TTM - QM - Solutions

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Abstract

Below are solution proposals to the exercises of *The Theoretical Minimum - Quantum Mechanics*, written by Leonard Susskind and Art Friedman. An effort has been so as to recall from the book all the referenced equations, and to be rather verbose regarding mathematical details, rather in line with the general tone of the serie.

Contents

1 Systems and Experiments

1.1 Inner Products

Exercise 1.
a) Using the axioms for inner products, prove

$$\Big(\langle A| + \langle B|\Big)|C\rangle = \langle A|C\rangle + \langle B|C\rangle$$

b) Prove $\langle A|A\rangle$ is a real number.

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a) Let us recall the two axioms in question:

Axiom 1.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 2.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

Let us recall also that if

- $\langle A|$ is the bra of $|A\rangle$
- $\langle B|$ is the bra of $|B\rangle$

Then $\langle A| + \langle B|$ is the bra of $|A\rangle + |B\rangle$.

Let us also observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$\begin{aligned}(a+b)^* &= (x_a + iy_a + x_b + iy_b)^* \\ &= x_a - iy_a + x_b - iy_b \\ &= a^* + b^*\end{aligned}$$

We thus have:

$$\begin{aligned}\Big(\langle A| + \langle B|\Big)|C\rangle &= \langle C|\Big(|A\rangle + |B\rangle\Big)^* \\ &= \Big(\langle C|A\rangle + \langle C|B\rangle\Big)^* \\ &= \langle C|A\rangle^* + \langle C|B\rangle^* \\ &= \langle A|C\rangle + \langle B|C\rangle \quad \Box\end{aligned}$$

b) Mainly from the second axiom:

$$\begin{aligned}x + iy &= \langle A|A\rangle \\ &= \langle A|A\rangle^* \\ &= x - iy \\ &\Rightarrow 2iy = 0 \\ &\Rightarrow y = 0 \\ &\Rightarrow \langle A|A\rangle = x \in \mathbb{R} \quad \Box\end{aligned}$$

Exercise 2.
Show that the inner product defined by Eq. 1.2 satisfies all the axioms of inner products.

Let us recall the two axioms in question:

Axiom 3.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 4.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

And let us recall Eq. 1.2 of the book:

$$\begin{aligned}\langle B|A\rangle &= \begin{pmatrix}\beta_1^* & \beta_2^* & \beta_3^* & \beta_4^* & \beta_5^*\end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5\end{pmatrix} \\ &= \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3 + \beta_4^*\alpha_4 + \beta_5^*\alpha_5\end{aligned}$$

For the first axiom, considering $\langle C| = (\gamma_i^*)$:

$$
\begin{aligned}
\langle C|\Big(|A\rangle + |B\rangle\Big) &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3+\beta_3\\ \alpha_4+\beta_4\\ \alpha_5+\beta_5\end{pmatrix}\\
&= \gamma_1^*(\alpha_1+\beta_1)+\gamma_2^*(\alpha_2+\beta_2)+\gamma_3^*(\alpha_3+\beta_3)+\gamma_4^*(\alpha_4+\beta_4)+\gamma_5^*(\alpha_5+\beta_5)\\
&= \big(\gamma_1^*\alpha_1+\gamma_2^*\alpha_2+\gamma_3^*\alpha_3+\gamma_4^*\alpha_4+\gamma_5^*\alpha_5\big)+\big(\gamma_1^*\beta_1+\gamma_2^*\beta_2+\gamma_3^*\beta_3+\gamma_4^*\beta_4+\gamma_5^*\beta_5\big)\\
&= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}+\begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}\\
&= \langle C|A\rangle + \langle C|B\rangle \quad \square
\end{aligned}
$$

Before diving into the second axiom, let us observe that for $(a,b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$
\begin{aligned}
(ab)^* &= \Big((x_a+iy_a)\times(x_b+iy_b)\Big)^*\\
&= \Big(x_ax_b - y_ay_b + i(x_by_a + x_ay_b)\Big)^*\\
&= x_ax_b - y_ay_b - i(x_by_a + x_ay_b)\\
&= (x_a - iy_a)\times(x_b - iy_b)\\
&= a^*b^*
\end{aligned}
$$

