

TTM - QM - L02E02

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Exercise 1.

Prove that $|i\rangle$ and $|o\rangle$ satisfy all of the conditions in Eqs. 2.7, 2.8 and 2.9. Are they unique in that respect?

Let us recall, in order, Eqs. 2.7, 2.8, 2.9, 2.10, which defines $|i\rangle$ and $|o\rangle$, and both 2.5 and 2.6 which defines $|r\rangle$ and $|l\rangle$:

$$\langle i|o\rangle = 0$$

$$\begin{aligned}\langle o|u\rangle \langle u|o\rangle &= \frac{1}{2} \\ \langle i|u\rangle \langle u|i\rangle &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\langle o|d\rangle \langle d|o\rangle &= \frac{1}{2} \\ \langle i|d\rangle \langle d|i\rangle &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\langle o|r\rangle \langle r|o\rangle &= \frac{1}{2} \\ \langle i|r\rangle \langle r|i\rangle &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\langle o|l\rangle \langle l|o\rangle &= \frac{1}{2} \\ \langle i|l\rangle \langle l|i\rangle &= \frac{1}{2}\end{aligned}$$

$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

For clarity, let us recall that $\langle u|A\rangle$ is the component of $|A\rangle$ on the orthonormal vector $|u\rangle$. This is because in a $(|i\rangle)_{i \in F}$ orthonormal basis we have:

$$\begin{aligned}|A\rangle &= \sum_{i \in F} \alpha_i |i\rangle \\ \Rightarrow \langle j|A\rangle &= \langle j| \sum_{i \in F} \alpha_i |i\rangle = \sum_{i \in F} \alpha_i \langle j|i\rangle = \alpha_j\end{aligned}$$

And to make better sense of those equations, let us recall that $\alpha_u^* \alpha_u = \langle A|u\rangle \langle u|A\rangle$ is the probability of a state vector $|A\rangle$ to be measured in the $|u\rangle$ state.

For Eq. 2.7, we have

$$\begin{aligned}
\langle i|o\rangle &= (\iota_u^* \quad \iota_d^*) \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= \iota_u^* o_u + \iota_d^* o_d \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \frac{-i}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0 \quad \square
\end{aligned}$$

For Eqs. 2.8, we can rely on the projection on orthonormal vector:

$$\begin{aligned}
\langle o|u\rangle \langle u|o\rangle &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle o|d\rangle \langle d|o\rangle &= \frac{i}{\sqrt{2}} \frac{-i}{\sqrt{2}} = \frac{1}{2} \quad \square \\
\langle i|u\rangle \langle u|i\rangle &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle i|d\rangle \langle d|i\rangle &= \frac{-i}{\sqrt{2}} \frac{i}{\sqrt{2}} = \frac{1}{2} \quad \square
\end{aligned}$$

For Eqs. 2.9, we need to rely on the column form of the inner-product:

$$\begin{aligned}
\langle o|r\rangle \langle r|o\rangle &= (o_u^* \quad o_d^*) \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} (\rho_u^* \quad \rho_d^*) \begin{pmatrix} o_u \\ o_d \end{pmatrix} & \langle o|l\rangle \langle l|o\rangle &= (o_u^* \quad o_d^*) \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} (\lambda_u^* \quad \lambda_d^*) \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \frac{1}{\sqrt{2}}) (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{-i}{\sqrt{2}}) & &= (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \frac{-1}{\sqrt{2}}) (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} \frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2}) (\frac{1}{2} - \frac{i}{2}) & &= (\frac{1}{2} - \frac{i}{2}) (\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4} (1+i)(1-i) & &= \frac{1}{4} (1-i)(1+i) \\
&= \frac{1}{4} (1+i-i+1) = \frac{1}{2} \quad \square & &= \frac{1}{4} (1-i+i+1) = \frac{1}{2} \quad \square \\
\langle i|r\rangle \langle r|i\rangle &= (\iota_u^* \quad \iota_d^*) \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} (\rho_u^* \quad \rho_d^*) \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} & \langle i|l\rangle \langle l|i\rangle &= (\iota_u^* \quad \iota_d^*) \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} (\lambda_u^* \quad \lambda_d^*) \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \frac{1}{\sqrt{2}}) (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{i}{\sqrt{2}}) & &= (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \frac{-1}{\sqrt{2}}) (\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} \frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2}) (\frac{1}{2} + \frac{i}{2}) & &= (\frac{1}{2} + \frac{i}{2}) (\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4} (1-i)(1+i) & &= \frac{1}{4} (1+i)(1-i) \\
&= \frac{1}{4} (1+i-i+1) = \frac{1}{2} \quad \square & &= \frac{1}{4} (1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}$$