Correlation and Convolution in 2 Dimensions

Motivation:

Correlation and convolution operations are commonly used in algorithms for processing of 2 dimensional function (eg. *images*).

The Python libraries `Numpy` and `Scipy` provide functions which support these operation. The `OpenCV` library has numerous functions for this task which are specifically aimed at image processing.

This notebook provides some background information and examples how to used these functions.

Links:

https://numpy.org/

https://scipy.org/

https://docs.opencv.org/3.4/d7/dbd/group__imgproc.html

Useful import statements

Here are some useful import statements for experiments with `Numpy`, `Scipy` and `OpenCV`

In [1]:
```
%matplotlib inline

import numpy as np
import scipy as sc
from scipy import signal
import cv2
from matplotlib import pyplot as plt
```

Some Conventions ...

In some books operations correlation and convolution are *loosely* defined by equations:

$$g_{corr}(x, y) = \sum_{u,v} f(x + u, y + v) \cdot h(u, v)$$

and

$$g_{conv}(x,y) = \sum_{u,v} f(x-u, y-v) \cdot h(u,v)$$

In these equations functions $f(x,y)$ and $h(u,v)$ are two dimensional (2D) functions. To be more specific some assumptions shall be made:

Function $f(x,y)$ is defined for a *finite* range of values x and y:

$$0 \le x \le N_x - 1$$

and

$$0 \le y \le N_y - 1$$

Function $h(u,v)$ is defined for a *finite* range of values u and v:

$$0 \le u \le N_u - 1$$

and

$$0 \le v < N_v - 1$$

Hence function $f(x,y)$ has $N_x \cdot N_y$ elements and function $h(u,v)$ has $N_u \cdot N_v$ elements

Correlation / Equation

The following equation shall be used for the explicit definition of correlation:

$$g_{corr}(x,y) = \sum_{u=0}^{N_u-1} \sum_{v=0}^{N_v-1} f(x+u, y+v) \cdot h(u,v)$$

boundary conditions

Function $f(x,y)$ has been defined on a finite range of values x and y. Outside this range $f(x,y)$ is undefined and in our case we make these assumptions:

$f(x,y) := 0$, if x and/or y are **not** in their definition range.

While this assumption is applied frequently there are other options, which define specific "fill-values" for $f(x,y)$ if x and/or y or outside the defined range.

Let us now look for which value pairs x, y the correlation function $g_{corr}(x,y)$ is defined.

From

$$\begin{aligned} 0 \le x + u \le N_x - 1 \\ -u \le x \le N_x - 1 - u \\ -N_u + 1 \le x \le N_x - 1 \end{aligned}$$

and

$$
\begin{aligned}
0 \leq y + v \leq N_y - 1 \\
- v \leq y \leq N_y - 1 - v \\
- N_v + 1 \leq y \leq N_y - 1
\end{aligned}
$$

