The Theoretical Minimum

Classical Mechanics - Solutions

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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January 6, 2025

Abstract

Below are solution proposals to the exercises of *The Theoretical Minimum - Classical Mechanics*, written by Leonard Susskind and George Hrabovsky. An effort has been so as to recall from the book all the referenced equations, and to be rather verbose regarding mathematical details, in line with the general tone of the series.

Contents

Lecture 1: The Nature of Classical Physics

What Is Classical Physics?

Exercise 1/3

Exercise 1. *Since the notion is so important to theoretical physics, think about what a closed system is, and speculate on whether closed systems can actually exist. What assumptions are implicit in establishing a closed system? What is an open system?*

The book defines a closed system as "a collection of objects [...] that is either the entire universe (1) or is so isolated from everything else that it behaves as if nothing else exists (2)". From everyday experience, we *know* there's a lot going on we're not aware of. Thus, we know we can't even hope to *truly* consider the entire universe, but in an abstract sense, at best.

Furthermore, to establish with absolute certainty (2), it is necessary to have a full understanding of everything that exists. For, even were we to build a system isolated from everything we know that could affect it, by the previous assumption, there may be some elements, unknown to us, affecting the system in various ways.

Hence, as for a lot of things in physics, a closed system is but a conceptual tool, a (very) convenient approximation of some aspects of what we think we experience of reality.

To establish our closed system, we must generally assume that it is closed *relatively* to what we are trying to measure/observe. That is, there will be no external factors noticeably/unexpectedly affecting the result of a a given observation/consideration.

As we can only define a closed system as an approximation, we won't bother trying to define an *open system* in absolute terms either. A opened system, is by contrast with a closed system, a collection of objects that isn't neither the entire Universe, nor so isolated from everything else that it behaves as if nothing else exists. More simply, it's a collection of objects affected by its environment.

Exercise 2/3

Exercise 2. *Can you think of a general way to classify the laws that are possible for a six-state system?*

From what precedes, we can think of classifying systems by their number of cycles. We could furthermore refine the classification, for instance by considering the distribution of states per cycle.

The official solutions[1] mention an interesting fact, not directly apparent from the book's examples: there could be cycles within the "main" cycles (and cycles within the "secondary" cycles, etc.), which can also be used to refine the classification.

Rules That Are Not Allowed: The Minus-First Law

Dynamical Systems with an Infinite Number of States

Exercise 3/3

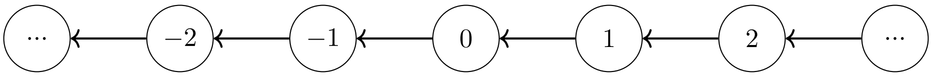
Exercise 3. *Determine which of the dynamical laws shown in Eq.s* (2) *through* (5) *are allowable.*

Let's recall that laws are said to be allowable if they are both deterministic (i.e. the future behavior of the system is completely determined by the initial state) and reversible (i.e. the law is still deterministic even by reversing the direction of all the arrows).

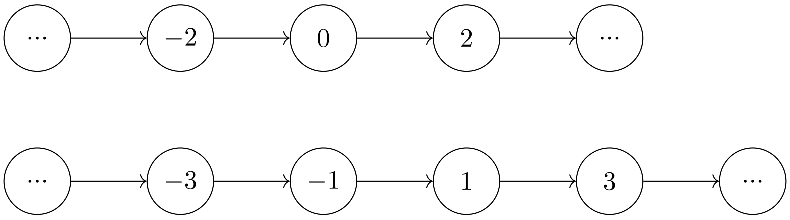
[1] `http://www.madscitech.org/tm/slns/l1e2.pdf`

Now, the dynamical laws referred to be the exercise are:

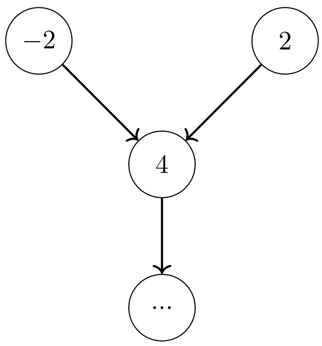
$$N(n+1) = N(n) - 1 \quad (2)$$
$$N(n+1) = N(n) + 2 \quad (3)$$
$$N(n+1) = N(n)^2 \quad (4)$$
$$N(n+1) = -1^{N(n)} N(n) \quad (5)$$



(2) is simply (1) (described by $N(n+1) = N(n) + 1$) with reversed arrows; and (1) has already be established as being allowed. Hence, $\boxed{\text{(2) is allowed}}$.

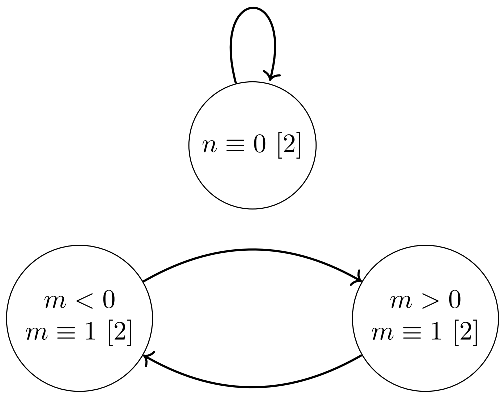


(3) is very similar to (1) or (2), only, it has two infinite cycles instead of one. Both are deterministic, and reversing the arrows lead to deterministic cycles. Hence, $\boxed{\text{(3) is allowed}}$.



We didn't plot (4) exhaustively, just enough to illustrate its irreversibility: if we start at either -2 or 2, we end up on 4. By reversing the arrows, we see that we can't decide where we came from. Furthermore, states such as -2 cannot be reached, because there's no number n such as $-2 = n^2$). Actually, even 2 cannot be reached, because there's no *integer* n such as $2 = n^2$.

Hence, $\boxed{\text{(4) is \textbf{not} allowed}}$.



By sketching a few cycles for (5), we see a pattern emerging, summed up in the diagram above:

- if we start on an even number, positive or negative, we'll loop on this number indefinitely;
- if we start with an odd number, positive or negative, we'll loop between the positive and negative version of that number;

Perhaps a mathematical subtlety would be in considering -1 raised to a negative power. But,

$$(\forall n \in \mathbb{N}^*), \quad -1^{-n} = (-1/1)^{-n} \equiv (1/-1)^n = (-1)^n$$

Hence, $\boxed{\text{(5) is allowed}}$.

Cycles and Conservation Laws

The Limits of Precision

Interlude 1: Spaces, Trigonometry, and Vectors

Coordinates

Exercise 1/6

Exercise 4. *Using a graphic calculator or a program like Mathematica, plot each of the following functions. See the next section if you are unfamiliar with the trigonometric functions.*

$$\begin{aligned} f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\ g(x) &= \sin x - \cos x \\ \theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\ x(t) &= \sin^2 t - \cos t \end{aligned}$$

Figure 1: $f(t) = t^4 + 3t^3 - 12t^2 + t - 6$

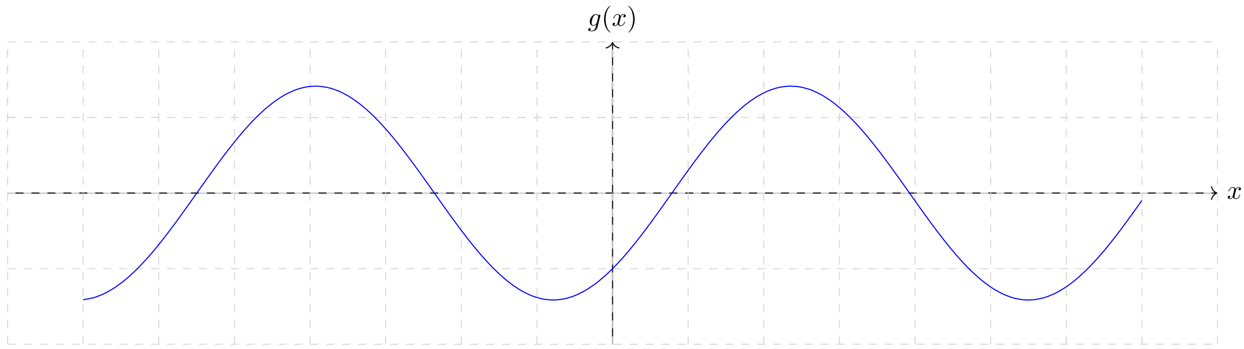


Figure 2: $g(x) = \sin x - \cos x$

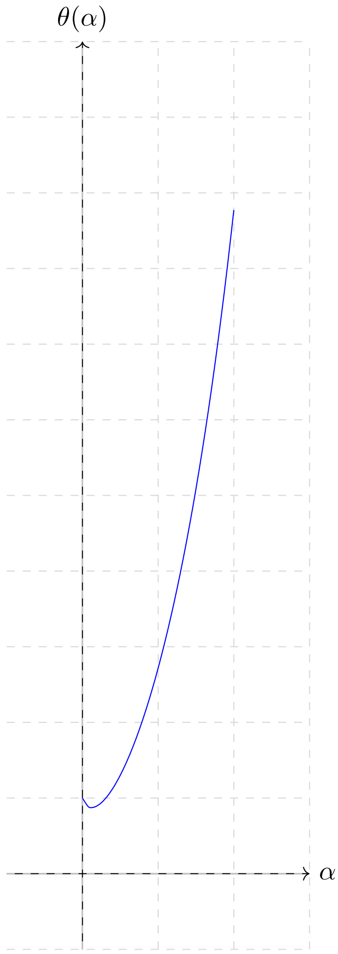


Figure 3: $\theta(\alpha) = e^{\alpha} + \alpha \ln \alpha$

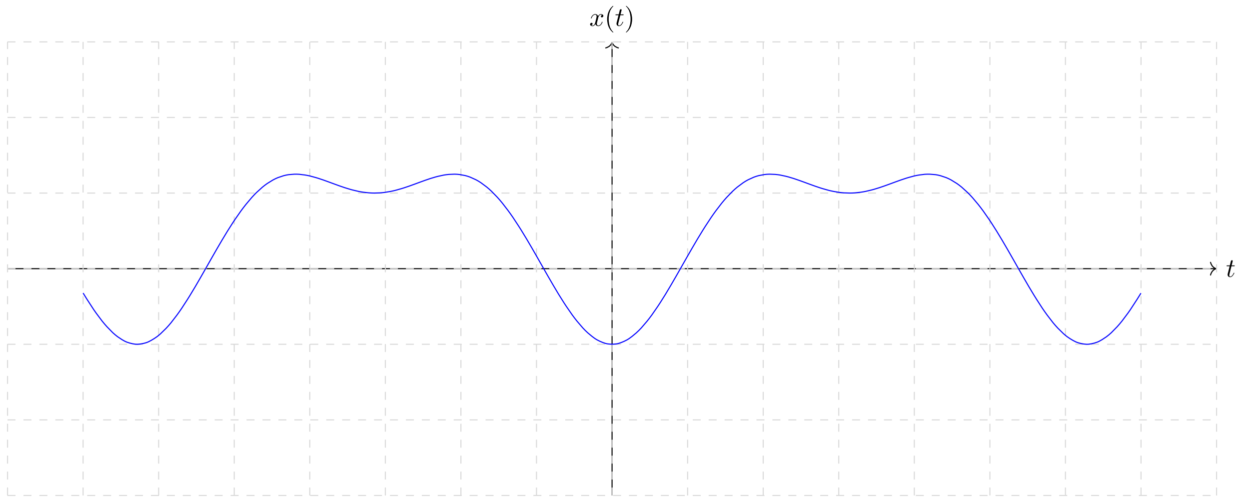


Figure 4: $x(t) = \sin^2 t - \cos t$

Remark 1. *All those plots were created using TiKz (with LaTeX then). For instance, here's the code for the last plot:*

```
\begin{figure}[H]
    \centering
    \begin{tikzpicture}
        \tikzmath{
            \xmin = -7;
            \xmax = 7;
            \ymin = -2;
            \ymax = 2;
        }
        \draw[->] (\xmin-1, 0) -- (\xmax+1, 0) node[right] {$t$};
        \draw[->] (0, \ymin-1) -- (0, \ymax+1) node[above] {$x(t)$};
        \draw[color=gray!30, dashed]
            (\xmin-1,\ymin-1) grid (\xmax+1,\ymax+1);
        \draw[scale=1, domain=\xmin:\xmax, smooth, samples=100, variable=\t, blue]
            plot ({\t}, {sin(\t r)^2 - cos(\t r)});
    \end{tikzpicture}
    \caption{$x(t) =  \sin^2t - \cos t$}
\end{figure}
```

Trigonometry

Vectors

Exercise 2/6

Exercise 5. *Work out the rule for vector subtraction.*

This exercise is about getting a (visual) feel for vector manipulation; it is *not* about vector coordinates manipulation. We were previously taught how to multiple vectors by a negative scalar:

> For example, $-2\vec{r}$ is the vector that is twice as long as $\vec{r}$, but points in the opposite direction.

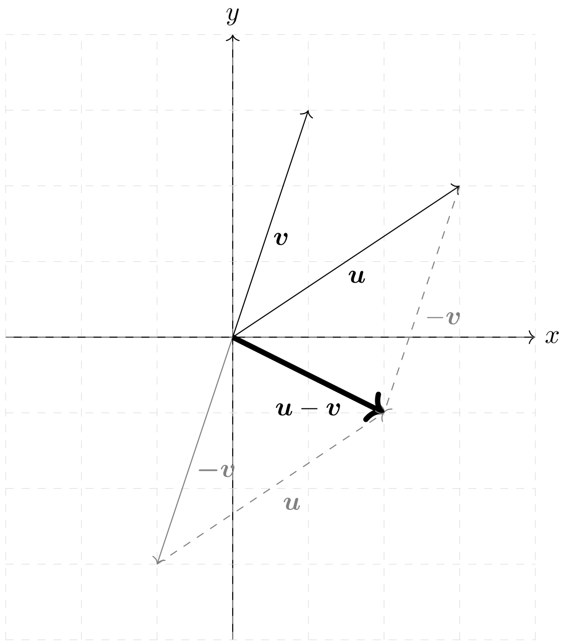
And how to add vectors:

> To add $\vec{A}$ and $\vec{B}$, place them as shown in Figure 13 to form a quadrilateral (this way the directions of the vectors are preserved). The sum of the vectors is the length and angle of the diagonal

So, by observing that (we'll use a **bold** font to denote vectors instead of arrows, e.g. $\boldsymbol{v}$ is a vector):

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-1\boldsymbol{v})$$

We conclude that we first need to reverse the direction of the vector to be subtracted, and add this to the other vector. Visually:



Exercise 3/6

Exercise 6. *Show that the magnitude of a vector satisfies* $|\vec{A}|^2 = \vec{A} \cdot \vec{A}$.

Remark 2. *We'll again use a* ***bold*** *font to denote vectors instead of arrows and use a slightly different symbol for the magnitude; the change can be summed up by stating:* $\|\boldsymbol{u}\| = |\vec{u}|\ (= u)$.

Let's recall that the magnitude of a vector was *defined* as:

$$\|\boldsymbol{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

And the dot product between two vectors as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_x v_x + u_y v_y + u_z v_z$$

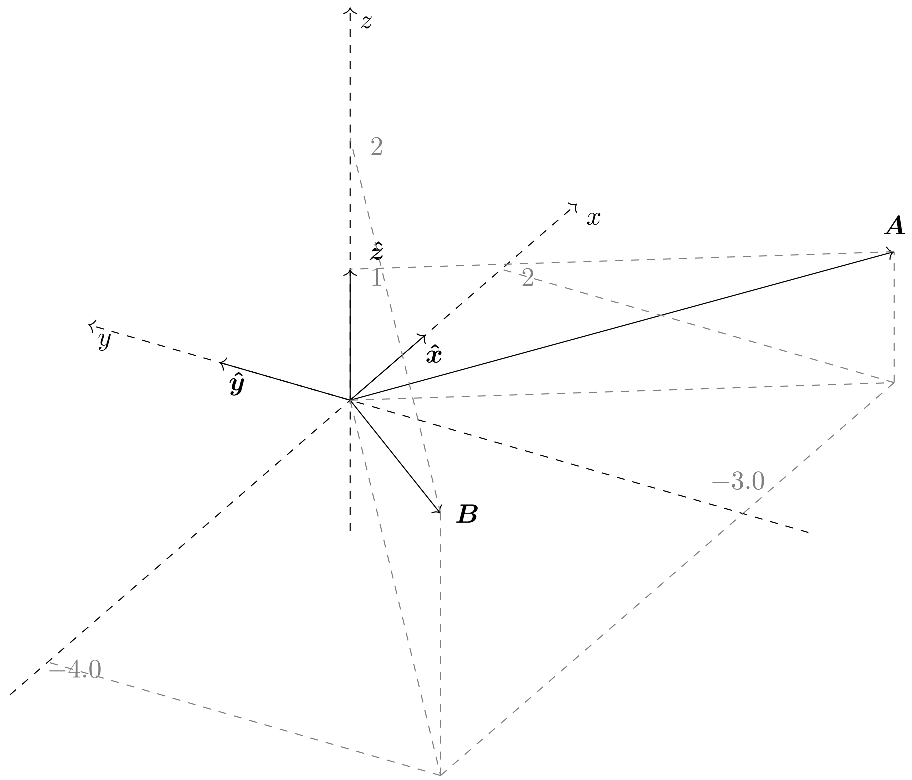
From there, we quickly reach the expected result:

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{u} &= u_x u_x + u_y u_y + u_z u_z \\
&= u_x^2 + u_y^2 + u_z^2 \\
&= \left(\sqrt{u_x^2 + u_y^2 + u_z^2}\right)^2 \\
&= \|\boldsymbol{u}\|^2 \quad \square
\end{aligned}$$

Exercise 4/6

Exercise 7. *Let* $(A_x = 2, A_y = -3, A_z = 1)$ *and* $(B_x = -4, B_y = -3, B_z = 2)$. *Compute the magnitude of* $\vec{A}$ *and* $\vec{B}$, *their dot product, and the angle between them.*

This is an immediate application of the formulas. Let's start by plotting those vectors:



Using slightly different notations, let us then recall first how the magnitude of a vector $\boldsymbol{u}$ is defined:

$$\|\boldsymbol{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2} = u \tag{1}$$

There are two formulas for the dot product: one involving the magnitudes and the angle between the vectors, and the other one involving the components:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_x v_x + u_y v_y + u_z v_z \\ &= \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta_{uv} \end{aligned} \tag{2}$$

By applying (1), we have:

$$\begin{aligned} A &= \sqrt{2^2 + (-3)^2 + 1^2} & \qquad B &= \sqrt{(-4)^2 + (-3)^2 + 2^2} \\ &= \sqrt{4+9+1} & &= \sqrt{16+9+4} \\ &= \boxed{\sqrt{14}} & &= \boxed{\sqrt{29}} \end{aligned}$$

We can also compute the dot product from the vectors' components, using the first form of (2):

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= 2(-4) + (-3)(-3) + 1 \times 2 \\ &= \boxed{3} \end{aligned}$$

From the second form of (2), we can deduce a formula for the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$, θ_{AB}:

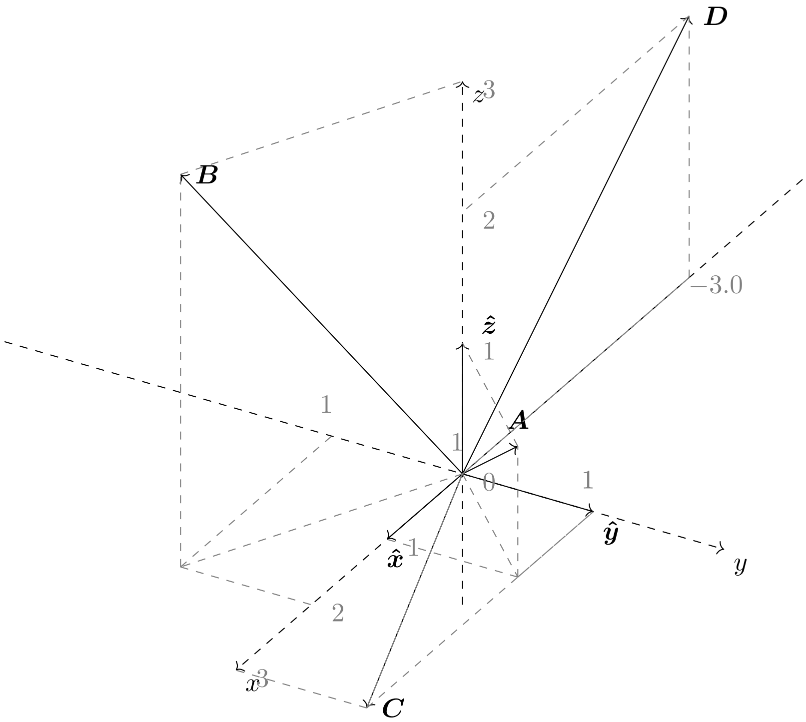
$$\begin{aligned} & \cos\theta_{AB} = & & \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\|\boldsymbol{A}\| \|\boldsymbol{B}\|} \\ \Leftrightarrow \quad & \theta_{AB} = & & \cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\|\boldsymbol{A}\| \|\boldsymbol{B}\|}\right) \\ \Leftrightarrow \quad & = & & \cos^{-1}\left(\frac{3}{\sqrt{14 \times 29}}\right) \\ \Leftrightarrow \quad & = & & \boxed{81.43753893^\circ} \end{aligned}$$

Remark 3. *Looking at our plots, the angle feels to be something a little less than* 90°*, which is coherent with what we've found.*

Exercise 5/6

Exercise 8. *Determine which pair of vectors are orthogonal.* $(1,1,1),(2,-1,3),(3,1,0),(-3,0,2)$.

This is again an immediate application of dot product formula. We'll respectively name the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$; let's start by plotting them:



Right before this exercise, the authors wrote:

> An important property of the dot product is that it is zero if the vectors are *orthogonal.*

Let us recall the "components-based" dot product formula we'll be using:

$$\boldsymbol{u}\cdot\boldsymbol{v} \quad = \quad u_xv_x+u_yv_y+u_zv_z$$

Then, it's just a matter of crunching numbers:

$$\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{B} &= A_xB_x+A_yB_y+A_zB_z \\
&= (1\times 2)+(1\times(-1))+(1\times 3) \\
&= \boxed{4} \\
\boldsymbol{A}\cdot\boldsymbol{D} &= A_xD_x+A_yD_y+A_zD_z \\
&= (1\times(-3))+(1\times 0)+(1\times 2) \\
&= \boxed{-1} \\
\boldsymbol{B}\cdot\boldsymbol{D} &= B_xD_x+B_yD_y+B_zD_z \\
&= (2\times(-3))+(-1\times 0)+(3\times 2) \\
&= \boxed{0}
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{C} &= A_xC_x+A_yC_y+A_zC_z \\
&= (1\times 3)+(1\times 1)+(1\times 0) \\
&= \boxed{4} \\
\boldsymbol{B}\cdot\boldsymbol{C} &= B_xC_x+B_yC_y+B_zC_z \\
&= (2\times 3)+(-1\times 1)+(3\times 0) \\
&= \boxed{5} \\
\boldsymbol{D}\cdot\boldsymbol{C} &= D_xC_x+D_yC_y+D_zC_z \\
&= (-3\times 3)+(0\times 1)+(2\times 0) \\
&= \boxed{-9}
\end{aligned}$$

Hence, the only two orthogonal vectors are $\boldsymbol{B}$ and $\boldsymbol{D}$, or $(2,-1,3)$ and $(-3,0,2)$.

Exercise 6/6

Exercise 9. *Can you explain why the dot product of two vectors that are orthogonal is* 0*?*

The dot product between two vectors is defined in two ways in the book; one of them involves the magnitudes of the vectors and the angle between them:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos \theta_{uv}$$

By definition, two vectors are orthogonal if the angle between them is $\pi/2$, or $90°$. But, $\cos(\pi/2) = 0$, so it follows from the previous dot product formula that, for two orthogonal vectors, their dot product must be 0.

Lecture 2: Motion

Mathematical Interlude: Differential Calculus

Exercise 1/8

Exercise 10. *Calculate the derivatives of each of these functions.*

$$\begin{aligned} f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\ g(x) &= \sin x - \cos x \\ \theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\ x(t) &= \sin^2 t - \cos t \end{aligned}$$

Remark 4. *Those are exactly the functions graphed in I01E01*

$\boldsymbol{f(t) = t^4 + 3t^3 - 12t^2 + t - 6}$

,

To find the derivative of f, we need to apply four rules that were mentioned (the first one was proved) in the book:

1. the formula for the derivative of a general power; for $n \in \mathbb{N}$:

$$\frac{d}{dt}(t^n) = nt^{n-1}$$

2. the fact that the derivative of a constant is zero; for $c \in \mathbb{R}$:

$$\frac{d}{dt}c = 0$$

3. the fact that the derivative of a constant times a function is the same as the constant times the derivative of the function; for $c \in \mathbb{R}$ and φ a function of t:

$$\frac{d}{dt}(c\varphi) = c\frac{d}{dt}\varphi$$

4. the *sum rule* (also referred to as the *linearity of differentiation* in mathematics); for both φ and ψ functions of t:

$$\frac{d}{dt}(\varphi + \psi) = \frac{d}{dt}\varphi + \frac{d}{dt}\psi$$

Remark 5. *Two simpler notations are common to denote the derivative of a function of a single variable:*

$$\frac{d}{dt}\varphi = \varphi' = \dot{\varphi}$$

While the first one is used mostly in mathematics, for abstract functions, the second one is used almost exclusively in physics, to denote a time derivative. We'll use them both in such ways from now on. For instance, as this exercise is rather mathematical, we'll use the prime notation.

While we're here, bear in mind that the prime notation can also be used to denote the derivative of a more or less complex expression, wrapped in parenthesis, e.g.:

$$(\varphi(x) + \psi(x) + \cos^2 x)' = \frac{d}{dx}(\varphi(x) + \psi(x) + \cos^2 x)$$

By application of the *sum rule*, the rule regarding the derivative of a constant, and the rule regarding a function multiplied by a constant:

$$f'(t) = (t^4)' + 3(t^3)' - 12(t^2)' + (t)' - 0$$

Then, it's just a matter of applying the formula for the derivative of a general power to each individual term:

$$\boxed{f'(t) = 4t^3 + 9t^2 - 24t + 1}$$

$g(x) = \sin x - \cos x$

,

In addition the the previously mentioned *sum rule*, we will also need two more rules to compute the derivative of g, also presented in the book, regarding the derivative of sin and cos:

$$\cos'(t) \quad = -\sin(t); \qquad \sin'(t) \quad = \cos(t)$$

We then have successively:

$$\begin{aligned} g'(x) &= \sin'(x) - \cos'(x) \\ &= \boxed{\cos(x) + \sin(x)} \end{aligned}$$

$\theta(\alpha) = e^\alpha + \alpha \ln \alpha$

,

To compute θ', in addition to the *sum rule* and the formula for a general power (with $n = 1$), we need two additional rules pertaining to the derivatives of both the exponential and the log:

$$(e^t)' \quad = \quad e^t; \qquad \ln'(t) \quad = \quad \frac{1}{t}$$

And the *product rule*; for φ and ψ functions of t:

$$\frac{d}{dt}(\varphi\psi) = \varphi'\psi + \varphi\psi'$$

Where all those rules were mentioned in the book. We then obtain:

$$\begin{aligned} \theta'(\alpha) &= (e^\alpha)' + (\alpha \ln(\alpha))' \\ &= e^\alpha + (\frac{d}{d\alpha}\alpha)\ln(\alpha) + \alpha \ln'(\alpha) \\ &= \boxed{e^\alpha + \ln(\alpha) + 1} \end{aligned}$$

$x(t) = (\sin t)^2 - \cos t$

,

Besides the *sum rule*, the rules regarding the derivative of cos and sin, and the formula for the derivative of a general power, we will only need a single new rule to compute x', the *chain rule*; for φ a function of t, and ψ and function whose domain (input) is the codomain (output) of φ:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then,

$$\begin{aligned} x'(t) &= ((\sin t)^2)' - (\cos t)' \\ &= (\sin t)'(u \mapsto u^2)'(\sin t) + \sin t \\ &= \cos t(u \mapsto 2u)(\sin t) + \sin t \\ &= 2\cos t \sin t + \sin t \\ &= \boxed{(1 + 2\cos t)\sin t} \end{aligned}$$

Remark 6. *We used a bit of mathematical notation to avoid us the need to explicitly name the function which squares its argument. More explicitly, to compute the derivative of $\mu(t) = \sin^2 t$, we could have define $\nu(u) = u^2$. Then, $\mu(t) = \nu(\sin(t))$, and $\nu'(u) = 2u$; by the chain rule*

$$(\sin^2 t)' = \mu'(t) = (\sin t)'\nu'(\sin(t)) = 2\cos t \sin t$$

Remark 7. *Instead of using the chain rule to compute the derivative of* $\sin^2 t$*, we could instead have used the product rule:*

$$(\sin^2 t)' = (\sin t \times \sin t)' = (\sin t)' \sin t + \sin t (\sin t)' = 2 \sin t (\sin t)' = 2 \sin t \cos t$$

Exercise 2/8

Exercise 11. *The derivative of a derivative is called the second derivative and is written* $\frac{d^2 f(t)}{dt^2}$*. Take the second derivative of each of the functions listed above.*

Remark 8. *As for the first derivative, there are two common notations for second derivatives:*

$$\frac{d^2}{dt^2}\varphi = \varphi'' = \ddot{\varphi}$$

Again, the dot notation is used in physics to denote time differentiation, while the prime notation is often used in mathematics, in a more abstract context.

