COSECTION LOCALIZED VIRTUAL FUNDAMENTAL CLASSES OF D-MANIFOLDS AND (-2)-SHIFTED SYMPLECTIC DERIVED SCHEMES, REVISITED

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ABSTRACT. We construct reduced virtual fundamental classes for derived manifolds with surjective cosections and cosection localized virtual fundamental classes for (-2)-shifted symplectic derived schemes in full generality. Moreover, using recent work of Oh–Thomas, we show that the algebraic and differential geometric constructions of reduced and cosection localized virtual fundamental classes of (-2)-shifted symplectic derived schemes yield the same result in homology. We obtain applications towards the construction and integrality of reduced invariants in Donaldson–Thomas theory of Calabi–Yau fourfolds.

Contents

1. Introduction

Coherent sheaves and vector bundles are objects of fundamental importance in algebraic geometry. From an enumerative perspective, the construction and study of invariants counting sheaves on algebraic varieties satisfying given constraints plays a foundational role with a prominent case of interest in theoretical physics and string theory being sheaf-theoretic enumerative invariants of Calabi–Yau manifolds, commonly referred to as Donaldson–Thomas invariants.

Let W be a smooth, projective complex Calabi-Yau manifold and M a proper moduli scheme parameterizing stable sheaves on W. While Donaldson–Thomas invariants were defined in the '90s [Tho00] when $\dim W = 3$ by integrating cohomology classes on M against a virtual fundamental cycle $[M]^{\mathrm{vir}}$ [BF97, LT98], the fourfold case required a different strategy to produce a virtual cycle which took a while to develop.

In [CC14], the authors introduced a gauge-theoretic approach which yields invariants in certain cases. Subsequently, Borisov–Joyce [BJ17] used the existence of a (-2)-shifted derived enhancement of M [PTVV13] together with an orientation [CGJ20] to carry out an intricate truncation procedure that produces a compact, oriented d-manifold in the sense of [Joy], which admits a well-defined virtual fundamental class $[M]^{\mathrm{vir}} \in H_*(M, \mathbb{Z})$. This gave access to the desired virtual fundamental class, not of algebraic, but rather differential geometric flavor.

The algebraic counterpart $[M]^{\mathrm{vir}} \in A_*(M, \mathbb{Z}[1/2])$ was constructed by Oh–Thomas [OT20] using the derived enhancement of M as well as a localized version of the square root Euler class of Edidin–Graham [EG95]. In [OT23], it was further proved that the differential geometric and algebraic virtual class are compatible with each other.

Donaldson–Thomas theory of Calabi–Yau fourfolds (DT4 theory) is an area of intense ongoing research activity in recent years (cf. [BKP22, CK20, CKM19, CKM22, CT20, CT22, Cao20, CMT18, CMT22, COT22a, COT22b] for a partial list of results).

One of the main tools used traditionally in the construction and study of enumerative invariants is localization of virtual cycles by cosection, introduced by Kiem–Li [KL13]. In the setting of DT4 theory, the differential geometric version of cosection localization à la Borisov–Joyce was developed in [Sav22], while the algebraic version à la Oh–Thomas was established in [KP20].

Building on the results of [OT23], the purpose of this paper is to complement the results of [Sav22] by establishing stronger, more general statements and simplifying their proofs and prove the compatibility of these two approaches. Namely, we do the following:

(1) Construct reduced virtual fundamental classes for d-manifolds with surjective cosections. Given the utility of reduced invariants in [COT22a, COT22b, BKP22], this provides access to integral homological versions of these invariants.

(2) Define integral homological cosection localized and reduced virtual fundamental classes for (-2)-shifted symplectic schemes in full generality, generalizing the results of [Sav22].

(3) Prove that the differential geometric and algebraic versions of these virtual classes agree whenever they are both defined.

Statement of results. We now give summarized statements of the main results of the paper for the convenience of the reader. For full details, we refer to the main text.

We begin with reduced virtual fundamental classes of d-manifolds with surjective cosections.

Theorem (Theorem 3.6, Subsection 3.4)**.** *Let $\mathcal{X}$ be a compact, oriented d-manifold of virtual dimension n with underlying topological space X, equipped with a surjective morphism $\underline{\sigma} \colon h^1(\mathbb{T}_{\mathcal{X}}) \to \mathbb{R}_X^k$. Then $[\mathcal{X}]^{\mathrm{vir}} = 0 \in H_n(X, \mathbb{Z})$ and there exists a well-defined reduced virtual fundamental class $[\mathcal{X}]_{\mathrm{red}}^{\mathrm{vir}} \in H_{n+k}(X, \mathbb{Z})$.*

We move on to (-2)-shifted symplectic derived algebraic geometry.

Theorem (Theorem 4.2, Theorem 4.4). *Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted symplectic derived scheme of virtual dimension $n = \operatorname{rk} \mathbb{L}_{\underline{\mathcal{X}}}|_X$, equipped with a morphism $\sigma \colon \mathbb{T}_{\underline{\mathcal{X}}}|_X[1] \to \mathcal{O}_X$.*

(1) *If $h^0(\sigma)$ fails to be surjective or is undefined on the closed subset $i \colon X(\sigma) \to X$, there exists a localized virtual fundamental class $[\underline{\mathcal{X}}]^{\mathrm{vir}}_{\mathrm{loc},\sigma} \in H_n(X(\sigma), \mathbb{Z})$ satisfying $i_*[\underline{\mathcal{X}}]^{\mathrm{vir}}_{\mathrm{loc},\sigma} = [\underline{\mathcal{X}}]^{\mathrm{vir}} \in H_n(X, \mathbb{Z})$.*

(2) *If σ is globally defined, surjective and isotropic with respect to the symplectic form $\omega_{\underline{\mathcal{X}}}$, then $[\underline{\mathcal{X}}]^{\mathrm{vir}} = 0 \in H_n(X, \mathbb{Z})$ and there exists a reduced virtual fundamental class $[\underline{\mathcal{X}}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+2}(X, \mathbb{Z})$.*

(3) *If σ is globally defined, surjective and non-degenerate with respect to the symplectic form $\omega_{\underline{\mathcal{X}}}$, then $[\underline{\mathcal{X}}]^{\mathrm{vir}} = 0 \in H_n(X, \mathbb{Z})$ and there exists a reduced virtual fundamental class $[\underline{\mathcal{X}}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+1}(X, \mathbb{Z})$.*

Using the machinery of complex Kuranishi charts of [OT23], we are able to show that the differential geometric and algebraic localized or reduced virtual classes are equal in the appropriate sense.

Theorem (Theorem 5.4, Theorem 5.6, Theorem 5.8). *Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted symplectic derived scheme, equipped with a morphism $\sigma \colon \mathbb{T}_{\underline{\mathcal{X}}}|_X[1] \to \mathcal{O}_X$.*

Then, in each of the three scenarios (1)-(3) of the previous theorem, the image of the respective algebraic virtual fundamental cycle, constructed in [KP20, BKP22], under the cycle class map $A_*(X, \mathbb{Z}[1/2]) \to H_{2*}(X, \mathbb{Z}[1/2])$ is equal to the image of the homological virtual fundamental class under the natural map $H_*(X, \mathbb{Z}) \to H_*(X, \mathbb{Z}[1/2])$.

In Section 6, we apply these results to construct reduced invariants and obtain integrality properties for surface counting invariants and cosection localized virtual classes of Calabi–Yau fourfolds, as well as Gopakumar–Vafa type invariants of holomorphic symplectic fourfolds.

Organization of the paper. In §2, we recall useful background on d-manifolds, (-2)-shifted symplectic derived schemes, their local properties and virtual classes. §3 is devoted to reduced virtual fundamental classes of d-manifolds, while §4 treats cosection localized and reduced virtual fundamental classes of (-2)-shifted symplectic derived schemes. In §5, we prove the compatibility between the differential geometric and algebraic virtual classes in all cases. Finally, §6 explores applications of our results to virtual classes and enumerative invariants in DT4 theory.

Notation and conventions. For topological purposes, we work with Borel-Moore homology and singular (co)homology with integer coefficients. These are equal for compact spaces and afford us the necessary flexibility to work with fundamental classes of (non)compact spaces. We refer the reader to [Bre97, Ive72] for a comprehensive treatment. To eliminate possible confusion, we always specify the coefficients we use for homology groups or Chow groups.

We work with the theory of $\mathcal{C}^\infty$-schemes and derived manifolds developed by Joyce [Joy]. In analogy to usual smooth manifolds, our $\mathcal{C}^\infty$-schemes will be separated, second countable and locally fair, meaning that their underlying topological space is Hausdorff, second countable and they are locally modelled by the spectrum of a finitely generated $\mathcal{C}^\infty$-ring. By definition, such $\mathcal{C}^\infty$-schemes can be the underlying scheme of a d-manifold. Unless otherwise stated, the underlying topological spaces of $\mathcal{C}^\infty$-schemes and d-manifolds are assumed to be Euclidean Neighbourhood Retracts (ENR) for simplicity. This is automatically the case for the complex analytic spaces or algebraic schemes appearing throughout the paper. For the purpose of working with virtual classes and cosections, this assumption may be removed and is thus not restrictive (see [Sav22, Section 5.2]).

All algebraic and derived schemes are defined over the field of complex numbers $\mathbb{C}$. Classical and derived algebraic schemes and complex analytic spaces are separated and of finite type. The reader can consult [Toë14] for more general background on derived algebraic geometry and shifted symplectic structures.

$\underline{\mathcal{X}}$ will typically denote a derived scheme and $\omega_{\underline{\mathcal{X}}}$ a (-2)-shifted symplectic form on $\underline{\mathcal{X}}$. $\mathcal{X}$ will typically denote a d-manifold and $X_{\mathcal{C}^\infty}$ its underlying $\mathcal{C}^\infty$-scheme. X will always denote the underlying topological space of $\underline{\mathcal{X}}$ or $\mathcal{X}$ and sometimes, by abuse of notation, the classical truncation of $\underline{\mathcal{X}}$ as well. In that case, X^{an} denotes the complex analytic space associated to classical truncation X of $\underline{\mathcal{X}}$. These distinctions will be explicitly stated or should be clear from context throughout the paper. In general, we typically use the notation $X_{\mathcal{C}^\infty}$ when we need to use the existence of quasicoherent sheaves on a $\mathcal{C}^\infty$-scheme, whereas we only consider homology groups of the underlying topological spaces.

2. Some background

In this section, we collect some requisite background and introduce terminology that will be used repeatedly in the rest of the paper. A slightly more detailed account of some overlapping material is given in [Sav22, Section 2].

2.1. $\mathcal{C}^\infty$-schemes and d-manifolds. The main references for the theory of $\mathcal{C}^\infty$-schemes and d-manifolds are the books [Joy19] and [Joy] by Joyce.

Roughly speaking, under the conventions of this paper, a $\mathcal{C}^\infty$-scheme is a locally ringed space that is locally isomorphic to the (real) spectrum of the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R}^n)/I$, where I is an ideal of smooth functions on $\mathbb{R}^n$. $\mathcal{C}^\infty$-schemes admit a theory of (quasi)coherent sheaves and vector bundles, which resembles that of sheaves on algebraic schemes.

