

# The Theoretical Minimum

## Classical Mechanics - Solutions

L03E04

M. Bivert

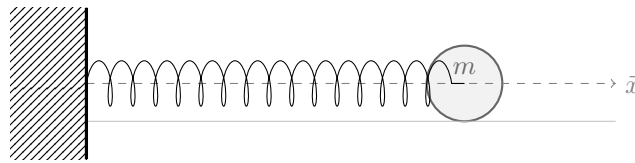
September 9, 2022

**Exercise 1.** Show by differentiation that the general solution to Eq. (6) is given in terms of two constants  $A$  and  $B$  by

$$x(t) = A \cos \omega t + B \sin \omega t$$

Determine the initial position and velocity at time  $t = 0$  in terms of  $A$  and  $B$ .

We're in the case of a 1-dimensional harmonic oscillator, depicted below as a mass  $m$  set on a frictionless ground attached to a fixed element to the left by a horizontally positioned spring of constant  $k$ .



**Remark 1.** Almost equivalently, we could have worked in a vertical setting, which, on one hand, would have avoided us the need for a frictionless ground, but on the other hand, would make the equation slightly more complicated.

There are interesting aspects to both cases, so because the difference between both is so small in the end, we'll discuss both systems.

Harmonic oscillators play a major role in physics, as explained by Feynman in Chapter 21 of the first volume of his *Lectures on Physics*:

The harmonic oscillator, which we are about to study, has close analogs in many other fields; although we start with a mechanical example of a weight on a spring, or a pendulum with a small swing, or certain other mechanical devices, we are really studying a certain differential equation. This equation appears again and again in physics and in other sciences, and in fact it is a part of so many phenomena that its close study is well worth our while.

Some of the phenomena involving this equation are the oscillations of a mass on a spring; the oscillations of charge flowing back and forth in an electrical circuit; the vibrations of a tuning fork which is generating sound waves; the analogous vibrations of the electrons in an atom, which generate light waves; the equations for the operation of a servosystem, such as a thermostat trying to adjust a temperature; complicated interactions in chemical reactions; the growth of a colony of bacteria in interaction with the food supply and the poisons the bacteria produce; foxes eating rabbits eating grass, and so on; all these phenomena follow equations which are very similar to one another, and this is the reason why we study the mechanical oscillator in such detail.

Thus, we'll take the time to analyze it more precisely than what is expected here: this is the occasion to present some aspects not covered, or not covered as explicitly in the book.

Despite everything happening in one-dimension, we'll use a vector position  $\mathbf{x}$ , which is a function of time. Hence, the vectors for speed and acceleration will respectively be  $\mathbf{v} = \dot{\mathbf{x}} = \frac{d}{dt}\mathbf{x}(t)$  and  $\mathbf{a} = \ddot{\mathbf{x}} = \frac{d^2}{dt^2}\mathbf{x}(t)$ .

Note that we're representing vectors with bold font instead of arrows; bold font is rather common in physics. This should be the only additional thing needed to follow through for someone having read the book up to this stage.

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### Contact force, normal force, friction force

The contact force is the force resulting from the contact of two objects. It is generally decomposed into vertical and horizontal components. The horizontal component perhaps is the most intuitive: it corresponds to friction forces. For instance, after having given a gentle push to an object on an ordinary table, it will only move up to a certain point: it is progressively stopped by the friction between the object and the table. This friction is caused in part because at a microscopic scale, both the object and the surface are far, far from being perfectly plane, even more so at an atomic scale.

The normal force is the vertical component of the contact force. An object on a table is affected by the Earth gravity, but naturally resists going through the table, unless of course, the object is really massive and/or the table very weak. There's a bunch of complicated interactions at the atomic level from which this situation occurs: at a macroscopic scale, we simplify things and wrap this complexity by abstracting it as a (macroscopic) force.