Let's start by recalling the functions:

$$\begin{aligned}
f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\
g(x) &= \sin x - \cos x \\
\theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\
x(t) &= \sin^2 t - \cos t
\end{aligned}$$

And their derivatives, computed in the previous exercise:

$$\begin{aligned}
f'(t) &= 4t^3 + 9t^2 - 24t + 1 \\
g'(x) &= \cos(x) + \sin(x) \\
\theta'(\alpha) &= e^\alpha + \ln(\alpha) + 1 \\
x'(t) &= (1 + 2\cos t)\sin t
\end{aligned}$$

While in the previous exercise (L02E01) the derivation were rather slow and detailed, because the process is essentially the same, we're going to go (much) faster here, the most difficult part being in the application of the product rule for x''.

$$\begin{aligned}
f''(t) &= \boxed{12t^2 + 18t - 24} \\
g''(x) &= \boxed{\cos(x) - \sin(x) = -g(x)} \\
\theta''(\alpha) &= \boxed{e^\alpha + \frac{1}{\alpha}} \\
x''(t) &= (1 + 2\cos t)\cos t - 2\sin t \sin t \\
&= \boxed{\cos t + 2(\cos^2 t - \sin^2 t)}
\end{aligned}$$

Remark 9. x'' *could be slightly improved by using the trigonometric identity* $\cos^2 x - \sin^2 x = \cos 2x$*, which hasn't been introduced in the book.*

Exercise 3/8

Exercise 12. *Use the chain rule to find the derivatives on each of the following functions:*

$$\begin{aligned}
g_0(t) &= \sin(t^2) - \cos(t^2) \\
\theta_0(\alpha) &= e^{3\alpha} + 3\alpha \ln(3\alpha) \\
x_0(t) &= \sin^2(t^2) - \cos(t^2)
\end{aligned}$$

Remark 10. *We've slightly altered the functions names compared to what they have in the book. The reason will be apparent very soon.*

Remark 11. *As in previous exercises (e.g. L02E01), we'll be using some additional mathematical notation to avoid us the need to define functions. The meaning should be obvious from the context.*

We will also use the prime notation to denote differentiation, as we also did in earlier exercises.

There are two main ways of proceeding: either applying the *chain rule* to each individual term of each function, for instance to compute g', we would fist compute $(\sin(t^2))' = (\sin((u \mapsto u^2)(t)))'$, using the chain rule, then $(\cos(t^2))'$ in a similar fashion, etc.

Or, and we're going to use this approach as it's likely to have been expected one, we can consider each function "globally" as a composition of two smaller functions. While doing so, we will find that g_0, θ_0 and x_0 are respectively created by composing functions f, θ and x of an earlier exercise, with two simple functions $(v \mapsto v^2)$ and $(v \mapsto 3v)$:

$$\begin{aligned}
g_0(t) &= (\underbrace{(u \mapsto \sin(u) - \cos(u))}_{g} \circ (v \mapsto v^2))(t) \\
\theta_0(\alpha) &= (\underbrace{(u \mapsto e^u + u\ln(u))}_{\theta} \circ (v \mapsto 3v))(t) \\
x_0(t) &= (\underbrace{(u \mapsto \sin^2(u) - \cos(u))}_{x} \circ (v \mapsto v^2))(t)
\end{aligned}$$

Let's remember the derivative of g, θ and x, that we've computed in L02E01:

$$\begin{aligned}
g'(x) &= \cos(x) + \sin(x) \\
\theta'(\alpha) &= e^\alpha + \ln(\alpha) + 1 \\
x'(t) &= (1 + 2\cos t)\sin t
\end{aligned}$$

Finally, let's recall the *chain rule*:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then, the derivative of $(v \mapsto v^2)$ and $(v \mapsto 3v)$ being respectively $(v \mapsto 2v)$ and $(v \mapsto 3)$ (constant function), we have the following derivatives for our functions:

$$\begin{aligned}
g_0'(t) &= \boxed{2t(\cos t^2 + \sin t^2)} \\
\theta_0'(\alpha) &= \boxed{3(e^{3\alpha} + \ln(3\alpha) + 1)} \\
x_0'(t) &= \boxed{2t((1 + 2\cos t^2)\sin t^2)}
\end{aligned}$$

Exercise 4/8

Exercise 13. *Prove the sum rule (fairly easy), the product rule (easy if you know the trick), and the chain rule (fairly easy).*

You may want to refer to a *real* mathematical textbook for a finer, more rigorous treatment of those exercises. We'll give reasonably solid proofs, that should be sufficient in the context of an introductory physics textbook. A interesting middle-ground, that we won't explore here, would be to study the proofs that one can obtain from an alternative (but equivalent) formulation of the derivative known as the Carathéodory's derivative, which allows for simple proofs of such results.

Let's start by recalling how differentiation is defined.

Definition 1. *A function $\varphi : E \to \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:*

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon \to 0} \frac{\varphi(e + \epsilon) - \varphi(e)}{\epsilon}}$$

If this limit exists for all points x of E (we note, $(\forall x \in E)$), then φ is said to be differentiable on E, or simply differentiable. The function which associate to each points of E this limit is called the derivative of φ, and is called φ.

Theorem 1 (sum rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi+\psi)' = \varphi' + \psi'}$$

Proof. We have, by definition of the differentiation, and after re-ordering the terms

$$\begin{aligned}
(\forall x \in E), \quad (\varphi+\psi)'(x) &= \lim_{\epsilon\to 0} \frac{(\varphi+\psi)(x+\epsilon) - (\varphi+\psi)(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) + \psi(x+\epsilon) - \varphi(x) - \psi(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \left(\frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} + \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \right) \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} + \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \\
&= \boxed{\varphi'(x) + \psi'(x)}
\end{aligned}$$

□

Remark 12. *As a rigorous proof is a bit tedious*[2]*, we* <u>*assumed*</u> *for the last step that a limit of a sum is the sum of the limits,* <u>*when all the involved limits exist*</u> *(which is the case here, because those limits are equivalent to saying our functions are differentiable, which they are, per hypothesis)*

$$\lim_{x\to a} (\varphi(x) + \psi(x)) = \lim_{x\to a} \varphi(x) + \lim_{x\to a} \psi(x))$$

Theorem 2 (product rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi\psi)' = \varphi'\psi + \varphi\psi'}$$

Proof. This is a simple and often used theorem, but unfortunately, the proof of it is a bit "magic": if we start by applying the definition of the differentiation to $(\varphi\psi)'$, we have to introduce a well-crafted term (in the form $-a + a = 0$) so as to factorize things to meet our goal. We would furthermore be implicitly assuming that $(\varphi\psi)'$ exists, but we have no guarantee of it.

We can solve those issues by starting from the definition of the differentiation of $\varphi'\psi + \varphi\psi'$, but assuming we already know the result we're trying to prove, is conceptually clumsy. Perhaps calling this a "verification" rather than a proof would be more correct then.

Using the former derivation, we have:

$$\begin{aligned}
(\forall x \in E), \quad (\varphi\psi)'(x) &= \lim_{\epsilon\to 0} \frac{(\varphi\psi)(x+\epsilon) - (\varphi\psi)(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon)\psi(x+\epsilon) - \varphi(x)\psi(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon)\psi(x+\epsilon) - \varphi(x)\psi(x) \overbrace{-\varphi(x+\epsilon)\psi(x) + \varphi(x+\epsilon)\psi(x)}^{=0}}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon)(\psi(x+\epsilon) - \psi(x)) + \psi(x)(\varphi(x+\epsilon) - \varphi(x))}{\epsilon} \\
&= \lim_{\epsilon\to 0} \varphi(x+\epsilon) \frac{(\psi(x+\epsilon) - \psi(x))}{\epsilon} + \psi(x) \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} \\
&= \left(\lim_{\epsilon\to 0} \varphi(x+\epsilon)\right) \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} + \psi(x) \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} \\
&= \boxed{\varphi(x)\psi'(x) + \psi(x)\varphi'(x)}
\end{aligned}$$

□

[2] if you want one, have a look at *Paul's Online Notes*: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

Remark 13. *We <u>assumed</u> another result on limits* [3]*, again assuming individual limits exists (and they do in our case, as our functions are differentiable, and (thus) continuous):*

$$\lim_{x\to a}(\varphi(x)\psi(x)) = \lim_{x\to a}\varphi(x)\times\lim_{x\to a}\psi(x))$$

Theorem 3 (chain rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi\circ\psi)' = \psi' \times (\varphi'\circ\psi)}$$

Proof. Again, the proof is a bit "magical" in that we're going to multiply by a well-crafted term, of the form $a/a = 1$:

$$\begin{aligned}
(\forall x \in E),\ (\varphi\circ\psi)(x)) &= \lim_{\epsilon\to 0}\frac{(\varphi\circ\psi)(x+\epsilon)-(\varphi\circ\psi)(x))}{\epsilon} \\
&= \lim_{\epsilon\to 0}\left(\frac{\varphi(\psi(x+\epsilon))-\varphi(\psi(x)))}{\epsilon}\times\overbrace{\frac{\psi(x+\epsilon)-\psi(x)}{\psi(x+\epsilon)-\psi(x)}}^{=1}\right) \\
&= \lim_{\epsilon\to 0}\left(\frac{\varphi(\psi(x+\epsilon))-\varphi(\psi(x)))}{\psi(x+\epsilon)-\psi(x)}\times\frac{\psi(x+\epsilon)-\psi(x)}{\epsilon}\right) \\
&= \lim_{\epsilon\to 0}\frac{\varphi(\psi(x+\epsilon))-\varphi(\psi(x)))}{\psi(x+\epsilon)-\psi(x)}\times\underbrace{\lim_{\epsilon\to 0}\frac{\psi(x+\epsilon)-\psi(x)}{\epsilon}}_{\psi'(x)}
\end{aligned}$$

Again for that last step, we've used the aforementioned rule on products of existing limits. To conclude, we need to compute the first limit: let's define $h = \psi(x+\epsilon)-\psi(x) \Leftrightarrow \psi(x+\epsilon) = \psi(x)+h$. Note that $\epsilon \to 0 \Rightarrow h \to 0$. So the first limit can be rewritten:

$$\lim_{h\to 0}\frac{\varphi(\psi(x)+h)-\varphi(\psi(x)))}{h} \triangleq \varphi'(\psi(x))$$

□

Now a problem with this previous proof is that it is invalid if ψ is a constant function for example, because $\psi(x+\epsilon)-\psi(x)$ is zero, and we're dividing by zero when performing our magical "multiplication" by 1. If you're interested, you can find an alternative proof using the other, equivalent form of the derivative here: `https://www.youtube.com/watch?v=COLwYhEAt7Q`.

Exercise 5/8

Remark 14. *This is a WIP; some intermediate results are missing.*

Exercise 14. *Prove each of the formulas in Eq.s* (2)*. Hint: Look up trigonometric identities and limit properties in a reference book.*

Let's recall the formulas of Eq.s (2):

$$\begin{aligned}
\frac{d}{dt}(\sin t) &= \cos t \\
\frac{d}{dt}(\cos t) &= \sin t \\
\frac{d}{dt}(e^t) &= e^t \\
\frac{d}{dt}(\ln t) &= \frac{1}{t}
\end{aligned}$$

Remark 15. *Interestingly, there are multiple ways of defining those functions*[4]*. As a result, there are different ways to compute the derivatives, depending on which definitions we choose.*

As the definitions given in the book for sin, cos *and the exponential are rather standard, we'll recall them and use those. The natural logarithm hasn't been clearly defined though, so we'll have to do it.*

[3] If you want a rigorous proof of it, you can refer to the same resource as before: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

[4] For instance, consider this page containing 6 equivalent definitions of the exponential: `https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function`

Remark 16. *The book suggests to look up trigonometric identities and common properties in some reference material, but we're going to take the time to prove almost all intermediate results here.*

As far as I can tell, to compute the derivative of either sin *or* cos*, starting from a geometrical (triangle-based) definition, while keeping the definition of* π *as a measure of angle in radians, one need to start from basic Euclidean geometry, derive some specific limits, and lightly touch on elementary integration results. This is a good occasion to refresh basic real analysis.*

To say it otherwise, we will go ***far beyond*** *what is expected for this exercise by the authors. That being said, take what follows with a grain of salt: there's a lot of results, which can be subtle, so you may want to refer to a more thorough treatment by real mathematicians in case of doubts.*

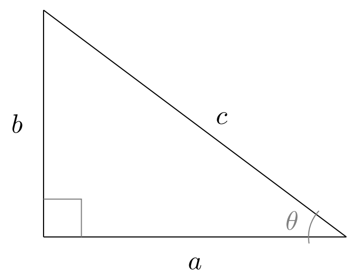
Let's start by recalling that a function $\varphi : E \to \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon\to 0} \frac{\varphi(e+\epsilon) - \varphi(e)}{\epsilon}}$$

---

d* sin *t*/*dt

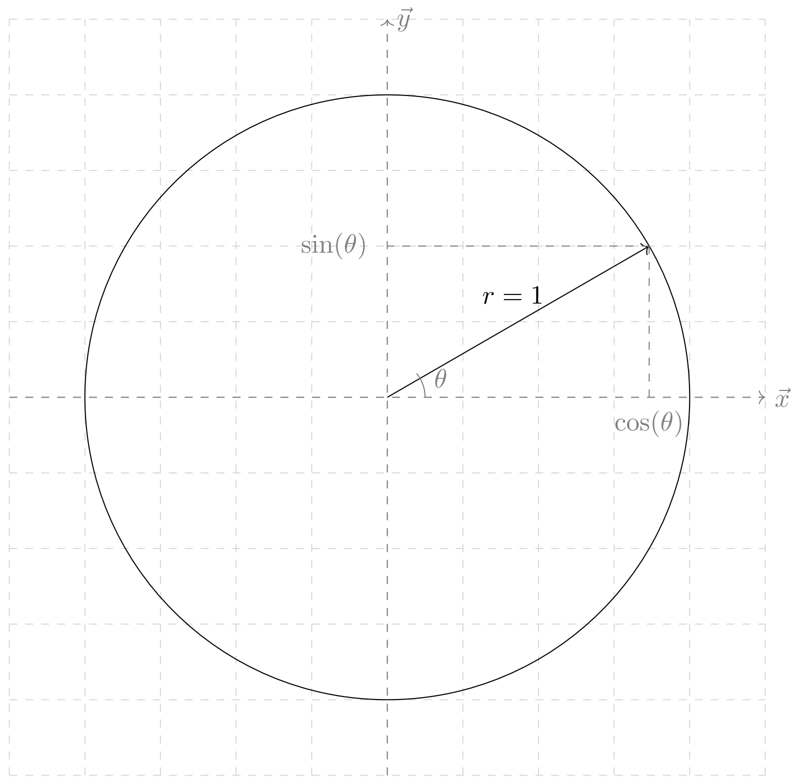
Then, let's remind ourselves of the common definitions of cos and sin, in the context of a right triangle:



$$\boxed{\cos\theta \triangleq \frac{c}{a}}$$

$$\boxed{\sin\theta \triangleq \frac{b}{c}}$$

In particular, we can identify points on the unit-circle by the angle between the x axis and the radius connecting the center of the circle to such points. Then, each point will then be located in the xy-plane as $(\cos\theta, \sin\theta)$, where θ is the angle previously described, associated to the point.



Note that we have:

Theorem 4.

$$(\forall x \in \mathbb{R}), \quad \boxed{\sin^2 x + \cos^2 x = 1}$$

Proof. This follows immediately from the Pythagorean theorem applied to the right triangle formed by $r = 1$, $\cos\theta$ and $\sin\theta$. □

We'll need this later: this is but a variant of the previous result where the circle isn't restricted to being unitary:

Theorem 5 (equation of a circle)**.** *The points $(x, y) \in \mathbb{R}^2$ describing a circle of radius r centered at the origin O are tied by the following equation*

$$\boxed{x^2 + y^2 = r^2}$$

Proof. This follows directly from Pythagorean's theorem □

Remark 17. *In particular, as r is a constant, this mean we can express y as a function of x:*

$$\boxed{y(x) = \sqrt{r^2 - x^2}}; \ x \in [-r, r]$$

In order to establish $\sin'$, we will need a few intermediate results that we're going to prove now. First will be to find a formula for $\sin(\alpha+\beta)$. Indeed, if you try to apply the definition of the derivative to sin, you should see a $\sin(x + \epsilon)$: we will need to have it expressed differently to develop the proof.

$$\sin'(x) \triangleq \lim_{\epsilon \to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon}$$

Theorem 6 ($\sin(\alpha + \beta)$)**.**

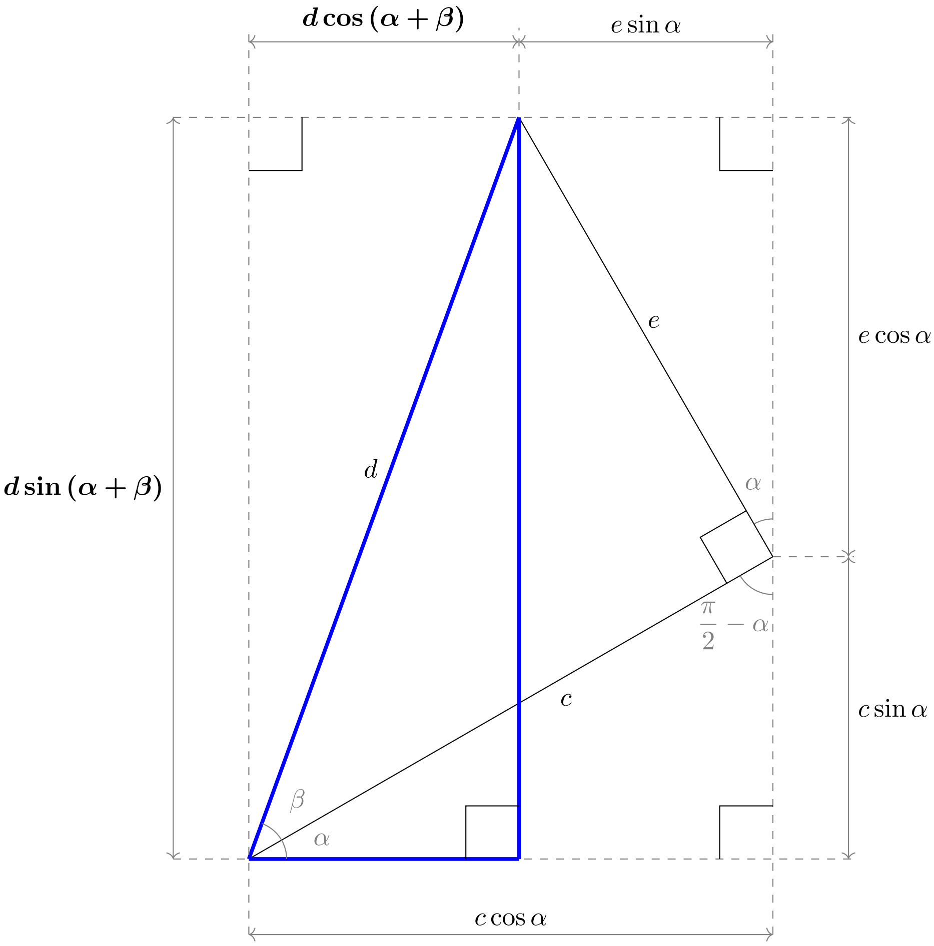
$$(\forall(\alpha, \beta) \in \mathbb{R}^2), \quad \boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \tag{3}$$

There's a also a formula for the cosine of a sum of angles, that we will need later, for the derivative of cos, but that will be rather immediate to prove along the one regarding the sine of a sum of angles.

Theorem 7 ($\cos(\alpha + \beta)$)**.**

$$(\forall(\alpha, \beta) \in \mathbb{R}^2), \quad \boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta} \tag{4}$$

Proof. This will be a "visual proof". Besides the aforementioned definitions of sin and cos, we will also use the "fact" that sum of angles in a right triangle is π, which is actually the *triangle postulate*, an axiom of Euclidean geometry, equivalent to the parallel postulate.



In the previous picture, considering the right triangle formed by c, d and e, we have:

$$c = d\cos\beta; \qquad e = d\sin\beta$$

If we look at the blue/thick triangle right triangle (of hypotenuse d, with an angle of $\beta+\alpha$, and whose other sides are created by projecting the point formed by d and e down to the bottom), we find a new relation, to which we can inject our previous results for c and e:

$$\begin{aligned} & d\sin(\alpha+\beta) &=& \; c\sin\alpha + e\cos\alpha \\ \Leftrightarrow & &=& \; (d\cos\beta)\sin\alpha + (d\sin\beta)\cos\alpha \\ \Leftrightarrow & \sin(\alpha+\beta) &=& \; \boxed{\cos\beta\sin\alpha + \sin\beta\cos\alpha} \end{aligned}$$

In the same blue/thick triangle, we can also establish a relation for $\cos(\alpha+\beta)$, using the same definition of c and e as before to conclude:

$$\begin{aligned} & d\cos(\alpha+\beta) &=& \; c\cos\alpha - e\sin\alpha \\ \Leftrightarrow & &=& \; (d\cos\beta)\cos\alpha - (d\sin\beta)\sin\alpha \\ \Leftrightarrow & \cos(\alpha+\beta) &=& \; \boxed{\cos\beta\cos\alpha - \sin\beta\sin\alpha} \end{aligned}$$

□

Here's an immediate consequence that we'll need in the future.

Theorem 8 (trigonometric shifts)**.** *Let* $x \in \mathbb{R}$.

$$\boxed{\sin(x+\frac{\pi}{2}) = \cos x; \qquad \cos(x+\frac{\pi}{2}) = -\sin x}$$

Proof. If we apply our previous formulas (3) and (4) regarding respectively the sine and cosine of a sum of two angles, in the case where one angle is $\pi/2$, we have:

$$\begin{aligned}
(\forall x \in \mathbb{R}), \quad \sin(x + \frac{\pi}{2}) &= \sin x \underbrace{\cos\frac{\pi}{2}}_{=0} + \cos x \underbrace{\sin\frac{\pi}{2}}_{=1} \\
&= \boxed{\cos x} \\
(\forall x \in \mathbb{R}), \quad \cos(x + \frac{\pi}{2}) &= \cos x \underbrace{\cos\frac{\pi}{2}}_{=0} - \sin x \underbrace{\sin\frac{\pi}{2}}_{=1} \\
&= \boxed{-\sin x}
\end{aligned}$$

□

Remark 18. *We could derive more similar formulas, but those are the only ones we'll need.*

Now if you try to write down $\sin'$ as previously suggested, and if you decompose $\cos(x + \epsilon)$ with the formula (3), you see that yields two limits:

$$\begin{aligned}
\sin'(x) &\triangleq \lim_{\epsilon \to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\sin x \cos\epsilon + \cos x \sin\epsilon - \sin x}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left(\frac{\sin x \cos\epsilon - 1}{\epsilon} + \frac{\cos x \sin\epsilon}{\epsilon} \right)
\end{aligned}$$

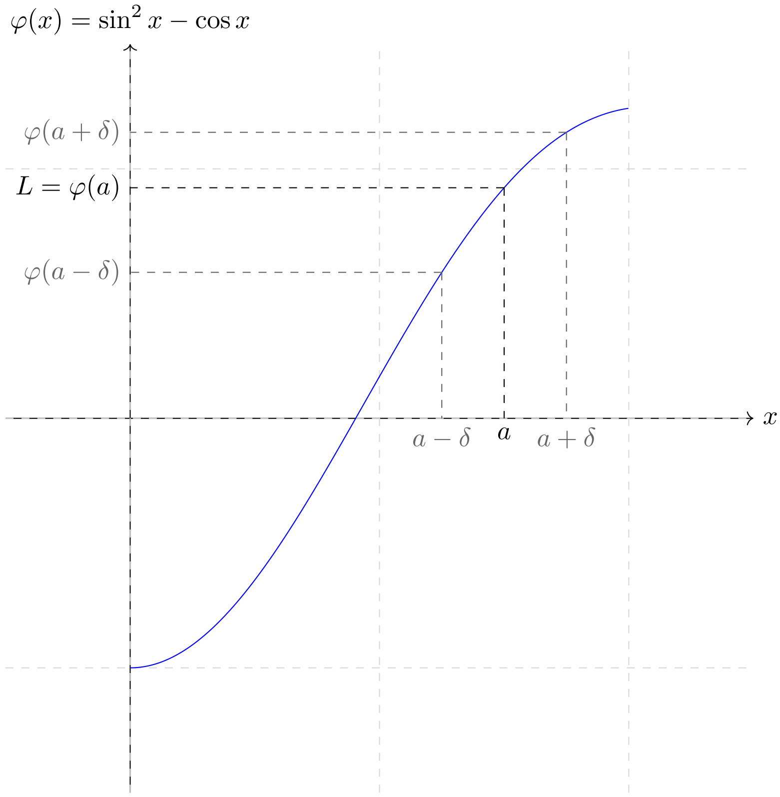
As we've already explained in L02E04, we can recursively split the previous limits, assuming each individual limit exists. Let's digress and review how to prove such results on limits. Starting with the definition of a limit:

Definition 2 ((ϵ, δ)-definition of a limit)**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$. Saying that $\lim_{x \to a} \varphi(x) = L$ is the same as saying:*

$$(\forall \epsilon > 0), (\exists \delta \in \mathbb{R}), (\forall x \in \mathbb{R})\, (|x - a| < \delta) \Rightarrow (|\varphi(x) - L| < \epsilon)$$

The definition is dense, so let's unpack it. Let's fix ϵ to some value close to zero, say 0.1, or 0.0001.

Now the idea is that, for this ϵ, we will always be able to find a distance δ such that if we pick an x between $a - \delta$ and $a + \delta$ (i.e. $|x - a| < \delta$), then $\varphi(x)$ will be between $L - \epsilon$ and $L + \epsilon$ (i.e. $|\varphi(x) - L| < \epsilon$).



But, this is true for all strictly positive ϵ. So in particular, this is true for an ever so smaller ϵ. In other words, regardless of how close we want $\varphi(x)$ and L to be, we will always be able to achieve it if we bring x and a close enough.

Alright, there's just one more thing we need before proving the sum rule, and that's the triangle inequality. This inequality is rooted in euclidean geometry: it states that the sum of the length of any two sides of a triangle is greater or equal than the length of the remaining side.

Theorem 9 (triangle inequality)**.**

$$\boxed{(\forall (x,y) \in \mathbb{R}^2), \quad |x+y| \leq |x| + |y|}$$

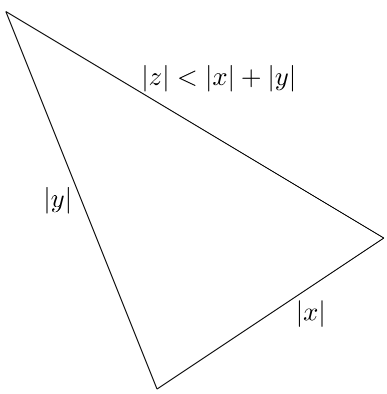


Figure 5: Unless the triangle is degenerate (i.e. all its edges are colinear), the length of the longest side of a triangle is strictly smaller than the length of the two other sides. It's trivially true for the shorter sides. If we allow triangles to be degenerates, then it's true for any sides of any triangle.

