

The Theoretical Minimum

Classical Mechanics - Solutions

I02E01

M. Bivert

October 16, 2022

Exercise 1. *Determine the indefinite integral of each of the following expressions by reversing the process of differentiation and adding a constant.*

Because this is the first integration exercise, we'll go slow. We will "implement" the reversing of the process of differentiation by applying the *fundamental theorem of calculus*, on a few previously established differentiation results; let's recall those:

$$\begin{aligned}\frac{d}{dt}t^n &= nt^{n-1} \\ \frac{d}{dt}\sin t &= \cos t\end{aligned}$$

Let's start by integrating both sides of each equation:

$$\begin{aligned}\int \frac{d}{dt}t^n dt &= \int nt^{n-1} dt \\ \int \frac{d}{dt}\sin t dt &= \int \cos t dt\end{aligned}$$

Let's then recall the second form of the *fundamental theorem of calculus* given in the book:

$$\int \frac{d}{dt}f dt = f(t) + c, \quad c \in \mathbb{R}$$

So our previous equations can be rewritten as:

$$\begin{aligned}t^n + c &= \int nt^{n-1} dt, \quad c \in \mathbb{R} \\ \sin t + c &= \int \cos t dt, \quad c \in \mathbb{R}\end{aligned}$$

Which are, to syntactical differences, the formulas given in the book. In addition to those, we will also rely on the *linearity of the integration*, which essentially is the combination of the *sum rule for integration* and *multiplication by a constant rule for integration*, both being analogues of what we had for differentiation, and which can be summed up by:

Theorem 1 (linearity of integration).

$$(\forall(\alpha, \beta) \in \mathbb{R}^2), (\forall(\varphi, \psi) \in (C^0)^2) \quad \boxed{\int \alpha\varphi + \beta\psi = \alpha \int \varphi + \beta \int \psi}$$

Remark 1. C^0 refers to the class ("set") of continuous functions; actually, mathematically-wise, it would suffice for the functions to be "partially continuous" so as to be integrable; in the context of physics, requiring them to be continuous is reasonable.

Remark 2. We're using the following "shortcut" notation:

$$\int \varphi = \int \varphi(t) dt$$

or for a more involved expression:

$$\int \alpha\varphi + \beta\psi = \int (\alpha\varphi(t) + \beta\psi(t)) dt$$

Proof. We can establish this result, again to syntactical differences, for instance through a similar process as we've just used for t^n and \cos , that is, by integrating differentiation results:

$$\begin{aligned} \frac{d}{dt}(\alpha\varphi + \beta\psi)(t) &= \alpha\varphi'(t) + \beta\psi'(t) \\ \Leftrightarrow \int \frac{d}{dt}(\alpha\varphi + \beta\psi)(t) &= \int \alpha\varphi'(t) + \beta\psi'(t) dt \\ \Leftrightarrow (\alpha\varphi + \beta\psi)(t) &= \int \alpha\varphi'(t) + \beta\psi'(t) dt \\ \Leftrightarrow \int \alpha\varphi' + \beta\psi' &= \alpha \int \varphi' + \beta \int \psi' \end{aligned}$$

□

$$f(t) = t^4$$

This is a simple application of:

$$\int nt^{n-1} dt = t^n + c, \quad c \in \mathbb{R}$$

with $n = 4$; using the *linearity of integration*:

$$\begin{aligned} \int 5t^{5-1} dt &= t^5 + c, \quad c \in \mathbb{R} \\ \Leftrightarrow \int t^4 dt &= \boxed{\frac{1}{5}t^5 + c} \end{aligned}$$

Remark 3. We can check the result by differentiating it

$$f(t) = \cos t$$

An even more direct application of the formulas established earlier:

$$\int \cos t dt = \boxed{\sin t + c}, \quad c \in \mathbb{R}$$

Remark 4. Again, we can check the result using differentiation: we know from earlier that the derivative of a constant is zero, that of sine is cosine, and that the derivative of a sum is the sum of the derivatives.

$$f(t) = t^2 - 2$$

Note that there's a special case for:

$$\int nt^{n-1} dt = t^n + c, \quad c \in \mathbb{R}$$

when $n = 1$:

$$\int 1 \times t^0 dt = \int dt = t^1 + c = t + c, \quad c \in \mathbb{R}$$

More generally, by *linearity of the integration*:

$$(\forall \alpha \in \mathbb{R}) \quad \int \alpha dt = \alpha \int dt = \alpha t + c, \quad c \in \mathbb{R}$$

And so we have:

$$\int t^2 - 2 dt = \int t^2 dt - 2 \int dt = \boxed{\frac{1}{3}t^3 - 2t + c}, \quad c \in \mathbb{R}$$

Which again is elementary to verify by differentiation.