

GENERALIZED DONALDSON-THOMAS INVARIANTS VIA KIRWAN BLOWUPS

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ABSTRACT. We develop a virtual cycle approach towards generalized Donaldson-Thomas theory of Calabi-Yau threefolds. Let \mathcal{M} be the moduli stack of Gieseker semistable sheaves of fixed topological type on a Calabi-Yau threefold W .

We construct an associated Deligne-Mumford stack $\widetilde{\mathcal{M}}$ with an induced semi-perfect obstruction theory of virtual dimension zero and define the generalized Donaldson-Thomas invariant of W via Kirwan blowups to be the degree of the virtual cycle $[\widetilde{\mathcal{M}}]^{\text{vir}}$. We show that it is invariant under deformations of the complex structure of W .

1. INTRODUCTION

1.1. Background, history and results. Donaldson-Thomas (in short DT) invariants were first introduced by Thomas in [Tho00] as enumerative invariants counting stable sheaves with fixed Chern character γ on Calabi-Yau threefolds, when the coarse moduli of such stable sheaves is proper. Following the virtual cycle construction of DT invariants, they are invariant under deformation of the Calabi-Yau threefold.

DT invariants constitute an important enumerative theory of Calabi-Yau threefolds and are conjecturally (and in many cases provably so, cf. [MNOP06b, MNOP06a]) related to other enumerative invariants, such as Gromov-Witten invariants, stable pairs [PT09, PT10] and Gopakumar-Vafa invariants [MT16].

Later, Joyce-Song constructed generalized Donaldson-Thomas invariants (in short GDT invariants) for moduli of semistable sheaves of Chern character γ (cf. [JS12]). Their construction relies on Behrend's motivic interpretation of DT invariants, which makes working with GDT relatively easy and their construction generalizable to moduli of complexes. However, their proof that the GDT invariants are invariant under the deformation of complex structure of the Calabi-Yau threefold is indirect and seems difficult to cover more general cases.

The purpose of this work is to provide a new construction of GDT that makes the deformation invariance transparent. Our approach is first using Kirwan's partial desingularization scheme to transform the moduli stack \mathcal{M} of Gieseker semistable sheaves of Chern character γ to a DM stack $\widetilde{\mathcal{M}}$, while at the same time “lift” certain 2-term complexes on \mathcal{M} to a (semi-)perfect

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obstruction theory of $\widetilde{\mathcal{M}}$, thus obtaining a virtual cycle $[\widetilde{\mathcal{M}}]^{\text{vir}}$. We call the so defined generalized DT invariant the generalized DT invariant via Kirwan blowup (in short DTK invariant).

We state our main theorem. Let W be a smooth projective Calabi-Yau threefold; let γ be as before and let \mathcal{M} be the stack of Gieseker semistable sheaves on W of Chern character γ , rigidified by the automorphisms generated by homothesis of every sheaf, so that the open substack of stable sheaves $\mathcal{M}^s \subset \mathcal{M}$ is a DM stack.¹

Theorem 1.1 (Theorem-Definition 6.5). *Let γ and $\mathcal{M}^s \subset \mathcal{M}$ be as stated. Then Kirwan’s partial desingularization constructs a proper DM stack $\widetilde{\mathcal{M}}$ and a morphism $\widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ so that $\widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ is isomorphic over $\mathcal{M}^s \subset \mathcal{M}$, and $\widetilde{\mathcal{M}}$ is equipped with a semi-perfect obstruction theory of virtual dimension zero, extending that of \mathcal{M}^s . This semi-perfect obstruction theory constructs a virtual cycle $[\widetilde{\mathcal{M}}]^{\text{vir}} \in A_0\widetilde{\mathcal{M}}$, whose degree $\deg[\widetilde{\mathcal{M}}]^{\text{vir}}$ is invariant under deformation of the complex structure of W .*

We define

$$\text{DT}_\gamma(\mathcal{M}) = \deg[\widetilde{\mathcal{M}}]^{\text{vir}}$$

to be the generalized DT via Kirwan blowups, or DTK invariant.

It will be desirable to compare the GDT invariant via Kirwan blowup with the GDT invariant constructed by Joyce-Song. We believe that the GDT invariant defined in this paper can be expressed as a “universal” combination of the GDT invariants of Joyce-Song. The confirmation of such a relation will add to our understanding of the GDT invariants of Calabi-Yau threefolds. This is particularly so as much of this construction will be generalized to cover the moduli of derived objects (complexes), by the third named author.

Besides the mentioned work of Joyce-Song in [JS12], there have been other works on constructing (G)DT invariants, notably that of Kontsevich-Soibelman in [KS10], and more recently by Behrend-Ronagh in [BR16a, BR16b].

1.2. Sketch of the construction. Let \mathcal{M} be an Artin stack. We assume that \mathcal{M} is of DT type, meaning that it is the truncation of a (-1) -shifted symplectic derived Artin stack, and assume \mathcal{M} is a GIT global quotient stack

$$\mathcal{M} = [X/G],$$

meaning that G is reductive and X fits into a chain of G -invariant closed subschemes $X \subset P \subset (\mathbb{P}^N)^{ss}$ so that P is smooth and G acts on \mathbb{P}^N via a homomorphism $G \rightarrow GL_{N+1}$.

As $P \subset (\mathbb{P}^N)^{ss}$ is smooth and closed, we can apply Kirwan’s partial desingularization to get a G -equivariant $\pi : \widetilde{P} \rightarrow P$ so that the GIT quotient $\widetilde{P} // G$, which exists and is projective, is a geometric quotient with only possibly finite quotient singularities. In this paper, we call both \widetilde{P} and the quotient $\widetilde{P} // G$ the Kirwan partial desingularizations of P and $[P // G]$, respectively.

¹All stacks of (semi)stable sheaves in this paper are rigidified as such.

As $X \subset P$ is a closed subscheme, possibly singular, adopting the intrinsic blowup introduced by the first and second named authors [KL13b], and applying the Kirwan partial desingularization procedure, we obtain a closed G -equivariant subscheme $\tilde{X} \subset \tilde{P}$ fitting into the squares

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\subset} & X \times_P \tilde{P} & \xrightarrow{\subset} & \tilde{P} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xlongequal{\quad} & X & \xrightarrow{\subset} & P. \end{array}$$

We define \tilde{X} and

$$\tilde{\mathcal{M}} = [\tilde{X}/G]$$

to be the Kirwan (intrinsic) partial desingularizations of X and \mathcal{M} , respectively. We comment that \tilde{X} only depends on the G -structure on X , not the embedding $X \subset P$, and the inclusion $\tilde{X} \subset X \times_P \tilde{P}$ is strict when X is singular along where the stabilizer is positive dimensional. Thus $\tilde{\mathcal{M}}$ is canonically defined.

To construct the virtual cycle of $\tilde{\mathcal{M}}$, we use the theory of d-critical loci of Joyce, developed in [Joy15]. \mathcal{M} being of DT type implies that X is a G -invariant d-critical locus. Then for every $x \in X$ such that $G \cdot x$ is closed in X and H is the stabilizer of x in G (hence reductive), we have G -invariant affine Zariski open $x \in U \subset X, x \in V \subset P$ and locally closed affine subschemes $T \subset U, S \subset V$ granted by Luna's étale slice theorem such that we have a diagram

$$(1.1) \quad \begin{array}{ccc} [T/H] & \longrightarrow & [U/G] \\ \downarrow & & \downarrow \\ [S/H] & \longrightarrow & [V/G] \end{array}$$

with étale horizontal arrows and $T = (df = 0) \subseteq S$ for $f: S \rightarrow \mathbb{A}^1$ an H -invariant regular function on S .

Thus we have the following H -equivariant 4-term complex

$$(1.2) \quad \mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_S|_T \xrightarrow{d(df)^\vee} F_S|_T = \Omega_S|_T \longrightarrow \mathfrak{h}^\vee.$$

For $x \in T$ with finite stabilizer, this is quasi-isomorphic to a 2-term complex which provides a perfect obstruction theory of $[T/H]$ and thus of $[U/G]$ near x .

In general, let $\tilde{x} \in \tilde{X}$ be lying over $x \in X$ with stabilizer R . Then we can lift (1.2) canonically and find an étale neighborhood $[\tilde{T}/R] \rightarrow [\tilde{S}/R] \rightarrow [\tilde{P}/G]$ of $\tilde{x} \in \tilde{P}$, a vector bundle $F_{\tilde{S}}$ over \tilde{S} with an invariant section $\omega_{\tilde{S}} \in H^0(\tilde{S}, F_{\tilde{S}})$ such that $\tilde{T} = (\omega_{\tilde{S}} = 0) \subset \tilde{S}$, and a divisor $D_{\tilde{S}}$, all R -equivariant, such that (1.2) lifts canonically to a sequence

$$(1.3) \quad \mathfrak{r} = \mathrm{Lie}(R) \longrightarrow T_{\tilde{S}}|_{\tilde{T}} \longrightarrow F_{\tilde{S}}|_{\tilde{T}} \longrightarrow \mathfrak{r}^\vee(-D_{\tilde{S}})$$

whose first arrow is injective and last arrow is surjective. Therefore, (1.3) is quasi-isomorphic to a 2-term complex

$$(1.4) \quad (d\omega_{\tilde{S}})^\vee : T_{[\tilde{S}/R]}|_{\tilde{T}} \longrightarrow F_{\tilde{S}}^{\mathrm{red}}|_{\tilde{T}},$$

where $F_{\tilde{S}}^{\text{red}}$ is the kernel of the last arrow in (1.3). Quotienting by R , we get

$$(1.5) \quad (d(\omega_{\tilde{S}}^{\text{red}})^{\vee})^{\vee} : T_{[\tilde{S}/R]|\tilde{T}/R} \longrightarrow F_{[\tilde{S}/R]|\tilde{T}/R}^{\text{red}}.$$

We will show that the collection of the cokernels $\text{coker}(d(\omega_{\tilde{S}}^{\text{red}})^{\vee})^{\vee}$ patch to a coherent sheaf $\mathcal{O}b_{\tilde{\mathcal{M}}}$ of $\mathcal{O}_{\tilde{\mathcal{M}}}$ -modules, and show that the symmetric obstruction theories of the various T defined by $(df = 0)$ induce a semi-perfect obstruction theory on $\tilde{\mathcal{M}} = [\tilde{X}/G]$, with obstruction sheaf $\mathcal{O}b_{\tilde{\mathcal{M}}}$. (See [CL11] for the definition and properties of semi-perfect obstruction theories.)

Using this semi-perfect obstruction theory in the case of Gieseker semistable sheaves of Chern character γ , we obtain a virtual cycle $[\tilde{\mathcal{M}}]^{\text{vir}}$ whose degree we define to be the generalized DT invariant via Kirwan blowups

$$(1.6) \quad \text{DT}_{\gamma} = \deg [\tilde{X}/G]^{\text{vir}}.$$

Finally, the relative version of the above construction can be constructed along parallel lines, using the machinery of derived symplectic geometry. This implies that DT_{γ} is invariant under the deformation of the underlying Calabi-Yau threefold W .

1.3. Outline of the paper. In Section 2 we review Kirwan's partial desingularization for GIT quotients of smooth projective schemes and intrinsic blowups and their role in generalizing Kirwan's procedure to obtain the so called Kirwan partial desingularization of general projective GIT quotients. Section 3 contains relevant material about d-critical loci, while Section 4 addresses semi-perfect obstruction theories. In Section 5, we gather several local computations and formalize several definitions that are helpful in the rest of the paper. Finally, in Section 6 we use all of the above to construct generalized DT invariants. In Section 7 we combine the absolute case and the machinery of derived algebraic geometry and (-1) -shifted symplectic structures to show their deformation invariance.

1.4. Notation and conventions. Here are the various notations and other conventions that we use throughout the paper:

- All varieties and schemes are defined over \mathbb{C} .
- C denotes a smooth quasi-projective curve over \mathbb{C} .
- G, H denote complex reductive groups. Usually, H will be a subgroup of G .
- If U is a scheme over C , then $d_{U/C}$ denotes the relative de Rham differential for U . When C is clear from context or suppressed in the notation, we often just write d_U .
- If $U \hookrightarrow V$ is an embedding, $I_{U \subseteq V}$ or I_U (when V is clear from context) denotes the ideal sheaf of U in V .
- For a morphism $\rho: U \rightarrow V$ and a sheaf E on V , we often use $E|_U$ to denote ρ^*E .
- If V is a G -scheme, V^G and Z_G are both used to denote the fixed point locus of G in V .
- If U is a scheme with a G -action, then \hat{U} is used to denote the Kirwan blowup of U with respect to G . \tilde{U} denotes the Kirwan partial desingularization of U .

- If V is a G -scheme with an equivariant bundle F_V and an invariant global section ω_V , $F_{\widehat{V}}$ and $\omega_{\widehat{V}}$ denote the induced bundle and global section (see Section 2) on the Kirwan blowup \widehat{V} . Note that we follow the convention that the Kirwan partial desingularization is the result of a sequence of Kirwan blowups.
- We often use the abbreviations DT, DM and GIT for Donaldson-Thomas, Deligne-Mumford and Geometric Invariant Theory respectively.

2. KIRWAN PARTIAL DESINGULARIZATION

In this section, we start by briefly recalling Kirwan's desingularization procedure for smooth Geometric Invariant Theory (in short GIT) quotients. Throughout this paper we work with the following situation. Let G be a reductive group; G acts on a \mathbb{P}^N via $G \rightarrow GL_{N+1}$, and

$$X \subset P \subset (\mathbb{P}^N)^{ss}$$

are G -invariant closed subschemes where P is smooth. Here by $(\mathbb{P}^N)^{ss}$ (resp. $(\mathbb{P}^N)^s$) we mean the open subscheme of GIT semistable (resp. stable) points under the linear G -action. (For GIT, see [MFK94].) We assume $P^s \neq \emptyset$.

We then proceed to generalize the procedure to possibly singular GIT quotients, by adapting the intrinsic blowups introduced in [KL13b].

2.1. Kirwan's blowup algorithm. We review Kirwan's partial desingularization \widetilde{P} of P .

As the stabilizer groups of points in P^s are all finite, $P^s // G = P^s / G$ can have at worst finite quotient singularities. In stack language, this means that the quotient stack $[P^s / G]$ is a DM stack. When $P^s \neq P$, the GIT quotient $P // G$ will have worse than finite quotient singularities, and the quotient stack is not DM. In [Kir85], Kirwan produced a canonical procedure to blow up P in order to produce a DM stack out of the Artin stack $[P / G]$.

If the orbit of an $x \in P$ is closed in P , then the stabilizer G_x of x is a reductive subgroup of G . Let us fix a representative of each conjugacy class of subgroups R of G that appear as the identity component of the stabilizer G_x of an $x \in P$ with $G \cdot x$ closed in P . Let $\mathfrak{R}(P)$ denote the set of such representatives. By [Kir85], $\mathfrak{R}(P)$ is finite and $\mathfrak{R}(P) = \{1\}$ if and only if $P = P^s$.

Let $R \in \mathfrak{R}(P)$ be an element of maximal dimension and let Z_R be the fixed locus by the action of R , which is smooth. Then $GZ_R = G \times_{N^R} Z_R$ is smooth in P , where N^R is the normalizer of R in G .

We let $\pi : \text{bl}_R(P) \rightarrow P$ be the blowup of P along $GZ_R = G \times_{N^R} Z_R$. Then $L = \pi^* \mathcal{O}_P(1)(-\epsilon E)$ is ample for $\epsilon > 0$ sufficiently small, where E denotes the exceptional divisor of π . The action of G on P induces a linear action of G on $\text{bl}_R(P)$ with respect to L . The semistable points in the closure of $\text{bl}_R(P)$ inside the projective space given by the embedding induced by the ample line bundle L all lie in $\text{bl}_R(P)$. The unstable points in $\text{bl}_R(P)$ are precisely those points whose orbit closure meets the unstable points in $E = \mathbb{P}N_{GZ_R/P}$, and the unstable points of $\text{bl}_R(P)$ lying in the fiber $E|_x = \mathbb{P}N_{GZ_R/P}|_x$ for $x \in Z_R$ are precisely the unstable points of the projective space $\mathbb{P}N_{GZ_R/P}|_x$.

with respect to the linear action of R . We define

$$\widehat{P} = (\mathrm{bl}_R(P))^{ss}.$$

By [Kir85], $\mathfrak{R}(\widehat{P}) = \mathfrak{R}(P) - \{R\}$.

Definition 2.1. *The scheme \widehat{P} (resp. \widehat{P}/G) is called the Kirwan blowup of P (resp. $P//G$) with respect to the group R .*

Therefore repeating the Kirwan blowup finitely many times, once for each element of $\mathfrak{R}(P)$ in order of decreasing dimension, we end up with a G -equivariant morphism

$$\widetilde{P} \longrightarrow P$$

which induces a projective morphism

$$\widetilde{P}/G \longrightarrow P//G.$$

As \widetilde{P} is smooth, \widetilde{P}/G has at worst finite quotient singularities.

Definition 2.2. *The scheme \widetilde{P} (resp. \widetilde{P}/G) is called the Kirwan partial desingularization of P (resp. $P//G$).*

Remark 2.3. *We note here that in the Kirwan blowup, we can detect which points on the exceptional divisors that occur are unstable just by looking at the action of R on $\mathbb{P}N_{GZ_R/P}$. Furthermore, a point off the exceptional divisor is unstable if the closure of its orbit meets the unstable locus of the exceptional divisor. Thus for any smooth affine G -scheme V , we can define its Kirwan blowup \widehat{V} associated to any $R \in \mathfrak{R}(V)$ of maximal dimension.*

2.2. Intrinsic blowups. Suppose that U is a scheme with an action of a reductive group G . Suppose moreover that we have an equivariant embedding $U \rightarrow V$ into a smooth G -scheme V and let I be the ideal defining U . Since $U \subset V$ is G -equivariant, G acts on I and we have a decomposition $I = I^{fix} \oplus I^{mv}$ into the fixed part of I and its complement as G -representations.

Let V^G be the fixed point locus of G inside V and $\pi: \mathrm{bl}_G(V) \rightarrow V$ the blowup of V along V^G . Let $E \subset \mathrm{bl}_G(V)$ be its exceptional divisor and $\xi \in \Gamma(\mathcal{O}_{\mathrm{bl}_G(V)}(E))$ the tautological defining equation of E . We claim that

$$(2.1) \quad \pi^{-1}(I^{mv}) \subset \xi \cdot \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}.$$

Let $\mathcal{R}: I \rightarrow I^{fix}$ be the Reynolds operator. Then for any $\zeta \in I^{mv}$, $\mathcal{R}(\zeta) = 0$. Let $x \in V^G$ be any closed point. Since \mathcal{O}_x is fixed by G , we have $\zeta|_x = \mathcal{R}(\zeta)|_x = 0$. This proves that all elements in I^{mv} vanish along V^G , hence (2.1).

Consequently, $\xi^{-1}\pi^{-1}(I^{mv}) \subset \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}$. We define $I^{intr} \subset \mathcal{O}_{\mathrm{bl}_G(V)}$ to be

$$(2.2) \quad I^{intr} = \text{ideal generated by } \pi^{-1}(I^{fix}) \text{ and } \xi^{-1}\pi^{-1}(I^{mv}).$$

Definition 2.4. (Intrinsic blowup) *The G -intrinsic blowup of U is the subscheme $U^{intr} \subset \mathrm{bl}_G(V)$ defined by the ideal I^{intr} .*

Lemma 2.5. *The G -intrinsic blowup of U is independent of the choice of G -equivariant embedding $U \subset V$, and hence is canonical.*

Proof sketch. The proof is identical to [KL13b, Section 3.1]. We give a very brief account of the main steps involved.

One firstly establishes the claim in the case of U being a formal affine scheme such that the fixed locus U^G is an affine scheme which has the same support as U does.

For an affine scheme U with a G -action, one can then take the formal completion U^c of U along U^G and check that the G -intrinsic blowup of U^c glues naturally with $U - U^G$ to yield the G -intrinsic blowup of U .

Finally, one can show by taking a cover by (Zariski or étale) open affine schemes that the intrinsic blowup is well-defined for a general scheme or DM stack with a (representable) action of G . \square

Remark 2.6. *The following remarks on the proof of Lemma 2.5 are in order:*

- (1) *If U is smooth, then the G -intrinsic blowup coincides with the blowup of U along U^G .*
- (2) *Since the core of the proof relies on working first in the formal completion of V^G inside V and proving the Lemma 2.5 in the case of formal schemes, this enables us to perform local calculations formally (or analytically) locally.*

Suppose U is an affine G -scheme, then we can think all points of U are semistable. By remark 2.3, we can make sense of semistable points in U^{intr} without ambiguity.

