The Theoretical Minimum

Quantum Mechanics - Solutions

L07E02

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Exercise 1. *Calculate the matrix elements of $\sigma_z \otimes \tau_x$ by forming inner products as we did in Eq. 7.2.*

This is essentially the same exercise as the previous one, but with a different composite operator. To check for errors, I'll still do the computation using the two approaches.

We'll start with the approach suggested in the exercise's statement: let's first start by recalling the portion of interest from the multiplication table computed in L06E04:

$$
\begin{aligned}
\sigma_z|uu\rangle &= \quad |uu\rangle; & \tau_x|uu\rangle &= |ud\rangle \\
\sigma_z|ud\rangle &= \quad |ud\rangle; & \tau_x|ud\rangle &= |uu\rangle \\
\sigma_z|du\rangle &= -|du\rangle; & \tau_x|du\rangle &= |dd\rangle \\
\sigma_z|dd\rangle &= -|dd\rangle; & \tau_x|dd\rangle &= |du\rangle
\end{aligned}
$$

Then, Eq. 7.2 applied to $\sigma_z \otimes \tau_x$ will give:

$$
\begin{aligned}
\sigma_z \otimes \tau_x \text{"}=\text{"} & \begin{pmatrix}
\langle uu|(\sigma_z \otimes \tau_x)|uu\rangle & \langle uu|(\sigma_z \otimes \tau_x)|ud\rangle & \langle uu|(\sigma_z \otimes \tau_x)|du\rangle & \langle uu|(\sigma_z \otimes \tau_x)|dd\rangle \\
\langle ud|(\sigma_z \otimes \tau_x)|uu\rangle & \langle ud|(\sigma_z \otimes \tau_x)|ud\rangle & \langle ud|(\sigma_z \otimes \tau_x)|du\rangle & \langle ud|(\sigma_z \otimes \tau_x)|dd\rangle \\
\langle du|(\sigma_z \otimes \tau_x)|uu\rangle & \langle du|(\sigma_z \otimes \tau_x)|ud\rangle & \langle du|(\sigma_z \otimes \tau_x)|du\rangle & \langle du|(\sigma_z \otimes \tau_x)|dd\rangle \\
\langle dd|(I \otimes \tau_x)|uu\rangle & \langle dd|(\sigma_z \otimes \tau_x)|ud\rangle & \langle dd|(\sigma_z \otimes \tau_x)|du\rangle & \langle dd|(\sigma_z \otimes \tau_x)|dd\rangle
\end{pmatrix} \\
\text{"}=\text{"} & \begin{pmatrix}
\langle uu|\sigma_z|ud\rangle & \langle uu|\sigma_z|uu\rangle & \langle uu|\sigma_z|dd\rangle & \langle uu|\sigma_z|du\rangle \\
\langle ud|\sigma_z|ud\rangle & \langle ud|\sigma_z|uu\rangle & \langle ud|\sigma_z|dd\rangle & \langle ud|\sigma_z|du\rangle \\
\langle du|\sigma_z|ud\rangle & \langle du|\sigma_z|uu\rangle & \langle du|\sigma_z|dd\rangle & \langle du|\sigma_z|du\rangle \\
\langle dd|\sigma_z|ud\rangle & \langle dd|\sigma_z|uu\rangle & \langle dd|\sigma_z|dd\rangle & \langle dd|\sigma_z|du\rangle
\end{pmatrix} \\
\text{"}=\text{"} & \begin{pmatrix}
\langle uu|ud\rangle & \langle uu|uu\rangle & -\langle uu|dd\rangle & -\langle uu|du\rangle \\
\langle ud|ud\rangle & \langle ud|uu\rangle & -\langle ud|dd\rangle & -\langle ud|du\rangle \\
\langle du|ud\rangle & \langle du|uu\rangle & -\langle du|dd\rangle & -\langle du|du\rangle \\
\langle dd|ud\rangle & \langle dd|uu\rangle & -\langle dd|dd\rangle & -\langle dd|du\rangle
\end{pmatrix} \\
\text{"}=\text{"} & \boxed{\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}}
\end{aligned}
$$

Let's verify our computation using the second approach, relying on Eq. 7.6 of the book:

$$
A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}
$$

Recall the Pauli matrices:

$$
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

Which then yields:

$$\begin{aligned}
\sigma_z \otimes \tau_x \text{"} = \text{"} & \begin{pmatrix} 1 \times \tau_x & 0 \times \tau_x \\ 0 \times \tau_x & -1 \times \tau_x \end{pmatrix} \\
\text{"} = \text{"} & \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \\
\text{"} = \text{"} & \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}
\end{aligned}$$

Which agrees with our previous result.