Applications of 1D Discrete Fourier Transform

In [34]:
```
%matplotlib inline
import numpy as np
import scipy as sc
from scipy import signal
from matplotlib import pyplot as plt
```

Harmonic function

Defining a test function

To demonstrate some possible applications of the 1D Discrete Fourier transform a test function $g(t)$ is defined. The test function is first defined as a continuous function from which a discrete version will be derived.

Properties of the test function:

1. Superposition of several harmonic functions
2. Duration of test function shall be an integer of the duration a the harmonic with the lowest frequency
3. A higly oversampled discrete version of test function $g(t)$ is used

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The continous function is composed of a DC component A_0 and three complex harmonics with amplitudes A_1, A_2, A_3 and frequencies f_1, f_2, f_3. τ denotes a time shift.

$$g(t-\tau) = A_0 + A_1 \cdot \exp(j \cdot 2\pi \cdot f_1 \cdot (t-\tau)) + A_2 \cdot \exp(j \cdot 2\pi \cdot f_2 \cdot (t-\tau)) + A_3 \cdot \exp(j \cdot$$

Frequencies f_1, f_2, f_3 shall be related to a fundamental frequency f_s (sometimes referred to a frequency increment) by integer multiples m_1, m_2, m_3.

$$f_1 = m_1 \cdot f_s$$
$$f_2 = m_2 \cdot f_s$$
$$f_3 = m_3 \cdot f_s$$

$$g(t) = A_0 + A_1 \cdot \exp(j \cdot 2\pi \cdot m_1 \cdot f_s \cdot (t-\tau)) + A_2 \cdot \exp(j \cdot 2\pi \cdot m_2 \cdot f_s \cdot (t-\tau)) + A_3 \cdot \mathrm{ex}$$

$$g(t) = A_0 + A_1 \cdot \exp(-j \cdot 2\pi \cdot m_1 \cdot f_s \cdot \tau) \cdot \exp(j \cdot 2\pi \cdot m_1 \cdot f_s \cdot t) + A_2 \cdot \exp(-j \cdot 2\pi \cdot m_2$$
$$(j \cdot 2\pi \cdot m_3 \cdot f_s \cdot t)$$

Function $g(t)$ is periodc with period $T_p = \frac{1}{f_s}$. The continous function $g(t)$ is converted into a time discrete version by evaluating/sampling $g(t)$ on timing instants $k \cdot t_s$.

t_s denotes the sampling duration. k are integer values running from $0, \ldots, N-1$.

Time shift τ is expressed by $a \cdot t_s$.

A period T_p of $g(t)$ consists of N samples. Hence the sampling duration t_s is therfore $t_s = \frac{T_p}{N}$.

With these definition we get an equation for the time discrete function $g(k \cdot t_s)$:

$$g(k \cdot t_s) = A_0 + A_1 \cdot \exp(-j \cdot 2\pi \cdot m_1 \cdot f_s \cdot t_s \cdot a) \cdot \exp(j \cdot 2\pi \cdot m_1 \cdot f_s \cdot k \cdot t_s) + A_2 \cdot \exp(-$$
$$(-j \cdot 2\pi \cdot m_3 \cdot f_s \cdot t_s \cdot a) \cdot \exp(j \cdot 2\pi \cdot m_3$$

Using

$$t_s \cdot f_s = \frac{1}{N}$$

$$g(k \cdot t_s) = A_0 + A_1 \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot m_1 \cdot a\right) \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot m_1 \cdot k\right) + A_2 \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot\right.$$
$$\left(j \cdot \frac{2\pi}{N} \cdot m_3 \cdot k\right)$$

Finally the choice of N depends on the highest frequency $m_3 \cdot f_s$. With an integer oversampling factor $osr \geq 2$, the number of samples N is choosen like this:

$$N = osr \cdot m_3$$

Note

It is fairly instructive to see how a time shift by $\tau = a \cdot t_s$ affects the Fourier coefficients A_1, A_2 and A_3. They are just multiplied by a complex exponential $\exp\left(-j \cdot \frac{2\pi}{N} \cdot m_x \cdot a\right)$ (where m_x denotes the frequency index. Time shift τ need not be an integer multiple of the sampling duration t_s. Any real value a will do.