Building on $\mathcal{C}^\infty$-geometry, d-manifolds form a theory of derived differential geometry being analogous to quasi-smooth derived or dg-schemes. Their local structure is described by [Joy, Example 1.4.4] as follows.

A principal d-manifold $\mathcal{X} = \mathcal{S}_{Y,E,s}$ is determined by local Kuranishi charts, i.e. the data (Y, E, s) of a smooth manifold Y, a smooth vector bundle E on Y and a smooth section $s \in \mathcal{C}^\infty(E)$: The underlying $\mathcal{C}^\infty$-scheme $X_{\mathcal{C}^\infty}$ of $\mathcal{X}$ is the affine scheme obtained as the spectrum of the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(Y)/I$, where $I = (s)$ is the ideal generated by the section s. The higher data are given by the morphism $E^\vee|_{X_{\mathcal{C}^\infty}} \xrightarrow{s^\vee} I/I^2$ of $\mathcal{O}_{X_{\mathcal{C}^\infty}}$-modules, so that the underlying topological space X is the zero locus of the section s. The tangent complex (or virtual tangent bundle) of X is the two-term complex

$$(2.1) \qquad \mathbb{T}_{\mathcal{X}} = [T_Y|_{X_{\mathcal{C}^\infty}} \xrightarrow{ds} E|_{X_{\mathcal{C}^\infty}}].$$

where for any choice of connection ∇ on $E^\vee$, ds is the restriction of $\nabla s \colon T_Y \to E$ to $X_{\mathcal{C}^\infty}$.

In general, every d-manifold $\mathcal{X}$ is locally equivalent to a principal dmanifold. When $\mathcal{X}$ is compact (i.e. its underlying topological space X is compact), Joyce proves that this is true globally.

Theorem-Definition 2.1. [Joy, Theorems 4.29, 4.34] *Let $\mathcal{X}$ be a compact d-manifold. Then $\mathcal{X} \simeq \mathcal{S}_{Y,E,s}$ where Y is an open subset of some Euclidean space $\mathbb{R}^N$. We refer to this equivalence as a presentation of $\mathcal{X}$ as a principal d-manifold or a global Kuranishi chart for $\mathcal{X}$ (or X when $\mathcal{X}$ is clear from context). The virtual dimension of $\mathcal{X}$ is then equal to $\mathrm{vd} = N - \mathrm{rk}\, E$.*

The obstruction sheaf of $\mathcal{X}$ is the coherent sheaf $\mathcal{O}b_{\mathcal{X}} := h^1(\mathbb{T}_{\mathcal{X}}) \in \mathrm{Coh}(X_{\mathcal{C}^\infty})$.

A real cosection on $\mathcal{X}$ is a morphism $\sigma \colon \mathcal{O}b_{\mathcal{X}} \to \mathbb{R}_X$ of coherent sheaves on $X_{\mathcal{C}^\infty}$, where $\mathbb{R}_X$ denotes the trivial real line bundle on X. Analogously, a complex cosection is a morphism $\sigma \colon \mathcal{O}b_{\mathcal{X}} \to \mathbb{C}_X$.

Finally, there is a notion of orientation for a d-manifold $\mathcal{X}$, which in the case of a principal d-manifold $\mathcal{S}_{Y,E,s}$ roughly amounts to an orientation of the determinant line bundle $\det \mathbb{T}_{\mathcal{X}}$ of the tangent complex (2.1). As in the case of manifolds, compact, oriented d-manifolds admit a virtual fundamental class.

Theorem 2.2. [Joy, Section 13] *Let $\mathcal{X}$ be a compact, oriented d-manifold with underlying topological space X. Then there exists a well-defined virtual fundamental class $[\mathcal{X}]^{\mathrm{vir}} \in H_{\mathrm{vd}}(X, \mathbb{Z})$, which only depends on the bordism class of $\mathcal{X}$.*

By [Sav22], we can give a simple description of $[\mathcal{X}]^{\mathrm{vir}}$ using a global Kuranishi chart $\mathcal{X} \simeq \mathcal{S}_{Y,E,s}$. Namely, we can associate a normal cone $C(s) \subseteq E|_X$ to the section s together with a homological fundamental class $[C(s)] \in H_N(C(s))$. Then, intersecting with the zero section of the oriented bundle $E|_X$ yields the virtual fundamental class so that $[\mathcal{X}]^{\mathrm{vir}} = 0^!_{E|_X}[C(s)] \in H_{\mathrm{vd}}(X, \mathbb{Z})$.

2.2. (-2)-shifted symplectic derived schemes. For an introduction to derived algebraic geometry, we refer the reader to [Toë14]. Shifted symplectic structures were introduced in the seminal paper [PTVV13].

In a nutshell, a derived scheme $\underline{\mathcal{X}}$ is locally modelled by the (derived) spectrum $\mathbf{Spec}\, A$ where A is a commutative differential (negatively) graded $\mathbb{C}$-algebra. A (-2)-shifted symplectic structure on $\underline{\mathcal{X}}$ is given by a 2-form

$$\omega_{\underline{\mathcal{X}}}\colon \mathbb{T}_{\underline{\mathcal{X}}} \wedge \mathbb{T}_{\underline{\mathcal{X}}} \longrightarrow \mathcal{O}_{\underline{\mathcal{X}}}[-2]$$

which is non-degenerate and is equipped with extra data that make it closed.

A (-2)-shifted symplectic scheme is a pair $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ consisting of a derived scheme $\underline{\mathcal{X}}$ and a symplectic structure on it. There is also an appropriate notion of orientation on a (-2)-shifted symplectic scheme (cf. [BJ17]).

Given the above, the obstruction sheaf of $\underline{\mathcal{X}}$ is the sheaf $\mathcal{O}b_{\underline{\mathcal{X}}} = h^1(\mathbb{T}_{\underline{\mathcal{X}}}|_X)$ and the obstruction space at a point $x \in \underline{\mathcal{X}}$ is the vector space $\mathcal{O}b(\underline{\mathcal{X}}, x) = h^1(\mathbb{T}_{\underline{\mathcal{X}}}|_x)$. A cosection is a morphism $\sigma\colon \mathbb{T}_{\underline{\mathcal{X}}}|_X[1] \to \mathcal{O}_X$. We will often work with the induced map on cohomology $\mathcal{O}b_{\underline{\mathcal{X}}} \to \mathcal{O}_X$.

In particular, the symplectic form $\omega_{\underline{\mathcal{X}}}$ induces a family of non-degenerate quadratic forms $q_x\colon \mathcal{O}b(\underline{\mathcal{X}}, x) \to \mathbb{C}$.

In [BBJ19], the local structure of $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ around every point $x \in X$ is described by Kuranishi charts in Darboux form as follows.

Theorem-Definition 2.3. [BBJ19, Example 5.16] *Let $x \in \underline{\mathcal{X}}$ be a point of a (-2)-shifted symplectic derived scheme $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$. Then there exists a Zariski open neighbourhood $\underline{U} = \mathbf{Spec}\, A \to \underline{\mathcal{X}}$ around x such that:*

(1) *(A, δ) is a commutative differential graded algebra, whose degree zero part A^0 is a smooth $\mathbb{C}$-algebra of dimension N, with a set of étale coordinates $\{x_i\}_{i=1}^N$.*

(2) *A is freely generated over A^0 by variables $\{y_j\}_{j=1}^r$ and $\{z_i\}_{i=1}^N$ in degrees -1 and -2 respectively.*

(3) *Let ω_A be the pullback of $\omega_{\underline{\mathcal{X}}}$ to $\mathbf{Spec}\, A$. There exist invertible elements $q_1, ..., q_r \in A^0$ such that*

$$\omega_A = \sum_{i=1}^N \mathrm{d}z_i \mathrm{d}x_i + \sum_{j=1}^r \mathrm{d}(q_j y_j)\mathrm{d}y_j.$$

(4) *The differential δ is determined by the equations*

$$\delta x_i = 0, \quad \delta y_j = s_j, \quad \delta z_i = \sum_{j=1}^r y_j\left(2q_j\frac{\partial s_j}{\partial x_i} + s_j\frac{\partial q_j}{\partial x_i}\right)$$

where the elements $s_j \in A^0$ satisfy

$$q_1 s_1^2 + ... + q_r s_r^2 = 0.$$

We let $Y = \mathrm{Spec}\, A^0$, E the trivial vector bundle of rank r whose dual has basis given by the variables y_j and F the trivial vector bundle whose dual has basis given by the variables z_i. We refer to all of the above data as a derived algebraic Darboux chart *for $\underline{\mathcal{X}}$ at x.*

At the classical level, the truncation $U = h^0(\underline{U}) \subseteq X$ is equipped with the following data:

(1) *A smooth affine scheme Y of dimension N.*

(2) *A trivial vector bundle E on V of rank r with a non-degenerate quadratic form q which is given by*

$$q_u(y_1, ..., y_r) = q_1(u)y_1^2 + ... + q_r(u)y_r^2$$

on each fiber of E over $u \in U$. Thus, E is naturally an orthogonal bundle, i.e. a $O(r, \mathbb{C})$-bundle.

(3) A q-isotropic section $s \in \Gamma(E)$ whose scheme-theoretic zero locus is $U \subseteq V$.

(4) A three-term perfect complex of amplitude $[0, 2]$

$$\mathbb{E}_\bullet := \mathbb{T}_{\underline{U}}|_U = [T_Y|_U \xrightarrow{ds} E|_U \xrightarrow{t} F|_U].$$

such that q_u gives a non-degenerate quadratic form on each obstruction space $\mathcal{O}b(\mathcal{X}, u)$.

We have a natural morphism $\phi\colon \mathbb{E}^\bullet = (\mathbb{E}_\bullet)^\vee \to \mathbb{L}_U$, which is surjective in degree -1 and an isomorphism in degree 0. $\mathbb{E}^\bullet$ is symmetric in the sense that $\mathbb{E}^\bullet \cong \mathbb{E}_\bullet[2]$ via q and the morphism ϕ is called the induced (three-term) symmetric obstruction theory on U. We write $\mathrm{vd} = \frac{1}{2} \mathrm{rk}\, \mathbb{E}_\bullet$ for the virtual dimension of U.

We refer to the data (Y, E, s) as an algebraic Darboux chart or algebraic, orthogonal Kuranishi chart for X at x.

By working in the complex analytic category, since the functions q_j are non-zero at x, by possibly shrinking V around x, they admit square roots $q_j = r_j^2$ and after re-parametrizing $y_j \mapsto \frac{y_j}{r_j}$ we may assume that $q_j(u) = 1$. We then refer to the data (Y, E, s) as an analytic Darboux chart or analytic, orthogonal Kuranishi chart for X at x.

2.3. **Virtual fundamental classes.** Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a proper (-2)-shifted symplectic derived scheme. As in the previous subsection, let $\mathbb{E}^\bullet = \mathbb{L}_{\underline{\mathcal{X}}}|_X \to \mathbb{L}_X$ be the induced (three-term) symmetric obstruction theory on X with virtual dimension $\mathrm{vd} = \mathrm{rk}\, \mathbb{E}_\bullet$.