Proof. Let's start by recalling the definition of the absolute value:

$$\boxed{x \in \mathbb{R}; \quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}}$$

It follows that:

$$|x+y| = \begin{cases} x+y & \text{if } (x+y) \geq 0 \\ -(x+y) = -x-y & \text{otherwise} \end{cases}$$

And that, for any $(x,y) \in \mathbb{R}^2$

$$x <= |x|; \quad -x \leq |x|; \quad y <= |y|; \quad -y \leq |y|;$$

So:

$$\Big(x+y \leq |x|+|y|; \quad -(x+y) = -x-y \leq |x|+|y|\Big) \Leftrightarrow \boxed{|x+y| \leq |x|+|y|}$$

□

Let's jump into the sum rule:

Theorem 10 (sum rule for limits)**.** *Assuming the two following limits exists:*

$$\lim_{x\to a} \varphi(x); \qquad \lim_{x\to a} \psi(x)$$

Then:

$$\boxed{\lim_{x\to a} (\varphi(x)+\psi(x)) = \lim_{x\to a} \varphi(x) + \lim_{x\to a} \psi(x)}$$

Proof. Most limits proofs are presented in a "confusing" way, starting with unexplained values that ends up doing exactly what we want. That happens when mathematicians have thought and drafted the proof in reverse order, but present it in the "right" order. We're going to use the "wrong" order here, for clarity; it should be immediate to check that there's no logical issues anyway. Just bear in mind that "reversed implications" are rather unorthodox.

Let's start by defining a few things:

$$\lim_{x\to a} \varphi(x) = L_1; \qquad \lim_{x\to a} \psi(x) = L_2$$

$$\lim_{x\to a} (\varphi(x)+\psi(x)) = L$$

Let's explicit the two first limits via the $(\epsilon-\delta)$-definition:

$$(\forall \epsilon_1 > 0),\ (\exists \delta_1 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x-a| < \delta_1) \Rightarrow (|\varphi(x) - L_1| < \epsilon_1)$$

$$(\forall \epsilon_2 > 0),\ (\exists \delta_2 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x-a| < \delta_2) \Rightarrow (|\psi(x) - L_2| < \epsilon_2)$$

Essentially, what we want to prove is then $L = L_1 + L_2$. If this is true, this means means that for an $\epsilon \in \mathbb{R}$, we should be able to find a δ such that:

$$\begin{array}{rrll} & |\varphi(x)+\psi(x)-L| & <\epsilon & \\ \Leftrightarrow & |\varphi(x)+\psi(x)-(L_1+L_2)| & <\epsilon & \text{(assumption)} \\ \Leftrightarrow & |(\varphi(x)-L_1)+(\psi(x)-L_2)| & <\epsilon & \\ \Leftarrow & |\varphi(x)-L_1|+|\psi(x)-L_2| & <\epsilon & \text{(triangular inequality)} \\ \Leftrightarrow & |\varphi(x)-L_1|+|\psi(x)-L_2| & <\epsilon_1+\epsilon_2 & (\epsilon = \epsilon_1+\epsilon_2) \end{array}$$

$$\Leftarrow \begin{cases} |\varphi(x)-L_1| < \epsilon_1 = \epsilon/2 \\ |\psi(x)-L_2| < \epsilon_2 = \epsilon/2 \end{cases} \qquad \text{(all numbers are positive)}$$

$$\Leftarrow \Big(|x-a| < \delta = \min(\delta_1, \delta_2)\Big)$$

To make things very clear, if we choose such a δ, then:

$$|x-a| < \delta \Leftrightarrow \begin{cases} |x-a| < \delta_1 \\ |x-a| < \delta_2 \end{cases}$$

Which means both limits will hold. Furthermore, given any real number, say ϵ_1, we can always choose to represent it as $\epsilon/2$, for $\epsilon \in \mathbb{R}$ too.

And this concludes the proof: if you look at the beginning and end of the previous derivation, we've found:

$$(\forall \epsilon > 0), (\exists \delta \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta) \Rightarrow \left(|\varphi(x) + \psi(x) - \underbrace{L_1 + L_2}_{L}| < \epsilon \right)$$

Which by definition means:

$$\lim_{x \to a} (\varphi(x) + \psi(x)) = L_1 + L_2$$

□

The following is not necessary here, but it's more involved, so good practice:

Theorem 11 (product rule for limits)**.** *Assuming the two following limits exists:*

$$\lim_{x \to a} \varphi(x); \qquad \lim_{x \to a} \psi(x)$$

Then:

$$\boxed{\lim_{x \to a} (\varphi(x)\psi(x)) = \lim_{x \to a} \varphi(x) \lim_{x \to a} \psi(x)}$$

Proof. We'll use the same unorthodox presentation; again, let's start by defining a few things:

$$\lim_{x \to a} \varphi(x) = L_1; \qquad \lim_{x \to a} \psi(x) = L_2$$

$$\lim_{x \to a} (\varphi(x)\psi(x)) = L$$

What we want to prove this time is that $L = L_1 L_2$. If this is true, this means means that for an $\epsilon \in \mathbb{R}$, we should be able to find a δ such that:

$$\begin{array}{rll} & |\varphi(x)\psi(x) - L| & < \epsilon \\ \Leftrightarrow & |\varphi(x)\psi(x) - L_1 L_2| & < \epsilon \end{array}$$

Well, this is embarrassing: we can't really follow through as we did before. Can we find an expression that we could algebraically connect to $\varphi(x)\psi(x) - L_1 L_2$?

Let's make a guess, and try the following product $(\varphi(x) - L_1)(\psi(x) - L_2)$. When developed, clearly it contains the previous expression that we desperately try to find a path to, plus two other terms:

$$\varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1 L_2$$

And we can see that in the context of x being arbitrarily close to a, we have, by the sum rule:

$$L_1\psi(x) \to L_1 L_2; \qquad L_2\varphi(x) \to L_2 L_1$$

Which means, when $x \to a$, this product should be equal to $\varphi(x)\psi(x) - L_1 L_2$. In summary, if we can prove that $(\varphi(x) - L_1)(\psi(x) - L_2)$ can be made as small as we want as $x \to a$, we should have a proof. Let's, then study the corresponding limit (that we expect to be zero):

$$\lim_{x \to a} (\varphi(x) - L_1)(\psi(x) - L_2)$$

First, note that we have:

$$\begin{array}{rcccl} \lim_{x \to a} (\varphi(x) - L_1) & = & (\lim_{x \to a} \varphi(x)) - L_1 & = & 0 \\ \lim_{x \to a} (\psi(x) - L_2) & = & (\lim_{x \to a} \psi(x)) - L_2 & = & 0 \end{array}$$

Which means, translated in the $(\epsilon - \delta)$ formalism (as before, we choose the same ϵ so as to control both limits at once)

$$(\forall \epsilon > 0),\ (\exists \delta_1 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x-a| < \delta_1) \Rightarrow (|\varphi(x) - L_1 - 0| < \epsilon)$$
$$(\forall \epsilon > 0),\ (\exists \delta_2 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x-a| < \delta_2) \Rightarrow (|\psi(x) - L_2 - 0| < \epsilon)$$

Let $\epsilon > 0$; let's choose $\delta = \min(\delta_1, \delta_2)$. Then

$$|x-a| < \delta \Leftrightarrow \begin{cases} |x-a| < \delta_1 \\ |x-a| < \delta_2 \end{cases}$$

Which means we have both:

$$\begin{cases} |\varphi(x) - L_1| < \epsilon \\ |\psi(x) - L_2| < \epsilon \end{cases}$$

All involved numbers are positive, so let's multiply both lines:

$$\begin{aligned} &|\varphi(x) - L_1| \times |\psi(x) - L_2| < \epsilon^2 \\ \Leftrightarrow &|(\varphi(x) - L_1)(\psi(x) - L_2)| < \epsilon^2 \end{aligned}$$

But if ϵ can be chosen arbitrarily in $\mathbb{R}_+^*$, then so can $\epsilon' = \epsilon^2$. So we've just proved that

$$(\forall \epsilon' > 0),\ (\exists \delta \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x-a| < \delta) \Rightarrow (|(\varphi(x) - L_1)(\psi(x) - L_2) - 0| < \epsilon')$$

That is:

$$\lim_{x \to a} (\varphi(x) - L_1)(\psi(x) - L_2) = 0$$

From what we've already said, this concludes the proof, but let's make it clear:

$$\begin{aligned}
& & \varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1L_2 &= (\varphi(x) - L_1)(\psi(x) - L_2) \\
\Leftrightarrow & & \lim_{x \to a} \Big(\varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1L_2\Big) &= \underbrace{\lim_{x \to a} (\varphi(x) - L_1)(\psi(x) - L_2)}_{0} \\
\Leftrightarrow & & \lim_{x \to a} (\varphi(x)\psi(x)) - L_1 \underbrace{\lim_{x \to a} (\psi(x))}_{L_2} - L_2 \underbrace{\lim_{x \to a} (\varphi(x))}_{L_1} + \lim_{x \to a} (L_1L_2) &= 0 \\
\Leftrightarrow & & \lim_{x \to a} (\varphi(x)\psi(x)) &= \boxed{L_1L_2}
\end{aligned}$$

□

Remark 19. *As the proof is rather elementary, we've assumed that for a constant $k \in \mathbb{R}$,*

$$\lim_{x \to a} (k\varphi(x)) = k \lim_{x \to a} (\varphi(x))$$

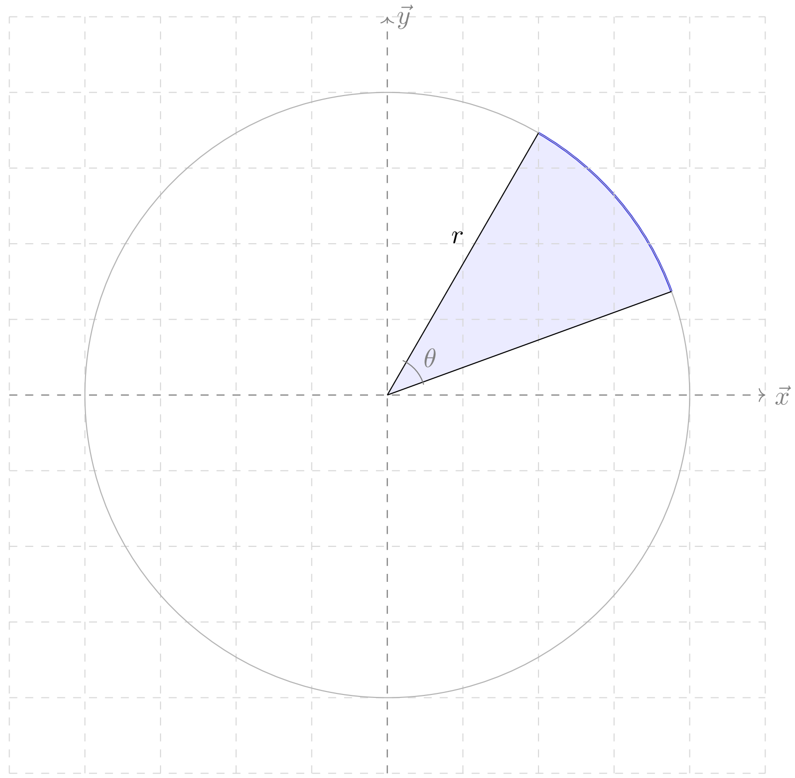
We've also assumed that the product of two absolute values is the absolute values of the product. Again, the proof is elementary.

So, what we did was applying the definition of the derivative on $x \mapsto \sin x$; this yields the limit of a sum. We know we can split it into sums of limits, provided the two individual limits exist. So we must now try to compute those two limits.

We'll first need to define what a *circular sector* is, and how to express its *area*, from which, you'll see, computing the area of a circle is but a special case. We will also need to establish another important result on limits: the *squeeze theorem*.

Let's start with circular sector:

Definition 3 (circular arc, circular sector)**.** *A* <u>*circular arc*</u> *is a portion of the circle between two points of that circle. A* <u>*circular sector*</u> *is a portion of disk enclosed between two (usually distinct) radii*[5] *and a circular arc.*



Remark 20. *A circle then, is but a circular sector of angle* 2π*.*

Unfortunately, if we want to compute the area of a circular sector, we either need some integral calculus to compute it directly, or indirectly, via the area of a circle. In addition to the definition of an integral, to compute the actual integral representing the area of a circle (or that of a sector), we will need the u-substitution rule, which demands the fundamental theorem of calculus.

A rigorous, extensive treatment of Riemann integration would require more work than what we aim to achieve here; you may want to refer to a full real analysis course for more.

Definition 4 ((Riemann) integral)**.** *TODO*

Theorem 12 (Fundamental theorem of calculus)**.** *TODO*

Proof. TODO □

Theorem 13 (u-substitution/reversed chain-rule (single variable))**.** *Let* $U \subseteq \mathbb{R}$*,* $(a,b) \in \mathbb{R}^2$*. Let* $\varphi : U \to \mathbb{R}$ *and* $\nu : [a,b] \to U$ *be respectively* C^0 *and* C^1 *functions. Then,*

$$\boxed{\int_a^b \varphi(\nu(x))\nu'(x)\,dx = \int_{\nu(a)}^{\nu(b)} \varphi(u)\,du}$$

Proof. Let's assume here that continuous functions are integrable. Then, the existence of both integrals is guaranteed by the restrictions imposed on φ and ν.

Then, let us note Φ the anti-derivative of φ, i.e. $\Phi' = \varphi$, which by the same assumption as before must exists, since φ is continuous. Let us then apply the chain-rule to $\Phi \circ \nu$:

$$(\Phi \circ \nu)'(x) = \nu'(x)\Phi'(\nu(x)) = \varphi(\nu(x))\nu'(x)$$

[5]"radii" is the plural of "radius"

Finally, by integrating both sides and repeatedly applying the fundamental theorem of calculus twice, we obtain:

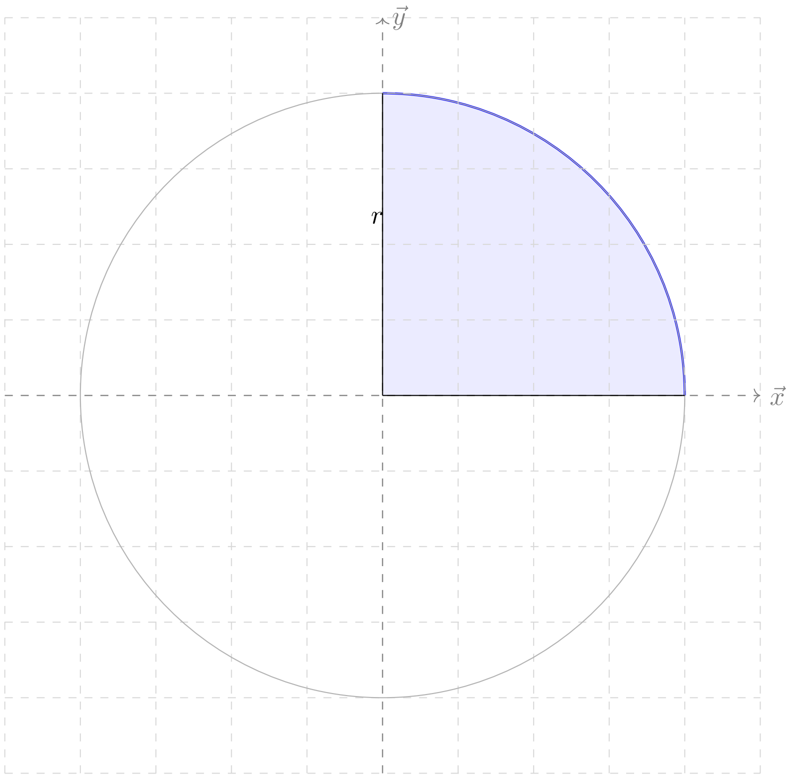
$$\begin{aligned}\int_a^b \varphi(\nu(x))\nu'(x)\,dx = & \int_a^b (\Phi\circ\nu)'(x)dx \\ =&(\Phi\circ\nu)(b) - (\Phi\circ\nu)(a) \\ =&\Phi(\nu(b)) - \Phi(\nu(a)) \\ =&\int_{\nu(a)}^{\nu(b)} \varphi(u)\,du\end{aligned}$$

□

Theorem 14 (circle area)**.** *The area of a circle of radius r is given by:*

$$\boxed{\pi r^2}$$

Proof. The problem of the area of a circle can be reduced via symmetry to the integration of a positive curve on an interval. More precisely, let's consider a circle centered at the origin of radius r. To compute its area, suffice to compute the area of a quadrant of it, say the first quadrant (the blueish one; starting from this one, the four quadrant are enumerated following an anti-clockwise/trigonometric direction)



By symmetry, the area of a circle A_S is four times that of any quadrant A_{Q_i}. Furthermore, the first quadrant's area A_{Q_1} is given by the area under the curve describing a circle that we saw earlier in 17, restricted to $[0, r]$. More precisely:

$$A_S = 4A_{Q_i} = 4A_{Q_1} = 4\int_0^r \sqrt{r^2 - x^2}\,dx$$

This integral can be considered in the form:

$$\int_{\nu(\theta_0)}^{\nu(\theta_1)} \varphi(x)\,dx$$

Let's perform a first change of variable by setting:

$$\nu(\theta) = r\sin\theta; \qquad \nu'(\theta) = r\cos\theta$$

It follows that:

$$\nu(\theta_0) = 0 \Rightarrow \theta_0 = 0; \qquad \nu(\theta_1) = 0 \Rightarrow \theta_1 = \frac{\pi}{2}$$

The integral can now be rewritten:

$$\begin{aligned}
A_S &= 4\int_0^{\pi/2} \phi(\nu(\theta))\nu'(\theta)\,d\theta \\
&= 4\int_0^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2}r\cos\theta\,d\theta \\
&= 4r^2\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta \qquad (r > 0) \\
&= 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \qquad (\cos\theta > 0, \text{ Pythagorean theorem})
\end{aligned}$$

However, with (8) we have:

$$4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \quad = \quad 4r^2\int_0^{\pi/2} (\sin(\theta + \frac{\pi}{2}))^2\,d\theta$$

We can perform a second change of variable on the rightmost integral:

$$\mu(u) = \frac{\pi}{2} - u; \qquad \mu'(u) = -1$$

Then:

$$\mu(u_0) = 0 \Rightarrow u_0 = \frac{\pi}{2}; \qquad \mu(u_1) = \frac{\pi}{2} \Rightarrow u_1 = 0$$

The integral becomes:

$$4r^2\int_0^{\pi/2} (\sin(\theta + \frac{\pi}{2}))^2\,d\theta \quad = \quad -4r^2\int_{\pi/2}^{0} \sin(u)^2\,du \quad = \quad 4r^2\int_0^{\pi/2} \sin^2\theta\,d\theta$$

Referring back to our previous integral equality, this means we have:

$$S_A \quad = \quad 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \quad = \quad 4r^2\int_0^{\pi/2} \sin^2\theta\,d\theta$$

However, by linearity of integration, and by the Pythagorean theorem, again, we have

$$\int_0^{\pi/2} \cos^2\theta\,d\theta + \int_0^{\pi/2} \sin^2\theta\,d\theta = \int_0^{\pi/2} \underbrace{\cos^2\theta + \sin^2\theta}_{=1}\,d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2} = 2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

Hence,

$$S_A = 2r^2\,\underbrace{2\int_0^{\pi/2} \cos^2\theta\,d\theta)}_{=\pi/2} = \boxed{\pi r^2}$$

□

Theorem 15 (circular sector area)**.** *The area of a circular sector, of angle θ, in a circle of radius r is given by:*

$$\boxed{\frac{1}{2}r^2\theta}$$

Proof. Because the area is evenly distributed on a circle, this is a simple cross-multiplication[6] involving the area of a circle:

$$\begin{aligned}
2\pi &\rightarrow \pi r^2 \\
\theta &\rightarrow A_\theta = \frac{\theta\pi r^2}{2\pi} = \boxed{\frac{1}{2}r^2\theta}
\end{aligned}$$

□

[6] https://en.wikipedia.org/wiki/Cross-multiplication

Remark 21. *We could also have proved it directly with an integral, as we did for the circle.*

Theorem 16 (squeeze theorem)**.** *Let φ, ψ, ϕ be three real-valued functions defined on an interval $I \subset \mathbb{R}$, and a be a point of I. If, $(\forall x \in I/\{a\})$ $(x \in I$ but $x \neq a)$, we have:*

$$\varphi(x) \leq \psi(x) \leq \phi(x)$$

With:

$$\lim_{x \to a} \varphi(x) = \lambda = \lim_{x \to a} \phi(x)$$

Then:

$$\boxed{\lim_{x \to a} \psi(x) = \lambda}$$

Proof. Let's translate our limits into their $(\epsilon - \delta)$ form, considering a single ϵ

$$(\forall \epsilon > 0), (\exists \delta_1 \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta_1) \Rightarrow (|\varphi(x) - \lambda| < \epsilon)$$

$$(\forall \epsilon > 0), (\exists \delta_2 \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta_2) \Rightarrow (|\phi(x) - \lambda| < \epsilon)$$

So, let $\epsilon > 0$, and let $\delta = \min(\delta_1, \delta_2)$. For the same reasons as with previous limits proofs, this δ implies both the inequalities involving ϵ, which can be restated, by definition of the absolute value, as:

$$-\epsilon < \varphi(x) - \lambda < \epsilon$$
$$-\epsilon < \phi(x) - \lambda < \epsilon$$

But we also have:

$$\varphi(x) \leq \psi(x) \leq \phi(x)$$
$$\Leftrightarrow \varphi(x) - \lambda \leq \psi(x) - \lambda \leq \phi(x) - \lambda$$

So:

$$-\epsilon < \varphi(x) - \lambda \leq \psi(x) - \lambda \leq \phi(x) - \lambda < \epsilon$$
$$\Leftrightarrow |\psi(x) - \lambda| < \epsilon$$

Which concludes the proof, as we now have an $(\epsilon - \delta)$ statement on ψ, equivalent to a limit.

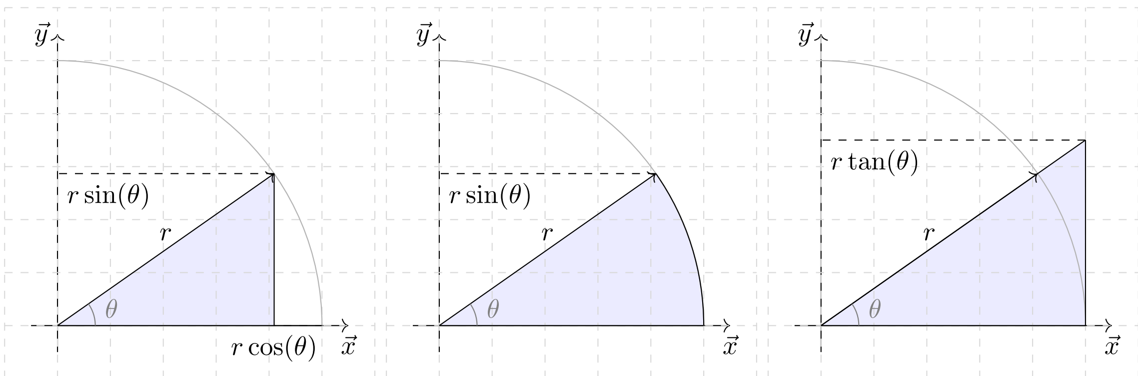
□

We now have everything we need to start computing the limits involved in the differentiation of sine:

Theorem 17.

$$\boxed{\lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon} = 1}$$

Proof. Consider the three following blueish areas:



The three areas are definitely ordered from smaller to bigger (left to right), and we can also determine them: the middle one is that of a sector, while the two side ones are right triangles (so their areas is half of the corresponding rectangle). We then have the following inequalities:

$$\begin{array}{rrcl}
 & \frac{1}{2}r^2\cos\theta\sin\theta \leq & \frac{1}{2}r^2\theta \leq & \frac{1}{2}r^2\tan\theta \\
\Leftrightarrow & \cos\theta\sin\theta \leq & \theta \leq & \frac{\sin\theta}{\cos\theta} \\
\Leftrightarrow & \cos\theta \leq & \frac{\theta}{\sin\theta} \leq & \frac{1}{\cos\theta} \\
\Leftrightarrow & \frac{1}{\cos\theta} \geq & \frac{\sin\theta}{\theta} \geq & \cos\theta
\end{array}$$

But, as θ goes to zero, the two extremes of this inequalities become:

$$\lim_{\theta\to 0}\frac{1}{\cos\theta} = 1; \qquad \lim_{\theta\to 0}\cos\theta = 1$$

Hence by the *squeeze theorem*, it follows that we *must* have:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$$

□

As for the other limit:

Theorem 18.

$$\boxed{\lim_{\epsilon\to 0}\frac{\cos\epsilon - 1}{\epsilon} = 0}$$

Proof. We will rely on the previous limit; this will be a "proof with a trick" (multiplying by $a/a = 1$; in the present context, a will always be non-zero). Note also at the end that we can apply the product rules for limits given the existence of both sublimits.

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\cos\epsilon - 1}{\epsilon} &= \lim_{\epsilon\to 0}\left(\frac{\cos\epsilon - 1}{\epsilon} \times \frac{\cos\epsilon + 1)}{\cos\epsilon + 1)}\right) \\
&= \lim_{\epsilon\to 0}\left(\frac{\cos^2\epsilon - 1}{\epsilon(\cos\epsilon + 1))}\right) \\
&= -\lim_{\epsilon\to 0}\left(\frac{\sin^2\epsilon}{\epsilon(\cos\epsilon + 1))}\right) \\
&= -\lim_{\epsilon\to 0}\left(\frac{\sin\epsilon}{\epsilon} \times \frac{\sin\epsilon}{\cos\epsilon + 1}\right) \\
&= -\underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\epsilon})}_{\to 1} \times \underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\cos\epsilon + 1})}_{\to 0/2=0} \\
&= \boxed{0}
\end{aligned}$$

□

We now have everything to conclude: let's recapitulate all the intermediate steps to compute $\sin'$:

Theorem 19 (sine derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\sin'(x) = \cos(x)}$$

Proof.

$$\begin{aligned}(\forall x \in \mathbb{R}), \quad \sin'(x) &\triangleq \lim_{\epsilon \to 0} \frac{\sin(x+\epsilon) - \sin x}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\sin x \cos \epsilon + \cos x \sin \epsilon - \sin x}{\epsilon} \\ &= \lim_{\epsilon \to 0} \left(\frac{\sin x(\cos \epsilon - 1)}{\epsilon} + \frac{\cos x \sin \epsilon}{\epsilon} \right) \\ &= \sin x \underbrace{\lim_{\epsilon \to 0} \frac{\cos \epsilon - 1}{\epsilon}}_{=0} + \cos x \underbrace{\lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon}}_{=1} \\ &= \boxed{\cos x}\end{aligned}$$

□

$d \cos t / dt$

Theorem 20 (cosine derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\cos'(x) = -\sin(x)}$$

Proof. The results follow from the shifts formulas **??**

$$\begin{aligned}(\forall x \in \mathbb{R}), \quad \cos'(x) &= \sin'(x + \frac{\pi}{2}) \\ &= (\sin \circ (y \mapsto y + \frac{\pi}{2}))'(x) \\ &= \cos(x + \frac{\pi}{2}) \\ &= \boxed{-\sin(x)}\end{aligned}$$

□

---

de^t / dt

This one, as mentioned in the book, is "trivial" when we define the exponential function to be precisely the function which is equal to its derivative (and such as $e^0 = 1$).

