RESEARCH STATEMENT

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My work in pure mathematics is in algebraic geometry broadly construed, with a focus on the foundations and structural properties of moduli problems and their associated invariants. Moduli problems arise naturally in many fields of mathematics: A fundamental approach of investigating the properties of a mathematical, e.g. geometric, object of certain type is to consider the parameter space of all possibilities, called a moduli space. Its points correspond to isomorphism classes of objects and its structure tells us how they can vary in families. An example of a big success story is the moduli space of Riemann surfaces $\mathcal{M}_g$, which is ubiquitous across mathematics, and whose geometry and cohomological invariants have historically led to key insights and breakthroughs in diverse fields. A significant motivation behind my research stems from the theory of moduli spaces parametrizing sheaves on algebraic varieties, fundamental algebro-geometric objects which are a singular generalization of vector bundles.

From an analytic perspective, moduli of sheaves are a natural holomorphic analogue of compactifications of moduli of vector bundles with connections on orientable manifolds, a subject with long history in geometric analysis, gauge theory, symplectic topology and mathematical physics. Understanding the singularities and possible limits of solution spaces of instanton equations, i.e. systems of partial differential equations (PDE) suited to the kind of connection or bundle of interest, is typically a difficult task and has led to important advances, including Chern-Simons theory in dimension 3 and Donaldson/Seiberg-Witten theory in dimension 4. For example, instantons were used by Donaldson to study smooth structures on four-manifolds and Seiberg-Witten invariants now appear centrally in formulations of Floer homology and mirror symmetry, counts of J-holomorphic curves and theoretical physics.

From an algebraic point of view, moduli of sheaves have been used in enumerative geometry as the basis of Donaldson-Thomas theory, a major approach towards counting holomorphic curves and, more generally, sheaves on complex orientable manifolds (in analogy with the real case), called Calabi-Yau manifolds. By Yau's proof of the Calabi conjecture, these admit a Ricci flat metric in every Kähler class and are thus a natural candidate for compactifications of spacetimes in string theory. Moreover, moduli of sheaves give examples of hyperkähler manifolds and more general holomorphic symplectic objects in derived algebraic geometry and are prime characters in birational geometry, stability conditions and geometric representation theory, e.g. through quiver representations.

My research explores this rich picture in several directions, organized as follows:

- **Moduli theory and enumerative invariants (§1)**: In the common scenario where the objects of interest have positive dimensional automorphism groups, the natural formulation of a moduli problem leads to the notion of stacks, that are not spaces in the sense of an algebraic variety and on which (co)homological tools, like integration, do not work. In [1, 41, 58, 94], I construct a canonical algorithm which eliminates stackiness in classical and derived algebraic geometry, together with a theory of derived moduli spaces. These results are of independent interest in the theory of stacks and moduli spaces and their main enumerative ramification is the first direct, geometric construction of generalized invariants counting sheaves on surfaces and Calabi–Yau (and Fano) threefolds. Subsection 1.5 discusses this program and cosection localization results in dimension 4 [92, 93] using derived differential geometry.
- **Invariants in K-theory (§2):** An important goal in moduli theory is the construction of refined invariants beyond numbers that better reflect the structure of a moduli problem. In [60, 61, 95], I develop a new general framework to define and study invariants in algebraic K-theory, applicable to many settings of interest, including K-theoretic Donaldson-Thomas invariants.
- **The Quot scheme and quivers (§3):** The interplay between the Quot scheme, d-critical loci and quiver representations has been central in understanding moduli spaces of sheaves. In [91], I explicitly investigate the local structure as d-critical loci and related properties of Quot schemes on Calabi-Yau threefolds coming from representations of quivers and derived algebraic symplectic geometry, bridging a gap in the literature.

Throughout, I indicate open questions and future research ideas, as well as some research projects for graduate students and potential projects I can work on with postdocs.

Finally **(§4)**, I also describe an ongoing research project in artificial intelligence, whose aim is to import categorical structures in order to build robust and safe theoretical foundations. In [10], I introduce a general categorical formalism for Markov Decision Processes, a fundamental model in reinforcement learning, where an agent learns to perform a task by interacting with and getting feedback from their environment. Among other things, the formalism incorporates and extends existing regimes, such as group symmetry, and provides a natural setting for compositionality, the combination of learned behaviors on subtasks to perform a composite task.