Definition 2.7 (Kirwan blowup). *We define the Kirwan blowup of a possibly singular affine G -scheme U associated with G to be $\widehat{U} = (U^{\text{intr}})^{\text{ss}}$.*

2.3. Blowup bundle and section. Let V be a smooth affine G -scheme, F_V a G -vector bundle on V and $\omega_V \in \Gamma(V, F_V)^G$ a G -invariant section. Then $U = (\omega_V = 0)$ is a G -invariant subscheme of V . Since G is reductive, we have a decomposition

$$(2.3) \quad F_V|_{V^G} = F_V|_{V^G}^{\text{fix}} \oplus F_V|_{V^G}^{\text{mv}}.$$

Definition 2.8. (Blowup bundle) *Let $\pi : \widehat{V} \rightarrow V$ be the Kirwan blowup of V associated with G . The blowup bundle of F_V , denoted by $F_{\widehat{V}}$, is*

$$F_{\widehat{V}} := \ker(\pi^* F_V \longrightarrow \pi^*(F_V|_{V^G}) \longrightarrow \pi^*(F_V|_{V^G}^{\text{mv}})).$$

The blowup section

$$\omega_{\widehat{V}} \in \Gamma(\widehat{V}, F_{\widehat{V}}),$$

is the lift of ω_V , which exists since π^ω_V maps to zero in $\pi^*(F_V|_{V^G}^{\text{mv}})$.*

Proposition 2.9. *Let $U' \subset \widehat{V}$ be defined by the vanishing of $\omega_{\widehat{V}}$. Then U' is the Kirwan blowup \widehat{U} of U .*

Proof. Let I be the ideal of U in V , generated by the section ω . We need to check that the ideal I^{intr} given by (2.2) coincides with the ideal generated by $\omega_{\widehat{V}}$. By the above, it suffices to work locally. But in local coordinates $\omega_{\widehat{V}}$ is obtained from $\pi^*\omega_V$ by multiplying the moving components with ξ^{-1} , which immediately implies the claim. \square

2.4. Kirwan partial desingularization for singular schemes. We continue working with the G -triple $X \subset P \subset (\mathbb{P}^N)^{ss}$ mentioned at the beginning of this paper. We list

$$\mathfrak{R}(P^{ss}) = \{R_1, \dots, R_m, \{1\}\}$$

in order of decreasing dimension.

We begin with $R = R_1 \in \mathfrak{R}(P^{ss})$. For any $x \in Z_R \subset P$, let S be an étale affine slice for x in P , provided by Luna's étale slice theorem (cf. [Dré04, Theorem 5.3]), and let $T = S \times_P X$. (In case $x \notin X$, we can choose S so that $T = \emptyset$.)

As S is smooth, affine and R -invariant, we let \hat{S} be the Kirwan blowup of S associated with R . As $T \subset S$ is closed and R -invariant, we let $\hat{T} \subset \hat{S}$ be the Kirwan blowup of T associated with R . They fit into a commutative diagram

$$(2.4) \quad \begin{array}{ccccc} G \times_R \hat{T} & \longrightarrow & G \times_R \hat{S} & \longrightarrow & \hat{P} \\ \downarrow & & \downarrow & & \downarrow \\ G \times_R T & \longrightarrow & G \times_R S & \longrightarrow & P \\ \downarrow & \nearrow & & & \\ X & & & & \end{array}$$

This collection of étale maps $G \times_R S \rightarrow P$ cover the locus GZ_R inside P . Let $E \subset \hat{P}$ be the exceptional divisor of $\hat{P} \rightarrow P$. Because $P - GZ_R = \hat{P} - E$, $P - GZ_R$ can be viewed as an open subscheme of \hat{P} . Consequently, the collection of étale maps $G \times_R \hat{S} \rightarrow \hat{P}$ together with $P - E$ form an étale covering of \hat{P} .

We next consider the collection of all possible $G \times_R \hat{T} \subset G \times_R \hat{S}$.

Proposition 2.10. *The collection of $G \times_R \hat{T} \rightarrow \hat{P}$ just mentioned together with $X - GZ_R \subset \hat{P} - E$ form a closed subscheme $\hat{X} \subset \hat{P}$, called the Kirwan blowup of X . Further, \hat{X} is canonical in that it is independent of the choice of slices or choice of projective embedding.*

Proof. We first show the independence from the particular choice of slices.

Let S_1, S_2 be two étale slices in P , such that $T_1 = S_1 \cap X$, $T_2 = S_2 \cap X$ are the induced slices for X . Let I_1, I_2 be the ideals of $T_1 \subset S_1$ and $T_2 \subset S_2$ respectively.

Near every point in P covered by S_1 and S_2 , we can find a common étale refinement S_{12} . This can be seen as follows: Since $[P/G]$ has affine diagonal, the fiber product $S_1 \times_{[P/G]} S_2$ is an affine scheme with a $(R \times R)$ -action. For any point z fixed by R we may take a slice S_{12} for z in $S_1 \times_{[P/G]} S_2$.

Consider the composition $p_i: S_{12} \rightarrow S_1 \times_{[P/G]} S_2 \rightarrow S_i$. Since S_{12} and S_i have the same dimension and it induces an isomorphism on tangent spaces at z , it must be étale (up to shrinking). It is also evidently R -equivariant and hence is indeed an étale refinement of S_1 and S_2 . It follows that $p_1^* I_1 \simeq p_2^* I_2 \simeq I_{12}$ as ideal sheaves on S_{12} , defining $T_{12} \subset S_{12}$.

Taking Kirwan blowups commutes with étale base change. We thus obtain induced étale maps $\widehat{p}_i: \widehat{S}_{12} \rightarrow \widehat{S}_i$, such that $\widehat{p}_1^* I_1^{intr} \simeq \widehat{p}_2^* I_2^{intr} \simeq I_{12}^{intr}$, defining the subscheme $\widehat{T}_{12} \subset \widehat{S}_{12}$.

Since we may cover $(G \times_R S_1) \times_P (G \times_R S_2)$ by étale opens of the form $G \times_R S_{12}$ around the fixed points of R , étale descent implies that we obtain a well-defined closed subscheme $\widehat{X} \subset \widehat{P}$.

Regarding the choice of projective embedding, we may cover X by G -invariant affine opens $U_\alpha \subset X$. Let $U_\alpha \rightarrow V_\alpha$ be equivariant embeddings into smooth G -schemes. Using those, we may define $\widehat{U}_\alpha \subset \widehat{V}_\alpha$ by first taking intrinsic blowups and restricting to semistable points. We observe that by Remark 2.3 the latter restriction is unambiguous, as the unstable points of the intrinsic blowup are the ones whose G -orbit closure intersects the unstable locus of the exceptional divisor. Therefore, by the canonical nature of intrinsic blowups, for each α the Kirwan blowup \widehat{U}_α is independent of the local embedding $U_\alpha \rightarrow V_\alpha$. Since we may choose those to come from a G -invariant open cover $V_\alpha \subset P$ of any projective embedding $X \subset P$, it follows that \widehat{X} is independent of the choice of projective embedding. \square

Remark 2.11. *When X^s is dense in X , then $\widehat{X} \rightarrow X$ is birational. The other extreme case is when $X^G = X$, then $\widehat{X} = \emptyset$. In general there are cases when $(X^G)_{red} = X_{red}$ and $\widehat{X} \neq \emptyset$.*

We let $P_1 = \widehat{P}$ and $X_1 = \widehat{X}$ be their respective Kirwan blowups associated with R_1 . Then $X_1 \subset P_1$, and $\mathfrak{R}(P_1) = \{R_2, \dots, R_m, \{1\}\}$. We let $X_2 \subset P_2$ be $\widehat{X}_1 \subset \widehat{P}_1$, the Kirwan blowups associated with R_2 , and so on, until we obtain $X_m \subset P_m$, having the property $\mathfrak{R}(P_m) = \{1\}$.

We denote

$$\widetilde{X} = X_m, \quad \widetilde{P} = P_m.$$

Definition 2.12. *We call \widetilde{X} and $\widetilde{\mathcal{M}} = [\widetilde{X}/G]$ the Kirwan partial desingularization of X and the Artin stack $\mathcal{M} = [X/G]$, respectively.*

Note that when X^s is dense in X , $\widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ is proper and birational.

3. D-CRITICAL LOCI

In this section, we recall Joyce's theory of d-critical loci, as developed in [Joy15], and establish some notation.

We comment that there is also a parallel theory of critical virtual manifolds, developed in [KL12], which is equivalent to the theory of d-critical loci for the cases considered in this paper and could alternatively be used as well.

3.1. d-critical schemes. We begin by defining the notion of d-critical charts.

Definition 3.1. (d-critical chart) *A d-critical chart for M is the data of (U, V, f, i) such that: $U \subseteq M$ is Zariski open, V is a smooth scheme, $f: V \rightarrow \mathbb{A}^1$ is a regular function on V and $U \xrightarrow{i} V$ is an embedding so that $U = (d_V f = 0) = \text{Crit}(f) \subseteq V$.*

If $x \in U$, then we say that the d-critical chart (U, V, f, i) is centered at x .

Joyce defines a canonical sheaf \mathcal{S}_M of \mathbb{C} -vector spaces with the property that for any Zariski open $U \subseteq M$ and an embedding $U \hookrightarrow V$ into a smooth scheme V with ideal I , \mathcal{S}_M fits into an exact sequence

$$(3.1) \quad 0 \longrightarrow \mathcal{S}_M|_U \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d_V} \Omega_V^1/I \cdot \Omega_V^1.$$

Example 3.2. For a d -critical chart (U, V, f, i) of M , the element $f + I^2 \in \Gamma(V, \mathcal{O}_V/I^2)$ gives a section of $\mathcal{S}_M|_U$.

Definition 3.3. (d -critical scheme) A d -critical structure on a scheme M is a section $s \in \Gamma(M, \mathcal{S}_M)$ such that M admits a cover by d -critical charts (U, V, f, i) and $s|_U$ is given by $f + I^2$ as above on each such chart. We refer to the pair (M, s) as a d -critical scheme.

For a d -critical scheme M and a Zariski open $U \subseteq M$, any embedding $U \hookrightarrow V$ into a smooth scheme can be locally made into a d -critical chart.

Proposition 3.4. [Joy15, Proposition 2.7] Let M be a d -critical scheme, $U \subseteq M$ Zariski open and $i: U \hookrightarrow V$ a closed embedding into a smooth scheme V . Then for any $x \in U$, there exist Zariski open $x \in U' \subseteq U$, $i(U') \subseteq V' \subseteq V$ and a regular function $f': V' \rightarrow \mathbb{A}^1$ such that $(U', V', f', i|_{U'})$ is a d -critical chart centered at x .

In order to compare different d -critical charts, we need the notion of an embedding.

Definition 3.5. Let (U, V, f, i) and (R, W, g, j) be two d -critical charts for a d -critical scheme (M, s) with $U \subseteq R$ Zariski open. We call a locally closed embedding $\Phi: V \rightarrow W$ an embedding between the two charts if $f = g \circ \Phi: V \rightarrow \mathbb{A}^1$ and the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ \downarrow & & \downarrow \Phi \\ R & \xrightarrow{j} & W \end{array}$$

By abuse of notation, we use $\Phi: (U, V, f, i) \rightarrow (R, W, g, j)$ to denote this data.

We then have the following way to compare different overlapping d -critical charts.

Proposition 3.6. [Joy15, Theorem 2.20] Let (U, V, f, i) and (S, W, g, j) be two d -critical charts centered at x for a d -critical scheme (M, s) . Then, after possibly (Zariski) shrinking V and W around x , there exists a d -critical chart (T, Z, h, k) centered at x and embeddings $\Phi: (U, V, f, i) \rightarrow (T, Z, h, k)$, $\Psi: (S, W, g, j) \rightarrow (T, Z, h, k)$.

3.2. Equivariant d -critical loci. For our purposes, we need equivariant analogues of the results of Section 3.1. The theory works in parallel as before (cf. [Joy15, Section 2.6]).

Definition 3.7. (Good action) Let G be an algebraic group acting on a scheme M . We say that the action is good if M has a cover $\{U_\alpha\}_{\alpha \in A}$ where every $U_\alpha \subseteq M$ is an invariant open affine subscheme of M .

Remark 3.8. *If M is obtained by GIT so that it is the semistable locus of a projective scheme with a linearized G -action, then the action of G on M is good.*

It is straightforward to extend Definitions 2.1, 2.2 and 2.3 and Proposition 3.4 in the equivariant setting (cf. [Joy15, Definition 2.40]).

Proposition 3.9. [Joy15, Remark 2.47] *Let G be a complex reductive group with a good action on a scheme M . Suppose that (M, s) is an invariant d -critical scheme. Then the following hold:*

- (1) *For any $x \in M$ fixed by G , there exists an invariant d -critical chart (U, V, f, i) centered at x , i.e. an invariant open affine $U \ni x$, a smooth scheme V with a G -action, an invariant regular function $f: V \rightarrow \mathbb{A}^1$ and an equivariant embedding $i: U \rightarrow V$ so that $U = \text{Crit}(f) \subseteq V$.*
- (2) *Let (U, V, f, i) and (S, W, g, j) be two invariant d -critical charts centered at the fixed point $x \in M$. Then, after possibly shrinking V and W around x , there exists an invariant d -critical chart (T, Z, h, k) centered at x and equivariant embeddings $\Phi: (U, V, f, i) \rightarrow (T, Z, h, k)$, $\Psi: (S, W, g, j) \rightarrow (T, Z, h, k)$.*

Remark 3.10. *If G is a torus $(\mathbb{C}^*)^k$, then Proposition 3.9 is true without the assumption that x is a fixed point of G .*

Remark 3.11. *There is a notion of d -critical locus for Artin stacks \mathcal{M} (cf. [Joy15, Section 2.8]). Then (cf. [Joy15, Example 2.55]) d -critical structures on quotient stacks $[M/G]$ are in bijective correspondence with invariant d -critical structures on M .*

Moreover, one may pull back d -critical structures along smooth morphisms between stacks.

4. SEMI-PERFECT OBSTRUCTION THEORIES

This section contains necessary material about semi-perfect obstruction theories, as developed in [CL11].

Let $U \rightarrow C$ be a morphism, where U is a scheme of finite type and C a smooth quasi-projective curve. We first recall the definition of perfect obstruction theory [BF97, LT98].

Definition 4.1. (Perfect obstruction theory [BF97]) *A (truncated) perfect (relative) obstruction theory consists of a morphism $\phi: E \rightarrow L_{U/C}^{\geq -1}$ in $D^b(\text{Coh } U)$ such that*

- (1) *E is of perfect amplitude, contained in $[-1, 0]$.*
- (2) *$h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.*

We refer to $\mathcal{O}b_\phi := H^1(E^\vee)$ as the obstruction sheaf of ϕ .

Definition 4.2. (Infinitesimal lifting problem) *Let $\iota: \Delta \rightarrow \bar{\Delta}$ be an embedding with $\bar{\Delta}$ local Artinian, such that $I \cdot \mathfrak{m} = 0$ where I is the ideal of Δ and \mathfrak{m} the closed point of $\bar{\Delta}$. We call $(\Delta, \bar{\Delta}, \iota, \mathfrak{m})$ a small extension. Given*

a commutative square

$$(4.1) \quad \begin{array}{ccc} \Delta & \xrightarrow{g} & U \\ \downarrow \iota & \nearrow \bar{g} & \downarrow \\ \bar{\Delta} & \longrightarrow & C \end{array}$$

such that the image of g contains a point $p \in U$, the problem of finding $\bar{g}: \bar{\Delta} \rightarrow U$ making the diagram commutative is the “infinitesimal lifting problem of U/C at p ”.

Definition 4.3. (Obstruction spaces) For a point $p \in U$, the intrinsic obstruction space to deforming p is $T_{p,U/C}^1 =: H^1\left((L_{U/C}^{\geq -1})^\vee|_p\right)$. The obstruction space with respect to a perfect obstruction theory ϕ is $\text{Ob}(\phi, p) =: H^1(E^\vee|_p)$.

Given an infinitesimal lifting problem of U/C at a point p , there exists by the standard theory of the cotangent complex a canonical element

$$(4.2) \quad \omega(g, \Delta, \bar{\Delta}) \in \text{Ext}^1\left(g^*L_{U/C}^{\geq -1}|_p, I\right) = T_{p,U/C}^1 \otimes_{\mathbb{C}} I$$

whose vanishing is necessary and sufficient for the lift \bar{g} to exist.

Definition 4.4. (Obstruction assignment) For an infinitesimal lifting problem of U/C at p and a perfect obstruction theory ϕ the obstruction assignment at p is the element

$$(4.3) \quad ob_U(\phi, g, \Delta, \bar{\Delta}) = h^1(\phi^\vee)\left(\omega(g, \Delta, \bar{\Delta})\right) \in \text{Ob}(\phi, p) \otimes_{\mathbb{C}} I.$$

Suppose now that U is given by the vanishing of a global section $s \in \Gamma(V, F)$ where F is a vector bundle on a scheme V which is smooth over C . Let J denote the ideal sheaf of U in V and $j: U \rightarrow V$ the embedding. Then we have a perfect obstruction theory given by the diagram

$$(4.4) \quad \begin{array}{ccc} E & \xlongequal{\quad} & [F^\vee|_U \xrightarrow{d_{V/C}s^\vee} \Omega_{V/C}|_U] \\ \downarrow \phi & & \downarrow s^\vee \quad \parallel \\ L_{U/M}^{\geq -1} & \xlongequal{\quad} & [J/J^2 \xrightarrow{d_{V/C}} \Omega_{V/C}|_U]. \end{array}$$

Since V is smooth over C we can find a lift $g': \bar{\Delta} \rightarrow V$ of the composition $j \circ g$. Composing with the section $s: V \rightarrow F$ we obtain a morphism $s \circ g': \bar{\Delta} \rightarrow (g')^*F$. Since $g = g'|_\Delta$ factors through U , we must have $s \circ g' \in I \otimes_{\mathbb{C}} F|_p$.

Let $\rho: I \otimes_{\mathbb{C}} F|_p \rightarrow I \otimes_{\mathbb{C}} \text{Ob}_\phi|_p = I \otimes_{\mathbb{C}} \text{Ob}(\phi, p)$ be the natural projection map.

Lemma 4.5. [KL12, Lemma 1.28] $ob_U(\phi, g, \Delta, \bar{\Delta}) = \rho(s \circ g')$.

Proof. $ob_U(\phi, g, \Delta, \bar{\Delta})$ is given by the composition

$$(4.5) \quad g^*E \longrightarrow g^*L_{U/C}^{\geq -1} \longrightarrow L_{\Delta/\bar{\Delta}}^{\geq -1} \longrightarrow I[1].$$

This fits into a commutative diagram

$$(4.6) \quad \begin{array}{ccccc} g^*\Omega_{V/C} & \xlongequal{\quad} & g^*\Omega_{V/C} & & \\ \downarrow & & \downarrow & \searrow^{ob_V} & \\ g^*E & \xrightarrow{\phi} & g^*L_{\bar{U}/C}^{\geq -1} & \longrightarrow & I[1] \\ \downarrow & & \nearrow_{\text{dashed}} & & \\ g^*F^\vee[1] & & & & \end{array}$$

Since V is smooth over C , the map ob_V must be zero. Using the distinguished triangle of the first column, we get a long exact sequence in cohomology

$$(4.7) \quad \begin{aligned} \mathrm{Hom}(g^*\Omega_{V/C}, I) &= I \otimes_{\mathbb{C}} T_{V/C}|_p \xrightarrow{(d_{V/C}s^\vee)^\vee} \mathrm{Hom}(g^*F^\vee, I) = I \otimes_{\mathbb{C}} F|_p \\ &\longrightarrow \mathrm{Ext}^1(g^*E, I) \longrightarrow \mathrm{Ext}^1(g^*\Omega_{V/C}, I). \end{aligned}$$

Now, the fact that ob_V is zero implies that $ob_U(\phi, g, \Delta, \bar{\Delta})$ lies in the cokernel $I \otimes_{\mathbb{C}} \mathrm{Ob}(\phi, p)$ of the map $(d_{V/C}s^\vee)^\vee$ in (4.7). It is now easy to see using the diagram

$$\begin{array}{ccccccc} g^*\Omega_{V/C}|_U & \longrightarrow & g^*E & \longrightarrow & g^*F^\vee|_U[1] & & \\ \parallel & & \downarrow^{g^*\phi} & & \downarrow & \searrow^{(g')^*s^\vee|_\Delta} & \\ g^*L_{V/C}^{\geq -1}|_U & \longrightarrow & g^*L_{\bar{U}/C}^{\geq -1} & \longrightarrow & g^*L_{\bar{U}/V}^{\geq -1} & \longrightarrow & g^*J/J^2[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_{\bar{\Delta}/C}^{\geq -1}|_\Delta & \longrightarrow & L_{\bar{\Delta}/C}^{\geq -1} & \longrightarrow & L_{\bar{\Delta}/\bar{\Delta}}^{\geq -1} & \longrightarrow & I[1], \end{array}$$

that indeed $ob_U(\phi, g, \Delta, \bar{\Delta}) = \rho(s \circ g')$. \square

Definition 4.6. Let $\phi: E \rightarrow L_{\bar{U}/C}^{\geq -1}$ and $\phi': E' \rightarrow L_{\bar{U}/C}^{\geq -1}$ be two perfect obstruction theories and $\psi: \mathcal{O}b_\phi \rightarrow \mathcal{O}b_{\phi'}$ be an isomorphism. We say that the obstruction theories give the same obstruction assignment via ψ if for any infinitesimal lifting problem of U/C at p

$$(4.8) \quad \psi(ob_U(\phi, g, \Delta, \bar{\Delta})) = ob_U(\phi', g, \Delta, \bar{\Delta}) \in \mathrm{Ob}(\phi', p) \otimes_{\mathbb{C}} I.$$

We are now ready to give the definition of a semi-perfect obstruction theory.

