The Theoretical Minimum
Classical Mechanics - Solutions

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October 11, 2022

Abstract

Below are solution proposals to the exercises of *The Theoretical Minimum - Classical Mechanics*, written by Leonard Susskind and George Hrabovsky. An effort has been so as to recall from the book all the referenced equations, and to be rather verbose regarding mathematical details, rather in line with the general tone of the series.

Contents

Lecture 1: The Nature of Classical Physics

What Is Classical Physics?

Exercise 1/3

Exercise 1. *Since the notion is so important to theoretical physics, think about what a closed system is, and speculate on whether closed systems can actually exist. What assumptions are implicit in establishing a closed system? What is an open system?*

The book defines a closed system as "a collection of objects [...] that is either the entire universe (1) or is so isolated from everything else that it behaves as if nothing else exists (2)". From everyday experience, we *know* there's a lot going on we're not aware of. Thus, we know we can't even hope to *truly* consider the entire universe, but in an abstract sense, at best.

Furthermore, to establish with absolute certainty (2), it is necessary to have a full understanding of everything that exists. For, even were we to build a system isolated from everything we know that could affect it, by the previous assumption, there may be some elements, unknown to us, affecting the system in various ways.

Hence, as for a lot of things in physics, a closed system is but a conceptual tool, a (very) convenient approximation of some aspects of what we think we experience of reality.

To establish our closed system, we must generally assume that it is closed *relatively* to what we are trying to measure/observe. That is, there will be no external factors noticeably/unexpectedly affecting the result of a a given observation/consideration.

As we can only define a closed system as an approximation, we won't bother trying to define an *open system* in absolute terms either. A opened system, is by contrast with a closed system, a collection of objects that isn't neither the entire Universe, nor so isolated from everything else that it behaves as if nothing else exists. More simply, it's a collection of objects affected by its environment.

Exercise 2/3

Exercise 2. *Can you think of a general way to classify the laws that are possible for a six-state system?*

From what precedes, we can think of classifying systems by their number of cycles. We could furthermore refine the classification, for instance by considering the distribution of states per cycle.

The official solutions[1] mentions an interesting fact, not directly apparent from the book's examples: there could be cycles within the "main" cycles (and cycles within the "secondary" cycles, etc.), which can also be used to refine the classification.

Rules That Are Not Allowed: The Minus-First Law

Dynamical Systems with an Infinite Number of States

Exercise 3/3

Exercise 3. *Determine which of the dynamical laws shown in Eq.s (2) through (5) are allowable.*

Let's recall that laws are said to be allowable if they are both deterministic (i.e. the future behavior of the system is completely determined by the initial state) and reversible (i.e. the law is still deterministic even by reversing the direction of all the arrows).

---

[1] http://www.madscitech.org/tm/slns/l1e2.pdf

Now, the dynamical laws refered to be the exercise are:

$$
\begin{array}{rcll}
N(n+1) & = & N(n) - 1 & (2) \\
N(n+1) & = & N(n) + 2 & (3) \\
N(n+1) & = & N(n)^2 & (4) \\
N(n+1) & = & -1^{N(n)} N(n) & (5)
\end{array}
$$
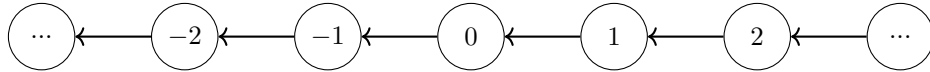
(2) is simply (1) (described by $N(n+1) = N(n)+1$) with reversed arrows; and (1) has already be established as being allowed. Hence, $\boxed{\text{(2) is allowed}}$.
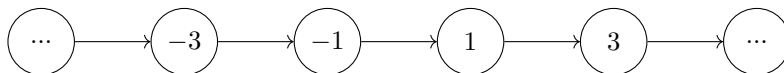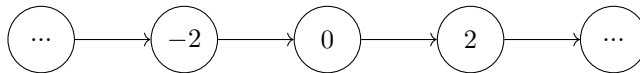
(3) is very similar to (1) or (2), only, it has two infinite cycles instead of one. Both are deterministic, and reversing the arrows lead to deterministic cycles. Hence, $\boxed{\text{(3) is allowed}}$.
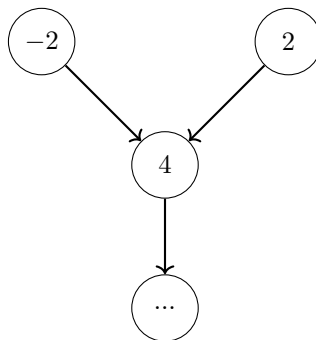
We didn't plot (4) exhaustively, just enough to illustrate its irreversibility: if we start at either -2 or 2, we end up on 4. By reversing the arrows, we see that we can't decide where we came from. Furthermore, states such as -2 cannot be reached, because there's no number n such as $-2 = n^2$). Actually, even 2 cannot be reached, because there's no *integer* n such as $2 = n^2$.

Hence, $\boxed{\text{(4) is \textbf{not} allowed}}$.
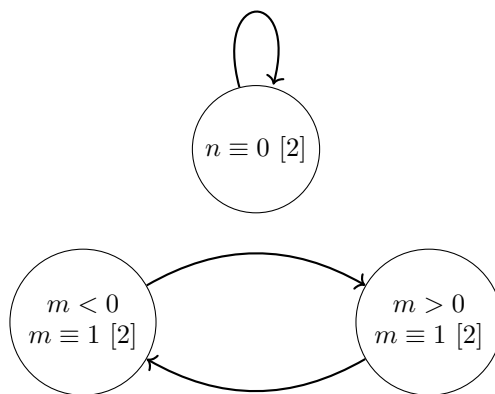
By sketching a few cycles for (5), we see a pattern emerging, summed up in the diagram above:

- if we start on an even number, positive or negative, we'll loop on this number indefinitely;

- if we start with an odd number, positive or negative, we'll loop between the positive and negative version of that number;

Perhaps a mathematical subtetly would be in considering -1 raised to a negative power. But,

$$(\forall n \in \mathbb{N}^*), \quad -1^{-n} = (-1/1)^{-n} \equiv (1/-1)^n = (-1)^n$$

Hence, $\boxed{\text{(5) is allowed}}$.

Cycles and Conservation Laws

The Limits of Precision

Interlude 1: Spaces, Trigonometry, and Vectors

Coordinates

Exercise 1/6

Exercise 4. *Using a graphic calculator or a program like Mathematica, plot each of the following functions. See the next section if you are unfamiliar with the trigonometric functions.*

$$
\begin{aligned}
f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\
g(x) &= \sin x - \cos x \\
\theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\
x(t) &= \sin^2 t - \cos t
\end{aligned}
$$
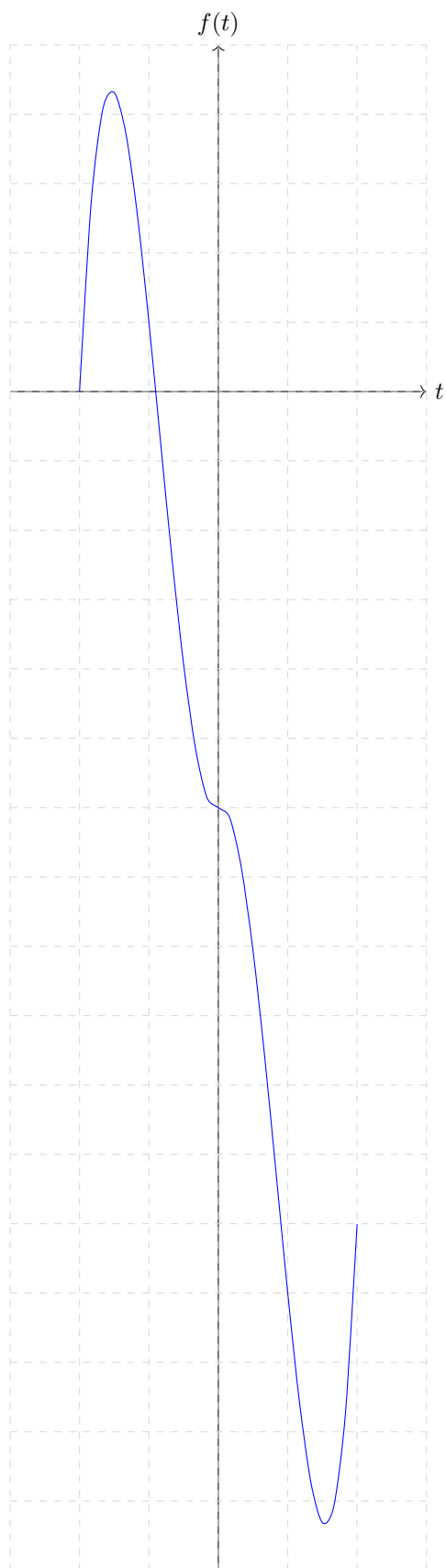
Figure 1: $f(t) = t^4 + 3t^3 - 12t^2 + t - 6$
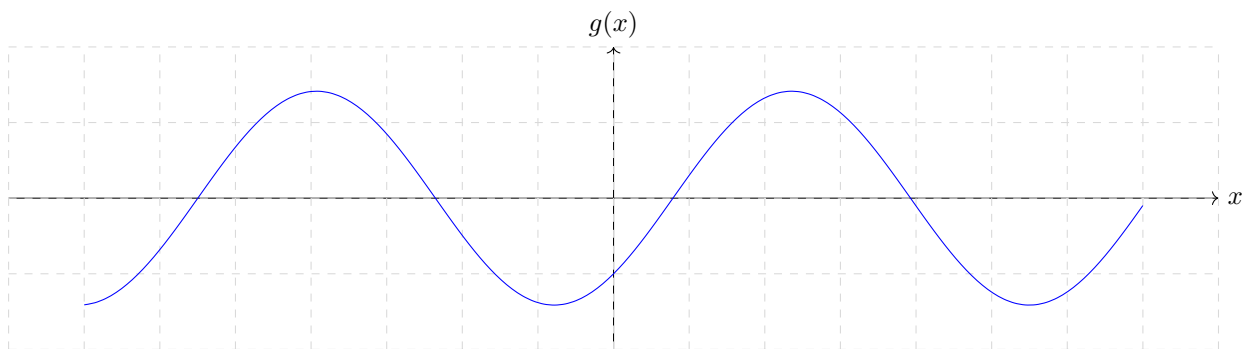
Figure 2: $g(x) = \sin x - \cos x$
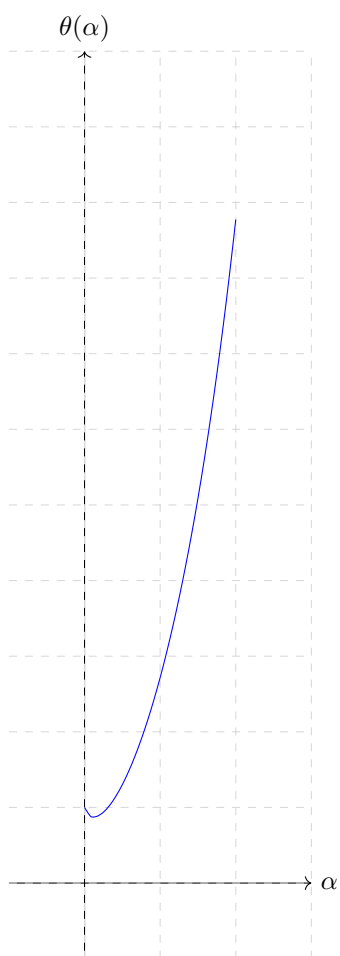
Figure 3: $\theta(\alpha) = e^{\alpha} + \alpha \ln \alpha$
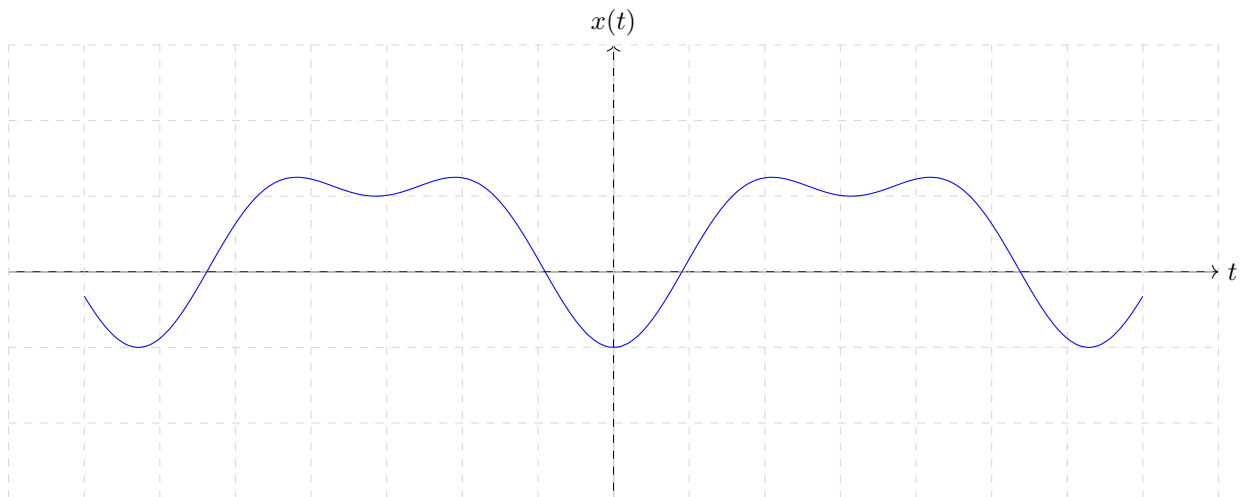
Figure 4: $x(t) = \sin^2 t - \cos t$