And this is usually the way the exponential function will be first introduced to students. You may want to have a look at other equivalent characterization of the function[7]. Trying to compute an exponential defined on a development in infinite series carries a certain aesthetic for instance.

$$\boxed{\frac{d}{dt} e^t \triangleq e^t}$$

$d \ln t / dt$

As for the exponential, there can be some variety here depending on how we *characterize* the ln function [8]. Usually, it will be introduced as the *inverse function* of the exponential:

Definition 5 (natural logarithm)**.** *The natural logarithm function is defined as the function* ln *such that:*

$$(\forall x \in \mathbb{R}), \quad \boxed{e^{\ln(x)} = x}$$

Remark 22. *To rigorously establish this definition, would have needed to prove that the exponential is invertible.*

[7] https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function

[8] https://en.wikipedia.org/wiki/Natural_logarithm#Definitions

Theorem 21 (natural logarithm derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\ln'(x) = \frac{1}{x}}$$

Proof. The proof develops from the previous definition of the logarithm by integrating both side and then applying the chain rule:

$$\begin{array}{lrcl} & (\forall x \in \mathbb{R}), \quad e^{\ln(x)} & = & x \\ \Leftrightarrow & \frac{d}{dx} e^{\ln(x)} & = & \frac{d}{dx} x \\ \Leftrightarrow & \ln'(x) \underbrace{e^{\ln(x)}}_{=x} & = & 1 \\ \Leftrightarrow & \ln'(x) & = & \boxed{\frac{1}{x}} \end{array}$$

□

Remark 23. *For the sake of completeness, some authors*[9]*, will for instance start by defining the logarithm as an integral, and then define the exponential as the inverse of the logarithm. From which they can prove that the derivative of the exponential is the exponential.*

Particle Motion

Example of Motion

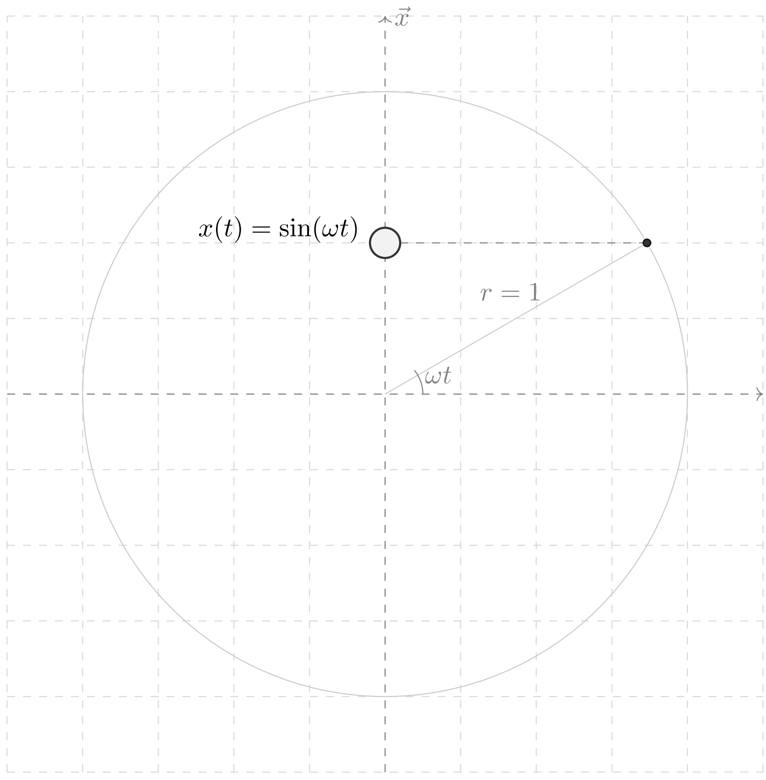
Exercise 6/8

Exercise 15. *How long does it take for the oscillating particle to go through one full cycle of motion?*

We're in the case of a particle oscillating in one dimension. Its motion, known as the *simple harmonic motion*, is described by:

$$x(t) = \sin(\omega t)$$

Essentially, $x(t)$ will correspond to the vertical component of a point moving on the unit circle, located by an angle ωt.



[9] https://www.whitman.edu/mathematics/calculus_late_online/section09.02.html

To fix things, consider the case of a particle starting at an extreme position, say $x = 1$ (at the top of the north hemisphere of the unit circle). It will need to go down to $x = -1$, and then back up to $x = 1$. In the mean time, the corresponding point on the unit circle would have walked a full circle, or 2π radians.

So we're looking for the time T that it will take for us to move by an angle 2π, knowing that we move at a speed of ω radians per unit of time (i.e. $\omega_{t=0} = 0$, $\omega_{t=1} = \omega$, $\omega_{t=2} = 2\omega$, ...):

$$\omega T = 2\pi \Leftrightarrow \boxed{T = \frac{2\pi}{\omega}}$$

Remark 24. *T is commonly called the period of motion.*

Exercise 7/8

Exercise 16. *Show that the position and velocity vectors from Eq.s* (3) *are orthogonal.*

Actually, Eq.s (3) refers to velocity and acceleration, not position. Because of this ambiguity, let's look for which pair of vectors are orthogonal among the three. Let's start by recalling how they have been defined (knowing that velocity and acceleration are obtained by differentiating position respectively once and twice):

$$\begin{aligned} r_x(t) &= R\cos(\omega t); & r_y(t) &= R\sin(\omega t) \\ v_x(t) &= -R\omega\sin(\omega t); & v_y(t) &= R\omega\cos(\omega t) \\ a_x(t) &= -R\omega^2\cos(\omega t); & a_y(t) &= -R\omega^2\sin(\omega t) \end{aligned}$$

We've established in I01E06 that two vectors are orthogonal if their dot product is zero; where the dot product has been defined as:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_x v_x + u_y v_y + u_z v_z \\ &= \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta_{uv} \end{aligned}$$

Then, let's compute a few dot products (we're in the plane, so the z components must be zero):

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{v} &= r_x v_x + r_y v_y \\ &= R\cos(\omega t) \times (-R\omega\sin(\omega t)) + R\sin(\omega t) \times R\omega\cos(\omega t) \\ &= \boxed{0} \\ \boldsymbol{r} \cdot \boldsymbol{a} &= r_x a_x + r_y a_y \\ &= R\cos(\omega t) \times (-R\omega^2\cos(\omega t)) + R\sin(\omega t) \times (-R\omega^2\sin(\omega t)) \\ &= -R^2\omega^2 \underbrace{(\cos^2(\omega t) + \sin^2(\omega t))}_{=1} \\ &= \boxed{-(R\omega)^2 \neq 0} \\ \boldsymbol{v} \cdot \boldsymbol{a} &= v_x a_x + v_y a_y \\ &= (-R\omega\sin(\omega t)) \times (-R\omega^2\cos(\omega t)) + R\omega\cos(\omega t) \times (-R\omega^2\sin(\omega t)) \\ &= \boxed{0} \end{aligned}$$

Hence both position and acceleration are orthogonal with velocity.

Remark 25. *Regarding $\boldsymbol{r} \cdot \boldsymbol{a}$, we could also have observed that $\boldsymbol{a} = -\omega^2\boldsymbol{r}$: the vectors are collinear, so they simply can't be orthogonal. From there, were we to already have established $\boldsymbol{r} \cdot \boldsymbol{v} = 0$, we could have inferred $\boldsymbol{v} \cdot \boldsymbol{a} = -\omega^2\boldsymbol{v} \cdot \boldsymbol{r} = 0$, using the (bi)linearity of the dot product.*

Exercise 8/8

Exercise 17. *Calculate the velocity, speed and acceleration for each of the following position vectors. If you have graphing software, plot each position vector, each velocity vector, and each acceleration vector.*

$$\begin{aligned} \vec{r} &= (\cos\omega t, e^{\omega t}) \\ \vec{r} &= (\cos(\omega t - \phi), \sin(\omega t - \phi)) \\ \vec{r} &= (c\cos^3 t, c\sin^3 t) \\ \vec{r} &= (c(t - \sin t), c(1 - \cos t)) \end{aligned}$$

Let's recall that each component of the velocity and acceleration vectors are defined respectively as the derivative and second derivative of the corresponding component of the position vector:

$$
\begin{aligned}
\boldsymbol{r}(t) &= (x(t), y(t)) \\
\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) &= (\dot{x}(t), \dot{y}(t)) \\
\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) &= (\ddot{x}(t), \ddot{y}(t))
\end{aligned}
$$

So this is just a differentiation exercise in disguise. We'll be using a fast pace here (80% of the work is about applying the chain rule); if you need a slower approach, see for instance L02E01, where we go in-depth on how to apply common differentiation rules.

$\boldsymbol{r_0(t) = (\cos(\omega t), e^{\omega t})}$

$$
\begin{aligned}
\boldsymbol{r_0}(t) &= (\cos(\omega t), e^{\omega t}) \\
\boldsymbol{v_0}(t) = \dot{\boldsymbol{r_0}}(t) &= \boxed{(-\omega \sin(\omega t), \omega e^{\omega t})} \\
\boldsymbol{a_0}(t) = \dot{\boldsymbol{v_0}}(t) = \ddot{\boldsymbol{r_0}}(t) &= \boxed{(-\omega^2 \cos(\omega t), \omega^2 e^{\omega t})}
\end{aligned}
$$

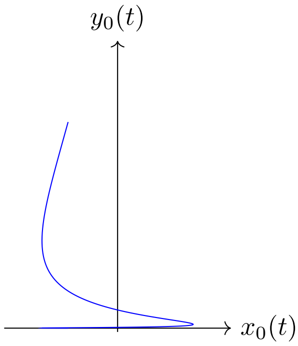


Figure 6: $\omega = 1$;
$\boldsymbol{r}_0(t) = (\cos(\omega t), e^{\omega t})$

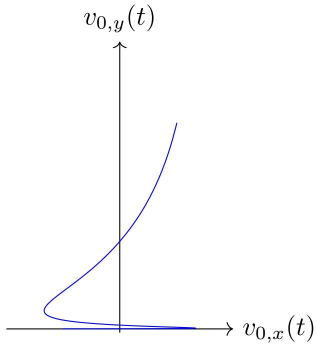


Figure 7: $\omega = 1$;
$\boldsymbol{v}_0(t) = (-\omega \sin(\omega t), \omega e^{\omega t})$

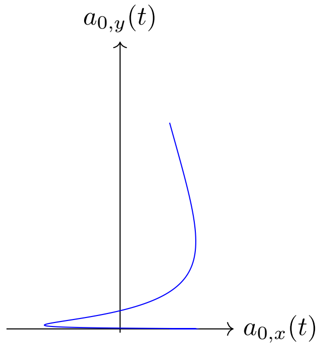


Figure 8: $\omega = 1$;
$\boldsymbol{a}_0(t) = (-\omega^2 \cos(\omega t), \omega^2 e^{\omega t})$

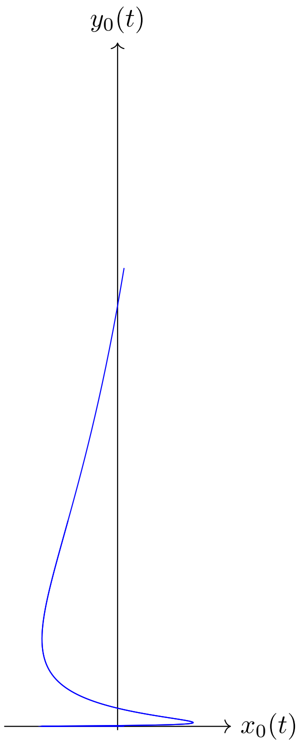


Figure 9: $\omega = 1.2$;
$\boldsymbol{r}_0(t) = (\cos(\omega t), e^{\omega t})$

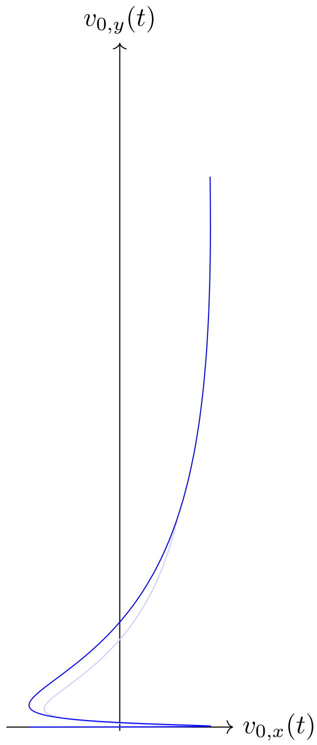


Figure 10: $\omega = 1.2$;
$\boldsymbol{v}_0(t) = (-\omega \sin(\omega t), \omega e^{\omega t})$

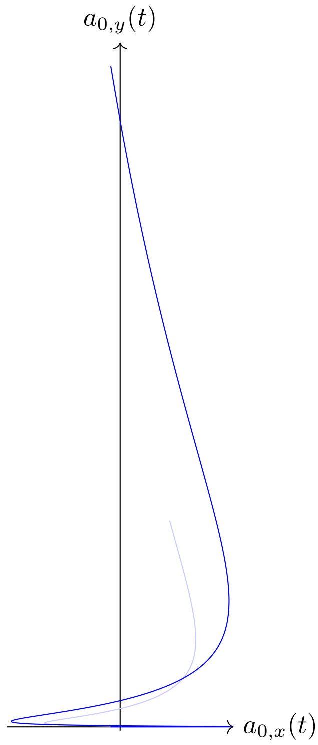


Figure 11: $\omega = 1.2$;
$\boldsymbol{a}_0(t) = (-\omega^2 \cos(\omega t), \omega^2 e^{\omega t})$

Remark 26. *Increasing ω will:*

- *For the position, increase the distance at which we travel in the y direction; the distance in the x direction will be the same, because it's constrained by a* cos*, but we'll get there faster:*
- *For the velocity, we will go faster in both the x and y directions; we've plotted in a fainted blue on the second graph the $\boldsymbol{v}_0(t)$ for $\omega = 1$ for comparison, because the effect in the x direction is small;*
- *And obviously if the velocity increases, the acceleration must increase accordingly, which it does, quadratically, both in the x and y directions (again, we've plotted in a fainted blue on the second graph $\boldsymbol{a}_0(t)$ for $\omega = 1$ for comparison).*

$\boldsymbol{r_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))}$

$$\begin{aligned}
\boldsymbol{r_1}(t) &= (\cos(\omega t - \phi), \sin(\omega t - \phi)) \\
\boldsymbol{v_1}(t) = \dot{\boldsymbol{r_1}}(t) &= \boxed{(-\omega \sin(\omega t - \phi), \omega \cos(\omega t - \phi))} \\
\boldsymbol{a_1}(t) = \dot{\boldsymbol{v_1}}(t) = \ddot{\boldsymbol{r_1}}(t) &= \boxed{(-\omega^2 \cos(\omega t - \phi), -\omega^2 \sin(\omega t - \phi))} \quad = -\omega^2 \boldsymbol{r}_1(t)
\end{aligned}$$

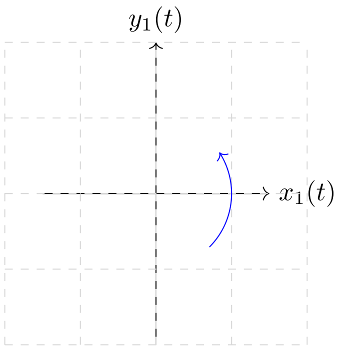


Figure 12: $\omega = 1, \varphi = 0$; $\boldsymbol{r}_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

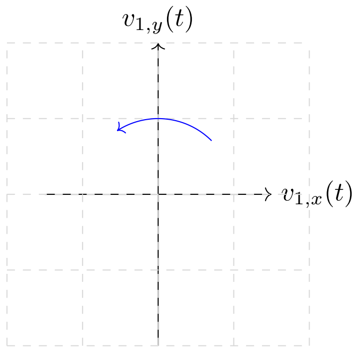


Figure 13: $\omega = 1, \varphi = 0$; $\boldsymbol{v}_1(t) = \omega(-\sin(\omega t - \phi), \cos(\omega t - \phi))$

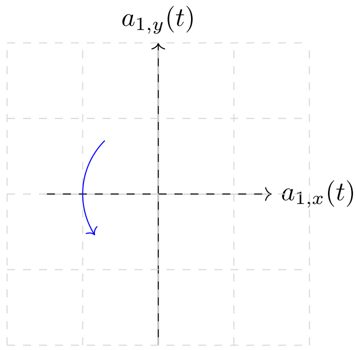


Figure 14: $\omega = 1, \varphi = 0$; $\boldsymbol{a}_1(t) = -\omega^2 \boldsymbol{r}_1(t))$

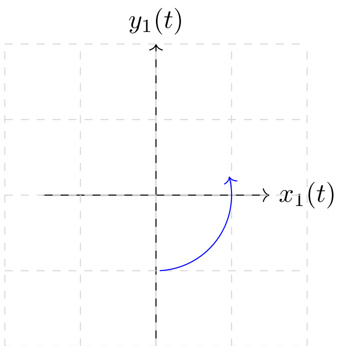


Figure 15: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{r}_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

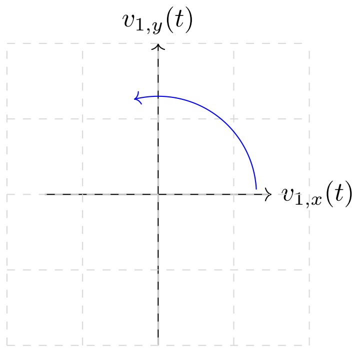


Figure 16: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{v}_1(t) = \omega(-\sin(\omega t - \phi), \cos(\omega t - \phi))$

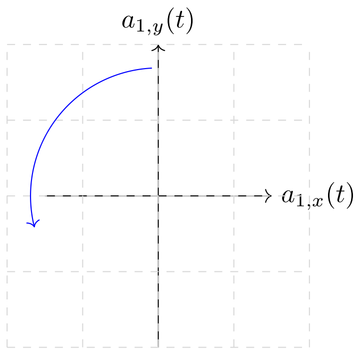


Figure 17: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{a}_1(t) = -\omega^2 \boldsymbol{r}_1(t))$

Remark 27. *All those plots were made with $t \in [-\pi/4, \pi/3]$ so as to make more visible the effect of changing the phase ϕ, which only alters our starting/ending point. The alteration would have been hidden were t to have gone through an interval wider or equal than 2π. An arrow has been added to indicate the ending point.*

ω is the angular velocity, or the number of radians the particle move per unit of time. Naturally, if it's increased, the particle will go further, faster; the increase in speed will demand a corresponding increase in acceleration.

$\boldsymbol{r_2(t) = c(\cos t)^3, c(\sin t)^3)}$

$\boldsymbol{a_2}$ is the most complex derivative for this exercise. We start by applying the product rule ($uv = u'v + uv'$), and the chain rule on one of the resulting factor.

$$\begin{aligned}
\boldsymbol{r_2}(t) &= (c\cos^3 t, c\sin^3 t) \\
\boldsymbol{v_2}(t) = \dot{\boldsymbol{r_2}}(t) &= \boxed{3c(-\sin t\cos^2 t, \cos t\sin^2 t)} \\
\boldsymbol{a_2}(t) = \dot{\boldsymbol{v_2}}(t) = \ddot{\boldsymbol{r_2}}(t) &= 3c(-\cos t\cos^2 t + (-\sin t)(-\sin t)(2\cos t), -\sin t\sin^2 t + \cos t\cos t 2\sin t) \\
&= \boxed{3c((\cos t)(2\sin^2 t - \cos^2 t), (\sin t)(2\cos^2 t - \sin^2 t))}
\end{aligned}$$

Remark 28. *We may be able to simplify the expression of the acceleration $\boldsymbol{a}_2$.*

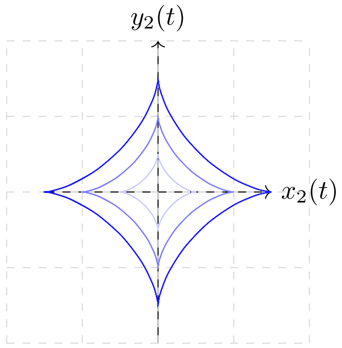


Figure 18: $c = 0.5,\ 1,\ 1.5; \boldsymbol{r}_2(t) = (c\cos^3 t, c\sin^3 t)$

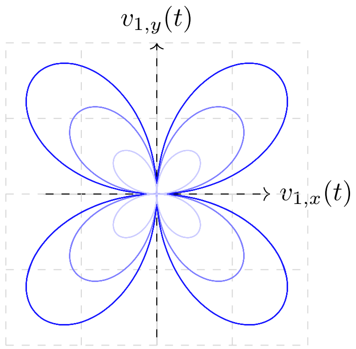


Figure 19: $c = 0.5,\ 1,\ 1.5;$
$\boldsymbol{v}_2(t) = 3c(-\sin t\cos^2 t, \cos t\sin^2 t)$

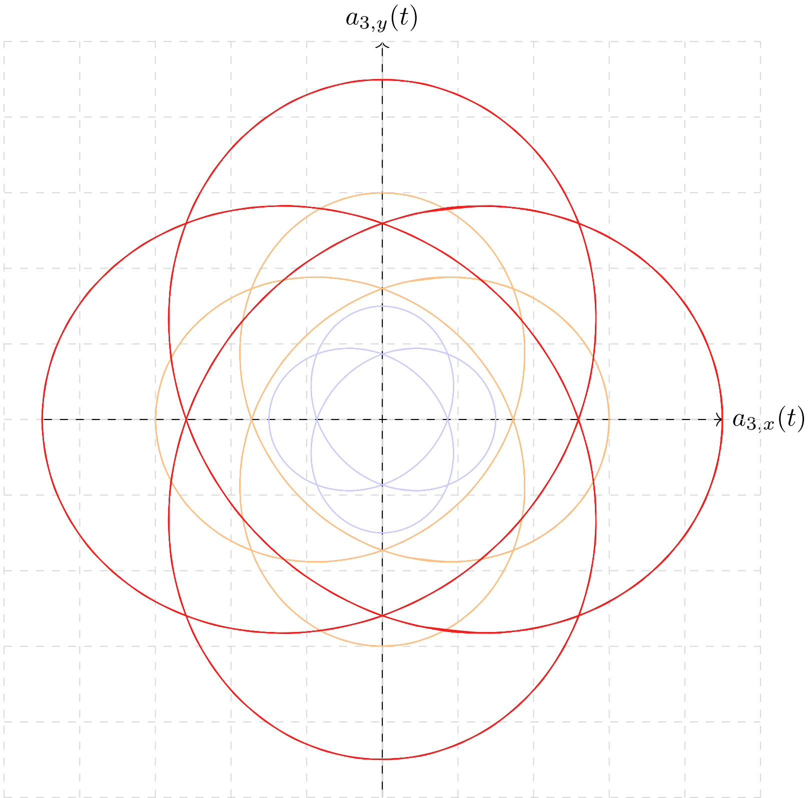


Figure 20: $c = 0.5,\ 1,\ 1.5$ (blue, orange, red); $\boldsymbol{a}_2(t) = 3c((\cos t)(2\sin^2 t - \cos^2 t), (\sin t)(2\cos^2 t - \sin^2 t))$

Remark 29. *We see from the equations that c is a scaling factor, operating on both axes. If we increase it, we will go higher (y-axis) and further away (x-axis) in the same amount of time t, hence we'll need greater speed, in both axis, and greater acceleration, again on both axes.*

$\boldsymbol{r_3(t) = (c(t - \sin t), c(1 - \cos t))}$

$$\begin{aligned}
\boldsymbol{r_3}(t) &= (c(t - \sin t), c(1 - \cos t)) \\
\boldsymbol{v_3}(t) = \dot{\boldsymbol{r_3}}(t) &= \boxed{c(1 - \cos t, \sin t)} \\
\boldsymbol{a_3}(t) = \dot{\boldsymbol{v_3}}(t) = \ddot{\boldsymbol{r_3}}(t) &= \boxed{c(\sin t, \cos t)}
\end{aligned}$$

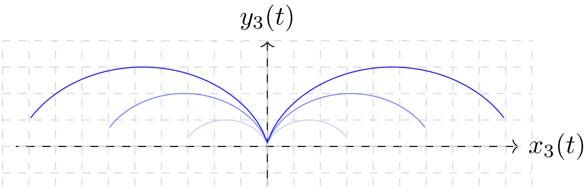


Figure 21:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{r}_3(t) = (c(t - \sin t), c(1 - \cos t))$

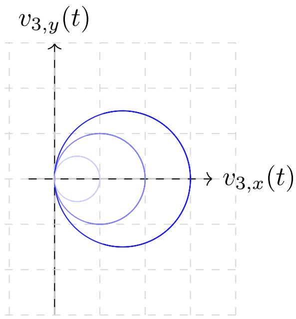


Figure 22:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{v}_3(t) = c(1 - \cos t, \sin t)$

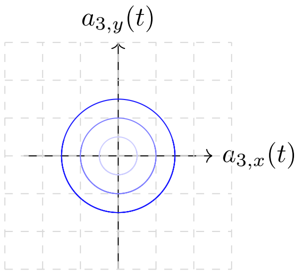


Figure 23:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{a}_3(t) = c(\sin t, \cos t)$

Remark 30. *As for the previous exercise, c is a scaling factor, with the same kind of impact as before.*

Interlude 2: Integral Calculus

Integral Calculus

Exercise 1/4

Exercise 18. *Determine the indefinite integral of each of the following expressions by reversing the process of differentiation and adding a constant.*

$$\begin{aligned} f(t) &= t^4 \\ f(t) &= \cos t \\ f(t) &= t^2 - 2 \end{aligned}$$

Because this is the first integration exercise, we'll go slow. We will "implement" the reversing of the process of differentiation by applying the *fundamental theorem of calculus*, on a few previously established differentiation results; let's recall those:

$$\begin{aligned} \frac{d}{dt} t^n &= nt^{n-1} \\ \frac{d}{dt} \sin t &= \cos t \end{aligned}$$

Let's start by integrating both sides of each equation:

$$\begin{aligned} \int \frac{d}{dt} t^n \, dt &= \int nt^{n-1} \, dt \\ \int \frac{d}{dt} \sin t \, dt &= \int \cos t \, dt \end{aligned}$$

Let's then recall the second form of the *fundamental theorem of calculus* given in the book:

$$\int \frac{d}{dt} f \, dt = f(t) + c, \quad c \in \mathbb{R}$$

So our previous equations can be rewritten as:

$$\begin{aligned} t^n + c &= \int nt^{n-1} \, dt, \qquad c \in \mathbb{R} \\ \sin t + c &= \int \cos t \, dt, \qquad c \in \mathbb{R} \end{aligned}$$

Which are, to syntactical differences, the formulas given in the book. In addition to those, we will also rely on the *linearity of the integration*, which essentially is the combination of the *sum rule for integration* and *multiplication by a constant rule for integration*, both being analogues of what we had for differentiation, and which can be summed up by:

Theorem 22 (linearity of integration)**.**

$$\left(\forall(\alpha,\beta)\in\mathbb{R}^2\right),\left(\forall(\varphi,\psi)\in(C^0)^2\right)\quad\boxed{\int\alpha\varphi+\beta\psi=\alpha\int\varphi+\beta\int\psi}$$

Remark 31. *C^0 refers to the class ("set") of continuous functions; actually, mathematically-wise, it would suffice for the functions to be "partially continuous" so as to be integrable; in the context of physics, requiring them to be continuous is reasonable.*

Remark 32. *We're using the following "shortcut" notation:*

$$\int\varphi=\int\varphi(t)\,dt$$

or for a more involved expression:

$$\int\alpha\varphi+\beta\psi=\int\left(\alpha\varphi(t)+\beta\psi(t)\right)\,dt$$

Proof. We can establish this result, again to syntactical differences, for instance through a similar process as we've just used for t^n and cos, that is, by integrating differentiation results:

$$\begin{array}{rrcl}
 & \frac{d}{dt}(\alpha\varphi+\beta\psi)(t) & = & \alpha\varphi'(t)+\beta\psi'(t)\\
\Leftrightarrow & \int\frac{d}{dt}(\alpha\varphi+\beta\psi)(t) & = & \int\alpha\varphi'(t)+\beta\psi'(t)dt\\
\Leftrightarrow & (\alpha\varphi+\beta\psi)(t) & = & \int\alpha\varphi'(t)+\beta\psi'(t)dt\\
\Leftrightarrow & \int\alpha\varphi'+\beta\psi' & = & \alpha\int\varphi'+\beta\int\psi'
\end{array}$$

□

$f(t)=t^4$
This is a simple application of:

$$\int nt^{n-1}\,dt=t^n+c,\quad c\in\mathbb{R}$$

with $n=4$; using the *linearity of integration*:

$$\begin{array}{rcl}
\int 5t^{5-1}\,dt & = & t^5+c,\quad c\in\mathbb{R}\\
\Leftrightarrow\int t^4\,dt & = & \boxed{\frac{1}{5}t^5+c}
\end{array}$$

Remark 33. *We can check the result by differentiating it*

$f(t)=\cos t$
An even more direct application of the formulas established earlier:

$$\int\cos t\,dt=\boxed{\sin t+c},\quad c\in\mathbb{R}$$

Remark 34. *Again, we can check the result using differentiation: we know from earlier that the derivative of a constant is zero, that of sine is cosine, and that the derivative of a sum is the sum of the derivatives.*

$f(t) = t^2 - 2$
Note that there's a special case for:

$$\int nt^{n-1}\,dt = t^n + c, \quad c \in \mathbb{R}$$

when $n = 1$:

$$\int 1 \times t^0\,dt = \int dt = t^1 + c = t + c, \quad c \in \mathbb{R}$$

More generally, by *linearity of the integration*:

$$(\forall \alpha \in \mathbb{R}) \quad \int \alpha\,dt = \alpha \int dt = \alpha t + c, \quad c \in \mathbb{R}$$

And so we have:

$$\int t^2 - 2\,dt = \int t^2\,dt - 2\int dt = \boxed{\frac{1}{3}t^3 - 2t + c}, \quad c \in \mathbb{R}$$

Which again is elementary to verify by differentiation.