1. Moduli theory and enumerative invariants

1.1. **Moduli problems in algebraic geometry.** Ever since the groundbreaking paper of Deligne and Mumford [37], it has been understood that, in order to study families of geometric objects, one needs to work with more general objects, called algebraic stacks, which go beyond varieties, and are nowadays at the forefront of algebraic geometry with a vast literature. At the same time, even for classical applications, recent breakthroughs in derived algebraic geometry [100] and shifted symplectic structures [87] have made derived stacks increasingly important. These can be thought of as (co)homological thickenings of classical stacks and materialize the hidden smoothness principle of Kontsevich [64], with singular classical stacks being truncations of better behaved derived stacks.

The presence of possibly positive dimensional automorphism groups of objects, referred to as stabilizers, implies that basic tools, such as integration, cannot be directly applied. It is thus often desirable to "destackify", i.e. reduce the stabilizer dimension to zero, in which case stacks behave similarly to algebraic varieties. This has been successfully carried out for smooth stack quotients $[X/G]$ of group actions obtained from Geometric Invariant Theory (GIT) [79] by Kirwan [63] with a large number of applications, e.g. in homological calculations [62] and moduli theory [52, 53], and recently generalized by Edidin-Rydh [38] to any Artin stack with a good moduli space [2]. However, their constructions are not well-suited to enumerative applications and to being generalized in derived algebraic geometry. To rectify this issue for classical stacks, my collaborators and I have given a new, alternative construction.

Theorem 1.1. (a) [58, Kiem–Li–**S**] Let $\mathcal{M} = [X/G]$ be a GIT quotient stack. There exists a canonical proper, Deligne-Mumford GIT quotient stack $\pi\colon \widetilde{\mathcal{M}} = [\widetilde{X}/G] \to \mathcal{M}$, called the intrinsic stabilizer reduction of $\mathcal{M}$.

(b) [94, **S**] Let $\mathcal{M}$ be an Artin stack with a good moduli space. Then $\mathcal{M}$ admits an intrinsic stabilizer reduction $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$, where $\widetilde{\mathcal{M}}$ is a proper Deligne-Mumford stack with a good moduli space. ♣

The proof uses a powerful recent slicing theorem [5] to leverage the case of a quotient stack and define a modified blowup operation, called the Kirwan blowup, generalizing a construction given in special cases by Kiem-Li [55]. In recent work, I have further obtained a fully derived upgrade by proving that the Kirwan blowup is the classical truncation of an equivariant derived blowup. This gives a conceptual explanation of its intrinsic nature and suitability for enumerative applications.

Theorem 1.2. [41, Hekking–Rydh–**S**] Let $\boldsymbol{\mathcal{M}}$ be a derived Artin stack whose classical truncation $\mathcal{M}$ admits a good moduli space. There exists a canonical, derived stabilizer reduction $\boldsymbol{\pi}\colon \widetilde{\boldsymbol{\mathcal{M}}} \to \boldsymbol{\mathcal{M}}$ whose classical truncation is the intrinsic stabilizer reduction $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$. ♣

The existence of a good moduli space is a fundamental feature in the above constructions, both as a starting point and a preserved property. Good moduli spaces generalize GIT quotients $X/\!\!/G$, have a rich literature and nowadays play a significant role for a wide range of moduli, importantly those parametrizing objects in abelian categories [6] and K-semistable Fano varieties in the minimal model program [4]. In ongoing work, we establish the foundations of the theory within derived algebraic geometry and obtain several applications, including proving that derived stabilizer reductions admit good moduli spaces, a derived étale slice theorem and natural derived enhancements of classical moduli spaces, such as moduli spaces of stable maps.

Theorem 1.3. [1, Ahlqvist–Hekking–Pernice–**S**] There exists a robust theory of derived good moduli spaces for derived Artin stacks, which is a derived enhancement of the classical theory. ♣

Problem 1.4. (a) Develop a theory of adequate moduli spaces [3] for derived stacks over charasteristic $p > 0$.

(b) Prove relative versions of the derived étale slice theorem and stabilizer reduction over any derived base scheme S, including mixed characteristic.

