The Theoretical Minimum

Quantum Mechanics - Solutions

L03E02

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Exercise 1. *Prove that Eq.* 3.16 *is the unique solution to Eqs.* 3.14 *and* 3.15.

Let's recall all the equations, 3.14, 3.15 and 3.16

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

By developing the matrix product and identifying the vectors components, the first two equations make a system of four equations involving four unknowns $(\sigma_z)_{11}$, $(\sigma_z)_{12}$, $(\sigma_z)_{21}$ and $(\sigma_z)_{22}$:

$$\begin{cases} 1(\sigma_z)_{11} + 0(\sigma_z)_{12} &= 1 \\ 1(\sigma_z)_{21} + 0(\sigma_z)_{22} &= 0 \\ 0(\sigma_z)_{11} + 1(\sigma_z)_{12} &= 0 \\ 0(\sigma_z)_{21} + 1(\sigma_z)_{22} &= -1 \end{cases} \Leftrightarrow \begin{cases} (\sigma_z)_{11} &= 1 \\ (\sigma_z)_{21} &= 0 \\ (\sigma_z)_{12} &= 0 \\ (\sigma_z)_{22} &= -1 \end{cases} \Leftrightarrow \boxed{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \quad \square \tag{4}$$