Definition 4.7. (Semi-perfect obstruction theory [CL11]) Let $\mathcal{M} \rightarrow C$ be a morphism, where \mathcal{M} is a DM stack, proper over C , of finite presentation and C is a smooth quasi-projective curve. A semi-perfect obstruction theory ϕ consists of an étale covering $\{U_\alpha\}_{\alpha \in A}$ of \mathcal{M} and perfect obstruction theories $\phi_\alpha: E_\alpha \rightarrow L_{U_\alpha/C}^{\geq -1}$ such that

- (1) For each pair of indices α, β , there exists an isomorphism

$$\psi_{\alpha\beta}: \mathcal{O}b_{\phi_\alpha}|_{U_{\alpha\beta}} \longrightarrow \mathcal{O}b_{\phi_\beta}|_{U_{\alpha\beta}}$$

so that the collection $(\mathcal{O}b_{\phi_\alpha}, \psi_{\alpha\beta})$ gives descent data of a sheaf on \mathcal{M} .

- (2) For each pair of indices α, β , the obstruction theories $E_\alpha|_{U_{\alpha\beta}}$ and $E_\beta|_{U_{\alpha\beta}}$ give the same obstruction assignment via $\psi_{\alpha\beta}$ (as in Definition 4.6).

Remark 4.8. The obstruction sheaves $\{\mathcal{O}b_{\phi_\alpha}\}_{\alpha \in A}$ glue to define a sheaf $\mathcal{O}b_\phi$ on \mathcal{M} . This is the obstruction sheaf of the semi-perfect obstruction theory ϕ .

Suppose now that $\mathcal{M} \rightarrow C$ is as above and admits a semi-perfect obstruction theory. Then, for each $\alpha \in A$, we have

$$\mathcal{C}_{U_\alpha/C} \subset N_{U_\alpha/C} = h^1/h^0((L_{U_\alpha/C}^{\geq -1})^\vee) \xrightarrow{h^1/h^0(\phi_\alpha^\vee)} h^1/h^0(E_\alpha^\vee) \subset h^1(E_\alpha^\vee),$$

where $\mathcal{C}_{U_\alpha/C}$ and $N_{U_\alpha/C}$ denote the intrinsic normal cone stack and intrinsic normal sheaf stack respectively, where by abuse of notation we identify a sheaf \mathcal{F} on \mathcal{M} with its sheaf stack.

We therefore obtain a cycle class $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ by taking the pushforward of the cycle $[\mathcal{C}_{U_\alpha/C}] \in Z_*N_{U_\alpha/C}$.

Theorem-Definition 4.9. [CL11, Theorem-Definition 3.7] *Let \mathcal{M} be a DM stack, proper over C , of finite presentation and C a smooth quasi-projective curve, such that $\mathcal{M} \rightarrow C$ admits a semi-perfect obstruction theory ϕ . The classes $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ glue to define an intrinsic normal cone cycle $[\mathfrak{c}_\phi] \in Z_*\mathcal{O}b_\phi$. Let s be the zero section of the sheaf stack $\mathcal{O}b_\phi$. The virtual cycle of \mathcal{M} is defined to be*

$$[\mathcal{M}, \phi]^{\text{vir}} := s^![\mathfrak{c}_\phi] \in A_*\mathcal{M},$$

where $s^!: Z_*\mathcal{O}b_\phi \rightarrow A_*\mathcal{M}$ is the Gysin map. This virtual cycle satisfies all the usual properties, such as deformation invariance.

Remark 4.10. One can also consider étale covers of \mathcal{M} by DM quotient stacks $[U_\alpha/G_\alpha]$, where G_α acts on U_α with finite stabilizers. Then there is a natural generalization of the notion of semi-perfect obstruction theory and Theorem 4.9 in this setting. This will be used in Section 6 in order to glue the intrinsic normal cone cycles obtained by perfect obstruction theories on a cover of this form.

5. LOCAL CALCULATIONS

In this section, we collect a series of lemmas and propositions that will be useful in the following section of the paper.

5.1. Local models and standard forms. Let V be a smooth affine G -scheme. The action of G on V induces a morphism $\mathfrak{g} \otimes \mathcal{O}_V \rightarrow T_V$ and its dual $\sigma_V: \Omega_V \rightarrow \mathfrak{g}^\vee \otimes \mathcal{O}_V$.

We consider the following tuple on V .

Setup-Definition 5.1. *The quadruple (V, F_V, ω_V, D_V) , where F_V is a G -equivariant vector bundle on V , ω_V an invariant section with zero locus $U = (\omega_V = 0) \subset V$ and $D_V \subset V$ an effective invariant divisor, satisfying:*

- (1) $\sigma_V(-D_V) : \Omega_V(-D_V) \rightarrow \mathfrak{g}^\vee(-D_V)$ factors through a morphism ϕ_V as shown

$$(5.1) \quad \Omega_V(-D_V) \longrightarrow F_V \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V).$$

- (2) The composition $\phi_V \circ \omega_V$ vanishes identically.
 (3) Let R be the identity component of the stabilizer group of a closed point in V with closed orbit. Let V^R denote the fixed point locus of R . Then $\phi_V|_{V^R}$ composed with the projection $\mathfrak{g}^\vee(-D_V) \rightarrow \mathfrak{t}^\vee(-D_V)$ is zero, where \mathfrak{t} is the Lie algebra of R .

gives rise to data

$$\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$$

on V . We say that these data give a weak local model structure for V . We also say that U is in weak standard form.

Remark 5.2. Note that if $f : V \rightarrow \mathbb{A}^1$ is a G -invariant function on V , then $(U, V, \Omega_V, df, 0, \sigma_V)$ give a weak local model for V , being equivalent to an invariant d -critical chart (U, V, f, i) for U . Therefore, an invariant d -critical locus is a particular case of weak standard form.

The next lemma states that the structure of a weak local model behaves well under taking Kirwan blowups and slices thereof.

Lemma 5.3. Let $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ be as in Setup 5.1. Let $\pi : \widehat{V} \rightarrow V$ be the Kirwan blowup of V associated with G . Then we have induced data $\Lambda_{\widehat{V}} = (\widehat{U}, \widehat{V}, F_{\widehat{V}}, \omega_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}})$, where $F_{\widehat{V}}$ is the blowup bundle of F_V , $\omega_{\widehat{V}}$ the blowup section and $D_{\widehat{V}} = \pi^*D_V + 2E$, that give a weak local model structure for \widehat{V} .

Moreover, for a slice S of a closed point x in \widehat{V} with closed G -orbit and stabilizer H , we obtain induced data $\Lambda_S = (T, S, F_S, \omega_S, D_S, \phi_S)$, where F_S is an H -equivariant bundle on S with a section ω_S and the conditions of Setup 5.1 are satisfied for S as well.

Proof. By pulling back via $\pi : \widehat{V} \rightarrow V$ the factorization in (5.1), we obtain

$$\pi^*\Omega_V(-D_V) \longrightarrow \pi^*F_V \xrightarrow{\pi^*\phi_V} \mathfrak{g}^\vee(-D_V).$$

Let $E \subset \widehat{V}$ be the exceptional divisor. By slight abuse of notation, we use D_V to also denote the pull-back of D_V to the blow-up. Then, applying (3) with $R = G$, $\pi^*\phi_V$ factors through $\mathfrak{g}^\vee(-E - D_V)$. Using the obvious inclusion $\Omega_{\widehat{V}}(-E) \rightarrow \pi^*\Omega_V$, we get that the morphism $\Omega_{\widehat{V}}(-E - D_V) \rightarrow \mathfrak{g}^\vee(-E - D_V)$ induced by the action of G , factors as

$$\Omega_{\widehat{V}}(-E - D_V) \longrightarrow \pi^*\Omega_V(-D_V) \longrightarrow \pi^*F_V \longrightarrow \mathfrak{g}^\vee(-E - D_V).$$

We have the following diagram

$$(5.2) \quad \begin{array}{ccccc} \Omega_{\widehat{V}}(-E - D_V) & \longrightarrow & \pi^* F_V & \longrightarrow & \mathfrak{g}^\vee(-E - D_V) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{\widehat{V}}(-E - D_V)|_E & \longrightarrow & (\pi^* F_V)|_E & \longrightarrow & \mathfrak{g}^\vee(-E - D_V)|_E \\ & & \downarrow & \nearrow & \\ & & \pi^*(F_V|_{V^G}^{mv}) & & \end{array}$$

Note that

$$(5.3) \quad (\pi^* F_V)|_E \cong \pi^*(F_V|_{V^G}^{fix}) \oplus \pi^*(F_V|_{V^G}^{mv})$$

and by equivariance we see that $\pi^*\phi_V|_E$ maps $\pi^*(F_V|_{V^G}^{fix})$ to zero inside $\mathfrak{g}^\vee(-E - D_V)|_E$. This induces the last up-right arrow of the diagram. Therefore by taking kernels, we get a factorization

$$(5.4) \quad \sigma_{\widehat{V}} : \Omega_{\widehat{V}}(-D_{\widehat{V}}) \longrightarrow F_{\widehat{V}} \xrightarrow{\phi_{\widehat{V}}} \mathfrak{g}^\vee(-D_{\widehat{V}}),$$

where $D_{\widehat{V}} = 2E + D_V$. This shows (1) for \widehat{V} .

By looking at the diagram, we can easily see that it follows from the identical vanishing of $\phi_V \circ \omega_V$ on V that $\phi_{\widehat{V}} \circ \omega_{\widehat{V}}$ vanishes identically on \widehat{V} . This is (2).

Let us check (3) on \widehat{V} . Away from the exceptional divisor E , $\phi_{\widehat{V}}$ is the same as ϕ_V and hence we have the same vanishing on $\widehat{V}^R - E$. On the other hand, on \widehat{V} , Kirwan's general theory in [Kir85] guarantees that no new R can arise from the blow-up procedure and $(\widehat{V})^R$ is the proper transform of V^R . It readily follows that, since $\phi_{\widehat{V}}|_{(\widehat{V})^R}$ composed with the projection onto $\mathfrak{t}^\vee(-D_{\widehat{V}})$ is vanishing on $\widehat{V} - E$, it vanishes on $(\widehat{V})^R$ as desired.

Next we restrict (5.4) to a slice S in \widehat{V} . The fibration $G \times_H S \rightarrow G/H$ with fiber S gives an exact sequence

$$0 \longrightarrow (\mathfrak{g}/\mathfrak{h})^\vee \longrightarrow \Omega_{\widehat{V}}|_S \longrightarrow \Omega_S \longrightarrow 0,$$

where \mathfrak{h} is the Lie algebra of H . The composition of the first arrow $(\mathfrak{g}/\mathfrak{h})^\vee \rightarrow \Omega_{\widehat{V}}|_S$ with the homomorphism $\sigma_{\widehat{V}} : \Omega_{\widehat{V}}|_S \rightarrow \mathfrak{g}^\vee$ induced by the action of G is the inclusion $(\mathfrak{g}/\mathfrak{h})^\vee \hookrightarrow \mathfrak{g}^\vee$. Therefore $(\mathfrak{g}/\mathfrak{h})^\vee(-D_{\widehat{V}})|_S$ is a subbundle of $\Omega_{\widehat{V}}(-D_{\widehat{V}})|_S$ and $\pi^*F_V|_S$ as well as $\mathfrak{g}^\vee(-D_{\widehat{V}})|_S$. If we take the quotient of (5.4) restricted to S by $(\mathfrak{g}/\mathfrak{h})^\vee(-D_{\widehat{V}})|_S$, we obtain a factorization

$$(5.5) \quad \sigma_S : \Omega_S(-D_S) \longrightarrow F_S \xrightarrow{\phi_S} \mathfrak{h}^\vee(-D_S)$$

of the morphism σ_S induced by the action of H on S , where D_S is the restriction of $D_{\widehat{V}}$ to S . This shows (1) for S . Finally, it is not hard to verify that (2) and (3) are also true. \square

We may now give the following definition, which will be useful when we introduce the concept of Ω -compatibility later in the paper.

Definition 5.4. We say that the data $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ give a local model for V if either they are the data of a d -critical chart or are obtained by such after a sequence of Kirwan blowups and/or taking slices of closed points with closed orbit. We also say that U is then in standard form.

5.2. Obstruction theory of local model. Suppose that U is in weak standard form for data Λ_V of a local model on V .

Lemma 5.5. The following sequence is a complex:

$$(5.6) \quad K_V = [\mathfrak{g} \xrightarrow{\sigma_V^\vee} T_V|_U \xrightarrow{(d_V \omega_V^\vee)^\vee} F_V|_U \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V)].$$

Proof. Since ω_V is G -invariant, the composition $(d_V \omega_V^\vee)^\vee \circ \sigma_V = 0$. Moreover, since $\phi_V \circ \omega_V = 0$, by differentiating we obtain $\phi_V \circ (d_V \omega_V^\vee)^\vee = -\omega_V^\vee \circ d_V \phi_V$, which is zero when restricted to $U = (\omega_V = 0)$. This proves the lemma. \square

Definition 5.6. (Reduced tangent and obstruction sheaf) We define the reduced tangent sheaf and reduced obstruction sheaf of V to be

$$T_V^{\text{red}} := \text{coker } \sigma_V^\vee, \quad \text{and} \quad F_V^{\text{red}} := \ker \phi_V.$$

The section ω_V induces a section of F_V^{red} , denoted by ω_V^{red} .

The restriction of the complex (5.6) to the stable part U^s of U gives rise (and is quasi-isomorphic) to a two-term complex

$$K_V^{\text{red}} = [(d_V(\omega_V^{\text{red}})^\vee)^\vee : T_V^{\text{red}}|_{U^s} \longrightarrow F_V^{\text{red}}|_{U^s}].$$

We denote $\text{coker}(d_V(\omega_V^{\text{red}})^\vee)^\vee|_{U^s}$ by $\mathcal{O}b_V^{\text{red}}$. One can easily check that there are natural isomorphisms

$$(5.7) \quad H^1(K_V)|_{U^s} \cong H^1(K_V|_{U^s}) \cong \mathcal{O}b_V^{\text{red}}.$$

We refer to $\mathcal{O}b_V^{\text{red}}$ as the reduced obstruction sheaf. This is validated by the following proposition.

Proposition 5.7. (Reduced obstruction theory) The dual of K_V^{red} induces a perfect obstruction theory on the DM stack $[U^s/G]$.

Proof. On V^s , σ_V^\vee is injective and ϕ_V is surjective. The latter follows from the fact that the surjective morphism $\sigma_V^\vee(-D_V)$ factors through ϕ_V . In particular, the two terms of K_V^{red} are bundles and U^s is the zero locus of $\omega_V^{\text{red}}|_{V^s}$.

Let $q_V : V^s \rightarrow [V^s/G]$ be the quotient morphism. We have the exact triangle of truncated cotangent complexes

$$q_V^* L_{[V^s/G]}^{\geq -1} \longrightarrow L_{V^s}^{\geq -1} \longrightarrow \mathfrak{g}^\vee \longrightarrow q_V^* L_{[V^s/G]}^{\geq -1}[1],$$

from which we deduce that $T_V^{\text{red}}|_{V^s} = q_V^* T_{[V^s/G]}^{\leq 1}$. Therefore, using the same triangle for U^s , we find that

$$q_U^* L_{[U^s/G]}^{\geq -1} = [I^s/(I^s)^2 \longrightarrow \Omega_V^{\text{red}}|_{U^s}],$$

where I^s is the ideal of U^s in V^s , $\Omega_V^{\text{red}}|_{V^s} = (T_V^{\text{red}}|_{V^s})^\vee = \ker \sigma_V|_{V^s} = q_V^* \Omega_{[V^s/G]} = q_V^* L_{[V^s/G]}^{\geq -1}$ and that there is an arrow

$$\begin{array}{ccc} K_V^{\text{red}}|_{U^s}^\vee & \xlongequal{\quad} & [F_V^{\text{red}}|_{U^s}^\vee \xrightarrow{d_V(\omega_V^{\text{red}})^\vee} \Omega_V^{\text{red}}|_{U^s}] \\ \downarrow & & \downarrow (\omega_V^{\text{red}})^\vee \quad \parallel \\ q_U^* L_{[U^s/G]}^{\geq -1} & \xlongequal{\quad} & [I^s/(I^s)^2 \xrightarrow{d_V} \Omega_V^{\text{red}}|_{U^s}]. \end{array}$$

Therefore $K_V^{\text{red}}|_{U^s}$ descends to a perfect obstruction theory on $[U^s/G]$. \square

Remark 5.8. Since the rank of F_V is equal to $\dim V$, in order to obtain a zero-dimensional virtual cycle, we need to replace $F_V|_{U^s}$ by $F_V^{\text{red}}|_{U^s}$, which has rank $\dim V - \dim G = \dim[V^s/G]$. Note that the surjective morphism $F_V|_{U^s} \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V)$ induces a twisted cosection $\mathcal{O}b_{U^s} \rightarrow \mathfrak{g}^\vee(-D_V)$ (cf. [KL13a]), which enables us to make the perfect obstruction theory 0-dimensional.

5.3. Ω -equivalence. We introduce the following definition.

Definition 5.9. Let V be a smooth affine G -scheme and F_V a G -equivariant bundle on V . We say that two invariant sections $\omega_V, \bar{\omega}_V \in \Gamma(V, F_V)$ are Ω -equivalent if

- (1) $(\omega_V) = (\bar{\omega}_V) =: I_U$ as ideals in \mathcal{O}_V , and
- (2) there exist equivariant morphisms $A, B: F_V \rightarrow T_V$ such that

$$(5.8) \quad \omega_V^\vee = \bar{\omega}_V^\vee + \bar{\omega}_V^\vee \circ A^\vee \circ (d\bar{\omega}_V^\vee) \pmod{I_U^2},$$

$$(5.9) \quad \bar{\omega}_V^\vee = \omega_V^\vee + \omega_V^\vee \circ B^\vee \circ (d\omega_V^\vee) \pmod{I_U^2}.$$

The reason for introducing this notion is the following proposition.

Proposition 5.10. Let U be an affine G -invariant d -critical scheme and (U, V, f, i) and (U, V, g, i) two invariant d -critical charts with V an affine, smooth G -scheme. Then $\omega_f = df$, $\omega_g = dg$ are Ω -equivalent sections of Ω_V .

Proof. We may assume that $f - g \in I_U^2$, where I_U is the ideal of U in V .

Let x_1, \dots, x_n be étale coordinates on V . Let us write f_i for $\frac{\partial f}{\partial x_i}$, f_{ij} for $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and H_f for the Hessian of f for convenience (and similarly for g). Then we have $I_U = (f_i)_{i=1}^n = (g_i)_{i=1}^n$. Moreover $f - g \in I_U^2$ implies that

$$(5.10) \quad f = g + \sum_{k,l} a^{kl} \cdot g_k \cdot g_l, \quad a^{kl} \in \Gamma(\mathcal{O}_V).$$

Differentiating, we obtain for any i and pair (i, j) the relations

$$(5.11) \quad \begin{aligned} f_i &= g_i + \sum_{k,l} a^{kl} \cdot g_{ki} \cdot g_l + \sum_{k,l} a^{kl} \cdot g_k \cdot g_{li} \pmod{I_U^2}, \\ f_{ij} &= g_{ij} + \sum_{k,l} a^{kl} \cdot g_{lj} \cdot g_{ki} + \sum_{k,l} a^{kl} \cdot g_{kj} \cdot g_{li} \pmod{I_U}. \end{aligned}$$

Note that we may re-write (5.11) as

$$\omega_f = \omega_g + H_g \circ A \circ dg \pmod{I_U^2},$$

where A is a morphism $A: \Omega_V \rightarrow T_V$.

Since df , dg and H_g are invariant, applying the Reynolds operator we can assume that A is equivariant. Hence

$$(5.12) \quad \omega_f^\vee = \omega_g^\vee + \omega_g^\vee \circ A^\vee \circ (d_V \omega_g^\vee) \pmod{I_U^2},$$

and similarly for g we have

$$(5.13) \quad \omega_g^\vee = \omega_f^\vee + \omega_f^\vee \circ B^\vee \circ (d_V \omega_f^\vee) \pmod{I_U^2}.$$

This proves the proposition. \square

The following two lemmas show that Ω -equivalence is preserved by the operations of Kirwan blowup and taking slices of closed orbits.