(c) Pursue a theory of good moduli spaces and the above for non-connective derived stacks using [16].

Moreover, resolutions of stabilizers have been used to study the Grothendieck ring of stacks, a universal motivic invariant under cut-and-paste operations. For Deligne-Mumford stacks, [17] produces a Bittner presentation with smooth varieties as generators and relations determined by smooth blowups, using weak factorization of proper, birational morphisms between smooth DM stacks into sequences of stacky blowups and blowdowns.

Postdoc Research Project 1.5. (a) Prove weak factorization theorems for morphisms between smooth, classical Artin stacks with good moduli spaces and morphisms between derived stacks, e.g derived DM stacks or quasi-smooth stacks, using derived blowups.

(b) Construct a Bittner presentation of the Grothendieck ring of Artin stacks with good moduli spaces.

1.2. **Sheaf-counting in dimensions** 2 **and** 3**.** A main motivation for the development of the intrinsic stabilizer reduction lies in enumerative geometry and Donaldson-Thomas (DT) theory. DT theory is a holomorphic analogue of Chern-Simons theory in dimension 3: Let W be a Calabi-Yau (CY) threefold and $\Omega \in \Omega_W^{3,0}$ a holomorphic volume form on W. A holomorphic vector bundle consists of the data of a fixed underlying smooth complex vector bundle E together with a holomorphic structure, i.e. a semi-connection $\bar\partial_E \colon C^\infty(E) \to C^\infty(E \otimes_{\mathbb{C}} \Omega_W^{0,1})$ which satisfies the flatness condition $\bar\partial_E^2 = 0$. Any other semi-connection is of the form $\bar\partial_E + A$ with $A \in C^\infty(\mathrm{End}(E) \otimes_{\mathbb{C}} \Omega_W^{0,1})$, and $(\bar\partial_E + A)^2 = 0$ is equivalent to the equation $\mathrm{dCS}(\bar\partial_E + A) = 0$ where CS is the Chern-Simons functional

$$\mathrm{CS}(\bar\partial_E + A) = \frac{1}{4\pi^2}\int_W \mathrm{tr}\left(\frac{1}{2}(\bar\partial_E A) + \frac{1}{3}A \wedge A \wedge A\right) \wedge \Omega. \tag{1.1}$$

The gauge-theoretic local description of the moduli of holomorphic bundles as a d-critical locus provides crucial connections to Morse homology and Floer theory, used in symplectic geometry. Since the seminal paper of Thomas [99], this conceptual picture has been algebraized [20] and greatly generalized [46] with DT invariants forming a major sheaf theoretic approach towards counting curves in a CY threefold and more generally (semi)stable sheaves and complexes. They correspond to counts of D-branes in string theory, and are intimately related to a wealth of other invariants, such as Gromov-Witten [12], Gopakumar-Vafa [56, 77] and Stable Pair [86] invariants.

Letting $\mathcal{M}$ denote a moduli stack parametrizing stable sheaves, Thomas constructed a 0-dimensional virtual fundamental cycle $[\mathcal{M}]^{\mathrm{vir}} \in H_0(\mathcal{M})$ by studying the two-term deformation-obstruction theory of objects in $\mathcal{M}$ and applying the perfect obstruction theory formalism of [14, 71]. A stability condition here is a choice of extra data, necessary to make $\mathcal{M}$ well-behaved. The classical DT invariant is then defined intersection-theoretically as

$$\mathrm{DT}(\mathcal{M}) := \deg\, [\mathcal{M}]^{\mathrm{vir}} = \int_{[\mathcal{M}]^{\mathrm{vir}}} 1 \in \mathbb{Q}, \tag{1.2}$$

a (virtual) count of sheaves parametrized by $\mathcal{M}$ with the key property of being invariant under deforming the complex structure of W. In a surprising breakthrough, an alternative, motivic description was established by Behrend [13].

Both of these approaches work under the assumption that all semistable sheaves are stable, so a natural and important question is what happens when semistability does not coincide with stability. $\mathcal{M}$ is then an Artin stack with positive dimensional stabilizers and no longer a DM stack and the above techniques do not apply.