Or, perhaps more simply using complex numbers' exponential's form:

$$
\begin{aligned}
(ab)^* &= \Big(r_ar_be^{i(\theta_a+\theta_b)}\Big)^*\\
&= r_ar_be^{-i(\theta_a+\theta_b)}\\
&= a^*b^*
\end{aligned}
$$

Hence, regarding the second axiom:

$$
\begin{aligned}
\langle B|A\rangle &= \Big(\big(\,\langle B|A\rangle\,\big)^*\Big)^*\\
&= \Big(\big(\beta_1^*\alpha_1+\beta_2^*\alpha_2+\beta_3^*\alpha_3+\beta_4^*\alpha_4+\beta_5^*\alpha_5\big)^*\Big)^*\\
&= \Big(\beta_1\alpha_1^*+\beta_2\alpha_2^*+\beta_3\alpha_3^*+\beta_4\alpha_4^*+\beta_5\alpha_5^*\Big)*\\
&= \Big(\alpha_1^*\beta_1+\alpha_2^*\beta_2+\alpha_3^*\beta_3+\alpha_4^*\beta_4+\alpha_5^*\beta_5\Big)*\\
&= \left(\begin{pmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* & \alpha_5^*\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}\right)^*\\
&= \langle A|B\rangle^* \quad \square
\end{aligned}
$$

2 Quantum States

2.1 Along the x Axis

Exercise 3.
Prove that the vector $|r\rangle$ in Eq. 2.5 is orthogonal to vector $|l\rangle$ in Eq. 2.6.

Let us recall respectively Eq. 2.5 and Eq. 2.6:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \qquad\qquad |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

Orthogonality can be detected with the inner-product: $|l\rangle$ and $|r\rangle$ are orthogonals $\Leftrightarrow \langle r|l\rangle = \langle l|r\rangle = 0$.

Remark 1.
The nullity of either inner-product is sufficient, because of the $\langle A|B\rangle = \langle B|A\rangle^*$ *axiom.*

For instance:

$$\begin{aligned}
\langle l|r\rangle &= \begin{pmatrix}\lambda_u^* & \lambda_d^*\end{pmatrix}\begin{pmatrix}\rho_u \\ \rho_d\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\
&= 0 \quad \square
\end{aligned}$$

Or, similarly:

$$\begin{aligned}
\langle r|l\rangle &= \begin{pmatrix}\rho_u^* & \rho_d^*\end{pmatrix}\begin{pmatrix}\lambda_u \\ \lambda_d\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\
&= 0 \quad \square
\end{aligned}$$

2.2 Along the y Axis

Exercise 4.
Prove that $|i\rangle$ *and* $|o\rangle$ *satisfy all of the conditions in Eqs.* 2.7, 2.8 *and* 2.9. *Are they unique in that respect?*

Let us recall, in order, Eqs. 2.7, 2.8, 2.9, 2.10, which defines $|i\rangle$ and $|o\rangle$, and both 2.5 and 2.6 which defines $|r\rangle$ and $|l\rangle$:

$$\langle i|o\rangle = 0$$

$$\begin{aligned}
\langle o|u\rangle\langle u|o\rangle &= \frac{1}{2} \qquad\qquad & \langle o|d\rangle\langle d|o\rangle &= \frac{1}{2} \\
\langle i|u\rangle\langle u|i\rangle &= \frac{1}{2} & \langle i|d\rangle\langle d|i\rangle &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\langle o|r\rangle\langle r|o\rangle &= \frac{1}{2} \qquad\qquad & \langle o|l\rangle\langle l|o\rangle &= \frac{1}{2} \\
\langle i|r\rangle\langle r|i\rangle &= \frac{1}{2} & \langle i|l\rangle\langle l|i\rangle &= \frac{1}{2}
\end{aligned}$$

$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \qquad |i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \qquad |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

