

The Theoretical Minimum

Quantum Mechanics - Solutions

L02E03

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Exercise 1. *For the moment, forget that Eqs. 2.10 give us working definitions for $|i\rangle$ and $|o\rangle$ in terms of $|u\rangle$ and $|d\rangle$, and assume that the components α, β, γ and δ are unknown:*

$$|o\rangle = \alpha|u\rangle + \beta|d\rangle \qquad |i\rangle = \gamma|u\rangle + \delta|d\rangle$$

a) Use Eqs. 2.8 to show that

$$\alpha^* \alpha = \beta^* \beta = \gamma^* \gamma = \delta^* \delta = \frac{1}{2}$$

b) Use the above results and Eqs. 2.9 to show that

$$\alpha^* \beta + \alpha \beta^* = \gamma^* \delta + \gamma \delta^* = 0$$

c) Show that $\alpha^* \beta$ and $\gamma^* \delta$ must each be pure imaginary.

If $\alpha^* \beta$ is pure imaginary, then α and β cannot both be real. The same reasoning applies to $\gamma^* \delta$.

Let's start by recalling Eqs. 2.8, 2.9 and 2.10, which are respectively:

$$\begin{aligned} \langle o|u\rangle \langle u|o\rangle &= \frac{1}{2} & \langle o|d\rangle \langle d|o\rangle &= \frac{1}{2} \\ \langle i|u\rangle \langle u|i\rangle &= \frac{1}{2} & \langle i|d\rangle \langle d|i\rangle &= \frac{1}{2} \end{aligned} \tag{1}$$

$$\begin{aligned} \langle o|r\rangle \langle r|o\rangle &= \frac{1}{2} & \langle o|l\rangle \langle l|o\rangle &= \frac{1}{2} \\ \langle i|r\rangle \langle r|i\rangle &= \frac{1}{2} & \langle i|l\rangle \langle l|i\rangle &= \frac{1}{2} \end{aligned} \tag{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \qquad |o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \tag{3}$$

a) Let's start by recalling that the inner-product in a Hilbert space is defined between a bra and a ket, and that it should satisfy the following two axioms:

$$\langle C|\{ |A\rangle + |B\rangle \} = \langle C|A\rangle + \langle C|B\rangle \text{ (linearity)}$$

$$\langle B|A\rangle = \langle A|B\rangle^* \text{ (complex conjugation)}$$

Furthermore, the scalar-multiplication of a ket is linear:

$$z \in \mathbb{C}, \qquad |zA\rangle = z|A\rangle$$

Then we can multiply $|o\rangle = \alpha|u\rangle + \beta|d\rangle$ to the left by $\langle u|$ to compute $\langle u|o\rangle$, using the linearity of the inner-product/scalar multiplication, and the fact that $|u\rangle$ and $|d\rangle$ are, by definition, unitary orthogonal vectors (meaning, $\langle u|d\rangle = 0$ and $\langle u|u\rangle = \langle d|d\rangle = 1$)

$$\langle u|o\rangle = \alpha \langle u|u\rangle + \beta \langle u|d\rangle = \alpha$$

Because of the complex conjugation rule, we have

$$\langle o|u\rangle = \langle u|o\rangle^* = \alpha^*$$

And so by Eqs. 2.8 and the previous computation we have

$$\frac{1}{2} = \underbrace{\langle o|u\rangle}_{\alpha} \underbrace{\langle u|o\rangle}_{\alpha^*} = \alpha\alpha^* \quad \square$$

The process is very similar to prove $\beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$:

$$\begin{aligned} \frac{1}{2} &= \langle o|d\rangle \langle d|o\rangle \\ &= (\langle d|o\rangle)^* \langle d|o\rangle \\ &= \left(\langle d|\{\alpha|u\rangle + \beta|d\rangle\} \right)^* \left(\langle d|\{\alpha|u\rangle + \beta|d\rangle\} \right) \\ &= \left(\underbrace{\alpha \langle d|u\rangle}_{=0} + \underbrace{\beta \langle d|d\rangle}_{=1} \right)^* \left(\underbrace{\alpha \langle d|u\rangle}_{=0} + \underbrace{\beta \langle d|d\rangle}_{=1} \right) \\ &= \beta^*\beta \quad \square \end{aligned}$$