My research program directly defines an intersection-theoretic generalized DT invariant in this case, giving a novel approach towards generalized DT theory of sheaves and complexes on CY threefolds in all cases of interest. The following technical statement basically says that we can carry out standard intersection-theoretic methods after replacing $\mathcal{M}$ by its intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$.

Theorem 1.6. [58, Kiem–Li–**S**] [94, **S**] Let $\mathcal{M}$ be a moduli stack parametrizing semistable sheaves or perfect complexes on a smooth, projective CY threefold, where stability includes Gieseker stability [42] for sheaves and Bridgeland [69, 88] and polynomial stability [74, 75] for complexes.

Then $\mathcal{M}$ admits an intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ equipped with a semi-perfect obstruction theory in the sense of Chang-Li [34] of virtual dimension zero. This induces a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in H_0(\widetilde{\mathcal{M}})$. The generalized DT invariant via Kirwan blowups (DTK invariant) is then defined by

$$\mathrm{DTK}(\mathcal{M}) := \deg\, [\widetilde{\mathcal{M}}]^{\mathrm{vir}} = \int_{[\widetilde{\mathcal{M}}]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

and is invariant under deformations of the complex structure of W.

In the classical case, $\widetilde{\mathcal{M}} = \mathcal{M}$ and $\mathrm{DTK}(\mathcal{M}) = \mathrm{DT}(\mathcal{M})$, recovering (1.2). ♣

Joyce-Song [48] and Kontsevich-Soibelman [65] have also produced generalized DT invariants by using Hall algebras to generalize Behrend's motivic approach. However, their invariants are not directly enumerative in nature and their deformation invariance is not transparent and only obtainable via wall-crossing, properties enjoyed by the DTK invariant.

The above theorem also applies in a direct fashion to surfaces and Fano threefolds, whose derived moduli stacks $\boldsymbol{\mathcal{M}}$ of sheaves and complexes are automatically quasi-smooth. In [41], we show that the stabilizer reduction of a quasi-smooth derived Artin stack is a quasi-smooth derived DM stack. Such stacks always admit natural virtual fundamental cycles, which is thus the case for the stabilizer reductions $\widetilde{\boldsymbol{\mathcal{M}}}$. Integrating against them allows us to construct generalized Donaldson-type invariants via Kirwan blowups (DTK invariants by abuse of abbreviation).

1.3. **Derived symplectic geometry.** By [87], the stack $\mathcal{M}$ is the classical truncation of a derived Artin stack $\boldsymbol{\mathcal{M}}$ with a (-1)-shifted symplectic structure $\boldsymbol{\omega}$. Shifted symplectic stacks and their geometry vastly generalize holomorphic symplectic varieties (algebraic hyperkähler manifolds) and traditional symplectic notions, such as Lagrangians, Hamiltonians etc., and have been a subject of intense research activity.

An important consequence of the existence of $(\boldsymbol{\mathcal{M}}, \boldsymbol{\omega})$, which we utilize in the proof of Theorem 1.6, is that, by the derived Darboux theorem [15], $\mathcal{M}$ is (smooth) locally the critical locus of an algebraic function $f\colon V \to \mathbb{C}$ on a smooth variety V. The comparison with the Chern-Simons picture (1.1) is open and would be an interesting beginning project for a graduate student, bridging gauge theory and derived geometry.

Graduate Research Project 1.7. Are the critical loci of the Chern-Simons functional and f compatible in the sense of the theory of d-critical loci introduced in [46]?

In [41], we show that the obstruction theory of $\widetilde{\mathcal{M}}$ in Theorem 1.6 can be canonically obtained by the tangent complex of its derived enhancement $\boldsymbol{\pi}\colon \widetilde{\boldsymbol{\mathcal{M}}} \to \boldsymbol{\mathcal{M}}$. This provides a fully derived perspective on generalized DT theory in dimension 3, framing it at the crossroads of derived symplectic algebraic geometry and stabilizer reduction. An important immediate corollary is that we can apply the DTK formalism and torus localization to directly construct generalized Vafa-Witten invariants [97, 98] of surfaces enumerating semistable Higgs pairs on a projective surface.