Notation 5.11. *In what follows*

$$\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V), \quad \bar{\Lambda}_V = (U, V, F_V, \bar{\omega}_V, D_V, \phi_V)$$

will denote data of a weak local model structure on V . Similarly on the Kirwan blowup we write

$$\Lambda_{\hat{V}} = (\hat{U}, \hat{V}, F_{\hat{V}}, \omega_{\hat{V}}, D_{\hat{V}}, \phi_{\hat{V}}), \quad \bar{\Lambda}_{\hat{V}} = (\hat{U}, \hat{V}, F_{\hat{V}}, \bar{\omega}_{\hat{V}}, D_{\hat{V}}, \phi_{\hat{V}})$$

and on an étale slice thereof

$$\Lambda_S = (T, S, F_S, \omega_S, D_S, \phi_S), \quad \bar{\Lambda}_S = (T, S, F_S, \bar{\omega}_S, D_S, \phi_S).$$

Lemma 5.12. *Let V be a smooth affine G -scheme and $\Lambda_V, \bar{\Lambda}_V$ as above, such that $\omega_V, \bar{\omega}_V$ are Ω -equivalent with A, B as in (5.8) and (5.9). Then the blowup sections $\omega_{\hat{V}}, \bar{\omega}_{\hat{V}}$ are Ω -equivalent (via induced equivariant morphisms $\hat{A}, \hat{B}: F_{\hat{V}} \rightarrow T_{\hat{V}}$).*

Proof. Let $i_E: E \rightarrow \hat{V}$ be the exceptional divisor of the blowup $\pi: \hat{V} \rightarrow V$. We have $N_{V^G/V} = T_V|_{V^G}^{mv}$ and therefore the relative Euler sequence

$$(5.14) \quad 0 \longrightarrow \mathcal{O}_E(E) \longrightarrow \pi^*(T_V|_{V^G}^{mv}) \xrightarrow{\alpha} T_{E/V^G}(E) \longrightarrow 0.$$

We also have the tangent sequence

$$(5.15) \quad 0 \longrightarrow T_{\hat{V}} \xrightarrow{d\pi} \pi^*T_V \xrightarrow{\beta} i_{E*}T_{E/V^G}(E) \longrightarrow 0.$$

Using these and the definition of $F_{\hat{V}}$ we have the following commutative diagram

$$(5.16) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_{\hat{V}} & \xrightarrow{\gamma} & \pi^*F_V & \longrightarrow & \pi^*F_V|_E \longrightarrow \pi^*(F_V|_{V^G}^{mv}) \\ & & \downarrow \hat{A} & & \downarrow \pi^*A & & \downarrow \\ 0 & \longrightarrow & T_{\hat{V}} & \xrightarrow{d\pi} & \pi^*T_V & \longrightarrow & \pi^*T_V|_E \longrightarrow \pi^*(T_V|_{V^G}^{mv}) \\ & & & & \searrow \beta & & \downarrow \alpha \\ & & & & & & T_{E/V^G}(E). \end{array}$$

By (5.14), (5.15) and (5.16) it follows that the composition $\beta \circ \pi^*A \circ \gamma$ is zero and therefore we obtain an equivariant morphism $\hat{A}: F_{\hat{V}} \rightarrow T_{\hat{V}}$ induced by π^*A .

It remains to check that \widehat{A} satisfies (5.8) for the blowup sections $\omega_{\widehat{V}}$ and $\bar{\omega}_{\widehat{V}}$. We can dualize and pull back (5.8) to obtain

$$(5.17) \quad \pi^* \omega_V = \pi^* \bar{\omega}_V + \pi^* (d_V \bar{\omega}_V^\vee)^\vee \circ \pi^* A \circ \pi^* \bar{\omega}_V \pmod{\pi^* I_U^2}.$$

From the diagram

$$(5.18) \quad \begin{array}{ccccccc} \mathcal{O}_{\widehat{V}} & \xrightarrow{\bar{\omega}_{\widehat{V}}} & F_{\widehat{V}} & \xrightarrow{\widehat{A}} & T_{\widehat{V}} & \xrightarrow{(d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee)^\vee} & F_{\widehat{V}} \\ & \searrow \pi^* \bar{\omega}_V & \downarrow \gamma & & \downarrow d\pi & & \downarrow \gamma \\ & & \pi^* F_V & \xrightarrow{\pi^* A} & \pi^* T_V & \xrightarrow{\pi^* (d_V \bar{\omega}_V^\vee)^\vee} & \pi^* F_V \end{array}$$

we see that

$$(5.19) \quad \pi^* (d_V \bar{\omega}_V^\vee)^\vee \circ \pi^* A \circ \pi^* \bar{\omega}_V = \gamma \circ (d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee)^\vee \circ \widehat{A} \circ \bar{\omega}_{\widehat{V}}.$$

Therefore, we may re-write (5.17) as

$$(5.20) \quad \gamma \circ \left(\omega_{\widehat{V}} - \bar{\omega}_{\widehat{V}} - (d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee)^\vee \circ \widehat{A} \circ \bar{\omega}_{\widehat{V}} \right) \in \Gamma \left(\widehat{V}, \pi^* I_U^2 \cdot \pi^* F_V \right).$$

By equivariance, we can un-twist by γ (which is multiplication by ξ , the equation of the exceptional divisor, on the moving part of $\pi^* F_V$) to obtain

$$(5.21) \quad \omega_{\widehat{V}} = \bar{\omega}_{\widehat{V}} + (d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee)^\vee \circ \widehat{A} \circ \bar{\omega}_{\widehat{V}} \pmod{I_{\widehat{U}}^2}.$$

Here we are using the fact that if η is an invariant section of $I_U^2 \cdot F_V$, then the induced section $\widehat{\eta}$ factors through $I_{\widehat{U}}^2 \cdot F_{\widehat{V}}$.

Dualizing the latter, we obtain relation (5.8) for the blowup. (5.9) for the blowup follows by the same argument. \square

Lemma 5.13. *Let S be an étale slice of a closed point $x \in \widehat{V}$ with closed orbit and stabilizer H . Let $\Lambda_{\widehat{V}}, \bar{\Lambda}_{\widehat{V}}$ be data of a weak local model on \widehat{V} such that $\omega_{\widehat{V}}, \bar{\omega}_{\widehat{V}}$ are Ω -equivalent. Let $\omega_S, \bar{\omega}_S$ be the two sections obtained as part of the local setup induced on S using these two choices of data and Lemma 5.3. Then $\omega_S, \bar{\omega}_S$ are also Ω -equivalent.*

Proof. Let \widehat{U} be the zero locus of $\bar{\omega}_{\widehat{V}}$ (or equivalently $\omega_{\widehat{V}}$) and T be the zero locus of $\bar{\omega}_S$ (equivalently ω_S). The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee & \longrightarrow & \Omega_{\widehat{V}}|_S & \xrightarrow{q_S} & \Omega_S \longrightarrow 0 \\ & & \parallel & & \downarrow \sigma_{\widehat{V}}^\vee & & \downarrow \sigma_S^\vee \\ 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee & \longrightarrow & \mathfrak{g}^\vee & \longrightarrow & \mathfrak{h}^\vee \longrightarrow 0 \end{array}$$

induces an isomorphism of kernels $\Omega_{\widehat{V}}|_T^{\text{red}} \xrightarrow{q_S|_T} \Omega_S|_T^{\text{red}}$ of $\sigma_{\widehat{V}}^\vee|_T$ and $\sigma_S^\vee|_T$ respectively. Since both exact sequences are locally split we may shrink S around x and find a (H -equivariant) right inverse $r_S: \Omega_S \rightarrow \Omega_{\widehat{V}}|_S$ for q_S , which is then also an inverse for $q_S|_T$ when restricted to $\Omega_S|_T^{\text{red}}$.

By the same argument for the commutative diagram (cf. the definition of F_S in the proof of Lemma 5.3)

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee(-D_S) & \longrightarrow & F_{\widehat{V}}|_S & \xrightarrow{\gamma_S} & F_S \longrightarrow 0 \\ & & \parallel & & \downarrow \phi_{\widehat{V}} & & \downarrow \phi_S \\ 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee(-D_S) & \longrightarrow & \mathfrak{g}^\vee(-D_S) & \longrightarrow & \mathfrak{h}^\vee(-D_S) \longrightarrow 0 \end{array}$$

we may find a right inverse $\delta_S: F_S \rightarrow F_{\widehat{V}}|_S$, which is an inverse for γ_S when restricted to $F_S|_T^{\text{red}}$, the kernel of $\phi_S|_T$.

Consider the (non-commutative) diagram

$$\begin{array}{ccccccc} F_{\widehat{V}}^\vee|_T & \xrightarrow{d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^\vee} & \Omega_{\widehat{V}}|_T & \xrightarrow{\widehat{A}^\vee} & F_{\widehat{V}}^\vee|_T & \xrightarrow{\bar{\omega}_{\widehat{V}}^\vee} & I_{\widehat{U}}/I_{\widehat{U}}^2|_T \\ \gamma_S^\vee \uparrow & & \downarrow q_S & & \gamma_S^\vee \uparrow & & \downarrow q_S \\ F_S^\vee|_T & \xrightarrow{d_S\bar{\omega}_S^\vee} & \Omega_S|_T & \xrightarrow{\widehat{A}_S^\vee} & F_S^\vee|_T & \xrightarrow{\bar{\omega}_S^\vee} & I_T/I_T^2, \end{array}$$

where we define $\widehat{A}_S^\vee := \delta_S^\vee \circ \widehat{A}^\vee \circ r_S$. The left- and rightmost squares are commutative.

We check now that the composition $q_S \circ \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^\vee \circ \gamma_S^\vee$ is equal to $\bar{\omega}_S^\vee \circ \widehat{A}_S^\vee \circ d_S\bar{\omega}_S^\vee$.

By Lemma 5.3 and Lemma 5.5, the above diagram factors through the sub-diagram

$$\begin{array}{ccccccc} \Omega_{\widehat{V}}|_T^{\text{red}} & \longrightarrow & \Omega_{\widehat{V}}|_T & \xrightarrow{\widehat{A}^\vee} & F_{\widehat{V}}^\vee|_T & \longrightarrow & (F_{\widehat{V}}|_T^{\text{red}})^\vee \\ \downarrow q_S & & \downarrow q_S & & \gamma_S^\vee \uparrow & & \gamma_S^\vee \uparrow \\ \Omega_S|_T^{\text{red}} & \longrightarrow & \Omega_S|_T & \xrightarrow{\widehat{A}_S^\vee} & F_S^\vee|_T & \longrightarrow & (F_S|_T^{\text{red}})^\vee. \end{array}$$

Observe now that by the definition of \widehat{A}_S^\vee the outer square in this diagram commutes. This immediately implies that indeed

$$(5.22) \quad q_S \circ \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^\vee \circ \gamma_S^\vee = \bar{\omega}_S^\vee \circ \widehat{A}_S^\vee \circ d_S\bar{\omega}_S^\vee.$$

Since $\omega_S^\vee = q_S \circ \omega_{\widehat{V}}^\vee \circ \gamma_S^\vee$ and $\bar{\omega}_S^\vee = q_S \circ \bar{\omega}_{\widehat{V}}^\vee \circ \gamma_S^\vee$, applying $q_S \circ (\bullet) \circ \gamma_S^\vee$ to the restriction of (5.8) to the slice S and using (5.22) we obtain

$$\omega_S^\vee = \bar{\omega}_S^\vee + \bar{\omega}_S^\vee \circ \widehat{A}_S^\vee \circ d_S\bar{\omega}_S^\vee \pmod{I_T^2}.$$

The exact same argument can be used to show the existence of \widehat{B}_S . \square

The next lemma states that for the purposes of comparing obstruction sheaves and assignments we may replace a section by any Ω -equivalent section without any effect.

Given two Ω -equivalent $\omega_V, \bar{\omega}_V \in H^0(F_V)$, denoting $U = (\omega_V = 0) = (\bar{\omega}_V = 0)$, we obtain

$$(d\omega_V^\vee)^\vee|_U, (d\bar{\omega}_V^\vee)^\vee|_U: T_V|_U \longrightarrow F_V|_U.$$

Lemma 5.14. *Let V be a smooth affine G -scheme and $\Lambda_V, \bar{\Lambda}_V$ data of a weak local model on V , such that $\omega_V, \bar{\omega}_V$ are Ω -equivalent. Then*

$$(5.23) \quad \text{coker} (d\omega_V^\vee)^\vee|_U = \text{coker} (d\bar{\omega}_V^\vee)^\vee|_U.$$

Moreover, the two obstruction theories on U induced by ω_V and $\bar{\omega}_V$ give the same obstruction assignments via the morphism (5.21).

Proof. We check that $\text{im} (d\omega_V^\vee)^\vee|_U = \text{im} (d\bar{\omega}_V^\vee)^\vee|_U$. Since $\omega_V, \bar{\omega}_V$ are Ω -equivalent, there exist equivariant morphisms $A, B: F \rightarrow T_V$ such that

$$(5.24) \quad \begin{aligned} \omega_V^\vee &= \bar{\omega}_V^\vee + \bar{\omega}_V^\vee \circ A^\vee \circ (d\bar{\omega}_V^\vee) \pmod{I_U^2}, \\ \bar{\omega}_V^\vee &= \omega_V^\vee + \omega_V^\vee \circ B^\vee \circ (d\omega_V^\vee) \pmod{I_U^2}. \end{aligned}$$

Differentiating (5.24) and dualizing, we obtain

$$(5.25) \quad (d\omega_V^\vee)^\vee = (d\bar{\omega}_V^\vee)^\vee + (d\bar{\omega}_V^\vee)^\vee \circ A \circ (d\bar{\omega}_V^\vee)^\vee \pmod{I_U}.$$

This implies that $\text{im} (d\omega_V^\vee)^\vee|_U \subseteq \text{im} (d\bar{\omega}_V^\vee)^\vee|_U$. By the same argument, using the second equation in (5.24), the first claim follows.

For the obstruction assignments, consider an infinitesimal lifting problem of U at p . Let $\mathcal{O}b = \text{coker} (d\omega_V^\vee)^\vee|_U$ and

$$\rho: I \otimes_{\mathbb{C}} F_V|_p \rightarrow I \otimes_{\mathbb{C}} H^1(E|_p) = I \otimes_{\mathbb{C}} H^1(\bar{E}|_p)$$

be the quotient morphism. Then by Lemma 4.5, we need to show that $\rho(\omega_V \circ g') = \rho(\bar{\omega}_V \circ g')$. But this holds, since dualizing (5.24) and composing with g' we have that $\omega_V \circ g' - \bar{\omega}_V \circ g' \in I \otimes_{\mathbb{C}} \text{im} (d\omega_V^\vee)^\vee|_p$. \square

5.4. Ω -compatibility. We begin with the following lemma, describing how normal bundles behave with respect to Kirwan blowups. It will be used repeatedly in the rest of this subsection.

Lemma 5.15. *Let $\Phi: V \hookrightarrow W$ be a G -equivariant embedding of smooth affine schemes. Let $N_{V/W}$ be the normal bundle of V in W , and let $\text{bl}(N_{V/W})$ as in Definition 2.8. Then there is a natural isomorphism $\text{bl}(N_{V/W}) \cong N_{\widehat{V}/\widehat{W}}$.*

Proof. Let $x \in V^G$. Up to shrinking, we have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & T_x V = \mathbb{A}^n \\ \downarrow \Phi & & \downarrow d\Phi \\ W & \longrightarrow & T_x W = \mathbb{A}^{n+m}, \end{array}$$

where the maps $V \rightarrow T_x V$, $W \rightarrow T_x W$ are equivariant étale and the G -action on the tangent spaces is linear. Since G is reductive, we may pick coordinates x_1, \dots, x_n on \mathbb{A}^n on which G acts linearly and extra coordinates x_{n+1}, \dots, x_{n+m} with a linear G -action on \mathbb{A}^{n+m} such that the embedding $\mathbb{A}^n \rightarrow \mathbb{A}^{n+m}$ takes the canonical form $(x_1, \dots, x_n) \rightarrow (x_1, \dots, x_n, 0, \dots, 0)$. In particular, we get étale coordinates $x_1|_W, \dots, x_{n+m}|_W$ on W , $x_1|_V, \dots, x_n|_V$ on V and may also arrange that $I_V = (x_{n+1}|_W, \dots, x_{n+m}|_W)$.

In what follows, we often write just x_i in place of $x_i|_W$ or $x_i|_V$ by abuse of notation.

Let us assume that x_1, \dots, x_p and x_{n+1}, \dots, x_{n+q} are moving and x_{p+1}, \dots, x_n and $x_{n+q+1}, \dots, x_{n+m}$ are fixed by G .

Since the question is local, we may localize at x and assume that V and $W = \operatorname{Spec} A$ are local and the maximal ideal of A is

$$\mathfrak{m} = (x_1, \dots, x_p, x_{n+1}, \dots, x_{n+q}).$$

Now \widehat{W} is covered by open affines of the form (for $k = 1, \dots, p+q$)

$$(5.26) \quad R_k^n = \operatorname{Spec} A[T_{k,1}, \dots, T_{k,p+q}][\xi_k]/(T_{k,k} - 1),$$

where $x_i = \xi_k T_{k,i}$ if $i \leq p$, $x_{n-p+i} = \xi_k T_{k,i}$ if $i > p$ and $\xi_k = 0$ is the exceptional divisor in R_k^n . It is easy to see that \widehat{V} is covered by such affines for $k \leq p$ and in each such we have that

$$\begin{aligned} I_{\widehat{V}}|_{R_k^n} &= \left(\frac{1}{\xi_k} x_{n+1}, \dots, \frac{1}{\xi_k} x_{n+q}, x_{n+q+1}, \dots, x_{n+m} \right) \\ &= (T_{k,p+1}, \dots, T_{k,p+q}, x_{n+q+1}, \dots, x_{n+m}). \end{aligned}$$

In particular, $N_{\widehat{V}/\widehat{W}}|_{R_k^n}$ has a basis of sections given by $\frac{\partial}{\partial T_{k,p+i}}, \frac{\partial}{\partial x_{n+q+j}}$.

Since $N_{V/W}$ has a basis by $\frac{\partial}{\partial x_{n+i}}$, we see that $\operatorname{bl}(N_{V/W})$ has a basis by $\xi_k \frac{\partial}{\partial x_{n+i}}$ for $1 \leq i \leq q$ and $\frac{\partial}{\partial x_{n+j}}$ for $q+1 \leq j \leq m$. But $x_{n+i} = \xi_k T_{k,p+i}$ implies that $\frac{1}{\xi_k} d_{\widehat{W}} x_{n+i} = d_{\widehat{W}} T_{k,p+i} \pmod{I_{\widehat{V}}}$ and thus by dualizing $\xi_k \frac{\partial}{\partial x_{n+i}} = \frac{\partial}{\partial T_{k,p+i}} \pmod{I_{\widehat{V}}}$. We see that the frames of the two bundles $N_{\widehat{V}/\widehat{W}}$ and $\operatorname{bl}(N_{V/W})$ match. This concludes the proof. \square

For the purposes of comparing obstruction theories obtained by embeddings of d -critical charts and their Kirwan blowups and slices thereof, we introduce the following definition.

Definition 5.16. Let $U \subset V \xrightarrow{\Phi} W$ be a sequence of G -equivariant embeddings of affine G -schemes such that U is in standard form for data $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ and $\Lambda_W = (W, F_W, \omega_W, D_W, \phi_W)$. We say that ω_V and ω_W are Ω -compatible via Φ if the following hold:

- (1) D_W pulls back to D_V under Φ .
- (2) The embedding $\Phi: V \rightarrow W$ induces a surjective equivariant morphism $\eta_\Phi: F_W|_V \rightarrow F_V$, compatible with $\phi_W|_V$ and ϕ_V , such that $\eta_\Phi(\omega_W|_V)$ is Ω -equivalent to ω_V .
- (3) Let $\mathcal{O}b_W := \operatorname{coker}(d_W \omega_W^\vee)^\vee|_U$, $\mathcal{O}b_V := \operatorname{coker}(d_V \omega_V^\vee)^\vee|_U$. Then η_Φ induces an isomorphism

$$\Phi^{\operatorname{ck}}: \mathcal{O}b_W \longrightarrow \mathcal{O}b_V.$$

Remark 5.17. It makes sense to talk about an induced surjection η_Φ , since the local data we are considering arise either as the data of an embedding of d -critical charts or are obtained by such an embedding by performing Kirwan blowups and taking slices. In the former case η_Φ is just pullback of differential forms and in the latter it is canonically induced starting from pulling back differential forms and then blowing up or taking slices.

The motivation behind the above definition is the following lemma.

Lemma 5.18. Let $(U, V, f, i) \xrightarrow{\Phi} (R, W, g, j)$ be a G -equivariant embedding of invariant d -critical charts, where $V \subset W$ as before is a pair of smooth G -schemes. Then $\omega_g = dg$ and $\omega_f = df$ are Ω -compatible.