**Remark 2.** *In the case of the vertical setup, with a mass attached to a vertical spring, the mass is not in contact with a ground surface, so there would have been no need to discuss contact forces.*

**Remark 3.** *A static object on a flat surface corresponds to a special case where the horizontal component of the contact force (friction) is null. Hence, the contact force and the normal force are, in such a special case, one and the same. The exact same thing would happen in the case of a frictionless surface.*

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### Newton's laws

Let us start by recalling Newton's laws of motion:

1. **Principle of inertia:** Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it;
2. **" $\mathbf{F} = m\mathbf{a}$ ":** The change of motion [*momentum*] of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed;
3. **"action  $\Rightarrow$  reaction":** To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

**Remark 4.** *A force always is applied from an object, to another object; with the restriction that an object cannot apply a force to itself.*

**Remark 5.** *The second law is actually a statement about momentum, from which we can derive  $\mathbf{F} = m\mathbf{a}$ . More precisely, the momentum, called "motion" by Newton, is defined as  $\mathbf{p} = m\mathbf{v}$ , which literally captures the idea of the speed  $\mathbf{v}$  of a certain amount of matter  $m$ . Then, it follows that the general case,  $\mathbf{p}$  is as  $\mathbf{v}$  a function of time.*

*The "change of motion" then refers to an infinitesimal change of the momentum over time, mathematically captured by the time derivative  $\frac{d}{dt}\mathbf{p}$ . The law can then be progressively written, assuming the mass  $m$  is constant over time:*

$$\mathbf{F} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}m\mathbf{v} = m\frac{d}{dt}\mathbf{v} = m\mathbf{a} \quad \square$$

**Remark 6.** *It may seem weird to mention in the previous Remark 5 that we assume the mass  $m$  to be constant over time. In the current context, this will be the case, but in general, likely contrary to what any reasonable human would expect, they are exceptions. More precisely, in special relativity, the mass is a function of the velocity  $v$ :*

$$m_{\text{rel}} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

*We won't delve into the details here, but note that if  $v = \|\mathbf{v}\|$  is much smaller than  $c$ , the speed of light, then the relative mass  $m_{\text{rel}}$  and  $m$  are very close.*

*To say it otherwise, this assumption doesn't hold at high velocities.*

**Remark 7.** In the second law, the force impressed (" $\mathbf{F}$ ") is actually the force resulting from all the other forces applied to an object,  $\sum_i \mathbf{F}_i$ , which is sometimes called  $\mathbf{F}_{\text{net}}$ .

For instance, if you push a cart,  $\mathbf{F}_{\text{net}}$  should contain at least the force you are exerting, the gravity exerted by Earth, and the contact force generated by the ground on the cart's wheel. The sum of all the forces applied to the cart that will ultimately influence its motion.

As a result, in practice, when analyzing a situation so as to establish the equation of motion of an object, one should start by identifying all the external forces applied to this object.

**Remark 8.** A common special case of the second law is to consider objects that, within a certain frame of reference, do not move. Hence, by definition, their speed  $\mathbf{v}$  would be 0, and so would be their acceleration. That is,  $\sum_i \mathbf{F}_i = \mathbf{0}$ .

Another special case for the second law is when things move at constant speed. Again, the acceleration would then be  $\mathbf{0}$  and thus still  $\sum_i \mathbf{F}_i = \mathbf{0}$ .

**Remark 9.** It is interesting to spend a moment to ponder what the mass  $m$  of an object really "is". Intuitively, for a non-physicist mass is another word for weight, but not to a physicist.

Instead, in the context of the second law, physicists will observe that the more mass there is, the more force will be needed to accelerate an object. Conversely, the less mass there is, the less force will be needed to alter the motion of an object. Thus, they would conceptualize the mass as a **measure of resistance to acceleration** (change of movement).

This is the second time we meet a subtlety regarding masses, the first one being Remark 6 about how mass is a function of the velocity. We'll come back later once again in a moment at some more subtlety regarding mass.