$$\begin{aligned} \frac{1}{2} &= \langle i|u\rangle \langle u|i\rangle \\ &= (\langle u|i\rangle)^* \langle u|i\rangle \\ &= \left(\langle u|\{\gamma|u\rangle + \delta|d\rangle\} \right)^* \left(\langle u|\{\gamma|u\rangle + \delta|d\rangle\} \right) \\ &= \left(\underbrace{\gamma \langle u|u\rangle}_{=1} + \underbrace{\delta \langle u|d\rangle}_{=0} \right)^* \left(\underbrace{\gamma \langle u|u\rangle}_{=1} + \underbrace{\delta \langle u|d\rangle}_{=0} \right) \\ &= \gamma^*\gamma \quad \square \end{aligned}$$

$$\begin{aligned} \frac{1}{2} &= \langle i|d\rangle \langle d|i\rangle \\ &= (\langle d|i\rangle)^* \langle d|i\rangle \\ &= \left(\langle d|\{\gamma|u\rangle + \delta|d\rangle\} \right)^* \left(\langle d|\{\gamma|u\rangle + \delta|d\rangle\} \right) \\ &= \left(\underbrace{\gamma \langle d|u\rangle}_{=0} + \underbrace{\delta \langle d|d\rangle}_{=1} \right)^* \left(\underbrace{\gamma \langle d|u\rangle}_{=0} + \underbrace{\delta \langle d|d\rangle}_{=1} \right) \\ &= \delta^*\delta \quad \square \end{aligned}$$

b) I don't think we can conclude here without recalling the definition of $|r\rangle$:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

Let's start with a piece from Eqs. 2.9, arbitrarily (we could use $\langle i|l\rangle \langle l|i\rangle = \frac{1}{2}$, but I think we'd still need the previous definition of $|r\rangle$):

$$\langle i|r\rangle \langle r|i\rangle = \frac{1}{2}$$

But:

$$\langle r|i\rangle = \langle r|\{\alpha|u\rangle + \beta|d\rangle\} = \alpha \langle r|u\rangle + \beta \langle r|d\rangle$$

And:

$$\langle i|r \rangle = (\langle r|i \rangle)^* = (\alpha \langle r|u \rangle + \beta \langle r|d \rangle)^* = \alpha^* \langle u|r \rangle + \beta^* \langle d|r \rangle$$

So

$$\begin{aligned} \langle i|r \rangle \langle r|i \rangle &= \frac{1}{2} \\ \Leftrightarrow (\alpha^* \langle u|r \rangle + \beta^* \langle d|r \rangle) (\alpha \langle r|u \rangle + \beta \langle r|d \rangle) &= \frac{1}{2} \\ \Leftrightarrow \underbrace{\alpha^* \alpha}_{=1/2} \langle u|r \rangle \langle r|u \rangle + \alpha^* \beta \langle u|r \rangle \langle r|d \rangle + \beta^* \alpha \langle d|r \rangle \langle r|u \rangle + \underbrace{\beta^* \beta}_{=1/2} \langle d|r \rangle \langle r|d \rangle &= \frac{1}{2} \\ \Leftrightarrow \frac{1}{2} (\underbrace{\langle u|r \rangle \langle r|u \rangle + \langle d|r \rangle \langle r|d \rangle}_{=1}) + \alpha^* \beta \langle u|r \rangle \langle r|d \rangle + \beta^* \alpha \langle d|r \rangle \langle r|u \rangle &= \frac{1}{2} \end{aligned}$$

Now if $|r \rangle = \rho_u |u \rangle + \rho_d |d \rangle$, then

$$\langle u|r \rangle \langle r|u \rangle + \langle d|r \rangle \langle r|d \rangle = \rho_u \rho_u^* + \rho_d \rho_d^* = 1$$

As $\rho_u \rho_u^*$ would be the probability of $|r \rangle$ to be up, and $\rho_d \rho_d^*$ would be the probability of $|r \rangle$ to be down, which are two orthogonal states in a two-states setting, and so the sum of their probability must be 1.