Remark 1. *All those plots were created using TiKz (with LaTeX then). For instance, here's the code for the last plot:*

```
\begin{figure}[H]
    \centering
    \begin{tikzpicture}
        \tikzmath{
            \xmin = -7;
            \xmax = 7;
            \ymin = -2;
            \ymax = 2;
        }
        \draw[->] (\xmin-1, 0) -- (\xmax+1, 0) node[right] {$t$};
        \draw[->] (0, \ymin-1) -- (0, \ymax+1) node[above] {$x(t)$};
        \draw[color=gray!30, dashed]
            (\xmin-1,\ymin-1) grid (\xmax+1,\ymax+1);
        \draw[scale=1, domain=\xmin:\xmax, smooth, samples=100, variable=\t, blue]
            plot ({\t}, {sin(\t r)^2 - cos(\t r)});
    \end{tikzpicture}
    \caption{$x(t) =  \sin^2t - \cos t$}
\end{figure}
```

Trigonometry

Vectors

Exercise 2/6

Exercise 5. *Work out the rule for vector substraction.*

This exercise is about getting a (visual) feel for vector manipulation; it is *not* about vector coordinates manipulation. We were previously taught how to multiple vectors by a negative scalar:

> For example, $-2\vec{r}$ is the vector that is twice as long as $\vec{r}$, but points in the opposite direction.

And how to add vectors:

> To add $\vec{A}$ and $\vec{B}$, place them as shown in Figure 13 to form a quadrilateral (this way the directions of the vectors are preserved). The sum of the vectors is the length and angle of the diagonal

So, by observing that (we'll use a **bold** font to denote vectors instead of arrows, e.g. $\boldsymbol{v}$ is a vector):

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-1\boldsymbol{v})$$

We conclude that we first need to reverse the direction of the vector to be substracted, and add this to the other vector. Visually:
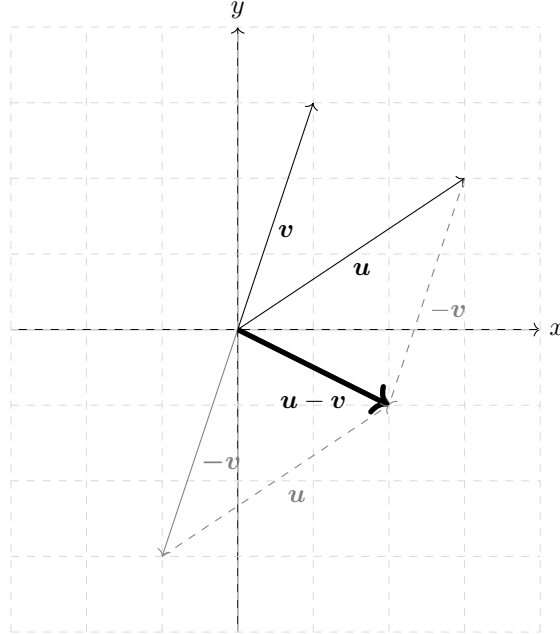
Exercise 3/6

Exercise 6. *Show that the magnitude of a vector satisfies $|\vec{A}|^2 = \vec{A} \cdot \vec{A}$.*

Remark 2. *We'll again use a **bold** font to denote vectors instead of arrows and use a sligthly different symbol for the magnitude; the change can be summed up by stating: $\|\boldsymbol{u}\| = |\vec{u}| (= u)$.*

Let's recall that the magnitude of a vector was *defined* as:

$$\|\boldsymbol{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

And the dot product between two vectors as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_x v_x + u_y v_y + u_z v_z$$

From there, we quickly reach the expected result:

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{u} &= u_x u_x + u_y u_y + u_z u_z \\
&= u_x^2 + u_y^2 + u_z^2 \\
&= \left(\sqrt{u_x^2 + u_y^2 + u_z^2}\right)^2 \\
&= \|\boldsymbol{u}\|^2 \quad \square
\end{aligned}$$

Exercise 4/6

Exercise 7. *Let $(A_x = 2, A_y = -3, A_z = 1)$ and $(B_x = -4, B_y = -3, B_z = 2)$. Compute the magnitude of $\vec{A}$ and $\vec{B}$, their dot product, and the angle between them.*

This is an immediate application of the formulas. Let's start by plotting those vectors:
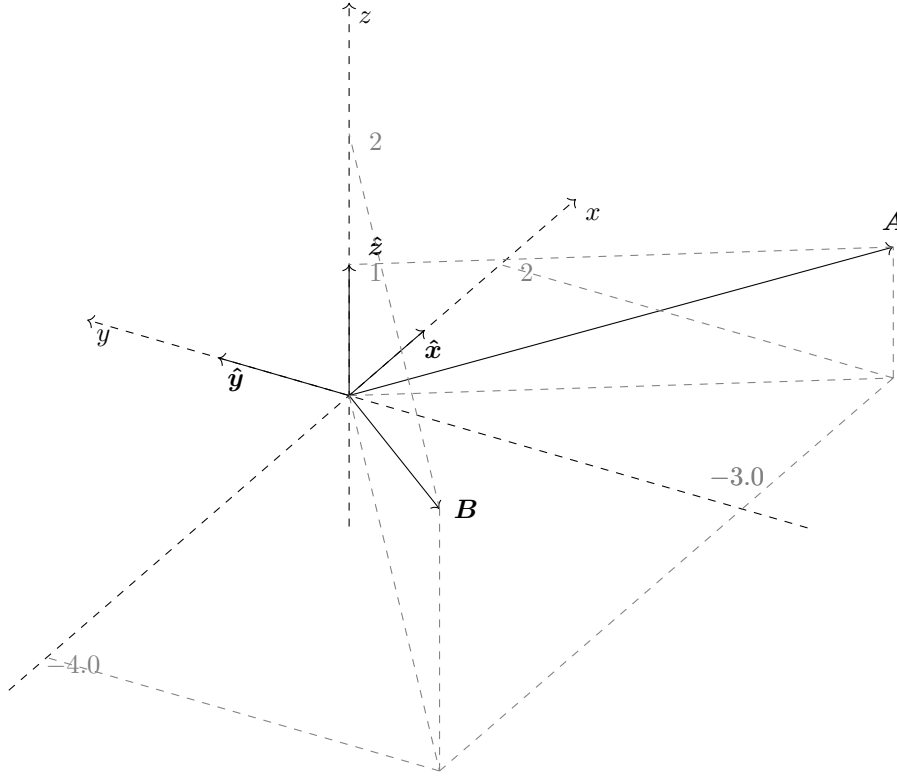
Uusing slightly different notations, let us then recall first how the magnitude of a vector $\boldsymbol{u}$ is defined:

$$\|\boldsymbol{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2} = u \tag{1}$$

There are two formulas for the dot product: one involving the magnitudes and the angle between the vectors, and the other one involving the components:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_x v_x + u_y v_y + u_z v_z \\ &= \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos \theta_{uv} \end{aligned} \tag{2}$$

By applying (1), we have:

$$\begin{aligned} A &= \sqrt{2^2 + (-3)^2 + 1^2} & B &= \sqrt{(-4)^2 + (-3)^2 + 2^2} \\ &= \sqrt{4 + 9 + 1} & &= \sqrt{16 + 9 + 4} \\ &= \boxed{\sqrt{14}} & &= \boxed{\sqrt{29}} \end{aligned}$$

We can also compute the dot product from the vectors' components, using the first form of (2):

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= 2(-4) + (-3)(-3) + 1 \times 2 \\ &= \boxed{3} \end{aligned}$$

From the second form of (2), we can deduce a formula for the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, θ_{AB}:

$$\begin{aligned} \cos \theta_{AB} &= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\|\boldsymbol{A}\|\|\boldsymbol{B}\|} \\ \Leftrightarrow \quad \theta_{AB} &= \cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\|\boldsymbol{A}\|\|\boldsymbol{B}\|}\right) \\ \Leftrightarrow \quad &= \cos^{-1}\left(\frac{3}{\sqrt{14 \times 29}}\right) \\ \Leftrightarrow \quad &= \boxed{81.43753893°} \end{aligned}$$

Remark 3. *Looking at our plots, the angle feels to be something a little less than $90°$, which is coherent with what we've found.*

Exercise 8. *Determine which pair of vectors are orthogonal.* $(1, 1, 1), (2, -1, 3), (3, 1, 0), (-3, 0, 2)$.

This is again an immediate application of dot product formula. We'll respectively name the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$; let's start by plotting them:
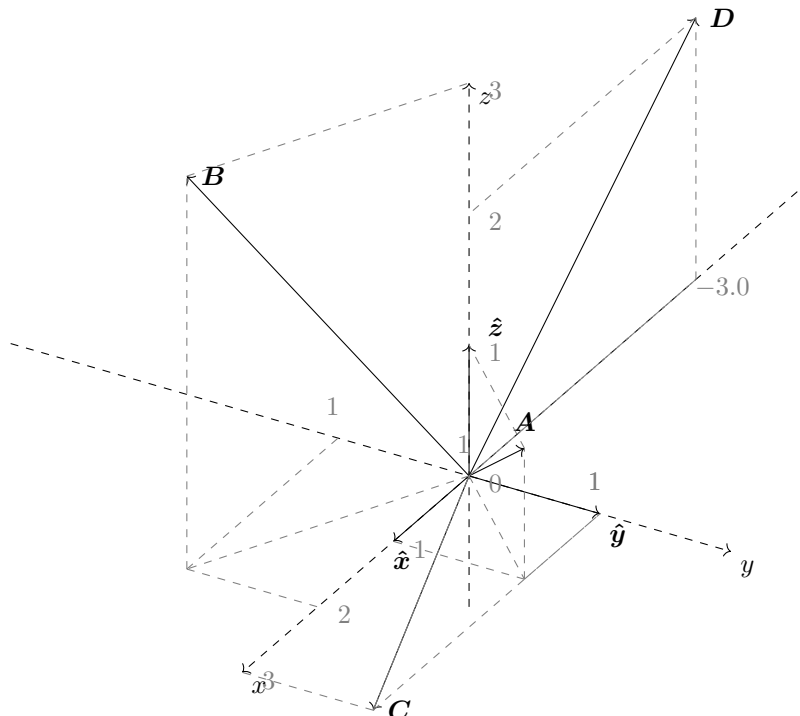
Right before this exercise, the authors wrote:

An important property of the dot product is that it is zero if the vectors are *orthogonal*.

Let us recall the "components-based" dot product formula we'll be using:

$$\boldsymbol{u} \cdot \boldsymbol{v} \quad = \quad u_x v_x + u_y v_y + u_z v_z$$

Then, it's just a matter of crunching numbers:

$$
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{B} &= A_x B_x + A_y B_y + A_z B_z \\
&= (1 \times 2) + (1 \times (-1)) + (1 \times 3) \\
&= \boxed{4} \\
\boldsymbol{A} \cdot \boldsymbol{D} &= A_x D_x + A_y D_y + A_z D_z \\
&= (1 \times (-3)) + (1 \times 0) + (1 \times 2) \\
&= \boxed{-1} \\
\boldsymbol{B} \cdot \boldsymbol{D} &= B_x D_x + B_y D_y + B_z D_z \\
&= (2 \times (-3)) + (-1 \times 0) + (3 \times 2) \\
&= \boxed{0}
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{C} &= A_x C_x + A_y C_y + A_z C_z \\
&= (1 \times 3) + (1 \times 1) + (1 \times 0) \\
&= \boxed{4} \\
\boldsymbol{B} \cdot \boldsymbol{C} &= B_x C_x + B_y C_y + B_z C_z \\
&= (2 \times 3) + (-1 \times 1) + (3 \times 0) \\
&= \boxed{5} \\
\boldsymbol{D} \cdot \boldsymbol{C} &= D_x C_x + D_y C_y + D_z C_z \\
&= (-3 \times 3) + (0 \times 1) + (2 \times 0) \\
&= \boxed{-9}
\end{aligned}
$$

Hence, the only two orthogonal vectors are $\boldsymbol{B}$ and $\boldsymbol{D}$, or $(2, -1, 3)$ and $(-3, 0, 2)$.