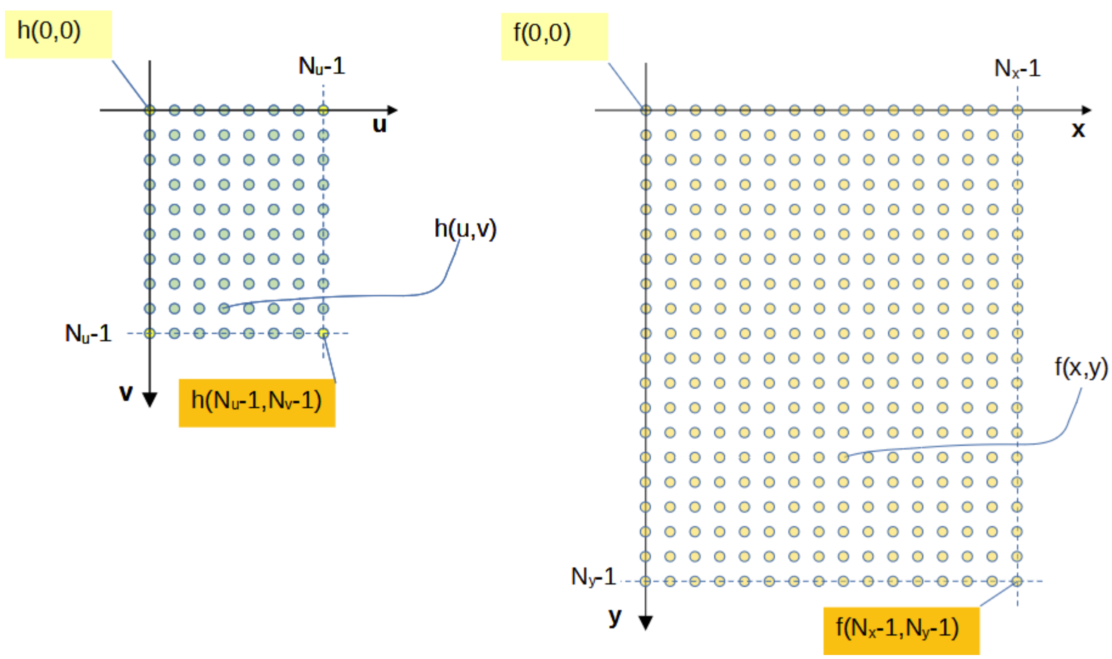
we see that the correlation function is defined for the extended range of x and y values:

$$
\begin{aligned}
-N_u + 1 \leq x \leq N_x - 1 \\
- N_v + 1 \leq y \leq N_y - 1
\end{aligned}
$$

Thus the full correlation has $N_x + N_u - 1$ x values and $N_y + N_v - 1$ y values.

A figure below gives more insight into the range of values where functions $h(u, v)$ and $f(x, y)$ are defined.

Visualisation



Depending on the choice of x, y we distinguish three scenarios:

scenario#1

values x, y have been chosen such that $f(x + u, y + v)$ is undefined for the full range of values u and y. Thus the correlation $g_{corr}(x, y)$ is undefined as well.

