

Shrinkage and regression to the mean

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Shrinkage of results can be seen
to be a necessary fact of life.

(Stephen Senn)

1. Shrinkage is a fact of life

Assume true diastolic blood pressure, τ , at baseline is measured with error ϵ so that

$$X = \tau + \epsilon \quad (1)$$

is the *observed* blood pressure. Let the *true* mean difference between patients be Δ and the *observed* mean difference D , then the expectation of D is

$$E(D \mid \Delta = \delta) = \delta, \quad (2)$$

that is, we have “classical” *unbiasedness*, since $\beta_{D|\Delta} = \frac{\text{Cov}(\Delta, D)}{\sigma_\Delta^2} = 1$.

However, the contrary is not true. We have for the expectation of Δ , given an observed difference d ,

$$|E(\Delta \mid D = d)| < |d|, \quad (3)$$

since $\beta_{\Delta|D} = \frac{\text{Cov}(\Delta, D)}{\sigma_D^2} < 1$, and we have *regression to the mean*¹.

1

In a Bayesian approach, shrinking is *natural* and we have *inverse unbiasedness*. Most bayesians are rather unconcerned about unbiasedness (at least in the formal sampling-theory sense above) of their estimates. For example, Gelman et al (1995) write: “From a Bayesian perspective, the principle of unbiasedness is reasonable in the limit of large samples, but otherwise it is potentially misleading. Unbiasedness as conventionally understood is not a necessary property of good inferences”. Assume without loss of generality $E(\Delta) = 0$ and $\hat{\Delta}$ an unbiased estimate of a given effect Δ and $\hat{\Delta}_{shrunk}$ a shrunk estimate. Although $\hat{\Delta}_{shrunk}$ is not unbiased in the classic forward sense, $E(\hat{\Delta}_{shrunk} \mid \Delta = \delta) \neq \delta$ it is unbiased in the Bayesian backward sense: $E(\Delta \mid \hat{\Delta}_{shrunk} = \hat{\delta}_{shrunk}) = \hat{\delta}_{shrunk}$.