Exercise 9. *Can you explain why the dot product of two vectors that are orthogonal is* 0*?*

The dot product between two vectors is defined in two ways in the book; one of them involves the magnitudes of the vectors and the angle between them:

$$\boldsymbol{u} \cdot \boldsymbol{v} \quad = \quad \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos \theta_{uv}$$

By definition, two vectors are orthogonals if the angle between them is $\pi/2$, or $90°$. But, $\cos(\pi/2) = 0$, so it follows from the previous dot product formula that, for two orthogonal vectors, their dot product must be 0.

Lecture 2: Motion

Mathematical Interlude: Differential Calculus

Exercise 1/8

Exercise 10. *Calculate the derivatives of each of these functions.*

$$\begin{aligned} f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\ g(x) &= \sin x - \cos x \\ \theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\ x(t) &= \sin^2 t - \cos t \end{aligned}$$

Remark 4. *Those are exactly the functions graphed inI01E01*

$f(t) = t^4 + 3t^3 - 12t^2 + t - 6$

,
To find the derivative of f, we need to apply four rules that were mentioned (the first one was proved) in the book:

1. the formula for the derivative of a general power; for $n \in \mathbb{N}$:

$$\frac{d}{dt}(t^n) = nt^{n-1}$$

2. the fact that the derivative of a constant is zero; for $c \in \mathbb{R}$:

$$\frac{d}{dt}c = 0$$

3. the fact that the derivative of a constant times a function is the same as the constant times the derivative of the function; for $c \in \mathbb{R}$ and φ a function of t:

$$\frac{d}{dt}(c\varphi) = c\frac{d}{dt}\varphi$$

4. the *sum rule* (also referred to as the *linearity of differentiation* in mathematics); for both φ and ψ functions of t:

$$\frac{d}{dt}(\varphi + \psi) = \frac{d}{dt}\varphi + \frac{d}{dt}\psi$$

Remark 5. *Two simpler notations are common to denote the derivative of a function of a single variable:*

$$\frac{d}{dt}\varphi = \varphi' = \dot{\varphi}$$

While the first one is used mostly in mathematics, for abstract functions, the second one is used almost exclusively in physics, to denote a time derivative. We'll use them both in such ways from now on. For instance, as this exercise is rather mathematical, we'll use the prime notation.

While we're here, bear in mind that the prime notation can also be used to denote the derivative of a more or less complex expression, wrapped in parenthesis, e.g.:

$$(\varphi(x) + \psi(x) + \cos^2 x)' = \frac{d}{dx}(\varphi(x) + \psi(x) + \cos^2 x)$$

By application of the *sum rule*, the rule regarding the derivative of a constant, and the rule regarding a function multiplied by a constant:

$$f'(t) = (t^4)' + 3(t^3)' - 12(t^2)' + (t)' - 0$$

Then, it's just a matter of applying the formula for the derivative of a general power to each individual term:

$$\boxed{f'(t) = 4t^3 + 9t^2 - 24t + 1}$$

$g(x) = \sin x - \cos x$

,

In addition the the previously mentioned *sum rule*, we will also need two more rules to compute the derivative of g, also presented in the book, regarding the derivative of sin and cos:

$$\cos'(t) \;=\; -\sin(t); \qquad \sin'(t) \;=\; \cos(t)$$

We then have successively:

$$g'(x) \;=\; \sin'(x) - \cos'(x)$$
$$=\; \boxed{\cos(x) + \sin(x)}$$

$\theta(\alpha) = e^\alpha + \alpha \ln$

,

To compute θ', in addition to the *sum rule* and the formula for a general power (with $n = 1$), we need two additional rules pertaining to the derivatives of both the exponential and the log:

$$(e^t)' \;=\; e^t; \qquad \ln'(t) \;=\; \frac{1}{t}$$

And the *product rule*; for φ and ψ functions of t:

$$\frac{d}{dt}(\varphi\psi) = \varphi'\psi + \varphi\psi'$$

Where all those rules were mentioned in the book. We then obtain:

$$\theta'(\alpha) \;=\; (e^\alpha)' + (\alpha \ln(\alpha))'$$
$$=\; e^\alpha + (\frac{d}{d\alpha}\alpha)\ln(\alpha) + \alpha \ln'(\alpha)$$
$$=\; \boxed{e^\alpha + \ln(\alpha) + 1}$$

$x(t) = (\sin t)^2 - \cos t$

,

Besides the *sum rule*, the rules regarding the derivative of cos and sin, and the formula for the derivative of a general power, we will only need a single new rule to compute x', the *chain rule*; for φ a function of t, and ψ and function whose domain (input) is the codomain (output) of φ:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then,

$$x'(t) \;=\; ((\sin t)^2)' - (\cos t)'$$
$$=\; (\sin t)'(u \mapsto u^2)'(\sin t) + \sin t$$
$$=\; \cos t(u \mapsto 2u)(\sin t) + \sin t$$
$$=\; 2\cos t \sin t + \sin t$$
$$=\; \boxed{(1 + 2\cos t)\sin t}$$

Remark 6. *We used a bit of mathematical notation to avoid us the need to explicitly name the function which squares its argument. More explicitly, to compute the derivative of $\mu(t) = \sin^2 t$, we could have define $\nu(u) = u^2$. Then, $\mu(t) = \nu(\sin(t))$, and $\nu'(u) = 2u$; by the chain rule*

$$(\sin^2 t)' = \mu'(t) = (\sin t)'\nu'(\sin(t)) = 2\cos t \sin t$$

Remark 7. *Instead of using the chain rule to compute the derivative of $\sin^2 t$, we could instead have used the product rule:*

$$(\sin^2 t)' = (\sin t \times \sin t)' = (\sin t)' \sin t + \sin t (\sin t)' = 2\sin t (\sin t)' = 2\sin t \cos t$$

Exercise 2/8

Exercise 11. *The derivative of a derivative is called the second derivative and is written $\dfrac{d^2 f(t)}{dt^2}$. Take the second derivative of each of the functions listed above.*

Remark 8. *As for the first derivative, there are two common notations for second derivatives:*

$$\frac{d^2}{dt^2}\varphi = \varphi'' = \ddot{\varphi}$$

Again, the dot notation is used in physics to denote time differentiation, while the prime notation is often used in mathematics, in a more abstract context.

Let's start by recalling the functions:

$$
\begin{aligned}
f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\
g(x) &= \sin x - \cos x \\
\theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\
x(t) &= \sin^2 t - \cos t
\end{aligned}
$$

And their derivatives, computed in the previous exercise:

$$
\begin{aligned}
f'(t) &= 4t^3 + 9t^2 - 24t + 1 \\
g'(x) &= \cos(x) + \sin(x) \\
\theta'(\alpha) &= e^\alpha + \ln(\alpha) + 1 \\
x'(t) &= (1 + 2\cos t)\sin t
\end{aligned}
$$

While in the previous exercise (L02E01) the derivation were rather slow and detailled, because the process is essentially the same, we're going to go (much) faster here, the most difficult part being in the application of the product rule for x''.

$$
\begin{aligned}
f''(t) &= \boxed{12t^2 + 18t - 24} \\
g''(x) &= \boxed{\cos(x) - \sin(x) = -g(x)} \\
\theta''(\alpha) &= \boxed{e^\alpha + \frac{1}{\alpha}} \\
x''(t) &= (1 + 2\cos t)\cos t - 2\sin t \sin t \\
&= \boxed{\cos t + 2(\cos^2 t - \sin^2 t)}
\end{aligned}
$$

Remark 9. *x'' could be slightly improved by using the trigonometric identity $\cos^2 x - \sin^2 x = \cos 2x$, which hasn't been introduced in the book.*

Exercise 3/8

Exercise 12. *Use the chain rule to find the derivatives on each of the following functions:*

$$
\begin{aligned}
g_0(t) &= \sin(t^2) - \cos(t^2) \\
\theta_0(\alpha) &= e^{3\alpha} + 3\alpha \ln(3\alpha) \\
x_0(t) &= \sin^2(t^2) - \cos(t^2)
\end{aligned}
$$

Remark 10. *We've slightly altered the functions names compared to what they have in the book. The reason will be apparent very soon.*

Remark 11. *As in previous exercises (e.g. L02E01), we'll be using some additional mathematical notation to avoid us the need to define functions. The meaning should be obvious from the context.*

We will also use the prime notation to denote differentiation, as we also did in earlier exercises.

There are two main ways of proceeding: either applying the *chain rule* to each individual term of each function, for instance to compute g', we would fist compute $(\sin(t^2))' = (\sin((u \mapsto u^2)(t)))'$, using the chain rule, then $(\cos(t^2))'$ in a similar fashion, etc.

Or, and we're going to use this approach as it's likely to have been expected one, we can consider each function "globally" as a composition of two smaller functions. While doing so, we will find that g_0, θ_0 and x_0 are respectively created by composing functions f, θ and x of an earlier exercise, with two simple functions $(v \mapsto v^2)$ and $(v \mapsto 3v)$:

$$g_0(t) = (\underbrace{(u \mapsto \sin(u) - \cos(u))}_{g} \circ (v \mapsto v^2))(t)$$

$$\theta_0(\alpha) = (\underbrace{(u \mapsto e^u + u\ln(u))}_{\theta} \circ (v \mapsto 3v))(t)$$

$$x_0(t) = (\underbrace{(u \mapsto \sin^2(u) - \cos(u))}_{x} \circ (v \mapsto v^2))(t)$$

Let's remember the derivative of g, θ and x, that we've computed in L02E01:

$$g'(x) = \cos(x) + \sin(x)$$
$$\theta'(\alpha) = e^\alpha + \ln(\alpha) + 1$$
$$x'(t) = (1 + 2\cos t)\sin t$$

Finally, let's recall the *chain rule*:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then, the derivative of $(v \mapsto v^2)$ and $(v \mapsto 3v)$ being respectively $(v \mapsto 2v)$ and $(v \mapsto 3)$ (constant function), we have the following derivatives for our functions:

$$g_0'(t) = \boxed{2t(\cos t^2 + \sin t^2)}$$

$$\theta_0'(\alpha) = \boxed{3(e^{3\alpha} + \ln(3\alpha) + 1)}$$

$$x_0'(t) = \boxed{2t((1 + 2\cos t^2)\sin t^2)}$$

Exercise 4/8

Exercise 13. *Prove the sum rule (fairly easy), the product rule (easy if you know the trick), and the chain rule (fairly easy).*

You may want to refer to a *real* mathematical textbook for a finer, more rigorous treatment of those exercises. We'll give reasonably solid proofs, that should be sufficient in the context of an introductory physics textbook. A interesting middle-ground, that we won't explore here, would be to study the proofs that one can obtain from an alternative (but equivalent) formulation of the derivative known as the Carathéodory's derivative, which allows for simple proofs of such results.

Let's start by recalling how differentiation is defined.

Definition 1. *A function $\varphi : E \to \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:*

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon \to 0} \frac{\varphi(e + \epsilon) - \varphi(e)}{\epsilon}}$$

If this limit exists for all points x of E (we note, $(\forall x \in E)$), then φ is said to be differentiable on E, or simply differentiable. The function which associate to each points of E this limit is called the derivative of φ, and is called φ.

Theorem 1 (sum rule). *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi + \psi)' = \varphi' + \psi'}$$

Proof. We have, by definition of the differentiation, and after re-ordering the terms

$$
\begin{aligned}
(\forall x \in E), \quad (\varphi + \psi)'(x) &= \lim_{\epsilon \to 0} \frac{(\varphi + \psi)(x + \epsilon) - (\varphi + \psi)(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon) + \psi(x + \epsilon) - \varphi(x) - \psi(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left(\frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon} + \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon} + \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon} \\
&= \boxed{\varphi'(x) + \psi'(x)}
\end{aligned}
$$

$\square$

Remark 12. *As a rigorous proof is a bit tedious[2], we <u>assumed</u> for the last step that a limit of a sum is the sum of the limits, <u>when all the involved limits exist</u> (which is the case here, because those limits are equivalent to saying our functions are differentiable, which they are, per hypothesis)*

$$\lim_{x \to a} (\varphi(x) + \psi(x)) = \lim_{x \to a} \varphi(x) + \lim_{x \to a} \psi(x))$$

Theorem 2 (product rule). *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi\psi)' = \varphi'\psi + \varphi\psi'}$$

Proof. This is a simple and often used theorem, but unfortunately, the proof of it is a bit "magic": if we start by applying the definition of the differentiation to $(\varphi\psi)'$, we have to introduce a well-crafted term (in the form $-a + a = 0$) so as to factorize things to meet our goal. We would furthermore be implicitly assuming that $(\varphi\psi)'$ exists, but we have no guarantee of it.

We can solve those issues by starting from the definition of the differentiation of $\varphi'\psi + \varphi\psi'$, but assuming we already know the result we're trying to prove, is conceptually clumsy. Perhaps calling this a "verification" rather than a proof would be more correct then.