**Remark 10.** In the case of a static object laying on a flat surface, one may think that the gravitational force exerted by the Earth on the object and the normal force/contact force arising from the contact of the object with the surface are two opposite forces, in the sense of Newton's third law: after all, they are indeed of equal intensity and opposite directions.

But this is incorrect: in such a situation, we actually have two pairs of opposite forces, in the sense of Newton's third law:

1.  $\mathbf{F}_{o,e}$  and  $\mathbf{F}_{e,o}$ , respectively the gravitational force exerted by the object on Earth, and the gravitational force exerted by Earth on the object;
2.  $\mathbf{F}_{o,t}$  and  $\mathbf{F}_{t,o}$ , respectively the contact force exerted by the object on the table, and the contact force exerted by the table on the object.

Now to be more complete, there's even a third pair of forces, which would be in almost all practical situations negligible, namely, the gravitational forces exerted by the surface on the object, and by the object on the surface.

A good exercise to check one understanding of this topic would be to try to identify forces in the case of a hand pressed on a (static) wall, or in the case of two hands pressed against each other.

**Remark 11.** Newton's third law is actually a consequence of the principle of conservation of momentum: essentially, in a closed system, the total momentum is a constant over time.

This is a key idea in modern physics, and will be key later in the book.

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### Hooke's law

Hooke's law, as Newton's laws, is an *empirical law*, that is, which has been found by repeated experimentation. Essentially, it states that the force needed to extend or compress a spring by a given distance varies linearly with the distance during which the spring is extended/compressed. That linear factor  $k$  is the spring's constant. This force is thus a function of that distance/displacement  $\delta$ :

$$\mathbf{F}_s(\delta) = k\delta$$

The equation could be read as something like:

- If I don't compress/stretch the spring ( $\delta = 0$ ), no force is exerted;
- The more I compress/stretch the spring, the greater the force, where the relation between both varies linearly.

**Remark 12.** In most practical cases,  $\delta$  will be a function of time  $\delta(t)$ ; thus, the force itself will be a function of time:  $\mathbf{F}_s(\delta(t))$ .

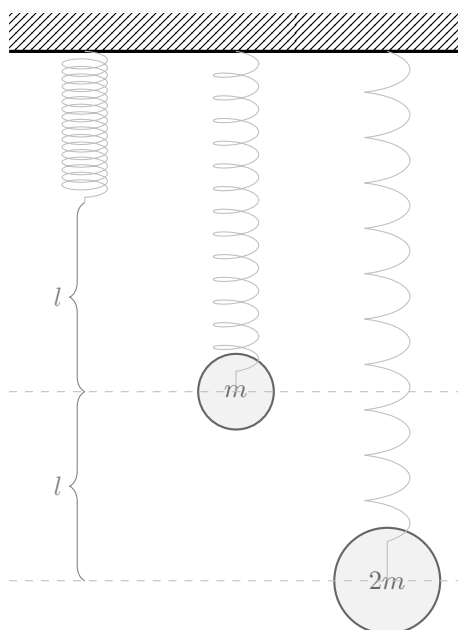
**Remark 13.** By Newton's third law, if there's an object exerting a force on the spring, then the spring also exerts a force on the object, of equal intensity, but in the opposite direction. Hence, equivalently, Hooke's law could be reformulated in the form presented in the book:

$$\mathbf{F}_s(\delta) = -k\delta$$

**Remark 14.** You may sometimes find that last version of the law, as is the case in the book, written as  $\mathbf{F}_s(\mathbf{x}) = -k\mathbf{x}$ . This is a convenient choice that physicists can make, where, by identifying the origin of the coordinate system as the rest position of the mass, the measure of displacement  $\delta$  will indeed always match the position vector  $\mathbf{x}$ .

This can be summarized by saying that  $\mathbf{F}_s$  is the force pulling the mass back to its rest position.

**Remark 15.** If you have a spring or two around (architect lamps are a good source) and some regular weight (small tableware, such as iron tea cups would do), you should be able to easily verify this law experimentally. In this case, the objects' weight will be the source of a gravitational force (we'll come back to the gravitational force in a moment): doubling the mass (attaching two cups) should double the deformation of the spring.



