In [1]:
```
%matplotlib inline
import numpy as np
from matplotlib import pyplot as plt
```

# Properties of 1D Fourier Transform

definition of Fourier transform an its inverser transformation

Fourier transform of periodic function

Relationship with discrete Fourier transformation

Some applications

Interpolation

Wavelets

# Definitions

Source:

**Signal Analysis**

Anasthasios Papoulis; McGRAW-Hill (international student edition)

The Fourier transformation of function $f(t)$ is defined by equation:

$$F(f) = \int_{-\infty}^{\infty} f(t) \cdot \exp[-j \cdot 2\pi \cdot f \cdot t] \cdot dt$$

In a signal processing context variable $t$ is often referred to as **time** and $f$ is then referred to as **frequency**. And even if the meaning of variable $t$ and $f$ changes with the application context we still use the notion *time* and *frequency* throughout.

If the Fourier transform $F(f)$ exists there is an *inverse* Fourier transform defined by equation:

$$f(t) = \int_{-\infty}^{\infty} F(f) \cdot \exp[j \cdot 2\pi \cdot f \cdot t] \cdot df$$

**Proof**

$$f(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) \cdot \exp\left[j \cdot 2\pi \cdot f \cdot \left(t - t'\right)\right] \cdot df \cdot dt'$$

$$f(t) = \int_{-\infty}^{\infty} f(t) \underbrace{\int_{-\infty}^{\infty} \exp\left[j \cdot 2\pi \cdot f \cdot \left(t - t'\right)\right] \cdot df}_{\delta\left(t-t'\right)} \cdot dt'$$

Observing that in this equation the second integral is just the *delta* function:

$$\delta\left(t - t'\right) = \int_{-\infty}^{\infty} \exp\left[j \cdot 2\pi \cdot f \cdot \left(t - t'\right)\right] \cdot df$$

We obtain:

$$f(t) = \int_{-\infty}^{\infty} f(t) \cdot \delta\left(t - t'\right) \cdot dt' := f(t)$$

---

# Excursion / Delta function

In the definition of the delta function only the real part of $\exp\left[j \cdot 2\pi \cdot f \cdot \left(t - t'\right)\right]$ contributes to the integral. Hence the delta function is real function.

$$\delta\left(t - t'\right) = \int_{-\infty}^{\infty} \cos\left[2\pi \cdot f \cdot \left(t - t'\right)\right] \cdot df$$

Another approach defines the delta function as the limit of an integral like this:

$$\delta\left(t - t'\right) = \lim_{a \to \infty} \int_{-a}^{a} \exp\left[j \cdot 2\pi \cdot f \cdot \left(t - t'\right)\right] \cdot df$$

$$\delta\left(t - t'\right) = \lim_{a \to \infty} \frac{\sin\left(2\pi \cdot a \cdot \left(t - t'\right)\right)}{\pi \cdot \left(t - t'\right)}$$

For finite value of $a$ and the limiting case $\left(t - t'\right) \to 0$ we obtain:

$$\lim_{\left(t-t'\right) \to 0} \frac{\sin\left(2\pi \cdot a \cdot \left(t - t'\right)\right)}{\pi \cdot \left(t - t'\right)} = 2 \cdot a$$