Problem 1.8. (a) Develop a theory of shifted quasi-symplectic derived Artin stacks that includes pairs of the intrinsic stabilizer reduction $\widetilde{\boldsymbol{\mathcal{M}}}$ of a shifted symplectic Artin stack $\boldsymbol{\mathcal{M}}$ and its pullback form $\boldsymbol{\pi}^*\boldsymbol{\omega}$.

(b) Study the derived cotangent complex $\mathbb{L}_{\widetilde{\boldsymbol{\mathcal{M}}}}$ of $\widetilde{\boldsymbol{\mathcal{M}}}$ and its duality properties under the degenerate pairing induced by $\boldsymbol{\pi}^*\boldsymbol{\omega}$ when $(\boldsymbol{\mathcal{M}}, \boldsymbol{\omega})$ is a n-shifted symplectic derived Artin stack.

1.4. **Further questions and future work.** The definition of DTK invariants is only the beginning of my research program on generalized DT theory.

Problem 1.9. Prove a wall-crossing formula governing the behavior of DTK invariants under change of stability condition and a universal formula that relates DTK invariants with Joyce-Song (JS) invariants in dimension 3 and Donaldson invariants [47, 78] and Khan's virtual fundamental classes [50] in dimension 2.

Establishing wall-crossing formulas for intersection-theoretic invariants is an intriguing problem in its own right. Such formulas have mainly been obtained so far by variation of GIT [96] and by constructing master spaces [47, 78] and applying virtual torus localization [55]. In ongoing and future work, I plan to combine these ideas with the intrinsic stabilizer reduction to prove a general wall-crossing formula for DTK-type invariants. Moreover, a comparison formula with the JS invariant will bridge the gap between intersection theory and motivic methods, generalizing the foundational result of Behrend [13] in the classical case.

Apart from wall crossing, a primary tool for computing classical DT invariants is degeneration [72]: One deforms a smooth CY variety W into a union of two smooth, irreducible components W_1, W_2 intersecting along a common smooth divisor D. A degeneration formula then expresses the DT invariants of W in terms of those of the hopefully simpler geometries W_1, W_2 and D. Li-Wu's degeneration formula uses the stack of expanded degenerations [70] and works for ideal sheaves and stable pairs, as stability needs to be preserved under restriction from W_i to D. Intrinsic stabilizer reductions should provide a way to address this limitation. A potentially stronger approach will be to combine intrinsic stabilizer reduction with a framework of logarithmic DT invariants, e.g. as in [76], which still uses expanded degenerations, or the more general theory currently in development by Siebert-Talpo-Thomas.

Postdoc Research Project 1.10. (a) Prove a degeneration formula for generalized DT invariants.
(b) Define generalized logarithmic DT invariants via logarithmic Kirwan blowups.

Graduate Research Project 1.11. Perform explicit computations of DTK invariants, for example, for zero-dimensional sheaves (see also Section 3).

The zero-dimensional case reduces to the concrete problem of performing Kirwan blowups on schemes of pairwise commuting matrices, whose singularities and birational geometry are of independent interest.

Finally, desingularizations of moduli spaces of sheaves on curves and surfaces have been explicitly studied in [51, 52, 81]. In this vein, I ultimately hope to give a moduli-theoretic interpretation of the intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ to understand the enumerative nature of DTK invariants.

1.5. **Donaldson-Thomas theory in dimension 4.** Let M a proper moduli scheme parametrizing stable sheaves on a Calabi-Yau fourfold W. In this case, the deformation-obstruction theory of M is not two-term and hence the standard machinery used to define virtual cycles in the threefold case does not apply.

Motivated by gauge theory considerations by Cao-Leung [22], Borisov-Joyce [18] first defined DT invariants by using (-2)-shifted symplectic structures [87] and derived differential geometry [45]: M is the truncation of a (-2)-shifted symplectic derived scheme $\boldsymbol{M}$. One can then perform a real differential geometric truncation of the algebraic local models of $\boldsymbol{M}$ afforded by [20], and use compatibility data coming from derived geometry to show that they glue to define a compact derived manifold M_{dm}. Given a choice of orientation on $\boldsymbol{M}$ (such exists by [23]), M_{dm} is oriented and by [45] admits a virtual fundamental class $[M_{dm}]^{\mathrm{vir}} \in H_*(M, \mathbb{Z})$. This is independent of the choices made in the construction and defined to be the virtual fundamental class of $\boldsymbol{M}$.

