The Theoretical Minimum

Classical Mechanics - Solutions

L06E03

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Exercise 1. *Use the Euler-Lagrange equations to derive the equations of motions from the Lagrangian in Eq.* (12)*.*

Again, let us recall the general form of Euler-Lagrange equations for a configuration space of size $n \in \mathbb{N}$: $(\forall i \in [\![1, n]\!])$,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}L\right) = \frac{\partial}{\partial x_i}L \tag{1}$$

In the case of this exercise, the Lagrangian L is defined in Eq. (12) as:

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega(\dot{X}Y - \dot{Y}X)$$

Let's compute the partial derivatives of L on $\dot{X}$, X, $\dot{Y}$ and Y:

$$\begin{aligned}
\frac{\partial}{\partial \dot{X}}L &= \frac{\partial}{\partial \dot{X}}\left(\frac{m}{2}\dot{X}^2 + m\omega\dot{X}Y\right) & \qquad \frac{\partial}{\partial X}L &= \frac{\partial}{\partial X}\left(\frac{m\omega^2}{2}X^2 - m\omega\dot{Y}X\right) \\
&= m\dot{X} + m\omega Y & &= m\omega^2 X - m\omega\dot{Y} \\
\frac{\partial}{\partial \dot{Y}}L &= \frac{\partial}{\partial \dot{Y}}\left(\frac{m}{2}\dot{Y}^2 - m\omega\dot{Y}X\right) & \qquad \frac{\partial}{\partial Y}L &= \frac{\partial}{\partial Y}\left(\frac{m\omega^2}{2}Y^2 + m\omega\dot{X}Y\right) \\
&= m\dot{Y} - m\omega X & &= m\omega^2 Y + m\omega\dot{X}
\end{aligned} \tag{2}$$

Finally, by plugging (2) into (1), we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(m\dot{X} + m\omega Y\right) = m\omega^2 X - m\omega\dot{Y} & \qquad &\frac{d}{dt}\left(m\dot{Y} - m\omega X\right) = m\omega^2 Y + m\omega\dot{X} \\
\Leftrightarrow &\boxed{m\ddot{X} = m\omega^2 X - 2m\omega\dot{Y}} & \qquad \Leftrightarrow &\boxed{m\ddot{Y} = m\omega^2 Y + 2m\omega\dot{X}} \quad \square
\end{aligned}$$

Remark 1. *Those results indeed matches the equations proposed in the book just slightly before this exercise.*