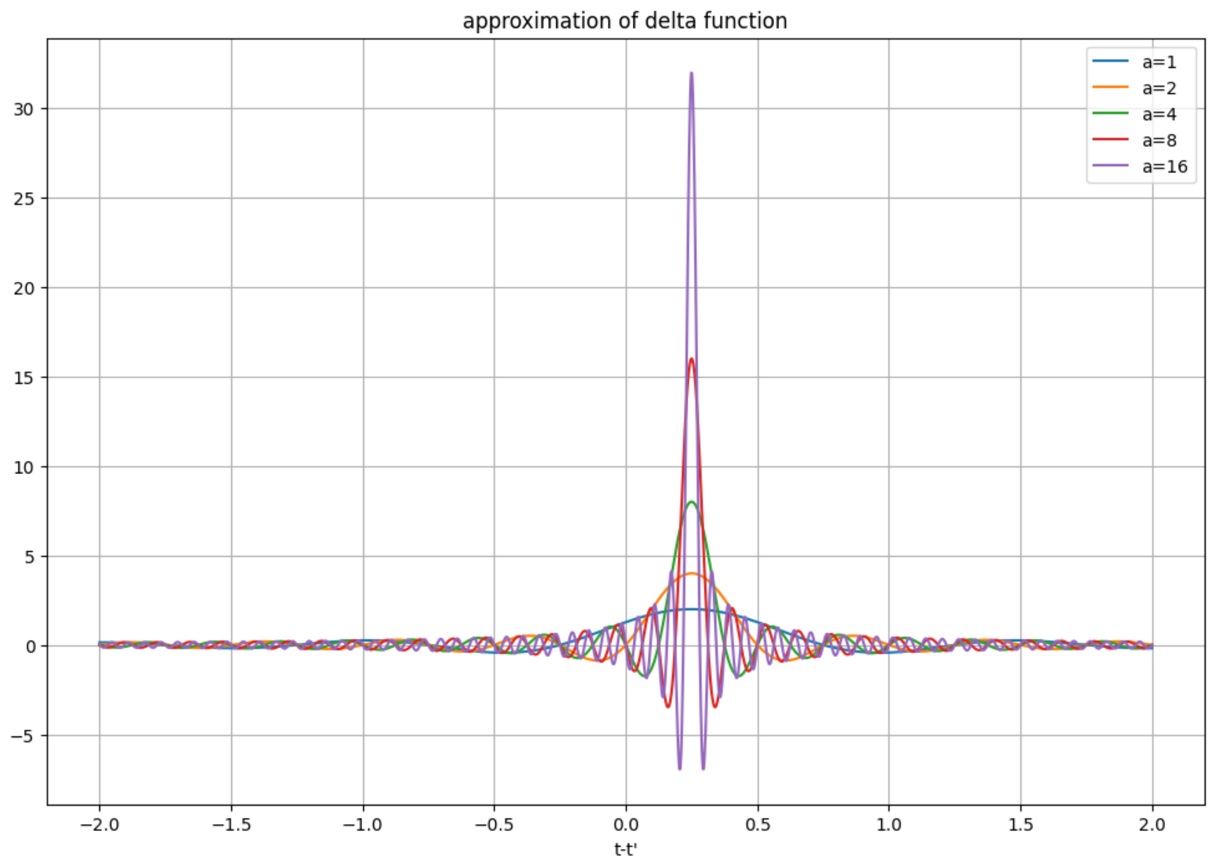
Below it is shown how function $\frac{\sin\left(2\pi \cdot a \cdot \left(t-t'\right)\right)}{\pi \cdot \left(t-t'\right)}$ becomes more and more localised around $t = t'$ as factor $a$ increases.

In [2]:
```
def sinDelta(a, t, ts):
    z0 = 1e-6
```

```
    args = np.pi * (t - ts)
    args[np.abs(args) < z0] = z0
    return np.sin(2*a*args)/args

t = np.linspace(-2,2, 2000)
delta_a_1 = sinDelta(1, t, 0.25)
delta_a_2 = sinDelta(2, t, 0.25)
delta_a_4 = sinDelta(4, t, 0.25)
delta_a_8 = sinDelta(8, t, 0.25)
delta_a_16 = sinDelta(16, t, 0.25)

# graphics
fig1 = plt.figure(1, figsize=[12, 8])
ax_f1 = fig1.add_subplot(1, 1, 1)
ax_f1.plot(t, delta_a_1, label="a=1")
ax_f1.plot(t, delta_a_2, label="a=2")
ax_f1.plot(t, delta_a_4, label="a=4")
ax_f1.plot(t, delta_a_8, label="a=8")
ax_f1.plot(t, delta_a_16, label="a=16")
ax_f1.legend()
ax_f1.grid(True)
ax_f1.set_xlabel('t-t\'')
ax_f1.set_title('approximation of delta function');
```




# Convolution

Definition

$$g(t) = \int_{-\infty}^{\infty} f(v) \cdot h(t - v) \cdot dv$$

What is the Fourier transform $G(f)$ of the convolution function $g(t)$ ?

$$G(f) = \int_{-\infty}^{\infty} g(t) \cdot \exp[-j \cdot 2\pi \cdot f \cdot t] \cdot dt$$

$$G(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v) \cdot h(t - v) \cdot \exp[-j \cdot 2\pi \cdot f \cdot t] \cdot dt \cdot dv$$