A practical example

$$A_0 = 1.5; A_1 = 1.0; A_2 = 4.0; A_3 = 6.0$$
$$m_1 = 5; m_2 = 15; m_3 = 42$$
$$\tau = a \cdot t_s = 0$$
$$osr = 10 \rightarrow N = 420$$

In [35]:
```
# the definition
def tstFunc(A_0, A_1, A_2, A_3, m_1, m_2, m_3, osr, a):
    N = osr*m_3
```

```
    kv = np.arange(0, N)
    gfunc = A_0 + A_1*np.exp(-2j*np.pi*m_1*a/N)*np.exp(2j*np.pi*m_1*kv/N) +\
    A_2 * np.exp(-2j*np.pi*m_2*a/N) * np.exp(2j*np.pi*m_2*kv/N) +\
    A_3 * np.exp(-2j*np.pi*m_3*a/N) * np.exp(2j*np.pi*m_3*kv/N)
    return gfunc, kv, N
```

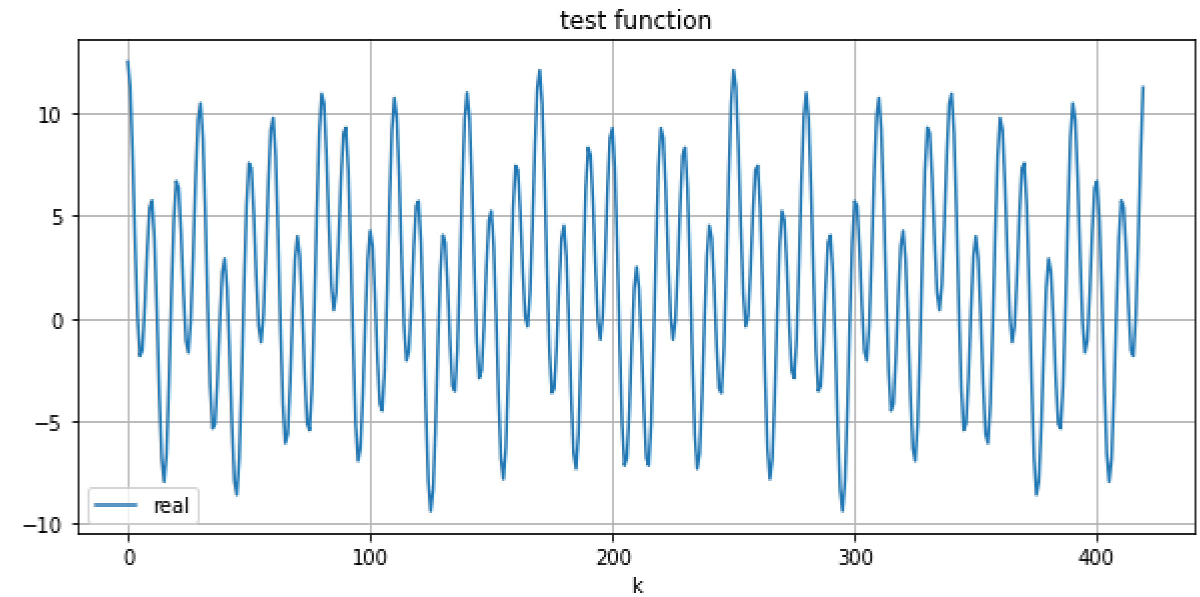
In [36]:

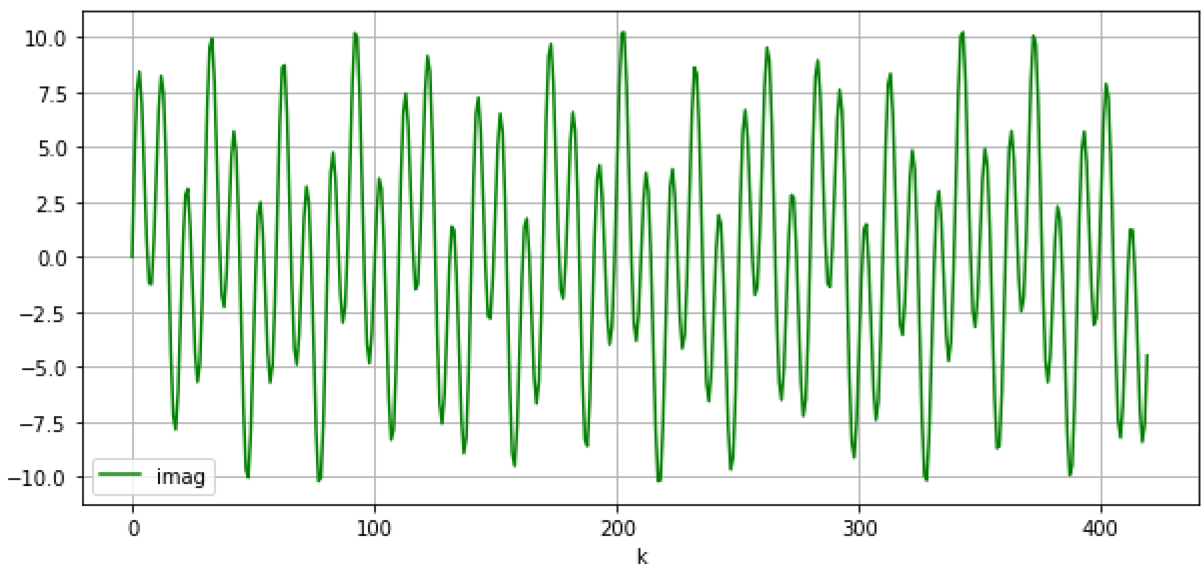
```
# function evaluation
A_0 = 1.5
A_1 = 1.0
A_2 = 4.0
A_3 = 6.0
m_1 = 5
m_2 = 15
m_3 = 42
a = 0
osr = 10

gfunc, kv, N = tstFunc(A_0, A_1, A_2, A_3, m_1, m_2, m_3, osr, a)
```

The graphics below shows the real- and imaginary part in two subplots.

In [37]:

```
# graphics
fig1 = plt.figure(1, figsize=[10, 10])
ax_f11 = fig1.add_subplot(2, 1, 1)
ax_f11.plot(kv, gfunc.real, label="real")
ax_f11.legend(loc='lower left')
ax_f11.grid(True)
ax_f11.set_xlabel('k')
ax_f11.set_title('test function')

ax_f12 = fig1.add_subplot(2, 1, 2)
ax_f12.plot(kv, gfunc.imag, color='g', label="imag")
ax_f12.legend(loc='lower left')
ax_f12.grid(True)
ax_f12.set_xlabel('k');
```





Applying the DFT

Function `fft` of libraries `Scipy/Numpy` implements the DFT according to this formula:

$$A_n = \sum_{k=0}^{N-1} a_k \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot n \cdot k\right)$$

To get the actual amplitudes A_0, A_1, A_2, A_1 a correction factor of $\frac{1}{N}$ must be applied.