Proof. η_Φ is pullback of differential forms and $\eta_\Phi(\omega_g|_V) = \omega_f$. Moreover, we have exact sequences

$$\begin{array}{ccccccc} T_W|_U & \xrightarrow{(d_W\omega_g^\vee)^\vee} & \Omega_W|_U & \longrightarrow & \Omega_U & \longrightarrow & 0 \\ & & \downarrow \eta_\Phi & & \parallel & & \\ T_V|_U & \xrightarrow{(d_V\omega_f^\vee)^\vee} & \Omega_V|_U & \longrightarrow & \Omega_U & \longrightarrow & 0 \end{array}$$

from which we deduce that $\text{coker}(d_W\omega_g^\vee)^\vee|_U = \text{coker}(d_V\omega_f^\vee)^\vee|_U = \Omega_U$. We obtain an isomorphism on cokernels induced naturally by η_Φ . \square

In the rest of this subsection, we consider the following situation:

Notation 5.19. Let $U \subset V \xrightarrow{\Phi} W$ be a sequence of G -equivariant embeddings of affine G -schemes such that U is in standard form for data $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ and $\Lambda_W = (U, W, F_W, \omega_W, D_W, \phi_W)$ and ω_V and ω_W are Ω -compatible.

We now check that one may compare obstruction theories given by different data of a local model with Ω -compatible sections.

Lemma 5.20. Let Λ_V, Λ_W be as above. Consider the two complexes E_W, E_V

$$\begin{aligned} (L_U^{\geq -1})^\vee &\longrightarrow E_W = [T_W|_U \xrightarrow{(d_W\omega_W^\vee)^\vee} F_W|_U], \\ (L_U^{\geq -1})^\vee &\longrightarrow E_V = [T_V|_U \xrightarrow{(d_V\omega_V^\vee)^\vee} F_V|_U], \end{aligned}$$

on U . The obstruction theories induced by the dual complexes E_W^\vee, E_V^\vee give the same obstruction assignments via Φ^{ck} .

Proof. Suppose we have an infinitesimal lifting problem at a closed point $p \in U$. Let

$$\rho_W : I \otimes F_W|_p \rightarrow I \otimes \mathcal{O}b_W|_p \quad \text{and} \quad \rho_V : I \otimes F_V|_p \rightarrow I \otimes \mathcal{O}b_V|_p$$

be the induced quotient morphisms. By Lemma 4.5, we need to show

$$(5.27) \quad \Phi^{\text{ck}}(\rho_W(\omega_W \circ \Phi \circ g')) = \rho_V(\omega_V \circ g')$$

Since $\eta_\Phi(\omega_W|_V)$ and ω_V are Ω -equivalent, by Lemma 5.14, we may assume that $\eta_\Phi(\omega_W|_V) = \omega_V$.

Because of $\eta_\Phi \circ \omega_W \circ \Phi = \omega_V$, $\Phi^{\text{ck}} \circ \rho_W = \rho_V \circ \eta_\Phi$, and the commutative diagram

$$\begin{array}{ccccccc} \bar{\Delta} & \xrightarrow{g'} & V & \xrightarrow{\omega_V} & F_V & \xrightarrow{\rho_V} & \mathcal{O}b_V \\ \uparrow & \nearrow g & \downarrow \Phi & & \uparrow \eta_\Phi & & \uparrow \Phi^{\text{ck}} \\ \Delta & & W & \xrightarrow{\omega_W} & F_W & \xrightarrow{\rho_W} & \mathcal{O}b_W \end{array}$$

(5.27) follows. \square

We now show that Ω -compatibility is preserved under taking Kirwan blowups and slices thereof. We begin with a preparatory lemma.

Lemma 5.21. *Let Λ_V, Λ_W as above. Then the induced embedding of Kirwan blowups $\widehat{\Phi}: \widehat{V} \rightarrow \widehat{W}$ induces a morphism $\widehat{\Phi}^{\text{ck}}: \mathcal{O}b_{\widehat{W}} \rightarrow \mathcal{O}b_{\widehat{V}}$. The same is true for an étale slice of a closed point of \widehat{U} with closed orbit.*

Proof. We need to show that $\eta_{\widehat{\Phi}}$ maps $\text{im}(d_{\widehat{W}}\omega_{\widehat{W}}^{\vee})^{\vee}|_{\widehat{U}}$ to $\text{im}(d_{\widehat{V}}\omega_{\widehat{V}}^{\vee})^{\vee}|_{\widehat{U}}$. Since $\eta_{\widehat{\Phi}}$ maps $T_{\widehat{V}}|_{\widehat{U}}$ into $\text{im}(d_{\widehat{V}}\omega_{\widehat{V}}^{\vee})^{\vee}|_{\widehat{U}}$, it suffices to show that the same is true for $N_{\widehat{V}/\widehat{W}}|_{\widehat{U}}$, for any local splitting $T_{\widehat{W}}|_{\widehat{V}} = T_{\widehat{V}} \oplus N_{\widehat{V}/\widehat{W}}$.

η_{Φ} satisfies the same requirement, so by the same reasoning we may find a morphism $\alpha: N_{V/W}|_U \rightarrow T_V|_U$ such that the following diagram commutes (for a splitting $T_W|_V = T_V \oplus N_{V/W}$)

$$\begin{array}{ccccc} N_{V/W}|_U & \longrightarrow & T_W|_U & \xrightarrow{(d_W\omega_W^{\vee})^{\vee}} & F_W|_U \\ & \searrow \alpha & & & \downarrow \eta_{\Phi} \\ & & T_V|_U & \xrightarrow{(d_V\omega_V^{\vee})^{\vee}} & F_V|_U. \end{array}$$

By Lemma 5.12 and Lemma 5.15, we have that $\text{bl}(T_V) \subset T_{\widehat{V}}$, $\text{bl}(T_W|_V) \subset T_{\widehat{W}}|_{\widehat{V}}$ are sub-bundles and $\text{bl}(N_{V/W}) = N_{\widehat{V}/\widehat{W}}$. We therefore obtain a commutative diagram

$$\begin{array}{ccccccc} N_{\widehat{V}/\widehat{W}}|_U & \longrightarrow & \text{bl}(T_W|_U) & \longrightarrow & T_{\widehat{W}}|_{\widehat{U}} & \xrightarrow{(d_{\widehat{W}}\omega_{\widehat{W}}^{\vee})^{\vee}} & F_{\widehat{W}}|_{\widehat{U}} \\ & \searrow \widehat{\alpha} & & & & & \downarrow \eta_{\widehat{\Phi}} \\ & & \text{bl}(T_V|_U) & \longrightarrow & T_{\widehat{V}}|_{\widehat{U}} & \xrightarrow{(d_{\widehat{V}}\omega_{\widehat{V}}^{\vee})^{\vee}} & F_{\widehat{V}}|_{\widehat{U}}, \end{array}$$

where the composition $N_{\widehat{V}/\widehat{W}}|_{\widehat{U}} \rightarrow \text{bl}(T_W|_U) \rightarrow T_{\widehat{W}}|_{\widehat{U}}$ comes from the induced splitting $T_{\widehat{W}}|_{\widehat{V}} = T_{\widehat{V}} \oplus N_{\widehat{V}/\widehat{W}}$ and the desired conclusion follows.

Let now S be an étale slice for \widehat{W} of a closed point $x \in \widehat{V} \xrightarrow{\Phi} \widehat{W}$ with stabilizer H and $T = \widehat{V} \cap S$, $R = \widehat{U} \cap T = \widehat{U} \cap S$. Then we have induced data Λ_T, Λ_S of a local model on T and S (cf. Lemma 5.3) respectively such that R is in standard form.

By the definition of ω_S, ω_T we see that the two horizontal compositions in the diagram

$$\begin{array}{ccccccc} T_S|_R & \longrightarrow & T_{\widehat{W}}|_R & \longrightarrow & F_{\widehat{W}}|_R & \longrightarrow & F_S|_R \xrightarrow{\rho_S} \mathcal{O}b_S \\ \uparrow & & \uparrow & & \downarrow \eta_{\Phi} & & \downarrow \eta_{\Psi} \\ T_T|_R & \longrightarrow & T_{\widehat{V}}|_R & \longrightarrow & F_{\widehat{V}}|_R & \longrightarrow & F_T|_R \xrightarrow{\rho_T} \mathcal{O}b_T \end{array}$$

equal $(d_S\omega_S^{\vee})^{\vee}$ and $(d_T\omega_T^{\vee})^{\vee}$. By Lemma 5.5 and an identical argument to Lemma 5.13, in order to show that η_{Ψ} maps the image $\text{im}(d_S\omega_S^{\vee})^{\vee}|_R$ into $\text{im}(d_T\omega_T^{\vee})^{\vee}|_R$ we may replace all bundles by their reduced analogues and get

a commutative diagram of sheaves

$$\begin{array}{ccccccc}
T_S|_R^{\text{red}} & \longrightarrow & T_{\widehat{W}}|_R^{\text{red}} & \longrightarrow & F_{\widehat{W}}|_R^{\text{red}} & \longrightarrow & F_S|_R^{\text{red}} \\
\uparrow & & \uparrow & & \downarrow \eta_\Phi & & \downarrow \eta_\Psi \\
T_T|_R^{\text{red}} & \longrightarrow & T_{\widehat{V}}|_R^{\text{red}} & \longrightarrow & F_{\widehat{V}}|_R^{\text{red}} & \longrightarrow & F_T|_R^{\text{red}}
\end{array}$$

where all horizontal arrows except those in the middle are isomorphisms. It follows immediately that we obtain an induced morphism $\Psi^{\text{ck}}: \mathcal{O}b_S \rightarrow \mathcal{O}b_T$. \square

Lemma 5.22. *Let Λ_V, Λ_W be as above. Then the induced sections $\omega_{\widehat{V}}$ and $\omega_{\widehat{W}}$ (cf. Lemma 5.3) are Ω -compatible.*

Proof. By Lemma 5.12, it readily follows that conditions (1) and (2) in the definition of Ω -compatibility hold for the respective Kirwan blowups.

It remains to check conditions (3). By Lemma 5.21, we see that $\eta_{\widehat{\Phi}}$ induces a morphism $\widehat{\Phi}^{\text{ck}}: \mathcal{O}b_{\widehat{W}} \rightarrow \mathcal{O}b_{\widehat{V}}$. Moreover, since $\eta_{\widehat{\Phi}}$ is surjective, $\widehat{\Phi}^{\text{ck}}$ is surjective. We need to check that it is an isomorphism.

$$\begin{array}{ccccccc}
(5.28) \quad T_{\widehat{W}}|_{\widehat{U}} & \xrightarrow{(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee} & F_{\widehat{W}}|_{\widehat{U}} & \xrightarrow{\rho_W} & \mathcal{O}b_{\widehat{W}} & \longrightarrow & 0 \\
& & \downarrow \eta_{\widehat{\Phi}} & & \downarrow \widehat{\Phi}^{\text{ck}} & & \\
T_{\widehat{V}}|_{\widehat{U}} & \xrightarrow{(d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee} & F_{\widehat{V}}|_{\widehat{U}} & \xrightarrow{\rho_V} & \mathcal{O}b_{\widehat{V}} & \longrightarrow & 0.
\end{array}$$

Consider a morphism $\Delta \rightarrow \widehat{U}$, where Δ is the spectrum of a local Artinian ring. We will show that $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\mathcal{O}b_{\widehat{V}}|_\Delta)$. Since length is additive in exact sequences, we have

$$(5.29) \quad \ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(F_{\widehat{W}}|_\Delta) - \ell(\ker \rho_W|_\Delta)$$

We have an exact sequence

$$T_{\widehat{W}}|_\Delta \xrightarrow{(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta} F_{\widehat{W}}|_\Delta \xrightarrow{\rho_W|_\Delta} \mathcal{O}b_{\widehat{W}}|_\Delta \longrightarrow 0.$$

Thus $\ker \rho_W|_\Delta = \text{im}(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta$ and it follows that

$$(5.30) \quad \ell(\ker \rho_W|_\Delta) = \ell(\text{im}(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta) = \ell(T_{\widehat{W}}|_\Delta) - \ell(\ker(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta)$$

Notice now that

$$(5.31) \quad \ker(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta = \text{coker}\left(d_{\widehat{W}}\omega_{\widehat{W}}^\vee|_\Delta\right)^\vee = \Omega_{\widehat{U}}|_\Delta^\vee$$

Therefore combining (5.29), (5.30) and (5.31) we get, since $F_{\widehat{W}}$ and $T_{\widehat{W}}$ have the same rank, $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\Omega_{\widehat{U}}|_\Delta^\vee)$.

An identical argument shows that $\ell(\mathcal{O}b_{\widehat{V}}|_\Delta) = \ell(\Omega_{\widehat{U}}|_\Delta^\vee)$.

Since $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\mathcal{O}b_{\widehat{V}}|_\Delta)$ for all such $\Delta \rightarrow \widehat{U}$ and $\widehat{\Phi}^{\text{ck}}$ is surjective, we conclude that $\widehat{\Phi}^{\text{ck}}$ is an isomorphism. \square

Remark 5.23. *It might seem counterintuitive that there is an induced map $\widehat{\Phi}^{\text{ck}}$, given that in the diagram (5.28) the derivative arrow $T_{\widehat{V}}|_{\widehat{U}} \rightarrow T_{\widehat{W}}|_{\widehat{U}}$ goes “the wrong way”. However, the fact that we begin with d -critical charts and then perform Kirwan blowups allows us to show that this is possible.*

The corresponding situation for taking étale slices also owes to the special properties of Kirwan blowups.

Moreover, the isomorphism property of $\widehat{\Phi}^{\text{ck}}$ depends crucially on the fact that the 2-term complexes in question induce perfect obstruction theories on \widehat{U} . This enables us to show that the kernels of $(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_\Delta$ and $(d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee|_\Delta$ are the same and hence obtain an equality of lengths.

Lemma 5.24. *Let Λ_V, Λ_W be as above. Let S be an étale slice for \widehat{W} of a closed point $x \in \widehat{V} \xrightarrow{\Phi} \widehat{W}$ with stabilizer H and $T = \widehat{V} \cap S$, $R = \widehat{U} \cap T = \widehat{U} \cap S$. Then we have induced data Λ_T, Λ_S of a local model on T and S (cf. Lemma 5.3) respectively such that R is in standard form and ω_T and ω_S are Ω -compatible.*

Proof. By Lemma 5.13, $\eta_\Psi(\omega_S|_T)$ and ω_T are Ω -equivalent, since $\eta_\Phi(\omega_{\widehat{W}}|_{\widehat{V}})$ and $\omega_{\widehat{V}}$ are. Thus, by Lemma 5.21, conditions (1) and (2) of the definition of Ω -compatibility hold. Condition (3) follows by an identical argument involving lengths as in the previous lemma. \square

The induced map Φ^{ck} is independent of the particular choice of embedding Φ .

Lemma 5.25. *Let $U \subset V \xrightarrow{\Phi, \Psi} W$ be G -equivariant embeddings of affine G -schemes such that U is in standard form for data Λ_V and Λ_W , ω_V and $\eta_\Phi(\omega_W|_V)$ are Ω -equivalent, ω_V and $\eta_\Psi(\omega_W|_V)$ are Ω -equivalent and the diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \Phi, \Psi \\ & & W \end{array}$$

commutes. If $\Phi^{\text{ck}} = \Psi^{\text{ck}}$, then the induced morphisms on the Kirwan blowups satisfy $\widehat{\Phi}^{\text{ck}} = \widehat{\Psi}^{\text{ck}}$. The same holds if we take slices of these blowups.

Furthermore, if Λ_V and Λ_W are data of d -critical charts for U , then we indeed have $\Phi^{\text{ck}} = \Psi^{\text{ck}}$ for any two choices of embedding $V \rightarrow W$.

Proof. It suffices to show that if $\eta_\Phi - \eta_\Psi$ maps $F_W|_U$ to $\text{im}(d_V\omega_V^\vee)^\vee|_U$ then the same is true for $\widehat{\Phi} - \widehat{\Psi}$. Let us denote $\alpha = \eta_\Phi - \eta_\Psi$ for brevity.

Since $\alpha|_U$ maps $F_W|_U$ to $\text{im}(d_V\omega_V^\vee)^\vee|_U$, it follows that α factors as

$$F_W|_U \xrightarrow{\beta} T_V|_U \xrightarrow{(d_V\omega_V^\vee)^\vee} F_V|_U.$$

Similarly to Lemma 5.21, we obtain a commutative diagram

$$\begin{array}{ccccc} F_{\widehat{W}}|_{\widehat{U}} & \xrightarrow{\widehat{\alpha}} & F_{\widehat{V}}|_{\widehat{U}} & \xrightarrow{\gamma} & \pi^*F_V|_{\widehat{U}} \\ & \searrow \widehat{\beta} & & & \uparrow \pi^*(d_V\omega_V^\vee)^\vee \\ & & T_{\widehat{V}}|_{\widehat{U}} & \xrightarrow{d\pi} & \pi^*T_V|_{\widehat{U}}. \end{array}$$

Hence $\gamma \circ \widehat{\alpha} = \pi^*(d_V\omega_V^\vee)^\vee \circ d\pi \circ \widehat{\beta} = \gamma \circ (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta}$ on \widehat{U} , from which we deduce that $\gamma \circ (\widehat{\alpha} - (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta}) \in \pi^*I_U \cdot \pi^*F_V$, which in turn yields

that $\widehat{\alpha} - (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta} \in I_{\widehat{U}} \cdot F_{\widehat{V}}$ and therefore $\widehat{\alpha}|_{\widehat{U}} = (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee|_{\widehat{U}} \circ \widehat{\beta}|_{\widehat{U}}$, which implies what we want.

The proof for slices proceeds along the same lines of Lemma 5.13 and Lemma 5.21 and we omit it.

Finally if the two sets of data give d-critical charts of U , we have $\Phi^* - \Psi^*: \mathcal{O}_W \rightarrow \text{im } \omega_V^\vee \subset \mathcal{O}_V$ for the map on coordinate rings and therefore by symmetry $\eta_\Phi - \eta_\Psi: \Omega_W|_U \rightarrow \text{im } (d_V\omega_V^\vee)|_U = \text{im } (d_V\omega_V^\vee)^\vee|_U \subset \Omega_V|_U$, which implies the desired equality $\Phi^{\text{ck}} = \Psi^{\text{ck}}$. \square

The following lemma asserts that taking a slice gives compatible reduced obstruction sheaves and assignments and concludes this section.

Lemma 5.26. *Let $(U, V, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model on V . Let $\Phi: S \rightarrow V$ be an étale slice of a closed point of V with closed G -orbit and stabilizer H and $(T, S, F_S, \omega_S, D_S, \phi_S)$ be the induced data on S .*

Consider the diagram

$$(5.32) \quad \begin{array}{ccccccc} K_V|_T & = & [\mathfrak{g} & \longrightarrow & T_V|_T & \xrightarrow{(d_V\omega_V^\vee)^\vee} & F_V|_T \longrightarrow \mathfrak{g}^\vee(-D_S)] \\ & & \uparrow & & \uparrow & & \downarrow \\ K_S & = & [\mathfrak{h} & \longrightarrow & T_S|_T & \xrightarrow{(d_S\omega_S^\vee)^\vee} & F_S|_T \longrightarrow \mathfrak{h}^\vee(-D_S)] \end{array}$$

The surjection $\eta_\Phi: F_V|_S \rightarrow F_S$ induces an isomorphism $\Phi^b: H^1(K_V|_T) \rightarrow H^1(K_S)$ which restricts to an isomorphism $\mathcal{O}b_V^{\text{red}}|_{T^s} \rightarrow \mathcal{O}b_S^{\text{red}}$, via which the dual complexes $(K_V^{\text{red}}|_T)^\vee, (K_S^{\text{red}})^\vee$ give the same obstruction assignments on T .

Proof. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & T_S|_T & \longrightarrow & T_V|_T & \longrightarrow & \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \end{array}$$

inducing an isomorphism $T_S|_T^{\text{red}} \rightarrow T_V|_T^{\text{red}}$.

