The Theoretical Minimum

Quantum Mechanics - Solutions

L07E05

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Exercise 1. *a) Show that*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

b) Now, suppose

$$\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Calculate

$$\rho^2$$

$$Tr(\rho)$$

$$Tr(\rho^2)$$

c) If ρ is a density matrix, does it represent a pure state or a mixed state?

The exercise is fairly trivial.

a)

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \boxed{\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}} \quad \square$$

b) By application of the previous result,

$$\rho^2 = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}^2 = \begin{pmatrix} (1/3)^2 & 0 \\ 0 & (2/3)^2 \end{pmatrix} = \boxed{\begin{pmatrix} 1/9 & 0 \\ 0 & 4/9 \end{pmatrix}}$$

Recall that there's a result alluded to by the authors in a footnote page 195 (section 7.2) that the trace of an operator is the sum of the diagonal elements of any matrix representation of this operator. Hence:

$$\text{Tr}(\rho) = \frac{1}{3} + \frac{2}{3} = \boxed{1}; \qquad \text{Tr}(\rho^2) = \frac{1}{9} + \frac{4}{9} = \boxed{\frac{5}{9}}$$

c) We just saw in the book some properties of density matrices. In particular, for a pure state, and a density matrix ρ, we *must* have:

$$\rho^2 = \rho \text{ and } \text{Tr}(\rho)^2 = 1$$

While for a mixed state, we *must* have:

$$\rho^2 \neq \rho \text{ and } \text{Tr}(\rho)^2 < 1$$

Clearly, in our case, $\boxed{\rho \text{ represents a mixed state.}}$.