Applying variable substitution

$t^{'} = t - v$ and therefore $t = t^{'} + v$

yields

$$G(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v) \cdot h(t^{'}) \cdot \exp\left[-j \cdot 2\pi \cdot f \cdot t^{'}\right] \cdot \exp[-j \cdot 2\pi \cdot f \cdot v] \cdot dt^{'} \cdot dv$$

$$G(f) = \underbrace{\int_{-\infty}^{\infty} f(v) \cdot \exp[-j \cdot 2\pi \cdot f \cdot v] \cdot dv}_{F(f)} \cdot \underbrace{\int_{-\infty}^{\infty} h(t^{'}) \cdot \exp\left[-j \cdot 2\pi \cdot f \cdot t^{'}\right] \cdot dt^{'}}_{H(f)}$$

$$G(f) = F(f) \cdot H(f)$$

# Periodic repetitions

Some useful properties of periodic functions are discussed here. We start with a function which is periodic in the time domain variable $t$.

Then we repeat this process with a frequency periodic function.

## Time Domain

A function $g(t)$ that is repeated with period $T_p$ may be expressed by an infinite summation.

$$g_p(t) = \sum_{n=-\infty}^{\infty} g\left(t + n \cdot T_p\right)$$

Since $g_p(t)$ is periodic it may be expressed by a Fourier series:

$$g_p(t) = \sum_{n=-\infty}^{\infty} C_n \cdot \exp\left(j \cdot 2\pi \cdot n \cdot \frac{t}{T_p}\right)$$

Denoting the fundamental frequency $f_s$ by $f_s = \frac{1}{T_p}$ a new equation for the Fourier series results:

$$g_p(t) = \sum_{n=-\infty}^{\infty} C_n \cdot \exp\left(j \cdot 2\pi \cdot n \cdot f_s \cdot t\right)$$

The coefficients $C_n$ of the Fourier series are computed from:

$$\int_0^{T_p} g_p(t) \cdot \exp\left(-j \cdot 2\pi \cdot m \cdot f_s \cdot t\right) \cdot dt = \sum_{n=-\infty}^{\infty} C_n \cdot \int_0^{T_p} \exp\left(j \cdot 2\pi \cdot (n-m) \cdot f_s \cdot t\right) \cdot dt$$

The integral on the right hand side of the equation is different from $0$ only if $m = n$.

$$\int_0^{T_p} \exp\left(j \cdot 2\pi \cdot (n-m) \cdot f_s \cdot t\right) \cdot dt = \begin{cases} 0 & m \neq n \\ T_p & m == n \end{cases}$$

Therefore Fourier coefficients $C_m$ are computed from:

$$C_m = \frac{1}{T_p} \int_0^{T_p} g_p(t) \cdot \exp\left(-j \cdot 2\pi \cdot m \cdot f_s \cdot t\right) \cdot dt$$

---

# Frequency Domain

Repeating Fourier transform $G(f)$ with period $f_p$ gives a periodic function $U_p(f)$:

$$U_p(f) = \sum_{n=-\infty}^{\infty} G\left(f + n \cdot f_p\right)$$

Since $U_p(f)$ is periodic it may be expressed by a Fourier series:

$$U_p(f) = \sum_{n=-\infty}^{\infty} D_n \cdot \exp\left(-j \cdot 2\pi \cdot n \cdot \frac{f}{f_p}\right)$$

Denoting the fundamental time increment $t_s$ by $t_s = \frac{1}{f_p}$ a new equation for the Fourier series results:

$$U_p(f) = \sum_{n=-\infty}^{\infty} D_n \cdot \exp\left(-j \cdot 2\pi \cdot n \cdot t_s \cdot f\right)$$

The coefficients $U_n$ of the Fourier series are computed from:

$$\int_0^{f_p} U_p(f) \cdot \exp\left(j \cdot 2\pi \cdot m \cdot t_s \cdot f\right) \cdot df = \sum_{n=-\infty}^{\infty} D_n \cdot \int_0^{f_p} \exp\left(j \cdot 2\pi \cdot (m-n) \cdot t_s \cdot f\right) \cdot df$$

The integral on the right hand side of the equation is different from $0$ only if $m = n$.

$$\int_0^{f_p} \exp\left(j \cdot 2\pi \cdot (m-n) \cdot t_s \cdot f\right) \cdot df = \begin{cases} 0 & m \neq n \\ f_p & m == n \end{cases}$$