By the definition of F_S , we have a diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee(-D_T) & \longrightarrow & F_V|_T & \longrightarrow & F_S|_T \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^\vee(-D_T) & \longrightarrow & \mathfrak{g}^\vee(-D_T) & \longrightarrow & \mathfrak{h}^\vee(-D_T) \longrightarrow 0 \end{array}$$

and similarly we deduce that $F_V|_T^{\text{red}} \rightarrow F_S|_T^{\text{red}}$ is an isomorphism. Then the central square of (5.32) factors through the commutative diagram

$$\begin{array}{ccc} T_V|_T^{\text{red}} & \longrightarrow & F_V|_T^{\text{red}} \\ \uparrow & & \downarrow \\ T_S|_T^{\text{red}} & \longrightarrow & F_S|_T^{\text{red}}, \end{array}$$

where the vertical arrows are isomorphisms. We obtain an induced isomorphism $H^1(K_V|_T) \rightarrow H^1(K_S)$ restricting to an isomorphism $\mathcal{O}b_V^{\text{red}}|_{T^s} \rightarrow \mathcal{O}b_S^{\text{red}}$ by (5.7) and the obstruction assignments of the reduced complexes must match by standard arguments as in the proof of Proposition 5.20. \square

Remark 5.27. *All of the above results on Ω -compatibility are true if one replaces locally closed embeddings $\Phi: V \rightarrow W$ by unramified morphisms and Zariski open embeddings by étale maps. This is because for our purposes it suffices to work étale locally and then Zariski open maps correspond to étale maps and locally closed embeddings to unramified morphisms. We will tacitly use this observation in the following sections of the paper.*

One may alternatively choose to work in the complex analytic topology where everything works verbatim.

6. GENERALIZED DONALDSON-THOMAS INVARIANTS

This section leads up to the main result of the paper, the construction of the generalized DT invariant via Kirwan blowups.

6.1. Obstruction theory of Kirwan partial desingularization of equivariant d-critical loci. As in Section 2, let $\mathcal{M} = [X/G]$ be a quotient stack obtained by GIT. We have therefore equivariant embeddings $X \subset P \subset (\mathbb{P}^N)^{ss}$ with G acting on \mathbb{P}^N via a homomorphism $G \rightarrow \text{GL}(N+1)$.

Suppose also that \mathcal{M} admits a d-critical structure $s \in \Gamma(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$ so that (\mathcal{M}, s) is a d-critical stack. By Remark 3.11 this is equivalent to a G -invariant d-critical structure on X .

In this section, we carefully follow the steps of Section 2 tailored to the case of a d-critical locus to show that the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ admits a semi-perfect obstruction theory.

Let $\mathfrak{R}(X) = \{R_1, \dots, R_m, \{1\}\}$ in order of decreasing dimension.

Let $x \in X$ be a closed point with closed G -orbit, fixed by R_1 . Since the action of G is good, we have a G -invariant affine open $x \in U \subset X$. Therefore, we may apply the étale slice theorem [Dré04, Theorem 5.3] to get a locally closed affine $x \in T \subset U$ such that $[T/R_1] \rightarrow [U/G] \subset [X/G]$ is étale. We thus obtain an étale cover

$$\coprod [T_{\alpha}^1/R_1] \coprod [(X - GZ_{R_1})/G] \longrightarrow [X/G] = \mathcal{M},$$

where each T_{α}^1 is an R_1 -invariant d-critical locus. In particular, for each α there exist data of a local model $(T_{\alpha}^1, S_{\alpha}^1, \Omega_{S_{\alpha}^1}^1, df_{\alpha}, 0, \sigma_{S_{\alpha}^1})$ with $T_{\alpha}^1 = (df_{\alpha} = 0) \subset S_{\alpha}^1$ in standard form and we may take S_{α}^1 to be affine, locally closed in P .

Let $X_1 = \widehat{X}$ be the Kirwan blowup of X_1 associated with R_1 and set $X_1^{\circ} = X - GZ_{R_1}$. Then we obtain an induced étale cover

$$\coprod [\widehat{T}_{\alpha}^1/R_1] \coprod [X_1^{\circ}/G] \longrightarrow [X_1/G] = \mathcal{M}_1 \subset \mathcal{P}_1 := [P_1/G].$$

For each α , we have induced data $(\widehat{T}_{\alpha}^1, \widehat{S}_{\alpha}^1, F_{\widehat{S}_{\alpha}^1}, \omega_{\widehat{S}_{\alpha}^1}, D_{\widehat{S}_{\alpha}^1}, \phi_{\widehat{S}_{\alpha}^1})$ with $\widehat{T}_{\alpha}^1 = (\omega_{\widehat{S}_{\alpha}^1} = 0) \subset \widehat{S}_{\alpha}^1 \subset \widehat{P} =: P_1$. Note also that $[X_1^{\circ}/G] \longrightarrow \mathcal{M}_1$ factors through \mathcal{M} as an open immersion.

Now $R_2 \in \mathfrak{R}(X_1)$ is of maximal dimension. Let $x \in X_1$ be a closed point with closed G -orbit, fixed by R_2 . Then its orbit will lie in X_1° or it will be contained in the image of some $[\widehat{T}_\alpha^1/R_1]$. Applying the same reasoning, we get an induced étale cover

$$\coprod [\widehat{T}_\beta^2/R_2] \coprod [(\widehat{T}_\alpha^1)^\circ/R_1] \coprod [X_2^\circ/G] \longrightarrow [X_2/G] = \mathcal{M}_2 \subset \mathcal{P}_2 := [P_2/G],$$

where $(\widehat{T}_\alpha^1)^\circ = \widehat{T}_\alpha^1 - \widehat{T}_\alpha^1 \cap GZ_{R_2}$, $X_2^\circ = X_1^\circ - GZ_{R_2}$ and the various $T_\beta^2 \subset S_\beta^2$ are étale slices of $G \times_{R_1} \widehat{T}_\alpha^1$ or X_1° in standard form, where we may take S_β^2 to be slices in $P_2 = \widehat{P}_1$.

Continuing inductively, we have for any n an étale cover

$$\coprod [\widehat{T}_\gamma^n/R_n] \coprod \cdots \coprod [(\widehat{T}_\alpha^1)^\circ/R_1] \coprod [X_n^\circ/G] \rightarrow \mathcal{M}_n \subset \mathcal{P}_n := [P_n/G],$$

where by abuse of notation we write

$$(\widehat{T}_\alpha^i)^\circ = \widehat{T}_\alpha^i - GZ_{R_{i+1}} \cap \widehat{T}_\alpha^i - \cdots - GZ_{R_n} \cap \widehat{T}_\alpha^i$$

and so on, and the T_α^n are appropriate slices of the elements of the étale cover for \mathcal{M}_{n-1} . Note also that for each i , $[(\widehat{T}_\alpha^i)^\circ/R_i]$ factors through an étale morphism to \mathcal{M}_i .

We see that $\widetilde{\mathcal{M}} = \mathcal{M}_m = [X_m/G] \subset [P_m/G]$, a DM stack, is the Kirwan partial desingularization of \mathcal{M} . We formalize the above procedure in the following proposition, where we also keep track of obstruction sheaves and their comparison data.

Lemma 6.1. *For each $n \geq 0$, let $\mathcal{E}_n^X, \mathcal{E}_n^P$ be the union of all exceptional divisors in \mathcal{M}_n and \mathcal{P}_n respectively. There exist collections of étale morphisms $[T_\alpha/R_\alpha] \rightarrow \mathcal{M}_n$ and $[S_\alpha/R_\alpha] \rightarrow \mathcal{P}_n$ such that:*

- (1) *For each α , $R_\alpha \in \{R_1, \dots, R_n\}$.*
- (2) *Each T_α is in standard form for data $(T_\alpha, S_\alpha, F_{S_\alpha}, \omega_{S_\alpha}, D_{S_\alpha}, \phi_{S_\alpha})$ of a local model on a smooth affine R_α -scheme $S_\alpha \subset P_n$.*
- (3) *The collections cover \mathcal{E}_n^X and \mathcal{E}_n^P respectively.*
- (4) *The identity components of stabilizers that occur in \mathcal{M}_n lie, up to conjugacy, in the set $\{R_{n+1}, \dots, R_m\}$.*
- (5) *The data $(T_\alpha, S_\alpha, F_{S_\alpha}, \omega_{S_\alpha}, D_{S_\alpha}, \phi_{S_\alpha})$ restricted to the complement of $\mathcal{E}_n^P \subset \mathcal{P}_n$ are the same as those of a d -critical chart on T_α .*
- (6) *For α, β and $q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta]$ whose stabilizer has identity component conjugate to R_q , there exist an affine $T_{\alpha\beta}$ in standard form for data $(T_{\alpha\beta}, S_{\alpha\beta}, F_{S_{\alpha\beta}}, \omega_{S_{\alpha\beta}}, D_{S_{\alpha\beta}}, \phi_{S_{\alpha\beta}})$ of a local model and an equivariant commutative diagram*

$$(6.1) \quad \begin{array}{ccccc} & T_{\alpha\beta} & \longrightarrow & S_{\alpha\beta} & \longleftarrow & T_{\alpha\beta} \\ & \swarrow i_\alpha & & \swarrow \theta_\alpha & & \searrow \theta_\beta \\ T_\alpha & \longrightarrow & S_\alpha & & S_\beta & \longleftarrow & T_\beta \\ & \searrow & \searrow & & \swarrow & \swarrow & \swarrow \\ & & \mathcal{M}_n & \longrightarrow & \mathcal{P}_n & \longleftarrow & \mathcal{M}_n \end{array}$$

inducing a commutative diagram consisting of étale maps on the corresponding quotient stacks for arrows pointing downwards. $T_{\alpha\beta}, S_{\alpha\beta}$

are étale slices for both T_α, T_β and S_α, S_β respectively. All horizontal arrows are embeddings and $\theta_\alpha, \theta_\beta$ are unramified and R_q -equivariant. $\eta_{\theta_\alpha}(\omega_{S_\alpha})$ and $\eta_{\theta_\beta}(\omega_{S_\beta})$ are Ω -equivalent to $\omega_{\alpha\beta}$.

(7) For each index α , consider the 4-term complex

$$K_\alpha = [\mathfrak{r}_\alpha \longrightarrow T_{S_\alpha}|_{T_\alpha} \xrightarrow{(d\omega_{S_\alpha}^\vee)^\vee} F_{S_\alpha}|_{T_\alpha} \xrightarrow{\phi_{S_\alpha}} \mathfrak{r}_\alpha^\vee(-D_{S_\alpha})],$$

where by convention we place $F_{S_\alpha}|_{T_\alpha}$ in degree 1. θ_α induces an isomorphism $\theta_\alpha^b: \mathcal{O}b_{S_\alpha}^{\text{red}}|_{T_{\alpha\beta}^s} \rightarrow \mathcal{O}b_{S_{\alpha\beta}}^{\text{red}}$. This does not change if we replace ω_{S_α} by any Ω -equivalent section. Analogous statements are true for the index β .

(8) We obtain comparison isomorphisms

$$\theta_{\alpha\beta}^b := (\theta_\beta^b)^{-1} \theta_\alpha^b: \mathcal{O}b_{S_\alpha}^{\text{red}}|_{T_{\alpha\beta}^s} \rightarrow \mathcal{O}b_{S_\beta}^{\text{red}}|_{T_{\alpha\beta}^s}.$$

These give the same obstruction assignments for the complexes K_α^{red} , K_β^{red} on $T_{\alpha\beta}^s$.

(9) Let now q be a point in the triple intersection

$$q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta] \times_{\mathcal{M}_n} [T_\gamma/R_\gamma]$$

with stabilizer in class R_q . Then we have $S_{\alpha\beta}$ as in (6) for the indices α, β , $S_{\beta\gamma}$ as in (6) for the indices β, γ and $S_{\gamma\alpha}$ for the indices γ, α with a common R_q -invariant étale neighbourhood $S_{\alpha\beta\gamma}$ fitting on top of the diagrams of the form (6.1). The descents of $\theta_{\alpha\beta}^b, \theta_{\beta\gamma}^b, \theta_{\gamma\alpha}^b$ to $[T_{\alpha\beta\gamma}^s/R_q]$ satisfy the cocycle condition.

Proof. For $n = 0$ there is nothing to show, as we may take an empty set of étale morphisms. We proceed by induction. Suppose the claim is true for n .

Consider the locus of closed points $x \in \mathcal{M}_n$ whose stabilizer has identity component conjugate to R_{n+1} . Then, either x is in the image of some $[T_\alpha/R_\alpha]$ or not.

Let's examine the first case. Take $[T_\alpha/R_\alpha]$ and consider all such closed points $x \in [T_\alpha/R_\alpha]$. For each x , let $[T_\alpha^x/R_{n+1}] \rightarrow [T_\alpha/R_\alpha]$ be induced by an étale slice $[S_\alpha^x/R_{n+1}] \rightarrow [S_\alpha/R_\alpha]$. Then we consider the collection of étale maps $[\widehat{T}_\alpha^x/R_{n+1}] \rightarrow \mathcal{M}_{n+1}$ together with $[T_\alpha^\circ/R_\alpha] \rightarrow \mathcal{M}_{n+1}$ where T_α° is the complement in T_α of the locus of closed points with stabilizer whose identity component is conjugate to R_{n+1} . We may repeat this process for all $[T_\alpha/R_\alpha] \rightarrow \mathcal{M}_n$ and x . We say that these étale maps are of type I.

So, consider $x \in \mathcal{M}_n$ such that x does not lie in the image of any $[T_\alpha/R_\alpha]$. In particular, by (3) x does not lie in \mathcal{E}_n^X and thus we may assume that it is a closed point of $[X/G]$. Then there exist slices $T \subset S \subset P$ such that T is in standard form for data $(T, S, \Omega_S, df_S, 0, \sigma_S)$ of a local model on S and we have étale morphisms $[T/R_{n+1}] \rightarrow \mathcal{M}_n$, $[S/R_{n+1}] \rightarrow \mathcal{P}_n$. Then, we may take étale maps $[\widehat{T}/R_{n+1}] \rightarrow \mathcal{M}_{n+1}$, $[\widehat{S}/R_{n+1}] \rightarrow \mathcal{P}_{n+1}$. We may repeat this process for all such x . We say that these maps are of type II.

We have thus produced a collection of étale morphisms for \mathcal{M}_{n+1} and \mathcal{P}_{n+1} . It is clear by our choice and the inductive hypothesis that (1)-(5) are automatically satisfied by Lemma 5.3 and the properties of Kirwan blowups.

To check (6), the fact that the maps are unramified follows from the slice property since the derivatives are injective around the point of interest. Furthermore, since by (5) the restriction of a map of type I to the complement of \mathcal{E}_n^X clearly yields rise to a map of type II for \mathcal{M}_{n+1} it suffices to produce comparison data for maps of the same type.

Now, for two maps of type I, we may assume that $S_{\alpha\beta}$ in (6.1) factors through S_α^x and S_β^y and therefore we may use $\widehat{\theta}_\alpha$ and $\widehat{\theta}_\beta$ to get comparison data, which satisfy the requirements, since Ω -equivalence is preserved by Kirwan blowups and taking slices by Lemma 5.12 and Lemma 5.13.

For two maps of type II, coming from two choices of $[T/R_{n+1}] \rightarrow \mathcal{M}$ and $[S/R_{n+1}] \rightarrow \mathcal{P}$, we may find (up to shrinking), as in the proof of Proposition 2.10 and using the properties of d-critical loci, a common étale refinement $T_{\alpha\beta} \rightarrow T_\alpha$, $T_{\alpha\beta} \rightarrow T_\beta$ with $T_{\alpha\beta} = (\omega_{\alpha\beta} = df_{\alpha\beta} = 0) \subset S_{\alpha\beta}$ and commutative comparison diagrams

$$(6.2) \quad \begin{array}{ccc} T_{\alpha\beta} & \longrightarrow & S_{\alpha\beta} \\ \downarrow & & \downarrow \theta_\alpha \\ T_\alpha & \longrightarrow & S_\alpha \\ & & \downarrow f_\alpha \\ & & \mathbb{A}^1 \end{array}$$

fitting in a diagram of the form (6.1) for $\mathcal{M}_0 := \mathcal{M}$ and $\mathcal{P}_0 := \mathcal{P}$, such that $\theta_\alpha, \theta_\beta$ are étale and $df_\alpha|_{S_{\alpha\beta}}, df_\beta|_{S_{\alpha\beta}}$ are Ω -equivalent to $df_{\alpha\beta}$. Since Ω -equivalence is preserved for Kirwan blowups and taking slices, we see that (6) is satisfied in this case as well and exceptional divisors pull back to exceptional divisors (cf. Lemma 5.3).

Finally (7) and (8) are an immediate consequence of the inductive hypothesis and Lemma 5.26 of the preceding section. The existence of $S_{\alpha\beta\gamma}$ in (9) can be seen, using the inductive hypothesis for maps of type I, and the d-critical structure for maps of type II. The cocycle condition follows from the commutativity of the diagrams of quotient stacks induced by (6.1) and the fact that by construction θ_α^b is induced by pullback, is functorial with respect to compositions and both the reduced obstruction sheaves $\mathcal{O}b_{S_\alpha}^{\text{red}}$ and the morphisms θ_α descend to the level of quotient stacks. We note here that all the results of Section 6 hold true if one replaces embeddings by unramified morphisms, so they apply to the present situation as well (cf. Remark 5.27). \square

We are now in position to show that $\widetilde{\mathcal{M}} = \mathcal{M}_m$ is equipped with a semi-perfect obstruction theory of dimension 0. Let $[T_\alpha/R_\alpha] \rightarrow \widetilde{\mathcal{M}}$ be the cover granted by the previous proposition. Each $[T_\alpha/R_\alpha]$ is a DM stack and together they cover the strictly semistable locus of \mathcal{M} . Then for any $x \in \mathcal{M}^s$, as in the proof of the preceding proposition, we have an étale map $T_x \rightarrow \mathcal{M}$, where T_x is a d-critical locus with $T_x \subset S_x \rightarrow P$. The stable locus \mathcal{M}^s is not affected during the partial desingularization procedure and so \mathcal{M}^s is an open substack of $\widetilde{\mathcal{M}}$ and we get étale maps $T_x \rightarrow \widetilde{\mathcal{M}}$. We obtain an étale

surjective cover

$$\coprod [T_\alpha/R_\alpha] \coprod T_x \longrightarrow \widetilde{\mathcal{M}}.$$

It is easy to check that the obstruction sheaves and assignments of the T_x are compatible with those of the cover $[T_\alpha/R_\alpha] \rightarrow \widetilde{\mathcal{M}}$ by an identical argument as in the above proof. We thus see that the reduced obstruction theories indeed give a semi-perfect obstruction theory of dimension 0 on $\widetilde{\mathcal{M}}$.

We have proved our main theorem.

Theorem 6.2. *Let $\mathcal{M} = [X/G]$, where X is a quasi-projective scheme, which is the semistable part of a projective scheme X^\dagger with a linearized G -action. Suppose also that \mathcal{M} admits a d -critical structure $s \in \Gamma(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$ so that (\mathcal{M}, s) is a d -critical stack. Then the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ is a proper DM stack endowed with a canonical semi-perfect obstruction theory of dimension zero induced by s , giving rise to a 0-dimensional virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\text{vir}}$.*

Remark 6.3. *Since at any point of our construction, we are working with stabilizers of closed points and slices thereof, $\widetilde{\mathcal{M}}$ and its obstruction theory are independent of the particular presentation of \mathcal{M} as a quotient stack $[X/G]$.*

Remark 6.4. *If we replace the d -critical structure s by $s' = c \cdot s$, where $c \in \mathbb{C}^*$, the virtual cycle stays the same. This is because rescaling the d -critical structure is equivalent to replacing every d -critical chart (U, V, f, i) of X by (U, V, cf, i) and this does not affect the intrinsic normal cone of the obstruction theory of $\widetilde{\mathcal{M}}$.*

6.2. Generalized DT invariants via Kirwan blowups. Let \mathcal{M} be the moduli space of Gieseker semistable sheaves with fixed Chern character γ on a Calabi-Yau threefold W . Let $\kappa: \mathcal{O}_W \rightarrow K_W$ be a trivialization of K_W .

The space \mathcal{M} is the truncation of a derived Artin stack \mathbf{M} . Moreover, by [PTVV13] κ induces a (-1) -shifted symplectic structure on \mathbf{M} and therefore by [BBBBJ15] a d -critical structure on \mathcal{M} . Since $\mathcal{M} = [X/G]$ is obtained by GIT (cf. [HL10, Section 4]), the conditions of Theorem 6.2 hold and we have the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ and its virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\text{vir}}$.

Since choosing a different trivialization amounts to rescaling κ and subsequently the d -critical structure, by Remarks 6.3 and 6.4, the virtual cycle does not depend on the choice of κ or presentation as a quotient stack.

Theorem-Definition 6.5. (Generalized DT invariant) *We define the generalized DT invariant via Kirwan blowups of Chern character γ to be*

$$\text{DT}_\gamma(W) := \deg[\widetilde{\mathcal{M}}]^{\text{vir}}.$$

7. RELATIVE THEORY AND DEFORMATION INVARIANCE

In this section, we show that Kirwan blowups behave well over a smooth base curve C and use this fact together with the relative versions of the results of the paper, obtained via derived symplectic geometry, to conclude

that the generalized Donaldson-Thomas invariant via Kirwan blowup is invariant under deformations of the complex structure of the Calabi-Yau three-fold W .

