The Theoretical Minimum

Quantum Mechanics - Solutions

L04E05

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Exercise 1. *Take any unit* 3*-vector* $\boldsymbol{n}$ *and form the operator*

$$H = \frac{\hbar\omega}{2}\sigma \cdot \boldsymbol{n}$$

Find the energy eigenvalues and eigenvectors by solving the time-independent Schrödinger equation. Recall that Eq. 3.23 *gives* $\sigma \cdot \boldsymbol{n}$ *in component form.*

Let's recall Eq. 3.23, which is general form of the spin 3-vector operator:

$$\sigma_n = \sigma \cdot \boldsymbol{n} = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

And the *time-independent Schrödinger equation*[1]:

$$H|E_j\rangle = E_j|E_j\rangle$$

In an earlier exercise (L03E04), we actually diagonalized σ_n: this gave us two eigenvalues $+1$ and -1, and two eigenvectors:

$$|+1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \exp(i\phi)\sin(\theta/2) \end{pmatrix}; \qquad |-1\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi)\cos(\theta/2) \end{pmatrix}$$

Where $\boldsymbol{n}$ was a regular unitary 3-vector expressed in spherical coordinates:

$$\boldsymbol{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Let's see how we can leverage this previous work to our advantage: such an $\boldsymbol{n}$ vector still fit our purpose here. Furthermore, we know that the eigenvalues of σ_n are the only solutions to:

$$\sigma_n|F_j\rangle = F_j|F_j\rangle$$

But if we multiply both sides of this equation by $\frac{\hbar\omega}{2}$, we get exactly the equation we want to solve:

$$\underbrace{\frac{\hbar\omega}{2}\sigma_n}_{H}|F_j\rangle = \left(\frac{\hbar\omega}{2}F_j\right)|F_j\rangle$$

Multiplying the equation by a constant doesn't change the eigenvectors: they still are the only solutions, but the associated eigenvalues are now different:

$$\boxed{\lambda_1 = \frac{\hbar\omega}{2}; \qquad |\lambda_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \exp(i\phi)\sin(\theta/2) \end{pmatrix}}$$

$$\boxed{\lambda_2 = -\frac{\hbar\omega}{2}; \qquad |\lambda_2\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi)\cos(\theta/2) \end{pmatrix}}$$

[1] That's quite a fancy name for describing the eigenvectors of an operator, by comparison with the "iconic" Schrödinger equation. . .