The Theoretical Minimum

Classical Mechanics - Solutions

L06E01

M. Bivert

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Exercise 1. *Show that Eq.* (4) *is just another form of Newton's equation of motion* $F = ma$.

Where Eq. (4) are the freshly derived Euler-Lagrange equations of motions:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \tag{1}$$

In the context of a single particle moving in one dimension, with kinetic and potential energy given by:

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ V &= V(x) \end{aligned}$$

From which results the Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}^2 - V(x) \end{aligned} \tag{2}$$

Let us recall that we also have the *potential energy principle*, stated in one-dimension as Eq. (1) of the previous chapter, *Lecture 5: Energy*:

$$F(x) = -\frac{d}{dx}V(x) \tag{3}$$

Which is also stated more generally in that same chapter, for an abstract configuration space $\{x\} = \{x_i\}$, as Eq. (5):

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\})$$

Thus, deriving each part of (1) with our Lagrangian (2), and considering the *definition* of a potential energy $V(x)$ (3) yields:

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{x}}L &= \frac{d}{dt}m\dot{x} \\ &= m\ddot{x} \end{aligned} \qquad\qquad \begin{aligned} \frac{\partial}{\partial x}L &= \frac{\partial}{\partial x}V(x) \\ &= -F \end{aligned}$$

Then indeed, Euler-Lagrange equations become equivalent to Newton's law of motion:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0 \\ \Leftrightarrow\, &m\ddot{x} - (-F) = 0 \\ \Leftrightarrow\, &\boxed{F = m\ddot{x} = ma} \quad \square \end{aligned}$$