Suppose also that $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ admits an orientation as defined in [BJ17], which basically amounts to a trivialization of the determinant of $\mathbb{E}_\bullet$.

Under these assumptions, Borisov–Joyce [BJ17] define a virtual fundamental class $[\underline{\mathcal{X}}]^{\mathrm{vir}} \in H_{\mathrm{vd}}(X, \mathbb{Z})$. Their method involves taking real smooth truncations of a cover of $\underline{\mathcal{X}}$ by orthogonal Kuranishi charts and performing homotopical gluing to construct an associated compact, oriented d-manifold $\mathcal{X}_{dm}$ of virtual dimension vd. While $\mathcal{X}_{dm}$ is not canonical, Borisov–Joyce show that it is well-defined up to bordism and hence so is its virtual fundamental class $[\mathcal{X}_{dm}]^{\mathrm{vir}} \in H_{\mathrm{vd}}(X, \mathbb{Z})$. Thus, one can unambiguously define $[\underline{\mathcal{X}}]^{\mathrm{vir}} := [\mathcal{X}_{dm}] \in H_{\mathrm{vd}}(X, \mathbb{Z})$.

Now, suppose moreover that $\mathrm{vd} \in 2\mathbb{Z}$ and $\mathbb{E}_\bullet$ admits a global resolution

$$\mathbb{E}_\bullet \cong [B \xrightarrow{d} E \cong E^\vee \xrightarrow{d^\vee} B^\vee],$$

so that the intrinsic normal cone $\mathcal{C}_X$ of X embeds as a closed substack of the vector bundle stack $[E/B]$. For example, this is the case when X is projective by [OT20, Section 4]. The choice of orientation of $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ then implies that E is a special orthogonal bundle, i.e. an $SO(2r, \mathbb{C})$-bundle for some integer r.

With these at hand, Oh–Thomas [OT20] use the fact that the cone $C = \mathcal{C}_X \times_{[E/B]} E \subseteq E$ is isotropic and a localized version of the square root Euler class $\sqrt{e}$ of Edidin–Graham [EG95] to define an algebraic virtual fundamental cycle $[X]^{\mathrm{vir}} = \sqrt{e}(E|_C, \tau)[C] \in A_{\mathrm{vd}/2}(X, \mathbb{Z}[1/2])$. Here τ denotes the tautological section of the bundle $E|_C$.

Assuming that X is projective, in their follow-up work [OT23], Oh–Thomas show that the image of the virtual cycle $[X]^{\mathrm{vir}}$ under the cycle class map $A_{\mathrm{vd}/2}(X, \mathbb{Z}[1/2]) \to H_{\mathrm{vd}}(X, \mathbb{Z}[1/2])$ agrees with the image of $[\underline{\mathcal{X}}]^{\mathrm{vir}}$ under the natural map $H_{\mathrm{vd}}(X, \mathbb{Z}) \to H_{\mathrm{vd}}(X, \mathbb{Z}[1/2])$.

They achieve this by showing that there exists a *global* orthogonal, complex Kuranishi chart (Y, E, s) of X in the sense of Theorem-Definition 2.3, which is suited to the (-2)-shifted symplectic structure of $\underline{\mathcal{X}}$ and is analytic upon restriction on X but smooth on the complement $Y \setminus X$. The following properties of the chart will be relevant for us:

(1) Y is a complex manifold, E is an oriented smooth bundle over Y and s is a smooth section of E such that $X \cong Z(s) \subseteq Y$.

(2) E admits a smooth, non-degenerate quadratic form $q \colon E \otimes E \to \mathbb{C}_Y$ with respect to which s is isotropic.

(3) These data are compatible with the Darboux charts used by Borisov–Joyce [BJ17].

(4) $E|_X$ is naturally a holomorphic vector bundle on the complex analytic space X, the derivative $\mathrm{d}s \colon T_Y|_X \to E|_X$ is a holomorphic morphism of holomorphic bundles on X, and the shifted tangent complex $\mathbb{T}_{\underline{\mathcal{X}}}|_X[1]$ is given by

$$(2.2) \qquad \mathbb{T}_{\underline{\mathcal{X}}}|_X[1] \simeq [T_Y|_X \xrightarrow{\mathrm{d}s} E|_X \simeq E|_X^{\vee} \xrightarrow{\mathrm{d}s^{\vee}} \Omega_Y|_X],$$

where the middle isomorphism is induced by the quadratic form q, which is also holomorphic when restricted to X.

Since E is an $SO(2r, \mathbb{C})$-bundle, we may take $E \cong E^+ \otimes_{\mathbb{R}} \mathbb{C} = E^+ \oplus E^-$, a splitting into the real and imaginary parts of E, on which $\mathrm{Re}(q)$ is positive and negative definite respectively. E^+ inherits a natural orientation. Write s^+ for the projection of s onto E^+ whose zero locus is $X \subseteq Y$, since s is isotropic. The properties of the chart (Y, E, s) then imply that we may take $\mathcal{X}_{dm} = \mathcal{S}_{Y, E^+, s^+}$ and we have the equality

$$[X]^{\mathrm{vir}} = [\mathcal{X}_{dm}] \in H_{\mathrm{vd}}(X, \mathbb{Z}[1/2]).$$

3. Reduced virtual fundamental classes

Let $\mathcal{X}$ be a compact, oriented d-manifold of virtual dimension n with underlying topological space X, whose truncation is the $\mathcal{C}^\infty$-scheme $X_{\mathcal{C}^\infty}$. Throughout this section, $\sigma \colon \mathcal{O}b_{\mathcal{X}} \to \mathbb{R}_X$ will be a surjective real cosection for $\mathcal{X}$.

To fix more notation, by Theorem-Definition 2.1, $\mathcal{X}$ is equivalent to a principal d-manifold $\mathcal{S}_{Y, E, s}$, where Y is an open subset of some Euclidean space $\mathbb{R}^N$, E is a smooth oriented vector bundle on Y and s a smooth section of E. Therefore, a model for the tangent complex of $\mathcal{X}$ is given by (2.1).

3.1. **Reduced classes via reduced d-manifold structure.** We may assume that Y retracts onto X so that we have an isomorphism

$$(3.1) \qquad\qquad f_* \colon H_i(X) \longrightarrow H_i(Y)$$

is an isomorphism for any i, where $f \colon X \to Y$ denotes the inclusion map.

Given these data, we obtain a surjection $\sigma_E \colon E|_{X_{\mathcal{C}^\infty}} \to \mathcal{O}b_{\mathcal{X}} \to \mathbb{R}_X$ of sheaves on $X_{\mathcal{C}^\infty}$. In particular, after possibly shrinking Y around X, we

may assume that σ_E extends to a surjective morphism of vector bundles $\sigma_E \colon E \to \mathbb{R}_Y$ on Y. Write $E(\sigma)$ for the kernel bundle of this map so that we have an exact sequence of vector bundles

$$(3.2) \qquad 0 \longrightarrow E(\sigma) \longrightarrow E \longrightarrow \mathbb{R}_Y \longrightarrow 0.$$

Note that the orientation of E induces a natural orientation on $E(\sigma)$.

After picking a smooth metric g on E, we obtain a smooth splitting of (3.2)

$$(3.3) \qquad E \cong_g E(\sigma) \oplus \mathbb{R}_Y.$$

Let $s(\sigma)_g$ be the first component of s under this isomorphism. We obtain an induced principal d-manifold $\mathcal{X}_g = \mathcal{S}_{Y,E(\sigma),s(\sigma)_g}$ of virtual dimension $n+1$.

Since $E(\sigma)$ is oriented, $\mathcal{X}_g$ is an oriented d-manifold. Moreover, as $\mathcal{X}$ is compact, after possibly shrinking Y around X, we may assume that E and s extend to the closure $\overline{Y}$ of Y in $\mathbb{R}^N$ and, in addition, the vanishing locus of $s(\sigma)_g$ does not meet the boundary $\overline{Y} \setminus Y$. This implies that the underlying topological space X_g of $\mathcal{X}_g$ is compact, as a closed and bounded subset of $\overline{Y}$, which is disjoint from $\overline{Y} \setminus Y$. We conclude that $\mathcal{X}_g$ is a compact, oriented d-manifold.

Write $f_g \colon X_g \to Y$ for the inclusion map. We may now give the following definition.

Definition 3.1. *The* reduced J-virtual fundamental class *of the d-manifold* $\mathcal{X}$ *is defined as the J-virtual fundamental class of the d-manifold* $\mathcal{X}_g$, *that is,*

$$(3.4) \qquad [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J} := (f_*)^{-1}(f_g)_*[\mathcal{X}_g]^{\mathrm{vir}} \in H_{n+1}(X).$$

A priori, $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J}$ depends on the choices of the principal model $\mathcal{S}_{Y,E,s}$ of $\mathcal{X}$ as well as the metric g. We will see later on that it is independent of choices and hence well-defined.

Remark 3.2. *Given a choice of presentation* $\mathcal{X} \simeq \mathcal{S}_{Y,E,s}$, *it is easy to see that different choices of metric* g_1 *and* g_2 *result in bordant d-manifolds* $\mathcal{X}_{g_1}$ *and* $\mathcal{X}_{g_2}$ *over* Y *by interpolating between the metrics via the path* $g_t = (1-t)g_1 + tg_2$. *Since the virtual fundamental class of a d-manifold only depends on its bordism class, it follows that* $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J}$ *is independent of the choice of metric and can only possibly depend on the choice of principal model* $\mathcal{S}_{Y,E,s}$.

3.2. **Reduced classes via cone reduction.** We now give an alternative definition of the reduced class $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}}$. We continue using the notation of the previous subsection.

Recall that by the smooth cone reduction criterion [Sav22, Proposition 4.1], the normal cone $C(s)$ is contained entirely in the kernel bundle $E(\sigma)|_X$.

Definition 3.3. *The* reduced BF-virtual fundamental class *of the d-manifold* $\mathcal{X}$ *is defined as*

$$(3.5) \qquad [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF} := 0^!_{E(\sigma)|_X}[C(s)] \in H_{n+1}(X).$$

Again, we shall soon see that $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF}$ is independent of the choice of principal model $\mathcal{S}_{Y,E,s}$ of $\mathcal{X}$.

3.3. **Reduced classes are well-defined.** In this subsection, we check that the definitions of reduced classes we have given are consistent with each other and independent of all choices involved, as promised.

Proposition 3.4. $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J} = [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF}.$

Proof. By (3.4) and (3.5), we need to show that

$$(3.6) \qquad (f_g)_*[\mathcal{X}_g]^{\mathrm{vir}} = f_* 0^!_{E(\sigma)|_X}[C(s)] \in H_{n+1}(Y).$$

Let s' be a small perturbation of s which intersects the zero section Y of E transversely. In particular, this implies that $s'(\sigma)_g$ is a small perturbation of $s(\sigma)_g$, which intersects the zero section of $E(\sigma)$ transversely.

Write X' and X'_g for the zero loci of s' and $s'(\sigma)_g$ respectively. Let $f' \colon X' \to Y$ and $f'_g \colon X'_g \to Y$ be the inclusion maps.