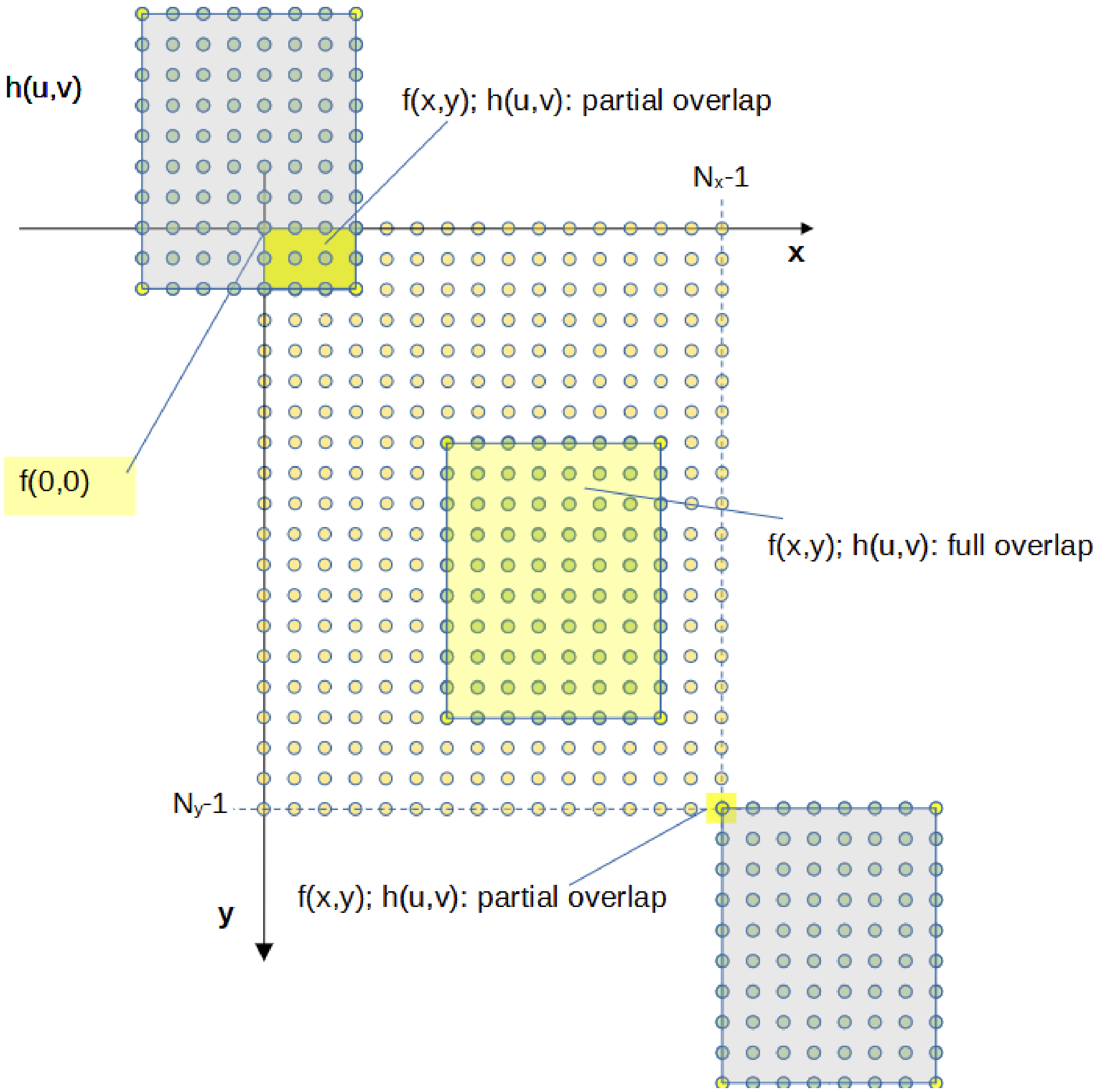
scenario#2

values x, y have been chosen such that $f(x + u, y + v)$ is only undefined for some values of u and y. The correlation $g_{corr}(x, y)$ is defined. (partial overlap of functions $f(x, y)$ and $h(u, v)$)

scenario#3

values x, y have been chosen such that $f(x+u, y+v)$ is defined for all values of u and y. The correlation $g_{corr}(x, y)$ is defined. (full overlap of functions $f(x, y)$ and $h(u, v)$)

The figure below shows two cases where there is a partial overlap of functions $f(x, y)$ and $h(u, v)$) and a case where there is full overlap.



Examples

Some examples shall demonstrate how to use `Scipy's` 2D correlation function `correlate2d`.

First two 2D arrays `fa` and `ha` are defined. With these arrays the experiments are computed.

creating 2D arrays

In [2]:
```python
fa = np.array([[1,2,3,4,5,6],
                 [2,3,4,5,6,7],
                 [3,4,5,6,7,8],
                 [4,5,6,7,8,9],
                 [5,6,7,8,9,10]], dtype=np.uint8)


ha = np.array([[3,4],
               [4,5],
               [5,6],
               [4,5]], dtype=np.uint8)

N_y, N_x = np.shape(fa)
N_v, N_u = np.shape(ha)
print(f"shape(fa): {np.shape(fa)}; N_x: {N_x} N_y: {N_y}")
print(f"shape(ha): {np.shape(ha)}; N_u: {N_u} N_v: {N_v}")
```

```
shape(fa): (5, 6); N_x: 6 N_y: 5
shape(ha): (4, 2); N_u: 2 N_v: 4
```

The correlation is then calculated with `Scipy` function `correlate2d` with several values of function parameter `mode`

`mode='full'`

`out = scipy.signal.correlate2d(in1, in2, mode='full', boundary='fill', fillvalue=0)`

