The Theoretical Minimum
Quantum Mechanics - Solutions
L01E02

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Exercise 1. *Show that the inner product defined by Eq. 1.2 satisfies all the axioms of inner products.*

Let us recall the two axioms in question:

Axiom 1.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 2.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

And let us recall Eq. 1.2 of the book:

$$\langle B|A\rangle = \begin{pmatrix} \beta_1^* & \beta_2^* & \beta_3^* & \beta_4^* & \beta_5^* \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}$$

$$= \beta_1^* \alpha_1 + \beta_2^* \alpha_2 + \beta_3^* \alpha_3 + \beta_4^* \alpha_4 + \beta_5^* \alpha_5$$

For the first axiom, considering $\langle C| = (\gamma_i^*)$:

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \begin{pmatrix} \gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^* \end{pmatrix} \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \\ \alpha_4 + \beta_4 \\ \alpha_5 + \beta_5 \end{pmatrix}$$

$$= \gamma_1^*(\alpha_1 + \beta_1) + \gamma_2^*(\alpha_2 + \beta_2) + \gamma_3^*(\alpha_3 + \beta_3) + \gamma_4^*(\alpha_4 + \beta_4) + \gamma_5^*(\alpha_5 + \beta_5)$$

$$= \big(\gamma_1^*\alpha_1 + \gamma_2^*\alpha_2 + \gamma_3^*\alpha_3 + \gamma_4^*\alpha_4 + \gamma_5^*\alpha_5\big) + \big(\gamma_1^*\beta_1 + \gamma_2^*\beta_2 + \gamma_3^*\beta_3 + \gamma_4^*\beta_4 + \gamma_5^*\beta_5\big)$$

$$= \begin{pmatrix} \gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^* \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} + \begin{pmatrix} \gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}$$

$$= \langle C|A\rangle + \langle C|B\rangle \quad \square$$

Before diving into the second axiom, let us observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$
\begin{aligned}
(ab)^* &= \left((x_a + iy_a) \times (x_b + iy_b) \right)^* \\
&= \left(x_a x_b - y_a y_b + i(x_b y_a + x_a y_b) \right)^* \\
&= x_a x_b - y_a y_b - i(x_b y_a + x_a y_b) \\
&= (x_a - iy_a) \times (x_b - iy_b) \\
&= a^* b^*
\end{aligned}
$$

Or, perhaps more simply using complex numbers' exponential's form:

$$
\begin{aligned}
(ab)^* &= \left(r_a r_b e^{i(\theta_a + \theta_b)} \right)^* \\
&= r_a r_b e^{-i(\theta_a + \theta_b)} \\
&= a^* b^*
\end{aligned}
$$

Hence, regarding the second axiom:

$$
\begin{aligned}
\langle B|A \rangle &= \left(\left(\langle B|A \rangle \right)^* \right)^* \\
&= \left(\left(\beta_1^* \alpha_1 + \beta_2^* \alpha_2 + \beta_3^* \alpha_3 + \beta_4^* \alpha_4 + \beta_5^* \alpha_5 \right)^* \right)^* \\
&= \left(\beta_1 \alpha_1^* + \beta_2 \alpha_2^* + \beta_3 \alpha_3^* + \beta_4 \alpha_4^* + \beta_5 \alpha_5^* \right)* \\
&= \left(\alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \alpha_3^* \beta_3 + \alpha_4^* \beta_4 + \alpha_5^* \beta_5 \right)* \\
&= \left(\begin{pmatrix} \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* & \alpha_5^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \right)^* \\
&= \langle A|B \rangle^* \quad \square
\end{aligned}
$$