7.1. Kirwan blowups in families. Kirwan blowups behave well in families over a smooth curve.

Lemma 7.1. *Let G be a reductive group; let V be a smooth G -scheme, C a smooth curve and $\pi: V \rightarrow C$ a smooth G -equivariant morphism, where G acts on C trivially. Then V^G is smooth over C .*

Proof. This is a standard fact as we are considering over \mathbb{C} . First, as V is smooth, V^G is smooth. To prove that V^G is smooth over C , we need to show that for any closed $x \in V^G$, the projection $d\pi(x): T_x V^G \rightarrow T_{\pi(x)} C$ is surjective.

Indeed, as both V and V^G are smooth, $T_x V^G = (T_x V)^G$. Let $v \in T_x V$ so that $d\pi(x)(v) \neq 0$. Applying the Reynolds operator \mathcal{R} , we get that the G -invariant part of v , namely $\mathcal{R}(v) \in (T_x V)^G$, has $d\pi(x)(\mathcal{R}(v)) = d\pi(x)(v)$, thus $d\pi(x): T_x V^G \rightarrow T_{\pi(x)} C$ is surjective. \square

The same proof gives the following result on étale slices.

Lemma 7.2. *Let $\pi: V \rightarrow C$ be as in Lemma 7.1. If x is a closed point in V with reductive stabilizer H and S is an étale slice for x , then $\pi: S \rightarrow C$ is smooth.*

Corollary 7.3. *Let $\pi: V \rightarrow C$ be as in Lemma 7.1. Then for any point $c \in C$ we have a canonical isomorphism $(\widehat{V})_c \cong \widehat{V}_c$.*

Proof. This follows immediately from the fact that V^G is smooth over C . \square

We obtain the following result on intrinsic partial desingularizations.

Proposition 7.4. *Let $X = (X^\dagger)^{ss}$ with $X \subset P \subset \mathbb{P}^N \times C$ be closed G -schemes as before (cf. the beginning of Subsection 2.4) except where C is a smooth curve and G acts on \mathbb{P}^N via a homomorphism $G \rightarrow GL(N+1)$. Let \widetilde{X} be the Kirwan partial desingularization of X . Then for any closed $c \in C$, $(\widetilde{X})_c = \widetilde{X}_c$.*

Proof. Let \widehat{X} be the Kirwan blowup of X with respect to G . By the construction of Kirwan partial desingularization, the lemma follows from that $(\widehat{X})_c = \widehat{X}_c$.

We are considering the case where X comes with an equivariant C -embedding $X \subset V$ where $V \rightarrow C$ is smooth. Let $I \subset \mathcal{O}_V$ be the ideal sheaf of $X \subset V$. Then, applying the Kirwan blowup $\pi: \widehat{X} \rightarrow X$ we have a short exact sequence

$$0 \longrightarrow I^{intr} \longrightarrow \mathcal{O}_{\widehat{V}} \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow 0.$$

Let \mathbb{C}_c be the residue field at $c \in C$. We then have

$$I^{intr} \otimes_{\mathcal{O}_C} \mathbb{C}_c \longrightarrow \mathcal{O}_{(\widehat{V})_c} \longrightarrow \mathcal{O}_{(\widehat{X})_c} \longrightarrow 0.$$

This fits into a diagram of exact sequences

$$\begin{array}{ccccccc} I^{intr} \otimes_{\mathcal{O}_C} \mathbb{C}_c & \longrightarrow & \mathcal{O}_{(\widehat{V})_c} & \longrightarrow & \mathcal{O}_{(\widehat{X})_c} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (I \otimes_{\mathcal{O}_C} \mathbb{C}_c)^{intr} & \longrightarrow & \mathcal{O}_{\widehat{V}_c} & \longrightarrow & \mathcal{O}_{\widehat{X}_c} & \longrightarrow & 0. \end{array}$$

By the preceding corollary, the middle arrow is an isomorphism. Moreover, if $I = I^{fix} \oplus I^{mv}$ is the decomposition of I into its fixed and moving parts, since G is reductive,

$$I \otimes_{\mathcal{O}_C} \mathbb{C}_c = (I^{fix} \otimes_{\mathcal{O}_C} \mathbb{C}_c) \oplus (I^{mv} \otimes_{\mathcal{O}_C} \mathbb{C}_c)$$

is the decomposition into fixed and moving parts since the action of G on C is trivial. We conclude that the leftmost horizontal arrows have the same image under the identification $\mathcal{O}_{(\widehat{V})_c} \simeq \mathcal{O}_{\widehat{V}_c}$ and thus we have a natural isomorphism $(\widehat{X})_c \simeq \widehat{X}_c$. \square

7.2. Derived symplectic geometry: local picture. We assume that $C = \text{Spec } S$ is a smooth affine curve over \mathbb{C} . In what follows, we consider commutative differential graded algebras (A^\bullet, δ) (cdga's) over S , which are negatively graded. Also, whenever we refer to a reductive group G , we assume that it acts trivially on C . Results and properties of derived affine schemes that are quoted in this subsection can be found in the exposition of [BBJ13, Sections 2,3].

Definition 7.5. (Standard form cdga) *We say that a cdga (A^\bullet, δ) is of standard form if A^0 is a smooth S -algebra and, as a graded algebra, it is freely generated over A^0 by finitely many generators in each negative degree (i.e. it is quasifree).*

Now, a cdga (A^\bullet, δ) gives rise to the affine derived scheme $\mathbf{Spec } A^\bullet$ with underlying classical truncation the affine scheme $\text{Spec } A^\bullet = \text{Spec } H^0(A^\bullet)$. $\mathbf{Spec } A^\bullet$ and $\text{Spec } H^0(A^\bullet)$ have the same underlying topological space.

If A^\bullet is of standard form, then its derived cotangent complex \mathbb{L}_{A^\bullet} is represented by the Kähler differentials together with the internal differential $(\Omega_{A^\bullet}, \delta)$. We also have the usual de Rham differential d on Ω_{A^\bullet} , so that we obtain a mixed complex. Moreover, $\mathbb{L}_{A^\bullet} \otimes H^0(A^\bullet)$ is a complex of free $H^0(A^\bullet)$ -modules.

Definition 7.6. (Minimality) *Let (A^\bullet, δ) be of standard form and $x \in \mathbf{Spec } A^\bullet$. We say that A^\bullet is minimal at x if all the differentials in $\mathbb{L}_{A^\bullet}|_x$ are zero.*

If G is a reductive group acting on (A^\bullet, δ) (and by our convention trivially on C), then we have analogous equivariant statements. We will consider minimality at x only when x is a fixed point of G .

We proceed to give a brief account of (-1) -shifted symplectic forms on $\mathbf{Spec } A^\bullet$, introduced in [PTVV13].

Definition 7.7. ((-1) -shifted symplectic form) *We say that $\omega = (\omega_0, \omega_1, \dots)$ is a (-1) -shifted symplectic form on $\mathbf{Spec } A^\bullet$ if $\omega_i \in (\wedge^{2+i} \Omega_{A^\bullet})^{-1-i}$ such that*

- (1) ω_0 gives a quasi-isomorphism $\mathbb{L}_{A^\bullet} \rightarrow \mathbb{T}_{A^\bullet}[1]$,
- (2) $\delta\omega_0 = 0$ and $d\omega_i + \delta\omega_{i+1} = 0$ for $i \geq 0$.

We refer to condition (1) as the non-degeneracy property and condition (2) as the closedness property.

Two forms ω, ω' are equivalent if there exist $\alpha_i \in (\wedge^{2+i}\Omega_{A^\bullet})^{-2-i}$ such that $\omega_0 - \omega'_0 = \delta\alpha_0$ and for all $i \geq 0$, $\omega_{i+1} - \omega'_{i+1} = d\alpha_i + \delta\alpha_{i+1}$.

A (-1) -shifted symplectic form can be defined on a derived stack by smooth descent. Suppose now that the derived quotient stack $[\mathbf{Spec} A^\bullet/G]$ has a (-1) -shifted symplectic form ω , where A^\bullet is in standard form and minimal at a fixed point x . Then ω_0 induces a quasi-isomorphism

$$\mathbb{L}_{[\mathbf{Spec} A^\bullet/G]}|_{H^0(A^\bullet)} \rightarrow \mathbb{T}_{[\mathbf{Spec} A^\bullet/G]}|_{H^0(A^\bullet)}[1].$$

This implies that \mathbb{L}_{A^\bullet} must have Tor-amplitude $[-2, 0]$ and therefore A^\bullet is freely generated over A^0 in degrees -1 and -2 by generators $y_j \in A^{-1}$ and $w_k \in A^{-2}$ respectively. We may write the above quasi-isomorphism as the following equivariant morphism of complexes

$$(7.1) \quad \begin{array}{ccccccc} V^{-2} & \longrightarrow & V^{-1} & \longrightarrow & \Omega_{A^0} & \longrightarrow & \mathfrak{g}^\vee \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & T_{A^0} & \longrightarrow & (V^{-1})^\vee & \longrightarrow & (V^{-2})^\vee. \end{array}$$

Since A^\bullet is minimal at the fixed point x , we may localize G -invariantly around x and assume that the vertical arrows are isomorphisms. In particular, we obtain an isomorphism $V^{-1} \simeq T_{A^0}$. Let x_i be a set of étale coordinates for A^0 . We may now choose generators $y_i \in A^{-1}$ such that $dy_i \in V^{-1}$ are the dual basis to $dx_i \in \Omega_{A^0}$ under this isomorphism. We may thus identify A^{-1} and T_{A^0} as A^0 -modules. Therefore the differential $\delta: T_{A^0} \rightarrow A^0$ induces an invariant section ω of Ω_{A^0} whose zero locus is precisely $\mathbf{Spec} H^0(A^\bullet)$.

Definition 7.8. We say that a cdga (A^\bullet, δ) with a G -action, minimal at a fixed point x of G , is special if it is freely generated over A^0 in degrees -1 and -2 by generators y_i and w_j respectively, together with an identification $A^{-1} = V^{-1}$ (where V^{-1} is as in (7.1)) mapping y_i to dy_i .

We denote $U = \mathbf{Spec} H^0(A^\bullet)$ and $V = \mathbf{Spec} A^0$.

Every finitely presented affine derived scheme is (up to Zariski shrinking) equivalent to $\mathbf{Spec} A^\bullet$ with A^\bullet a standard form cdga and this is also true in the equivariant setting around a fixed point x of G . For more details, we refer to [BBJ13, Theorem 4.1] and its proof, which is valid in the equivariant setting as well. We deduce the following proposition.

Proposition 7.9. Let U be an affine G -scheme over C , which is the classical scheme associated to a derived affine G -scheme \mathbf{U} such that the stack $[\mathbf{U}/G]$ is (-1) -shifted symplectic with form ω . Moreover, let $x \in U$ be a fixed point of G . Then, up to equivariant Zariski shrinking, \mathbf{U} is equivalent to $\mathbf{Spec} A^\bullet$, where (A^\bullet, δ) is a special cdga, minimal at x .

In particular, there exists a smooth affine G -scheme $V \rightarrow C$, a G -equivariant embedding $U \rightarrow V$ over C minimal at x and an induced invariant 1-form $\omega \in H^0(\Omega_{V/C})$ such that $U = Z(\omega) \subset V$ is the zero locus of ω .

We can use the above to understand the local structure of quotient stacks that arise as the truncation of (-1) -shifted symplectic derived stacks. The following proposition can be deduced by work of Halpern-Leistner² [HL].

Proposition 7.10. *Let $\mathcal{M} \rightarrow C$ be a (-1) -shifted symplectic derived stack whose truncation $\mathcal{M} = [X/G] \rightarrow C$ is a quotient stack such that the action of G on X is good (and trivial on C). Let $x \in \mathcal{M}$ be a closed point with reductive stabilizer R . Then there exists an étale morphism $\mathbf{f}: [\mathbf{T}/R] \rightarrow \mathcal{M}$, a point $t \in [\mathbf{T}/R]$ fixed by R , where \mathbf{T} is equivalent to $\mathbf{Spec} A^\bullet$ with (A^\bullet, δ) a special cdga, minimal at t , mapping t to x and inducing the inclusion $R \subset G$ on stabilizer groups.*

At the classical level, we get an étale morphism $f: [T/R] \rightarrow \mathcal{M}$. There exists a smooth affine R -scheme $S \rightarrow C$, a G -equivariant embedding $T \rightarrow S$ over C minimal at t and an induced invariant 1-form $\omega \in H^0(\Omega_{S/C})$ such that $T = Z(\omega) \subset S$ is the zero locus of ω .

Proof. Let $x \in U$ be a G -invariant affine open in X (such exists since the G -action on X is good). Since U is affine and G is reductive, by [HL, Lemma 2.4], there exists an affine derived G -scheme $\mathbf{U} = \mathbf{Spec} B^\bullet$ such that we have a fiber diagram

$$\begin{array}{ccc} [U/G] & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ [\mathbf{U}/G] & \longrightarrow & \mathcal{M}. \end{array}$$

Let $V = \mathbf{Spec} B^0$. By Luna's étale slice theorem, we may pick an affine slice $x \in S$ in V and obtain an étale map $[T/R] \rightarrow [U/G]$, where $T = U \cap S$, induced from the étale map $[S/R] \rightarrow [V/G]$. Using $S \rightarrow V$, there exists a derived affine scheme $\mathbf{T} = \mathbf{Spec} C^\bullet$ with an R -action and $C^0 = B^0$, whose classical truncation is T , fitting in a fiber diagram

$$\begin{array}{ccc} [T/R] & \longrightarrow & [U/G] \\ \downarrow & & \downarrow \\ [\mathbf{T}/R] & \longrightarrow & [\mathbf{U}/G]. \end{array}$$

where the lower horizontal arrow is étale. Therefore $[\mathbf{T}/R]$ is (-1) -shifted symplectic and we may apply Proposition 7.9 to deduce that \mathbf{T} is equivalent to $\mathbf{Spec} A^\bullet$, where (A^\bullet, δ) is special and minimal at t . \square

In analogy with the absolute case, we give the following definition.

Definition 7.11. *We say that the tuple $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ gives a relative local model structure on U if V is a smooth G -equivariant scheme over C , in addition to the rest of the data satisfying Setup-Definition 5.1 and one of the following:*

- (1) $F_V = \Omega_{V/C}$, $D_V = 0$ and $\phi_V: \Omega_{V/C} \rightarrow \mathfrak{g}^\vee$ is the dual of the G -action. In this case, we call Λ_V a quasi-critical chart on V .

²At the time of writing of this paper, it is available online at http://www.math.columbia.edu/~danh1/derived_equivalences_2016_09_18.pdf.

- (2) Λ_V is obtained by a quasi-critical chart by a sequence of Kirwan blowups and taking étale slices of closed points with closed orbit.

We then say that the C -scheme U is in relative standard form.

Remark 7.12. One of the main results of [BBJ13] is that, in the absolute case, if A^\bullet is a standard form \mathbb{C} -cdga with a (-1) -shifted symplectic form ω , then up to equivalence and possible shrinking we have $\omega = (\omega_0, 0, 0, \dots)$, where $\omega_0 = d\phi$ and $\delta\phi = d\Phi$ for $\phi \in (\Omega_A^\bullet)^{-1}$ and $\Phi \in A^0$. In particular, this implies that $\text{Spec } H^0(A^\bullet)$ is the critical locus of Φ inside $\text{Spec } A^0$. This also works equivariantly as in the above.

7.3. Comparison of local presentations. We first examine how the 1-form ω changes if one moves ω within its equivalence class.

Proposition 7.13. Let (A^\bullet, δ) be a special cdga, minimal at x . Suppose that ω, η are equivalent (-1) -shifted symplectic forms on $[\text{Spec } A^\bullet/G]$. Then, up to equivariant shrinking of V around x , these induce 1-forms $\omega, \eta \in H^0(\Omega_{V/C})$ which are Ω -equivalent:

- (1) We have an equality of ideals in A^0 , $(\omega) = (\eta) = I_U$.
- (2) There exist equivariant morphisms $B, C: \Omega_{V/C} \rightarrow T_{V/C}$ such that

$$\omega^\vee - \eta^\vee = \eta^\vee B^\vee d\eta^\vee \text{ mod } I_U^2$$

and

$$\eta^\vee - \omega^\vee = \omega^\vee C^\vee d\omega^\vee \text{ mod } I_U^2.$$

Proof. Let x_i be an étale basis for V over C . As in the discussion preceding Definition 7.8, we may write $\omega_0 = \sum_i dy_i^\omega dx_i$ and $\eta_0 = \sum_i dy_i^\eta dx_i$, where y_i^ω and y_i^η are bases for A^{-1} , for the 0-part of the pullbacks of ω and η to $\text{Spec } A^\bullet$. Thus we have

$$y_i^\omega = \sum_k J_{ik}^\omega y_k, \quad y_i^\eta = \sum_k J_{ik}^\eta y_k, \quad J_{ik}^\omega, J_{ik}^\eta \in A^0$$

The induced 1-forms are then given by

$$(7.2) \quad \begin{aligned} \omega &= \sum_i \delta y_i^\omega dx_i = \sum_{i,k} J_{ik}^\omega s_k dx_i, \\ \eta &= \sum_i \delta y_i^\eta dx_i = \sum_{i,k} J_{ik}^\eta s_k dx_i, \end{aligned}$$

where we write $s_i = \delta y_i$ for convenience. Since $y_i, y_i^\omega, y_i^\eta$ are all bases for A^{-1} it is clear that $I_U = (\delta y_i) = (\delta y_i^\omega) = (\delta y_i^\eta)$, which proves (1).

Let $\delta w_j = \sum_i W_{ji} y_i$, where $W_{ji} \in A^0$. Since $\delta^2(w_j) = 0$ we have

$$(7.3) \quad \sum_{i,j} W_{ji} s_i = 0$$

Now ω, η are equivalent as symplectic forms so in particular we have $\omega_0 - \eta_0 = \delta\alpha_0$ for some $\alpha_0 \in (\wedge^2 \Omega_{A^\bullet})^{-2}$. By degree considerations, we may write

$$\alpha_0 = \sum_{ij} E_{ij} dy_i dy_j + \sum_{ik} F_{ik} dw_k dx_i + \alpha'_0, \quad E_{ij}, F_{ik} \in A^0,$$

where α'_0 is a 2-form, whose every term is divisible by some of the y_i , and we may assume without loss of generality that E_{ij} is symmetric. Thus $\delta\alpha'_0 \in I_U \cdot (\wedge^2 \Omega_{A^\bullet}^1)^{-1}$ and we have

$$\begin{aligned} \delta\alpha_0 &= -2 \sum_{i,j} E_{ij} ds_i dy_j - \sum_{i,k} F_{ik} d(\delta w_k) dx_i \pmod{I_U} \\ &= -2 \sum_{i,j} E_{ij} ds_i dy_j - \sum_{i,j,k} F_{ik} W_{kj} dx_i dy_j \pmod{I_U}. \end{aligned}$$

We have also

$$\begin{aligned} \omega_0 - \eta_0 &= \sum_i dy_i^\omega dx_i - \sum_i dy_i^\eta dx_i \\ &= \sum_{i,k} dJ_{ik}^\omega y_k dx_i - \sum_{i,k} J_{ik}^\omega dy_k dx_i - \sum_{i,k} dJ_{ik}^\eta y_k dx_i + \sum_{i,k} J_{ik}^\eta dy_k dx_i. \end{aligned}$$

By comparing the coefficients of each dy_j , we obtain a relation

$$\sum_i J_{ij}^\omega dx_i - \sum_i J_{ij}^\eta dx_i = 2 \sum_i E_{ij} ds_i + \sum_{i,k} F_{ik} W_{kj} dx_i \pmod{I_U}$$

for each index j . Then, using (7.2),

$$\begin{aligned} (7.4) \quad \omega - \eta &= \sum_{i,k} (J_{ik}^\omega dx_i - J_{ik}^\eta dx_i) s_k \\ &= 2 \sum_{i,k} E_{ik} ds_i s_k + \sum_{i,k,j} F_{ij} W_{jk} s_k dx_i \pmod{I_U^2} \\ &= 2 \sum_{i,k} E_{ik} ds_i s_k \pmod{I_U^2}, \end{aligned}$$

where in the second expression the second term is zero using (7.3).

Note that in order to derive (7.4), we only used the fact that y_i is a basis for A^{-1} . So we could repeat the exact same analysis and obtain a similar equation with s_i replaced by $s_i^\omega = \delta y_i^\omega$ or $s_i^\eta = \delta y_i^\eta$ and E_{ik} by E_{ik}^ω or E_{ik}^η respectively. It is then easy to see that these exactly imply (2), with the coefficients of B, C being determined by $E_{ik}^\omega, E_{ik}^\eta$ after averaging over G to make the morphism equivariant ($\omega - \eta$ is already invariant). \square

Suppose now that we have étale morphisms $\mathbf{f}_\alpha: [\mathbf{T}_\alpha/R_\alpha] \rightarrow \mathcal{M}$ and $\mathbf{f}_\beta: [\mathbf{T}_\beta/R_\beta] \rightarrow \mathcal{M}$ as in Proposition 7.10, where \mathbf{T}_α is equivalent to $\mathbf{Spec} A^\bullet$ and \mathbf{T}_β is equivalent to $\mathbf{Spec} B^\bullet$, with A^\bullet, B^\bullet special equivariant cdgas. Let $z \in [\mathbf{T}_\alpha/R_\alpha] \times_{\mathcal{M}} [\mathbf{T}_\beta/R_\beta]$ be a closed point with stabilizer H .