In [3]:
```python
g_corr_a = signal.correlate2d(fa, ha, mode='full', boundary='fill', fillvalue=0)

N_y_g, N_x_g = np.shape(g_corr_a)
print(f"shape(g_corr_a): {np.shape(g_corr_a)}; N_x_g: {N_x_g} N_y_g: {N_y_g}\n")
print(f"g_corr_a:\n{g_corr_a}")
```

```
shape(g_corr_a): (8, 7); N_x_g: 7 N_y_g: 8

g_corr_a:
[[  5  14  23  32  41  50  24]
 [ 16  40  60  80 100 120  58]
 [ 32  74 103 132 161 190  91]
 [ 52 114 150 186 222   2 122]
 [ 72 150 186 222   2  38 138]
 [ 62 127 154 181 208 235 110]
 [ 41  82  98 114 130 146  67]
 [ 20  39  46  53  60  67  30]]
```

`mode='valid'`

`out = scipy.signal.correlate2d(in1, in2, mode='valid', boundary='fill', fillvalue=0)`

In [4]:
```
g_corr_a = signal.correlate2d(fa, ha, mode='valid', boundary='fill', fillvalue=0)

N_y_g, N_x_g = np.shape(g_corr_a)
print(f"shape(g_corr_a): {np.shape(g_corr_a)}; N_x_g: {N_x_g} N_y_g: {N_y_g}\n")
print(f"g_corr_a:\n{g_corr_a}")
```

```
shape(g_corr_a): (2, 5); N_x_g: 5 N_y_g: 2

g_corr_a:
[[114 150 186 222   2]
 [150 186 222   2  38]]
```

`mode='same'`

`out = scipy.signal.correlate2d(in1, in2, mode='same', boundary='fill', fillvalue=0)`

In [5]:
```
g_corr_a = signal.correlate2d(fa, ha, mode='same', boundary='fill', fillvalue=0)

N_y_g, N_x_g = np.shape(g_corr_a)
print(f"shape(g_corr_a): {np.shape(g_corr_a)}; N_x_g: {N_x_g} N_y_g: {N_y_g}\n")
print(f"g_corr_a:\n{g_corr_a}")
```

```
shape(g_corr_a): (5, 6); N_x_g: 6 N_y_g: 5

g_corr_a:
[[ 74 103 132 161 190  91]
 [114 150 186 222   2 122]
 [150 186 222   2  38 138]
 [127 154 181 208 235 110]
 [ 82  98 114 130 146  67]]
```

Convolution / Equation

The following equation shall be used for the explicit definition of convolution:

$$g_{conv}(x,y) = \sum_{u=0}^{N_u-1}\sum_{v=0}^{N_v-1} f(x-u, y-v)\cdot h(u,v)$$

As in the case of correlation the function $f(x,y)$ shall be defined for range of values x and y :

$$0 \le x \le N_x - 1$$
$$0 \le y \le N_y - 1$$

Outside this range the product $f(x-u, y-v)\cdot h(u,v)$ in the double sum of the equation for convolution is not defined. Hence the product contributes to the convolution only if:

$$0 \le x-u \le N_x - 1$$
$$0 \le y-v \le N_y - 1$$

Finally we observe that the convolution function $g_{conv}(x, y)$ is defined for x and y in the range:

$$0 \le x \le N_x - 1 + N_u - 1$$
$$0 \le y \le N_y - 1 + N_v - 1$$

---

convolution vs. correlation

Next it will be shown under which condition convolution can be evaluated by correlation.

We will first introduce variables u' and v' and require that the following relationships hold:

$$u' + u = N_u - 1 \rightarrow u = N_u - 1 - u'$$
$$v' + v = N_v - 1 \rightarrow v = N_v - 1 - v'$$

With this change of variables the convolution equation is expressed as:

$$g_{conv}(x, y) = \sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x - N_u + 1 + u', y - N_v + 1 + v') \cdot h(N_u - 1 - u', N_v - 1 - v')$$

Defining function $h_{flip-ud-lr}(u', v')$ by:

$$h_{flip-ud-lr}(u', v') = h(N_u - 1 - u', N_v - 1 - v')$$

convolution may be expressed by:

$$g_{conv}(x, y) = \sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x - N_u + 1 + u', y - N_v + 1 + v') \cdot h_{flip-ud-lr}(u', v')$$

