The Theoretical Minimum

Quantum Mechanics - Solutions

L06E07

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Exercise 1. *Next, Charlie prepares the spins in a different state, called $|T_1\rangle$, where*

$$|T_1\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle + |du\rangle\right)$$

In these examples, T stands for triplet. These triplet states are completely different from the states in the coin and die examples. What are the expectation values of the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$?

What a difference a sign can make!

This is the same kind of computations there were done in the previous exercise, and earlier in the book. As usual, recall the Pauli matrices:

$$\tau_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \tau_y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \tau_z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Also recall, from L06E04, the rules for acting on composite state vectors[1]:

$$\begin{array}{rrlrl}
\sigma_z|uu\rangle = & |uu\rangle; & \tau_z|uu\rangle & = & |uu\rangle \\
\sigma_z|ud\rangle = & |ud\rangle; & \tau_z|ud\rangle & = & -|ud\rangle \\
\sigma_z|du\rangle = & -|du\rangle; & \tau_z|du\rangle & = & |du\rangle \\
\sigma_z|dd\rangle = & -|dd\rangle; & \tau_z|dd\rangle & = & -|dd\rangle \\
\hline
\sigma_x|uu\rangle = & |du\rangle; & \tau_x|uu\rangle & = & |ud\rangle \\
\sigma_x|ud\rangle = & |dd\rangle; & \tau_x|ud\rangle & = & |uu\rangle \\
\sigma_x|du\rangle = & |uu\rangle; & \tau_x|du\rangle & = & |dd\rangle \\
\sigma_x|dd\rangle = & |ud\rangle; & \tau_x|dd\rangle & = & |du\rangle \\
\hline
\sigma_y|uu\rangle = & i|du\rangle; & \tau_y|uu\rangle & = & i|ud\rangle \\
\sigma_y|ud\rangle = & i|dd\rangle; & \tau_y|ud\rangle & = & -i|uu\rangle \\
\sigma_y|du\rangle = & -i|uu\rangle; & \tau_y|du\rangle & = & i|dd\rangle \\
\sigma_y|dd\rangle = & -i|ud\rangle; & \tau_y|dd\rangle & = & -i|du\rangle
\end{array}$$

We now have everything we need to compute the expectation values.

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[1] You have the same in the book's appendix

$$
\begin{aligned}
\langle\sigma_z\tau_z\rangle :=&\quad \langle T_1|\sigma_z\tau_z|T_1\rangle \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_z\tau_z\left(|ud\rangle + |du\rangle\right) \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_z\left(-|ud\rangle + |du\rangle\right) \\
=&\quad -\frac{1}{\sqrt{2}}\langle T_1|\left(|ud\rangle + |du\rangle\right) \\
=&\quad -\frac{1}{2}(\langle ud| + \langle du|)(|ud\rangle + |du\rangle) \\
=&\quad -\frac{1}{2}\left(\underbrace{\langle ud|ud\rangle}_{1} + \underbrace{\langle ud|du\rangle}_{0} + \underbrace{\langle du|ud\rangle}_{0} + \underbrace{\langle du|du\rangle}_{1}\right) \\
=&\quad \boxed{-1}
\end{aligned}
$$

For the last step, remember, as for the previous exercise, that $|du\rangle$ and $|ud\rangle$ are orthonormal basis vectors.

$$
\begin{aligned}
\langle\sigma_x\tau_x\rangle :=&\quad \langle T_1|\sigma_x\tau_x|T_1\rangle \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_x\tau_x\left(|ud\rangle + |du\rangle\right) \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_x\left(|uu\rangle + |dd\rangle\right) \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\left(|du\rangle + |ud\rangle\right) \\
=&\quad \frac{1}{2}(\langle ud| + \langle du|)(|du\rangle + |ud\rangle) \\
=&\quad -\frac{1}{2}\left(\underbrace{\langle ud|du\rangle}_{0} + \underbrace{\langle ud|ud\rangle}_{1} + \underbrace{\langle du|du\rangle}_{1} + \underbrace{\langle du|ud\rangle}_{0}\right) \\
=&\quad \boxed{+1}
\end{aligned}
$$

$$
\begin{aligned}
\langle\sigma_y\tau_y\rangle :=&\quad \langle T_1|\sigma_y\tau_y|T_1\rangle \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_y\tau_y\left(|ud\rangle + |du\rangle\right) \\
=&\quad \frac{1}{\sqrt{2}}\langle T_1|\sigma_y\left(-i|uu\rangle + i|dd\rangle\right) \\
=&\quad \frac{i}{\sqrt{2}}\langle T_1|\left(-i|du\rangle - i|ud\rangle\right) \\
=&\quad \frac{1}{2}(\langle ud| + \langle du|)(|du\rangle + |ud\rangle) \\
=&\quad -\frac{1}{2}\left(\underbrace{\langle ud|du\rangle}_{0} + \underbrace{\langle ud|ud\rangle}_{1} + \underbrace{\langle du|du\rangle}_{1} + \underbrace{\langle du|ud\rangle}_{0}\right) \\
=&\quad \boxed{+1}
\end{aligned}
$$