Using the former derivation, we have:

$$
\begin{aligned}
(\forall x \in E), \quad (\varphi\psi)'(x) &= \lim_{\epsilon \to 0} \frac{(\varphi\psi)(x + \epsilon) - (\varphi\psi)(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon)\psi(x + \epsilon) - \varphi(x)\psi(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon)\psi(x + \epsilon) - \varphi(x)\psi(x) \overbrace{-\varphi(x + \epsilon)\psi(x) + \varphi(x + \epsilon)\psi(x)}^{=0}}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon)(\psi(x + \epsilon) - \psi(x)) + \psi(x)(\varphi(x + \epsilon) - \varphi(x))}{\epsilon} \\
&= \lim_{\epsilon \to 0} \varphi(x + \epsilon) \frac{(\psi(x + \epsilon) - \psi(x))}{\epsilon} + \psi(x) \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon} \\
&= \left(\lim_{\epsilon \to 0} \varphi(x + \epsilon) \right) \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon} + \psi(x) \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon) - \varphi(x)}{\epsilon} \\
&= \boxed{\varphi(x)\psi'(x) + \psi(x)\varphi'(x)}
\end{aligned}
$$

$\square$

---

[2]if you want one, have a look at *Paul's Online Notes*: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

Remark 13. *We <u>assumed</u> another result on limits [3], again assuming individual limits exists (and they do in our case, as our functions are differentiable, and (thus) continuous):*

$$\lim_{x \to a} (\varphi(x)\psi(x)) = \lim_{x \to a} \varphi(x) \times \lim_{x \to a} \psi(x)$$

Theorem 3 (chain rule). *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi \circ \psi)' = \psi' \times (\varphi' \circ \psi)}$$

Proof. Again, the proof is a bit "magical" in that we're going to multiply by a well-crafted term, of the form $a/a = 1$:

$$
\begin{aligned}
(\forall x \in E),\ (\varphi \circ \psi)(x)) &= \lim_{\epsilon \to 0} \frac{(\varphi \circ \psi)(x + \epsilon) - (\varphi \circ \psi)(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left(\frac{\varphi(\psi(x + \epsilon)) - \varphi(\psi(x)))}{\epsilon} \times \overbrace{\frac{\psi(x + \epsilon) - \psi(x)}{\psi(x + \epsilon) - \psi(x)}}^{=1} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{\varphi(\psi(x + \epsilon)) - \varphi(\psi(x)))}{\psi(x + \epsilon) - \psi(x)} \times \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \frac{\varphi(\psi(x + \epsilon)) - \varphi(\psi(x)))}{\psi(x + \epsilon) - \psi(x)} \times \underbrace{\lim_{\epsilon \to 0} \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon}}_{\psi'(x)}
\end{aligned}
$$

Again for that last step, we've used the aforementioned rule on products of existing limits. To conclude, we need to compute the first limit: let's define $h = \psi(x + \epsilon) - \psi(x) \Leftrightarrow \psi(x + \epsilon) = \psi(x) + h$. Note that $\epsilon \to 0 \Rightarrow h \to 0$. So the first limit can be rewritten:

$$\lim_{h \to 0} \frac{\varphi(\psi(x) + h) - \varphi(\psi(x)))}{h} \triangleq \varphi'(\psi(x))$$

<div align="right">□</div>

Now a problem with this previous proof is that it is invalid if ψ is a constant function for example, because $\psi(x + \epsilon) - \psi(x)$ is zero, and we're dividing by zero when performing our magical "multiplication" by 1. If you're interested, you can find an alternative proof using the other, equivalent form of the derivative here: `https://www.youtube.com/watch?v=COLwYhEAt7Q`.

Exercise 5/8

Particle Motion

Example of Motion

Exercise 6/8

Exercise 14. *How long does it take for the oscillating particle to go through one full cycle of motion?*

We're in the case of a particle oscillating in one dimension. Its motion, known as the *simple harmonic motion*, is described by:

$$x(t) = \sin(\omega t)$$

Essentially, $x(t)$ will correspond to the vertical component of a point moving on the unit circle, located by an angle ωt.

---

[3] If you want a rigorous proof of it, you can refer to the same resource as before: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`
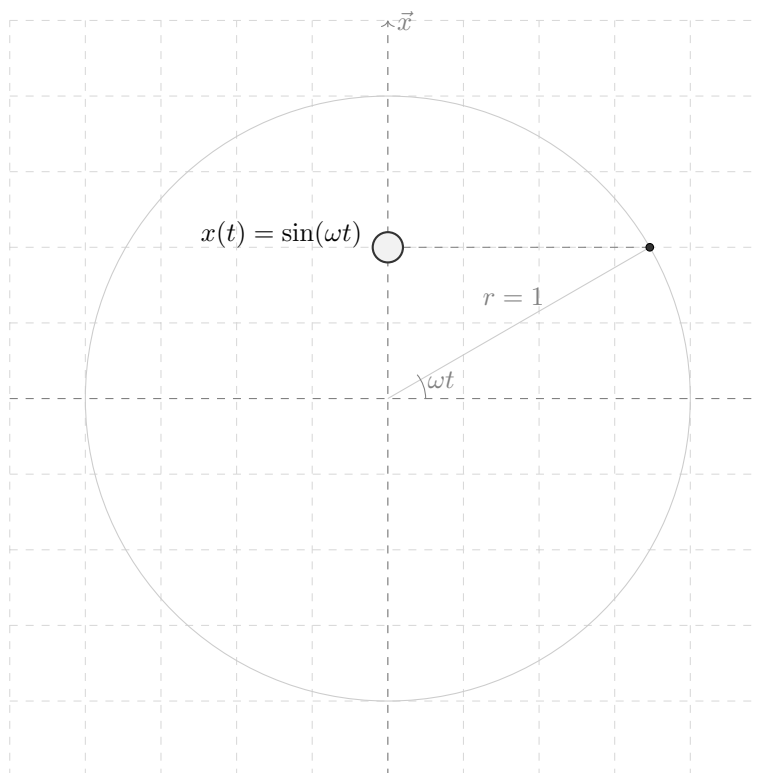
To fix things, consider the case of a particle starting at an extreme position, say $x = 1$ (at the top of the north hemisphere of the unit circle). It will need to go down to $x = -1$, and then back up to $x = 1$. In the mean time, the corresponding point on the unit circle would have walked a full circle, or 2π radians.

So we're looking for the time T that it will take for us to move by an angle 2π, knowing that we move at a speed of ω radians per unit of time (i.e. $\omega_{t=0} = 0$, $\omega_{t=1} = \omega$, $\omega_{t=2} = 2\omega$, ...):

$$\omega T = 2\pi \Leftrightarrow \boxed{T = \frac{2\pi}{\omega}}$$

Remark 14. *T is commonly called the period of motion.*

Exercise 7/8

Exercise 8/8

Interlude 2: Integral Calculus

Lecture 3: Dynamics

Aristotle's Law of Motion

Exercise 1/4

Exercise 15. *Given a force that varies with time according to $F = 2t^2$, and with the initial condition at time zero, $x(0) = \pi$, use Aristotle's law to find $x(t)$ at all times.*

Let us recall that Aristotle's law of motion is defined, for a one-dimensional particle (otherwise, $F(t)$ and $x(t)$ would be vector-values functions $\boldsymbol{F}(t)$ and $\boldsymbol{x}(t)$) earlier in the book as:

$$\frac{d}{dt}x(t) = \frac{F(t)}{m}$$

And that by integrating both sides, thanks to the fundamental theorem of calculus[4], assuming the mass is constant over time, we obtain:

$$x(t) = \frac{1}{m} \int F(t)\, dt$$

Which is our case, for $F(t) = 2t^2$, develops in:

$$x(t) = \frac{1}{m} \int 2t^2\, dt$$
$$= \frac{2}{3m} t^3 + c,\ c \in \mathbb{R}$$

The initial condition $x(0) = \pi$ implies that $c = \pi$, hence the position at all time would be:

$$\boxed{x(t) = \frac{2}{3m} t^3 + \pi} \quad \square$$

Mass, Acceleration and Force

Some Simple Examples of Solving Newton's Equations

Exercise 2/4

Exercise 16. *Integrate this equation. Hint: Use definite integrals.*

The equation in question resulting from Newton's second law in the case of a constant force F_z being applied to an object of mass m following the z-axis:

$$\dot{v}_z = \dot{v}_z(t) = \frac{F_z}{m}$$

By integrating both sides, thanks to the fundamental theorem of calculus, assuming the mass is constant over time, we obtain:

$$v_z(t) = \int \frac{F_z}{m}\, dt$$
$$= \frac{F_z}{m} \int dt$$
$$= \frac{F_z}{m} t + c,\ c \in \mathbb{R}$$

Generally, c would be determine from an initial condition $v_z(0)$, which is our case, would precisely be c, hence:

$$\boxed{v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

Which is exactly the solution proposed in the book.

Exercise 3/4

Exercise 17. *Show by differentiation that this satisfies the equation of motion.*

Contrary to the previous exercice, instead of integrating to find the solution, we start from the solution and climb back to our original equation of motion, which are, in the case of a constant force F_z applied to a mass m following the z-axis:

$$v_z(t) = \dot{z}(t) = v_z(0) - \frac{F_z}{m} t$$

The proposed solution is:

$$z(t) = z_0 + v_z(0) t + \frac{F_z}{2m} t^2$$

Immediately, by derivation, constants goes to 0, t becomes 1 and t^2 becomes $2t$, we indeed obtain:

$$\boxed{\frac{d}{dt} z(t) = \dot{z}(t) = v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

[4] https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

Exercise 4/4

Exercise 18. *Show by differentiation that the general solution to Eq. (6) is given in terms of two constants A and B by*

$$x(t) = A\cos\omega t + B\sin\omega t$$

Determine the initial position and velocity at time $t = 0$ in terms of A and B.

We're in the case of a 1-dimensional harmonic oscillator, depicted below as a mass m set on a frictionless ground attached to a fixed element to the left by a horizontally positioned spring of constant k.
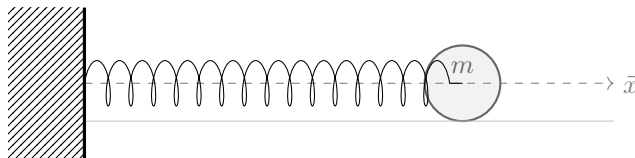
Remark 15. *Almost equivalently, we could have worked in a vertical setting, which, on one hand, would have avoided us the need for a frictionless ground, but on the other hand, would make the equation slightly more complicated.*

There are interesting, complementary aspects to both cases. Because the differences are minute, we can explore both of them here.

Harmonic oscillators play a major role in physics, as explained by Feynman in Chapter 21 of the first volume of his *Lectures on Physics*:

> The harmonic oscillator, which we are about to study, has close analogs in many other fields; although we start with a mechanical example of a weight on a spring, or a pendulum with a small swing, or certain other mechanical devices, we are really studying a certain differential equation. This equation appears again and again in physics and in other sciences, and in fact it is a part of so many phenomena that its close study is well worth our while.
>
> Some of the phenomena involving this equation are the oscillations of a mass on a spring; the oscillations of charge flowing back and forth in an electrical circuit; the vibrations of a tuning fork which is generating sound waves; the analogous vibrations of the electrons in an atom, which generate light waves; the equations for the operation of a servosystem, such as a thermostat trying to adjust a temperature; complicated interactions in chemical reactions; the growth of a colony of bacteria in interaction with the food supply and the poisons the bacteria produce; foxes eating rabbits eating grass, and so on; all these phenomena follow equations which are very similar to one another, and this is the reason why we study the mechanical oscillator in such detail.

Thus, we'll take the time to analyze it more precisely than what is expected here: this is the occasion to present some aspects not covered, or not covered as explicitly in the book.

Despite everything happening in one-dimension, we'll use a vector position $\boldsymbol{x}$, which is a function of time. Hence, the vectors for speed and acceleration will respectively be $\boldsymbol{v} = \dot{\boldsymbol{x}} = \dfrac{d}{dt}\boldsymbol{x}(t)$ and $\boldsymbol{a} = \ddot{\boldsymbol{x}} = \dfrac{d^2}{dt^2}\boldsymbol{x}(t)$. Note that we're representing vectors with bold font instead of arrows; bold font is rather common in physics. This should be the only additional thing needed to follow through for someone having read the book up to this stage.

Contact force, normal force, friction force
The contact force is the force resulting from the contact of two objects. It is generally decomposed into vertical and horizontal components. The horizontal component perhaps is the most intuitive: it corresponds to friction forces. For instance, after having given a gentle push to an object an on ordinary table, it will only move up to a certain point: it is progressively stopped by the friction between the object and the table. You may think that this friction is caused at least in part because at a microscopic scale, both the object and the surface are far, far from being perfectly plane, and even more so at an atomic scale.

But this is false. As a counter-example, consider gauge blocks, manufactured to be extremely flat: their surfaces, far from being frictionless to each other, cling!