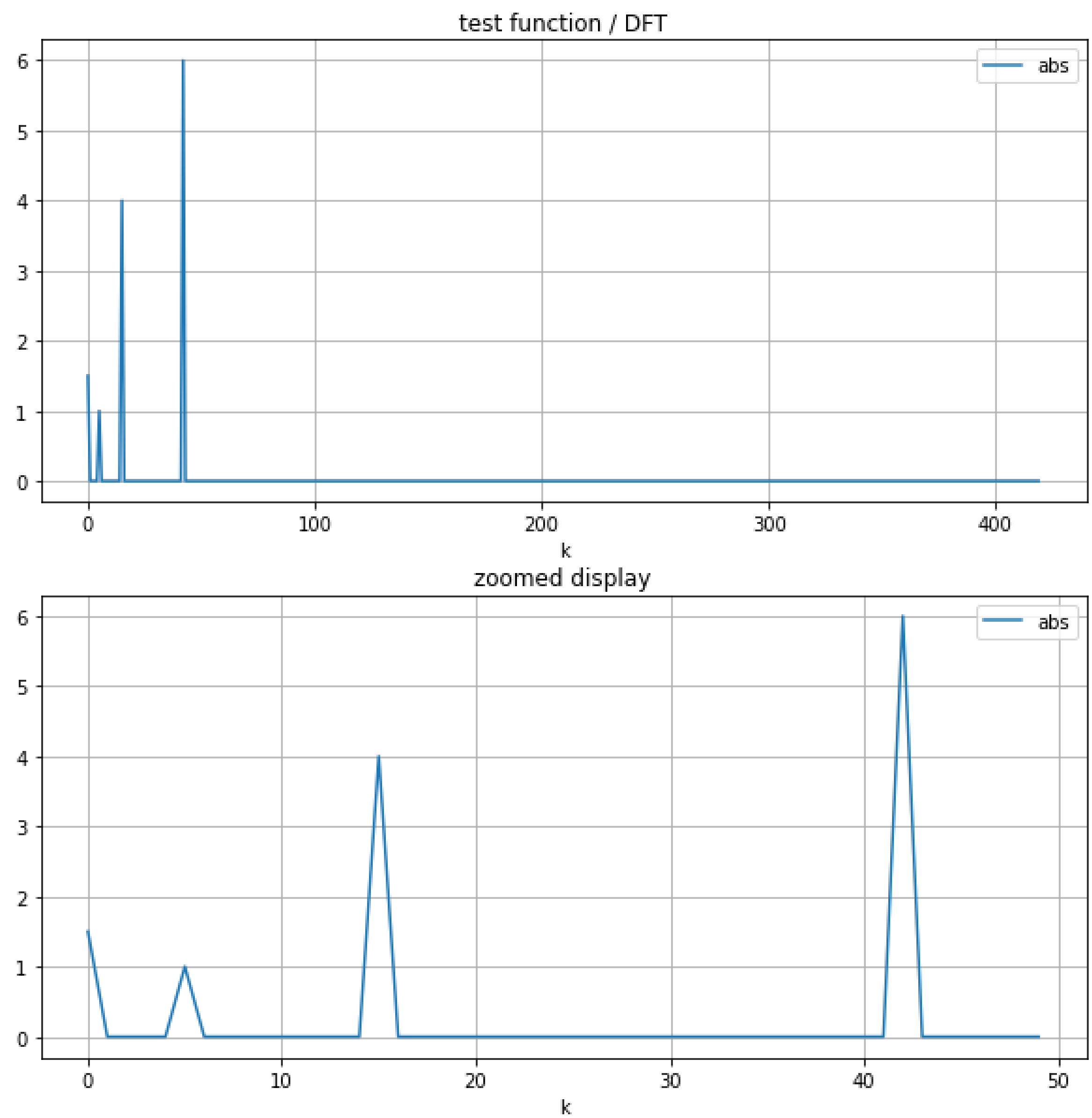
See example below.

In [38]:
```
# note the correction factor (1/N)
Gfft = (1/N)*np.fft.fft(gfunc)
```

In the plot below only the magnitude of the DFT is displayed. Amplitudes A_0, A_1, A_2, A_1 are accurately reproduce.

A subplot below displays a zoomed version just to show that amplitudes occur at their correct *frequency points* m_1, m_2, m_3

In [39]:
```python
# graphics
fig2 = plt.figure(2, figsize=[10, 10])
ax_f21 = fig2.add_subplot(2, 1, 1)
ax_f21.plot(kv, np.abs(Gfft), label="abs")
ax_f21.legend(loc='upper right')
ax_f21.grid(True)
ax_f21.set_xlabel('k')
ax_f21.set_title('test function / DFT')

ax_f22 = fig2.add_subplot(2, 1, 2)
ax_f22.plot(kv[0:50], np.abs(Gfft[0:50]), label="abs")
ax_f22.legend(loc='upper right')
ax_f22.grid(True)
ax_f22.set_xlabel('k')
ax_f22.set_title('zoomed display');
```



Applying the Inverse DFT (IDFT)

The examples show how to use the inverse discrete Fourier transform to obtain the sequence from the Fourier coefficients A_1, A_2, A_3.

