The Theoretical Minimum
Quantum Mechanics - Solutions
L06E02

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Exercise 1. *Show that if the two normalization conditions of Eqs. 6.4 are satisfied, then the state-vector of Eq. 6.5 is automatically normalized as well. In other words, show that for this product state, normalizing the overall state-vector does not put any additional constraints on the α's and the β's.*

Recall that we're in the context of two distinct state-spaces, each of them referring to a full-blown spin. Spin states for the first space (Alice's) are denoted:

$$\alpha_u|u\} + \alpha_d|d\}, \quad (\alpha_u, \alpha_d) \in \mathbb{C}^2$$

While spin states for the second space (Bob's) are denoted:

$$\beta_u|u\rangle + \beta_d|d\rangle, \quad (\beta_u, \beta_d) \in \mathbb{C}^2$$

Such states are, as usual, normalized: this is the condition referred to by Eqs. 6.4:

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

The two underlying state spaces (complex space, but really, Hilbert spaces) are glued by a tensor product: this allows the creation of new state space, called the *product state space*, which states can refer to both Alice's and Bob's state in a single expression.

Remark 1. *I encourage you to have a look at how Mathematicians formalize the notion of a tensor product of vector spaces: there is for instance a great introductory YouTube video[1] by Michael Penn on the topic.*

The core idea is to start with what is called a formal product of vector spaces, which is a new space built from the span of purely "syntactical" combinations of elements of two (or more) vector spaces. Equivalence classes are then used to constrain this span to be a vector space.

For instance, the three following elements would be distinct elements in the formal product of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$2\begin{pmatrix}1\\2\end{pmatrix} * \begin{pmatrix}3\\4\\5\end{pmatrix}; \qquad \begin{pmatrix}2\\4\end{pmatrix} * \begin{pmatrix}3\\4\\5\end{pmatrix}; \qquad \begin{pmatrix}1\\2\end{pmatrix} * \begin{pmatrix}6\\8\\10\end{pmatrix}$$

But they would be identified by equivalence classes so as to be the same element in the tensor product of $\mathbb{R}^2$ and $\mathbb{R}^3$. We can keep identifying elements likewise until the operations (sum, scalar product) on the formal product space respect the properties the corresponding operations in a vector space.

Here's Eq. 6.5, the general form for such a product state, living in the tensor product space created from Alice's and Bob's state spaces (I've just named it Ψ so as to refer to it later on):

$$|\Psi >= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

The claim we have to prove is that this vector is naturally normalized, from the normalization constraints imposed on the individual state spaces.

[1] `https://www.youtube.com/watch?v=K7f2pCQ3p3U`

Let's start by computing the norm of product state (assuming an ordered basis $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$:

$$|\Psi|^2 = \langle\Psi|\Psi\rangle = \begin{pmatrix}(\alpha_u\beta_u)^* & (\alpha_u\beta_d)^* & (\alpha_d\beta_u)^* & (\alpha_d\beta_d)^*\end{pmatrix}\begin{pmatrix}\alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d\end{pmatrix}$$

We can develop it further, using the fact that for $(a, b) \in \mathbb{C}$, $(ab)^* = a^*b^*$:

$$\begin{aligned}
|\Psi|^2 = \; & \alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d + \alpha_d^*\beta_u^*\alpha_d\beta_u + \alpha_d^*\beta_d^*\alpha_d\beta_d \\
= \; & \alpha_u^*\alpha_u(\underbrace{\beta_u^*\beta_u + \beta_d^*\beta_d}_{=1}) + \alpha_d^*\alpha_d(\underbrace{\beta_u^*\beta_u + \beta_d^*\beta_d}_{=1}) \\
= \; & \underbrace{\alpha_u^*\alpha_u + \alpha_d^*\alpha_d}_{=1} \\
= \; & 1
\end{aligned}$$

But the norm is axiomatically positively defined (i.e. $(\forall\Psi \in \mathcal{H})$, $|\Psi| \geq 0$ with equality iff $\Psi = 0_{\mathcal{H}}$) so:

$$\boxed{|\Psi| = 1} \quad \square$$