From the algebraic point of view, Oh-Thomas gave a new algebro-geometric definition of DT invariants in [83, 82], which recovers the homological Borisov-Joyce DT invariant with $\mathbb{Z}\left[\frac{1}{2}\right]$-coefficients. Both approaches are valuable: the differential geometric works with $\mathbb{Z}$-coefficients and also applies when the real virtual dimension is odd, while the algebraic is more tractable and better adapted to computation.

Due to the significant interest in these invariants, see for example [21, 24, 25, 26, 27, 28, 29, 32], it is important to extend essential techniques to the study of virtual classes in this context. In [92], I proved the first cosection localization [54] results for the virtual fundamental class of d-manifolds and (-2)-shifted symplectic derived schemes, using techniques from derived geometry[1]. A description is as follows: Let $\mathcal{X}$ be a compact, oriented d-manifold with virtual tangent bundle $\mathbb{T}_{\mathcal{X}}$ and $(\boldsymbol{M}, \omega_{\boldsymbol{M}})$ be an oriented, (-2)-shifted symplectic derived scheme with derived tangent complex $\mathbb{T}_{\boldsymbol{M}}$. A cosection σ is roughly a morphism from the first homology sheaf of $\mathbb{T}_{\mathcal{X}}$ or $\mathbb{T}_{\boldsymbol{M}}$ to the trivial real or complex line bundle respectively. Cosection localization means that the virtual fundamental class in both cases is naturally obtained by a localized class supported on the locus where σ is non-zero.

Combining the methods of [59, 83, 92], in [93], I further showed that the differential geometric cosection localized virtual fundamental classes agree with their algebraic counterparts and constructed reduced virtual fundamental classes for d-manifolds and (-2)-shifted symplectic derived schemes with surjective cosections.

The results of [92, 93] have several applications, including the vanishing of stable pair invariants for hyperkähler fourfolds defined in [29] and the homological integrality of the cosection localized classes of [59], the reduced classes used to define Gopakumar–Vafa type invariants on holomorphic symplectic fourfolds in [30, 31] and the reduced classes used for surface counting on Calabi-Yau fourfolds in [8].

Another natural and exciting direction of research is the construction of new generalized DT invariants of CY fourfolds. At the moment, there is a conjectural, indirect theory for such invariants by Joyce [47] using wall-crossing.

Problem 1.12. Construct generalized DT invariants of CY fourfolds via Kirwan blowups.

[1] The corresponding algebro-geometric cosection localization formalism for the Oh-Thomas invariant was subsequently developed by Kiem-Park in [59].

With Young-Hoon Kiem, we have a preliminary approach which adapts the intrinsic stabilizer reduction formalism to the Borisov-Joyce setting of real derived differential geometry. The basic idea is to replace the complex algebraic Kirwan blowup with a real, derived equivariant blowup. So far, we can treat the case of rank 2 sheaves with primitive K-theory class. However, we expect that we will be able to successfully define invariants in all cases.

2. Virtual structure sheaves and K-theoretic invariants

Beyond numerical or intersection-theoretic invariants of moduli problems, it is often desirable to obtain refined invariants that reflect the structure of the problem at hand, for example motivic invariants or invariants in K-theory. In particular, K-theoretic enumerative invariants have attracted significant attention, motivated by considerations in geometric representation theory and theoretical physics [84, 85].

If a Deligne-Mumford stack $\mathcal{M}$ is the truncation of a quasi-smooth derived DM stack or a quasi-smooth dg-scheme $\boldsymbol{\mathcal{M}}$ [36], it naturally admits a virtual structure sheaf $[\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}] \in K_0(\mathcal{M})$, which is the K-theoretic analogue of the virtual fundamental cycle. More generally, by the work of Lee [67], a virtual structure sheaf exists when $\mathcal{M}$ is equipped with a perfect obstruction theory (POT) [14] that is a two-term complex of vector bundles on $\mathcal{M}$. However, in practice, we often encounter moduli stacks where it is not clear how to construct a POT or whose POT does not admit a global resolution by a two-term complex of locally free sheaves.