By definition, we then have that

$$(3.7) \qquad (f_g)_*[\mathcal{X}_g]^{\mathrm{vir}} = (f'_g)_*[X'_g] \in H_{n+1}(Y).$$

By the cartesian square

$$(3.8) \qquad \begin{array}{ccc} E(\sigma)|_X & \xrightarrow{\ h\ } & E(\sigma) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

we have $f_* 0^!_{E(\sigma)|_X} = 0^!_E h_*$ and hence

$$(3.9) \qquad f_* 0^!_{E(\sigma)|_X}[C(s)] = 0^!_{E(\sigma)} h_*[C(s)].$$

The proof of the cone reduction criterion [Sav22, Proposition 4.1] shows that $C(s) = C(s(\sigma)_g)$ as subsets of $E(\sigma)$. By the definition of the homological normal cone, it then follows that

$$(3.10) \qquad h_*[C(s)] = h_*[C(s(\sigma)_g)] = [\Gamma_{s(\sigma)_g}] = [\Gamma_{s'(\sigma)_g}] \in H_N(E(\sigma)),$$

where the last equality is due to the fact that $s'(\sigma)_g$ is a small perturbation of $s(\sigma)_g$.

Since $(f'_g)_*[X'_g] = 0^!_{E(\sigma)}[\Gamma_{s'(\sigma)_g}]$, combining (3.7), (3.9) and (3.10) finishes the proof. $\qquad\square$

It follows from the above proposition that $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J}$ does not depend on the choice of metric g, since $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF}$ does not involve the metric. From now on we write $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}}$ for the reduced virtual fundamental class of $\mathcal{X}$. To show that it is well-defined, we thus only need to check that it is independent of the principal d-manifold model $\mathcal{S}_{Y,E,s}$ we have used for $\mathcal{X}$.

Proposition 3.5. $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}}$ *is independent of the choice of presentation* $\mathcal{X} \simeq \mathcal{S}_{Y,E,s}$ *and thus well-defined.*

Proof. Suppose that $\mathcal{X} \simeq \mathcal{S}_{Y_1,E_1,s_1} \simeq \mathcal{S}_{Y_2,E_2,s_2}$. We work with the BF-virtual fundamental class in what follows.

Following the argument in the proof of [Sav22, Proposition 5.2], we may reduce to the case of an equivalence of the following standard form: There is

a submersion $h\colon Y_1 \to Y_2$ and a surjection of vector bundles $\widehat{h}\colon E_1 \to h^*E_2$ such that $\widehat{h} \circ s_1 = h^*s_2$ and a commutative diagram

$$
\begin{array}{ccccc}
T_{Y_1}|_X & \xrightarrow{\mathrm{d}s_1} & E_1|_X & & \\
\downarrow{\scriptstyle \mathrm{d}h} & & \downarrow{\scriptstyle \widehat{h}} & \searrow & \\
h^*T_{Y_2}|_X & \xrightarrow[\mathrm{d}(h^*s_2)]{} & E_2|_X & \longrightarrow & \mathcal{O}b_{\mathcal{X}} \xrightarrow{\sigma} \mathbb{R}_X,
\end{array}
$$

where the morphism of tangent complexes is a quasi-isomorphism on all fibers over points $x \in X$.

Since both derivatives $\mathrm{d}s_1$ and $\mathrm{d}s_2$ factor through the kernel bundles $E_1(\sigma)$ and $E_2(\sigma)$ we get an induced quasi-isomorphism (on fibers over $x \in X$) of two-term complexes

$$
\begin{array}{ccc}
T_{Y_1}|_X & \xrightarrow{\mathrm{d}s_1} & E_1(\sigma)|_X \\
\downarrow{\scriptstyle \mathrm{d}h} & & \downarrow{\scriptstyle \widehat{h}} \\
h^*T_{Y_2}|_X & \xrightarrow{\mathrm{d}(h^*s_2)} & E_2(\sigma)|_X.
\end{array}
$$

It follows that the projection $\widehat{h}$ induces a submersion $\widehat{h}(\sigma)\colon E_1(\sigma) \to E_2(\sigma)$, which makes $E_1(\sigma)$ into a vector bundle over $E_2(\sigma)$. We pick a splitting of $\widehat{h}$ so that $E_1(\sigma) \cong E_2(\sigma) \oplus E_3$, for some vector bundle E_3 on Y.

Now, $\widehat{h}$ and hence $\widehat{h}(\sigma)$ map $C(s_1)$ onto $C(s_2)$ such that $C(s_1)$ is a vector bundle over $C(s_2)$ and under the chosen splitting we actually have $C(s_1) \cong C(s_2) \oplus E_3|_X$.

Therefore, we conclude that

$$0^!_{E_1(\sigma)|_X}[C(s_1)] = 0^!_{E_2(\sigma)|_X \oplus E_3|_X}[C(s_2)] \oplus [E_3|_X] = 0^!_{E_2(\sigma)|_X}[C(s_2)],$$

which is what we want. $\square$

We may summarize as follows.

Theorem 3.6. *Let $\mathcal{X}$ be a compact, oriented d-manifold of virtual dimension n with underlying topological space X and $\sigma\colon \mathcal{O}b_{\mathcal{X}} \to \mathbb{R}_X$ a surjective real cosection. Then there exists an induced, well-defined reduced virtual fundamental class $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+1}(X, \mathbb{Z})$.*

3.4. **The case of multiple cosections.** We finally note that we can define reduced virtual fundamental classes in the presence of multiple cosections.

Namely, let $\underline{\sigma}\colon \mathcal{O}b_{\mathcal{X}} \to \mathbb{R}^k_X$ be a surjective morphism of sheaves on $X_{\mathcal{C}^\infty}$. Write $E(\underline{\sigma})$ for the kernel of the surjection $E \to \mathcal{O}b_{\mathcal{X}} \xrightarrow{\sigma} \mathbb{R}^k_X$, which is a vector bundle with a naturally induced orientation. Then, we may define a reduced virtual fundamental class $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+k}(X)$, exactly as before, in two ways:

(1) After picking a metric g on E, we have an isomorphism $E \cong E(\underline{\sigma}) \oplus \mathbb{R}^k_X$. Let $s(\underline{\sigma})_g$ be the first component of s under this isomorphism. Then, we obtain a compact, oriented d-manifold $\mathcal{X}_g = \mathcal{S}_{Y,E(\underline{\sigma}),s(\underline{\sigma})_g}$ of virtual dimension $n+k$ with underlying topological space X_g and we define

$$[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J} := (f_*)^{-1}(f_g)_*[\mathcal{X}_g]^{\mathrm{vir}} \in H_{n+k}(X),$$

where $f_g \colon X_g \to Y$ denotes the inclusion map.

(2) A straightforward generalization of the cone reduction criterion [Sav22, Proposition 4.1] shows that the normal cone $C(s)$ is contained in $E(\underline{\sigma})$. We can thus define

$$[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF} := 0^!_{E(\underline{\sigma})|_X}[C(s)] \in H_{n+k}(X).$$

All the previous results immediately generalize in this setting so that $[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},J} = [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red},BF}$ and the reduced virtual fundamental class is independent of the choice of presentation $\mathcal{X} \simeq \mathcal{S}_{E,Y,s}$ and metric g.

4. Cosection localization and reduced virtual fundamental classes for (-2)-shifted symplectic derived schemes

Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a proper, oriented, (-2)-shifted derived scheme with underlying classical scheme (and, by abuse of notation, topological space) X, equipped with a meromorphic cosection, i.e. a morphism $\sigma \colon \mathbb{T}_{\underline{\mathcal{X}}}|_X[1] \dashrightarrow \mathcal{O}_X$ defined on an open subscheme $\underline{\mathcal{U}}(\sigma) \subseteq \underline{\mathcal{X}}$ with underlying classical scheme (and topological space) $U(\sigma)$.

Write $\mathcal{X}_{dm}, \mathcal{U}(\sigma)_{dm}$ for the associated d-manifolds to the (-2)-shifted symplectic schemes $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ and $(\underline{\mathcal{U}}(\sigma), \omega_{\underline{\mathcal{X}}}|_{\underline{\mathcal{U}}(\sigma)})$ respectively. In [Sav22], we gave non-degeneracy conditions under which σ induces a surjective real cosection on $\underline{\mathcal{U}}_{dm}$. This in turn allowed us to localize the virtual fundamental class $[\mathcal{X}_{dm}]^{\mathrm{vir}}$ to the degeneracy locus $X(\sigma) = X \setminus U(\sigma)$ by the methods of [KL13, Appendix A].

In this section, we use the machinery of complex Kuranishi structures introduced recently by Oh–Thomas [OT23] to establish a similar result for *any* cosection. The global complex Kuranishi charts constructed by Oh–Thomas allow us to bypass any subtleties arising from possible different choices of local Darboux charts for $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ and simplify the arguments used in [Sav22].

In the algebraic setting, cosection localization for (-2)-shifted symplectic schemes with isotropic cosections was established by Kiem–Park in [KP20]. Thus our results match the expectation that there should be a differential geometric analogue. Conversely, the case of non-degenerate cosections was subsequently treated algebraically in [BKP22]. Hence, we now have a complete picture from both points of view. We will see in the next section that all definitions and different constructions are consistent with each other.

In addition to the above, we now assume that X is projective. Take a global orthogonal, complex Kuranishi chart (Y, E, s) of X, satisfying the properties (1)-(4) listed in Subsection 2.3. In particular, truncate this chart to obtain a principal d-manifold $\mathcal{S}_{Y,E^+,s^+}$, which is a valid choice of model for the compact, oriented d-manifold $\mathcal{X}_{dm}$ associated to $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$.

Theorem 4.1. *Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted derived scheme with a regular, i.e. globally defined, and surjective cosection σ. Then, given a global orthogonal, complex Kuranishi chart (Y, E, s) for X, σ induces a surjective real cosection σ^+ on the d-manifold $\mathcal{X}_{dm} = \mathcal{S}_{Y,E^+,s^+}$.*

Proof. Without loss of generality, by a standard argument (cf. [KP20, Lemma 7.4]) we may assume that σ is represented by an honest morphism

of complexes

$$(4.1) \qquad \begin{array}{ccccc} T_Y|_X & \xrightarrow{\mathrm{d}s} & E|_X \simeq E|_X^{\vee} & \xrightarrow{\mathrm{d}s^{\vee}} & \Omega_Y|_X \\ \downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

It follows that the map $\sigma \colon E|_X \to \mathcal{O}_X$ must be surjective.

σ^+ is then defined via the composition $E^+|_X \subseteq E|_X \xrightarrow{\sigma} \mathbb{C}_X \to \mathbb{R}_X$, where the last surjection will be defined shortly. It is evident that this induces a well-defined real cosection on $\mathcal{X}_{dm}$.

If $\sigma_x \colon E|_x \to \mathbb{C}_X|_x = \mathbb{C}$ satisfies that $\sigma_x^{\vee}$ defines a non-degenerate line in $E|_x^{\vee}$ with respect to the quadratic form $q|_x$, then, since σ_x is $\mathbb{C}$-linear, it is the complexification of a surjective map $\sigma_x^+ \colon E^+|_x \to \mathbb{R}$. We thus have a projection $\mathbb{C}_X \to \mathbb{R}_X$ defined on the open locus of $x \in X$ where σ_x is non-degenerate, such that the induced map $E^+|_X \to \mathbb{R}_X$ is surjective. We extend smoothly to the whole of X.

