The Theoretical Minimum

Classical Mechanics - Solutions

L10E02

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Exercise 1. *Hamilton's equations can be written in the form $\dot{q} = \{q, H\}$ and $\dot{p} = \{p, H\}$. Assume that the Hamiltonian has the form $H = \frac{1}{2m}p^2 + V(q)$. Using only the PB axioms, prove Newton's equations of motion.*

So, the goal of this exercise is to derive Newton's equations of motion, meaning, a "variant" of $F = ma$, *without* referring directly to the definition of the Poisson brackets, but rather, using its algebraic properties. Let's recall them for clarity.

Let A, B, and C be functions of qs and ps; $k \in \mathbb{R}$:

Anti-symmetry :

$$\{A, C\} = -\{C, A\};$$

Linearity :

$$\{kA, C\} = k\{A, C\};$$
$$\{A + B, C\} = \{A, C\} + \{B, C\};$$

"Product rule" :

$$\{AB, C\} = A\{B, C\} + B\{A, C\}$$

We'll also need the following:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0; \qquad \{q_i, p_j\} = \delta_i^j$$

And Eq. (14) and Eq. (15) of the book, which are respectively, for F a function of qs and ps:

$$\{F(q,p), p_i\} = \frac{\partial F(q,p)}{\partial q_i}$$

$$\{F(q,p), q_i\} = -\frac{\partial F(q,p)}{\partial p_i}$$

Alright, let's start by observing that we're in the case were $N = 1$: we have a single p and a single q. Then, let's begin by applying the anti-symmetry rule to $\dot{q} = \{q, H\} = -\{H, q\}$.

We have two options to go further:

1. Either we expand the expression of H and keep applying some rules further;
2. Or, as $H = H(p, q)$, we can also apply Eq. (15).

Let's try both, in this order (we should get the same result):

$$\begin{aligned}
\dot{q} &= \{q, H\} \\
&= -\{H, q\} \quad \text{(anti-symmetry)} \\
&= -\{\frac{1}{2m}p^2 + V(q), q\} \quad \text{(H's definition)} \\
&= -\frac{1}{2m}\{p^2, q\} + \{V(q), q\} \quad \text{(linearity)}
\end{aligned}$$

Using the product rule, we can develop

$$\{p^2, q\} = \{pp, q\} = p\{p, q\} + p\{p, q\} = 2p\{p, q\}$$

But then, this is just $\{q_i, p_j\} = \delta_i^j$, modulo some anti-symmetry (as we only have one p and one q, they always "match" as far as the Kronecker delta is concerned):

$$\{p^2, q\} = 2p\{p, q\} = -2p\{q, p\} = -2p$$

What about {V(q), q}? We can get there in two ways: either we consider that $V(q) = V(q, p)$ with no p, and thus by Eq. (15),

$$\{V(q), q\} = \{V(q, p), q\} = \frac{\partial V(q, p)}{\partial p} = 0$$

But we could also argue that $V(q)$ can be expressed as a polynomial in q; then, by linearity of the Poisson brackets on the terms of that polynomial, we would be able to apply the $\{q_i, q_j\} = 0$; systematically, and also get zero.

Finally, this leaves us with:

$$\dot{q} = -\frac{1}{2m}\underbrace{\{p^2, q\}}_{=-2p} + \underbrace{\{V(q), q\}}_{=0}$$

By re-arranging the terms a little, we get the definition of the moment:

$$\boxed{p = m\dot{q}}$$

We'll continue from here in a moment, but first, let's explore the second option we mentioned earlier, and use Eq. (15) directly after the application of the anti-symmetry on $\dot{q} = \{q, H\}$:

$$\begin{aligned}
\dot{q} &= \{q, H\} \\
&= -\{H, q\} \quad \text{(anti-symmetry)} \\
&= -\{H(p, q), q\} \\
&= \frac{\partial H(q, p)}{\partial p} \quad \text{(Eq. (15))} \\
&= \frac{\partial}{\partial p}\left(\frac{1}{2m}p^2 + V(q)\right) \quad \text{(H's definition)} \\
&= \frac{1}{m}p
\end{aligned}$$

Which indeed agrees with our previous result: $p = m\dot{q}$.

OK we've found back the definition of the moment, now what? We'd want to find a way to use $\dot{p} = \{p, H\}$, but we have no $\dot{p}$, so let's make one by deriving the definition of the moment:

$$p = m\dot{q} \Rightarrow \dot{p} = m\ddot{q}$$

We'll soon find ourselves in the same situation as before, where we can continue the derivation either by applying Eq. (14), or by following a more "manual" path; I'll go with the latter as this is a bit more verbose:

$$
\begin{aligned}
m\ddot{q} &= \dot{p} \\
&= \{p, H\} \\
&= -\{H, p\} && \text{(anti-symmetry)} \\
&= -\{\frac{1}{2m}p^2 + V(q), p\} && \text{(H's definition)} \\
&= -\frac{1}{2m}\{pp, p\} + \{V(q), p\} && \text{(linearity)} \\
&= -\frac{1}{2m}2p\underbrace{\{p, p\}}_{=0} + \{V(q), p\} && \text{(product rule)} \\
&= \{V(q), p\} && (\{p_i, p_j\} = 0) \\
&= \frac{\partial}{\partial q}V(q) && \text{(Eq. (14))} \\
&= \frac{\partial}{\partial q}V(q) && \text{(forces are derived from potential)} \\
&= F_q && \square
\end{aligned}
$$