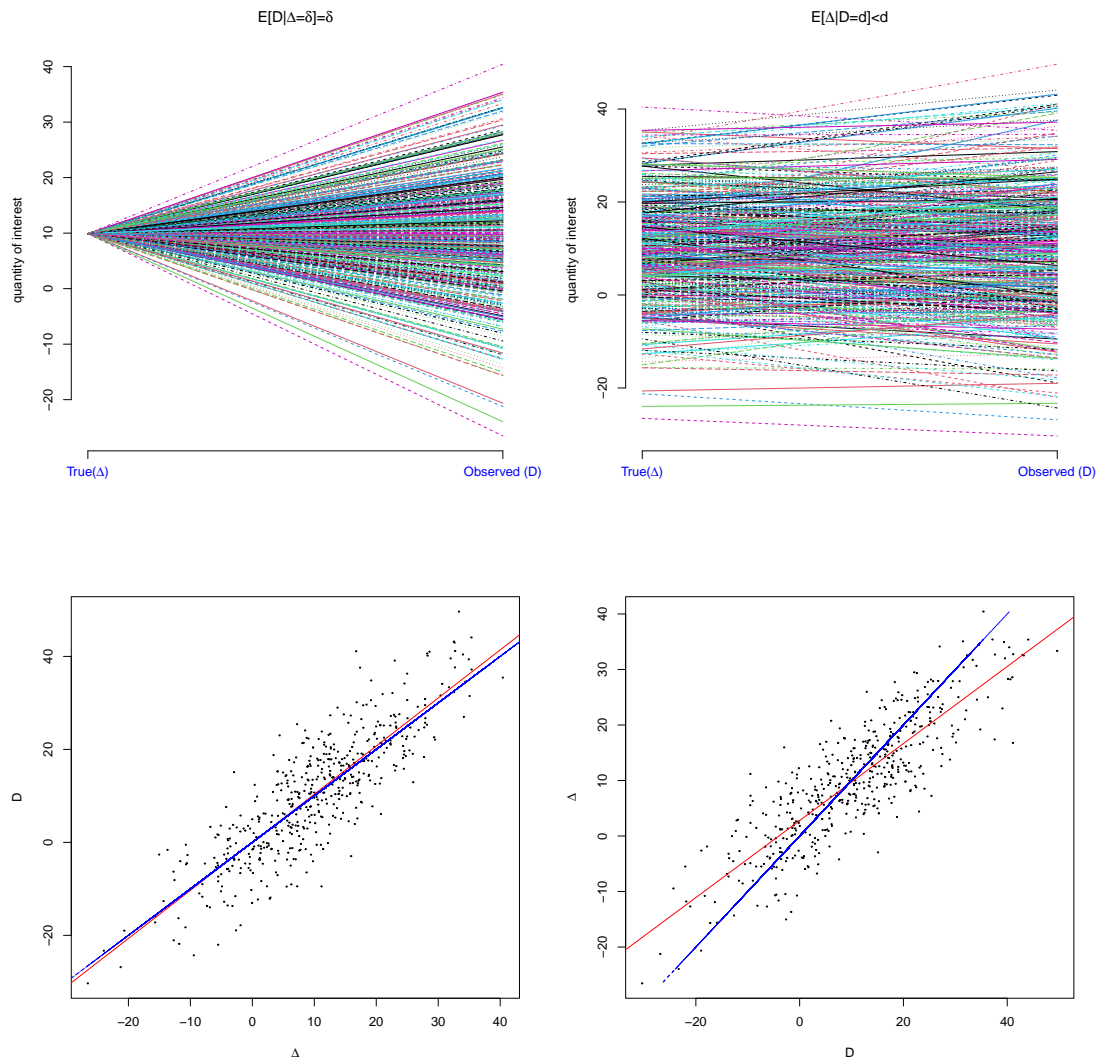


Figure 1: Shrinkage as a fact of life

Reliability as upper bound The maximal possible correlation between Δ and D is $\sqrt{rel_D}$.

2. Placebo as a statistical phenomenon

Placebo effects can often be interpreted as a purely statistical – not a psychological – phenomenon.

Assuming no true change. We simulate correlated pre-post diastolic blood pressure data assuming *no change* from baseline to follow-up: simulations from parameters: $\rho_{BL,FU} = 0.76, \mu_{BL} = \mu_{FU} = 90, \sigma_{BL} = \sigma_{FU} = 8$. Then let us look at the *subgroup* of “hypertensive at baseline” only. We have regression to the mean, since $\beta_{FU|BL} = \frac{\sigma_{FU,BL}}{\sigma_{BL}^2} = r \leq 1$.