Because a rigorous understanding of friction forces requires advanced mechanics knowledge, we can satisfy ourselves with Feynman's vulgarization, from Chapter 12 of the first volume of his *Lectures on Physics*:

> There is another kind of friction, called dry friction or sliding friction, which occurs when one solid body slides on another. In this case a force is needed to maintain motion. This is called a frictional force, and its origin, also, is a very complicated matter. Both surfaces of contact are irregular, on an atomic level. There are many points of contact where the atoms seem to cling together, and then, as the sliding body is pulled along, the atoms snap apart and vibration ensues; something like that has to happen. Formerly the mechanism of this friction was thought to be very simple, that the surfaces were merely full of irregularities and the friction originated in lifting the slider over the bumps; but this cannot be, for there is no loss of energy in that process, whereas power is in fact consumed. The mechanism of power loss is that as the slider snaps over the bumps, the bumps deform and then generate waves and atomic motions and, after a while, heat, in the two bodies.
>
> [...]
>
> It was pointed out above that attempts to measure μ by sliding pure substances such as copper on copper will lead to spurious results, because the surfaces in contact are not pure copper, but are mixtures of oxides and other impurities. If we try to get absolutely pure copper, if we clean and polish the surfaces, outgas the materials in a vacuum, and take every conceivable precaution, we still do not get μ. For if we tilt the apparatus even to a vertical position, the slider will not fall off—the two pieces of copper stick together! The coefficient μ, which is ordinarily less than unity for reasonably hard surfaces, becomes several times unity! The reason for this unexpected behavior is that when the atoms in contact are all of the same kind, there is no way for the atoms to "know" that they are in different pieces of copper. When there are other atoms, in the oxides and greases and more complicated thin surface layers of contaminants in between, the atoms "know" when they are not on the same part. When we consider that it is forces between atoms that hold the copper together as a solid, it should become clear that it is impossible to get the right coefficient of friction for pure metals.

The normal force is the vertical component of the contact force. An object on a table is affected by the Earth gravity, but naturally resists going through the table, unless of course, the object is really massive and/or the table very weak. There's a bunch of complicated interactions at the atomic level from which this situation occurs: at a macroscopic scale, we simplify things and wrap this complexity by abstracting it as a (macroscopic) force.

Remark 16. *In the case of the vertical setup, with a mass attached to a vertical spring, the mass is not in contact with a ground surface, so there would have been no need to discuss contact forces.*

Remark 17. *A static object on a flat surface corresponds to a special case where the horizontal component of the contact force (friction) is null. Hence, the contact force and the normal force are, in such a special case, one and the same. The exact same thing would happen in the case of a frictionless surface.*

Newton's laws

Let us start by recalling Newton's laws of motion:

1. **Principle of inertia**: Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it;

2. "$\boldsymbol{F} = m\boldsymbol{a}$": The change of motion [*momentum*] of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed;

3. "action $\Rightarrow$ reaction": To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Remark 18. *A force always is applied from an object, to another object; with the restriction that an object cannot apply a force to itself.*

Remark 19. *The second law is a actually a statement about momentum, from which we can derive $\boldsymbol{F} = m\boldsymbol{a}$. More precisely, the momentum, called "motion" by Newton, is defined as $\boldsymbol{p} = m\boldsymbol{v}$, which literally captures the idea of the speed $\boldsymbol{v}$ of a certain amount of matter m. Then, it follows that the general case, $\boldsymbol{p}$ is as $\boldsymbol{v}$ a function of time.*
The "change of motion" then refers to an infinitesimal change of the momentum over time, mathematically captured by the time derivative $\dfrac{d}{dt}\boldsymbol{p}$. The law can then be progressively written, assuming the mass m is constant over time:

$$\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p} = \frac{d}{dt}m\boldsymbol{v} = m\frac{d}{dt}\boldsymbol{v} = m\boldsymbol{a} \quad \square$$

Remark 20. *It may seem weird to mention in the previous Remark 19 that we assume the mass m to be constant over time. In the current context, this will be the case, but in general, likely contrary to what any reasonable human would expect, they are exceptions. More precisely, in special relativity, the mass is a function of the velocity v:*

$$m_{rel} = \frac{m}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

We won't delve into the details here, but note that if $v = \|\boldsymbol{v}\|$ is much smaller that c, the speed of light, then the relative mass m_{rel} and m are very close.

To say it otherwise, this assumption doesn't hold at high velocities.

Remark 21. *In the second law, the force impressed ("$\boldsymbol{F}$") is actually the force resulting from all the other forced applied to an object, $\sum_i \boldsymbol{F}_i$, which is sometimes called $\boldsymbol{F}_{net}$.*

For instance, if you push a cart, $\boldsymbol{F}_{net}$ should contain at least the force you are exerting, the gravity exerted by Earth, and the contact force generated by the ground on the cart's wheel. The sum of all the forces applied to the cart that will ultimately influence its motion.

As a result, in practice, when analyzing a situation so as to establish the equation of motion of an object, one should start by identifying all the external forces applied to this object.

Remark 22. *A common special case of the second law is to consider objects that, within a certain frame of reference, do not move. Hence, by definition, their speed $\boldsymbol{v}$ would be 0, and so would be their acceleration. That is, $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.*

Another special case for the second law is when things move at constant speed. Again, the acceleration would then be $\boldsymbol{0}$ and thus still $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.

Remark 23. *It is interesting to spend a moment to ponder what the mass m of an object really "is". Intuitively, for a non-physicist mass is another word for weight, but not to a physicist.*

*Instead, in the context of the second law, physicists will observe that the more mass there is, the more force will be needed to accelerate an object. Conversely, the less mass there is, the less force will be needed to alter the motion of an object. Thus, they would conceptualize the mass as a **measure of resistance to acceleration** (change of movement).*

This is the second time we meet a subtlety regarding masses, the first one being Remark 20 about how mass is a function of the velocity. We'll come back later once again in a moment at some more subtlety regarding mass.

Remark 24. *In the case of a static object laying on a flat surface, one may think that the gravitational force exerted by the Earth on the object and the normal force/contact force arising from the contact of the object with the surface are two opposite forces, in the sense of Newton's third law: after all, they are indeed of equal intensity and opposite directions.*

But this is incorrect: in such a situation, we actually have two pairs of opposite forces, in the sense of Newton's third law:

1. $\boldsymbol{F}_{o,e}$ and $\boldsymbol{F}_{e,o}$, respectively the gravitational force exerted by the object on Earth, and the gravitational force exerted by Earth on the object;

2. $\boldsymbol{F}_{o,t}$ and $\boldsymbol{F}_{t,o}$, respectively the contact force exerted by the object on the table, and the contact force exerted by the table on the object.

Now to be more complete, there's even a third pair of forces, which would be in almost all practical situations negligible, namely, the gravitational forces exerted by the surface on the object, and by the object on the surface.

A good exercise to check one understanding of this topic would be to try to identify forces in the case of a hand pressed on a (static) wall, or in the case of two hands pressed against each other.

Remark 25. *Newton's third law is actually a consequence of the principle of conservation of momentum: essentially, in a closed system, the total momentum is a constant over time.*

This is a key idea in modern physics, and will be key later in the book.

Hooke's law
Hooke's law, as Newton's laws, is an *empirical law*, that is, which has been found by repeated experimentation. Essentially, it states that the force needed to extend or compress a spring by a given distance varies linearly with the distance during which the spring is extended/compressed. That linear factor k *is* the spring's constant. This force is thus a function of that distance/displacement $\boldsymbol{\delta}$:

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = k\boldsymbol{\delta}$$

The equation could be read as something like:

- If I don't compress/stretch the spring ($\boldsymbol{\delta} = \boldsymbol{0}$), no force is exerted;

- The more I compress/stretch the spring, the greater the force, where the relation between both varies linearly.

Remark 26. *In most practical cases, $\boldsymbol{\delta}$ will be a function of time $\boldsymbol{\delta}(t)$; thus, the force itself will be a function of time: $\boldsymbol{F}_s(\boldsymbol{\delta}(t))$.*

Remark 27. *By Newton's third law, if there's an object exerting a force on the spring, then the spring also exerts a force on the object, of equal intensity, but in the opposite direction. Hence, equivalently, Hooke's law could be reformulated in the form presented in the book:*

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = -k\boldsymbol{\delta}$$

Remark 28. *You may sometimes find that last version of the law, as is the case in the book, written as $\boldsymbol{F}_s(\boldsymbol{x}) = -k\boldsymbol{x}$. This is a convenient choice that physicists can make, where, by identifying the origin of the coordinate system as the rest position of the mass, the measure of displacement $\boldsymbol{\delta}$ will indeed always match the position vector $\boldsymbol{x}$.*

This can be summarized by saying that bmF_s is the the force pulling the mass back to its rest position.

Remark 29. *If you have a spring or two around (architect lamps are a good source) and some regular weight (small tableware, such as iron tea cups would do), you should be able to easily verify this law experimentally. In this case, the objects' weight will be the source of a gravitational force (we'll come back to the gravitational force in a moment): doubling the mass (attaching two cups) should double the deformation of the spring.*
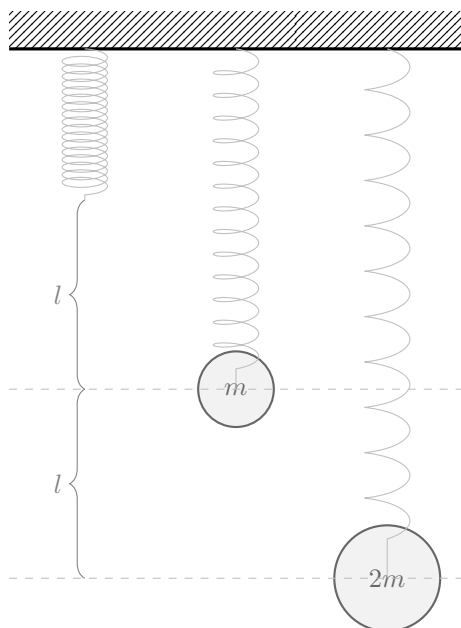
Remark 30. *Were you at least to imagine the previous experiment, you would realize that there must so some limits to this law: the spring could break or be irremediably deformed. Hooke's law is actually a first-order linear approximation, that works reasonably well in plenty of practical cases. There are more advanced models who generalize Hooke's law, but they are of no use to us in this context.*

Forces diagram

The goal of a force diagram, or a free body diagram, is to identify the forces acting on an object, with the intent of applying Newton's second law, so as to later determine an equation of movement for that object: we want to identify the sum of all the forces **acting on the object**, $\boldsymbol{F}_{\mathrm{net}} = \sum_i \boldsymbol{F}_i$, who are likely to contribute to the shape of its trajectory. Let's complete the diagram of our horizontal spring setup with:

$\boldsymbol{F}_{e,o}$ the gravitational force exerted by the earth on our object. Note that we haven't talked about gravitation yet, and as you'll see in a moment, we don't need to, but this will be necessary for the vertical spring setup;

$\boldsymbol{F}_{t,o}$ the normal force exerted by the table on our object. Because the surface is frictionless, the normal force *is* the contact force;

$\boldsymbol{F}_{s,o}$ the force exerted by the spring on our object, which can be described by Hooke's law.
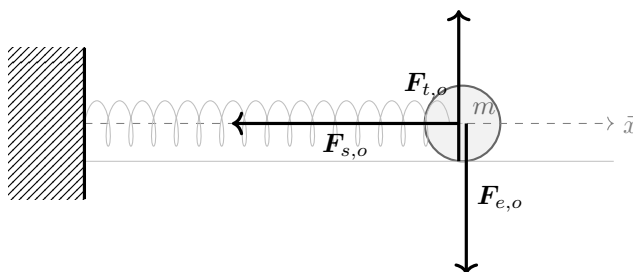
Placement of the origin of the $\vec{x}$-axis As we've already observed before, it can be interesting to identify the origin of our coordinate system to the position where the system will be at equilibrium/rest.

Intensities and signs Thanks the previous convention, as discussed in Remark 28, Hooke's law can be used to determine the force exerted by the spring on the object, and can be written as $\boldsymbol{F}_{s,o} = -k\boldsymbol{x}$.

We know, from Newton's second law, that the contact/normal force exerted by the table on the object is equal to $-\boldsymbol{F}_{o,t}$, where $\boldsymbol{F}_{o,t}$ is the force exerted by the object on the table, which itself results from the gravitational force $\boldsymbol{F}_{e,o}$, i.e.

$$\boldsymbol{F}_{t,o} = -\boldsymbol{F}_{o,t} = -\boldsymbol{F}_{e,o}$$

As we'll see quickly, we don't actually need to compute anything more, in this situation. We'll nevertheless show how to compute the gravitational force in the next section, when considering the similar case of a mass attached to a vertical spring.

Applying Newton's second law We can now use Newton's second law, $\sum_i \boldsymbol{F}_i = m\boldsymbol{a} = m\ddot{\boldsymbol{x}}$ to deduce an equation of motion:

$$\begin{aligned} & \boldsymbol{F}_{e,o} + \boldsymbol{F}_{t,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \boldsymbol{F}_{e,o} - \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & -k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \quad \square \end{aligned}$$

Which is a second order, linear differential equation, homogeneous, with constant coefficients. It's often rewritten, as is the case in the book, by defining $\omega^2 = \dfrac{k}{m}$:

$$\boxed{\ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x}}$$

Gravitational force

To handle the case of a vertical spring, we won't be able to brush the gravitational force away as we did previously, and we'll need to compute it.