The last equation is a 2D-correlation. It shows that with a flipped function $h_{flip-ud-lr}(u', v')$ convolution evaluates as correlation. Defining variables x' and y'

$$x' = x - N_u + 1$$
$$y' = y - N_v + 1$$

the last equation can be re-expressed like this:

$$g_{conv}(x', y') = \underbrace{\sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x' + u', y' + v') \cdot h_{flip-ud-lr}(u', v')}_{correlation}$$

---

correlation vs. convolution

Correlation has been defined before by:

$$g_{corr}(x,y) = \sum_{u=0}^{N_u-1} \sum_{v=0}^{N_v-1} f(x+u, y+v) \cdot h(u,v)$$

Again we will first introduce variables u' and v' and require that the following relationships hold:

$$u' + u = N_u - 1 \rightarrow u = N_u - 1 - u'$$
$$v' + v = N_v - 1 \rightarrow v = N_v - 1 - v'$$

With this change of variables the correlation equation is expressed as:

$$g_{corr}(x,y) = \sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x + N_u - 1 - u', y + N_v - 1 - v') \cdot h(N_u - 1 - u', N_v - 1 - v')$$

Defining function $h_{flip-ud-lr}(u', v')$ by:

$$h_{flip-ud-lr}(u', v') = h(N_u - 1 - u', N_v - 1 - v')$$

correlation may be expressed by:

$$g_{corr}(x,y) = \sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x + N_u - 1 - u', y + N_v - 1 - v') \cdot h_{flip-ud-lr}(u', v')$$

Defining variables x' and y'

$$x' = x - N_u + 1$$
$$y' = y - N_v + 1$$

the last equation can be re-expressed like this:

$$g_{corr}(x', y') = \underbrace{\sum_{u'=0}^{N_u-1} \sum_{v'=0}^{N_v-1} f(x' - u', y' - v') \cdot h_{flip-ud-lr}(u', v')}_{convolution}$$

conclusion

This equation defines a 2D convolution. Therefore 2D-correlation can be expressed as 2D-convolution with a modified function $h_{flip-ud-lr}(u', v')$ which is obtained from $h(u,v)$ by flipping rows (left-right) and columns (up-down).

Experiments with correlation and convolution

defining data for experiments

2D array `fa` is used as function $f(x,y)$

2D array `ha` is used as function $h(u, v)$

2D array `ha_ud_lr` obtained from `ha` by flipping rows and columns. It corresponds to $h_{flip-ud-lr}(u', v')$

In [6]:
```
fa = np.array([[1,2,3,4,5,6],
                 [2,3,4,5,6,7],
                 [3,4,5,6,7,8],
                 [4,5,6,7,8,9],
                 [5,6,7,8,9,10]], dtype=np.float64)

ha = np.array([[3,4,5],
               [4,5,6],
               [5,6,7],
               [6,7,8]], dtype=np.float64)

# flip array ha up-down and left-right
ha_ud_lr = np.flip(ha)

print(f"fa:\n{fa}\n")
print(f"ha:\n{ha}\n")
print(f"ha_ud_lr:\n{ha_ud_lr}")
```

```
fa:
[[ 1.  2.  3.  4.  5.  6.]
 [ 2.  3.  4.  5.  6.  7.]
 [ 3.  4.  5.  6.  7.  8.]
 [ 4.  5.  6.  7.  8.  9.]
 [ 5.  6.  7.  8.  9. 10.]]

ha:
[[3. 4. 5.]
 [4. 5. 6.]
 [5. 6. 7.]
 [6. 7. 8.]]

ha_ud_lr:
[[8. 7. 6.]
 [7. 6. 5.]
 [6. 5. 4.]
 [5. 4. 3.]]
```

Demonstration#1

Convolving `fa` with `ha` (result: `g1`) is shown to be identical to correlation `fa` with `ha_ud_lr` (result: `g2`).