Exercise 2/4

Exercise 19. *Use the fundamental theorem of calculus to evaluate each integral from Exercise* 1 *with limits of integration being* $t = 0$ *to* $t = T$.

We're going to build on the indefinite integrals we've just computed in I02E01, and simply evaluate the difference of the primitives between $t = T$ and $t = 0$.

Remark 35. *There are two common notations to evaluate a primitive between two values; I'll use the second one, out of habit. Let's recall the fundamental theorem of calculus along the way:*

$$\boxed{F(t)|_a^b = [F(t)]_a^b \triangleq \int_a^b F'(t)\,dt = F(b) - F(a)}$$

$f(t) = t^4$
The primitive was:

$$\frac{1}{5}t^5 + c, \quad c \in \mathbb{R}$$

Evaluated as expected gives:

$$\left[\frac{1}{5}t^5 + c\right]_0^T = \frac{1}{5}T^5 + c - (\frac{1}{5}0^5 + c) = \boxed{\frac{1}{5}T^5}$$

Remark 36. *Note how the constant of integration gets canceled. This will happen systematically here.*

$f(t) = \cos t$
The primitive was:

$$\sin t + c, \quad c \in \mathbb{R}$$

Evaluated as expected gives:

$$[\sin t + c]_0^T = \sin T - \underbrace{\sin 0}_{=0} = \boxed{\sin T}$$

$f(t) = t^2 - 2$
The primitive was:

$$\frac{1}{3}t^3 - 2t + c, \quad c \in \mathbb{R}$$

Evaluated as expected gives:

$$\left[\frac{1}{3}t^3 - 2t + c\right]_0^T = \boxed{\frac{1}{3}T^3 - 2T}$$

Exercise 3/4

Exercise 20. *Treat the expressions from Exercise* 1 *as expressions for the acceleration of a particle. Integrate them once, with respect to time, and determine the velocities, and a second time to determine the trajectories. Because we will use* t *as one of the limits of integration we will adopt the dummy integration variable* t'. *Integrate them from* $t' = 0$ *to* $t' = t$.

$$\begin{aligned} v(t) &= \int_0^t t'^4 \, dt' \\ v(t) &= \int_0^t \cos t' \, dt' \\ v(t) &= \int_0^t (t'^2 - 2) \, dt' \end{aligned}$$

Conceptually, the exercise seems to be about creating functions from integrals, by having a variable of the integration limit being a function parameter, which forces to use a different name for the integration variable; multiple choices are obviously possible, as long as the result is consistent:

$$v(t) = \int_0^t a(T) \, dT$$

$\boldsymbol{v(t) = \int_0^t t'^4 dt'}$

Let's recall our results from either I02E01 or I02E02. If we work from the former, we first would need to change the variable name from the primitive, say to t', and evaluate the indefinite integral between t and 0. Working from the later, we would need to replace T by t.

$$\begin{aligned} v(t) &= \int_0^t t'^4 dt' \\ &= \left[\frac{1}{5}t'^5 + c\right]_0^t, \quad c \in \mathbb{R} \\ &= \boxed{\frac{1}{5}t^5} \end{aligned}$$

Now we can repeat the same process to compute the position $x(t)$:

$$\begin{aligned} x(t) &= \int_0^t v(t') \, dt' \\ &= \int_0^t \frac{1}{5}t'^5 \, dt' \\ &= \left[\frac{1}{30}t'^6 + c\right]_0^t, \quad c \in \mathbb{R} \\ &= \boxed{\frac{1}{30}t^6} \end{aligned}$$

$\boldsymbol{v(t) = \int_0^t \cos t' dt'}$

Same exact process:

$$
\begin{aligned}
v(t) &= \int_0^t \cos t' \, dt' \\
&= [\sin t' + c]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{\sin t} \\
x(t) &= \int_0^t v(t') \, dt' \\
&= \int_0^t \sin t' \, dt' \\
&= [-\cos t' + c]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{-\cos t}
\end{aligned}
$$

$\boldsymbol{v(t) = \int_0^t (t'^2 - 2)\, dt'}$

No surprises here either:

$$
\begin{aligned}
v(t) &= \int_0^t (t'^2 - 2) \, dt' \\
&= \left[\frac{1}{3}t'^3 - 2t' + c\right]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{\frac{1}{3}t^3 - 2t} \\
x(t) &= \int_0^t v(t') \, dt' \\
&= \int_0^t \frac{1}{3}t'^3 - 2t' \, dt' \\
&= \left[\frac{1}{12}t'^4 - t'^2 + c\right]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{\left(\frac{1}{12}t^2 - 1\right) t^2}
\end{aligned}
$$

Integration by Parts

Exercise 4/4

Exercise 21. *Finish evaluating*

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

We will re-compute the integral from the beginning.

Let's start by recalling the formula for the integration by parts, given two real valued functions φ and ψ continuously differentiable on $[a, b] \subset \mathbb{R}$:

$$\boxed{\int_a^b \varphi'(x)\psi(x) \, dx = [\varphi(x)\psi(x)]_a^b - \int_a^b \varphi(x)\psi'(x) \, dx}$$

Now, in our integral, let's identify:

$$
\begin{aligned}
\varphi'(x) &= x \\
\psi(x) &= \cos x
\end{aligned}
$$

From which we have immediately:

$$
\begin{aligned}
\varphi(x) &= \frac{1}{2}x^2 \\
\psi'(x) &= \sin x
\end{aligned}
$$

Our integral then becomes:

$$\int_0^{\pi/2} x\cos x\,dx = \left[\frac{1}{2}x^2\cos x\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{2}x^2\sin x\,dx$$

Well, it didn't really helped us, didn't it? But this gives us a clue on what perhaps we should try: while the sine/cosine will essentially oscillate between each other when integrating/differentiating, the x will square when being integrated, but will vanish when being differentiated. So instead, let's identify things this way:

$$\begin{array}{rclrcl} \varphi'(x) &=& \cos x; & \varphi(x) &=& \sin x \\ \psi(x) &=& x; & \psi'(x) &=& 1 \end{array}$$

And now the integral becomes:

$$\begin{aligned} \int_0^{\pi/2} x\cos x\,dx &= [x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx \\ &= \frac{\pi}{2} + [\cos x]_0^{\pi/2} \\ &= \boxed{\frac{\pi}{2} - 1} \end{aligned}$$

Lecture 3: Dynamics

Aristotle's Law of Motion

Exercise 1/4

Exercise 22. *Given a force that varies with time according to $F = 2t^2$, and with the initial condition at time zero, $x(0) = \pi$, use Aristotle's law to find $x(t)$ at all times.*

Let us recall that Aristotle's law of motion is defined, for a one-dimensional particle (otherwise, $F(t)$ and $x(t)$ would be vector-values functions $\boldsymbol{F}(t)$ and $\boldsymbol{x}(t)$) earlier in the book as:

$$\frac{d}{dt}x(t) = \frac{F(t)}{m}$$

And that by integrating both sides, thanks to the fundamental theorem of calculus[10], assuming the mass is constant over time, we obtain:

$$x(t) = \frac{1}{m}\int F(t)\,dt$$

Which is our case, for $F(t) = 2t^2$, develops in:

$$\begin{aligned} x(t) &= \frac{1}{m}\int 2t^2\,dt \\ &= \frac{2}{3m}t^3 + c,\ c \in \mathbb{R} \end{aligned}$$

The initial condition $x(0) = \pi$ implies that $c = \pi$, hence the position at all time would be:

$$\boxed{x(t) = \frac{2}{3m}t^3 + \pi} \quad \square$$

[10] https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

Mass, Acceleration and Force

Some Simple Examples of Solving Newton's Equations

Exercise 2/4

Exercise 23. *Integrate this equation. Hint: Use definite integrals.*

The equation in question resulting from Newton's second law in the case of a constant force F_z being applied to an object of mass m following the z-axis:

$$\dot{v_z} = \dot{v_z(t)} = \frac{F_z}{m}$$

By integrating both sides, thanks to the fundamental theorem of calculus, assuming the mass is constant over time, we obtain:

$$\begin{aligned} v_z(t) &= \int \frac{F_z}{m}\, dt \\ &= \frac{F_z}{m} \int dt \\ &= \frac{F_z}{m} t + c,\ c \in \mathbb{R} \end{aligned}$$

Generally, c would be determine from an initial condition $v_z(0)$, which is our case, would precisely be c, hence:

$$\boxed{v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

Which is exactly the solution proposed in the book.

Exercise 3/4

Exercise 24. *Show by differentiation that this satisfies the equation of motion.*

Contrary to the previous exercise, instead of integrating to find the solution, we start from the solution and climb back to our original equation of motion, which are, in the case of a constant force F_z applied to a mass m following the z-axis:

$$v_z(t) = \dot{z}(t) = v_z(0) - \frac{F_z}{m} t$$

The proposed solution is:

$$z(t) = z_0 + v_z(0)t + \frac{F_z}{2m} t^2$$

Immediately, by derivation, constants goes to 0, t becomes 1 and t^2 becomes $2t$, we indeed obtain:

$$\boxed{\frac{d}{dt} z(t) = \dot{z}(t) = v_z(t) = v_z(0) + \frac{F_z}{m} t} \quad \square$$

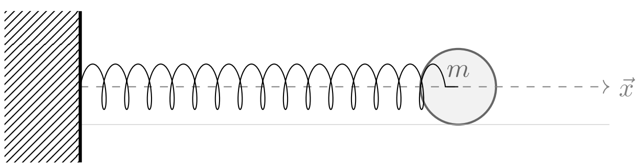
Exercise 4/4

Exercise 25. *Show by differentiation that the general solution to Eq.* (6) *is given in terms of two constants A and B by*

$$x(t) = A \cos \omega t + B \sin \omega t$$

Determine the initial position and velocity at time $t = 0$ in terms of A and B.

We're in the case of a 1-dimensional harmonic oscillator, depicted below as a mass m set on a frictionless ground attached to a fixed element to the left by a horizontally positioned spring of constant k.



Remark 37. *Almost equivalently, we could have worked in a vertical setting, which, on one hand, would have avoided us the need for a frictionless ground, but on the other hand, would make the equation slightly more complicated.*

There are interesting, complementary aspects to both cases. Because the differences are minute, we can explore both of them here.

Harmonic oscillators play a major role in physics, as explained by Feynman in Chapter 21 of the first volume of his *Lectures on Physics*:

> The harmonic oscillator, which we are about to study, has close analogs in many other fields; although we start with a mechanical example of a weight on a spring, or a pendulum with a small swing, or certain other mechanical devices, we are really studying a certain differential equation. This equation appears again and again in physics and in other sciences, and in fact it is a part of so many phenomena that its close study is well worth our while.
>
> Some of the phenomena involving this equation are the oscillations of a mass on a spring; the oscillations of charge flowing back and forth in an electrical circuit; the vibrations of a tuning fork which is generating sound waves; the analogous vibrations of the electrons in an atom, which generate light waves; the equations for the operation of a servosystem, such as a thermostat trying to adjust a temperature; complicated interactions in chemical reactions; the growth of a colony of bacteria in interaction with the food supply and the poisons the bacteria produce; foxes eating rabbits eating grass, and so on; all these phenomena follow equations which are very similar to one another, and this is the reason why we study the mechanical oscillator in such detail.

Thus, we'll take the time to analyze it more precisely than what is expected here: this is the occasion to present some aspects not covered, or not covered as explicitly in the book.

Despite everything happening in one-dimension, we'll use a vector position $\boldsymbol{x}$, which is a function of time. Hence, the vectors for speed and acceleration will respectively be $\boldsymbol{v} = \dot{\boldsymbol{x}} = \frac{d}{dt}\boldsymbol{x}(t)$ and $\boldsymbol{a} = \ddot{\boldsymbol{x}} = \frac{d^2}{dt^2}\boldsymbol{x}(t)$. Note that we're representing vectors with bold font instead of arrows; bold font is rather common in physics. This should be the only additional thing needed to follow through for someone having read the book up to this stage.

Contact force, normal force, friction force
The contact force is the force resulting from the contact of two objects. It is generally decomposed into vertical and horizontal components. The horizontal component perhaps is the most intuitive: it corresponds to friction forces. For instance, after having given a gentle push to an object an on ordinary table, it will only move up to a certain point: it is progressively stopped by the friction between the object and the table. You may think that this friction is caused at least in part because at a microscopic scale, both the object and the surface are far, far from being perfectly plane, and even more so at an atomic scale.

But this is false. As a counter-example, consider gauge blocks, manufactured to be extremely flat: their surfaces, far from being frictionless to each other, cling!

Because a rigorous understanding of friction forces requires advanced mechanics knowledge, we can satisfy ourselves with Feynman's vulgarization, from Chapter 12 of the first volume of his *Lectures on Physics*:

> There is another kind of friction, called dry friction or sliding friction, which occurs when one solid body slides on another. In this case a force is needed to maintain motion. This is called a frictional force, and its origin, also, is a very complicated matter. Both surfaces of contact are irregular, on an atomic level. There are many points of contact where the atoms seem to cling together, and then, as the sliding body is pulled along, the atoms snap apart and vibration ensues; something like that has to happen. Formerly the mechanism of this friction was thought to be very simple, that the surfaces were merely full of irregularities and the friction originated in lifting the slider over the bumps; but this cannot be, for there is no loss of energy in that process, whereas power is in fact consumed. The mechanism of power

> loss is that as the slider snaps over the bumps, the bumps deform and then generate waves and atomic motions and, after a while, heat, in the two bodies.
>
> [...]
>
> It was pointed out above that attempts to measure μ by sliding pure substances such as copper on copper will lead to spurious results, because the surfaces in contact are not pure copper, but are mixtures of oxides and other impurities. If we try to get absolutely pure copper, if we clean and polish the surfaces, outgas the materials in a vacuum, and take every conceivable precaution, we still do not get μ. For if we tilt the apparatus even to a vertical position, the slider will not fall off—the two pieces of copper stick together! The coefficient μ, which is ordinarily less than unity for reasonably hard surfaces, becomes several times unity! The reason for this unexpected behavior is that when the atoms in contact are all of the same kind, there is no way for the atoms to "know" that they are in different pieces of copper. When there are other atoms, in the oxides and greases and more complicated thin surface layers of contaminants in between, the atoms "know" when they are not on the same part. When we consider that it is forces between atoms that hold the copper together as a solid, it should become clear that it is impossible to get the right coefficient of friction for pure metals.

The normal force is the vertical component of the contact force. An object on a table is affected by the Earth gravity, but naturally resists going through the table, unless of course, the object is really massive and/or the table very weak. There's a bunch of complicated interactions at the atomic level from which this situation occurs: at a macroscopic scale, we simplify things and wrap this complexity by abstracting it as a (macroscopic) force.

Remark 38. *In the case of the vertical setup, with a mass attached to a vertical spring, the mass is not in contact with a ground surface, so there would have been no need to discuss contact forces.*

Remark 39. *A static object on a flat surface corresponds to a special case where the horizontal component of the contact force (friction) is null. Hence, the contact force and the normal force are, in such a special case, one and the same. The exact same thing would happen in the case of a frictionless surface.*

Newton's laws
Let us start by recalling Newton's laws of motion:

1. **Principle of inertia**: Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it;
2. "$\boldsymbol{F} = m\boldsymbol{a}$": The change of motion [*momentum*] of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed;
3. "action $\Rightarrow$ reaction": To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Remark 40. *A force always is applied from an object, to another object; with the restriction that an object cannot apply a force to itself.*

Remark 41. *The second law is a actually a statement about momentum, from which we can derive $\boldsymbol{F} = m\boldsymbol{a}$. More precisely, the momentum, called "motion" by Newton, is defined as $\boldsymbol{p} = m\boldsymbol{v}$, which literally captures the idea of the speed $\boldsymbol{v}$ of a certain amount of matter m. Then, it follows that the general case, $\boldsymbol{p}$ is as $\boldsymbol{v}$ a function of time.*
The "change of motion" then refers to an infinitesimal change of the momentum over time, mathematically captured by the time derivative $\frac{d}{dt}\boldsymbol{p}$. The law can then be progressively written, assuming the mass m is constant over time:

$$\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p} = \frac{d}{dt}m\boldsymbol{v} = m\frac{d}{dt}\boldsymbol{v} = m\boldsymbol{a} \quad \square$$

Remark 42. *It may seem weird to mention in the previous Remark 41 that we assume the mass m to be constant over time. In the current context, this will be the case, but in general, likely contrary to what any reasonable human would expect, they are exceptions. More precisely, in special relativity, the mass is a function of the velocity v:*

$$m_{rel} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We won't delve into the details here, but note that if $v = \|\boldsymbol{v}\|$ is much smaller that c, the speed of light, then the relative mass m_{rel} and m are very close.

To say it otherwise, this assumption doesn't hold at high velocities.

Remark 43. *In the second law, the force impressed (”$\boldsymbol{F}$”) is actually the force resulting from all the other forced applied to an object, $\sum_i \boldsymbol{F}_i$, which is sometimes called $\boldsymbol{F}_{net}$.*

For instance, if you push a cart, $\boldsymbol{F}_{net}$ should contain at least the force you are exerting, the gravity exerted by Earth, and the contact force generated by the ground on the cart's wheel. The sum of all the forces applied to the cart that will ultimately influence its motion.

As a result, in practice, when analyzing a situation so as to establish the equation of motion of an object, one should start by identifying all the external forces applied to this object.

Remark 44. *A common special case of the second law is to consider objects that, within a certain frame of reference, do not move. Hence, by definition, their speed $\boldsymbol{v}$ would be 0, and so would be their acceleration. That is, $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.*

Another special case for the second law is when things move at constant speed. Again, the acceleration would then be $\boldsymbol{0}$ and thus still $\sum_i \boldsymbol{F}_i = \boldsymbol{0}$.

Remark 45. *It is interesting to spend a moment to ponder what the mass m of an object really ”is”. Intuitively, for a non-physicist mass is another word for weight, but not to a physicist.*

Instead, in the context of the second law, physicists will observe that the more mass there is, the more force will be needed to accelerate an object. Conversely, the less mass there is, the less force will be needed to alter the motion of an object. Thus, they would conceptualize the mass as a ***measure of resistance to acceleration*** *(change of movement).*

This is the second time we meet a subtlety regarding masses, the first one being Remark 42 about how mass is a function of the velocity. We'll come back later once again in a moment at some more subtlety regarding mass.

Remark 46. *In the case of a static object laying on a flat surface, one may think that the gravitational force exerted by the Earth on the object and the normal force/contact force arising from the contact of the object with the surface are two opposite forces, in the sense of Newton's third law: after all, they are indeed of equal intensity and opposite directions.*

But this is incorrect: in such a situation, we actually have two pairs of opposite forces, in the sense of Newton's third law:

1. *$\boldsymbol{F}_{o,e}$ and $\boldsymbol{F}_{e,o}$, respectively the gravitational force exerted by the object on Earth, and the gravitational force exerted by Earth on the object;*
2. *$\boldsymbol{F}_{o,t}$ and $\boldsymbol{F}_{t,o}$, respectively the contact force exerted by the object on the table, and the contact force exerted by the table on the object.*

Now to be more complete, there's even a third pair of forces, which would be in almost all practical situations negligible, namely, the gravitational forces exerted by the surface on the object, and by the object on the surface.

A good exercise to check one understanding of this topic would be to try to identify forces in the case of a hand pressed on a (static) wall, or in the case of two hands pressed against each other.

Remark 47. *Newton's third law is actually a consequence of the principle of conservation of momentum: essentially, in a closed system, the total momentum is a constant over time.*

This is a key idea in modern physics, and will be key later in the book.

Hooke's law

Hooke's law, as Newton's laws, is an *empirical law*, that is, which has been found by repeated experimentation. Essentially, it states that the force needed to extend or compress a spring by a given distance varies linearly with the distance during which the spring is extended/compressed. That linear factor k *is* the spring's constant. This force is thus a function of that distance/displacement $\boldsymbol{\delta}$:

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = k\boldsymbol{\delta}$$

The equation could be read as something like:

- If I don't compress/stretch the spring ($\boldsymbol{\delta} = \boldsymbol{0}$), no force is exerted;
- The more I compress/stretch the spring, the greater the force, where the relation between both varies linearly.

Remark 48. *In most practical cases, $\boldsymbol{\delta}$ will be a function of time $\boldsymbol{\delta}(t)$; thus, the force itself will be a function of time: $\boldsymbol{F}_s(\boldsymbol{\delta}(t))$.*

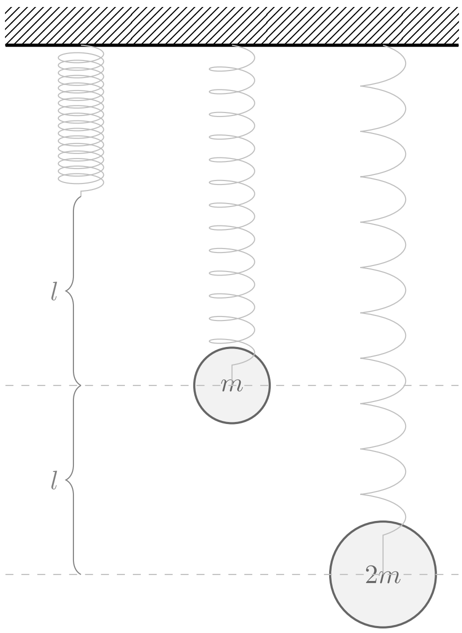
Remark 49. *By Newton's third law, if there's an object exerting a force on the spring, then the spring also exerts a force on the object, of equal intensity, but in the opposite direction. Hence, equivalently, Hooke's law could be reformulated in the form presented in the book:*

$$\boldsymbol{F}_s(\boldsymbol{\delta}) = -k\boldsymbol{\delta}$$

Remark 50. *You may sometimes find that last version of the law, as is the case in the book, written as $\boldsymbol{F}_s(\boldsymbol{x}) = -k\boldsymbol{x}$. This is a convenient choice that physicists can make, where, by identifying the origin of the coordinate system as the rest position of the mass, the measure of displacement $\boldsymbol{\delta}$ will indeed always match the position vector $\boldsymbol{x}$.*

This can be summarized by saying that bmF_s is the the force pulling the mass back to its rest position.

Remark 51. *If you have a spring or two around (architect lamps are a good source) and some regular weight (small tableware, such as iron tea cups would do), you should be able to easily verify this law experimentally. In this case, the objects' weight will be the source of a gravitational force (we'll come back to the gravitational force in a moment): doubling the mass (attaching two cups) should double the deformation of the spring.*



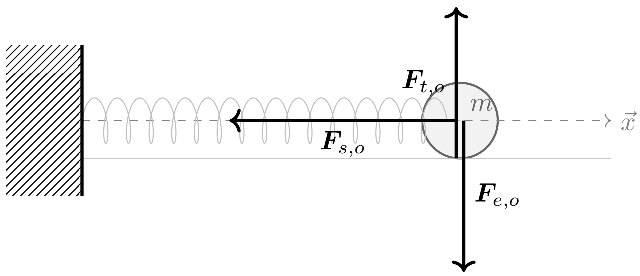
Remark 52. *Were you at least to imagine the previous experiment, you would realize that there must so some limits to this law: the spring could break or be irremediably deformed. Hooke's law is actually a first-order linear approximation, that works reasonably well in plenty of practical cases. There are more advanced models who generalize Hooke's law, but they are of no use to us in this context.*

Forces diagram
The goal of a force diagram, or a free body diagram, is to identify the forces acting on an object, with the intent of applying Newton's second law, so as to later determine an equation of movement for that object: we want to identify the sum of all the forces **acting on the object**, $\boldsymbol{F}_{\text{net}} = \sum_i \boldsymbol{F}_i$, who are likely to contribute to the shape of its trajectory. Let's complete the diagram of our horizontal spring setup with:

$\boldsymbol{F}_{e,o}$ the gravitational force exerted by the earth on our object. Note that we haven't talked about gravitation yet, and as you'll see in a moment, we don't need to, but this will be necessary for the vertical spring setup;

$\boldsymbol{F}_{t,o}$ the normal force exerted by the table on our object. Because the surface is frictionless, the normal force *is* the contact force;

$\boldsymbol{F}_{s,o}$ the force exerted by the spring on our object, which can be described by Hooke's law.



Placement of the origin of the $\vec{x}$-axis As we've already observed before, it can be interesting to identify the origin of our coordinate system to the position where the system will be at equilibrium/rest.

Intensities and signs Thanks the previous convention, as discussed in Remark 50, Hooke's law can be used to determine the force exerted by the spring on the object, and can be written as $\boldsymbol{F}_{s,o} = -k\boldsymbol{x}$.

We know, from Newton's second law, that the contact/normal force exerted by the table on the object is equal to $-\boldsymbol{F}_{o,t}$, where $\boldsymbol{F}_{o,t}$ is the force exerted by the object on the table, which itself results from the gravitational force $\boldsymbol{F}_{e,o}$, i.e.

$$\boldsymbol{F}_{t,o} = -\boldsymbol{F}_{o,t} = -\boldsymbol{F}_{e,o}$$

As we'll see quickly, we don't actually need to compute anything more, in this situation. We'll nevertheless show how to compute the gravitational force in the next section, when considering the similar case of a mass attached to a vertical spring.

Applying Newton's second law We can now use Newton's second law, $\sum_i \boldsymbol{F}_i = m\boldsymbol{a} = m\ddot{\boldsymbol{x}}$ to deduce an equation of motion:

$$\begin{aligned} & \boldsymbol{F}_{e,o} + \boldsymbol{F}_{t,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow\quad & \boldsymbol{F}_{e,o} - \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow\quad & \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow\quad & -k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \quad \square \end{aligned}$$

Which is a second order, linear differential equation, homogeneous, with constant coefficients. It's often rewritten, as is the case in the book, by defining $\omega^2 = \dfrac{k}{m}$:

$$\boxed{\ddot{\boldsymbol{x}} = -\omega^2 \boldsymbol{x}}$$

Gravitational force
To handle the case of a vertical spring, we won't be able to brush the gravitational force away as we did previously, and we'll need to compute it.

There are a few different ways to reach a formula for the gravitational force, but perhaps the simplest one, in the context of studying a mass attached to a vertical spring, would be to empirically study the fall of an object of mass m in Earth's gravitational field.

One could proceed for instance by video-taping a vertically falling object of known mass m. The video can then be discretized into a series of images, on which we could measure the evolution of the position of the mass over time. From there, we could compute the speed, as the variation of position, and the acceleration, as the variation of speed.

Then, neglecting the air resistance (well, we *could* perform the experiment in a vacuum), we can pretend that there's a single force acting on the mass, which *is* our gravitational force.

The experiment could obviously be performed multiple times to improve the accuracy. In the end, physicists have found that Earth's gravity is an acceleration vector $\boldsymbol{g}$ of intensity $9.81\, m.s^{-2}$, directed towards the center of the Earth, which experimentally gives a gravitational force of:

$$\boldsymbol{F}_{e,o} = m\boldsymbol{g}$$

Remark 53. *A home-made reenactment of the experiment would be difficult, especially if there's a need for strong precision, which would involve the presence of a vacuum. At least, more difficult than experimentally validating Hooke's law on a single spring.*

Historically, a simpler and more primitive setup was employed, for instance by Galileo, which would involved measuring the speed of a ball rolling on an inclined plane. A great way to lower the speed of the "fall" and thus to get more accurate time measurements.