Note that $\mathbf{f}_\alpha, \mathbf{f}_\beta$ are also affine. Then, since $T_\alpha, T_\beta \rightarrow \mathcal{M}$ are affine, the diagonal of \mathcal{M} is affine, and a derived scheme is affine if and only if its truncation is affine, we obtain that $\mathbf{T}_\alpha \times_{\mathcal{M}} \mathbf{T}_\beta$ is an affine derived scheme with an action of $R_\alpha \times R_\beta$ such that $[\mathbf{T}_\alpha \times_{\mathcal{M}} \mathbf{T}_\beta / R_\alpha \times R_\beta]$ is (-1) -shifted symplectic. Then, there exists a special cdga C^\bullet with an H -action, minimal at $t_{\alpha\beta}$, such that we have H -equivariant morphisms $\alpha: A^\bullet \rightarrow C^\bullet, \beta: B^\bullet \rightarrow$

C^\bullet and a commutative diagram of étale arrows

$$(7.5) \quad \begin{array}{ccc} [\mathbf{Spec} C^\bullet/H] & \xrightarrow{\theta_\beta} & [\mathbf{Spec} B^\bullet/R_\beta] \\ \theta_\alpha \downarrow & & \downarrow f_\beta \\ [\mathbf{Spec} A^\bullet/R_\alpha] & \xrightarrow{f_\alpha} & \mathcal{M}. \end{array}$$

Moreover, the morphism $[\mathbf{Spec} C^\bullet/H] \rightarrow [T_\alpha \times_{\mathcal{M}} T_\beta/R_\alpha \times R_\beta]$ is étale and maps $t_{\alpha\beta}$ to z .

We can recast the above data at the level of classical stacks and schemes. We have an étale map $f_\alpha: [T_\alpha/R_\alpha] \rightarrow \mathcal{M}$, where $T_\alpha \subset S_\alpha$ is the zero locus of an invariant section ω_α of $\Omega_{S_\alpha/C}$ and $S_\alpha \rightarrow C$ is smooth, R_α -equivariant. Similar data is obtained for the étale map $[T_\beta/R_\beta] \rightarrow \mathcal{M}$. The above diagram shows that we have the following comparison data:

- (1) We have an affine, smooth H -scheme $S_{\alpha\beta} \rightarrow C$ with an invariant section $\omega_{\alpha\beta}$ of $\Omega_{S_{\alpha\beta}/C}$ with zero locus $T_{\alpha\beta}$, minimal at a point $t_{\alpha\beta}$ fixed by H . Here $T_{\alpha\beta}$ is the truncation of the derived scheme $\mathbf{Spec} C^\bullet$.
- (2) There exist H -equivariant unramified morphisms $\theta_\alpha: S_{\alpha\beta} \rightarrow S_\alpha$ and $\theta_\beta: S_{\alpha\beta} \rightarrow S_\beta$, inducing unramified morphisms $T_{\alpha\beta} \rightarrow T_\alpha$ and $T_{\alpha\beta} \rightarrow T_\beta$.
- (3) $\eta_{\theta_\alpha}(\omega_\alpha)$, $\eta_{\theta_\beta}(\omega_\beta)$ are Ω -equivalent to $\omega_{\alpha\beta}$.
- (4) We have a commutative diagram with étale arrows

$$(7.6) \quad \begin{array}{ccc} [T_{\alpha\beta}/H] & \xrightarrow{\theta_\beta} & [T_\beta/R_\beta] \\ \theta_\alpha \downarrow & & \downarrow f_\beta \\ [T_\alpha/R_\alpha] & \xrightarrow{f_\alpha} & \mathcal{M}. \end{array}$$

Definition 7.14. (Common roof) *If the above four conditions hold, coming from a diagram (7.5), we say that the quasi-critical chart $\Lambda_{S_{\alpha\beta}}$ is a common roof for the quasi-critical charts Λ_{S_α} and Λ_{S_β} . More generally, the same definition applies to any two relative local models which are not necessarily quasi-critical charts.*

Remark 7.15. *Suppose that we have a common roof coming from a commutative diagram (7.5) where $\mathcal{M} = [\mathbf{Spec} D^\bullet/G]$ and $R_\alpha = R_\beta = H$. Moreover, assume that we have two compositions $g_\alpha: D^\bullet \rightarrow A^\bullet \rightarrow C^\bullet$ and $g_\beta: D^\bullet \rightarrow B^\bullet \rightarrow C^\bullet$ such that $\mathbf{Spec} g_\alpha$, $\mathbf{Spec} g_\beta$ become equivalent when composed with the quotient morphism $\mathbf{Spec} D^\bullet \rightarrow [\mathbf{Spec} D^\bullet/G]$ and induce the diagram (7.5). If $g_\alpha, g_\beta: D_0 \rightarrow C_0$ are the induced morphisms and we denote $V = \mathbf{Spec} D^0$,*

$$g_\alpha^b = \theta_\alpha^b \circ f_\alpha^b, \quad g_\beta^b = \theta_\beta^b \circ f_\beta^b: \mathcal{O}b_V^{\text{red}}|_{T_{\alpha\beta}^s} \longrightarrow \mathcal{O}b_{S_{\alpha\beta}}^{\text{red}}$$

are obtained by the corresponding maps from (7.5) on cotangent complexes of quotient stacks, as in Lemma 5.26. It follows that for the (non-commutative)

diagram of quotient stacks

$$\begin{array}{ccc} [\mathrm{Spec} C^0/H] & \xrightarrow{\theta_\beta} & [\mathrm{Spec} B^0/H] \\ \theta_\alpha \downarrow & & \downarrow f_\beta \\ [\mathrm{Spec} A^0/H] & \xrightarrow{f_\alpha} & [\mathrm{Spec} D^0/G]. \end{array}$$

which is compatible with (7.6), there is a natural equivalence between the compositions $\theta_\alpha^b \circ f_\alpha^b$ and $\theta_\beta^b \circ f_\beta^b$, which is also compatible with the commutativity of (7.6).

Another way to see this more concretely is when G is a Zariski open subscheme of an affine space. This is the case in our application to Donaldson-Thomas invariants, since closed points correspond to polystable sheaves whose stabilizers are products of general linear groups. The fact that g_α, g_β composed with $\mathrm{Spec} D^\bullet \rightarrow [\mathrm{Spec} D^\bullet/G]$ are equivalent implies then that there exists a morphism $h = \mathrm{Spec} h: \mathrm{Spec} C^\bullet \rightarrow G$ such that the composition

$$\mathrm{Spec} g'_\beta: \mathrm{Spec} C^\bullet \xrightarrow{(id, h)} \mathrm{Spec} C^\bullet \times G \xrightarrow{(\mathrm{Spec} g_\beta, id)} \mathrm{Spec} D^\bullet \times G \longrightarrow \mathrm{Spec} D^\bullet,$$

where the last arrow is given by the group action, gives a map of cdgas $D^\bullet \rightarrow C^\bullet$ which is homotopic to g_β . In particular, the induced morphisms $g_\beta: D^0 \rightarrow C^0$ and $g'_\beta: D^0 \rightarrow C^0$ satisfy $g_\beta - g'_\beta: D^0 \rightarrow \mathrm{im} C^{-1} = \mathrm{im} \omega_{\alpha\beta}^\vee$ and thus, as in Lemma 5.25, we get that

$$g_\beta^b = (g'_\beta)^b: \mathcal{O}b_V^{\mathrm{red}}|_{T_{\alpha\beta}^s} \longrightarrow \mathcal{O}b_{S_{\alpha\beta}}^{\mathrm{red}}$$

and the induced morphism $h: \mathrm{Spec} C^0 \rightarrow G$ gives the data for the equivalence mentioned above.

7.4. Obstruction theory in the relative case. As before, suppose that $\mathcal{M} \rightarrow C$ is a (-1) -shifted symplectic derived stack, whose truncation $\mathcal{M} = [X/G]$ is a quotient stack obtained by GIT, where X is a C -scheme and G acts trivially on C .

The Kirwan partial desingularization procedure goes through in exactly the same way as in the absolute case. We have the following modified version of Proposition 6.1 in this relative situation.

The main difference stems from the fact that we need to use local models, arising from special cdga's (cf. Definition 8.8), which satisfy a minimality property (cf. Definition 8.6). Thus we can no longer use slices S_α coming from a global embedding $\mathcal{M} \rightarrow \mathcal{P}$ into a smooth stack and we will need to use the notion of Ω -compatibility to address the issue of extra coordinates. In particular, in the analogue of diagram (6.1), we will be missing the arrows $S_\alpha \rightarrow \mathcal{P}_n$, $S_\beta \rightarrow \mathcal{P}_n$ and S_α, S_β can have extra coordinates.

Proposition 7.16. *For each $n \geq 0$, let \mathcal{E}_n^X be the union of all exceptional divisors in \mathcal{M}_n . There exist collections of étale morphisms $[T_\alpha/R_\alpha] \rightarrow \mathcal{M}_n$ such that:*

- (1) *For each α , $R_\alpha \in \{R_1, \dots, R_n\}$.*
- (2) *Each T_α is in relative standard form for data*

$$\Lambda_{S_\alpha} = (T_\alpha, S_\alpha, F_\alpha, \omega_\alpha, D_\alpha, \phi_\alpha)$$

of a relative local model on a smooth affine R_α -scheme S_α .

- (3) The collections cover \mathcal{E}_n^X respectively.
- (4) The identity components of stabilizers that occur in \mathcal{M}_n lie, up to conjugacy, in the set $\{R_{n+1}, \dots, R_m\}$.
- (5) Λ_{S_α} restricted to the complement of the union of exceptional divisors $\mathcal{E}_{S_\alpha} \subset S_\alpha$ are the same as those of a quasi-critical chart on T_α .
- (6) For indices α, β and $q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta]$ whose stabilizer has identity component conjugate to R_q , there exist an affine $T_{\alpha\beta}$ in relative standard form for data $\Lambda_{S_{\alpha\beta}} = (T_{\alpha\beta}, S_{\alpha\beta}, F_{\alpha\beta}, \omega_{\alpha\beta}, D_{\alpha\beta}, \phi_{\alpha\beta})$ of a relative local model and an equivariant commutative diagram

$$(7.7) \quad \begin{array}{ccccc} & S_{\alpha\beta} & \longleftarrow & T_{\alpha\beta} & \longrightarrow & S_{\alpha\beta} \\ & \theta_\alpha \swarrow & & \searrow i_\alpha & & \searrow i_\beta & & \searrow \theta_\beta \\ S_\alpha & \longleftarrow & T_\alpha & & T_\beta & \longrightarrow & S_\beta \\ & & \searrow & & \swarrow & & \\ & & \mathcal{M}_n & & \end{array}$$

such that $\Lambda_{S_{\alpha\beta}}$ is a common roof for Λ_{S_α} and Λ_{S_β} coming from a diagram of the form (7.5). Moreover, θ_α factors as $S_{\alpha\beta} \rightarrow S'_\alpha \rightarrow S_\alpha$ where S'_α is an étale slice for S_α at the image of q , $T_{\alpha\beta} \rightarrow T'_\alpha$ is étale and $S_{\alpha\beta} \rightarrow S'_\alpha$ is unramified. Analogous conditions hold for the index β .

- (7) For each index α , consider the 4-term complex

$$K_\alpha = [\mathfrak{r}_\alpha \longrightarrow T_{S_\alpha}|_{T_\alpha} \xrightarrow{(d\omega_{S_\alpha}^\vee)^\vee} F_\alpha|_{T_\alpha} \xrightarrow{\phi_{S_\alpha}} \mathfrak{r}_\alpha^\vee(-D_\alpha)],$$

where by convention we place $F_{S_\alpha}|_{T_\alpha}$ in degree 1. θ_α induces an isomorphism $\theta_\alpha^b: \mathcal{O}b_{S_\alpha}^{\text{red}}|_{T_{\alpha\beta}^s} \rightarrow \mathcal{O}b_{S_{\alpha\beta}}^{\text{red}}$. This also does not change if we replace ω_{S_α} by any Ω -equivalent section. Analogous statements are true for the index β .

- (8) We obtain comparison isomorphisms

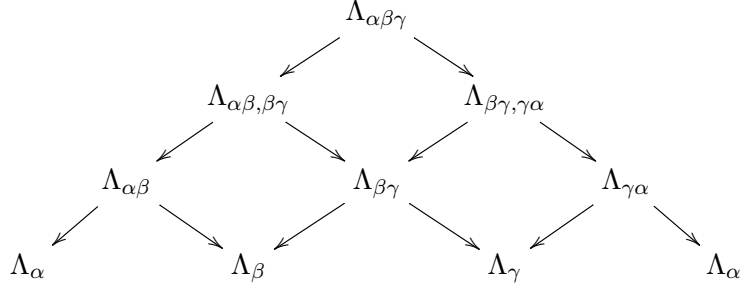
$$\theta_{\alpha\beta}^b := (\theta_\beta^b)^{-1} \theta_\alpha^b: \mathcal{O}b_{S_\alpha}^{\text{red}}|_{T_{\alpha\beta}^s} \rightarrow \mathcal{O}b_{S_\beta}^{\text{red}}|_{T_{\alpha\beta}^s}.$$

These give the same obstruction assignments for the complexes K_α^{red} , K_β^{red} on $T_{\alpha\beta}^s$.

- (9) Let now q be a point in the triple intersection

$$q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta] \times_{\mathcal{M}_n} [T_\gamma/R_\gamma]$$

with stabilizer in class R_q . We then have a commutative schematic R_q -equivariant diagram of common roofs



such that all the morphisms between local models $\Lambda_{\lambda} \rightarrow \Lambda_{\mu}$, where λ, μ are multi-indices, satisfy the properties of a roof, the morphisms $T_{\lambda} \rightarrow T_{\mu}$ are étale, and we have induced isomorphisms $\mathcal{O}b_{\mu}^{\text{red}}|_{T_{\lambda}^s} \rightarrow \mathcal{O}b_{\lambda}^{\text{red}}$. The descents of $\theta_{\alpha\beta}^b, \theta_{\beta\gamma}^b, \theta_{\gamma\alpha}^b$ to $[T_{\alpha\beta\gamma}^s/R_q]$ satisfy the cocycle condition.

Proof. In the same way as in the absolute case, all conditions (1)-(9) are preserved at each inductive step. So we only need to check that they hold in the beginning and also address the differences in (6), (7) and (9) from the absolute case.

By the previous subsection, roofs exist and they satisfy the conditions of (6) by construction, since we may first replace $\mathbf{T}_{\alpha}/R_{\alpha}$ by an étale slice \mathbf{T}_{α}'/H (and similarly for the index β) and then take a common roof. In particular, by Lemma 5.22 and Lemma 5.26 we immediately deduce that θ_{α}^b indeed induce isomorphisms on the reduced obstruction sheaves. It is at this point where we have to use the notion of Ω -compatibility to deal with possible extra coordinates.

Finally, the cocycle condition holds by applying Remark 7.15 and using Lemma 5.22, 5.24, 5.25 and 5.26, which are valid in the relative setting, and going through the appropriate diagrams of roofs granted by (9). We leave the details to the reader. \square

We have shown that we can follow the same steps as in the absolute case to obtain the following theorem.

Theorem 7.17. *Let $\mathcal{M} = [X/G] \rightarrow C$, where $X \rightarrow C$ is a quasi-projective scheme, which is the semistable part of a projective scheme $X^{\dagger} \rightarrow C$ with a linearized G -action and C is a smooth, quasi-projective curve with a trivial G -action. Suppose also that \mathcal{M} is the truncation of a relative (-1) -shifted symplectic derived stack \mathbf{M} over C . Then the Kirwan partial desingularization $\widetilde{\mathcal{M}} \rightarrow C$ is a proper DM stack over C endowed with a canonical relative semi-perfect obstruction theory induced by the relative (-1) -shifted symplectic form, giving rise to a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\text{vir}}$.*

Furthermore, the restriction of the obstruction theory to the fiber $\widetilde{\mathcal{M}}_c$ over a point $c \in C$ is the semi-perfect obstruction theory of Theorem 6.2, constructed in the absolute case.

7.5. Deformation invariance of DTK invariants. Let $W \rightarrow C$ be a family of Calabi-Yau threefolds over a smooth affine curve. The relative canonical bundle $K_{W/C}$ is then trivial.

Let \mathcal{M} be the moduli stack of relatively semistable sheaves over C of Chern character γ on W . As in the absolute case, a trivialization of $K_{W/C}$ induces a relative (-1) -shifted symplectic structure on the derived stack $\mathcal{M} \rightarrow C$. Then by Theorem 7.17, it follows that applying the same construction we obtain a Kirwan partial desingularization $\widetilde{\mathcal{M}} \rightarrow C$ with a relative semi-perfect obstruction theory, which induces the absolute semi-perfect obstruction theory on each fiber $\widetilde{\mathcal{M}}_c$ we constructed earlier.

Moreover, its fiber over $c \in C$ is the Kirwan partial desingularization of the moduli stack of semistable sheaves on W_c by the results of subsection 7.1. Therefore, by “conservation of number” for semi-perfect obstruction theories [CL11, Proposition 3.8], the generalized Donaldson-Thomas invariant stays the same along the family, i.e. $\mathrm{DT}_\gamma(W_c)$ is constant for $c \in C$.

Theorem 7.18. *The generalized DT invariant defined in Theorem-Definition 6.5 is invariant under deformation of the complex structure of the Calabi-Yau threefold.*

Remark 7.19. *Choosing a different trivialization of $K_{W/C}$ amounts to rescaling the induced trivializations of K_{W_c} on each fiber, so by Remark 6.4 our discussion does not depend on the specific choice of trivialization.*

REFERENCES

- [BBBBJ15] O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce. A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications. *Geom. Topol.*, 19(3):1287–1359, 2015.
- [BBJ13] C. Brav, V. Bussi, and D. Joyce. A ‘Darboux theorem’ for derived schemes with shifted symplectic structure. *ArXiv e-prints*, May 2013.
- [BF97] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.
- [BR16a] K. Behrend and P. Ronagh. The Eigenvalue Spectrum of the Inertia Operator. *ArXiv e-prints*, December 2016.
- [BR16b] K. Behrend and P. Ronagh. The inertia operator on the motivic Hall algebra. *ArXiv e-prints*, December 2016.
- [CL11] H.-L. Chang and J. Li. Semi-perfect obstruction theory and Donaldson-Thomas invariants of derived objects. *Comm. Anal. Geom.*, 19(4):807–830, 2011.
- [Dré04] J.-M. Drézet. Luna’s slice theorem and applications. In Jaroslaw A. Wisniewski, editor, *Algebraic group actions and quotients*, pages 39–90. Hindawi Publishing Corporation, 2004.
- [HL] D. Halpern-Leistner. The D-Equivalence Conjecture for Moduli Spaces of Sheaves.
- [HL10] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.
- [Joy15] D. Joyce. A classical model for derived critical loci. *J. Differential Geom.*, 101(2):289–367, 2015.
- [JS12] D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. *Mem. Amer. Math. Soc.*, 217(1020):iv+199, 2012.
- [Kir85] F. C. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Ann. of Math. (2)*, 122(1):41–85, 1985.

- [KL12] Y.-H. Kiem and J. Li. Categorification of Donaldson-Thomas invariants via Perverse Sheaves. *ArXiv e-prints*, December 2012.
- [KL13a] Y.-H. Kiem and J. Li. Localizing virtual cycles by cosections. *J. Amer. Math. Soc.*, 26(4):1025–1050, 2013.
- [KL13b] Y.-H. Kiem and J. Li. A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional. *Asian J. Math.*, 17(1):63–94, 2013.
- [KS10] M. Kontsevich and Y. Soibelman. Motivic Donaldson-Thomas invariants: summary of results. In *Mirror symmetry and tropical geometry*, volume 527 of *Contemp. Math.*, pages 55–89. Amer. Math. Soc., Providence, RI, 2010.
- [LT98] J. Li and G. Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- [MNOP06a] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006.
- [MNOP06b] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. II. *Compos. Math.*, 142(5):1286–1304, 2006.
- [MT16] D. Maulik and Y. Toda. Gopakumar-Vafa invariants via vanishing cycles. *ArXiv e-prints*, October 2016.
- [PT09] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.
- [PT10] R. Pandharipande and R. P. Thomas. Stable pairs and BPS invariants. *J. Amer. Math. Soc.*, 23(1):267–297, 2010.
- [PTVV13] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted symplectic structures. *Publications mathématiques de l’IHÉS*, 117(1):271–328, 2013.
- [Tho00] R. P. Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54(2):367–438, 2000.

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