Correlation and Convolution of Images

recommended reading:

Computer Vision: Algorithms and Applications 2nd Edition

Richard Szeliski

The book can be freely downloaded from

https://szeliski.org/Book

correlation and convolution is covered in chapter 3 of the book

In [2]:
```
%matplotlib inline
import pandoc
import cv2
import numpy as np
from matplotlib import pyplot as plt
```

Formulas / Correlation

In the computer vision book, correlation is defined by equation:

$$g(x,y) = \sum_{u,v} f(x+u, y+v) \cdot h(u,v)$$

Let us make some specific assumptions to get to a formulation which is more useful for programming or to get further insight:

$f(x,y)$ denotes the *original* image. It is defined on a set of value pairs x, y. For the original image let us assume a finite range for x and y. (For some applications it may be necessary to extend the image beyond this range. But currently let us assume, that $f(x,y) = 0$ outside the range of x and y defined above).

$$0 < x < N_x$$

and

$$0 < y < N_y$$

Function $h(u,v)$ the *kernel* function. Let u defined for

$$-k_{u,l} \leq u \leq k_{u,h}$$

and in a similar way for v

$$-k_{v,l} \leq v \leq k_{v,h}$$

Now the summation formula for the correlation operation can be rewritten more explictly:

$$g(x,y) = \sum_{u=-k}^{k} \sum_{v=-k}^{k} h(u,v) \cdot f(x+u, y+v)$$

This formula can be quite often found in computer vision literature.

Discussion

The specific choice setting the range of indices of the kernel function centered at $u = 0, v = 0$ means that the *central value* $h(0,0)$ always is the weighting factor for the image point $f(x,y)$.

Therefore correlation function $g(x,y)$ can be interpreted as the *weighted contribution* of image points in the vicinity of the original image point $f(x,x)$. How large this vicinity is, depends on the number of samples of the kernel $h(u,v)$.

Even under the assumption we made for the original image $f(x,y)$ being defined only for $x : [0, \ldots, N_x - 1]$ and $y : [0, \ldots, N_y - 1]$ (and 0 outside this range), the correlation $g(x,y)$ is defined for a larger range of x and y values.

But by what amount is it larger ?

From the *finite range condition* of the original image $f()$ we observe:

$$0 < x + u \leq N_x - 1$$

$$-u < x \leq N_x - 1 - u$$

$$0 < y + v \leq N_y - 1$$

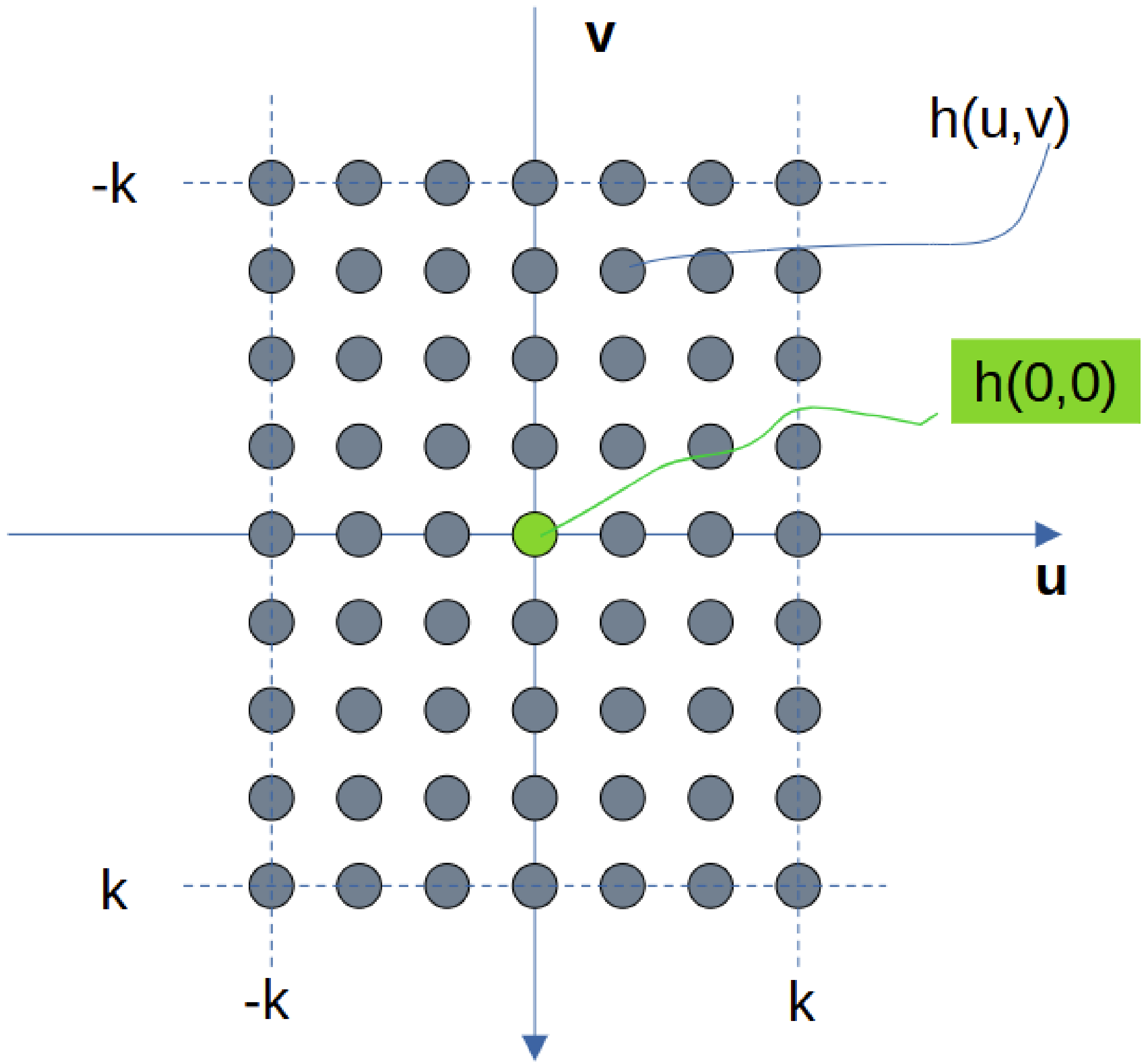
$$-v < y \leq N_y - 1 - v$$

Since $min(-u) = -k$ and $max(N_x - 1 - u) = N_x - 1 + k$ and $min(-v) = -k$ and $max(N_y - 1 - v) = N_y - 1 + k$ the correlation function $g(x,y)$ is defined in the extended range:

$$-k < x \leq N_x - 1 + k$$

$$-k < y \leq N_y - 1 + k$$

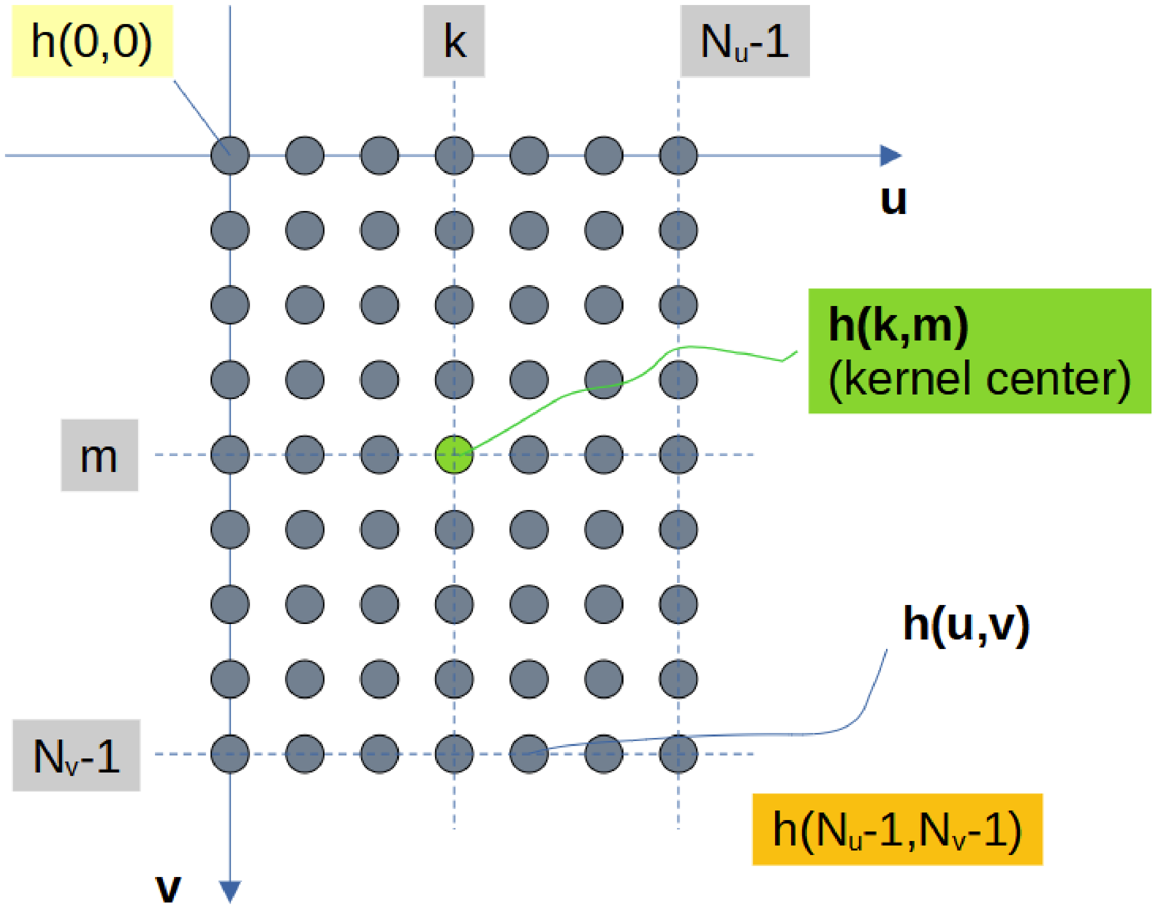
The figure shows the indexing schema of the kernel function with a center value $h(0,0)$. The kernel can interpreted as a matrix with $N_v = 2 \cdot k$ rows and $N_u = 2 \cdot k$ columns.



