

The Theoretical Minimum

Classical Mechanics - Solutions

L07E03

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Exercise 1. Show that the combination $aq_1 + bq_2$, along with the Lagrangian, is invariant under Equations (7).

Let us first recall the equations for the potential (Equations (3)):

$$V(q_1, q_2) = V(aq_1 - bq_2)$$

Which is meant to be considered in the case of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(aq_1 - bq_2) \quad (1)$$

Finally, the "Equations (7)" relate to the following change of coordinates:

$$\begin{aligned} q_1 &\rightarrow q_1 - b\delta \\ q_2 &\rightarrow q_2 + a\delta \end{aligned} \quad (2)$$

Remark 1. There are typos around here in the book. In my printed version, it is as previously described, but in an online version, it is given by (mind the signs):

$$\begin{aligned} q_1 &\rightarrow q_1 + b\delta \\ q_2 &\rightarrow q_2 - a\delta \end{aligned}$$

yet in that same online version, the potential is said to depend on $aq_1 + bq_2$ in accordance to Equations (3), but said Equations (3) actually make it depend on $aq_1 - bq_2$!

To summarize, with a $V(aq_1 + bq_2)$, the two previous transformations will keep the Lagrangian unchanged. But with a $V(aq_1 - bq_2)$, none of the previous transformations will keep the Lagrangian; those two will:

$$\begin{aligned} q_1 &\rightarrow q_1 - b\delta & q_1 &\rightarrow q_1 + b\delta \\ q_2 &\rightarrow q_2 - a\delta & q_2 &\rightarrow q_2 + b\delta \end{aligned} \quad (3)$$

In what follows, we will arbitrarily assume a $V(aq_1 - bq_2)$, and, say, the first transformation of (3).

Assuming a , b and δ are time-invariant, it follows that \dot{q}_1 and \dot{q}_2 are unchanged by this transformation, hence

$$\begin{aligned} \dot{q}_i &\rightarrow \dot{q}_i \\ \dot{q}_1^2 + \dot{q}_2^2 &\rightarrow \dot{q}_1^2 + \dot{q}_2^2 \end{aligned}$$

Injecting (2) into (1) gives us the following Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(a(q_1 - b\delta) - b(q_2 - a\delta)) \\ &= \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(aq_1 - bq_2) \end{aligned}$$

We can see that indeed, the Lagrangian is unchanged; because the \dot{q}_i are also unchanged, we would derive the exact same equation of motions as we did for the previous exercise.