The Theoretical Minimum

Classical Mechanics - Solutions

I03E01

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Exercise 1. *Compute all first and second partial derivatives —including mixed derivatives—of the following functions.*

$$x^2 + y^2 = \sin(xy)$$
$$\frac{x}{y}e^{(x^2+y^2)}$$
$$e^x \cos y$$

This is again a simple differentiation exercise. We're not going to go too much in details; you may want to refer to L02E01 if you need a more detailed treatment. The process is very mechanical: use linearity to isolate constants and propagate differentiation to individual terms, if there's a product of functions, use the product rule, and if you can represent an expression as a composition of functions, often by introducing intermediate functions, apply the chain rule.

Regarding partial differentiation, the key thing is to consider all arguments of a function to be constants but the one we're differentiating the function against.

$\boldsymbol{E(x,y) : x^2 + y^2 = \sin(xy)}$

This looks more like an expression than a function; we'll interpret its differentiation to be the differentiation of each part of the equality.

$$\boxed{\frac{\partial}{\partial x}E(x,y) : 2x = y\cos(xy);} \qquad \boxed{\frac{\partial}{\partial y}E(x,y) : 2y = x\cos(xy)}$$

We may now compute second order derivatives:

$$\boxed{\frac{\partial^2}{\partial x^2}E(x,y) : 2 = -y^2\sin(xy);} \qquad \boxed{\frac{\partial^2}{\partial y^2}E(x,y) : 2 = -x^2\sin(xy)}$$

And assuming the symmetry of second derivatives:

$$\frac{\partial^2}{\partial x \partial y}E(x,y) = \frac{\partial^2}{\partial y \partial x}E(x,y) : \boxed{2 = \cos(xy) - xy\sin(xy)}$$

Remark 1. *The fact that:*

$$\frac{\partial^2}{\partial x \partial y}\varphi = \frac{\partial^2}{\partial y \partial x}\varphi$$

Isn't so obvious, mathematically speaking: the result is called Clairaut's theorem, or Schwarz's theorem[1]*. It requires φ to have* ***continuous second partial derivatives****. In the context of classical mechanics, almost always we'll be dealing with smooth*[2] *functions of time (positions/velocities/accelerations, so we'll always assume it to be true.*

[1] `https://en.wikipedia.org/wiki/Symmetry_of_second_derivatives`
[2] `https://en.wikipedia.org/wiki/Smoothness`

$\varphi(x,y) = \frac{x}{y}e^{(x^2+y^2)}$
First order derivatives; we can go a little slower here. Essentially, reserve the constant $(1/y)$, apply the product rule followed by a chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi(x,y) &= \frac{1}{y}\frac{\partial}{\partial x}xe^{(x^2+y^2)} \\
&= \frac{1}{y}\left((\frac{\partial}{\partial x}x)e^{(x^2+y^2)} + x(\frac{\partial}{\partial x}e^{(x^2+y^2)})\right) \\
&= \frac{1}{y}\left(e^{(x^2+y^2)} + x(\frac{\partial}{\partial x}x^2+y^2)e^{(x^2+y^2)})\right) \\
&= \boxed{\frac{1}{y}(2x^2+1)e^{(x^2+y^2)}}
\end{aligned}$$

Remark 2. *As I don't think this has been encountered before, note that we'll use the following "identity":*

$$x^{-n} = \frac{1}{x^n}$$

to help compute the derivatives of x^{-n} using the rule to derivate x^n:

$$\frac{d}{dx}\frac{1}{x^n} = \frac{d}{dx}x^{-n} = -nx^{-n-1} = -n\frac{1}{x^{n+1}}$$

And so for the other first order-derivative:

$$\begin{aligned}
\frac{\partial}{\partial y}\varphi(x,y) &= x\frac{\partial}{\partial y}y^{-1}e^{(x^2+y^2)} \\
&= \boxed{xe^{(x^2+y^2)}(2-\frac{1}{y^2})}
\end{aligned}$$

Then for the (non-mixed) second order derivatives:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\varphi(x,y) &= \frac{1}{y}\frac{\partial^2}{\partial x^2}(2x^2+1)e^{(x^2+y^2)}; \\
&= \frac{1}{y}e^{(x^2+y^2)}(4x+(2x^2+1)2x); \\
&= \boxed{\frac{x}{y}(4x^2+6)e^{(x^2+y^2)};}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial y^2}\varphi(x,y) &= x\frac{\partial^2}{\partial y^2}e^{(x^2+y^2)}(2-y^{-2}) \\
&= xe^{(x^2+y^2)}((2-y^{-2})2y+2y^{-3}) \\
&= \boxed{2xe^{(x^2+y^2)}(2y-\frac{1}{y}+\frac{1}{y^3})}
\end{aligned}$$

Finally, for the mixed second derivatives:

$$\frac{\partial^2}{\partial x\partial y}\varphi(x,y) = (2x^2+1)e^{(x^2+y^2)}(-y^{-2}+y^{-1}2y) = \boxed{(2x^2+1)e^{(x^2+y^2)})(2-\frac{1}{y^2})}$$

Remark 3. *There's a common shortcut notation for partial derivatives that we will use from now on:*

$$\frac{\partial}{\partial x}\varphi = \varphi_x; \quad \frac{\partial^2}{\partial x^2}\varphi = \varphi_{x,x}; \quad \frac{\partial^2}{\partial y\partial x}\varphi = \varphi_{x,y}$$

$\phi(x,y) = e^x \cos y$

$$\begin{aligned}
\phi_x(x,y) &= \boxed{e^x\cos y;} & \phi_y(x,y) &= \boxed{-e^x\sin y} \\
\phi_{x,x}(x,y) &= \boxed{e^x\cos y;} & \phi_{y,y}(x,y) &= \boxed{-e^x\cos y}
\end{aligned}$$

$$\phi_{x,y}(x,y) = \phi_{y,x}(x,y) = \boxed{-e^x\sin y}$$