Therefore Fourier coefficients $D_m$ are computed from:

$$D_m = \frac{1}{f_p} \int_0^{f_p} U_p(f) \cdot \exp\left(j \cdot 2\pi \cdot m \cdot t_s \cdot f\right) \cdot df$$

# Poisson sum formula

## Time Domain

Inserting the definition of periodic function $g_p(t)$ into the equation of Fourier coefficients $C_m$ gives a more useful expression.

$$C_m = \frac{1}{T_p} \sum_{n=-\infty}^{\infty} \int_0^{T_p} g\left(t + n \cdot T_p\right) \cdot \exp\left(-j \cdot 2\pi \cdot m \cdot f_s \cdot t\right) \cdot dt$$

Doing variable substitution $t' = t + n \cdot T_p$:

$$C_m = \frac{1}{T_p} \sum_{n=-\infty}^{\infty} \int_{n \cdot T_p}^{(n+1) \cdot T_p} g\left(t'\right) \cdot \exp\left(-j \cdot 2\pi \cdot m \cdot f_s \cdot \left(t' - n \cdot T_P\right)\right) \cdot dt' C_m = \frac{1}{T_p} \sum_{n=-\infty}^{\infty} \int_{n \cdot T_p}^{(n+1) \cdot T_p} g\left(t'\right.$$

The combination of the *infinite* sum and the *finite* integral can be compactly expressed by an integral with infinite limits. The integral is just the value of the Fourier transform $G(f)$ at a the specific frequency $m \cdot f_s$:

$$C_m = \frac{1}{T_p} \underbrace{\int_{-\infty}^{\infty} g\left(t'\right) \cdot \exp\left(-j \cdot 2\pi \cdot m \cdot f_s \cdot t'\right) \cdot dt'}_{G(m \cdot f_s)} C_m = \frac{1}{T_p} \cdot G(m \cdot f_s)$$

Finally the periodic function can be written as infinite series

$$g_p(t) = \frac{1}{T_p} \cdot \sum_{n=-\infty}^{\infty} G(n \cdot f_s) \cdot \exp\left(j \cdot 2\pi \cdot n \cdot f_s \cdot t\right)$$

This equation is often referred to as **Poisson Sum Formula**. It follows that the sample values $G(n \cdot f_s)$ of the Fourier transform $G(f)$ equal the Fourier series coefficients of a periodic function $T_p \cdot g_p(t)$.

## Application

Applying a time shift $\tau$ results in a slightly modified set of Fourier series coefficients:

$$g_p(t-\tau) = \frac{1}{T_p} \cdot \sum_{n=-\infty}^{\infty} G(n \cdot f_s) \cdot \exp\left(-j \cdot 2\pi \cdot n \cdot f_s \cdot \tau\right) \cdot \exp\left(j \cdot 2\pi \cdot n \cdot f_s \cdot t\right)$$

# Frequency domain

Inserting the definition of periodic function $U_p(f)$ into the equation of Fourier coefficients $D_m$ gives a more useful expression.

$$D_m = \frac{1}{f_p} \sum_{n=-\infty}^{\infty} \int_0^{f_p} G\left(f + n \cdot f_p\right) \cdot \exp\left(j \cdot 2\pi \cdot m \cdot t_s \cdot f\right) \cdot df$$

Doing variable substitution $f^{'} = f + n \cdot f_p$:

$$D_m = \frac{1}{f_p} \sum_{n=-\infty}^{\infty} \int_{n \cdot f_p}^{(n+1) \cdot f_p} G\left(f^{'}\right) \cdot \exp\left(j \cdot 2\pi \cdot m \cdot t_s \cdot f^{'}\right) \cdot df^{'}$$

The combination of the *infinite* sum and the *finite* integral can be compactly expressed by an integral with infinite limits. The integral is just the value of the inverse Fourier transform $g(f)$ at a the specific time instant $m \cdot t_s$:

$$D_m = \frac{1}{f_p} \underbrace{\int_{-\infty}^{\infty} G\left(f^{'}\right) \cdot \exp\left(j \cdot 2\pi \cdot m \cdot t_s \cdot f^{'}\right) \cdot df^{'}}_{g\left(m \cdot t_s\right)}$$