For clarity, let us recall that $\langle u|A\rangle$ is the component of $|A\rangle$ on the orthonormal vector $|u\rangle$. This is because in a $(|i\rangle)_{i\in F}$ *orthonormal* basis we have:

$$\begin{aligned} |A\rangle &= \sum_{i\in F} \alpha_i |i\rangle \\ \Rightarrow \langle j|A\rangle &= \langle j| \sum_{i\in F} \alpha_i |i\rangle = \sum_{i\in F} \alpha_i \langle j|i\rangle = \alpha_j \end{aligned}$$

And to make better sense of those equations, let us recall that $\alpha_u^* \alpha_u = \langle A|u\rangle \langle u|A\rangle$ is the probability of a state vector $|A\rangle$ to be measured in the $|u\rangle$ state.
For Eq. 2.7, we have

$$\begin{aligned} \langle i|o\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\ &= \iota_u^* o_u + \iota_d^* o_d \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0 \quad \square \end{aligned}$$

For Eqs. 2.8, we can rely on the projection on orthonormal vector:

$$\begin{aligned} \langle o|u\rangle \langle u|o\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle o|d\rangle \langle d|o\rangle &= \frac{i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2} \quad \square \\ \langle i|u\rangle \langle u|i\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \square & \langle i|d\rangle \langle d|i\rangle &= \frac{-i}{\sqrt{2}}\frac{i}{\sqrt{2}} = \frac{1}{2} \quad \square \end{aligned}$$

For Eqs. 2.9, we need to rely on the column form of the inner-product:

$$\begin{aligned}
\langle o|r\rangle \langle r|o\rangle &= \begin{pmatrix} o_u^* & o_d^* \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} \begin{pmatrix} \rho_u^* & \rho_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle o|l\rangle \langle l|o\rangle &= \begin{pmatrix} o_u^* & o_d^* \end{pmatrix} \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} \begin{pmatrix} \lambda_u^* & \lambda_d^* \end{pmatrix} \begin{pmatrix} o_u \\ o_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{-i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1-i+i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle i|r\rangle \langle r|i\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_d \end{pmatrix} \begin{pmatrix} \rho_u^* & \rho_d^* \end{pmatrix} \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} - \frac{i}{2})(\frac{1}{2} + \frac{i}{2}) \\
&= \frac{1}{4}(1-i)(1+i) \\
&= \frac{1}{4}(1+i+i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

$$\begin{aligned}
\langle i|l\rangle \langle l|i\rangle &= \begin{pmatrix} \iota_u^* & \iota_d^* \end{pmatrix} \begin{pmatrix} \lambda_u \\ \lambda_d \end{pmatrix} \begin{pmatrix} \lambda_u^* & \lambda_d^* \end{pmatrix} \begin{pmatrix} \iota_u \\ \iota_d \end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{i}{\sqrt{2}}) \\
&= (\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2}) \\
&= \frac{1}{4}(1+i)(1-i) \\
&= \frac{1}{4}(1+i-i+1) = \frac{1}{2} \quad \square
\end{aligned}$$

Exercise 5.
For the moment, forget that Eqs. 2.10 *give us working definitions for* $|i\rangle$ *and* $|o\rangle$ *in terms of* $|u\rangle$ *and* $|d\rangle$*, and assume that the compoments* α, β, γ *and* δ *are unknown:*

$$|o\rangle = \alpha|u\rangle + \beta|d\rangle \qquad\qquad |i\rangle = \gamma|u\rangle + \delta|d\rangle$$

a) Use Eqs. 2.8 *to show that*

$$\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$$

b) Use the above results and Eqs. 2.9 *to show that*

$$\alpha^*\beta + \alpha\beta^* = \gamma^*\delta + \gamma\delta^* = 0$$

c) Show that $\alpha^*\beta$ *and* $\gamma^*\delta$ *must each be pure imaginary.*

If $\alpha^*\beta$ *is pure imaginary, then* α *and* β *cannot both be real. The same reasoning applies to* $\gamma^*\delta$*.*

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3 Principles of Quantum Mechanics

3.1 Mathematical Interlude: Linear Operators

3.1.1 Hermitian Operators and Orthonormal Bases