There are a few different ways to reach a formula for the gravitational force, but perhaps the simplest one, in the context of studying a mass attached to a vertical spring, would be to empirically study the fall of an object of mass m in Earth's gravitational field.

One could proceed for instance by video-taping a vertically falling object of known mass m. The video can then be discretized into a series of images, on which we could measure the evolution of the position of the mass over time. From there, we could compute the speed, as the variation of position, and the acceleration, as the variation of speed.

Then, neglecting the air resistance (well, we *could* perform the experiment in a vacuum), we can pretend that there's a single force acting on the mass, which *is* our gravitational force.

The experiment could obviously be performed multiple times to improve the accuracy. In the end, physicists have found that Earth's gravity is an acceleration vector $\boldsymbol{g}$ of intensity $9.81\,m.s^{-2}$, directed towards the center of the Earth, which experimentally gives a gravitational force of:

$$\boldsymbol{F}_{e,o} = m\boldsymbol{g}$$

Remark 31. *A home-made reenactment of the experiment would be difficult, especially if there's a need for strong precision, which would involve the presence of a vacuum. At least, more difficult than experimentally validating Hooke's law on a single spring.*

Historically, a simpler and more primitive setup was employed, for instance by Galileo, which would involved measuring the speed of a ball rolling on an inclined plane. A great way to lower the speed of the "fall" and thus to get more accurate time measurements.

Remark 32. *1) Note that $\boldsymbol{g}$ captures not only the gravitational force exerted by the Earth, but also captures the contribution of the Earth's rotation (centrifugal force).*

2) Because the Earth is not a perfect sphere, the value of $\|\boldsymbol{g}\|$ is not uniform everywhere on Earth; there are noticeable differences from city to city for instance.

Remark 33. *In Remark 23, we conceptualized the mass as a measure of resistance to acceleration. Physicists call this the inertial mass. And they conceptually distinguish it from what they call the gravitational mass, which is the property of matter that qualify how an object will interact with a gravitational field.*

We tacitly assumed that both are the same: this is the equivalence principle, a cornerstone of modern theories of gravities. So far, physicists haven't been able to distinguish the two masses experimentally either. But conceptually, they refer to two distinct notions.

Mathematically, this means that in the case of a falling object subject to a single force exerted by gravitation, by applying Newton's second law, we have:

$$\boldsymbol{F}_{net} = \boldsymbol{F}_{e,o} = m_i \boldsymbol{a}$$
$$\Leftrightarrow \qquad m_g \boldsymbol{g} = m_i \boldsymbol{a}$$
$$\Leftrightarrow \qquad \boldsymbol{g} = \boldsymbol{a}$$

Remark 34. *1) Another approach could have been to rely on Newton's law of universal gravitation.*

$$\boldsymbol{F}_{e,o} = -G \frac{m_e m_o}{\|\boldsymbol{r}_{e,o}\|^2} \hat{\boldsymbol{r}}_{e,o}$$

But not only is this law also empirical, but it would involve knowing the mass of the Earth m_e, the gravitational constant G, etc.

2) Similarly we could delve into Einstein's theory of general relativity, but the prerequisites would again have been more sophisticated than exposing a basic experiment.

Vertical spring

We know have all we need to study a variant of our previous setup, slightly more complicated mathematically, where a mass m is attached to the end side of a vertical spring of constant k, itself firmly attached to a fixed element at its top:
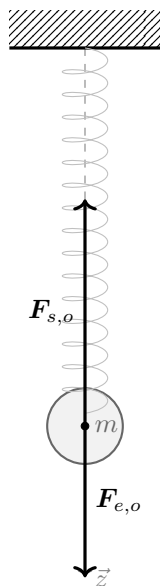
We'll choose the same convention regarding the origin of our coordinate system (see Remark 28). Contrary to the previous case, there's no other force to counterbalance the gravitational force $\boldsymbol{F}_{e,o}$; applying Newton's second law gives:

$$\sum_i \boldsymbol{F}_i = \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}}$$
$$\Leftrightarrow \qquad m\boldsymbol{g} - k\boldsymbol{x} = m\ddot{\boldsymbol{x}}$$
$$\Leftrightarrow \qquad \ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x} + \boldsymbol{g}$$

Again, with $\omega^2 = \dfrac{k}{m}$; so the only difference is that we get an additional constant $\boldsymbol{g}$, but this still is a second-order linear differential equation with constant coefficients.

Remark 35. *However, this differential equation is said to be non-homogeneous because of the non-zero constant $\boldsymbol{g}$, by opposition with the previous homogeneous equation obtained from the horizontal spring setup.*

Verifying the solution

Solving differential equations, mathematically, can be rather involved. For such simple equations, mathematicians themselves may find it sufficient to make an educated guess by tweaking an exponential-based function, despite having explored the subject with much more depth.

We won't discuss the details here, and will pragmatically satisfy ourselves with verifying that the proposed solution actually solves the equation:

$$x(t) = A\cos\omega t + B\sin\omega t$$

Remark 36. *We've been before expressing all our equations with vectors so far, but because we're working in one-dimension, we can divide everything by $\hat{\boldsymbol{x}}$ and work from the resulting scalar equations.*

And indeed, the proposed solution does solve the equation for the horizontal spring setup.

$$\begin{aligned}
\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\
&= -\omega^2(A\cos\omega t + B\sin\omega t) \\
&= -\omega^2 x(t) \quad \square
\end{aligned}$$

But it doesn't solve the vertical setup; we'll get there quickly, this isn't much more complicated. You may want to try by yourself to see if you can tweak the solution of the homogeneous equation to solve the non-homogeneous one.

Initial conditions

Let's start by computing the expression of the velocity:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Then, at $t = 0$, we have:

$$\begin{aligned}
x(0) &= A\cos(0) + B\sin(0) = \boxed{B} \\
\dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square
\end{aligned}$$

Equations and initial conditions for the vertical setup

There are general mathematical methods for reaching a solution to a non-homogeneous linear second-order differential equation with constant coefficient from a homogeneous, but again, suffice for us to verify that the following would work:

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}$$

Indeed:

$$\begin{aligned}
\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\
&= -A\omega^2\cos\omega t - B\omega^2\sin\omega t - \frac{\omega^2}{\omega^2}g + g \\
&= -\omega^2(A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}) + g \\
&= -\omega^2 x(t) + g \quad \square
\end{aligned}$$

From there, we can again express the initial position and speed as functions of A and B, first by computing the expression of the speed:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

And then by looking at what's happening at $t = 0$:

$$\begin{aligned}
x(0) &= A\cos(0) + B\sin(0) + \frac{g}{\omega^2} = \boxed{B + \frac{g}{\omega^2}} \\
\dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square
\end{aligned}$$

Interlude 3: Partial Differentiation

Lecture 4: Systems of More Than One Particle

Lecture 5: Energy

More than one dimension

Exercise 2/3

Exercise 19. *Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitly that the total energy is conserved.*

Equations of motion: For this system, the potential energy V is:

$$V = \frac{1}{2}k(x^2 + y^2) \tag{3}$$

By Newton's second law of motion, given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{4}$$

Or,

$$\begin{aligned} F_x &= & m\ddot{x} \\ F_y &= & m\ddot{y} \end{aligned} \tag{5}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{6}$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x}V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y}V(x, y) = -ky \end{aligned} \tag{7}$$

Combining (5) and (7):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \tag{8}$$

Which are known by *L03E04* to be differential equations associated to harmonic motion, and solved by a slightly more general solution that the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{9}$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$\begin{aligned} v(t) = & \dot{x}(t) = & \omega(-\alpha \sin(\omega t - \theta) + \beta \cos(\omega t - \theta) \\ a(t) = & \ddot{x}(t) = & -\omega^2(\alpha \cos(\omega t - \theta) + \beta \sin(\omega t - \theta)) \\ & = & -\omega^2 x(t) \end{aligned} \tag{10}$$

Where, to differentiate e.g. $\alpha \cos(\omega t - \theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \tag{11}$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t=0), \dot{x}(t=0), y(t=0), \dot{y}(t=0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$
\begin{aligned}
x(0) &= \alpha_x \cos(0) + \beta_x \sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha \sin(0) + \beta \cos(0)) \\
&= \omega\beta \\
&= \sqrt{\frac{k}{m}}\beta
\end{aligned}
\tag{12}
$$

Circular orbits: The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r, centered on point (a, b), with $(a, b, r) \in \mathbb{R}^3$ if:

$$(\forall t \geq 0), \ (x(t) - a)^2 + (y(t) - b)^2 = r^2 \tag{13}$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$\sin(\theta \pm \varphi) = \sin\theta \cos\varphi \pm \cos\theta \sin\varphi$$

Then, let's introduce two angles φ_x and φ_y such as:

$$
\begin{aligned}
\sin\varphi_x &= \alpha_x & \sin\varphi_y &= \alpha_y \\
\cos\varphi_x &= \beta_x & \cos\varphi_y &= \beta_y
\end{aligned}
$$

From, there, we can use the previous identity to rewrite our equations of motions (9) as:

$$
\begin{aligned}
x(t) &= \sin\varphi_x \cos(\omega t - \theta_x) + \cos\varphi_x \sin(\omega t - \theta_x) &= \sin(\omega t + \varphi_x - \theta_x) = \sin\Omega_x \\
y(t) &= \sin\varphi_y \cos(\omega t - \theta_y) + \cos\varphi_y \sin(\omega t - \theta_y) &= \sin(\omega t + \varphi_y - \theta_y) = \sin\Omega_y
\end{aligned}
\tag{14}
$$

Obvoiusly with $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$ and $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$.

Let us now develop (13) with those two versions of $x(t)$ and $y(t)$:

$$
\begin{aligned}
(x(t) - a)^2 + (y(t) - b)^2 &= r^2 \\
\Leftrightarrow (\sin\Omega_x - a)^2 + (\sin\Omega_y - b)^2 &= r^2 \\
\Leftrightarrow \sin^2\Omega_x + \sin^2\Omega_y - 2(a\sin\Omega_x + b\sin\Omega_y) + a^2 + b^2 &= r^2
\end{aligned}
$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a, b) = (0, 0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$\sin^2\Omega_x + \sin^2\Omega_y = r^2$$

Which we can rewrite a little bit using the fact that $\sin\varphi = \cos(\varphi - \pi/2)$:

$$\sin^2\Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2$$

But we know the pythagorean identity $\sin^2\varphi + \cos^2\varphi = 1$, hence we know there will be circular orbits when:

$$
\begin{cases} r = 1 \\ \Omega_x = \Omega_y - \frac{\pi}{2} \end{cases}
\Leftrightarrow
\begin{cases} r = 1 \\ \omega t + \varphi_x - \theta_x = \omega t + \varphi_y - \theta_y - \frac{\pi}{2} \end{cases}
\Leftrightarrow
\begin{cases} r = 1 \\ \varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2} \end{cases}
$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (14) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

Remark 37. *For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.*

Remark 38. *The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.*

Energy conservation: Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2} \sum_i m_i \dot{x_i}^2 \tag{15}$$

Which gives us for this system, expliciting the time-dependancies:

$$T(t) = \frac{1}{2} m \dot{x}(t)^2 + \frac{1}{2} m \dot{y}(t)^2 = \frac{1}{2} m (\dot{x}(t)^2 + \dot{y}(t)^2) \tag{16}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned}
\frac{d}{dt} T(t) &= \frac{1}{2} m (2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\
&= m(\dot{x}\ddot{x} + \dot{y}\ddot{y})
\end{aligned} \tag{17}$$

On the other hand, we can compute the variation of potential energy over time from (3), again using the chain rule:

$$\begin{aligned}
\frac{d}{dt} V(t) &= \frac{1}{2} k \left(\frac{d}{dt} x(t)^2 + \frac{d}{dt} y(t)^2 \right) \\
&= \frac{1}{2} k (2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\
&= k(x(t)\dot{x}(t) + y(t)\dot{y}(t))
\end{aligned} \tag{18}$$

From (8), we have:

$$\begin{aligned}
x(t) &= -\frac{m}{k} \ddot{x}(t) \\
y(t) &= -\frac{m}{k} \ddot{y}(t)
\end{aligned} \tag{19}$$

Injecting in (18):

$$\begin{aligned}
\frac{d}{dt} V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\
&= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y})
\end{aligned} \tag{20}$$

Thus from (17) and (20):

$$\frac{d}{dt} E(t) = \frac{d}{dt} T(t) + \frac{d}{dt} V(t) = 0 \quad \square \tag{21}$$

That is, total energy E over time doesn't change.