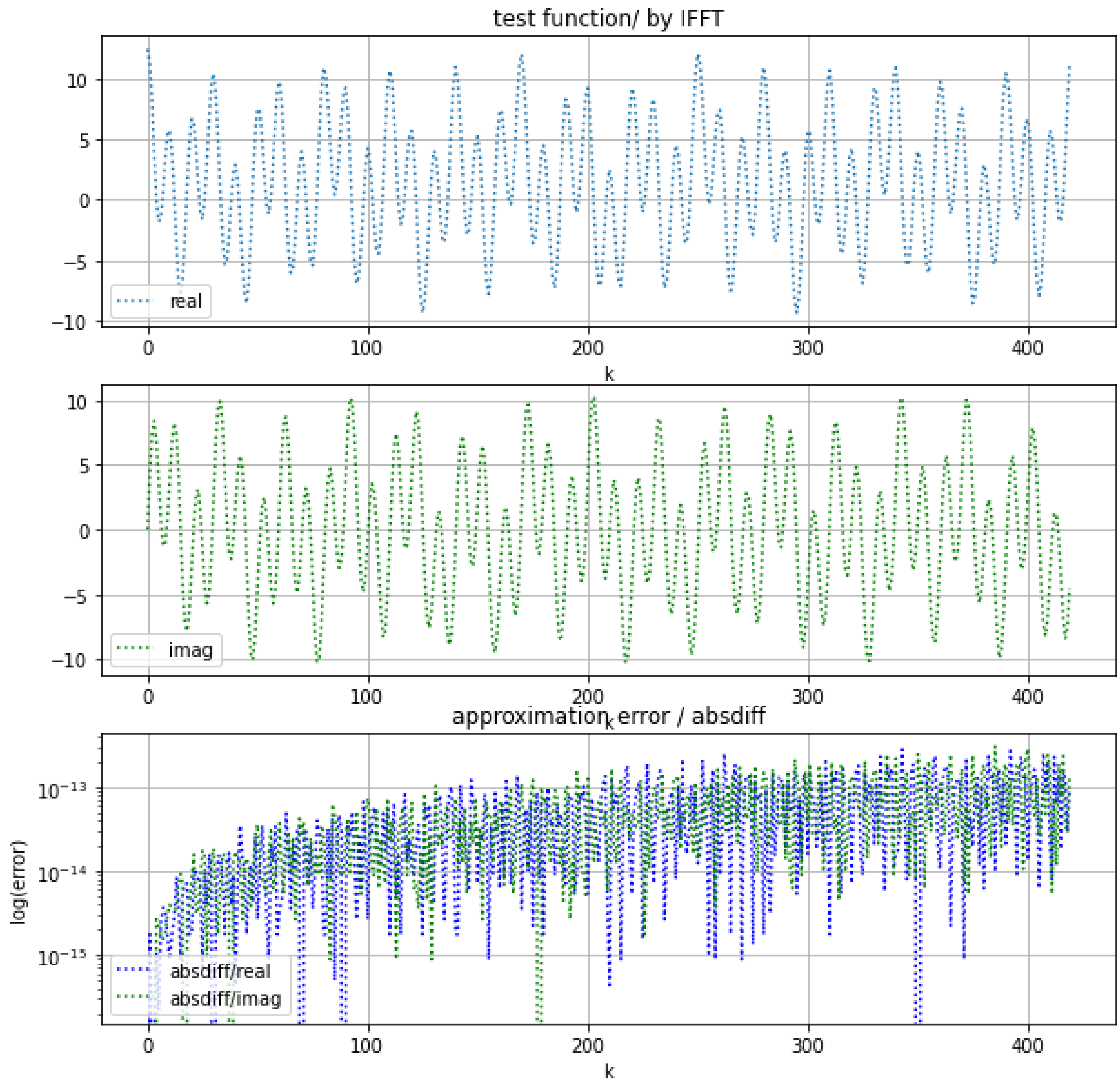
An array S of length N is initialised by zeros. The Fourier coefficients A_1, A_2, A_3 are then *plugged* into array S at the corresponding frequency indices m_1, m_2, m_3.

Finally the inverser DFT is applied by calling function `ifft` and applying a correction factor N. The resulting series `gIFFT` is displayed.

A third subplot shows the absolute error of the original sequence and the sequence obtained from applying the inverse transform. The error show on a logarithmic scale is quite small.