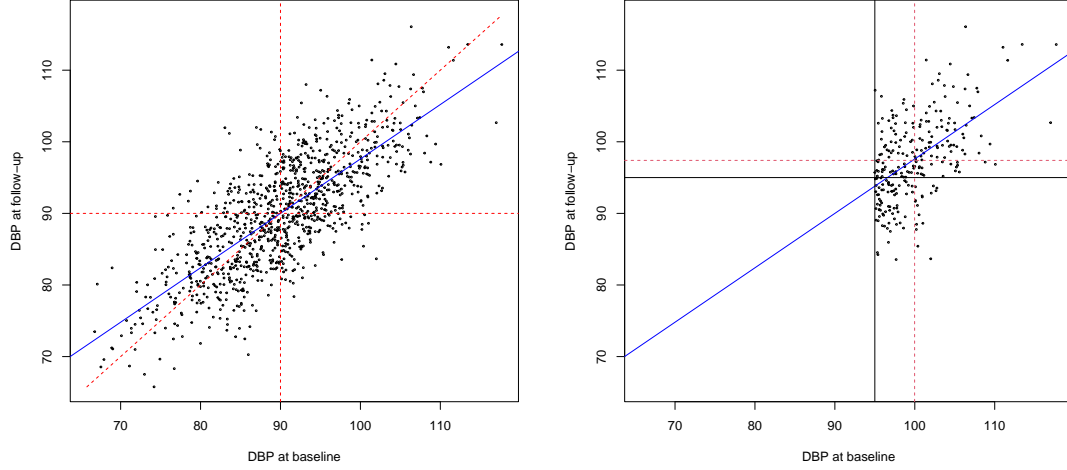
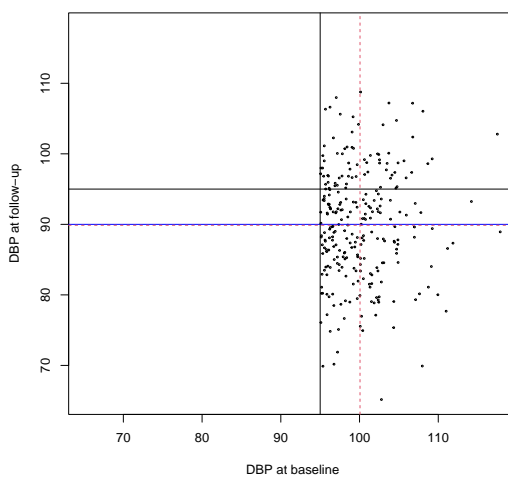
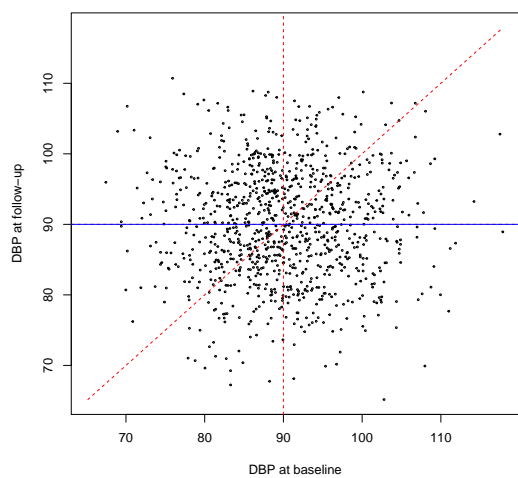
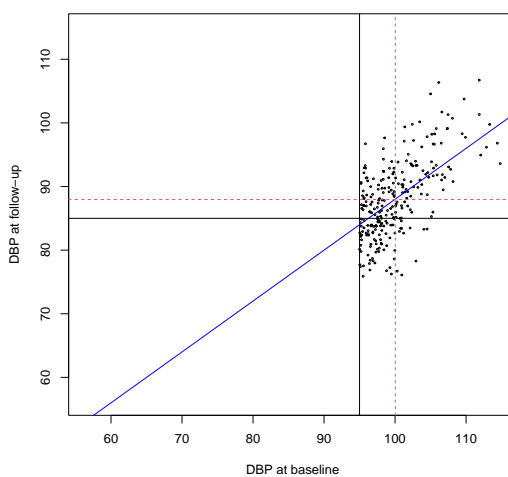
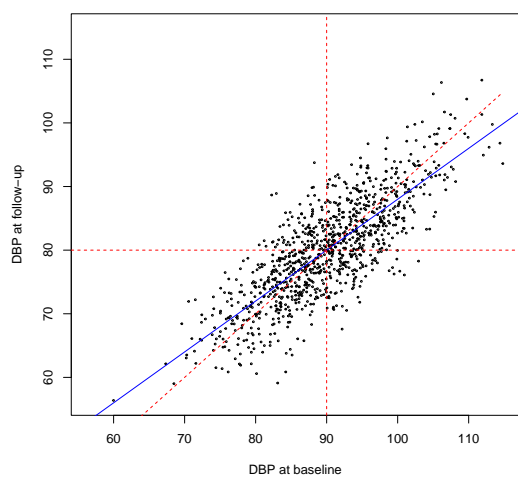


Figure 2: Simulation of diastolic blood pressure data. Simulations from parameters: $\rho_{BL,FU} = 0.76, \mu_{BL} = \mu_{FU} = 90, \sigma_{BL} = \sigma_{FU} = 8$. Left panel: Baseline versus Follow-up for diastolic blood pressure: no change in the mean. Right panel: Baseline versus Follow-up for “hypertensive at baseline” only. We observe an apparent change due to regression to the mean (Solid line: Regression of follow-up on baseline-measure (that is, by fixing baseline)). Dashed lines: mean values and equality lines.