Lecture 6: The Principle of Least Action

The Transition to Advanced Mechanics

Action and the Lagrangian

Derivation of the Euler-Lagrange Equation

Exercise 1/6

Exercise 20. *Show that Eq. (4) is just another form of Newton's equation of motion $F = ma$.*

Where Eq. (4) are the freshly derived Euler-Lagrange equations of motions:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \tag{22}$$

In the context of a single particle moving in one dimension, with kinetic and potential energy given by:

$$T = \frac{1}{2}m\dot{x}^2$$
$$V = V(x)$$

From which results the Lagrangian:

$$L = T - V$$
$$= \frac{1}{2}m\dot{x}^2 - V(x) \tag{23}$$

Let us recall that we also have the *potential energy principle*, stated in one-dimension as Eq. (1) of the previous chapter, *Lecture 5: Energy*:

$$F(x) = -\frac{d}{dx}V(x) \tag{24}$$

Which is also stated more generally in that same chapter, for an abstract configuration space $\{x\} = \{x_i\}$, as Eq. (5):

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\})$$

Thus, deriving each part of (22) with our Lagrangian (23), and considering the *definition* of a potential energy $V(x)$ (24) yields:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L = \frac{d}{dt}m\dot{x} \qquad\qquad \frac{\partial}{\partial x}L = \frac{\partial}{\partial x}V(x)$$
$$= m\ddot{x} \qquad\qquad\qquad\qquad = -F$$

Then indeed, Euler-Lagrange equations become equivalent to Newton's law of motion:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0$$
$$\Leftrightarrow m\ddot{x} - (-F) = 0$$
$$\Leftrightarrow \boxed{F = m\ddot{x} = ma} \quad \square$$

More Particles and More dimensions

Exercise 2/6

Exercise 21. *Show that Eq. (6) is just another form of Newton's equation of motion $F_i = m_i\ddot{x}_i$.*

Where Eq. (6) are the following set of equation, defined for all $i \in [\![1, n]\!]$:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}L\right) = \frac{\partial}{\partial x_i}L \tag{25}$$

Remark 39. *This exercice is simply a generalization of the previous exercice (L06E01) to a configuration space of size $n \in \mathbb{N}$.*

Then again, let us recall the Lagrangian defined slightly earlier in the related section of the book:

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} m_i \dot{x}_i^2 \right) - V(\{x\}) \tag{26}$$

Hence, $(\forall i \in [\![1, n]\!])$:

$$\frac{\partial}{\partial \dot{x}_i} L = \frac{\partial}{\partial \dot{x}_i} \sum_{j=1}^{n} \frac{1}{2} m_j \dot{x}_j^2 \qquad\qquad \frac{\partial}{\partial x_i} L = -\frac{\partial}{\partial x_i} V(\{x\})$$

$$= \sum_{j=1}^{n} m_j \dot{x}_j \delta_{ij}$$

$$= m_i \dot{x}_i \tag{27}$$

Again, we need the *potential energy principle*, stated as Eq. (5) of the previous chapter *Lecture 5: Energy*, for abstract configuration space $\{x\} = \{x_i\}$, as:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \tag{28}$$

From which we can conclude, by injecting (28) in the second half of (27), and connecting each side with Euler-Lagrange's equations (25), $(\forall i \in [\![1, n]\!])$:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L \right) = \frac{\partial}{\partial x_i} L$$

$$\Leftrightarrow \frac{d}{dt} m_i \dot{x}_i = F_i(\{x\})$$

$$\Leftrightarrow \boxed{F_i = m_i \ddot{x}_i} \quad \square$$

What's Good about Least Action?

Exercise 3/6

Exercise 22. *Use the Euler-Lagrange equations to derive the equations of motions from the Lagrangian in Eq. (12).*

Again, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: $(\forall i \in [\![1, n]\!])$,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L \right) = \frac{\partial}{\partial x_i} L \tag{29}$$

In the case of this exercice, the Lagrangian L is defined in Eq. (12) as:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X)$$

Let's compute the partial derivatives of L on $\dot{X}$, X, $\dot{Y}$ and Y:

$$\frac{\partial}{\partial \dot{X}} L = \frac{\partial}{\partial \dot{X}}\left(\frac{m}{2}\dot{X}^2 + m\omega\dot{X}Y \right) \qquad\qquad \frac{\partial}{\partial X} L = \frac{\partial}{\partial X}\left(\frac{m\omega^2}{2}X^2 - m\omega\dot{Y}X \right)$$

$$= m\dot{X} + m\omega Y \qquad\qquad\qquad\qquad = m\omega^2 X - m\omega\dot{Y}$$

$$\frac{\partial}{\partial \dot{Y}} L = \frac{\partial}{\partial \dot{Y}}\left(\frac{m}{2}\dot{Y}^2 - m\omega\dot{Y}X \right) \qquad\qquad \frac{\partial}{\partial Y} L = \frac{\partial}{\partial Y}\left(\frac{m\omega^2}{2}Y^2 + m\omega\dot{X}Y \right)$$

$$= m\dot{Y} - m\omega X \qquad\qquad\qquad\qquad = m\omega^2 Y + m\omega\dot{X} \tag{30}$$

Finally, by plugging (30) into (29), we obtain:

$$\frac{d}{dt}\left(m\dot{X} + m\omega Y\right) = m\omega^2 X - m\omega\dot{Y} \qquad\qquad \frac{d}{dt}\left(m\dot{Y} - m\omega X\right) = m\omega^2 Y + m\omega\dot{X}$$

$$\Leftrightarrow \boxed{m\ddot{X} = m\omega^2 X - 2m\omega\dot{Y}} \qquad\qquad\qquad \Leftrightarrow \boxed{m\ddot{Y} = m\omega^2 Y + 2m\omega\dot{X}} \quad \square$$

Remark 40. *Those results indeed matches the equations proposed in the book just slightly before this exercice.*

Exercise 4/6

Exercise 23. *Work out George's Lagrangian and Euler-Lagrange equations in polar coordinates.*

As always, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: ($\forall i \in [\![1, n]\!]$),

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \tag{31}$$

The original Lagrangian L in our case is defined by the Eq. (10) of this chapter as:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{32}$$

After the following coordinate shift (Eq. (9) of the book):

$$x = X\cos(\omega t) + Y\sin(\omega t) \qquad\qquad y = -X\sin(\omega t) + Y\cos(\omega t) \tag{33}$$

We obtained this Lagrangian (Eq. (12) of the book):

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X) \tag{34}$$

For the current exercice, the coordinate shift to polar equations is:

$$X = R\cos\theta \qquad\qquad\qquad Y = R\sin\theta \tag{35}$$

Where, implicitly, both R and θ are, as X and Y, functions of time.

Now, we have at least two ways of solving this exercice:

1. Either perform the coordinate shit (35) in (34): this will be a tedious but very similar developement as the one performed in the book to obtain (34) from (32) and (33);

2. or perform this new coordinate shift (35) directly in the first coordinate shift (33), and work from the first Lagrangian (32) instead: some trigonometric identities are likely to ease at least the beginning of the work here.

We will try both aproaches, and expect to find the exact same solutions in the end.

First approach

Let use start by computing the time derivative of X and Y as defined by (35), using the both the product [5] and the chain rule [6]:

$$\dot{X} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta \qquad\qquad \dot{Y} = \dot{R}\sin\theta + R\dot{\theta}\cos\theta \tag{36}$$

Remark 41. *For clarity, as a similar developement will happen a few times, let's go into details for the first one: the product rule for two functions u and v of a single variable, of respective derivatives u' and v' is*

$$(uv)' = u'v + uv'$$

---
[5] https://en.wikipedia.org/wiki/Product_rule
[6] https://en.wikipedia.org/wiki/Chain_rule

Now the chain rule is, again for the same kind of functions:

$$(u(v(x)))' = v'(x)u'(v(x))$$

In the present case, we have a $X(t)$ defined as the product of two functions: $X(t) = R(t)\cos(\omega(t))$, where the second one is itself a composition of two functions $\cos(\omega(t))$. Hence, by applying first the product rule, we obtain:

$$X'(t) = R'(t)\cos(\omega(t)) + R(t)\left(\cos(\omega(t))\right)'$$

While the chain rule gives us:

$$(\cos(\omega(t)))' = -\omega'(t)\sin(\omega(t))$$

Hence,

$$X'(t) = R'(t)\cos(\omega(t)) - R(t)\omega'(t)\sin(\omega(t))$$

Now, our goal will be to plug (35) and (36) into the Lagrangian (34) obtained after the first coordinate shift, but doing that transformation at once will gives a difficult to read equation. Instead, we'll work in smaller steps, simplying our results using trigonometric identities along the way.

Let us start with $X^2 + Y^2$, using the fact that $\sin^2\theta + \cos^2\theta = 1$:

$$
\begin{aligned}
X^2 + Y^2 &= R^2\cos^2\theta + R^2\sin^2\theta \\
&= R^2(\cos^2\theta + \sin^2\theta) \\
&= R^2
\end{aligned}
\tag{37}
$$

Now for $\dot{X}^2 + \dot{Y}^2$, using the same trigonometric identity:

$$
\begin{aligned}
\dot{X}^2 &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)^2 \\
&= \dot{R}^2\cos^2\theta - 2R\dot{R}\dot{\theta}\cos\theta\sin\theta + R^2\dot{\theta}^2\sin^2\theta
\end{aligned}
\qquad
\begin{aligned}
\dot{Y}^2 &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)^2 \\
&= \dot{R}^2\sin^2\theta + 2R\dot{R}\dot{\theta}\sin\theta\cos\theta + R^2\dot{\theta}^2\cos^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{X}^2 + \dot{Y}^2 &= \dot{R}^2(\cos^2\theta + \sin^2\theta) + R^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \\
&= \dot{R}^2 + R^2\dot{\theta}^2
\end{aligned}
\tag{38}
$$

Finally, for $\dot{X}Y - \dot{Y}X$:

$$
\begin{aligned}
\dot{X}Y &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)R\sin\theta \\
&= R\dot{R}\cos\theta\sin\theta - R^2\dot{\theta}\sin^2\theta
\end{aligned}
\qquad
\begin{aligned}
\dot{Y}X &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)R\cos\theta \\
&= R\dot{R}\cos\theta\sin\theta + R^2\dot{\theta}\cos^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{X}Y - \dot{Y}X &= -R^2\dot{\theta}(\sin^2\theta + \cos^2\theta) \\
&= -R^2\dot{\theta}
\end{aligned}
\tag{39}
$$

Now we're ready to plug (37), (38) and (39) into (34):

$$
L = \boxed{\frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}}
\tag{40}
$$

Now, let's compute the partial derivatives of our new Lagrangian:

$$
\begin{aligned}
\frac{\partial}{\partial\dot{R}}L &= \frac{\partial}{\partial\dot{R}}\left(\frac{m}{2}\dot{R}^2\right) \\
&= m\dot{R}
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{\partial}{\partial R}L &= \frac{\partial}{\partial R}\left(\frac{m}{2}R^2\dot{\theta}^2 + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}\right) \\
&= (\dot{\theta}^2 + \omega^2 - 2\omega\dot{\theta})mR \\
&= (\dot{\theta} - \omega)^2 mR
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\dot{\theta}}L &= \frac{\partial}{\partial\dot{\theta}}\left(\frac{m}{2}R^2\dot{\theta}^2 - m\omega R^2\dot{\theta}\right) \\
&= mR^2(\dot{\theta} - \omega)
\end{aligned}
\qquad\qquad
\frac{\partial}{\partial\theta}L = 0
\tag{41}
$$

And from there, plug (41) in Euler-Lagrange (31) to derive the equations of motion (again for the second one, we use a combination of the product and chain rules for derivatives):

$$\frac{d}{dt}\left(m\dot{R}\right) = (\dot{\theta} - \omega)^2 mR \qquad\qquad \frac{d}{dt}\left(mR^2(\dot{\theta} - \omega)\right) = 0$$

$$\Leftrightarrow \boxed{\ddot{R} = (\dot{\theta} - \omega)^2 R} \qquad\qquad \Leftrightarrow m\left((\dot{\theta} - \omega)2\dot{R}R + R^2\ddot{\theta}\right) = 0$$

$$\Leftrightarrow \boxed{R\ddot{\theta} = (\omega - \dot{\theta})2\dot{R}} \quad \square$$

Second approach

We'll now try to see if we can get a cleaner derivation, hopefully with the same results, by combining the two coordinate shifts (33) and (35) first, and then rely on the original Lagrangian (32).

