The Theoretical Minimum

Quantum Mechanics - Solutions

L04E06

M. Bivert

April 19, 2023

Exercise 1. *Carry out the Schrödinger Ket recipe for a single spin. The Hamiltonian is* $H = \frac{\omega\hbar}{2}\sigma_z$ *and the final observable is* σ_x*. The initial state is given as* $|u\rangle$ *(the state in which* $\sigma_z = +1$*).*

After time t*, an experiment is done to measure* σ_y*. What are the possible outcomes and what are the probabilities for those outcomes?*

Congratulations! You have now solved a real quantum mechanics problem for an experiment that can actually be carried out in the laboratory. Feel free to pat yourself on the back.

Remark 1. *There's a typo in the statement of this exercise: the final observable is said first to be* σ_x *and then* σ_y*. The French version of the book uses* σ_y *for both, so that's what I'll do here.*

1. *Derive, look up, guess, borrow, or steal the Hamiltonian operator* H;
   Well, let's take it from the authors:
   $$H = \frac{\omega\hbar}{2}\sigma_z = \frac{\omega\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

2. *Prepare an initial state* $|\psi(0)\rangle$;
   Again, from the exercise statement, let's prepare an up state:
   $$|\psi(0)\rangle = |u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

3. *Find the eigenvalues and eigenvectors of* H *by solving the time-independent Schrödinger equation,*
   $$H|E_j\rangle = E_j|E_j\rangle$$
   I don't recall us already diagonalizing σ_z before, so let's do it, but I'll be shorter than usual. The eigenvalues are given by the non-invertibility condition of $H - I\lambda$, as the solutions of
   $$\det(H - I\lambda) = (\frac{\omega\hbar}{2} - \lambda)(\lambda - \frac{\omega\hbar}{2}) = 0$$
   Hence the two eigenvalues:
   $$E_1 = \frac{\omega\hbar}{2}; \qquad E_2 = -\frac{\omega\hbar}{2}$$
   From which we can derive the two eigenvectors:
   $$\underbrace{\frac{\omega\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{H}|E_1\rangle = \frac{\omega\hbar}{2}|E_1\rangle$$
   Assuming an eigenvector of a general form $(a\ b)^T$ yields the following system:
   $$\Leftrightarrow \begin{cases} a = a \\ -b = b \end{cases}$$

So $b = 0$; furthermore, as $|E_1\rangle$ must be unitary (from the fundamental theorem/real spectral theorem, we know the eigenvectors of a Hermitian operator, which H most definitely is, are unitary, because the eigenvectors make an orthonormal basis), we must have $a = \pm 1$; let's chose more or less arbitrarily $a = 1$. Hence:

$$\boxed{|E_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Similarly for $|E_2\rangle$, assume a general form of $(c\ d)^T$, this yields the following system:

$$\Leftrightarrow \begin{cases} c = -c \\ -d = -d \end{cases}$$

By a similar argument, as before we find:

$$\boxed{|E_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Remark 2. *I'm not sure why we have an extra degree of freedom via the signs on the non-zero component of the eigenvectors; I can't think of an extra constraint.*

4. *Use the initial state-vector $|\psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3, to calculate the initial coefficients $\alpha_j(0)$:*

$$\alpha_j(0) = \langle E_j|\psi(0)\rangle$$

That's an elementary computation:

$$\alpha_1(0) = 1; \qquad \alpha_2(0) = 0$$

5. *Rewrite $|\psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $\alpha_j(0)$:*

$$|\psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$$

Again, quite elementary given the quantities involved:

$$|\psi(0)\rangle = 1|E_1\rangle = |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6. *In the above equation, replace each $\alpha_j(0)$ with $\alpha_j(t)$ to capture its time-dependence. As a result, $|\psi(0)\rangle$ becomes $|\psi(t)\rangle$:*

$$|\psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$$

Naturally:

$$|\psi(t)\rangle = \alpha_1(t)|E_1\rangle + \alpha_2(t)|E_2\rangle$$

7. *Using Eq. 4.30[1], replace each $\alpha_j(t)$ with $\alpha_j(0)\exp(-\frac{i}{\hbar}E_jt)$:*

$$|\psi(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_jt)|E_j\rangle$$

Because $\alpha_2(0) = 0$, it only remains:

$$\boxed{|\psi(t)\rangle = \exp(-\frac{i}{\hbar}t)|u\rangle}$$

[1] This equation corresponds exactly to what this step describes

OK, then the idea is that if we have an observable L, the probability to measure λ (where λ is then an eigenvalue of L) is given by:

$$P_\lambda(t) = |\langle\lambda|\psi(t)\rangle|^2$$

The authors are asking us to consider as an observable $L = \sigma_y$. Recall:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This is a matrix corresponding to the spin observable following the y-axis: we *must* expect its eigenvalues to be ± 1 and its eigenvectors to be $|i\rangle$ and $|o\rangle$, but let's compute them all anyway for practice:

$$\det(\sigma_y - I\lambda) = \lambda^2 + i^2 = 0 \Leftrightarrow \lambda = \pm 1$$

For the eigenvectors, again we can assume a general form and solve the corresponding system of equations:

$$\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} \begin{pmatrix} a \\ b \end{pmatrix} = (+1) \begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{cases} -ib = a \\ ia = b \end{cases}$$

Both equations are actually equivalent (multiply the first one by i to get the second). We furthermore have an additional constraint as the eigenvectors are supposed to be unitary, which yields:

$$|E_1\rangle = \begin{pmatrix} a \\ ia \end{pmatrix} \text{ and } a^2 + (ia)(-ia) = 1 \Leftrightarrow |E_1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = |i\rangle$$

Similarly:

$$\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} \begin{pmatrix} c \\ d \end{pmatrix} = (-1) \begin{pmatrix} c \\ d \end{pmatrix} \Leftrightarrow \begin{cases} -id = -c \\ ic = -d \end{cases}$$

Again, the two equations are equivalent (multiply the first by $-i$ to get the second one), but we have an additional constraint, as the vector must be unitary. In the end, this yields:

$$|E_2\rangle = \begin{pmatrix} c \\ -ic \end{pmatrix} \text{ and } c^2 + (ic)(-ic) = 1 \Leftrightarrow |E_1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = |o\rangle$$

We may now apply our previous probability formula (Principle 4):

$$P_{+1}(t) = |\langle i|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{2}} \exp(-\frac{it}{\hbar})|^2 = \boxed{\frac{1}{2}}$$

And either because the sum of probabilities must be 1, or by explicit computation:

$$P_{-1}(t) = |\langle o|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{2}} \exp(-\frac{it}{\hbar})|^2 = \boxed{\frac{1}{2}}$$