

L07E01

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Exercise 1. *Derive Equations (2) and explain the sign difference.*

Let us recall Equations (2):

$$\dot{p}_1 = -V'(q_1 - q_2) \qquad \dot{p}_2 = +V'(q_1 - q_2)$$

We have to derive them from the Lagrangian given in Equation (1), which represents a system of two generalized coordinates q_1 and q_2 :

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1 - q_2) \quad (1)$$

To retrieve the equations of motions from a Lagrangian, we need to use Euler-Lagrange's equations, for instance recalled as Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_i} L \right) = \frac{\partial}{\partial q_i} L$$

Let us also recall, again from previous chapter, right after Equation (13), that the conjugate momentum is defined by

$$p_i = \frac{\partial}{\partial \dot{q}_i} L$$

For our Lagrangian (1), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q}_1} L = \dot{q}_1 \qquad p_2 \equiv \frac{\partial}{\partial \dot{q}_2} L = \dot{q}_2 \quad (2)$$

$$\frac{d}{dt} p_1 = \dot{p}_1 = \ddot{q}_1 \qquad \frac{d}{dt} p_2 = \dot{p}_2 = \ddot{q}_2 \quad (3)$$

Using the chain rule¹ for the other half, with $\varphi(q_i) = q_1 - q_2$, we get:

$$\begin{aligned} \frac{\partial}{\partial q_1} L &= -\frac{\partial}{\partial q_1} V(\varphi(q_1)) & \frac{\partial}{\partial q_2} L &= -\frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -\frac{\partial}{\partial q_1} \varphi(q_1) \frac{\partial}{\partial q_1} V(\varphi(q_1)) & &= -\frac{\partial}{\partial q_2} \varphi(q_2) \frac{\partial}{\partial q_2} V(\varphi(q_2)) \\ &= -\left(\frac{\partial}{\partial q_1} V \right) (q_1 - q_2) & &= +\left(\frac{\partial}{\partial q_2} V \right) (q_1 - q_2) \end{aligned} \quad (4)$$

By noting $V' = \frac{\partial}{\partial q_i} V$, and combining equations (2), (3) and (4), we indeed obtain the expected equations of motion \square .

Remark 1. *That is, assuming, $\frac{\partial}{\partial q_1} V(q_1) = \frac{\partial}{\partial q_2} V(q_2)$: for all the energy potential presented earlier in the book, there's indeed such a symmetry, e.g.*

$$V = \frac{1}{2}k(x^2 + y^2), \quad p103$$

$$V = \frac{1}{2} \frac{k}{x^2 + y^2}, \quad p103$$

$$V = -m\omega^2(X^2 + Y^2), \quad p120$$

¹https://en.wikipedia.org/wiki/Chain_rule

*A similar tacit assumption seems to exist in Herbert Goldstein's Classical Mechanics*².

Mathematically, the sign difference comes from the fact that the potential depends on one side from q_1 and on the other from $-q_2$, which will persist when differentiating the potential V . Physically, it reflects that there's an order relation between the two "positions" q_1 and q_2 : one will come before the other, and our potential V depends on this ordering.

²<https://physics.stackexchange.com/a/107141>