Finally the periodic function $U_p(f)$ can be written as infinite series

$$U_p(f) = \frac{1}{f_p} \sum_{n=-\infty}^{\infty} g\left(n \cdot t_s\right) \cdot \exp\left(-j \cdot 2\pi \cdot n \cdot t_s \cdot f\right)$$

# From Fourier series to Discrete Fourier series

## Time Domain

The periodic function $g_p(t)$ is sampled at equidistant time instants $t = m \cdot t_s$. The sampling time $t_s$ is defined as an integer fraction of the period $T_p$. $t_s = \frac{T_p}{N}$. ($N$ samples per period $T_p$)

$$g_p\left(m \cdot t_s\right) = \frac{1}{T_p} \cdot \sum_{n=-\infty}^{\infty} G(n \cdot f_s) \cdot \exp\left(j \cdot 2\pi \cdot n \cdot m \cdot f_s \cdot t_s\right) g_p\left(m \cdot t_s\right) = \frac{1}{T_p} \cdot \sum_{n=-\infty}^{\infty} G(n \cdot f_s) \cdot \exp\left(j \cdot 2\right.$$

---

Using indices

$$n = k + r \cdot N k = 0, \ldots, N-1 r = \ldots, -1, 0, 1, \ldots$$

samples $g_p\left(m \cdot t_s\right)$ are given by:

$$g_p\left(m \cdot t_s\right) = \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} \sum_{r=-\infty}^{\infty} G\left((k + r \cdot N) \cdot f_s\right) \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot (k + r \cdot N) \cdot m\right) g_p\left(m \cdot t_s\right) = \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} \sum_{r=-\infty}^{\infty}$$

Here $G_a\left(k \cdot f_s\right)$ denote the *aliased* coefficients defined by equation:

$$G_a\left(k \cdot f_s\right) = \sum_{r=-\infty}^{\infty} G\left((k + r \cdot N) \cdot f_s\right)$$

Apparently the aliased coefficients $G_a$ are periodic with period $N \cdot f_s$:

$$G_a\left(k \cdot f_s\right) = G_a\left(k \cdot f_s + m \cdot N \cdot f_s\right)$$

$$g_p\left(m \cdot t_s\right) = \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} G_a\left(k \cdot f_s\right) \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot k \cdot m\right)$$

---

## How to get $G_a\left(k \cdot f_s\right)$ from $g_a\left(k \cdot t_s\right)$

### Recipe

$$\sum_{m=0}^{N-1} g_p\left(m \cdot t_s\right) \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot n \cdot m\right) = \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} G_a\left(k \cdot f_s\right) \underbrace{\sum_{m=0}^{N-1} \exp\left(j \cdot \frac{2\pi}{N} \cdot (k-n) \cdot m\right)}_{N \text{ if k=n };0 \text{ otherwise}} G_a\left(k \cdot f_s\right) =$$

## Frequency Domain

The periodic function $U_p(f)$ is sampled at equidistant frequencies $f = m \cdot f_s$. Specifically we choose $f_s = \frac{1}{T_p}$. With this choice we have $t_s \cdot f_s = \frac{1}{N}$.

$$U_p(m \cdot f_s) = \frac{1}{f_p} \sum_{n=-\infty}^{\infty} g\left(n \cdot t_s\right) \cdot \exp\left(-j \cdot 2\pi \cdot n \cdot t_s \cdot m \cdot f_s\right) U_p(m \cdot f_s) = \frac{1}{f_p} \sum_{n=-\infty}^{\infty} g\left(n \cdot t_s\right) \cdot \exp\left(-j \cdot \frac{2}{N}\right.$$

---

Using indices

$$n = k + r \cdot N k = 0, \ldots, N-1 r = \ldots, -1, 0, 1, \ldots$$

samples $U_p(m \cdot f_s)$ are given by:

$$U_p(m \cdot f_s) = \frac{1}{f_p} \sum_{k=0}^{N-1} \underbrace{\sum_{r=-\infty}^{\infty} g\left(k \cdot t_s + r \cdot N \cdot t_s\right)}_{g_a\left(k \cdot t_s\right)} \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot k \cdot m\right)$$