On the other hand, if $\sigma_x \colon E|_x \to \mathbb{C}_X|_x = \mathbb{C}$ satisfies that $\sigma_x^{\vee}$ defines an isotropic line in $E|_x^{\vee}$ with respect to the quadratic form $q|_x$, then $E^+|_x \subseteq E|_x \to \mathbb{C}$ is surjective and hence so is the map $E^+|_x \to \mathbb{C} \to \mathbb{R}$ as well. If not, then $\mathrm{im}(\sigma_x^{\vee}) \cap (E^-|_x)^{\vee}$ would be non-empty, which is impossible. This concludes the proof. $\qquad \square$

As a corollary, we may now generalize [Sav22, Theorem 5.8].

Theorem 4.2. *Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted symplectic derived scheme of virtual dimension $n = \mathrm{rk}\,\mathbb{L}_{\underline{\mathcal{X}}}|_X$ with a meromorphic cosection σ. Then there exists a cosection localized virtual fundamental class $[\mathcal{X}_{dm}]^{\mathrm{vir}}_{\mathrm{loc},\sigma} \in H_n(X(\sigma), \mathbb{Z})$ satisfying*

$$i_*[\mathcal{X}_{dm}]^{\mathrm{vir}}_{\mathrm{loc},\sigma} = [\mathcal{X}_{dm}]^{\mathrm{vir}} \in H_n(X, \mathbb{Z}),$$

where $i \colon X(\sigma) \to X$ is the inclusion map.

Proof. By Theorem 4.1, σ induces a surjective cosection σ^+ on the open sub-d-manifold of $\mathcal{X}_{dm}$ with underlying topological space $U(\sigma)$. The proof of [Sav22, Theorem 5.8] then applies verbatim to show the statement, using [Sav22, Theorem-Definition 5.3, Remark 5.4]. $\qquad \square$

Remark 4.3. *For a meromorphic cosection σ on a (-2)-shifted symplectic scheme $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$, we say that σ is* isotropic *if the induced quadratic form q_x on $\mathcal{O}b(\underline{\mathcal{X}}, x)^{\vee}$ is zero on the image of $\sigma_x^{\vee}$ for all $x \in X$. Equivalently, the composition*

$$(4.2) \qquad \mathcal{O}_{U(\sigma)} \xrightarrow{\sigma^{\vee}} \mathbb{L}_{\underline{\mathcal{U}}(\sigma)}|_{U(\sigma)}[-1] \xrightarrow{\omega_{\underline{\mathcal{X}}}|_{U(\sigma)}} \mathbb{T}_{\underline{\mathcal{U}}(\sigma)}|_{U(\sigma)}[1] \xrightarrow{\sigma} \mathcal{O}_{U(\sigma)}$$

vanishes.

When X is proper and σ is regular, i.e. defined everywhere on X, then σ is either non-degenerate or isotropic depending on whether the morphism (4.2) is non-zero or vanishes respectively. These are the two most common cases in the literature. The theorem above allows for the greater flexibility of any meromorphic cosection σ.

The proof of Theorem 4.1 immediately implies the following statement on reduced classes.

Theorem 4.4. *Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted symplectic derived scheme of virtual dimension n with a regular, everywhere surjective cosection σ. Then:*

(1) *If σ is non-degenerate, it induces a surjective real cosection on $[\mathcal{X}_{dm}]$. Hence, there is a reduced virtual fundamental class $[\mathcal{X}_{dm}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+1}(X, \mathbb{Z})$.*

(2) *If σ is isotropic, the morphism $E^+ \subseteq E \xrightarrow{\sigma} \mathbb{C}_X$ induces a surjective complex cosection $\sigma^+ \colon \mathcal{O}b_{\mathcal{X}_{dm}} \to \mathbb{C}_X = \mathbb{R}^2_X$ on $[\mathcal{X}_{dm}]$. Hence, there is a reduced virtual fundamental class $[\mathcal{X}_{dm}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+2}(X, \mathbb{Z})$.*

5. COMPARISON BETWEEN DIFFERENTIAL GEOMETRIC AND COMPLEX ANALYTIC COSECTION LOCALIZED VIRTUAL FUNDAMENTAL CLASSES

We now proceed to extend the comparison result of [OT23] in the case of even virtual dimension, showing that the differential geometric and complex analytic versions of the virtual fundamental class of a (-2)-shifted symplectic scheme are equal in homology, to the cosection localized case.

5.1. The square root Euler class.
We start by giving a homological formula for the square root Euler class of Oh–Thomas [OT20] and Kiem–Park [KP20].

We set up some notation. Let E be an $SO(2r)$-bundle over a scheme Y together with two isotropic sections $s, \sigma^\vee$ satisfying $s \cdot \sigma^\vee = 0$ where $(-) \cdot (-)$ denotes the pairing induced by the quadratic form on E. Using the quadratic structure of E, the section $\sigma^\vee$ is equivalent to a cosection $\sigma \colon E \cong E^\vee \to \mathcal{O}_Y$.

Let X be the vanishing locus of s and $X(\sigma) \subseteq X$ the vanishing locus of $\sigma|_X$ and moreover assume that $s, \sigma^\vee$ are linearly independent away from a closed subset $Z \subseteq Y$ which contains the vanishing locus $\sigma^{-1}(0)$ and satisfies $X \cap Z = X(\sigma)$ (cf. [KP20, Definition 6.4]). From now on, let us write $t = \sigma^\vee$ for brevity and consistency with the notation of Kiem–Park [KP20].

Kiem–Park then define a cosection localized square root Euler class map

$$(5.1) \qquad \sqrt{e}(E, s; t) \colon A_*(Y, \mathbb{Z}) \longrightarrow A_{*-r}(X(\sigma), \mathbb{Z}[1/2]).$$

We establish some further notation and recall the definition in order to prove a formula in homology.

Let $\rho \colon \widetilde{Y} \to Y$ be the blowup of Y along X with exceptional divisor D. Write $L = \mathcal{O}_{\widetilde{Y}}(D)$ and $\widetilde{E} = L^\perp/L$ for the induced $SO(2r-2)$-bundle. Using the fact that $s \cdot t = 0$, the pullback $\rho^* t \in H^0(\widetilde{Y}, \rho^* E)$ induces an isotropic section $\widetilde{t}$ of $\widetilde{E}$. Write s_D for the tautological section of L.

We have a commutative diagram

$$(5.2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L^\perp & \longrightarrow & \widetilde{E} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \rho^* E & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences of vector bundles. By definition, $\rho^* s$ factors through the tautological section s_D.

We also have a commutative diagram

$$(5.3) \qquad \begin{array}{ccccccc} D(t) & \xrightarrow{\imath''} & D(\widetilde{t}) & \xrightarrow{\jmath'} & D & \xrightarrow{\jmath} & \widetilde{Y} \\ \downarrow & & \downarrow{\rho''} & & \downarrow{\rho'} & & \downarrow{\rho} \\ X(\sigma) & \xrightarrow[\imath'']{} & X(\sigma)^{\#} & \xrightarrow[\imath']{} & X & \xrightarrow[\imath]{} & Y, \end{array}$$

in which $X(\sigma)^{\#} = \rho'(D \cap \widetilde{t}^{-1}(0)) \cup X(\sigma) \subseteq X$ and the rightmost square is cartesian.

By our assumption on the independence of s and t, [KP20, Theorem 6.5] shows that $X(\sigma)^{\#}_{\mathrm{red}} \subseteq X(\sigma)_{\mathrm{red}}$, so, while in general the codomain of $\sqrt{e}(E, s; t)$ is the Chow group of $X(\sigma)^{\#}$, we may replace it by $X(\sigma)$, as we have already done in (5.1).

Let $\gamma \in A_*(Y, \mathbb{Z})$. Then there exist classes $\alpha \in A_*(\widetilde{Y}, \mathbb{Z}), \beta \in A_*(X, \mathbb{Z})$ such that $\gamma = \rho_* \alpha + \imath_* \beta$ and Kiem–Park [KP20] define

$$(5.4) \qquad \sqrt{e}(E, s; t) = \rho''_* \jmath^! \sqrt{e}(\widetilde{E}, \widetilde{t}) \alpha + \imath''_* \sqrt{e}(F, t) \beta,$$

where $\sqrt{e}$ is the square root Euler class defined by Oh–Thomas [OT20]. It is shown in [KP20] that this formula is well-defined and enjoys the expected properties of an Euler class.

Since E is an oriented, orthogonal bundle, write $E \cong E^+ \otimes_{\mathbb{R}} \mathbb{C}$ and s^+, t^+ for the real parts of s, t respectively. We reserve the same notation for other quadratic bundles and their sections, when clear from context.

Lemma 5.1. *Under the cycle class map $A_*(X(\sigma), \mathbb{Z}[1/2]) \to H_*(X(\sigma), \mathbb{Z}[1/2])$, we have*

$$(5.5) \qquad \sqrt{e}(E, s; t)[Y] = [C(s^+)] \cap [\Gamma_{t^+}],$$

where the intersection takes place inside E^+.

Proof. Using the notation preceding the lemma, since $[Y] = \rho_*[\widetilde{Y}]$, (5.4) implies that

$$(5.6) \qquad \begin{aligned} \sqrt{e}(E, s; t)(\alpha) &= \rho''_* \jmath^! \sqrt{e}(\widetilde{E}, \widetilde{t})[\widetilde{Y}] \\ &= \rho''_* \jmath^!([C(\widetilde{t})] \cap [\widetilde{E}^-]) \\ &= \rho''_* \jmath^!([C(\widetilde{t}^+)] \cap [0_{\widetilde{E}^+}]) \\ &= \rho''_* \jmath^!([\Gamma_{\widetilde{t}^+}] \cap [0_{\widetilde{E}^+}]), \end{aligned}$$

where the two intermediate equalities follow from [OT23, Theorem 10.2], [OT23, Theorem 7.23] and [OT23, Equation (7.25)], and the last equality from the fact that, as homology classes on $\widetilde{E}^+$, $[C(\widetilde{t}^+)] = [\Gamma_{\widetilde{t}^+}]$, whose intersection with the zero section is contained inside the zero locus $\widetilde{t}^{-1}(0)$.

We may now choose a $\mathbb{C}$-linear smooth splitting of the first row of (5.2) and consider $\widetilde{E}$ as a subbundle of $\rho^* E$, which intersects L trivially. In particular, we can write $\rho^* E = \widetilde{E} \oplus R$ for some vector bundle R, such that the inclusion map $L \to \rho^* E$ factors through R, and thus we have an induced splitting $\rho^* E^+ = \widetilde{E}^+ \oplus R^+$.

Note that since L is an isotropic (complex) line bundle in ρ^*E, the composite morphism $L \to \rho^*E \to R^+$ is an isomorphism of smooth (real) bundles which maps the section s_D to ρ^*s^+.

