

TTM - QM - L01E02

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Exercise 1.

Show that the inner product defined by Eq. 1.2 satisfies all the axioms of inner products.

Let us recall the two axioms in question:

Axiom 1.

$$\langle C | (|A\rangle + |B\rangle) \rangle = \langle C | A \rangle + \langle C | B \rangle$$

Axiom 2.

$$\langle B | A \rangle = \langle A | B \rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

And let us recall Eq. 1.2 of the book:

$$\begin{aligned} \langle B | A \rangle &= (\beta_1^* \quad \beta_2^* \quad \beta_3^* \quad \beta_4^* \quad \beta_5^*) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} \\ &= \beta_1^* \alpha_1 + \beta_2^* \alpha_2 + \beta_3^* \alpha_3 + \beta_4^* \alpha_4 + \beta_5^* \alpha_5 \end{aligned}$$

For the first axiom, considering $\langle C | = (\gamma_i^*)$:

$$\begin{aligned} \langle C | (|A\rangle + |B\rangle) \rangle &= (\gamma_1^* \quad \gamma_2^* \quad \gamma_3^* \quad \gamma_4^* \quad \gamma_5^*) \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \alpha_4 + \beta_4 \\ \alpha_5 + \beta_5 \end{pmatrix} \\ &= \gamma_1^* (\alpha_1 + \beta_1) + \gamma_2^* (\alpha_2 + \beta_2) + \gamma_3^* (\alpha_3 + \beta_3) + \gamma_4^* (\alpha_4 + \beta_4) + \gamma_5^* (\alpha_5 + \beta_5) \\ &= (\gamma_1^* \alpha_1 + \gamma_2^* \alpha_2 + \gamma_3^* \alpha_3 + \gamma_4^* \alpha_4 + \gamma_5^* \alpha_5) + (\gamma_1^* \beta_1 + \gamma_2^* \beta_2 + \gamma_3^* \beta_3 + \gamma_4^* \beta_4 + \gamma_5^* \beta_5) \\ &= (\gamma_1^* \quad \gamma_2^* \quad \gamma_3^* \quad \gamma_4^* \quad \gamma_5^*) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} + (\gamma_1^* \quad \gamma_2^* \quad \gamma_3^* \quad \gamma_4^* \quad \gamma_5^*) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \\ &= \langle C | A \rangle + \langle C | B \rangle \quad \square \end{aligned}$$

Before diving into the second axiom, let us observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$\begin{aligned}
(ab)^* &= \left((x_a + iy_a) \times (x_b + iy_b) \right)^* \\
&= \left(x_a x_b - y_a y_b + i(x_b y_a + x_a y_b) \right)^* \\
&= x_a x_b - y_a y_b - i(x_b y_a + x_a y_b) \\
&= (x_a - iy_a) \times (x_b - iy_b) \\
&= a^* b^*
\end{aligned}$$

Or, perhaps more simply using complex numbers' exponential's form:

$$\begin{aligned}
(ab)^* &= \left(r_a r_b e^{i(\theta_a + \theta_b)} \right)^* \\
&= r_a r_b e^{-i(\theta_a + \theta_b)} \\
&= a^* b^*
\end{aligned}$$

Hence, regarding the second axiom:

$$\begin{aligned}
\langle B|A \rangle &= \left(\left(\langle B|A \rangle \right)^* \right)^* \\
&= \left(\left(\beta_1^* \alpha_1 + \beta_2^* \alpha_2 + \beta_3^* \alpha_3 + \beta_4^* \alpha_4 + \beta_5^* \alpha_5 \right)^* \right)^* \\
&= \left(\beta_1 \alpha_1^* + \beta_2 \alpha_2^* + \beta_3 \alpha_3^* + \beta_4 \alpha_4^* + \beta_5 \alpha_5^* \right)^* \\
&= \left(\alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \alpha_3^* \beta_3 + \alpha_4^* \beta_4 + \alpha_5^* \beta_5 \right)^* \\
&= \left(\begin{pmatrix} \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* & \alpha_5^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \right)^* \\
&= \langle A|B \rangle^* \quad \square
\end{aligned}$$