Here $g_a\left(k \cdot t_s\right)$ denote the *aliased* coefficients defined by equation:

$$g_a\left(k \cdot t_s\right) = \sum_{r=-\infty}^{\infty} g\left(k \cdot t_s + r \cdot \underbrace{N \cdot t_s}_{T_p}\right) = \sum_{r=-\infty}^{\infty} g\left(k \cdot t_s + r \cdot T_p\right)$$

The aliased coefficients are periodic :

$$g_a\left(k \cdot t_s\right) = g_a\left(k \cdot t_s + M \cdot T_p\right)$$

Only if the function $g(t)$ is time limited for an interval of duration $\leq T_p$ the samples of $g_a\left(k \cdot t_s\right)$ can be used to *recover* the original samples $g\left(k \cdot t_s\right)$.

$$U_p(m \cdot f_s) = \frac{1}{f_p} \sum_{k=0}^{N-1} g_a\left(k \cdot t_s\right) \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot k \cdot m\right)$$

---

## How to get $g_a\left(k \cdot t_s\right)$ from $U_p(m \cdot f_s)$

**Recipe**

$$\sum_{m=0}^{N-1} U_p(m \cdot f_s) \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot n \cdot m\right) = \frac{1}{f_p} \sum_{k=0}^{N-1} g_a\left(k \cdot t_s\right) \cdot \underbrace{\sum_{m=0}^{N-1} \exp\left(j \cdot \frac{2\pi}{N} \cdot (n-k) \cdot m\right)}_{N \text{ if } k=n;\ \text{otherwise } 0} g_a\left(n \cdot t_s\right) = \frac{f_p}{N} \cdot \sum_{m}^{N}$$

**Summary:**

1. Samples of $U_p(m \cdot f_s)$ are related to samples $g_a\left(k \cdot t_s\right)$ via a discrete Fourier series.
2. If $g(t)$ is even time limited, samples $U_p(m \cdot f_s)$ are even directly related to the samples $g(\left(k \cdot t_s\right))$.
3. ccc

# 1D Discrete Fourier transform (DFT)

The 1D discrete Fourier transform defines a method to transform a discrete Fourier series into another discrete Fourier series.

Let us defines a series of $N$ coefficients $A_m$ with $m := [0, \ldots, N-1]$ and another series of $N$ coefficients $a_k$ with $k := [0, \ldots, N-1]$.

Let coefficients $a_k$ be defined by finite sum (*trigonometric polynomial*) like this:

$$a_k = \frac{1}{N} \cdot \sum_{m=0}^{N-1} A_m \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot m \cdot k\right)$$

A similar approach allows to compute coeffients $A_m$ from coefficients $a_k$:

$$\sum_{k=0}^{N-1} a_k \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot n \cdot k\right) = \frac{1}{N} \cdot \sum_{m=0}^{N-1} A_m \cdot \underbrace{\sum_{k=0}^{N-1} \exp\left(j \cdot \frac{2\pi}{N} \cdot (m-n) \cdot k\right)}_{N:\ \text{if n=m}\ ;\ 0:\ \text{otherwise}} A_n = \sum_{k=0}^{N-1} a_k \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot\right.$$

---

# 1D DFT in `Scipy / Numpy`

see: https://docs.scipy.org/doc/scipy/tutorial/fft.html

Function `fft` implements the DFT:

$$A_n = \sum_{k=0}^{N-1} a_k \cdot \exp\left(-j \cdot \frac{2\pi}{N} \cdot n \cdot k\right)$$

and `ifft` implements the inverse DFT (IDFT):

$$a_k = \frac{1}{N} \cdot \sum_{m=0}^{N-1} A_m \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot m \cdot k\right)$$

**Note**

By comparision we can see that the following expressions are *transform pairs*:

$$\underbrace{\left(f_p \cdot U_p(n \cdot f_s)\right)}_{A_n} = \sum_{k=0}^{N-1} \underbrace{g_a\left(k \cdot t_s\right)}_{a_k} \cdot \exp\left( -j \cdot \frac{2\pi}{N} \cdot k \cdot n\right)$$

and

$$\underbrace{g_a\left(k \cdot t_s\right)}_{a_k} = \frac{1}{N} \cdot \sum_{m=0}^{N-1} \underbrace{\left(f_p \cdot U_p(m \cdot f_s)\right)}_{A_m} \cdot \exp\left(j \cdot \frac{2\pi}{N} \cdot k \cdot m\right)$$

In [ ]:

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In [ ]:

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