Hence the previous expression becomes:

$$\alpha^* \beta \langle u|r \rangle \langle r|d \rangle + \beta^* \alpha \langle d|r \rangle \langle r|u \rangle = 0$$

Note that so far, we haven't needed the expression of $|r \rangle$, but I think we don't have a choice but to use it to conclude:

$$|r \rangle = \frac{1}{\sqrt{2}} |u \rangle + \frac{1}{\sqrt{2}} |d \rangle$$

So, as the coefficient are real numbers:

$$\langle u|r \rangle = \frac{1}{\sqrt{2}} = \langle r|u \rangle; \quad \langle d|r \rangle = \frac{1}{\sqrt{2}} = \langle r|d \rangle$$

Replacing in the previous expression we have:

$$\begin{aligned} \alpha^* \beta \underbrace{\langle u|r \rangle \langle r|d \rangle}_{=1/\sqrt{2}=1/\sqrt{2}} + \beta^* \alpha \underbrace{\langle d|r \rangle \langle r|u \rangle}_{=1/\sqrt{2}=1/\sqrt{2}} &= 0 \\ \Leftrightarrow \frac{1}{2} \alpha^* \beta + \frac{1}{2} \beta^* \alpha &= 0 \\ \Leftrightarrow \boxed{\alpha^* \beta + \beta^* \alpha = 0} &\quad \square \end{aligned}$$

The process is very similar to prove $\gamma^* \delta + \gamma \delta^* = 0$; one has to start again from a Eqs. 2.9, but this time, from another piece involving o , arbitrarily:

$$\begin{aligned} \langle o|r \rangle \langle r|o \rangle &= \frac{1}{2} \\ \Leftrightarrow (\langle r|o \rangle)^* \langle r|o \rangle &= \frac{1}{2} \\ \Leftrightarrow (\langle r|\{\gamma|u \rangle + \delta|d \rangle\})^* (\langle r|\{\gamma|u \rangle + \delta|d \rangle\}) &= \frac{1}{2} \\ \Leftrightarrow (\gamma^* \langle u|r \rangle + \delta^* \langle d|r \rangle) (\gamma \langle r|u \rangle + \delta \langle r|d \rangle) &= \frac{1}{2} \\ \Leftrightarrow \underbrace{\gamma^* \gamma}_{=1/2} \langle u|r \rangle \langle r|u \rangle + \gamma^* \delta \langle u|r \rangle \langle r|d \rangle + \delta^* \gamma \langle d|r \rangle \langle r|u \rangle + \underbrace{\delta^* \delta}_{=1/2} \langle d|r \rangle \langle r|d \rangle &= \frac{1}{2} \\ \Leftrightarrow \frac{1}{2} (\underbrace{\langle u|r \rangle \langle r|u \rangle + \langle d|r \rangle \langle r|d \rangle}_{=1}) + \gamma^* \delta \langle u|r \rangle \langle r|d \rangle + \delta^* \gamma \langle d|r \rangle \langle r|u \rangle &= \frac{1}{2} \\ \Leftrightarrow \gamma^* \delta \underbrace{\langle u|r \rangle \langle r|d \rangle}_{=1/2} + \delta^* \gamma \underbrace{\langle d|r \rangle \langle r|u \rangle}_{=1/2} &= 0 \\ \Leftrightarrow \boxed{\gamma^* \delta + \delta^* \gamma = 0} &\quad \square \end{aligned}$$

c) Let's assume $\alpha\beta^*$ is a complex number of the form:

$$\alpha\beta^* = a + ib, \quad (a, b) \in \mathbb{R}^2$$

But then:

$$\left(\alpha\beta^*\right)^* = a - ib = \alpha^*\beta$$

That's because, for two complex numbers $z = a + ib$ and $w = x + iy$, we have:

$$\left(zw\right)^* = z^*w^*$$

Indeed:

$$zw = (a + ib)(x + iy) = (ax - by) + i(bx + ya)$$

Hence:

$$\left(zw\right)^* = (ax - by) - i(bx + ya)$$

But:

$$z^*w^* = (a - ib)(x - iy) = (ax - by) - i(bx + ya)$$

Hence the result. Back to our α and β , we established in b) that:

$$\alpha^*\beta + \alpha\beta^* = 0$$

Which is equivalent from our previous little proof to:

$$\alpha^*\beta + \left(\alpha^*\beta\right)^* = 0$$

$$\Leftrightarrow (a + ib) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow \boxed{a = 0}$$

Which is the same as saying that the real part of $\alpha^*\beta$ is zero, or that it's a pure imaginary number. The exact same argument applies for $\gamma^*\delta$.