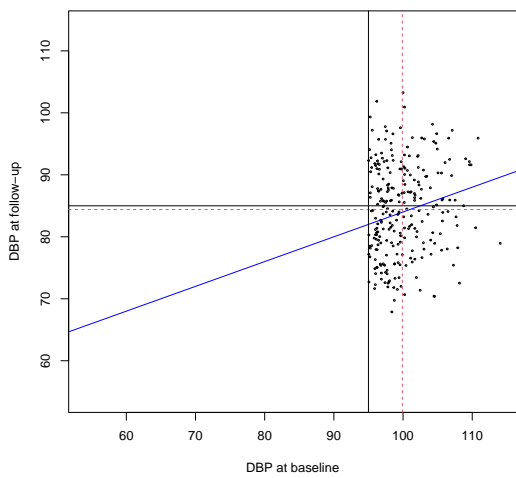
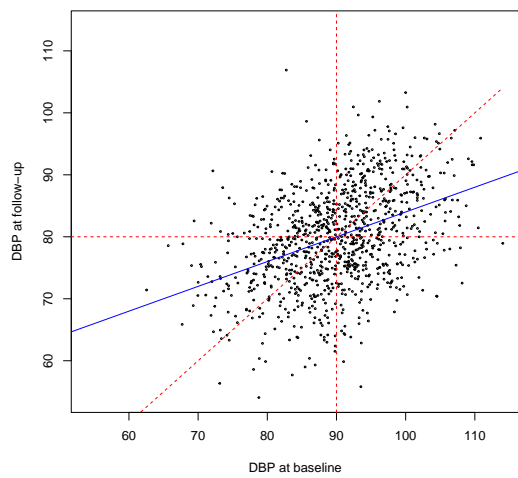
Extreme case: $\rho=0$



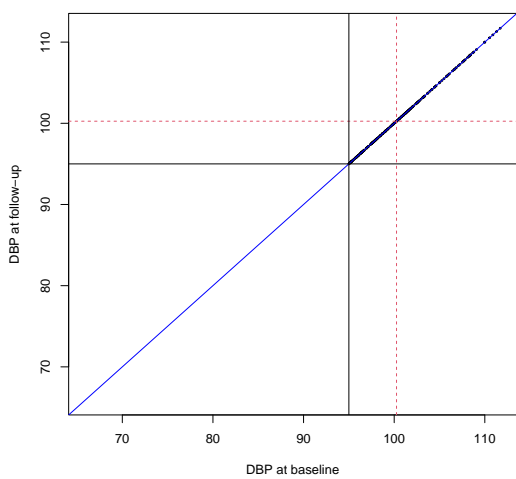
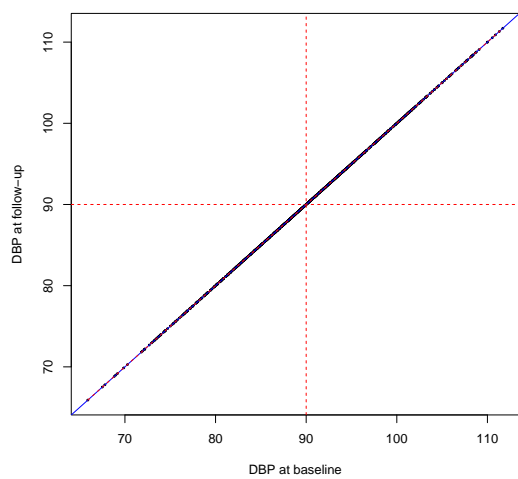
Including a true change of -10 and $\rho=.8$



Including a true change of -10 and $\rho=.4$



$\rho=1$ (Perfect reliability)



A. Maths

The conditional expected value of Y , given that X is t standard deviations above its mean (and that includes the case where it is below its mean, when $t < 0$), is ρ standard deviations above the mean of Y .

Since $|\rho| \leq 1$, Y is no farther from the mean than X is, as measured in the number of standard deviations. Hence, if $0 \leq \rho < 1$, then (X, Y) shows regression toward the mean.

$$\boxed{\frac{E(Y | X) - E(Y)}{\sigma_Y} = \rho \frac{X - E(X)}{\sigma_X}}, \quad (4)$$

that is, $z_{Y|X} = \rho z_x$, leading to $z_{Y|X} - z_X = (\rho - 1)z_x$. The amount of RTM is

$$\boxed{|z_{Y|X} - z_x| = (1 - \rho)z_x}. \quad (5)$$

The estimated fraction of RTM is given by $1 - \rho$, the fraction of variance that is due to within-subject variability. This quantity represents *unreliability*.