Alternate formulation of correlation function

Previously centered representation of the kernel function $h(u, v)$. But other representation should be equally possible. In fact the OpenCV library uses a indexing scheme of the kernel function which uses only positive indices and allows to choose the center / origin of the kernel function.

The figure below shows the indexing scheme used in OpenCV's function `filter2D`.



The kernel is described by indices u with $0 < u < N_u - 1$ and v with $0 < v < N_v - 1$. The center $h(k, m$ of the kernel can be chosen for k and m in the range:

$$0 < k < N_u - 1$$

$$0 < m < N_v - 1$$

Thus it is not required to choose the reference position of the kernel in the center of the kernel. We could if as well choose $k = 0, m = 0$ (upper left element of kernel) or $k = N_u - 1, m = N_v - 1$ (lower right element of kernel).

Correlation as used in OpenCv's function `filter2D`

$$g(x, y) = \sum_{u=0}^{N_u-1} \sum_{v=0}^{N_v-1} h(u, v) \cdot f(x - k + u, y - m + v)$$

To see why $h(k, m)$ is named the *kernel center* set $u = k$ and $v = m$ in the formula:

$$h(k, m) = h(k, m) \cdot f(x, y)$$

As before we explore the range of x and y for which the correlation $g(x, y)$ is defined:

$$0 < x - k + u < N_x - 1$$

$$k - u < x < N_x - 1 + k - u$$

and

$$0 < y - m + v < N_y - 1$$

$$m - v < y < N_y - 1 + m - v$$

The correlation function $g(x, y)$ is defined for the extended range:

$$k - N_u + 1 < x < N_x - 1 + k$$

$$m - N_v + 1 < y < N_y - 1 + m$$

While the original image $f(x, y)$ has N_x columns and N_y rows and represents a $N_y \cdot N_x$ pixel image the correlation operation represents an *image* with $\left(N_y + N_v - 1\right) \cdot (N_x + N_u - 1)$ pixel image.

Correlation as used Numpy's/Scipy's function correlation function

Both libraries do not include an explicit formula to compute the 2D-correlation of the 2D-arrays. Image and kernel are just represented as matrices. Nevertheless the correlation result is still represented by a matrix with $N_y + N_v - 1$ columns and $N_x + N_u - 1$ rows.

Formulas / Convolution

Recap / Correlation

In https://szeliski.org/Book correlation is defined by equation:

$$g(x, y) = \sum_{u,k} f(x + u, y + v) \cdot h(u, v)$$

and for convolution:

$$g(x, y) = \sum_{u,k} f(x - u, y - v) \cdot h(u, v)$$

A more practical formula for correlation is given in the description of OpenCv's function `filter2D`

$$g(x, y) = \sum_{u=0}^{N_u - 1} \sum_{v=0}^{N_v - 1} h(u, v) \cdot f(x - k + u, y - m + v)$$

Accordingly a similar equation can be provided for convolution:

$$g(x,y) = \sum_{u=0}^{N_u-1} \sum_{v=0}^{N_v-1} h(u,v) \cdot f(x+k-u, y+m-v)$$

In [ ]:

```

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