Remark 54. *1) Note that $\boldsymbol{g}$ captures not only the gravitational force exerted by the Earth, but also captures the contribution of the Earth's rotation (centrifugal force).*

2) Because the Earth is not a perfect sphere, the value of $\|\boldsymbol{g}\|$ is not uniform everywhere on Earth; there are noticeable differences from city to city for instance.

Remark 55. *In Remark 45, we conceptualized the mass as a measure of resistance to acceleration. Physicists call this the inertial mass. And they conceptually distinguish it from what they call the gravitational mass, which is the property of matter that qualify how an object will interact with a gravitational field.*

We tacitly assumed that both are the same: this is the equivalence principle, a cornerstone of modern theories of gravities. So far, physicists haven't been able to distinguish the two masses experimentally either. But conceptually, they refer to two distinct notions.

Mathematically, this means that in the case of a falling object subject to a single force exerted by gravitation, by applying Newton's second law, we have:

$$\begin{aligned} & \boldsymbol{F}_{net} = \boldsymbol{F}_{e,o} = m_i\boldsymbol{a} \\ \Leftrightarrow \quad & m_g\boldsymbol{g} = m_i\boldsymbol{a} \\ \Leftrightarrow \quad & \boldsymbol{g} = \boldsymbol{a} \end{aligned}$$

Remark 56. *1) Another approach could have been to rely on Newton's law of universal gravitation.*

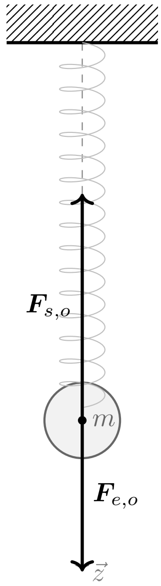
$$\boldsymbol{F}_{e,o} = -G\frac{m_e m_o}{\|\boldsymbol{r}_{e,o}\|^2}\hat{\boldsymbol{r}}_{e,o}$$

But not only is this law also empirical, but it would involve knowing the mass of the Earth m_e, the gravitational constant G, etc.

2) Similarly we could delve into Einstein's theory of general relativity, but the prerequisites would again have been more sophisticated than exposing a basic experiment.

Vertical spring

We know have all we need to study a variant of our previous setup, slightly more complicated mathematically, where a mass m is attached to the end side of a vertical spring of constant k, itself firmly attached to a fixed element at its top:



We'll choose the same convention regarding the origin of our coordinate system (see Remark 50). Contrary to the previous case, there's no other force to counterbalance the gravitational force $\boldsymbol{F}_{e,o}$; applying Newton's second law gives:

$$\begin{aligned} & \sum_i \boldsymbol{F}_i = \boldsymbol{F}_{e,o} + \boldsymbol{F}_{s,o} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & m\boldsymbol{g} - k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \\ \Leftrightarrow \quad & \ddot{\boldsymbol{x}} = -\omega^2\boldsymbol{x} + \boldsymbol{g} \end{aligned}$$

Again, with $\omega^2 = \dfrac{k}{m}$; so the only difference is that we get an additional constant $\boldsymbol{g}$, but this still is a second-order linear differential equation with constant coefficients.

Remark 57. *However, this differential equation is said to be non-homogeneous because of the non-zero constant* $\boldsymbol{g}$*, by opposition with the previous homogeneous equation obtained from the horizontal spring setup.*

Verifying the solution

Solving differential equations, mathematically, can be rather involved. For such simple equations, mathematicians themselves may find it sufficient to make an educated guess by tweaking an exponential-based function, despite having explored the subject with much more depth.

We won't discuss the details here, and will pragmatically satisfy ourselves with verifying that the proposed solution actually solves the equation:

$$x(t) = A\cos\omega t + B\sin\omega t$$

Remark 58. *We've been before expressing all our equations with vectors so far, but because we're working in one-dimension, we can divide everything by* $\hat{\boldsymbol{x}}$ *and work from the resulting scalar equations.*

And indeed, the proposed solution does solve the equation for the horizontal spring setup.

$$\begin{aligned}\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\ &= -\omega^2(A\cos\omega t + B\sin\omega t) \\ &= -\omega^2 x(t) \quad \square\end{aligned}$$

But it doesn't solve the vertical setup; we'll get there quickly, this isn't much more complicated. You may want to try by yourself to see if you can tweak the solution of the homogeneous equation to solve the non-homogeneous one.

Initial conditions
Let's start by computing the expression of the velocity:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Then, at $t = 0$, we have:

$$\begin{aligned}x(0) &= \quad A\cos(0) + B\sin(0) = \boxed{B} \\ \dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square\end{aligned}$$

Equations and initial conditions for the vertical setup
There are general mathematical methods for reaching a solution to a non-homogeneous linear second-order differential equation with constant coefficient from a homogeneous, but again, suffice for us to verify that the following would work:

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}$$

Indeed:

$$\begin{aligned}\ddot{x}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\ &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t - \frac{\omega^2}{\omega^2}g + g \\ &= -\omega^2(A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}) + g \\ &= -\omega^2 x(t) + g \quad \square\end{aligned}$$

From there, we can again express the initial position and speed as functions of A and B, first by computing the expression of the speed:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

And then by looking at what's happening at $t = 0$:

$$\begin{aligned}x(0) &= \quad A\cos(0) + B\sin(0) + \frac{g}{\omega^2} = \boxed{B + \frac{g}{\omega^2}} \\ \dot{x}(0) &= -A\omega\sin(0) + B\omega\cos(0) = \boxed{A\omega} \quad \square\end{aligned}$$

Interlude 3: Partial Differentiation

Partial Derivatives

Exercise 1/2

Exercise 26. *Compute all first and second partial derivatives —including mixed derivatives—of the following functions.*

$$x^2 + y^2 = \sin(xy)$$

$$\frac{x}{y}e^{(x^2+y^2)}$$

$$e^x\cos y$$

This is again a simple differentiation exercise. We're not going to go too much in details; you may want to refer to L02E01 if you need a more detailed treatment. The process is very mechanical: use linearity to isolate constants and propagate differentiation to individual terms, if there's a product of functions, use the product rule, and if you can represent an expression as a composition of functions, often by introducing intermediate functions, apply the chain rule.

Regarding partial differentiation, the key thing is to consider all arguments of a function to be constants but the one we're differentiating the function against.

$\boldsymbol{E(x,y) : x^2 + y^2 = \sin(xy)}$
This looks more like an expression than a function; we'll interpret its differentiation to be the differentiation of each part of the equality.

$$\boxed{\frac{\partial}{\partial x}E(x,y) : 2x = y\cos(xy);} \qquad \boxed{\frac{\partial}{\partial y}E(x,y) : 2y = x\cos(xy)}$$

We may now compute second order derivatives:

$$\boxed{\frac{\partial^2}{\partial x^2}E(x,y) : 2 = -y^2\sin(xy);} \qquad \boxed{\frac{\partial^2}{\partial y^2}E(x,y) : 2 = -x^2\sin(xy)}$$

And assuming the symmetry of second derivatives:

$$\frac{\partial^2}{\partial x \partial y}E(x,y) = \frac{\partial^2}{\partial y \partial x}E(x,y) : \boxed{2 = \cos(xy) - xy\sin(xy)}$$

Remark 59. *The fact that:*

$$\frac{\partial^2}{\partial x \partial y}\varphi = \frac{\partial^2}{\partial y \partial x}\varphi$$

Isn't so obvious, mathematically speaking: the result is called Clairaut's theorem, or Schwarz's theorem[11]*. It requires φ to have* ***continuous second partial derivatives****. In the context of classical mechanics, almost always we'll be dealing with smooth*[12] *functions of time (positions/velocities/accelerations, so we'll always assume it to be true.*

$\boldsymbol{\varphi(x,y) = \frac{x}{y}e^{(x^2+y^2)}}$
First order derivatives; we can go a little slower here. Essentially, reserve the constant $(1/y)$, apply the product rule followed by a chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi(x,y) &= \frac{1}{y}\frac{\partial}{\partial x}xe^{(x^2+y^2)} \\
&= \frac{1}{y}\left((\frac{\partial}{\partial x}x)e^{(x^2+y^2)} + x(\frac{\partial}{\partial x}e^{(x^2+y^2)})\right) \\
&= \frac{1}{y}\left(e^{(x^2+y^2)} + x(\frac{\partial}{\partial x}x^2 + y^2)e^{(x^2+y^2)})\right) \\
&= \boxed{\frac{1}{y}(2x^2+1)e^{(x^2+y^2)}}
\end{aligned}$$

Remark 60. *As I don't think this has been encountered before, note that we'll use the following "identity":*

$$x^{-n} = \frac{1}{x^n}$$

to help compute the derivatives of x^{-n} using the rule to differentiate x^n:

$$\frac{d}{dx}\frac{1}{x^n} = \frac{d}{dx}x^{-n} = -nx^{-n-1} = -n\frac{1}{x^{n+1}}$$

[11] https://en.wikipedia.org/wiki/Symmetry_of_second_derivatives
[12] https://en.wikipedia.org/wiki/Smoothness

And so for the other first order-derivative:

$$\begin{aligned}\frac{\partial}{\partial y}\varphi(x,y) &= x\frac{\partial}{\partial y}y^{-1}e^{(x^2+y^2)} \\ &= \boxed{xe^{(x^2+y^2)}(2-\frac{1}{y^2})}\end{aligned}$$

Then for the (non-mixed) second order derivatives:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\varphi(x,y) &= \frac{1}{y}\frac{\partial^2}{\partial x^2}(2x^2+1)e^{(x^2+y^2)}; \\ &= \frac{1}{y}e^{(x^2+y^2)}(4x+(2x^2+1)2x); \\ &= \boxed{\frac{x}{y}(4x^2+6)e^{(x^2+y^2)};}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial y^2}\varphi(x,y) &= x\frac{\partial^2}{\partial y^2}e^{(x^2+y^2)}(2-y^{-2}) \\ &= xe^{(x^2+y^2)}((2-y^{-2})2y+2y^{-3}) \\ &= \boxed{2xe^{(x^2+y^2)}(2y-\frac{1}{y}+\frac{1}{y^3})}\end{aligned}$$

Finally, for the mixed second derivatives:

$$\frac{\partial^2}{\partial x \partial y}\varphi(x,y) = (2x^2+1)e^{(x^2+y^2)}(-y^{-2}+y^{-1}2y) = \boxed{(2x^2+1)e^{(x^2+y^2)})(2-\frac{1}{y^2})}$$

Remark 61. *There's a common shortcut notation for partial derivatives that we will use from now on:*

$$\frac{\partial}{\partial x}\varphi = \varphi_x; \quad \frac{\partial^2}{\partial x^2}\varphi = \varphi_{x,x}; \quad \frac{\partial^2}{\partial y \partial x}\varphi = \varphi_{x,y}$$

$\boldsymbol{\phi(x,y) = e^x \cos y}$

$$\begin{aligned}\phi_x(x,y) &= \boxed{e^x\cos y;} & \phi_y(x,y) &= \boxed{-e^x\sin y} \\ \phi_{x,x}(x,y) &= \boxed{e^x\cos y;} & \phi_{y,y}(x,y) &= \boxed{-e^x\cos y}\end{aligned}$$

$$\phi_{x,y}(x,y) = \phi_{y,x}(x,y) = \boxed{-e^x\sin y}$$

Stationary Points and Minimizing Functions

Stationary Points in Higher Dimensions

Exercise 2/2

Lecture 4: Systems of More Than One Particle

Systems of Particles

The Space of States of a System of Particles

Momentum and Phase Space

Action, Reaction, and the Conservation of Momentum

Lecture 5: Energy

Force and Potential Energy

Exercise 1/3

Exercise 27. *Prove Eq.* (3)*. Hint: Use the product rule for differentiation.*

This is a very simple differentiation exercise. I'm going to be very slow, because using a "normal" pace would essentially mean stating the result, as it's been done in the book, hence negating the need for this exercise. If you can't compute this kind of derivative in your head at this stage, I would recommend practising until it becomes second nature. For instance, *Paul's Online Notes*[13] has plenty of corrected exercises on differentiation.

Let's recall the context a little bit: we're in the case of a single particle moving along the $\vec{x} - axis$ under the influence of a force $F(x)$. The *kinetic energy* of that particle, that is, the energy that the particle has because of its motion, is noted T and is "defined" as:

$$T = \frac{1}{2}mv^2$$

Eq. (3) refers to the time derivative of v^2, in the context of its usage in the formulation of kinetic energy. There are (at least) two ways to evaluate this derivative; let's start by using the authors' hint regarding the product rule; let's make the time dependency more obvious:

$$\frac{d}{dt}v(t)^2 = \frac{d}{dt}v(t)v(t)$$

Now let's recall the *product rule* is, for φ and ψ two real-valued functions of t:

$$(\varphi\psi)' = \varphi'\psi + \varphi\psi'$$

Hence, in the case of $\varphi = \psi = v$, we have:

$$\begin{aligned}\frac{d}{dt}v(t)^2 &= \frac{d}{dt}v(t)v(t) \\ &= \dot{v}(t)v(t) + v(t)\dot{v}(t) \\ &= \boxed{2v\dot{v}(t)} \quad \square\end{aligned}$$

Remark 62. *We could have also used the chain rule; for φ a function of t, and ψ and function whose domain (input) is the codomain (output) of φ:*

$$(\psi \circ \varphi)'(t) = (\psi(\varphi(t))' = \varphi'(t)\psi'(\varphi(t))$$

In this case, $\psi = (x \mapsto x^2)$ and $\varphi = v$, so:

$$\begin{aligned}\frac{d}{dt}v(t)^2 &= \frac{d}{dt}(x \mapsto x^2)(v(t)) \\ &= \dot{v}(t)(x \mapsto x^2)'(v(t)) \\ &= \dot{v}(t)(x \mapsto 2x)(v(t)) \\ &= \boxed{2v\dot{v}(t)} \quad \square\end{aligned}$$

Remark 63. *As a reminder, both the product rule and the chain rule have been proved in L02E04.*

More than one dimension

Exercise 2/3

Exercise 28. *Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitly that the total energy is conserved.*

Equations of motion
For this system, the potential energy V is:

[13] https://tutorial.math.lamar.edu/Problems/CalcI/DerivativeIntro.aspx

$$V = \frac{1}{2}k(x^2 + y^2) \tag{5}$$

By Newton's second law of motion, given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{6}$$

Or,

$$\begin{aligned} F_x =& \quad m\ddot{x} \\ F_y =& \quad m\ddot{y} \end{aligned} \tag{7}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{8}$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x}V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y}V(x, y) = -ky \end{aligned} \tag{9}$$

Combining (7) and (9):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \tag{10}$$

Which we known from L03E04: to be differential equations associated to harmonic motion, and (mathematically) solved by a slightly more general solution that the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{11}$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$\begin{aligned} v(t) = \quad \dot{x}(t) =& \quad \omega(-\alpha \sin(\omega t - \theta) + \beta \cos(\omega t - \theta) \\ a(t) = \quad \ddot{x}(t) =& \quad -\omega^2(\alpha \cos(\omega t - \theta) + \beta \sin(\omega t - \theta)) \\ =& \quad -\omega^2 x(t) \end{aligned} \tag{12}$$

Where, to differentiate e.g. $\alpha \cos(\omega t - \theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \tag{13}$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t = 0), \dot{x}(t = 0), y(t = 0), \dot{y}(t = 0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$\begin{aligned} x(0) &= \alpha_x \cos(0) + \beta_x \sin(0) \\ &= \alpha \\ \dot{x}(0) &= \omega(-\alpha \sin(0) + \beta \cos(0)) \\ &= \omega\beta \\ &= \sqrt{\frac{k}{m}}\beta \end{aligned} \tag{14}$$

Circular orbits

Remark 64. *This exercise was completed a few months before writing the present remark. The approach taken is mathematically interesting, but clumsy, from a physicist's point of view. Given the question, and given the fact that the simplest circular orbit has already been studied at the end of Chapter 2 (Motion), it would have been wiser to start by verifying this motion, rather than working from the most general solution to the involved differential equation.*

The quest for the general solution to the differential equation itself could and should, in the context of a physics exercise, have been polarized by a wiser choice of referential, to reduce the complexity of the solution, as a change of referential won't, fortunately, alter the laws of (classical) mechanics.

In short, the solution presented here was executed with a mathematician/computer scientist mindset (both being professionally conditioned to systematically handle special cases), while the present remark is written with a better understanding of physicists' approach.

You may want to refer to the next exercise, L05E03 for a more orthodox approach.

The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r, centered on point (a, b), with $(a, b, r) \in \mathbb{R}^3$ if:

$$(\forall t \geq 0),\ (x(t) - a)^2 + (y(t) - b)^2 = r^2 \tag{15}$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$$

Then, let's introduce two angles φ_x and φ_y such as:

$$\begin{array}{ll} \sin\varphi_x = \alpha_x & \sin\varphi_y = \alpha_y \\ \cos\varphi_x = \beta_x & \cos\varphi_y = \beta_y \end{array}$$

From, there, we can use the previous identity to rewrite our equations of motions (11) as:

$$\begin{aligned} x(t) &= \sin\varphi_x\cos(\omega t - \theta_x) + \cos\varphi_x\sin(\omega t - \theta_x) &&= \sin(\omega t + \varphi_x - \theta_x) = \sin\Omega_x \\ y(t) &= \sin\varphi_y\cos(\omega t - \theta_y) + \cos\varphi_y\sin(\omega t - \theta_y) &&= \sin(\omega t + \varphi_y - \theta_y) = \sin\Omega_y \end{aligned} \tag{16}$$

Obviously with $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$ and $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$.

Let us now develop (15) with those two versions of $x(t)$ and $y(t)$:

$$\begin{aligned} & (x(t) - a)^2 + (y(t) - b)^2 && = r^2 \\ \Leftrightarrow & (\sin\Omega_x - a)^2 + (\sin\Omega_y - b)^2 && = r^2 \\ \Leftrightarrow & \sin^2\Omega_x + \sin^2\Omega_y - 2(a\sin\Omega_x + b\sin\Omega_y)) + a^2 + b^2 && = r^2 \end{aligned}$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a, b) = (0, 0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$\sin^2\Omega_x + \sin^2\Omega_y = r^2$$

Which we can rewrite a little bit using the fact that $\sin\varphi = \cos(\varphi - \pi/2)$:

$$\sin^2\Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2$$

But we know the Pythagorean identity $\sin^2\varphi + \cos^2\varphi = 1$, hence we know there will be circular orbits when:

$$\left\{\begin{array}{lll} r & = & 1 \\ \Omega_x & = & \Omega_y - \dfrac{\pi}{2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{lll} r & = & 1 \\ \omega t + \varphi_x - \theta_x & = & \omega t + \varphi_y - \theta_y - \dfrac{\pi}{2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{lll} r & = & 1 \\ \varphi_x - \theta_x & = & \varphi_y - \theta_y - \dfrac{\pi}{2} \end{array}\right.$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (16) will be the same:

$$\boxed{T = \frac{2\pi}{\omega}}$$

Remark 65. *For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.*

Remark 66. *The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.*

Energy conservation
Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2}\sum_i m_i \dot{x_i}^2 \tag{17}$$

Which gives us for this system, expliciting the time-dependencies:

$$T(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}m\dot{y}(t)^2 = \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2) \tag{18}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned} \frac{d}{dt}T(t) &= \frac{1}{2}m(2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{19}$$

On the other hand, we can compute the variation of potential energy over time from (5), again using the chain rule:

$$\begin{aligned} \frac{d}{dt}V(t) &= \frac{1}{2}k(\frac{d}{dt}x(t)^2 + \frac{d}{dt}y(t)^2) \\ &= \frac{1}{2}k(2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= k(x(t)\dot{x}(t) + y(t)\dot{y}(t)) \end{aligned} \tag{20}$$

From (10), we have:

$$\begin{aligned} x(t) &= -\frac{m}{k}\ddot{x}(t) \\ y(t) &= -\frac{m}{k}\ddot{y}(t) \end{aligned} \tag{21}$$

Injecting in (20):

$$\begin{aligned} \frac{d}{dt}V(t) &= -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t)) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \end{aligned} \tag{22}$$

Thus from (19) and (22):

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square \tag{23}$$

That is, the total energy E doesn't change over time: it is indeed *conserved*.

Exercise 3/3

Exercise 29. *Rework Exercise 2 for the potential* $V = \frac{k}{2(x^2+y^2)}$. *Are there circular orbits? If so, do they all have the same period? Is the total energy conserved?*

Equations of motion

The approach is similar to what has been done for the previous exercise: for this system, the potential energy V is:

$$V = \frac{k}{2(x^2+y^2)} \tag{24}$$

By Newton's second law of motion[14], given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{25}$$

Or,

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \end{aligned} \tag{26}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{27}$$

As for the previous exercise, we make heavy use of the chain rule[15] for derivation:

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \tag{28}$$

To compute e.g. $F_x(x, y)$, we define $\phi(x) = x^2 + y^2$:

$$\begin{aligned} F_x(x,y) &= -\frac{\partial}{\partial x}V(x,y) \\ &= \frac{k}{2}\frac{d}{dx}\frac{1}{\phi(x)} \\ &= \frac{k}{2}\phi'(x)\frac{-1}{\sqrt{\phi(x)}} \\ &= \frac{kx}{(x^2+y^2)^2} \end{aligned} \tag{29}$$

Thus finally:

$$\begin{aligned} F_x(x,y) &= \frac{kx}{(x^2+y^2)^2} \\ F_y(x,y) &= \frac{ky}{(x^2+y^2)^2} \end{aligned} \tag{30}$$

Hence combining (30) and (26):

$$\begin{aligned} F_x(x,y) &= \boxed{m\ddot{x}(t) = k\frac{x(t)}{(x(t)^2+y(t)^2)^2}} \\ F_y(x,y) &= \boxed{m\ddot{y}(t) = k\frac{y(t)}{(x(t)^2+y(t)^2)^2}} \end{aligned} \tag{31}$$

Circular orbits

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[14] https://en.wikipedia.org/wiki/Newton%27s_laws_of_motion#Second

[15] https://en.wikipedia.org/wiki/Chain_rule

Let's make a guess, and see what would happen were we to plug the simplest circular motion, that we've already studied in the book at the end of Chapter 2 (Motion), given by:

$$x(t) = R\cos(\omega t); \qquad y(t) = R\sin(\omega t)$$

Which is very convenient for us, because if we try this solution in (31), the (common) denominator simplifies:

$$\left(x(t)^2 + y(t)^2\right)^2 = \left((R\cos(\omega t))^2 + (R\sin(\omega t))^2\right)^2 = R^4 \underbrace{(\cos^2(\omega t) + \sin^2(\omega t))^2}_{=1} = R^4$$

Let's now consider the velocities and accelerations we would obtain by differentiating our guess for $x(t)$ and $y(t)$:

$$\begin{aligned} \dot{x}(t) &= -R\omega\sin(\omega t); & \dot{y}(t) &= R\omega\cos(\omega t) \\ \ddot{x}(t) &= -R\omega^2\cos(\omega t); & \ddot{y}(t) &= -R\omega^2\sin(\omega t) \end{aligned}$$

There are two ways for this guess to actually work:

1. Either we set $\omega^2 = -k/mR^4$, which implies either:
   - $\boxed{k \text{ to be zero}}$ (trivial solution then);
   - or that mR to be close to infinite (unrealistic);
   - or $\boxed{\text{that } k \text{ is (strictly) negative;}}$
   - or that either m or R are negative (unrealistic);
   - or, mathematically, that ω is an imaginary (complex) number, which would be difficult to interpret, physically;
2. The other option would be for R to be negative, which again doesn't make a lot of sense, physically-wise.

Remark 67. *Note that our guess would have worked for a negated V:*

$$V = -\frac{k}{2(x^2 + y^2)}$$

Remark 68. *What is commonly referred to as "the trivial solution", especially in the context of differential equations, is the solution $x(t) = 0$, which is of little interest, mathematically and physically.*

We can conclude that, at least physically speaking, $\boxed{\textbf{there are no circular orbits}\text{, unless } k \text{ is negative}}$. This is because, if there were circular orbits, then they would be a coordinate change away from being in the form of our guess.

The only remaining issue is that k hasn't been clearly defined, physically speaking, so we can't really know for sure if assuming k to be negative (with a reminder that $k = 0$ leads to the trivial solution).

Remark 69. *Another approach, used for instance in the official solutions[16], relies on the polar coordinate (r, θ): the existence of a circular orbit then translate to r being a constant, or equivalently, $\dot{r} = 0$.*

We'll dive deeper into polar coordinates in a later exercise, alongside a bunch of other elements related to circular motion (L06E05, which involves a pendulum).

Energy conservation
Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$T = \frac{1}{2}\sum_i m_i \dot{x_i}^2 \tag{32}$$

[16] `http://www.madscitech.org/tm/slns/l5e3.pdf`

Which gives us for this system, expliciting the time-dependencies:

$$T(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}m\dot{y}(t)^2 = \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2) \tag{33}$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$\begin{aligned}\frac{d}{dt}T(t) &= \frac{1}{2}m(2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\ &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y})\end{aligned} \tag{34}$$

On the other hand, we can compute the variation of potential energy over time from (24). We'll use the chain rule again, with $\phi(t) = x(t)^2 + y(t)^2$ and thus:

$$\begin{aligned}\phi'(t) &= 2x'(t)x(t) + 2y'(t)y(t) \\ &= 2\dot{x}x + 2\dot{y}y\end{aligned}$$

It follows that:

$$\begin{aligned}\frac{d}{dt}V(t) &= \frac{d}{dt}\frac{k}{2(x(t)^2 + y(t)^2)} \\ &= \frac{k}{2}\frac{d}{dt}\phi(t)^{-1} \\ &= -\frac{k}{2}\phi'(t)\phi(t)^{-2} \\ &= -\frac{k}{2}\frac{2\dot{x}x + 2\dot{y}y}{(x(t)^2 + y(t)^2)} \\ &= -k\frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^2} \\ &= -k\frac{\dot{x}x + \dot{y}y}{\phi(t)^2}\end{aligned} \tag{35}$$

Then, from (31), we can extract

$$x(t) = \frac{m}{k}\ddot{x}\phi(t)^2; \qquad y(t) = \frac{m}{k}\ddot{y}\phi(t)^2$$

Injecting in (35) gives:

$$\begin{aligned}\frac{d}{dt}V(t) &= -\frac{k}{\phi(t)^2}\left(\dot{x}\frac{m}{k}\ddot{x}\phi(t)^2 + \dot{y}\frac{m}{k}\ddot{y}\phi(t)^2\right) \\ &= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y})\end{aligned} \tag{36}$$

And so by combining (36) and (34) we can indeed see that the energy is conserved:

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square$$

Lecture 6: The Principle of Least Action

The Transition to Advanced Mechanics

Action and the Lagrangian

Derivation of the Euler-Lagrange Equation

Exercise 1/6

Exercise 30. *Show that Eq.* (4) *is just another form of Newton's equation of motion $F = ma$.*

Where Eq. (4) are the freshly derived Euler-Lagrange equations of motions:

$$\frac{d}{dt}\frac{\partial}{\partial\dot{x}}L - \frac{\partial}{\partial x}L = 0 \tag{37}$$

In the context of a single particle moving in one dimension, with kinetic and potential energy given by:

$$T = \frac{1}{2}m\dot{x}^2$$
$$V = V(x)$$

From which results the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}^2 - V(x) \end{aligned} \tag{38}$$

Let us recall that we also have the *potential energy principle*, stated in one-dimension as Eq. (1) of the previous chapter, *Lecture 5: Energy*:

$$F(x) = -\frac{d}{dx}V(x) \tag{39}$$

Which is also stated more generally in that same chapter, for an abstract configuration space $\{x\} = \{x_i\}$, as Eq. (5):

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\})$$

Thus, deriving each part of (37) with our Lagrangian (38), and considering the *definition* of a potential energy $V(x)$ (39) yields:

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{x}}L &= \frac{d}{dt}m\dot{x} \\ &= m\ddot{x} \end{aligned} \qquad\qquad \begin{aligned} \frac{\partial}{\partial x}L &= \frac{\partial}{\partial x}V(x) \\ &= -F \end{aligned}$$

Then indeed, Euler-Lagrange equations become equivalent to Newton's law of motion:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \\ \Leftrightarrow &m\ddot{x} - (-F) = 0 \\ \Leftrightarrow &\boxed{F = m\ddot{x} = ma} \quad \square \end{aligned}$$

More Particles and More dimensions

Exercise 2/6

Exercise 31. *Show that Eq.* (6) *is just another form of Newton's equation of motion* $F_i = m_i\ddot{x}_i$.

