sing}\rangle$ is not a product state. $\square$