The Theoretical Minimum

Classical Mechanics - Solutions

L10E03

M. Bivert

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Exercise 1. *Using the definition of PB's and the axioms, work out the PB's in Equations* (19)*. Hint: In each expression, look for things in the parentheses that have non-zero Poisson Brackets with the coordinate x, y or z. For example, in the first PB, x has a nonzero PB with p_x.*

Let's start by recalling Equations (19):

$$\begin{aligned}
\{x, L_z\} &= \{x, (xp_y - yp_x)\} \\
\{y, L_z\} &= \{y, (xp_y - yp_x)\} \\
\{z, L_z\} &= \{z, (xp_y - yp_x)\}
\end{aligned}$$

Then, let's make things a little clearer/regular by renaming our coordinate variables:

$$x = q_x; \qquad y = q_y; \qquad z = q_z$$

So, for $k \in \{x, y, z\}$ (that's the set containing x, y and z, not a weird Poisson bracket), then all the Poisson brackets to compute are of the form:

$$\{q_k, L_z\} = \{q_k, (q_x p_y - q_y p_x)\}$$

Let's reduce it from the axioms:

$$\begin{aligned}
\{q_k, L_z\} &= \{q_k, (q_x p_y - q_y p_x)\} \\
&= -\{(q_x p_y - q_y p_x), q_k\} && \text{(anti-symmetry)} \\
&= -\left(\{q_x p_y, q_k\} - \{q_y p_x, q_k\}\right) && \text{(linearity)} \\
&= \{q_y p_x, q_k\} - \{q_x p_y, q_k\} \\
&= \Big(q_y\{p_x, q_k\} + p_x \underbrace{\{q_y, q_k\}}_{=0}\Big) - \Big(q_x\{p_y, q_k\} + p_y \underbrace{\{q_x, q_k\}}_{=0}\Big) && \text{(product rule)} \\
&= q_y\{p_x, q_k\} - q_x\{p_y, q_k\} && \{q_i, q_j\} = 0
\end{aligned}$$

Now suffice for us to evaluate that last expression with each value of k, and simplify the result with $\{q_i, p_j\} = \delta_i^j$:

$$\begin{aligned}
k = x &: \quad \{q_x, L_z\} = q_y\{p_x, q_x\} - q_x\{p_y, q_x\} = q_y \\
k = y &: \quad \{q_x, L_z\} = q_y\{p_x, q_y\} - q_x\{p_y, q_y\} = -q_x \\
k = z &: \quad \{q_x, L_z\} = q_y\{p_x, q_z\} - q_x\{p_y, q_z\} = 0
\end{aligned}$$

Or, with the original notations:

$$\boxed{\begin{aligned}
\{x, L_z\} &= y \\
\{y, L_z\} &= -x \\
\{z, L_z\} &= 0
\end{aligned}}$$

Remark 1. *Our solution slightly differs from the one in the book, as the latter contains a small sign error: the infinitesimal rotation is said to be:*

$$\begin{aligned} \delta_x &= -\epsilon y \\ \delta_y &= \epsilon x \end{aligned}$$

But earlier in the 7th lecture (p135), it was defined to be, small renaming aside:

$$\begin{aligned} \delta_x &= \epsilon y \\ \delta_y &= -\epsilon x \end{aligned}$$