Based on these observations, it follows by the definition of $\jmath^!$ that

$$(5.7) \qquad \jmath^!([\Gamma_{\widetilde{t}^+}] \cap [0_{\widetilde{E}^+}]) = [\Gamma_{\rho^*s^+}] \cap [\Gamma_{\widetilde{t}^+}],$$

where the intersection takes place inside ρ^*E^+.

Since the sections $\widetilde{t}^+$ and ρ^*t^+ are homotopic to each other and their graphs intersect the graph of ρ^*s^+ at the same subset of ρ^*E^+, we further obtain

$$(5.8) \qquad [\Gamma_{\rho^*s^+}] \cap [\Gamma_{\widetilde{t}^+}] = [\Gamma_{\rho^*s^+}] \cap [\Gamma_{\rho^*t^+}].$$

Finally, observe that $[\Gamma_{\rho^*t^+}] = \rho^*[\Gamma_{t^+}]$, where by abuse of notation ρ also denotes the map $\rho^*E^+ \to E^+$, so by the projection formula

$$
\begin{aligned}
(5.9) \qquad \rho_*\left([\Gamma_{\rho^*s^+}] \cap [\Gamma_{\rho^*t^+}]\right) &= \rho_*\left([\Gamma_{\rho^*s^+}] \cap \rho^*[\Gamma_{t^+}]\right) \\
&= [\Gamma_{s^+}] \cap [\Gamma_{t^+}] \\
&= [C(s^+)] \cap [\Gamma_{t^+}],
\end{aligned}
$$

again using the fact that $[\Gamma_{s^+}] = [C(s^+)]$, as homology classes on E^+.

Combining (5.6), (5.7), (5.8) and (5.9) finishes the proof. $\qquad\square$

Remark 5.2. *Observe that in the above, E, s and $\sigma^\vee$ do not need to be algebraic or holomorphic, given that X, Y are complex analytic spaces and the inclusions $X(\sigma) \to X$ and $X \to Y$ are complex analytic. It is sufficient that they are smooth or even continuous, and complex linear. This will be implicitly used in the following section.*

5.2. **Everything agrees.** Let $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$ be a projective, oriented (-2)-shifted symplectic derived scheme of (complex) virtual dimension $n = \mathrm{rk}\,\mathbb{L}_{\underline{\mathcal{X}}}|_X$ with an isotropic cosection σ, which is surjective on an open subset $U(\sigma)$ with complement $X(\sigma) = X \setminus U(\sigma)$.

Take (Y, E, s) a global complex Kuranishi chart for X with E an $SO(2r)$-bundle, s isotropic so that σ is represented by an honest morphism of complexes

$$
(5.10) \qquad
\begin{array}{ccccc}
T_Y|_X & \xrightarrow{\;\mathrm{d}s\;} & E|_X \simeq E|_X^\vee & \xrightarrow{\;\mathrm{d}s^\vee\;} & \Omega_Y|_X \\
\downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0,
\end{array}
$$

and extends smoothly to a morphism of smooth vector bundles which is surjective on an open neighborhood $U(\sigma) \subseteq V$ of $U(\sigma)$ in Y. Moreover, using a smooth metric on E and $|s|^2$ as a smooth function that vanishes on X and is non-zero on the complement $Y \setminus X$, up to a smooth change of σ off X and possible shrinking of Y around X, we may assume that $\sigma \circ s = 0$ identically on Y, $\sigma^\vee$ is isotropic and $s, \sigma^\vee$ are independent away from $X(\sigma) = Y \cap \sigma^{-1}(0)$. Thus, we have in particular that $s \cdot \sigma^\vee = 0$ on Y.

Fix an isomorphism $E \cong E^+ \oplus E^-$, as usual and write s^+, σ^+ for the real components of s, σ respectively.

Lemma 5.3. *The Kiem–Park cosection localized virtual fundamental class of X equals $\sqrt{e}(E, s; \sigma^\vee)[Y]$ in homology.*

Proof. Let τ be the tautological section of $E|_{C(s)}$ whose zero locus equals $X \subseteq C(s)$. By [KP20, Theorem 8.2], the algebraic cosection localized virtual fundamental class of X is equal to

$$\sqrt{e}(E|_{C(s)}, \tau; \sigma|_{C(s)}^\vee)[C(s)] \in H_n(X(\sigma), \mathbb{Z}[1/2]).$$

However, by deformation to the normal cone, the data (Y, E, s) deform to the tautological section $(C(s), E|_{C(s)}, \tau)$, together with the cosection σ which deforms to $\sigma|_{C(s)}$. Thus, by deformation invariance of the square root Euler class , it follows that

$$\sqrt{e}(E|_{C(s)}, \tau; \sigma|_{C(s)}^\vee)[C(s)] = \sqrt{e}(E, s; \sigma^\vee)[Y],$$

as we want. $\qquad\square$

Theorem 5.4. $[\mathcal{X}_{dm}]_{\mathrm{loc}, \sigma}^{\mathrm{vir}} = \sqrt{e}(E, s; \sigma^\vee)[Y] \in H_n(X(\sigma), \mathbb{Z}[1/2])$.

Proof. This follows immediately from (5.5) and Lemma 5.3, since, by [Sav22, Theorem-Definition 5.3], it holds by construction that $[\mathcal{X}_{dm}]_{\mathrm{loc}, \sigma}^{\mathrm{vir}} = [C(s^+)] \cap [\Gamma_{(\sigma^+)^\vee}]$. $\qquad\square$

Combining the above, we conclude that the algebraic and differential geometric cosection localized virtual fundamental classes agree in homology.

Remark 5.5. *Besides producing integral classes in homology, one advantage of the differential geometric approach to cosection localization is seen in the case of odd virtual dimension, where we still have potentially non-trivial cosection localized virtual classes, which however must be 2-torsion by the results of* [OT23]. *It is an interesting question to examine whether this phenomenon of non-trivial 2-torsion virtual fundamental classes (cosection localized or not) can indeed occur.*

5.3. **Equality of reduced classes: the isotropic case.** We continue to explain that our comparison result is also true for reduced classes defined in the presence of surjective cosections.

We work in the following slightly more general setup (cf. [KP20, Assumption 8.1]): Let X be a projective scheme with a symmetric obstruction theory $\mathbb{E}^\bullet \to \mathbb{L}_X$, where $\mathbb{E}^\bullet$ is perfect of amplitude $[-2, 0]$ and even rank $n = \mathrm{rk}\, \mathbb{E}^\bullet$ and there is a resolution $\mathbb{E}^\bullet \cong [B \xrightarrow{d} F \cong F^\vee \xrightarrow{d^\vee} B^\vee]$. Moreover, assume that we have an orientation $\mathcal{O}_X \cong \det \mathbb{E}^\bullet$ so that F is an $SO(2r)$-bundle and the cone $C = \mathcal{C}_X \times_{[F/B]} F \subseteq F$ is isotropic.

Let us now assume that there is a surjective, isotropic cosection $\sigma \colon \mathbb{E}_\bullet[1] = (\mathbb{E}^\bullet)^\vee[1] \to \mathcal{O}_X$, which is induced by a morphism $\sigma \colon F \to \mathcal{O}_X$.

By [OT23], given the above, there is a global, orthogonal complex Kuranishi chart (Y, E, s) for X such that $F = E|_X, d = \mathrm{d}s$, s is an isotropic section of E and σ extends smoothly to a morphism $\sigma \colon E \to \mathbb{C}_Y$. As in the previous subsection, we may assume that $\sigma^\vee$ is isotropic, $s \cdot \sigma^\vee = 0$ and $s, \sigma^\vee$ are independent away from $Y \cap \sigma^{-1}(0) = X(\sigma)$.

The associated compact, oriented d-manifold to these data is $\mathcal{X} = \mathcal{S}_{Y, E^+, s^+}$.

Consider the $SO(2r-2)$-bundle $\widetilde{E} = \langle\sigma^\vee\rangle^\perp/\langle\sigma^\vee\rangle$. By the orthogonal cone reduction lemma [KP20], there is a closed embedding $C(s) \subseteq \widetilde{E}$ and the reduced algebraic virtual fundamental cycle of X is defined by the formula [KP20, Definition 8.7]

$$(5.11) \qquad [X]^{\mathrm{vir}}_{\mathrm{red}} := \sqrt{e}(\widetilde{E}|_{C(s)}, \tau)[C(s)] \in A_{\frac{1}{2}n+1}(X, \mathbb{Z}[1/2]),$$

where τ denotes, as usual, the tautological section of $\widetilde{E}|_{C(s)}$ whose zero locus is X.

By the orthogonality and independence of s and $\sigma^\vee$, s induces a smooth isotropic section $\widetilde{s}$ of $\widetilde{E}$ with zero locus X. We thus obtain a principal compact, oriented d-manifold $\mathcal{X}_{\mathrm{red}} = \mathcal{S}_{Y,\widetilde{E}^+,\widetilde{s}^+}$.

It is easy to see that $C(s) = C(\widetilde{s})$, so by deformation to the normal cone, we obtain the equality

$$(5.12)$$
$$[X]^{\mathrm{vir}}_{\mathrm{red}} = \sqrt{e}(\widetilde{E}|_{C(s)}, \tau)[C(s)] = \sqrt{e}(\widetilde{E}, \widetilde{s})[Y] = [\mathcal{X}_{\mathrm{red}}]^{\mathrm{vir}} \in H_{n+2}(X, \mathbb{Z}[1/2]).$$

On the other hand, we may choose an orthogonal $\mathbb{C}$-linear smooth splitting $E \cong \widetilde{E} \oplus R$. Using the splitting, the surjection of smooth vector bundles $R \to \mathbb{C}_Y \cong \mathbb{R}^2_Y$ restricts to a surjection $\sigma^+ \colon R^+ \subseteq E \to \mathbb{R}^2_Y$. This follows from the fact that $\sigma^\vee$ is isotropic and factors through $R^\vee \cong R$, so its projection to R^+ is always non-zero. As in Theorem 4.4, σ^+ defines a surjective cosection of rank 2 on the d-manifold $\mathcal{X} = \mathcal{S}_{Y,E^+,s^+}$. By Subsection 3.4, the reduced virtual fundamental class of $\mathcal{X}$ is then by definition

$$(5.13) \qquad\qquad [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}} := [\mathcal{X}_{\mathrm{red}}]^{\mathrm{vir}} \in H_{n+2}(X, \mathbb{Z}).$$

Combining (5.12) and (5.13) yields the following result.

Theorem 5.6. *Let X be a projective scheme equipped with an oriented symmetric obstruction theory of amplitude $[-2,0]$, isotropic normal cone and isotropic surjective cosection σ, as above. Let $\mathcal{X}$ be the compact, oriented d-manifold associated to these data and σ^+ the induced surjective cosection on $\mathcal{X}$.*

Then the algebraic reduced virtual fundamental cycle of X and the reduced virtual fundamental class of $\mathcal{X}$ are equal in homology with $\mathbb{Z}[1/2]$-coefficients, that is,

$$[X]^{\mathrm{vir}}_{\mathrm{red}} = [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{red}} \in H_{n+2}(X, \mathbb{Z}[1/2]).$$

Remark 5.7. *Of course, the most natural scenario in which the conditions of the theorem arise is when X is the classical truncation of an oriented (-2)-shifted symplectic scheme $(\underline{\mathcal{X}}, \omega_{\underline{\mathcal{X}}})$.*

5.4. **Equality of reduced classes: the non-degenerate case.** We keep the notation of the previous subsection and move on to the case of a non-degenerate cosection $\underline{\sigma} \colon \mathbb{E}_\bullet[1] \to \mathcal{O}_X^{\oplus d}$.