**Remark 16.** Were you at least to imagine the previous experiment, you would realize that there must be some limits to this law: the spring could break or be irredeemably deformed. Hooke's law is actually a first-order linear approximation, that works reasonably well in plenty of practical cases. There are more advanced models who generalize Hooke's law, but they are of no use to us in this context.

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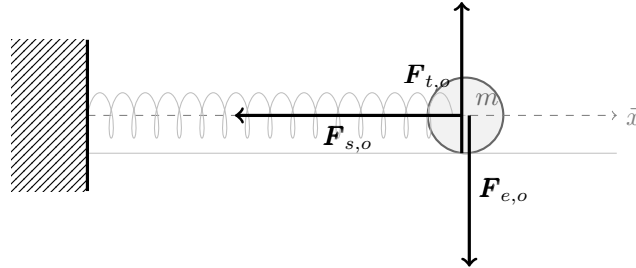
### Forces diagram

The goal of a force diagram, or a free body diagram, is to identify the forces acting on an object, with the intent of applying Newton's second law, so as to later determine an equation of movement for that object: we want to identify the sum of all the forces **acting on the object**,  $\mathbf{F}_{\text{net}} = \sum_i \mathbf{F}_i$ , who are likely to contribute to the shape of its trajectory. Let's complete the diagram of our horizontal spring setup with:

$\mathbf{F}_{e,o}$  the gravitational force exerted by the earth on our object. Note that we haven't talked about gravitation yet, and as you'll see in a moment, we don't need to, but this will be necessary for the vertical spring setup;

$\mathbf{F}_{t,o}$  the normal force exerted by the table on our object. Because the surface is frictionless, the normal force *is* the contact force;

$\mathbf{F}_{s,o}$  the force exerted by the spring on our object, which can be described by Hooke's law.



**Placement of the origin of the  $\vec{x}$ -axis** As we've already observed before, it can be interesting to identify the origin of our coordinate system to the position where the system will be at equilibrium/rest.

**Intensities and signs** Thanks the previous convention, as discussed in Remark 14, Hooke's law can be used to determine the force exerted by the spring on the object, and can be written as  $\mathbf{F}_{s,o} = -k\mathbf{x}$ .

We know, from Newton's second law, that the contact/normal force exerted by the table on the object is equal to  $-\mathbf{F}_{o,t}$ , where  $\mathbf{F}_{o,t}$  is the force exerted by the object on the table, which itself results from the gravitational force  $\mathbf{F}_{e,o}$ , i.e.

$$\mathbf{F}_{t,o} = -\mathbf{F}_{o,t} = -\mathbf{F}_{e,o}$$

As we'll see quickly, we don't actually need to compute anything more, in this situation. We'll nevertheless show how to compute the gravitational force in the next section, when considering the similar case of a mass attached to a vertical spring.

**Applying Newton's second law** We can now use Newton's second law,  $\sum_i \mathbf{F}_i = m\mathbf{a} = m\ddot{\mathbf{x}}$  to deduce an equation of motion:

$$\begin{aligned} \mathbf{F}_{e,o} + \mathbf{F}_{t,o} + \mathbf{F}_{s,o} &= m\ddot{\mathbf{x}} \\ \Leftrightarrow \mathbf{F}_{e,o} - \mathbf{F}_{e,o} + \mathbf{F}_{s,o} &= m\ddot{\mathbf{x}} \\ \Leftrightarrow \mathbf{F}_{s,o} &= m\ddot{\mathbf{x}} \\ \Leftrightarrow -k\mathbf{x} &= m\ddot{\mathbf{x}} \quad \square \end{aligned}$$

Which is a second order, linear differential equation, homogeneous, with constant coefficients. It's often rewritten, as is the case in the book, by defining  $\omega^2 = \frac{k}{m}$ :

$$\ddot{\mathbf{x}} = -\omega^2 \mathbf{x}$$

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### Gravitational force

To handle the case of a vertical spring, we won't be able to brush the gravitational force away as we did previously, and we'll need to compute it.