The combined coordinate shift is:

$$x = R\cos\theta\cos(\omega t) + R\sin\theta\sin(\omega t)$$
$$y = -R\cos\theta\sin(\omega t) + R\sin\theta\cos(\omega t)$$

We have the four following trigonometric identities[7]:

$$cos\theta\cos\varphi = \frac{\cos(\theta - \varphi) + cos(\theta + \varphi)}{2} \qquad\qquad sin\theta\sin\varphi = \frac{\cos(\theta - \varphi) - cos(\theta + \varphi)}{2}$$

$$cos\theta\sin\varphi = \frac{\sin(\theta + \varphi) - sin(\theta - \varphi)}{2} \qquad\qquad sin\theta\cos\varphi = \frac{\sin(\theta + \varphi) + sin(\theta - \varphi)}{2}$$

Hence the coordinate shift can be rewritten:

$$x = R\cos(\theta - \omega t)$$
$$y = R\sin(\theta - \omega t) \tag{42}$$

To inject it in the original Lagrangian (32), we need to compute $\dot{x}^2 + \dot{y}^2$. For the derivation, as previously, we'll rely on a combination of the product/chain rule; we'll note $\varphi = \theta - \omega t$:

$$\dot{x} = \dot{R}\cos\varphi - R(\dot{\theta} - \omega)\sin\varphi$$
$$\dot{y} = \dot{R}\sin\varphi + R(\dot{\theta} - \omega)\cos\varphi$$

$$\dot{x}^2 = \dot{R}^2\cos^2\varphi - 2R\dot{R}(\dot{\theta} - \omega)\cos\varphi\sin\varphi + R^2(\dot{\theta} - \omega)^2\sin^2\varphi$$
$$\dot{y}^2 = \dot{R}^2\sin^2\varphi + 2R\dot{R}(\dot{\theta} - \omega)\cos\varphi\sin\varphi + R^2(\dot{\theta} - \omega)^2\cos^2\varphi$$

Hence the Lagrangian becomes, again using the pythagorean trigonometric identity $\cos^2\theta + \sin^2\theta = 1$:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \boxed{\frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta} - \omega)^2\right)}$$
$$= \frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}^2 - 2\dot{\theta}\omega + \omega^2)\right)$$
$$= \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m}{2}R^2\omega^2 - m\omega R^2\dot{\theta}$$

Which is the same Lagrangian we had before in (40), from which we would obviously derive the exact same equation of motion. $\square$.

Remark 42. *As expected, the derivation is overall less tedious, but only because the complexity is now hidden behind the trigonometric identities.*

---

[7]https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities

Remark 43. *A little later in the book, a solution to this exercice is proposed: it starts with this Lagrangian:*

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Which is exactly our Lagrangian, however assuming for some reason that $\omega = 0$. From which follows the same equation of motions, again with the same assumption regarding ω:

$$\ddot{r} = r\dot{\theta}^2$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

Let's remind ourselves that ω represents the rotation of the polar coordinate system of the present exercice, a rotation which won't exist for a general polar coordinate system, hence the reason we have $\omega = 0$ in the general case.

Generalized Coordinates and Momenta

Exercise 5/6

Cyclic coordinates

Exercise 6/6

Exercise 24. *Explain how we derived this.*

Let us recall that "this" refers to the following expression for the kinetic energy:

$$T = m(\dot{x_+}^2 + \dot{x_-}^2)$$

Starting from the following Lagrangian, involving two particles moving on a line with respective position and velocity x_i, $\dot{x}_i$:

$$L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) - V(x_1 - x_2) \tag{43}$$

After having performed the following change of coordinates:

$$x_+ = \frac{x_1 + x_2}{2} \qquad\qquad x_- = \frac{x_1 - x_2}{2} \tag{44}$$

From the Lagrangian, (43) we have the kinetic energy:

$$T = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \tag{45}$$

By first both summing and substracting the two equations of (44), and then by linearity of the derivation, we get:

$$\begin{aligned} x_+ + x_- &= x_1 & x_+ - x_- &= x_2 \\ \dot{x_+} + \dot{x_-} &= \dot{x_1} & \dot{x_+} - \dot{x_-} &= \dot{x_2} \end{aligned} \tag{46}$$

It's now simply a matter of injecting (46) into (45):

$$\begin{aligned} T &= \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= \frac{m}{2}((\dot{x_+} + \dot{x_-})^2 + (\dot{x_+} - \dot{x_-})^2) \\ &= \frac{m}{2}(2\dot{x_+}^2 + 2\dot{x_-}^2 + 2\dot{x_+}\dot{x_-} - 2\dot{x_+}\dot{x_-}) \\ &= m(\dot{x_+}^2 + \dot{x_-}^2) \quad \square \end{aligned}$$

Lecture 7: Symmetries and Conservation Laws

Preliminaries

Exercise 1/7

Exercise 25. *Derive Equations (2) and explain the sign difference.*

Let us recall Equations (2):

$$\dot{p_1} = -V'(q_1 - q_2) \qquad\qquad \dot{p_2} = +V'(q_1 - q_2)$$

We have to derive them from the Lagrangian given in Equation (1), which represents a system of two generalized coordinates q_1 and q_2:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1 - q_2) \tag{47}$$

To retrieve the equations of motions from a Lagrangian, we need to use Euler-Lagrange's equations, for instance recalled as Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Let us also recall, again from previous chapter, right after Equation (13), that the conjugate momentum is defined by

$$p_i = \frac{\partial}{\partial \dot{q_1}}L$$

For our Lagrangian (47), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}}L = \dot{q_1} \qquad\qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}}L = \dot{q_2} \tag{48}$$

$$\frac{d}{dt}p_1 = \dot{p_1} = \ddot{q_1} \qquad\qquad \frac{d}{dt}p_2 = \dot{p_2} = \ddot{q_2} \tag{49}$$

Using the chain rule[8] for the other half, with $\varphi(q_i) = q_1 - q_2$, we get:

$$
\begin{aligned}
\frac{\partial}{\partial q_1}L &= -\frac{\partial}{\partial q_1}V(\varphi(q_1)) & \frac{\partial}{\partial q_2}L &= -\frac{\partial}{\partial q_2}V(\varphi(q_2))\\
&= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1}V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2}V(\varphi(q_2))\\
&= -(\frac{\partial}{\partial q_1}V)(q_1 - q_2) & &= +(\frac{\partial}{\partial q_2}V)(q_1 - q_2) \quad (50)
\end{aligned}
$$

By noting $V' = \dfrac{\partial}{\partial q_i}V$, and combining equations (48), (49) and (50), we indeed obtain the expected equations of motion $\square$.

Remark 44. *That is, assuming, $\dfrac{\partial}{\partial q_1}V(q_1) = \dfrac{\partial}{\partial q_2}V(q_2)$: for all the energy potential presented earlier in the book, there's indeed such a symmetry, e.g.*

$$V = \frac{1}{2}k(x^2 + y^2), \qquad\qquad p103$$

$$V = \frac{1}{2}\frac{k}{x^2 + y^2}, \qquad\qquad p103$$

$$V = -m\omega^2(X^2 + Y^2), \qquad\qquad p120$$

A similar tacit assumption seems to exists in Herbert Goldstein's Classical Mechanics[9].

---

[8] https://en.wikipedia.org/wiki/Chain_rule
[9] https://physics.stackexchange.com/a/107141

Mathematically, the sign difference comes from the fact that the potential depends on one side from q_1 and on the other from $-q_2$, which will persist when differentiating the potential V. Physically, it reflects that there's an order relation between the two "positions" q_1 and q_2: one will come before the other, and our potential V depends on this ordering.

Exercise 2/7

Exercise 26. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{51}$$

Because the question is a unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

As for the previous exercice, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Which was in the book followed by the definition of the conjugate momentum p_i:

$$p_i = \frac{\partial}{\partial \dot{q_1}}L$$

For our Lagrangian (51), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}}L = \dot{q_1} \qquad\qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}}L = \dot{q_2} \tag{52}$$

$$\frac{d}{dt}p_1 = \dot{p_1} = \ddot{q_1} \qquad\qquad \frac{d}{dt}p_2 = \dot{p_2} = \ddot{q_2} \tag{53}$$

Using the chain rule[10] for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\frac{\partial}{\partial q_1}L = -\frac{\partial}{\partial q_1}V(\varphi(q_1)) \qquad\qquad \frac{\partial}{\partial q_2}L = -\frac{\partial}{\partial q_2}V(\varphi(q_2))$$

$$= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1}V(\varphi(q_1)) \qquad\qquad = -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2}V(\varphi(q_2))$$

$$= -(a\frac{\partial}{\partial q_1}V)(aq_1 - bq_2) \qquad\qquad = +(b\frac{\partial}{\partial q_2}V)(aq_1 - bq_2) \tag{54}$$

As for the previous exercice, it seems that there a tacit assumption of a symmetry within the potential V so that we can write $V' = \frac{\partial}{\partial q_i}V$; then, combining (52), (53) and (54):

$$\dot{p_1} = -aV'(aq_1 - bq_2) \qquad\qquad \dot{p_2} = +bV'(aq_1 - bq_2)$$

As suggested, let's multiply the first equation by b, the second by a, and sum the result:

$$b\dot{p_1} + a\dot{p_2} = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is a indeed a constant over time, i.e. that it is conserved (over time).

---

[10]https://en.wikipedia.org/wiki/Chain_rule

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particules respectively by a and b, and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance

Examples of symmetries

Exercise 3/7

Exercise 27. *Show that the combination $aq_1 + bq_2$, along with the Lagrangian, is invariant under Equations (7).*

Let us first recall the equations for the potential (Equations (3)):

$$V(q_1, q_2) = V(aq_1 - bq_2)$$

Which is meant to be considered in the case of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{55}$$

Finally, the "Equations (7)" relate to the following change of coordinates:

$$q_1 \to q_1 - b\delta$$
$$q_2 \to q_2 + a\delta \tag{56}$$

Remark 45. *There are typos around here in the book. In my printed version, it is as previously described, but in an online version, it is given by (mind the signs):*

$$q_1 \to q_1 + b\delta$$
$$q_2 \to q_2 - a\delta$$

yet in that same online version, the potential is said to depend on $aq_1 + bq_2$ in accordance to Equations (3), but said Equations (3) actually make it depend on $aq_1 - bq_2$!

To summarize, with a $V(aq_1 + bq_2)$, the two previous transformations will keep the Lagrangian unchanged. But with a $V(aq_1 - bq_2)$, none of the previous transformations will keep the Lagrangian; those two will:

$$
\begin{array}{ll}
q_1 \to q_1 - b\delta & \qquad q_1 \to q_1 + b\delta \\
q_2 \to q_2 - a\delta & \qquad q_2 \to q_2 + b\delta
\end{array}
\tag{57}
$$

In what follows, we will arbitrarily assume a $V(aq_1 - bq_2)$, and, say, the first transformation of (57).

Assuming a, b and δ are time-invariant, it follows that $\dot{q_1}$ and $\dot{q_2}$ are unchanged by this transformation, hence

$$\dot{q_i} \to \dot{q_i}$$
$$\dot{q_1}^2 + \dot{q_2}^2 \to \dot{q_1}^2 + \dot{q_2}^2$$

Injecting (56) into (55) gives us the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(a(q_1 - b\delta) - b(q_2 - a\delta))$$
$$= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2)$$

We can see that indeed, the Lagrangian is unchanged; because the $\dot{q_i}$ are also unchanged, we would derive the exact same equation of motions as we did for the previous exercise.

Exercise 4/7

Exercise 28. *Show this to be true.*

Where "this" refers to the fact that this Lagrangian (Equation (8)):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \tag{58}$$

does not change to first order in δ, for the infinitesimal transformation described by e.g. Equations (12):

$$\delta_v x = y\delta$$
$$\delta_v y = -x\delta \tag{59}$$

The transformation to the derivatives over time of x and y has already been established in Equations (11):

$$\dot{x} \to \dot{x} + \dot{y}\delta$$
$$\dot{y} \to \dot{y} - \dot{x}\delta \tag{60}$$

Let's then perform the substitution described by (59) and (60) in the Lagrangian (58):

$$L = \frac{m}{2}((\dot{x} + \dot{y}\delta)^2 + (\dot{y} - \dot{x}\delta)^2) - V((x + y\delta)^2 + (y - x\delta)^2)$$

$$= \frac{m}{2}\left(\left(\dot{x}^2 + 2\dot{x}\dot{y}\delta + (\dot{y}\delta)^2\right) + \left(\dot{y}^2 - 2\dot{y}\dot{x}\delta + (\dot{x}\delta)^2\right)\right)$$

$$- V\left(\left(x^2 + 2xy\delta + (y\delta)^2\right) + \left(y^2 - 2yx\delta + (x\delta)^2\right)\right)$$

$$= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \delta^2(\dot{x}^2 + \dot{y}^2)\right) - V\left(x^2 + y^2 + \delta^2(x^2 + y^2)\right)$$

Now, as we care about first-order changes in δ only, changes proportional to $\delta^n|_{n \geq 2}$ will be negligible; it follows that the Lagrangian is indeed unchanged:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \quad \square$$

Back to examples

Exercise 5/7

Lecture 8: Hamiltonian Mechanics and Time-Translation Invariance

Time-Translation Symmetry

Energy Conservation

Phase Space and Hamilton's Equations

The Harmonic Oscillator Hamiltonian

Exercise 1/2

Exercise 2/2

Derivation of Hamilton's Equations