Following [BKP22, Sav22], non-degeneracy here means that the composition

$$\mathcal{O}_X^{\oplus d} \xrightarrow{\underline{\sigma}^\vee} \mathbb{E}^\bullet[-1] \cong \mathbb{E}_\bullet[1] \xrightarrow{\underline{\sigma}} \mathcal{O}_X^{\oplus d}$$

is an isomorphism. In particular, $\underline{\sigma}$ is surjective.

Let $\mathbb{E}_{\mathrm{red}}^\bullet$ be the cone of $\underline{\sigma}^\vee[1]$. Assume that

$$(\mathbb{E}_{\mathrm{red}}^\bullet)^\vee[2] \to \mathbb{E}_\bullet[2] \cong \mathbb{E}^\bullet \to \mathbb{E}_{\mathrm{red}}^\bullet$$

induces a symmetric form on $\mathbb{E}_{\mathrm{red}}^\bullet$ and there is a splitting of symmetric complexes $\mathbb{E}^\bullet \cong \mathbb{E}_{\mathrm{red}}^\bullet \oplus \mathcal{O}_X^{\oplus d}[1]$.

In the notation of the previous subsection, we may fix a global, orthogonal complex Kuranishi chart (Y, E, s) for X and further assume that $\underline{\sigma}$ is induced by a split surjection of orthogonal bundles $F \to \mathcal{O}_X^{\oplus d}$ which extends smoothly to a surjective $\mathbb{C}$-linear morphism $\underline{\sigma} \colon E \to \mathbb{C}_Y^d$ with kernel the oriented, orthogonal bundle E_{red}. Since $\underline{\sigma} \circ s = 0$, s factors through a section s_{red} of E_{red}.

Let $\mathcal{X} = \mathcal{S}_{Y, E^+, s^+}$ and $\mathcal{X}_{\mathrm{red}} = \mathcal{S}_{Y, E_{\mathrm{red}}^+, s_{\mathrm{red}}^+}$ be the associated compact, oriented d-manifolds.

By [BKP22, Proposition 4.15, Theorem 4.5], the isotropic normal cone $C(s) = C(s_{\mathrm{red}})$ is contained in E_{red} and the reduced virtual fundamental cycle of X is defined by the formula

$$
\begin{aligned}
(5.14) \qquad [X]_{\mathrm{red}}^{\mathrm{vir}} &:= \sqrt{e}(E_{\mathrm{red}}|_{C(s)}, \tau)[C(s)] \\
&= \sqrt{e}(E_{\mathrm{red}}|_{C(s_{\mathrm{red}})}, \tau)[C(s_{\mathrm{red}})] \\
&= \sqrt{e}(E_{\mathrm{red}}, s_{\mathrm{red}})[Y] = [\mathcal{X}_{\mathrm{red}}]^{\mathrm{vir}} \in H_{n+d}(X, \mathbb{Z}[1/2]),
\end{aligned}
$$

where the last equality follows from Theorem 5.4.

As in Theorem 4.4, we have a composition

$$\underline{\sigma}^+ \colon E^+ \subseteq E \xrightarrow{\sigma} \mathbb{C}_Y^d \to \mathbb{R}_Y^d,$$

which defines a surjective cosection on the d-manifold $\mathcal{X}$ and whose kernel bundle is exactly E_{red}^+. By Subsection 3.4, the reduced virtual fundamental class of $\mathcal{X}$ is then by definition

$$(5.15) \qquad\qquad [\mathcal{X}]_{\mathrm{red}}^{\mathrm{vir}} := [\mathcal{X}_{\mathrm{red}}]^{\mathrm{vir}} \in H_{n+d}(X, \mathbb{Z}).$$

Combining (5.12) and (5.13) yields the following result.

Theorem 5.8. *Let X be a projective scheme equipped with an oriented symmetric obstruction theory of amplitude $[-2, 0]$, isotropic normal cone and non-degenerate cosection $\underline{\sigma}$, as above. Let $\mathcal{X}$ be the compact, oriented d-manifold associated to these data and $\underline{\sigma}^+$ the induced surjective cosection on $\mathcal{X}$.*

Then the algebraic reduced virtual fundamental cycle of X and the reduced virtual fundamental class of $\mathcal{X}$ are equal in homology, that is,

$$[X]_{\mathrm{red}}^{\mathrm{vir}} = [\mathcal{X}]_{\mathrm{red}}^{\mathrm{vir}} \in H_{n+d}(X, \mathbb{Z}[1/2]).$$

6. Applications: Integrality and reduced invariants

We give a few (related) immediate applications of our results regarding integrality and reduced invariants in Donaldson–Thomas theory of Calabi–Yau and, in particular, holomorphic symplectic and hyperkähler fourfolds.

6.1. **Surface counting on Calabi–Yau fourfolds.** Let W be a Calabi-Yau fourfold, i.e. a smooth projective four-dimensional variety with trivial canonical bundle $\omega_W \colon \mathcal{O}_W \cong K_W$.

In [BKP22], the authors consider moduli spaces $P = P_v^{(q)}(W)$ of PT_q-pairs (F, s) satisfying

$$\mathrm{ch}(F) = (0, 0, \gamma, \beta, n - \gamma \cdot \mathrm{td}(W)) \in H^*(W, \mathbb{Q}),$$

where F is a coherent sheaf on W and $s \in H^0(W, F)$. For details on the stability conditions involved, we refer the reader to [BKP22].

By [BKP22, Theorem 1.4], P naturally admits a (-2)-shifted symplectic derived enhancement $\underline{\mathcal{P}}$ with a choice of orientation. Write $\mathbb{E}^\bullet \to \mathbb{L}_P$ for the induced three-term symmetric obstruction theory on P.

Now, given a surface class $\gamma \in H^2(W, \Omega_W^2)$ and $q \in \{-1, 0, 1\}$, let ρ_γ be the codimension of the Hodge locus of γ. There is then an induced non-degenerate cosection $\underline{\sigma}_\gamma \colon \mathbb{E}_\bullet[1] \to \mathcal{O}_P^{\rho_\gamma}$.

In [BKP22, Theorem 4.5], using these data, the authors define reduced fundamental cycles

$$[P]_{\mathrm{red}}^{\mathrm{vir}} = [P_v^{(q)}(W)]_{\mathrm{red}}^{\mathrm{vir}} \in A_{n - \frac{1}{2}\gamma^2 + \frac{1}{2}\rho_\gamma}(W, \mathbb{Z}[1/2]).$$

These cycles are essential in developing a surface counting theory on Calabi-Yau fourfolds and used to study the variational Hodge conjecture (cf. [BKP22, Conjecture 1.12]).

Applying Theorem 5.8, we can define these classes integrally after choosing an associated compact, oriented d-manifold model $\mathcal{P}$ to the oriented, (-2)-shifted symplectic scheme $\underline{\mathcal{P}}$.

Theorem 6.1. *There exist well-defined integral homological reduced virtual fundamental classes*

$$[P]_{\mathrm{red}}^{\mathrm{vir}} = [P_v^{(q)}(W)]_{\mathrm{red}}^{\mathrm{vir}} = [\mathcal{P}]_{\mathrm{red}}^{\mathrm{vir}} \in H_{2n - \gamma^2 + \rho_\gamma}(W, \mathbb{Z}).$$

Remark 6.2. *In fact, by [BKP22, Theorem 1.16], the class $[P]_{\mathrm{red}}^{\mathrm{vir}}$ may be alternatively defined directly by using an oriented, (-2)-shifted symplectic derived enhancement $\underline{\mathcal{P}}_{\mathrm{red}}$ of P which induces the reduced obstruction theory $\mathbb{E}_{\mathrm{red}}^\bullet$ on P. It is easy to check that a model for the associated compact, oriented d-manifold is $\mathcal{P}_{\mathrm{red}}$ and hence the virtual fundamental class is $[\mathcal{P}_{\mathrm{red}}]^{\mathrm{vir}}$, which by (5.15) equals $[\mathcal{P}]_{\mathrm{red}}^{\mathrm{vir}}$ in the above.*

6.2. **Gopakumar–Vafa type invariants of holomorphic symplectic fourfolds.** Let W be a holomorphic symplectic variety of dimension 4, i.e. a smooth projective four-dimensional variety with a non-degenerate holomorphic 2-form $\omega_W \in H^0(W, \Omega_W^2)$. Fix $\beta \in H_2(W, \mathbb{Z})$ and $n \in \mathbb{Z}$.

In [COT22a], the authors consider moduli spaces $P_n^t(W, \beta)$ of pairs (F, s), where F is an one-dimensional coherent sheaf on W with $[F] = \beta$ and $\chi(F) = n$ and $s \in H^0(W, F)$. These pairs are taken to be stable with respect to a stability condition that depends on the choice of real number $t \in \mathbb{R}$ and ample divisor on W.

For a general choice of $t \in \mathbb{R}$, the moduli space $P_n^t(W, \beta)$ is a projective scheme and admits a natural (-2)-shifted symplectic derived enhancement together with an orientation and a surjective isotropic cosection. Write

$\mathbb{I} = (\mathcal{O} \to \mathbb{F})$ for the universal stable pair and π_W, π_P for the two projections out of $W \times P_n^t(W, \beta)$.

In order to define counting invariants, consider the insertion operator

(6.1)
$$\tau \colon H^m(W, \mathbb{Z}) \longrightarrow H^{m-2}(P_n^t(W, \beta), \mathbb{Z})$$
$$\gamma \mapsto (\pi_P)_* \left(\pi_W^* \gamma \cup \mathrm{ch}_3(\mathbb{F})\right).$$

For generic $t \in \mathbb{R}$ and classes $\gamma_i \in H^{m_i}(W, \mathbb{Z})$ ($1 \leq i \leq l$), the Z_t-stable pair invariants are defined by the formula

$$P_{n,\beta}^t(\gamma_1, \ldots, \gamma_l) = \int_{[P_n^t(W, \beta)]_{\mathrm{red}}^{\mathrm{vir}}} \prod_{i=1}^l \tau(\gamma_i)$$

and, for $n = -1$, where no insertions are necessary,

$$P_{-1,\beta}^t = \int_{[P_{-1}^t(W, \beta)]_{\mathrm{red}}^{\mathrm{vir}}} 1.$$

In [COT22a, Conjecture 1.10], these invariants are conjectured to recover curve-counting invariants on W of Gopakumar–Vafa type. In particular, the conjecture necessitates that they are integral. In the formulation of the stable pair invariants in [COT22a], the authors used the algebraic reduced virtual fundamental cycle defined in [KP20]. By Theorem 5.6, we may instead equivalently use the integral homological virtual fundamental class $[P_n^t(W, \beta)]_{\mathrm{red}}^{\mathrm{vir}} \in H_*(P_n^t(W, \beta), \mathbb{Z})$. As a corollary, we obtain the following integrality result, which serves as a sanity check for the conjecture.

