The Theoretical Minimum

Quantum Mechanics - Solutions

L07E01

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Exercise 1. *Write the tensor product $I \otimes \tau_x$ as a matrix, and apply that matrix to each of the $|uu\rangle$, $|ud\rangle$, $|du\rangle$, and $|dd\rangle$ column vectors. Show that Alice's half of the state-vector is unchanged in each case. Recall that I is the 2×2 unit matrix.*

Recall that τ_x is a Pauli matrix, while I really is the identity matrix:

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We saw two different ways of building $I \otimes \tau_x$. Let's start with the first one: consider the usual ordered basis of the underlying composite space: $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$. Then, the elements of the matrix representation of $I \otimes \tau_x$ in this basis are given by:

$$(I \otimes \tau_x)_{ab,cd} = \langle ab|(I \otimes \tau_x)|cd\rangle$$

We can then use the multiplication table from either the appendix or from L06E04, where, remember, τ_x in this multiplication table was a shortcut notation for $I \otimes \tau_x$.

$$\begin{aligned} \tau_x|uu\rangle &= |ud\rangle; & \tau_x|ud\rangle &= |uu\rangle \\ \tau_x|du\rangle &= |dd\rangle; & \tau_x|dd\rangle &= |du\rangle \end{aligned}$$

And we're now ready to evaluate the operator's matrix form:

$$\begin{aligned} I \otimes \tau_x \text{"} = \text{"} & \begin{pmatrix} \langle uu|(I \otimes \tau_x)|uu\rangle & \langle uu|(I \otimes \tau_x)|ud\rangle & \langle uu|(I \otimes \tau_x)|du\rangle & \langle uu|(I \otimes \tau_x)|dd\rangle \\ \langle ud|(I \otimes \tau_x)|uu\rangle & \langle ud|(I \otimes \tau_x)|ud\rangle & \langle ud|(I \otimes \tau_x)|du\rangle & \langle ud|(I \otimes \tau_x)|dd\rangle \\ \langle du|(I \otimes \tau_x)|uu\rangle & \langle du|(I \otimes \tau_x)|ud\rangle & \langle du|(I \otimes \tau_x)|du\rangle & \langle du|(I \otimes \tau_x)|dd\rangle \\ \langle dd|(I \otimes \tau_x)|uu\rangle & \langle dd|(I \otimes \tau_x)|ud\rangle & \langle dd|(I \otimes \tau_x)|du\rangle & \langle dd|(I \otimes \tau_x)|dd\rangle \end{pmatrix} \\ \text{"} = \text{"} & \begin{pmatrix} \langle uu|ud\rangle & \langle uu|uu\rangle & \langle uu|dd\rangle & \langle uu|du\rangle \\ \langle ud|ud\rangle & \langle ud|uu\rangle & \langle ud|dd\rangle & \langle ud|du\rangle \\ \langle du|ud\rangle & \langle du|uu\rangle & \langle du|dd\rangle & \langle du|du\rangle \\ \langle dd|ud\rangle & \langle dd|uu\rangle & \langle dd|dd\rangle & \langle dd|du\rangle \end{pmatrix} \\ \text{"} = \text{"} & \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \end{aligned}$$

Let's move on to the second way, which consists in using Eq. 7.6 of the book:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}$$

Which then yields:

$$\begin{aligned}
I \otimes \tau_x \text{"} = \text{"} & \begin{pmatrix} 1 \times \tau_x & 0 \times \tau_x \\ 0 \times \tau_x & 1 \times \tau_x \end{pmatrix} \\
\text{"} = \text{"} & \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \\
\text{"} = \text{"} & \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}
\end{aligned}$$

Which is exactly what we've found earlier, albeit less tediously.

In our usual ordered basis $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$, the column representations of the basis vectors are as follow:

$$|uu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad |ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad |du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad |dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Remark 1. *Remember than the column notation is merely a syntactical shortcut over linear combinations of the basis vectors:*

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} := a|uu\rangle + b|ud\rangle + c|du\rangle + d|dd\rangle$$

Remark 2. *Note that we could also have used, as the authors did in the book, Eq.* 7.6 *to derive them.*

Then it's just a matter of computing some elementary matrix×vector products. As a shortcut, one can also recall from one's linear algebra class than such products, when they involve basis vectors, are simply a matter of extracting the columns of the matrix (which is fairly trivial to see):

$$(I \otimes \tau_x)|uu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |ud\rangle; \quad (I \otimes \tau_x)|ud\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |uu\rangle;$$

$$(I \otimes \tau_x)|du\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |dd\rangle; \quad (I \otimes \tau_x)|dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |du\rangle$$

Remark 3. *Naturally, this is consistent with the multiplication table we've recalled earlier; and Alice's part of the state is indeed kept unchanged, as expected.*