Where Eq. (6) are the following set of equation, defined for all $i \in [\![1, n]\!]$:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}L\right) = \frac{\partial}{\partial x_i}L \tag{40}$$

Remark 70. *This exercise is simply a generalization of the previous exercise (L06E01) to a configuration space of size* $n \in \mathbb{N}$.

Then again, let us recall the Lagrangian defined slightly earlier in the related section of the book:

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}m_i\dot{x}_i{}^2\right) - V(\{x\}) \tag{41}$$

Hence, $(\forall i \in [\![1, n]\!])$:

$$\begin{aligned} \frac{\partial}{\partial \dot{x}_i}L &= \frac{\partial}{\partial \dot{x}_i}\sum_{j=1}^{n}\frac{1}{2}m_j\dot{x}_j{}^2 \\ &= \sum_{j=1}^{n}m_j\dot{x}_j\delta_{ij} \\ &= m_i\dot{x}_i \end{aligned} \tag{42} \qquad\qquad \frac{\partial}{\partial x_i}L = -\frac{\partial}{\partial x_i}V(\{x\})$$

Again, we need the *potential energy principle*, stated as Eq. (5) of the previous chapter *Lecture 5: Energy*, for abstract configuration space $\{x\} = \{x_i\}$, as:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i} V(\{x\}) \tag{43}$$

From which we can conclude, by injecting (43) in the second half of (42), and connecting each side with Euler-Lagrange's equations (40), ($\forall i \in [\![1, n]\!]$):

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \\ \Leftrightarrow &\frac{d}{dt} m_i \dot{x}_i = F_i(\{x\}) \\ \Leftrightarrow &\boxed{F_i = m_i \ddot{x}_i} \quad \square \end{aligned}$$

What's Good about Least Action?

Exercise 3/6

Exercise 32. *Use the Euler-Lagrange equations to derive the equations of motions from the Lagrangian in Eq.* (12).

Again, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: ($\forall i \in [\![1, n]\!]$),

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \tag{44}$$

In the case of this exercise, the Lagrangian L is defined in Eq. (12) as:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X)$$

Let's compute the partial derivatives of L on $\dot{X}$, X, $\dot{Y}$ and Y:

$$\begin{aligned} \frac{\partial}{\partial \dot{X}} L &= \frac{\partial}{\partial \dot{X}}\left(\frac{m}{2}\dot{X}^2 + m\omega\dot{X}Y\right) & \frac{\partial}{\partial X} L &= \frac{\partial}{\partial X}\left(\frac{m\omega^2}{2}X^2 - m\omega\dot{Y}X\right) \\ &= m\dot{X} + m\omega Y & &= m\omega^2 X - m\omega\dot{Y} \\ \\ \frac{\partial}{\partial \dot{Y}} L &= \frac{\partial}{\partial \dot{Y}}\left(\frac{m}{2}\dot{Y}^2 - m\omega\dot{Y}X\right) & \frac{\partial}{\partial Y} L &= \frac{\partial}{\partial Y}\left(\frac{m\omega^2}{2}Y^2 + m\omega\dot{X}Y\right) \\ &= m\dot{Y} - m\omega X & &= m\omega^2 Y + m\omega\dot{X} \end{aligned} \tag{45}$$

Finally, by plugging (45) into (44), we obtain:

$$\begin{aligned} &\frac{d}{dt}\left(m\dot{X} + m\omega Y\right) = m\omega^2 X - m\omega\dot{Y} & &\frac{d}{dt}\left(m\dot{Y} - m\omega X\right) = m\omega^2 Y + m\omega\dot{X} \\ \Leftrightarrow &\boxed{m\ddot{X} = m\omega^2 X - 2m\omega\dot{Y}} & \Leftrightarrow &\boxed{m\ddot{Y} = m\omega^2 Y + 2m\omega\dot{X}} \quad \square \end{aligned}$$

Remark 71. *Those results indeed matches the equations proposed in the book just slightly before this exercise.*

Exercise 4/6

Exercise 33. *Work out George's Lagrangian and Euler-Lagrange equations in polar coordinates.*

As always, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: ($\forall i \in [\![1, n]\!]$),

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i} L\right) = \frac{\partial}{\partial x_i} L \tag{46}$$

The original Lagrangian L in our case is defined by the Eq. (10) of this chapter as:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{47}$$

After the following coordinate shift (Eq. (9) of the book):

$$x = X\cos(\omega t) + Y\sin(\omega t) \qquad\qquad y = -X\sin(\omega t) + Y\cos(\omega t) \tag{48}$$

We obtained this Lagrangian (Eq. (12) of the book):

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X) \tag{49}$$

For the current exercise, the coordinate shift to polar equations is:

$$X = R\cos\theta \qquad\qquad Y = R\sin\theta \tag{50}$$

Where, implicitly, both R and θ are, as X and Y, functions of time.

Now, we have at least two ways of solving this exercise:

1. Either perform the coordinate shift (50) in (49): this will be a tedious but very similar development as the one performed in the book to obtain (49) from (47) and (48);

2. or perform this new coordinate shift (50) directly in the first coordinate shift (48), and work from the first Lagrangian (47) instead: some trigonometric identities are likely to ease at least the beginning of the work here.

We will try both approaches, and expect to find the exact same solutions in the end.

First approach
Let use start by computing the time derivative of X and Y as defined by (50), using the both the product [17] and the chain rule [18]:

$$\dot{X} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta \qquad\qquad \dot{Y} = \dot{R}\sin\theta + R\dot{\theta}\cos\theta \tag{51}$$

Remark 72. *For clarity, as a similar development will happen a few times, let's go into details for the first one: the product rule for two functions u and v of a single variable, of respective derivatives u' and v' is*

$$(uv)' = u'v + uv'$$

Now the chain rule is, again for the same kind of functions:

$$(u(v(x))' = v'(x)u'(v(x))$$

In the present case, we have a $X(t)$ defined as the product of two functions: $X(t) = R(t)\cos(\omega(t))$, where the second one is itself a composition of two functions $\cos(\omega(t))$. Hence, by applying first the product rule, we obtain:

$$X'(t) = R'(t)\cos(\omega(t)) + R(t)\left(\cos(\omega(t))\right)'$$

While the chain rule gives us:

$$(\cos(\omega(t)))' = -\omega'(t)\sin(\omega(t))$$

Hence,

$$X'(t) = R'(t)\cos(\omega(t)) - R(t)\omega'(t)\sin(\omega(t))$$

Now, our goal will be to plug (50) and (51) into the Lagrangian (49) obtained after the first coordinate shift, but doing that transformation at once will gives a difficult to read equation. Instead, we'll work in smaller steps, simplifying our results using trigonometric identities along the way.

[17] `https://en.wikipedia.org/wiki/Product_rule`
[18] `https://en.wikipedia.org/wiki/Chain_rule`

Let us start with X^2+Y^2, using the fact that $\sin^2\theta+\cos^2\theta=1$:

$$\begin{aligned} X^2+Y^2 &= R^2\cos^2\theta + R^2\sin^2\theta \\ &= R^2(\cos^2\theta+\sin^2\theta) \\ &= R^2 \end{aligned} \tag{52}$$

Now for $\dot{X}^2+\dot{Y}^2$, using the same trigonometric identity:

$$\begin{aligned} \dot{X}^2 &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)^2 \\ &= \dot{R}^2\cos^2\theta - 2R\dot{R}\dot{\theta}\cos\theta\sin\theta + R^2\dot{\theta}^2\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}^2 &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)^2 \\ &= \dot{R}^2\sin^2\theta + 2R\dot{R}\dot{\theta}\sin\theta\cos\theta + R^2\dot{\theta}^2\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}^2+\dot{Y}^2 &= \dot{R}^2(\cos^2\theta+\sin^2\theta) + R^2\dot{\theta}^2(\cos^2\theta+\sin^2\theta) \\ &= \dot{R}^2 + R^2\dot{\theta}^2 \end{aligned} \tag{53}$$

Finally, for $\dot{X}Y-\dot{Y}X$:

$$\begin{aligned} \dot{X}Y &= (\dot{R}\cos\theta - R\dot{\theta}\sin\theta)R\sin\theta \\ &= R\dot{R}\cos\theta\sin\theta - R^2\dot{\theta}\sin^2\theta \end{aligned} \qquad \begin{aligned} \dot{Y}X &= (\dot{R}\sin\theta + R\dot{\theta}\cos\theta)R\cos\theta \\ &= R\dot{R}\cos\theta\sin\theta + R^2\dot{\theta}\cos^2\theta \end{aligned}$$

$$\begin{aligned} \dot{X}Y-\dot{Y}X &= -R^2\dot{\theta}(\sin^2\theta+\cos^2\theta) \\ &= -R^2\dot{\theta} \end{aligned} \tag{54}$$

Now we're ready to plug (52), (53) and (54) into (49):

$$L = \boxed{\frac{m}{2}(\dot{R}^2+R^2\dot{\theta}^2) + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}} \tag{55}$$

Now, let's compute the partial derivatives of our new Lagrangian:

$$\begin{aligned} \frac{\partial}{\partial\dot{R}}L &= \frac{\partial}{\partial\dot{R}}\left(\frac{m}{2}\dot{R}^2\right) \\ &= m\dot{R} \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial R}L &= \frac{\partial}{\partial R}\left(\frac{m}{2}R^2\dot{\theta}^2 + \frac{m\omega^2}{2}R^2 - m\omega R^2\dot{\theta}\right) \\ &= (\dot{\theta}^2+\omega^2-2\omega\dot{\theta})mR \\ &= (\dot{\theta}-\omega)^2 mR \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\dot{\theta}}L &= \frac{\partial}{\partial\dot{\theta}}\left(\frac{m}{2}R^2\dot{\theta}^2 - m\omega R^2\dot{\theta}\right) \\ &= mR^2(\dot{\theta}-\omega) \end{aligned} \qquad \frac{\partial}{\partial\theta}L = 0 \tag{56}$$

And from there, plug (56) in Euler-Lagrange (46) to derive the equations of motion (again for the second one, we use a combination of the product and chain rules for derivatives):

$$\begin{aligned} &\frac{d}{dt}\left(m\dot{R}\right) = (\dot{\theta}-\omega)^2 mR \\ \Leftrightarrow &\boxed{\ddot{R} = (\dot{\theta}-\omega)^2 R} \end{aligned} \qquad \begin{aligned} &\frac{d}{dt}\left(mR^2(\dot{\theta}-\omega)\right) = 0 \\ \Leftrightarrow &m\left((\dot{\theta}-\omega)2\dot{R}R + R^2\ddot{\theta}\right) = 0 \\ \Leftrightarrow &\boxed{R\ddot{\theta} = (\omega-\dot{\theta})2\dot{R}} \quad \square \end{aligned}$$

Second approach
We'll now try to see if we can get a cleaner derivation, hopefully with the same results, by combining the two coordinate shifts (48) and (50) first, and then rely on the original Lagrangian (47).

The combined coordinate shift is:

$$\begin{aligned} x &= R\cos\theta\cos(\omega t) + R\sin\theta\sin(\omega t) \\ y &= -R\cos\theta\sin(\omega t) + R\sin\theta\cos(\omega t) \end{aligned}$$

We have the four following trigonometric identities[19]:

$$cos\theta \cos\varphi = \frac{\cos(\theta-\varphi)+cos(\theta+\varphi)}{2} \qquad sin\theta \sin\varphi = \frac{\cos(\theta-\varphi)-cos(\theta+\varphi)}{2}$$
$$cos\theta \sin\varphi = \frac{\sin(\theta+\varphi)-sin(\theta-\varphi)}{2} \qquad sin\theta \cos\varphi = \frac{\sin(\theta+\varphi)+sin(\theta-\varphi)}{2}$$

Hence the coordinate shift can be rewritten:

$$\begin{aligned} x &= R\cos(\theta-\omega t) \\ y &= R\sin(\theta-\omega t) \end{aligned} \tag{57}$$

To inject it in the original Lagrangian (47), we need to compute $\dot{x}^2+\dot{y}^2$. For the derivation, as previously, we'll rely on a combination of the product/chain rule; we'll note $\varphi = \theta - \omega t$:

$$\begin{aligned} \dot{x} &= \dot{R}\cos\varphi - R(\dot{\theta}-\omega)\sin\varphi \\ \dot{y} &= \dot{R}\sin\varphi + R(\dot{\theta}-\omega)\cos\varphi \end{aligned}$$

$$\begin{aligned} \dot{x}^2 &= \dot{R}^2\cos^2\varphi - 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\sin^2\varphi \\ \dot{y}^2 &= \dot{R}^2\sin^2\varphi + 2R\dot{R}(\dot{\theta}-\omega)\cos\varphi\sin\varphi + R^2(\dot{\theta}-\omega)^2\cos^2\varphi \end{aligned}$$

Hence the Lagrangian becomes, again using the Pythagorean trigonometric identity $\cos^2\theta + \sin^2\theta = 1$:

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) \\ &= \boxed{\frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}-\omega)^2\right)} \\ &= \frac{m}{2}\left(\dot{R}^2 + R^2(\dot{\theta}^2 - 2\dot{\theta}\omega + \omega^2)\right) \\ &= \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{m}{2}R^2\omega^2 - m\omega R^2\dot{\theta} \end{aligned}$$

Which is the same Lagrangian we had before in (55), from which we would obviously derive the exact same equation of motion. $\square$.

Remark 73. *As expected, the derivation is overall less tedious, but only because the complexity is now hidden behind the trigonometric identities.*

Remark 74. *A little later in the book, a solution to this exercise is proposed: it starts with this Lagrangian:*

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Which is exactly our Lagrangian, however assuming for some reason that $\omega = 0$. From which follows the same equation of motions, again with the same assumption regarding ω:

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) &= 0 \end{aligned}$$

Let's remind ourselves that ω represents the rotation of the polar coordinate system of the present exercise, a rotation which won't exist for a general polar coordinate system, hence the reason we have $\omega = 0$ in the general case.

Generalized Coordinates and Momenta

Exercise 5/6

Cyclic coordinates

Exercise 6/6

Exercise 34. *Explain how we derived this.*

[19] `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Product-to-sum_and_sum-to-product_identities`

Let us recall that "this" refers to the following expression for the kinetic energy:

$$T = m(\dot{x_+}^2 + \dot{x_-}^2)$$

Starting from the following Lagrangian, involving two particles moving on a line with respective position and velocity x_i, $\dot{x}_i$:

$$L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) - V(x_1 - x_2) \tag{58}$$

After having performed the following change of coordinates:

$$x_+ = \frac{x_1 + x_2}{2} \qquad\qquad x_- = \frac{x_1 - x_2}{2} \tag{59}$$

From the Lagrangian, (58) we have the kinetic energy:

$$T = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \tag{60}$$

By first both summing and subtracting the two equations of (59), and then by linearity of the derivation, we get:

$$\begin{aligned} x_+ + x_- &= x_1 & \qquad x_+ - x_- &= x_2 \\ \dot{x_+} + \dot{x_-} &= \dot{x_1} & \qquad \dot{x_+} - \dot{x_-} &= \dot{x_2} \end{aligned} \tag{61}$$

It's now simply a matter of injecting (61) into (60):

$$\begin{aligned} T &= \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= \frac{m}{2}((\dot{x_+} + \dot{x_-})^2 + (\dot{x_+} - \dot{x_-})^2) \\ &= \frac{m}{2}(2\dot{x_+}^2 + 2\dot{x_-}^2 + 2\dot{x_+}\dot{x_-} - 2\dot{x_+}\dot{x_-}) \\ &= m(\dot{x_+}^2 + \dot{x_-}^2) \quad \square \end{aligned}$$

Lecture 7: Symmetries and Conservation Laws

Preliminaries

Exercise 1/7

Exercise 35. *Derive Equations* (2) *and explain the sign difference.*

Let us recall Equations (2):

$$\dot{p_1} = -V'(q_1 - q_2) \qquad\qquad \dot{p_2} = +V'(q_1 - q_2)$$

We have to derive them from the Lagrangian given in Equation (1), which represents a system of two generalized coordinates q_1 and q_2:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(q_1 - q_2) \tag{62}$$

To retrieve the equations of motions from a Lagrangian, we need to use Euler-Lagrange's equations, for instance recalled as Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Let us also recall, again from previous chapter, right after Equation (13), that the conjugate momentum is defined by

$$p_i = \frac{\partial}{\partial \dot{q_1}} L$$

For our Lagrangian (62), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}} L = \dot{q_1} \qquad\qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}} L = \dot{q_2} \tag{63}$$

$$\frac{d}{dt} p_1 = \dot{p_1} = \ddot{q_1} \qquad\qquad \frac{d}{dt} p_2 = \dot{p_2} = \ddot{q_2} \tag{64}$$

Using the chain rule[20] for the other half, with $\varphi(q_i) = q_1 - q_2$, we get:

$$\begin{aligned}
\frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\
&= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\
&= -(\frac{\partial}{\partial q_1} V)(q_1 - q_2) & &= +(\frac{\partial}{\partial q_2} V)(q_1 - q_2)
\end{aligned} \tag{65}$$

By noting $V' = \frac{\partial}{\partial q_i} V$, and combining equations (63), (64) and (65), we indeed obtain the expected equations of motion $\square$.

Remark 75. *That is, assuming, $\frac{\partial}{\partial q_1} V(q_1) = \frac{\partial}{\partial q_2} V(q_2)$: for all the energy potential presented earlier in the book, there's indeed such a symmetry, e.g.*

$$\begin{aligned}
V &= \frac{1}{2} k(x^2 + y^2), && p103 \\
V &= \frac{1}{2} \frac{k}{x^2 + y^2}, && p103 \\
V &= -m\omega^2 (X^2 + Y^2), && p120
\end{aligned}$$

A similar tacit assumption seems to exists in Herbert Goldstein's Classical Mechanics[21].

Mathematically, the sign difference comes from the fact that the potential depends on one side from q_1 and on the other from $-q_2$, which will persist when differentiating the potential V. Physically, it reflects that there's an order relation between the two "positions" q_1 and q_2: one will come before the other, and our potential V depends on this ordering.

Exercise 2/7

Exercise 36. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{66}$$

Because the question is a unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

[20] https://en.wikipedia.org/wiki/Chain_rule

[21] https://physics.stackexchange.com/a/107141

As for the previous exercise, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}} L\right) = \frac{\partial}{\partial q_i} L$$

Which was in the book followed by the definition of the conjugate momentum p_i:

$$p_i = \frac{\partial}{\partial \dot{q_1}} L$$

For our Lagrangian (66), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}} L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}} L = \dot{q_2} \tag{67}$$

$$\frac{d}{dt} p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt} p_2 = \dot{p_2} = \ddot{q_2} \tag{68}$$

Using the chain rule[22] for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\begin{aligned} \frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -(a\frac{\partial}{\partial q_1} V)(aq_1 - bq_2) & &= +(b\frac{\partial}{\partial q_2} V)(aq_1 - bq_2) \end{aligned} \tag{69}$$

As for the previous exercise, it seems that there a tacit assumption of a symmetry within the potential V so that we can write $V' = \frac{\partial}{\partial q_i} V$; then, combining (67), (68) and (69):

$$\dot{p_1} = -aV'(aq_1 - bq_2) \qquad \dot{p_2} = +bV'(aq_1 - bq_2)$$

As suggested, let's multiply the first equation by b, the second by a, and sum the result:

$$b\dot{p_1} + a\dot{p_2} = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is a indeed a constant over time, i.e. that it is conserved (over time).

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particles respectively by a and b, and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance

Examples of symmetries

Exercise 3/7

Exercise 37. *Show that the combination $aq_1 + bq_2$, along with the Lagrangian, is invariant under Equations* (7).

[22] https://en.wikipedia.org/wiki/Chain_rule

Let us first recall the equations for the potential (Equations (3)):

$$V(q_1, q_2) = V(aq_1 - bq_2)$$

Which is meant to be considered in the case of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{70}$$

Finally, the "Equations (7)" relate to the following change of coordinates:

$$\begin{aligned} q_1 &\to q_1 - b\delta \\ q_2 &\to q_2 + a\delta \end{aligned} \tag{71}$$

Remark 76. *There are typos around here in the book. In my printed version, it is as previously described, but in an online version, it is given by (mind the signs):*

$$\begin{aligned} q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 - a\delta \end{aligned}$$

yet in that same online version, the potential is said to depend on $aq_1 + bq_2$ in accordance to Equations (3)*, but said Equations* (3) *actually make it depend on $aq_1 - bq_2$!*

To summarize, with a $V(aq_1+bq_2)$, the two previous transformations will keep the Lagrangian unchanged. But with a $V(aq_1 - bq_2)$, none of the previous transformations will keep the Lagrangian; those two will:

$$\begin{aligned} q_1 &\to q_1 - b\delta & \qquad q_1 &\to q_1 + b\delta \\ q_2 &\to q_2 - a\delta & \qquad q_2 &\to q_2 + b\delta \end{aligned} \tag{72}$$

In what follows, we will arbitrarily assume a $V(aq_1 - bq_2)$, and, say, the first transformation of (72)*.*

Assuming a, b and δ are time-invariant, it follows that $\dot{q}_1$ and $\dot{q}_2$ are unchanged by this transformation, hence

$$\begin{aligned} \dot{q}_i &\to \dot{q}_i \\ \dot{q_1}^2 + \dot{q_2}^2 &\to \dot{q_1}^2 + \dot{q_2}^2 \end{aligned}$$

Injecting (71) into (70) gives us the following Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(a(q_1 - b\delta) - b(q_2 - a\delta)) \\ &= \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \end{aligned}$$

We can see that indeed, the Lagrangian is unchanged; because the $\dot{q}_i$ are also unchanged, we would derive the exact same equation of motions as we did for the previous exercise.

Exercise 4/7

Exercise 38. *Show this to be true.*

Where "this" refers to the fact that this Lagrangian (Equation (8)):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \tag{73}$$

does not change to first order in δ, for the infinitesimal transformation described by e.g. Equations (12):

$$\begin{aligned} \delta_v x &= y\delta \\ \delta_v y &= -x\delta \end{aligned} \tag{74}$$

The transformation to the derivatives over time of x and y has already been established in Equations (11):

$$\begin{aligned}\dot{x} &\to \dot{x} + \dot{y}\delta \\ \dot{y} &\to \dot{y} - \dot{x}\delta\end{aligned} \tag{75}$$

Let's then perform the substitution described by (74) and (75) in the Lagrangian (73):

$$\begin{aligned}L &= \frac{m}{2}((\dot{x} + \dot{y}\delta)^2 + (\dot{y} - \dot{x}\delta)^2) - V((x + y\delta)^2 + (y - x\delta)^2) \\ &= \frac{m}{2}\left(\left(\dot{x}^2 + 2\dot{x}\dot{y}\delta + (\dot{y}\delta)^2\right) + \left(\dot{y}^2 - 2\dot{y}\dot{x}\delta + (\dot{x}\delta)^2\right)\right) \\ &\quad - V\left(\left(x^2 + 2xy\delta + (y\delta)^2\right) + \left(y^2 - 2yx\delta + (x\delta)^2\right)\right) \\ &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \delta^2(\dot{x}^2 + \dot{y}^2)\right) - V\left(x^2 + y^2 + \delta^2(x^2 + y^2)\right)\end{aligned}$$

Now, as we care about first-order changes in δ only, changes proportional to $\delta^n|_{n\geq 2}$ will be negligible; it follows that the Lagrangian is indeed unchanged:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \quad \square$$

Back to examples

Exercise 5/7

Exercise 6/7

Exercise 7/7

Lecture 8: Hamiltonian Mechanics and Time-Translation Invariance

Time-Translation Symmetry

Energy Conservation

Phase Space and Hamilton's Equations

The Harmonic Oscillator Hamiltonian

Exercise 1/2

Exercise 39. *Start with the Lagrangian* $\frac{m\dot{x}^2}{2} - \frac{k}{2}x^2$ *and show that if you make the change in variables* $q = (km)^{1/4}x$*, the Lagrangian has the form of Eq.* (14)*. What is the connection among* k*,* m *and* ω *?*

Let's recall some context. We're in the case of a the harmonic oscillator, covered in-depth in L03E04.

More precisely, the authors just shown us how the harmonic oscillator can be used as an approximation in the case of an equilibrium state that is slightly disturbed. Physically, you can for instance consider the case of a pendulum: if the oscillations are kept small, then the mass of the pendulum will describe something that *locally* looks like the bottom of a quadratic polynomial (e.g. $ax^2 + bx + c = 0$). This is similar to how the derivative is a local linear approximation, except things jiggle a little more, so to speak.

Eq. (14) of the book refers to the following Lagrangian:

$$L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$$

Now let's simply perform the required change of variable. Both m and k are constants, so $\dot{q} = (km)^{1/4}\dot{x}$, and:

$$L = \frac{1}{2\omega}\sqrt{km}\dot{x}^2 - \frac{\omega}{2}\sqrt{km}x^2$$

Finally, if we can try to identify the previous expression with the one we're supposed to find, by finding a relation between ω, k and m. We have respectively for each term:

$$m = \frac{\sqrt{km}}{\omega}; \qquad k = \omega\sqrt{km}$$

$$\Leftrightarrow \omega = \frac{\sqrt{k}\sqrt{m}}{\sqrt{m}^2}; \qquad \omega = \frac{\sqrt{k}^2}{\sqrt{k}\sqrt{m}}$$

$$\Leftrightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

So we can consistently identify both expression by defining ω as previously stated. Such an ω furthermore matches the usual definition we have in a harmonic oscillator setting.

Exercise 2/2

Exercise 40. *Starting with Eq.* (14)*, calculate the Hamiltonian in terms of* p *and* q*.*

Again Eq. (14) of the book refers to the following Lagrangian:

$$L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$$

There are two different ways to proceed here, depending on what we take as our definition of the Hamiltonian. If we consider H to be the energy of the system, i.e. $K+V$, the sum of the kinetic and potential energies, then we need to identify them in the Lagrangian, typically defined as $K-V$.

Now we "know" the kinetic/potential energy formulation for a basic harmonic oscillator, and the previous change of variable we performed to get the Lagrangian doesn't really affect them, fundamentally, so we know that the first term of the Lagrangian is the kinetic energy, while the second is the potential energy.

So this gives us first:

$$H = \frac{1}{2\omega}\dot{q}^2 + \frac{\omega}{2}q^2$$

Then, we need to recall that p is defined by:

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{\omega}\dot{q}$$

Hence, $\dot{q} = \omega p$, which we can inject in the previous version of H to get:

$$H = \frac{1}{2\omega}(\omega p)^2 + \frac{\omega}{2}q^2 = \boxed{\frac{1}{2}\omega(p^2+q^2)}$$

As stated earlier, there's another approach, if we were to start from the definition of the Hamiltonian given by Eq. (4) of this chapter:

$$H = \sum_i (p_i\dot{q}_i) - L = p\dot{q} - L = p\dot{q} - \frac{1}{2\omega}\dot{q}^2 + \frac{\omega}{2}q^2$$

Then we can again inject $\dot{q} = \omega p$ to get:

$$H = p(p\omega) - \frac{1}{2\omega}(\omega p)^2 + \frac{\omega}{2}q^2 = \omega p^2 - \frac{1}{2}\omega p^2 + \frac{\omega}{2}q^2 = \frac{\omega}{2}p^2 + \frac{\omega}{2}q^2 = \boxed{\frac{\omega}{2}(p^2+q^2)}$$

Derivation of Hamilton's Equations

Lecture 9: The Phase Space Fluid and the Gibbs-Liouville Theorem

The Phase Space Fluid

A Quick Reminder

Flow and Divergence

Liouville's Theorem

Poisson Brackets

Lecture 10: Poisson Brackets, Angular Momentum, and Symmetries

An Axiomatic Formulation of Mechanics

Exercise 1/3

Exercise 41. *Prove Eq.* (14)

Eq. (14) of the book refers to:

$$\{F(q,p),p_i\} = \frac{\partial F(q,p)}{\partial q_i}$$

Where the brackets $\{.,.\}$ are the Poisson Brackets: for A and B each two functions of $2N$ variables $\{p_i\}_{1\leq i\leq N}$ and $\{q_i\}_{1\leq i\leq N}$, $(N\in\mathbb{N})$:

$$\{A,B\} = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

And $F(p,q)$ a function of q and p. There's a bit of ambiguity regarding what p and q are, which actually doesn't affect the derivation, but let's make things clear anyway. In the previous example, we proved that $\{q^n,p\} = nq^{n-1}$: in this case, $N=1$ and we had a single q and a single p.

