

The Theoretical Minimum

Quantum Mechanics - Solutions

L01E01

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Exercise 1. a) Using the axioms for inner products, prove

$$\left(\langle A| + \langle B| \right) |C\rangle = \langle A|C\rangle + \langle B|C\rangle$$

b) Prove $\langle A|A\rangle$ is a real number.

a) Let us recall the two axioms in question:

Axiom 1.

$$\langle C| \left(|A\rangle + |B\rangle \right) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 2.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

Let us recall also that if

- $\langle A|$ is the bra of $|A\rangle$
- $\langle B|$ is the bra of $|B\rangle$

Then $\langle A| + \langle B|$ is the bra of $|A\rangle + |B\rangle$.

Let us also observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$\begin{aligned} (a + b)^* &= (x_a + iy_a + x_b + iy_b)^* \\ &= x_a - iy_a + x_b - iy_b \\ &= a^* + b^* \end{aligned}$$

We thus have:

$$\begin{aligned} \left(\langle A| + \langle B| \right) |C\rangle &= \langle C| \left(|A\rangle + |B\rangle \right)^* \\ &= \left(\langle C|A\rangle + \langle C|B\rangle \right)^* \\ &= \langle C|A\rangle^* + \langle C|B\rangle^* \\ &= \langle A|C\rangle + \langle B|C\rangle \quad \square \end{aligned}$$

b) Mainly from the second axiom:

$$\begin{aligned} x + iy &= \langle A|A\rangle \\ &= \langle A|A\rangle^* \\ &= x - iy \\ &\Rightarrow 2iy = 0 \\ &\Rightarrow y = 0 \\ &\Rightarrow \langle A|A\rangle = x \in \mathbb{R} \quad \square \end{aligned}$$