Theorem 6.3. $P_{n,\beta}^t(\gamma_1, \ldots, \gamma_l), P_{-1,\beta}^t \in \mathbb{Z}$.

In a related vein, as in [COT22b], consider the (projective) moduli scheme M_β of one dimensional stable sheaves F on W satisfying $\mathrm{ch}_3(F) = \beta$ and $\chi(F) = 1$. Let $\mathbb{F}_{\mathrm{norm}}$ be the normalized universal sheaf, i.e. $\det(\pi_M)_* \mathbb{F}_{\mathrm{norm}} \cong \mathcal{O}_{M_\beta}$.

It is shown in [COT22b] that M_β admits a natural, oriented (-2)-shifted symplectic derived enhancement which is equipped with an isotropic cosection.

Using $\mathbb{F}_{\mathrm{norm}}$, there are natural insertion operators τ_0 defined by formula (6.1), and for any classes $\gamma_1, \ldots, \gamma_n \in H^*(W, \mathbb{Z})$ we have associated DT4 invariants (see equation (0.4) in [COT22b])

$$\langle \tau_0(\gamma_1), \ldots, \tau_0(\gamma_n) \rangle_\beta^{\mathrm{DT4}} = \int_{[M_\beta]_{\mathrm{red}}^{\mathrm{vir}}} \prod_{i=1}^n \tau_0(\gamma_i).$$

When β is an effective curve class, [COT22b, Conjecture 0.5] states that these DT4 invariants recover the genus 0 Gopakumar–Vafa type invariants of W, which are integral. Again, by Theorem 5.6, we can use the integral homological virtual fundamental class in the above formula to obtain the following integrality result.

Theorem 6.4. $\langle \tau_0(\gamma_1), \ldots, \tau_0(\gamma_n) \rangle_\beta^{\mathrm{DT4}} \in \mathbb{Z}$.

6.3. **Integrality of algebraic cosection localized virtual fundamental cycles.** We finally comment that using Theorems 4.2, 5.4, 5.6, one can define an integral cosection localized or reduced virtual fundamental class in homology, whenever a corresponding algebraic virtual fundamental cycle exists with $\mathbb{Z}[1/2]$-coefficients. In addition, these classes are well-defined when the virtual dimension is odd in constrast to their algebraic counterparts which vanish in that case.

Several such examples are given in [KP20, Section 9], among which one was also considered in [Sav22, Section 7].

We briefly sample a few for the convenience of the reader.

Let W be a Calabi–Yau fourfold with $\omega_W \in H^0(W, K_W)$ a nowhere vanishing form. Let M be a component of the moduli space of simple perfect complexes on W with fixed Chern character $c = (c_0, c_1, c_2, c_3, c_4)$. This is an algebraic space by [Ina02, Lie06], which we assume is a quasi-projective scheme in what follows. As usual, M admits an oriented, (-2)-shifted symplectic derived enhancement and an induced symmetric obstruction theory $\mathbb{E}_\bullet \to \mathbb{L}_M$ with (complex) virtual dimension vd.

We first consider [KP20, Corollary 9.6]. Given a $(3,1)$-form $\delta \in H^1(W, \Omega^3_W)$, there is an associated cosection $\sigma^\delta \colon \mathbb{E}_\bullet[1] \to \mathcal{O}_M$ defined in [KP20, Example 9.2]. If $c_2 = 0$ or a certain bilinear form on $H^1(W, T_W)$ (see equation (9.7) in [KP20, Corollary 9.6]) vanishes, then σ^δ is isotropic and we get an integral cosection localized virtual fundamental class $[M]^{\mathrm{vir}}_{\mathrm{loc}} \in H_{\mathrm{vd}}(M, \mathbb{Z})$.

On the other hand, if the bilinear form has positive rank $k > 0$, then $[M]^{\mathrm{vir}} = 0 \in H_{\mathrm{vd}}(M, \mathbb{Z})$ and we can define a reduced virtual fundamental class $[M]^{\mathrm{vir}}_{\mathrm{red}} \in H_{\mathrm{vd}+2k}(M, \mathbb{Z})$. This generalizes [KP20, Corollary 9.6] in the sense that the algebraic reduced virtual fundamental cycle is defined only for even values of k.

We may perform identical steps for cosections associated to $(0,2)$-forms $\gamma \in H^2(W, \mathcal{O}_W)$ to obtain an integral version of [KP20, Corollary 9.7] and drop the condition that $h^{0,2}$ is a positive and even number in order to define reduced integral classes.

For a second example, consider the moduli space $M = P_n(W, \beta)$ of stable pairs (F, s) on W satisfying $[F] = \beta$, $\chi(F) = n$ and assume that W is hyperkähler, as in [CMT22].

Then, by [Sav22, Theorem 7.2] and [KP20, Example 9.11(i)], the homological and algebraic virtual fundamental class of M vanish. Moreover, by [KP20, Example 9.11(i)], there is an algebraic reduced virtual fundamental cycle $[M]^{\mathrm{vir}}_{\mathrm{red}} \in A_{\frac{1}{2}\mathrm{vd}+1}(M, \mathbb{Z}[1/2])$, whose image in homology under the cycle class map equals the image of the integral homological virtual fundamental class $[M]^{\mathrm{vir}}_{\mathrm{red}} \in H_{\mathrm{vd}+2}(M, \mathbb{Z})$.

References

[BBJ19]   Christopher Brav, Vittoria Bussi, and Dominic Joyce. A Darboux theorem for derived schemes with shifted symplectic structure. *J. Amer. Math. Soc.*, 32(2):399–443, 2019.

[BF97]   Kai Behrend and Barbara Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[BJ17]   Dennis Borisov and Dominic Joyce. Virtual fundamental classes for moduli
             spaces of sheaves on Calabi-Yau four-folds. *Geom. Topol.*, 21(6):3231–3311,
             2017.

[BKP22]   Younghan Bae, Martijn Kool, and Hyeonjun Park. Counting surfaces on
             Calabi-Yau 4-folds I: Foundations. *arXiv e-prints*, page arXiv:2208.09474, Au-
             gust 2022.

[Bre97]   Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*.
             Springer-Verlag, New York, second edition, 1997.

[Cao20]   Yalong Cao. Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-
             folds II: Fano 3-folds. *Commun. Contemp. Math.*, 22(7):1950060, 25, 2020.

[CC14]   Yalong Cao and Naichung Conan Leung. Donaldson-Thomas theory for Calabi-
             Yau 4-folds. *arXiv e-prints*, page arXiv:1407.7659, July 2014.

[CGJ20]   Yalong Cao, Jacob Gross, and Dominic Joyce. Orientability of moduli spaces
             of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds. *Adv. Math.*,
             368:107134, 60, 2020.

[CK20]   Yalong Cao and Martijn Kool. Curve counting and DT/PT correspondence for
             Calabi-Yau 4-folds. *Adv. Math.*, 375:107371, 49, 2020.

[CKM19]   Yalong Cao, Martijn Kool, and Sergej Monavari. K-theoretic DT/PT corre-
             spondence for toric Calabi-Yau 4-folds. *arXiv e-prints*, page arXiv:1906.07856,
             June 2019.

[CKM22]   Yalong Cao, Martijn Kool, and Sergej Monavari. Stable pair invariants of local
             Calabi-Yau 4-folds. *Int. Math. Res. Not. IMRN*, (6):4753–4798, 2022.

[CMT18]   Yalong Cao, Davesh Maulik, and Yukinobu Toda. Genus zero Gopakumar-Vafa
             type invariants for Calabi-Yau 4-folds. *Adv. Math.*, 338:41–92, 2018.

[CMT22]   Yalong Cao, Davesh Maulik, and Yukinobu Toda. Stable pairs and
             Gopakumar-Vafa type invariants for Calabi-Yau 4-folds. *J. Eur. Math. Soc.
             (JEMS)*, 24(2):527–581, 2022.

[COT22a]   Yalong Cao, Georg Oberdieck, and Yukinobu Toda. Gopakumar-Vafa
             type invariants of holomorphic symplectic 4-folds. *arXiv e-prints*, page
             arXiv:2201.10878, January 2022.

[COT22b]   Yalong Cao, Georg Oberdieck, and Yukinobu Toda. Stable pairs and
             Gopakumar-Vafa type invariants on holomorphic symplectic 4-folds. *Adv.
             Math.*, 408:Paper No. 108605, 44, 2022.

[CT20]   Yalong Cao and Yukinobu Toda. Gopakumar-Vafa type invariants on Calabi-
             Yau 4-folds via descendent insertions. *arXiv e-prints*, page arXiv:2003.00787,
             March 2020.

[CT22]   Yalong Cao and Yukinobu Toda. Curve counting via stable objects in derived
             categories of Calabi-Yau 4-folds. *Adv. Math.*, 406:Paper No. 108531, 2022.

[EG95]   Dan Edidin and William Graham. Characteristic classes and quadric bundles.
             Duke Math. J., 78(2):277–299, 1995.

[Ina02]   Michi-aki Inaba. Toward a definition of moduli of complexes of coherent sheaves
             on a projective scheme. *J. Math. Kyoto Univ.*, 42(2):317–329, 2002.

[Ive72]   Birger Iversen. A fixed point formula for action of tori on algebraic varieties.
             Invent. Math., 16:229–236, 1972.

[Joy]   Dominic Joyce. D-manifolds and d-orbifolds: a theory of derived differential ge-
             ometry. Available online at `http://people.maths.ox.ac.uk/~joyce/dmbook.
             pdf`.

[Joy19]   Dominic Joyce. Algebraic geometry over C^∞-rings. *Mem. Amer. Math. Soc.*,
             260(1256):v+139, 2019.

[KL13]   Young-Hoon Kiem and Jun Li. Localizing virtual cycles by cosections. *J. Amer.
             Math. Soc.*, 26(4):1025–1050, 2013.

[KP20]   Young-Hoon Kiem and Hyeonjun Park. Localizing virtual cycles for
             Donaldson-Thomas invariants of Calabi-Yau 4-folds. *arXiv e-prints*, page
             arXiv:2012.13167, December 2020.

[Lie06]   Max Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.*,
             15(1):175–206, 2006.

[LT98]   Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

[OT20]   Jeongseok Oh and Richard P. Thomas. Counting sheaves on Calabi-Yau 4-folds, I. *arXiv e-prints*, page arXiv:2009.05542, September 2020.

[OT23]   Jeongseok Oh and Richard P. Thomas. Complex Kuranishi structures and counting sheaves on Calabi-Yau 4-folds, II. *arXiv e-prints*, page arXiv:2305.16441, May 2023.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[Sav22]   Michail Savvas. Cosection localization and vanishing for virtual fundamental classes of d-manifolds. *Adv. Math.*, 398:Paper No. 108232, 34, 2022.

[Tho00]   Richard P. Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54(2):367–438, 2000.

[Toë14]   Bertrand Toën. Derived algebraic geometry. *EMS Surv. Math. Sci.*, 1(2):153–240, 2014.

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