There are a few different ways to reach a formula for the gravitational force, but perhaps the simplest one, in the context of studying a mass attached to a vertical spring, would be to empirically study the fall of an object of mass  $m$  in Earth's gravitational field.

One could proceed for instance by video-taping a vertically falling object of known mass  $m$ . The video can then be discretized into a series of images, on which we could measure the evolution of the position of the mass over time. From there, we could compute the speed, as the variation of position, and the

acceleration, as the variation of speed.

Then, neglecting the air resistance (well, we *could* perform the experiment in a vacuum), we can pretend that there's a single force acting on the mass, which *is* our gravitational force.

The experiment could obviously be performed multiple times to improve the accuracy. In the end, physicists have found that Earth's gravity is an acceleration vector  $\mathbf{g}$  of intensity  $9.81 \text{ m.s}^{-2}$ , directed towards the center of the Earth, which experimentally gives a gravitational force of:

$$\mathbf{F}_{e,o} = m\mathbf{g}$$

**Remark 17.** *A home-made reenactment of the experiment would be difficult, especially if there's a need for strong precision, which would involve the presence of a vacuum. At least, more difficult than experimentally validating Hooke's law on a single spring.*

*Historically, a simpler and more primitive setup was employed, for instance by Galileo, which would involved measuring the speed of a ball rolling on an inclined plane. A great way to lower the speed of the "fall" and thus to get more accurate time measurements.*

**Remark 18.** *1) Note that  $\mathbf{g}$  captures not only the gravitational force exerted by the Earth, but also captures the contribution of the Earth's rotation (centrifugal force).*

*2) Because the Earth is not a perfect sphere, the value of  $\|\mathbf{g}\|$  is not uniform everywhere on Earth; there are noticeable differences from city to city for instance.*

**Remark 19.** *In Remark 9, we conceptualized the mass as a measure of resistance to acceleration. Physicists call this the inertial mass. And they conceptually distinguish it from what they call the gravitational mass, which is the property of matter that qualify how an object will interact with a gravitational field.*

*We tacitly assumed that both are the same: this is the equivalence principle, a cornerstone of modern theories of gravities. So far, physicists haven't been able to distinguish the two masses experimentally either. But conceptually, they refer to two distinct notions.*

*Mathematically, this means that in the case of a falling object subject to a single force exerted by gravitation, by applying Newton's second law, we have:*

$$\begin{aligned} \mathbf{F}_{net} &= \mathbf{F}_{e,o} = m_i \mathbf{a} \\ \Leftrightarrow \quad m_g \mathbf{g} &= m_i \mathbf{a} \\ \Leftrightarrow \quad \mathbf{g} &= \mathbf{a} \end{aligned}$$

**Remark 20.** *1) Another approach could have been to rely on Newton's law of universal gravitation.*

$$\mathbf{F}_{e,o} = -G \frac{m_e m_o}{\|\mathbf{r}_{e,o}\|^2} \hat{\mathbf{r}}_{e,o}$$

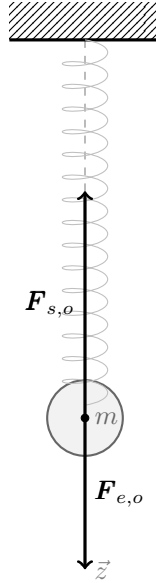
*But not only is this law also empirical, but it would involve knowing the mass of the Earth  $m_e$ , the gravitational constant  $G$ , etc.*

*2) Similarly we could delve into Einstein's theory of general relativity, but the prerequisites would again have been more sophisticated than exposing a basic experiment.*

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## Vertical spring

We now have all we need to study a variant of our previous setup, slightly more complicated mathematically, where a mass  $m$  is attached to the end side of a vertical spring of constant  $k$ , itself firmly attached to a fixed element at its top:



We'll choose the same convention regarding the origin of our coordinate system (see Remark 14). Contrary to the previous case, there's no other force to counterbalance the gravitational force  $\mathbf{F}_{e,o}$ ; applying Newton's second law gives:

$$\begin{aligned} \sum_i \mathbf{F}_i &= \mathbf{F}_{e,o} + \mathbf{F}_{s,o} = m\ddot{\mathbf{x}} \\ \Leftrightarrow \quad m\mathbf{g} - k\mathbf{x} &= m\ddot{\mathbf{x}} \\ \Leftrightarrow \quad \ddot{\mathbf{x}} &= -\omega^2\mathbf{x} + \mathbf{g} \end{aligned}$$

Again, with  $\omega^2 = \frac{k}{m}$ ; so the only difference is that we get an additional constant  $\mathbf{g}$ , but this still is a second-order linear differential equation with constant coefficients.

**Remark 21.** *However, this differential equation is said to be non-homogeneous because of the non-zero constant  $\mathbf{g}$ , by opposition with the previous homogeneous equation obtained from the horizontal spring setup.*

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### Verifying the solution

Solving differential equations, mathematically, can be rather involved. For such simple equations, mathematicians themselves may find it sufficient to make an educated guess by tweaking an exponential-based function, despite having explored the subject with much more depth.

We won't discuss the details here, and will pragmatically satisfy ourselves with verifying that the proposed solution actually solves the equation:

$$x(t) = A \cos \omega t + B \sin \omega t$$

**Remark 22.** *We've been before expressing all our equations with vectors so far, but because we're working in one-dimension, we can divide everything by  $\hat{\mathbf{x}}$  and work from the resulting scalar equations.*

And indeed, the proposed solution does solve the equation for the horizontal spring setup.

$$\begin{aligned} \ddot{x}(t) &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \\ &= -\omega^2 (A \cos \omega t + B \sin \omega t) \\ &= -\omega^2 x(t) \quad \square \end{aligned}$$

But it doesn't solve the vertical setup; we'll get there quickly, this isn't much more complicated. You may want to try by yourself to see if you can tweak the solution of the homogeneous equation to solve the non-homogeneous one.

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**Initial conditions**

Let's start by computing the expression of the velocity:

$$\dot{x}(t) = -A\omega \sin \omega t + B\omega \cos \omega t$$

Then, at  $t = 0$ , we have:

$$\begin{aligned}x(0) &= A \cos(0) + B \sin(0) = \boxed{B} \\ \dot{x}(0) &= -A\omega \sin(0) + B\omega \cos(0) = \boxed{A\omega} \square\end{aligned}$$

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**Equations and initial conditions for the vertical setup**

There are general mathematical methods for reaching a solution to a non-homogeneous linear second-order differential equation with constant coefficient from a homogeneous, but again, suffice for us to verify that the following would work:

$$x(t) = A \cos \omega t + B \sin \omega t + \frac{g}{\omega^2}$$

Indeed:

$$\begin{aligned}\ddot{x}(t) &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \\ &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t - \frac{\omega^2}{\omega^2}g + g \\ &= -\omega^2(A \cos \omega t + B \sin \omega t + \frac{g}{\omega^2}) + g \\ &= -\omega^2 x(t) + g \square\end{aligned}$$

From there, we can again express the initial position and speed as functions of  $A$  and  $B$ , first by computing the expression of the speed:

$$\dot{x}(t) = -A\omega \sin \omega t + B\omega \cos \omega t$$

And then by looking at what's happening at  $t = 0$ :

$$\begin{aligned}x(0) &= A \cos(0) + B \sin(0) + \frac{g}{\omega^2} = \boxed{B + \frac{g}{\omega^2}} \\ \dot{x}(0) &= -A\omega \sin(0) + B\omega \cos(0) = \boxed{A\omega} \square\end{aligned}$$