In [7]:
```
# showing that convolution and correlation yield same result if correlation uses fl
g1= signal.convolve2d(fa, ha, mode='full', boundary='fill', fillvalue=0)

# using the flipped 2D array
g2= signal.correlate2d(fa, ha_ud_lr, mode='full', boundary='fill', fillvalue=0)
```

```
print(f"g1 (convolution):\n{g1}\n")
print(f"g2 (correlation):\n{g2}")
```

```
g1 (convolution):
[[  3.  10.  22.  34.  46.  58.  49.  30.]
 [ 10.  30.  62.  89. 116. 143. 118.  71.]
 [ 22.  62. 123. 168. 213. 258. 209. 124.]
 [ 40. 108. 208. 274. 340. 406. 324. 190.]
 [ 58. 148. 274. 340. 406. 472. 372. 216.]
 [ 58. 143. 258. 312. 366. 420. 326. 187.]
 [ 49. 118. 209. 248. 287. 326. 250. 142.]
 [ 30.  71. 124. 145. 166. 187. 142.  80.]]

g2 (correlation):
[[  3.  10.  22.  34.  46.  58.  49.  30.]
 [ 10.  30.  62.  89. 116. 143. 118.  71.]
 [ 22.  62. 123. 168. 213. 258. 209. 124.]
 [ 40. 108. 208. 274. 340. 406. 324. 190.]
 [ 58. 148. 274. 340. 406. 472. 372. 216.]
 [ 58. 143. 258. 312. 366. 420. 326. 187.]
 [ 49. 118. 209. 248. 287. 326. 250. 142.]
 [ 30.  71. 124. 145. 166. 187. 142.  80.]]
```

Demonstration#2

Correlating `fa` with `ha` (result: `g3`) is shown to be identical to convolution of `fa` with `ha_ud_lr` (result: `g4`).

In [8]:
```
# showing that correlation and convolution yield same result if convolution uses fl
g3= signal.correlate2d(fa, ha, mode='full', boundary='fill', fillvalue=0)

# using the flipped 2D array
g4= signal.convolve2d(fa, ha_ud_lr, mode='full', boundary='fill', fillvalue=0)

print(f"g3 (correlation):\n{g3}\n")
print(f"g4 (convolution):\n{g4}")
```

```
g3 (correlation):
[[  8.  23.  44.  65.  86. 107.  72.  36.]
 [ 23.  58. 103. 142. 181. 220. 146.  72.]
 [ 44. 103. 174. 228. 282. 336. 220. 107.]
 [ 70. 156. 254. 320. 386. 452. 292. 140.]
 [ 96. 204. 320. 386. 452. 518. 332. 158.]
 [ 74. 154. 237. 282. 327. 372. 235. 110.]
 [ 50. 102. 154. 181. 208. 235. 146.  67.]
 [ 25.  50.  74.  86.  98. 110.  67.  30.]]

g4 (convolution):
[[  8.  23.  44.  65.  86. 107.  72.  36.]
 [ 23.  58. 103. 142. 181. 220. 146.  72.]
 [ 44. 103. 174. 228. 282. 336. 220. 107.]
 [ 70. 156. 254. 320. 386. 452. 292. 140.]
 [ 96. 204. 320. 386. 452. 518. 332. 158.]
 [ 74. 154. 237. 282. 327. 372. 235. 110.]
 [ 50. 102. 154. 181. 208. 235. 146.  67.]
 [ 25.  50.  74.  86.  98. 110.  67.  30.]]
```

Correlation / Convolution with `OpenCV`

Correlation and convolution in `OpenCV` is implemented by function `filter2D`

Contrary to `Scipy` functions `correlate2d` / `convolve2d` this function does **not** return the full correlation / convolution result.

The formula how function `filter2D` computes correlation is taken from the `OpenCV` reference:

$$g_{corr}(x,y) = \sum_{u=0}^{N_u-1}\sum_{v=0}^{N_v-1} f(x+u-m, y+v-k)\cdot h(u,v)$$

The correlation $g_{corr}(x,y)$ is only computed for values x and y for which the function $f(x,y)$ is defined; namely:

$$0 \le x \le N_x - 1$$
$$0 \le y \le N_y - 1$$

As a consequence of this definition matrices representing $f(x,y)$ and $g_{corr}(x,y)$ have identical dimensions.