In [61], with Young-Hoon Kiem, we develop an appropriate relaxation of the notion of POT, called almost perfect obstruction theory (APOT), which arises in many cases of interest, where the existence of a POT is not known, and define virtual structure sheaves and new K-theoretic invariants.

Theorem-Definition 2.1. [61, Kiem–**S**] A DM stack $\mathcal{M}$ equipped with an APOT admits an induced, deformation invariant virtual structure sheaf $[\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}] \in K_0(\mathcal{M})$. If $\mathcal{M}$ is proper, we define the β-twisted K-theoretic invariant of $\mathcal{M}$ for any class $\beta \in K^0(\mathcal{M})$ as the holomorphic Euler characteristic $\chi(\mathcal{M}, [\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}] \otimes \beta)$. ♣

[60, Kiem-S] and [95, S] establish natural generalizations of the techniques of virtual torus localization [89], cosection localization [57], their combination [33] as well as a virtual Riemann-Roch theorem [39, 90], to the setting of APOT, all fundamental tools in enumerative geometry.

Applications of the theory include d-critical loci [46], the Inaba-Lieblich moduli space of simple complexes on a CY threefold [43, 73] and the intrinsic stabilizer reductions of moduli stacks of sheaves and complexes on CY threefolds constructed in Section 1. In the latter two cases, we obtain new (β-twisted) K-theoretic generalized DT invariants as a consequence. We also prove a K-theoretic wall crossing formula for simple $\mathbb{C}^*$-wall crossings. Finally, the Jiang-Thomas dual obstruction cone [44] admits a $\mathbb{C}^*$-equivariant APOT and hence we have natural K-theoretic refinements of the Jiang-Thomas virtual signed Euler characteristics.

As conclusive steps of this program, I plan to investigate the following.

Problem 2.2. (a) Prove that the virtual structure sheaf induced by the APOT on the intrinsic stabilizer reductions $\widetilde{\mathcal{M}}$ in Section 1 equals the one induced by its derived enhancement $\widetilde{\boldsymbol{\mathcal{M}}}$.

(b) Generalize the virtual pullback formula $f^!\colon K_0(X) \to K_0(Y)$ of Qu [89] for a morphism $f\colon X \to Y$ of DM type equipped with a POT to the setting of APOT.

3. The structure of the Quot scheme

Hilbert and Quot schemes [80] are fundamental objects in moduli theory and their structure and properties have been long-standing subjects of great interest. Additionally, the relationships between the Quot scheme, d-critical loci and quiver representations have been central in understanding moduli spaces of sheaves.

Let $Q_{r,n}$ be the Quot scheme parametrizing surjective morphisms $[\mathcal{O}_{\mathbb{C}^3}^{\oplus r} \to F]$ where F is a zero dimensional sheaf of length n on complex affine space $\mathbb{C}^3$. $Q_{r,n}$ can be (locally) written as a d-critical locus, i.e. the vanishing locus of the derivative of a function on a smooth variety, in two different ways: In [11], Beentjes-Ricolfi give such a description with an explicit function $f_{r,n}$ by expressing $Q_{r,n}$ as a moduli space of stable representations of the Jacobi algebra of a quiver $\widetilde{L}_3$ with potential, depicted in Figure 1 and thus defining a d-critical structure s^{quiv} (cf. [46]). The case $r = 1$ recovers the familiar picture for the Hilbert scheme of length n subschemes of $\mathbb{C}^3$ as the vanishing locus of commutators of three $n \times n$ matrices. On the other hand, the moduli stack $\mathcal{M}_n$ of zero dimensional sheaves of

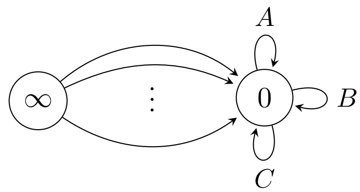

FIGURE 1. The r-framed 3-loop quiver $\widetilde{L}_3$ with potential $W = A[B, C] = ABC - ACB$.

length n on $\mathbb{C}^3$ is the truncation of a (-1)-shifted symplectic derived Artin stack $\boldsymbol{\mathcal{M}}_n$ [87]. Using [15], this induces a d-critical structure on $\mathcal{M}_n$ and hence a d-critical structure s^{der} on $Q_{r,n}$ by pulling back along the smooth forgetful morphism $Q_{r,n} \to \mathcal{M}_n$ which maps $[\mathcal{O}_{\mathbb{C}^3}^{\oplus r} \to F] \in Q_{r,n}$ to $[F] \in \mathcal{M}_n$.