In [44]:
```
S = np.zeros(N)
S[[0, m_1, m_2, m_3]] = np.array([A_0, A_1, A_2, A_3])

gIFFT = N*np.fft.ifft(S)
```

In [46]:
```
# graphics
fig3 = plt.figure(3, figsize=[10, 10])
ax_f31 = fig3.add_subplot(3, 1, 1)
ax_f31.plot(kv, gIFFT.real, linestyle=':', label="real")
ax_f31.legend(loc='lower left')
ax_f31.grid(True)
ax_f31.set_xlabel('k')
ax_f31.set_title('test function/ by IFFT')

ax_f32 = fig3.add_subplot(3, 1, 2)
ax_f32.plot(kv, gIFFT.imag, linestyle=':', color='g', label="imag")
ax_f32.legend(loc='lower left')
ax_f32.grid(True)
ax_f32.set_xlabel('k');

ax_f33 = fig3.add_subplot(3, 1, 3)
ax_f33.semilogy(kv, np.abs(gIFFT.real - gfunc.real), linestyle=':', color='b', labe
ax_f33.semilogy(kv, np.abs(gIFFT.imag - gfunc.imag), linestyle=':', color='g', labe
ax_f33.legend(loc='lower left')
ax_f33.grid(True)
ax_f33.set_title('approximation error / absdiff')
ax_f33.set_xlabel('k')
ax_f33.set_ylabel('log(error)');
```



Applying a time shift

By modifying the Fourier coefficents by a complex factor and applying the inverser DFT we obtain a time shifted series.

The example below demontrates this for a time shift $\tau = 4.5 \cdot t_s$.

For comparision the time shifted series has been computed using function `tstFunc` with a=4.5. The 3'rd subplot displays the absolute value of the difference. Again the error between direct computation and using inverse DFT is quite small due to rounding effects.

In [50]:
```
S_shift = np.zeros(N, dtype=np.complex64)

a= 4.5

# modifying the Fourier coefficients
A_1_shift = A_1 * np.exp(-2j*np.pi*m_1*a/N)
```