But now we're asked to prove a result involving $F(p,q)$ partially derived according to q_i, which implies, for the result not to be trivial, that F is a function of q_i, and thus that p and q are actually tuples of N p_i and N q_i.

So, let's expand the Poisson brackets to be evaluated, using the definition of the Poisson brackets:

$$\{F(q,p),p_i\} = \sum_{k=1}^{N} \frac{\partial}{\partial q_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial p_k} - \frac{\partial}{\partial p_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial q_k}$$

Now because p_i will never depends on q_k, as those are two distinct variables, $\dfrac{\partial p_i}{\partial q_k} = 0$, and the previous expression shrinks to:

$$\{F(q,p),p_i\} = \sum_{k=1}^{N} \frac{\partial}{\partial q_k}F(q_1,\cdots,q_N,p_1,\cdots,p_N)\frac{\partial p_i}{\partial p_k}$$

For similar reasons, $\dfrac{\partial p_i}{\partial p_k} = \delta_i^k$, and the previous expression shrinks again to:

$$\{F(q,p),p_i\} = \frac{\partial}{\partial q_i}F(q_1,\cdots,q_N,p_1,\cdots,p_N) = \boxed{\frac{\partial F(p,q)}{\partial q_i}}$$

Remark 77. *Eq.* (15) *of the book is to be proven as we did for Eq.* (14).

Remark 78. *Earlier in this section, the authors informally invited us to verify the properties of the Poisson brackets (anti-symmetry, linearity, product rule). I won't be doing it, because I think at this stage of the book, this should be elementary: you just have to replace the brackets by their definition, and re-arrange the terms, often using linearity of the differentiation/partial differentiation, and then switch back to expressions involving (the expected) Poisson brackets again.*

Exercise 2/3

Exercise 42. *Hamilton's equations can be written in the form* $\dot{q} = \{q, H\}$ *and* $\dot{p} = \{p, H\}$. *Assume that the Hamiltonian has the form* $H = \frac{1}{2m}p^2 + V(q)$. *Using only the PB axioms, prove Newton's equations of motion.*

So, the goal of this exercise is to derive Newton's equations of motion, meaning, a "variant" of $F = ma$, *without* referring directly to the definition of the Poisson brackets, but rather, using its algebraic properties. Let's recall them for clarity.

Let A, B, and C be functions of qs and ps; $k \in \mathbb{R}$:

Anti-symmetry :

$$\{A, C\} = -\{C, A\};$$

Linearity :

$$\{kA, C\} = k\{A, C\};$$
$$\{A + B, C\} = \{A, C\} + \{B, C\};$$

"Product rule" :

$$\{AB, C\} = A\{B, C\} + B\{A, C\}$$

We'll also need the following:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0; \qquad \{q_i, p_j\} = \delta_i^j$$

And Eq. (14) and Eq. (15) of the book, which are respectively, for F a function of qs and ps:

$$\{F(q,p), p_i\} = \frac{\partial F(q,p)}{\partial q_i}$$

$$\{F(q,p), q_i\} = -\frac{\partial F(q,p)}{\partial p_i}$$

Alright, let's start by observing that we're in the case were $N = 1$: we have a single p and a single q. Then, let's begin by applying the anti-symmetry rule to $\dot{q} = \{q, H\} = -\{H, q\}$.

We have two options to go further:

1. Either we expand the expression of H and keep applying some rules further;
2. Or, as $H = H(p, q)$, we can also apply Eq. (15).

Let's try both, in this order (we should get the same result):

$$\begin{aligned}
\dot{q} &= \{q, H\} \\
&= -\{H, q\} && \text{(anti-symmetry)} \\
&= -\{\frac{1}{2m}p^2 + V(q), q\} && \text{(H's definition)} \\
&= -\frac{1}{2m}\{p^2, q\} + \{V(q), q\} && \text{(linearity)}
\end{aligned}$$

Using the product rule, we can develop

$$\{p^2, q\} = \{pp, q\} = p\{p, q\} + p\{p, q\} = 2p\{p, q\}$$

But then, this is just $\{q_i, p_j\} = \delta_i^j$, modulo some anti-symmetry (as we only have one p and one q, they always "match" as far as the Kronecker delta is concerned):

$$\{p^2, q\} = 2p\{p, q\} = -2p\{q, p\} = -2p$$

What about {V(q), q}? We can get there in two ways: either we consider that $V(q) = V(q, p)$ with no p, and thus by Eq. (15),

$$\{V(q), q\} = \{V(q, p), q\} = \frac{\partial V(q, p)}{\partial p} = 0$$

But we could also argue that $V(q)$ can be expressed as a polynomial in q; then, by linearity of the Poisson brackets on the terms of that polynomial, we would be able to apply the $\{q_i, q_j\} = 0$; systematically, and also get zero.

Finally, this leaves us with:

$$\dot{q} = -\frac{1}{2m}\underbrace{\{p^2, q\}}_{=-2p} + \underbrace{\{V(q), q\}}_{=0}$$

By re-arranging the terms a little, we get the definition of the moment:

$$\boxed{p = m\dot{q}}$$

We'll continue from here in a moment, but first, let's explore the second option we mentioned earlier, and use Eq. (15) directly after the application of the anti-symmetry on $\dot{q} = \{q, H\}$:

$$\begin{aligned}
\dot{q} &= \{q, H\} \\
&= -\{H, q\} \quad \text{(anti-symmetry)} \\
&= -\{H(p, q), q\} \\
&= \frac{\partial H(q, p)}{\partial p} \quad \text{(Eq. (15))} \\
&= \frac{\partial}{\partial p}\left(\frac{1}{2m}p^2 + V(q)\right) \quad \text{(H's definition)} \\
&= \frac{1}{m}p
\end{aligned}$$

Which indeed agrees with our previous result: $p = m\dot{q}$.

OK we've found back the definition of the moment, now what? We'd want to find a way to use $\dot{p} = \{p, H\}$, but we have no $\dot{p}$, so let's make one by deriving the definition of the moment:

$$p = m\dot{q} \Rightarrow \dot{p} = m\ddot{q}$$

We'll soon find ourselves in the same situation as before, where we can continue the derivation either by applying Eq. (14), or by following a more "manual" path; I'll go with the latter as this is a bit more verbose:

$$
\begin{aligned}
m\ddot{q} &= \dot{p} \\
&= \{p, H\} \\
&= -\{H, p\} && \text{(anti-symmetry)} \\
&= -\{\frac{1}{2m}p^2 + V(q), p\} && \text{(H's definition)} \\
&= -\frac{1}{2m}\{pp, p\} + \{V(q), p\} && \text{(linearity)} \\
&= -\frac{1}{2m}2p\underbrace{\{p, p\}}_{=0} + \{V(q), p\} && \text{(product rule)} \\
&= \{V(q), p\} && (\{p_i, p_j\} = 0) \\
&= \frac{\partial}{\partial q}V(q) && \text{(Eq. (14))} \\
&= \frac{\partial}{\partial q}V(q) && \text{(forces are derived from potential)} \\
&= F_q && \square
\end{aligned}
$$

Angular Momentum

Exercise 3/3

Exercise 43. *Using the definition of PB's and the axioms, work out the PB's in Equations* (19)*. Hint: In each expression, look for things in the parentheses that have non-zero Poisson Brackets with the coordinate x, y or z. For example, in the first PB, x has a nonzero PB with p_x.*

Let's start by recalling Equations (19):

$$
\begin{aligned}
\{x, L_z\} &= \{x, (xp_y - yp_x)\} \\
\{y, L_z\} &= \{y, (xp_y - yp_x)\} \\
\{z, L_z\} &= \{z, (xp_y - yp_x)\}
\end{aligned}
$$

Then, let's make things a little clearer/regular by renaming our coordinate variables:

$$x = q_x; \qquad y = q_y; \qquad z = q_z$$

So, for $k \in \{x, y, z\}$ (that's the set containing x, y and z, not a weird Poisson bracket), then all the Poisson brackets to compute are of the form:

$$\{q_k, L_z\} = \{q_k, (q_x p_y - q_y p_x)\}$$

Let's reduce it from the axioms:

$$
\begin{aligned}
\{q_k, L_z\} &= \{q_k, (q_x p_y - q_y p_x)\} \\
&= -\{(q_x p_y - q_y p_x), q_k\} && \text{(anti-symmetry)} \\
&= -(\{q_x p_y, q_k\} - \{q_y p_x, q_k\}) && \text{(linearity)} \\
&= \{q_y p_x, q_k\} - \{q_x p_y, q_k\} \\
&= \Big(q_y\{p_x, q_k\} + p_x\underbrace{\{q_y, q_k\}}_{=0}\Big) - \Big(q_x\{p_y, q_k\} + p_y\underbrace{\{q_x, q_k\}}_{=0}\Big) && \text{(product rule)} \\
&= q_y\{p_x, q_k\} - q_x\{p_y, q_k\} && \{q_i, q_j\} = 0
\end{aligned}
$$

Now suffice for us to evaluate that last expression with each value of k, and simplify the result with $\{q_i, p_j\} = \delta_i^j$:

$$
\begin{aligned}
k = x &: \quad \{q_x, L_z\} = q_y\{p_x, q_x\} - q_x\{p_y, q_x\} = q_y \\
k = y &: \quad \{q_x, L_z\} = q_y\{p_x, q_y\} - q_x\{p_y, q_y\} = -q_x \\
k = z &: \quad \{q_x, L_z\} = q_y\{p_x, q_z\} - q_x\{p_y, q_z\} = 0
\end{aligned}
$$

Or, with the original notations:

$$\boxed{\begin{array}{rcr} \{x, L_z\} &=& y \\ \{y, L_z\} &=& -x \\ \{z, L_z\} &=& 0 \end{array}}$$

Remark 79. *Our solution slightly differs from the one in the book, as the latter contains a small sign error: the infinitesimal rotation is said to be:*

$$\begin{aligned} \delta_x &= -\epsilon y \\ \delta_y &= \epsilon x \end{aligned}$$

But earlier in the 7th lecture (p135), it was defined to be, small renaming aside:

$$\begin{aligned} \delta_x &= \epsilon y \\ \delta_y &= -\epsilon x \end{aligned}$$

Mathematical Interlude - The Levi-Civita Symbol

Back to Angular Momentum

Rotors and Precession

Symmetry and Conservation

Lecture 11: Electric and Magnetic Forces

Vector Fields

Mathematical Interlude: Del

Exercise 1/5

Exercise 44. *Confirm Eq.* (3)*. Also prove that*

$$V_i A_j - V_j A_i = \sum_k \epsilon_{ijk} \left(\vec{V} \times \vec{A}\right)_i$$

Let's recall that the Levi-Civita symbol is defined by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Eq. (3) refers to:

$$\left(\vec{V} \times \vec{A}\right)_i = \sum_j \sum_k \epsilon_{ijk} V_j A_k$$

So the idea is to express the components of the cross-product of two 3D vectors with the Levi-Civita symbol. Let's have a look at the components of the cross-product of two vectors:

$$\begin{aligned} \left(\vec{V} \times \vec{A}\right)_x &= V_y A_z - V_z A_y \\ \left(\vec{V} \times \vec{A}\right)_y &= V_z A_x - V_x A_z \\ \left(\vec{V} \times \vec{A}\right)_z &= V_x A_y - V_y A_x \end{aligned}$$

Remark 80. *There's a typo in the book the last term should contain an A_x, but the book says its an A_z. There's another typo in the exercise actually; we'll get to it in a moment.*

Observe that somehow, all those components are "equivalent", or "symmetric": for instance, we can get the second line from the first, by renaming in the first x by y, y by z and z by x.

This implies that to verify Eq. (3), we can satisfy ourselves with doing it only for one component, as the procedure would be exactly similar for the two others. So, let's get going, for instance by trying to prove the first line:

$$\begin{aligned}
\left(\vec{V}\times\vec{A}\right)_x &= \sum_j\sum_k \epsilon_{xjk}V_jA_k \\
&= \sum_k \underbrace{\epsilon_{xxk}}_{=0}V_xA_k + \sum_k \epsilon_{xyk}V_yA_k + \sum_k \epsilon_{xzk}V_zA_k \\
&= \sum_k \Big(\epsilon_{xyk}V_yA_k + \epsilon_{xzk}V_zA_k\Big) \\
&= \underbrace{\epsilon_{xyx}}_{=0}V_yA_x + \underbrace{\epsilon_{xzx}}_{=0}V_zA_x + \underbrace{\epsilon_{xyy}}_{=0}V_yA_y + \epsilon_{xzy}V_zA_y + \epsilon_{xyz}V_yA_z + \underbrace{\epsilon_{xzz}}_{=0}V_zA_z \\
&= \underbrace{\epsilon_{xzy}}_{=-1}V_zA_y + \underbrace{\epsilon_{xyz}}_{=1}V_yA_z \\
&= V_yA_z - V_zA_y \quad \square
\end{aligned}$$

For similar reasons (symmetry), we only have to consider the case e.g. $i = x$ of the remaining equation to be done with it, as the two others would be derived identically, but for some systematic renaming.

We then have three sub-cases, depending on the value of j. If $j = i(= x)$, then one one side:

$$V_iA_j - V_jA_i = V_iA_i - V_iA_i = 0$$

And on the other:

$$\sum_k \underbrace{\epsilon_{iik}}_{=0}\left(\vec{V}\times\vec{A}\right)_i = 0$$

And so the equation holds. Now let's consider the case where $j = y$. One one side we have:

$$V_iA_j - V_jA_i = V_xA_y - V_yA_x$$

And on the other:

$$\begin{aligned}
\sum_k \epsilon_{xjk}\left(\vec{V}\times\vec{A}\right)_x &= \sum_k \epsilon_{xjk}\Big(V_yA_z - V_zA_y\Big) \\
&= \sum_k \epsilon_{xyk}\Big(V_yA_z - V_zA_y\Big) \\
&= \Big(V_yA_z - V_zA_y\Big)(\underbrace{\epsilon_{xyx}}_{=0} + \underbrace{\epsilon_{xyy}}_{=0} + \underbrace{\epsilon_{xyz}}_{=1}) \\
&= V_yA_z - V_zA_y
\end{aligned}$$

Well, the computations are right, but obviously the result isn't! There's a typo in the book: we're expected to prove:

$$V_iA_j - V_jA_i = \sum_k \epsilon_{ijk}\left(\vec{V}\times\vec{A}\right)_k$$

So, let's start again the development of the right hand side:

$$\begin{aligned}
\sum_k \epsilon_{xjk}\left(\vec{V}\times\vec{A}\right)_k &= \sum_k \epsilon_{xyk}\left(\vec{V}\times\vec{A}\right) \\
&= \underbrace{\epsilon_{xyx}}_{=0}\left(\vec{V}\times\vec{A}\right)_x + \underbrace{\epsilon_{xyy}}_{=0}\left(\vec{V}\times\vec{A}\right)_y + \underbrace{\epsilon_{xyz}}_{=1}\left(\vec{V}\times\vec{A}\right)_z \\
&= \left(\vec{V}\times\vec{A}\right)_z \\
&= V_xA_y - V_yA_x \quad \square
\end{aligned}$$

Which corresponds to the left-hand side. Let's do it once more with $j = z$. On one side:

$$V_i A_j - V_j A_i = V_x A_z - V_z A_x$$

On the other:

$$\begin{aligned}
\sum_k \epsilon_{xjk}\left(\vec{V} \times \vec{A}\right)_k &= \sum_k \epsilon_{xzk}\left(\vec{V} \times \vec{A}\right) \\
&= \underbrace{\epsilon_{xzx}}_{=0}\left(\vec{V} \times \vec{A}\right)_x + \underbrace{\epsilon_{xzy}}_{=-1}\left(\vec{V} \times \vec{A}\right)_y + \underbrace{\epsilon_{xzz}}_{=0}\left(\vec{V} \times \vec{A}\right)_z \\
&= -\left(\vec{V} \times \vec{A}\right)_y \\
&= -(V_z A_x - V_x A_z) \\
&= V_x A_z - V_z A_x \quad \square
\end{aligned}$$

Exercise 2/5

Exercise 45. *Prove Eq.* (4).

Where Eq. (4) is the following, for V a scalar field:

$$\vec{\nabla} \times \left(\vec{\nabla} V(x)\right) = 0$$

If think we can agree that $V(x)$ is actually a $V(x, y, z)$.

And $\vec{\nabla}$ is the differentiation vector operator:

$$\vec{\nabla} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}$$

By this definition,

$$\vec{\nabla} V(x, y, z) = \begin{pmatrix} \dfrac{\partial V}{\partial x} \\ \dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial z} \end{pmatrix}$$

We also have previously established that for a field $F = (F_x, F_y, F_z)$:

$$\vec{\nabla} \times F = \begin{pmatrix} \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} \\ \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x} \\ \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y} \end{pmatrix}$$

And so,

$$\vec{\nabla} \times (\vec{\nabla} V(x, y, z)) = \begin{pmatrix} \dfrac{\partial}{\partial y}\dfrac{\partial V}{\partial z} - \dfrac{\partial}{\partial z}\dfrac{\partial V}{\partial y} \\ \dfrac{\partial}{\partial z}\dfrac{\partial V}{\partial x} - \dfrac{\partial}{\partial x}\dfrac{\partial V}{\partial z} \\ \dfrac{\partial}{\partial x}\dfrac{\partial V}{\partial y} - \dfrac{\partial}{\partial y}\dfrac{\partial V}{\partial x} \end{pmatrix} = \vec{0}$$

Where we can conclude because of Schwarz/Clairaut's theorem [23]. This means we consider V to have continuous second partial derivatives on its domain (or, at least in a neighborhood of a point x of its domain), which is often a reasonable assumption in Physics.

Magnetic Fields

Exercise 3/5

Exercise 46. *Show that the vector potentials in Equations* (8) *and Equations* (9) *both give the same uniform magnetic field. This means that the two differ by a gradient. Find the scalar whose gradient, when added to Equations* (8)*, gives Equations* (9)*.*

We're in the context of exploring how a magnetic field $\boldsymbol{B}$ *must* "derive" from vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

That is:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \\ \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{pmatrix}$$

Now the authors gave us two vector potential $\boldsymbol{A}$, $\boldsymbol{A'}$ in the referenced Equations (8) and (9):

$$\boldsymbol{A} = \begin{pmatrix} 0 \\ bx \\ 0 \end{pmatrix}; \qquad \boldsymbol{A'} = \begin{pmatrix} -by \\ 0 \\ 0 \end{pmatrix}$$

And we must prove that they correspond to an uniform magnetic field pointing in the z axis with intensity b (i.e $\boldsymbol{B} = (0, 0, b)$)
We just have to compute the curl of $\boldsymbol{A}$ and $\boldsymbol{A'}$:

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \\ \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ b-0 \end{pmatrix} = \boldsymbol{B} \quad \square$$

$$\boldsymbol{\nabla} \times \boldsymbol{A'} = \begin{pmatrix} \dfrac{\partial A'_z}{\partial y} - \dfrac{\partial A'_y}{\partial z} \\ \dfrac{\partial A'_x}{\partial z} - \dfrac{\partial A'_z}{\partial x} \\ \dfrac{\partial A'_y}{\partial x} - \dfrac{\partial A'_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ 0-(-b) \end{pmatrix} = \boldsymbol{B} \quad \square$$

Now the two vector fields must differ by gradient field generate from some scalar field $s(x, y, z)$:

$$\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla} s$$

Which means

$$\boldsymbol{\nabla} s = \boldsymbol{A'} - \boldsymbol{A} = \begin{pmatrix} -by \\ -bx \\ 0 \end{pmatrix} = -b \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial s}{\partial x} \\ \dfrac{\partial s}{\partial y} \\ \dfrac{\partial s}{\partial z} \end{pmatrix}$$

We can "see" that $s(x, y, z) = -bxy$ fits:

$$\frac{\partial s}{\partial x} = -by; \qquad \frac{\partial s}{\partial y} = -bx$$

[23] https://en.wikipedia.org/wiki/Symmetry_of_second_derivatives

The Force on a Charged Particle

The Lagrangian

Equations of Motion

The Hamiltonian

Exercise 4/5

Exercise 47. *Using the Hamiltonian, Eq.* (24)*, work out Hamilton's equations of motion and show that you just get back to the Newton-Lorentz equation of motion.*

The Hamiltonian is:

$$H = \frac{1}{2m}\sum_i \Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)^2$$

Hamilton's equation of motion are given by the pair:

$$\dot{q_j} = \frac{\partial H}{\partial p_j}; \qquad \dot{p_j} = -\frac{\partial H}{\partial q_j}$$

Let's get started with the first one:

$$\begin{aligned}
\dot{q_j} &= \frac{\partial H}{\partial p_j} \\
&= \frac{1}{2m}\frac{\partial}{\partial p_j}\sum_i \Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)^2 \\
&= \frac{1}{2m}\frac{\partial}{\partial p_j}\sum_i \Big(p_i^2 - 2\frac{e}{c}p_iA_i(\boldsymbol{q}) + (\frac{e}{c})^2A_i(\boldsymbol{q})^2\Big) \\
&= \frac{1}{2m}\sum_i \Big(\frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c}(\frac{\partial p_i}{\partial p_j}A_i(\boldsymbol{q}) + p_i\underbrace{\frac{\partial A_i(\boldsymbol{q})}{\partial p_j}}_{=0}) + (\frac{e}{c})^2\underbrace{\frac{\partial A_i(\boldsymbol{q})^2}{\partial p_j}}_{=0}\Big) \\
&= \frac{1}{2m}\sum_i \Big(\frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c}\frac{\partial p_i}{\partial p_j}A_i(\boldsymbol{q})\Big) \\
&= \frac{1}{2m}\Big(2p_j - 2\frac{e}{c}A_j(\boldsymbol{q})\Big) \\
&= \boxed{\frac{1}{m}(p_j - \frac{e}{c}A_j(\boldsymbol{q}))}
\end{aligned}$$

We've found back the expression of the moment, tweaked a little. Let's now take the time derivative of the previous equation to get:

$$\begin{aligned}
\ddot{q_j} &= \frac{1}{m}\Big(\dot{p_j} - \frac{e}{c}\dot{A}_j(\boldsymbol{q})\Big) \\
\Leftrightarrow \quad m\ddot{q_j} &= \dot{p_j} - \frac{e}{c}\dot{A}_j(\boldsymbol{q})
\end{aligned}$$

Which starts to look like an equation of motion. We'll develop $\dot{A}_j(\boldsymbol{q})$ using the chain rule later. For now, let's compute $\dot{p_j}$, but this time, let's make our life a bit simpler by using the chain rule:

$$
\begin{aligned}
\dot{p}_j &= -\frac{\partial H}{\partial q_j} \\
&= -\frac{1}{2m}\frac{\partial}{\partial q_j}\sum_i \Big(\underbrace{p_i - \frac{e}{c}A_i(\boldsymbol{q})}_{=\phi}\Big)^2 \\
&= -\frac{1}{2m}\sum_i 2\phi\frac{\partial \phi}{\partial q_j} \\
&= \sum_i \underbrace{\frac{1}{m}\Big(p_i - \frac{e}{c}A_i(\boldsymbol{q})\Big)}_{=\dot{q}_i}\frac{e}{c}\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} \\
&= \boxed{\frac{e}{c}\sum_i \dot{q}_i\frac{\partial A_i(\boldsymbol{q})}{\partial q_i}}
\end{aligned}
$$

Now let's use the multi-dimensional chain rule to develop $\dot{A}_j(\boldsymbol{q})$:

$$
\dot{A}_j(\boldsymbol{q}) = \dot{A}_j(q_x(t), q_y(t), q_z(t)) = \sum_i \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\dot{q}_i
$$

Finally, let's use this and the expression of $\dot{p}_j$ to rewrite our embryo of motion equation:

$$
\begin{aligned}
m\ddot{q}_j &= \dot{p}_j - \frac{e}{c}\dot{A}_j(\boldsymbol{q}) \\
&= \sum_i \dot{q}_i\frac{e}{c}\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} - \frac{e}{c}\sum_i \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\dot{q}_i \\
&= \frac{e}{c}\sum_i \dot{q}_i\Big(\frac{\partial A_i(\boldsymbol{q})}{\partial q_i} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_i}\Big) \\
&= \frac{e}{c}\Big(\dot{q}_j\underbrace{(\frac{\partial A_j(\boldsymbol{q})}{\partial q_j} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_j})}_{=0} + \dot{q}_k\underbrace{(\frac{\partial A_k(\boldsymbol{q})}{\partial q_k} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_k})}_{=B_l} + \dot{q}_l\underbrace{(\frac{\partial A_l(\boldsymbol{q})}{\partial q_l} - \frac{\partial A_j(\boldsymbol{q})}{\partial q_l})}_{=-B_k}\Big) \\
&= \frac{e}{c}\Big(B_l\dot{q}_k - B_k\dot{q}_l\Big)
\end{aligned}
$$

And so for j, k, l three distinct elements of $\{x, y, z\}$, we indeed have found our equations of Newton-Lorentz:

$$
\boxed{m\ddot{q}_j = \frac{e}{c}\Big(B_l\dot{q}_k - B_k\dot{q}_l\Big)}
$$

Motion in a Uniform Magnetic Field

Exercise 5/5

Exercise 48. *Show that in the x, y plane, the solution to Eq.* (25) *and the solution to Eq.* (26) *are a circular orbit with the center of the orbit being anywhere on the plane. Find the radius of the orbit in terms of the velocity.*

Let's recall Eq. (25) and Eq. (26):

$$
a_y = -\frac{eb}{mc}v_x; \qquad a_x = \frac{eb}{mc}v_y
$$

Let's rewrite them with dots instead:

$$
\ddot{y} = -\frac{eb}{mc}\dot{x}; \qquad \ddot{x} = \frac{eb}{mc}\dot{y}
$$

First, note that each equation was obtained from a different gauge. But, as the Hamiltonian is gauge-invariant, the equation of motions aren't affected by a gauge shift. So they do both describe the motion we're interested in and can be "combined".

Then, we've been using similar-looking differential equations (for instance when considering the harmonic oscillator), and we know they were solved by a sine/cosine-like function. So we're going to make some guess and calibrate a sine to have it work. We'll then have an expression for both $x(t)$ and $y(t)$, and if we can find a r such as:

$$(x(t) - a)^2 + (y(t) - b)^2 = r^2$$

We'll know that our coordinate function can draw a circle of radius r, centered at (a, b).

Alright so let's say $\dot{x}$ is a sin function; then

- x would be a $-\sin$;
- $\ddot{x}$ would be a sin;
- $\ddot{y}$ would also be a cos;
- $\dot{y}$ would be a $-\sin$;
- and y would be a $-\cos$.

Looks promising, as $x^2 + y^2$ would involve a $\cos^2 + \sin^2 = 1^2$. Now we'd just have to "caliber" the sin properly. Say, if it's a $\sin(\omega t)$, then by repeated differentiation, the ω would become a multiplicative factor, outside of the sin. So, the following feels reasonable:

$$\omega = \frac{eb}{mc}$$

Let's see what would happen to the first equation, if we start with a $\dot{x} = \sin(\omega t)$:

$$\ddot{y} = -\frac{eb}{mc}\dot{x} = -\omega \sin(\omega t)$$

From which we can derive both:

$$\dot{y} = \cos(\omega t); \qquad \ddot{x} = \omega \cos(\omega t)$$

And those two do validate the second equation:

$$\ddot{x} = \frac{eb}{mc}\dot{y}$$

So we've found a solution; note that we can shift both component by arbitrary constants a and b without affecting their correctness:

$$\boxed{y(t) = \frac{1}{\omega}\sin(\omega t) + a; \qquad x(t) = -\frac{1}{\omega}\cos(\omega t) + b}$$

And ω was well-named, as it correspond to the angular velocity. Does it describe an orbit around a point (a, b)? Let's find out:

$$(x(t) - a)^2 + (y(t) - b)^2 = \frac{1}{\omega^2}\Big(\sin^2(\omega t) + \cos^2(\omega t)\Big) = \Big(\frac{1}{\omega}\Big)^2$$

So they do draw a circle of radius:

$$\boxed{r = \frac{1}{\omega} = \frac{mc}{eb}}$$