With my collaborator Andrea Ricolfi, we prove that these two d-critical structures coincide, answering a question of Behrend and settling a gap in the literature[2].

Theorem 3.1. [91, Ricolfi–**S**] $s^{\mathrm{quiv}} = s^{\mathrm{der}}$ as étale (or complex analytic) d-critical structures. ♣

We also obtain an analytic local description of the Quot scheme $Q_{X,E,n}$ parametrizing quotients $[E \to F]$, where E is a locally free sheaf and F is a zero dimensional sheaf of length n on any Calabi-Yau threefold X. We also prove that the symmetric obstruction theory of the Hilbert scheme $Q_{1,n}$, induced by its universal family, is isomorphic to the symmetric obstruction theory induced by its structure as a d-critical locus, proving a conjecture of [40].

Since $Q_{r,n}$ is the truncation of a derived Quot scheme $\mathbf{Q}_{r,n}$ [35], whose quotient by an appropriate general linear group admits a (-1)-shifted symplectic structure [19], in ongoing work, we aim to establish the corresponding connections at the level of derived geometry. To this end, in [91, Ricolfi-S], we partially determine the relation between $\mathbf{Q}_{r,n}$, the derived critical locus $\mathbf{R}\mathrm{Crit}(f_{r,n})$ and the derived stack of representations of $\widetilde{L}_3$ in the Hilbert scheme case $r = 1$. Finally, I plan to obtain analogous general results that explicitly describe the local structure of moduli stacks of sheaves or complexes and may prove helpful in performing computations of DT invariants. Of particular interest are similar descriptions for $\mathbb{C}^4$ and the Calabi-Yau fourfold case of Subsection 1.5 more generally.

4. CATEGORICAL STRUCTURES IN ARTIFICIAL INTELLIGENCE

Artificial intelligence (AI) is an incredibly active field of research, having seen an unprecedented boom in recent years. It now enjoys a long list of impressive successes that include image recognition and computer vision, speech and natural language processing, and learning to solve complex tasks, to name a few. Due to its large importance, breadth of applications and related ethical considerations (e.g. concerning healthcare and privacy), the development of robust theoretical foundations that underpin AI and machine learning is becoming increasingly valuable.

With my collaborators Georgios Bakirtzis and Ufuk Topcu [10], we contribute in this direction by introducing a general categorical formalism for Markov Decision Processes (MDP), a fundamental model for reinforcement learning (RL). In RL, an agent learns to carry out a task by performing actions and taking feedback from their environment in terms of a reward or loss. RL is a prominent learning regime, for example see [7, 66, 101] for a list of applications, such as AlphaGo and robotic arm modeling in engineering and control.

More precisely, we define a category MDP with objects MDPs and natural operations (certain fiber products and pushouts) that model breaking up an MDP into subtasks and building an MDP by combining different components. Our theory unifies and extends existing constructions in the literature, such as safety for intelligent agents [68] and symmetry under a group action [102], and provides a natural framework for compositionality. This refers to combining learned behaviors on subtasks to perform a complex task and is an important endeavor in practice, e.g due to the potential for dimensionality reduction. It constitutes a challenge in the sense that there is no need in general for combined behavior to be optimal. However, we provide theoretical guarantees for optimality for simple sequential task completion. An example is a robotic arm tasked with fetching an object and then placing it at a specific position.

[2]Another proof of this result was subsequently obtained by Katz-Shi [49].

Our work is only the beginning of an ongoing research project that is the subject of a research grant proposal submission for which I am a co-PI (along with my usual grant applications in algebraic geometry). Data structures based on the category MDP have already been implemented in a Julia library [9]. Next steps in the project are the incorporation of time within MDP via a notion of S^1-equivariance, importing higher categorical notions via simplicial methods to enrich MDP and bisimulation modeling.

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