Values m and k are *offset values* to set a *center* value of the kernel $h(u,v)$. In the reference document the value pair m, k is denoted as *anchor*. Moreover m and k should lie *within* the kernel. Thus m, k should be in the range:

$$0 \le m \le N_u - 1$$
$$0 \le l \le N_v - 1$$

Choosing $m = -1, k = -1$ has a **special** meaning. The center of the kernel is chosen for this setting. While it is intuitively obvious what the *center* of the kernel $h(u, v)$ is if N_u and N_v are both odd numbers, the reference manual does not provide information what is regarded as *center* if one or both kernel dimension N_u, N_v is an even number.

Having done some experiments the center of the kernel is internally computed by:

$$m = floor\left(\frac{N_u}{2}\right)$$
$$k = floor\left(\frac{N_v}{2}\right)$$

example

$$N_u = 3$$
$$N_v = 4$$
$$\rightarrow m = 1, k = 2$$

Function `filter2D` is called like this in `Python`:

```
filter2D(src, ddepth, kernel[, dst[, anchor[, delta[, borderType]]]]    )
->      dst
```

`src` corresponds to $f(x, y)$

`ddepth` is chosen from a set of predefined values; it controls the numeric format of correlation result `dst`

`kernel` corresponds to $h(u, v)$

`dst` is the computed correlation $g_{corr}(x, y)$

`anchor`, `delta` and `borderType` are optional parameters. Frequently used are parameters are:

`anchor` which sets the center of the kernel $h(u, v)$

`borderType` offers some control over how **fill** values for function $f(x + u - m, y + v - k)$ are selected if $x + u - m$ and/or $y + v - k$ are **outside** their defined range:

$$0 \leq x + u - m \leq N_x - 1$$
$$0 \leq y + v - k \leq N_y - 1$$

In the examples (see below) we have chosen a centered anchor `anchor=(-1,-1)` and `borderType=cv2.BORDER_ISOLATED` (equivalent to `boundary='fill', fillvalue=0` in `Scipy` method `correlate2d / convolve2d`)

Optional parameter `delta` (default: 0) adds a fixed offset to the correlation result `dst`.

Some examples shall demonstrate how this function could be used.

In [10]:
```
# computing correlation
g3_filter2D = cv2.filter2D(fa, -1, ha, anchor=(-1,-1), borderType=cv2.BORDER_ISOLAT

print(f"g3_filter2D (correlation):\n{g3_filter2D}\n")
```

```
g3_filter2D (correlation):
[[ 58. 103. 142. 181. 220. 146.]
 [103. 174. 228. 282. 336. 220.]
 [156. 254. 320. 386. 452. 292.]
 [204. 320. 386. 452. 518. 332.]
 [154. 237. 282. 327. 372. 235.]]
```

In [17]:
```
N_v, N_u = ha.shape

# calculating the center (m,k) of the kernel explicitly
# it will be used as anchor=(m,k) in function filter2D as a replacement of anchor=(

m = N_u//2
k = N_v//2

print("dimensions of the kernel")
print(f"N_u: {N_u}")
print(f"N_v: {N_v}")
print(f"kernel center (m,k): ({m}, {k})")
```

```
dimensions of the kernel
N_u: 3
N_v: 4
kernel center (m,k): (1, 2)
```

In [16]:
```
# computing correlation (explicitly setting center of kernel)
g3_filter2D = cv2.filter2D(fa, -1, ha, anchor=(m,k), borderType=cv2.BORDER_ISOLATED

print(f"g3_filter2D (correlation):\n{g3_filter2D}\n")
```

```
g3_filter2D (correlation):
[[ 58. 103. 142. 181. 220. 146.]
 [103. 174. 228. 282. 336. 220.]
 [156. 254. 320. 386. 452. 292.]
 [204. 320. 386. 452. 518. 332.]
 [154. 237. 282. 327. 372. 235.]]
```

In [ ]:
```
```