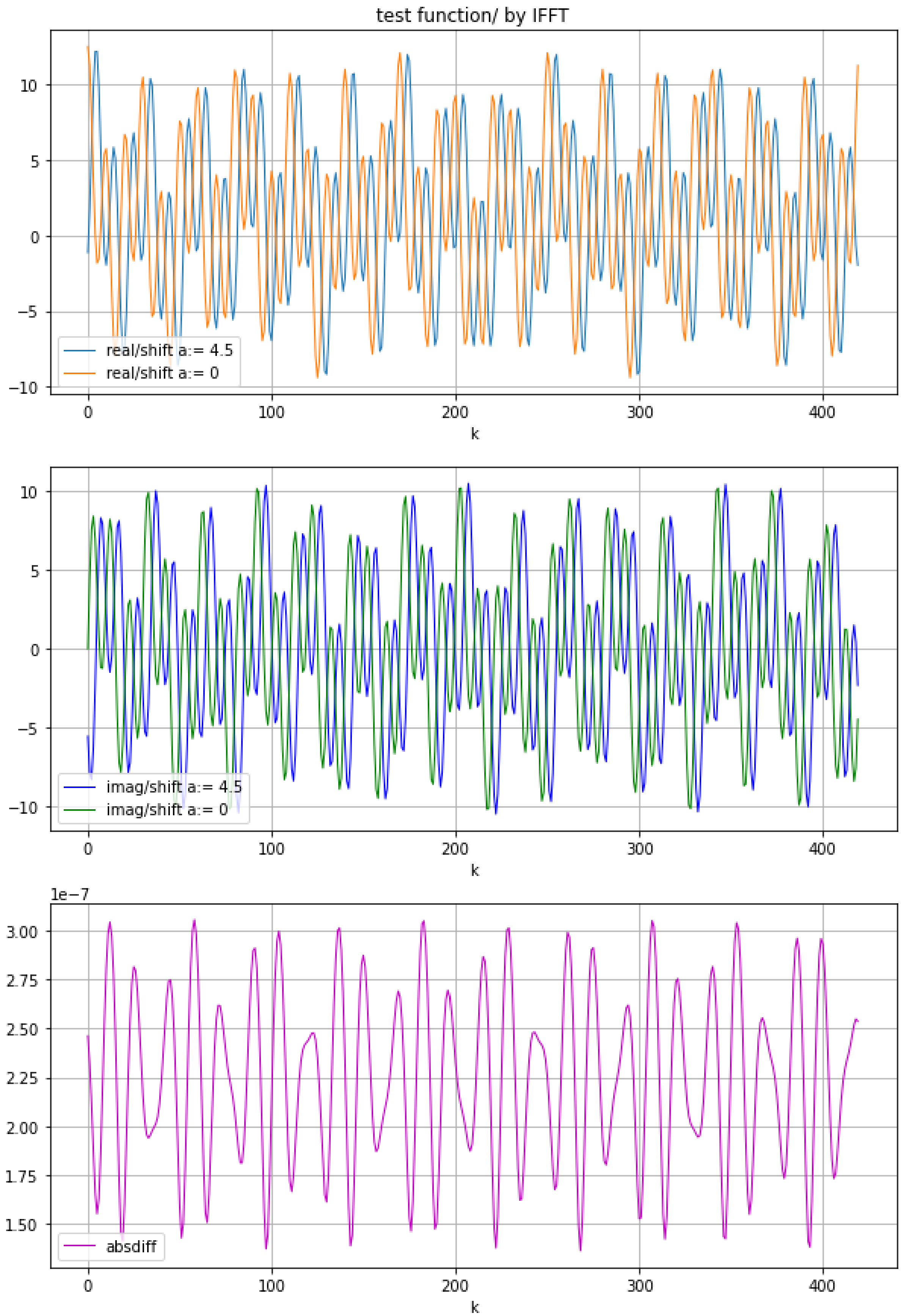
```
A_2_shift = A_2 * np.exp(-2j*np.pi*m_2*a/N)
A_3_shift = A_3 * np.exp(-2j*np.pi*m_3*a/N)

S_shift[[0, m_1, m_2, m_3]] = np.array([A_0, A_1_shift, A_2_shift, A_3_shift], dtyp

gIFFT_shift = N*np.fft.ifft(S_shift)

gfunc_shift, kv, N = tstFunc(A_0, A_1, A_2, A_3, m_1, m_2, m_3, osr, a)
```

Note

Plotting the time shifted series (real / imaginary parts) along with the unshifted series clearly shows the time shift.

More import however is the fact, that to applying a time shift to series amounts to modifying the Fourier coefficients by appropriately chosen factors. The inverse DFT then yields the samples of the time shifted series.

In [53]:
```
# graphics
fig4 = plt.figure(4, figsize=[10, 15])
ax_f41 = fig4.add_subplot(3, 1, 1)
ax_f41.plot(kv, gIFFT_shift.real, linewidth=1, label=f"real/shift a:= {a}")
ax_f41.plot(kv, gIFFT.real, linewidth=1, label=f"real/shift a:= 0")
ax_f41.legend(loc='lower left')
ax_f41.grid(True)
ax_f41.set_xlabel('k')
ax_f41.set_title('test function/ by IFFT')

ax_f42 = fig4.add_subplot(3, 1, 2)
ax_f42.plot(kv, gIFFT_shift.imag, linewidth=1, color='b', label=f"imag/shift a:= {a
ax_f42.plot(kv, gIFFT.imag, linewidth=1, color='g', label=f"imag/shift a:= 0")
ax_f42.legend(loc='lower left')
ax_f42.grid(True)
ax_f42.set_xlabel('k')

ax_f43 = fig4.add_subplot(3, 1, 3)
ax_f43.plot(kv, np.abs(gfunc_shift - gIFFT_shift), linewidth=1, color='m', label='a
ax_f43.legend(loc='lower left')
ax_f43.grid(True)
ax_f43.set_xlabel('k');
```


test function/ by IFFT
real/shift a:= 4.5
real/shift a:= 0
k
imag/shift a:= 4.5
imag/shift a:= 0
k
1e−7
absdiff
k

In [ ]: