GENERALIZED DONALDSON-THOMAS INVARIANTS OF DERIVED OBJECTS VIA KIRWAN BLOWUPS

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ABSTRACT. Let σ be a stability condition on the bounded derived category $D^b(\mathrm{Coh}W)$ of a Calabi-Yau threefold W and $\mathcal{M}$ a moduli stack parametrizing σ-semistable objects of fixed topological type. We define generalized Donaldson-Thomas invariants which act as virtual counts of objects in $\mathcal{M}$, fully generalizing the approach introduced by Kiem, Li and the author in the case of semistable sheaves.

We construct an associated proper Deligne-Mumford stack $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$, called the $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction of $\mathcal{M}$, with an induced semi-perfect obstruction theory of virtual dimension zero, and define the generalized Donaldson-Thomas invariant via Kirwan blowups to be the degree of the associated virtual cycle $[\widetilde{\mathcal{M}}^{\mathbb{C}^*}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}}^{\mathbb{C}^*})$. This stays invariant under deformations of the complex structure of W. Applications include Bridgeland stability, polynomial stability, Gieseker and slope stability.

CONTENTS

1. INTRODUCTION

1.1. **Brief historical background.** Donaldson-Thomas (abbreviated as DT from now on) invariants constitute one of the main approaches for curve counting on Calabi-Yau threefolds. They naturally appear in many enumerative problems of interest in algebraic geometry and string theory and are conjecturally equivalent with other counting invariants, such as Gromov-Witten invariants, Stable Pair invariants [PT09] and Gopakumar-Vafa invariants [MT18]. These relations have now been proven in many cases (for example, in [MNOP06a, MNOP06b, Tod10]).

Let W be a smooth, projective Calabi-Yau threefold and $\gamma \in H^*(W, \mathbb{Q})$. Classical DT theory was introduced in [Tho00] in order to obtain virtual

counts of stable sheaves on W of Chern character γ. More precisely, Thomas considered the (coarse) moduli space $M := M_L^{ss}(\gamma)$ parameterizing Gieseker semistable sheaves on W of positive rank, fixed determinant L and Chern character γ. Assuming that every semistable sheaf is stable, M is proper and admits a perfect obstruction theory in the sense of Li-Tian [LT98] or Behrend-Fantechi [BF97] of virtual dimension zero. Thus, there exists a virtual fundamental cycle $[M]^{\mathrm{vir}} \in A_0(M)$ and the classical DT invariant is defined as

$$\mathrm{DT}(M) := \deg [M]^{\mathrm{vir}}.$$

One of its important properties is its invariance under deformation of the complex structure of W.

Another important feature of DT invariants is their motivic nature. Behrend [Beh09] showed that the obstruction theory of M is symmetric in a certain sense and established an equality

$$(\dagger) \qquad \mathrm{DT}(M) = \deg [M]^{\mathrm{vir}} = \chi(M, \nu_M) = \sum_{n \in \mathbb{Z}} n \cdot \chi_{\mathrm{top}}(\nu_M^{-1}(n)),$$

where $\nu_M \colon M \to \mathbb{Z}$ is a canonical constructible function on M and the right-hand side a weighted Euler characteristic.

However, when stability and semistability of sheaves do not coincide, the above methods do not suffice and new techniques are required in order to define generalized DT invariants counting semistable sheaves. A main obstacle is that strictly semistable sheaves can have more automorphisms beyond $\mathbb{C}^*$-scaling and, as a result, the stacks $\mathcal{M}^{ss}(\gamma)$ parameterizing semistable sheaves are generally Artin and no longer Deligne-Mumford (after rigidifying $\mathbb{C}^*$-scaling, cf. Subsection 6.3). This is necessary for the standard machinery of perfect obstruction theory and virtual cycles to apply.

In [JS12], Joyce and Song constructed generalized DT invariants of moduli stacks $\mathcal{M}^{ss}(\gamma)$ taking advantage of the above motivic behaviour and using motivic Hall algebras to obtain a generalization of the right-hand side of ($\dagger$). Their DT invariant is easy to work with and amenable to computation. However, the proof of its deformation invariance is indirect and proceeds via wall-crossing to a stable pairs theory where semistability and stability coincide and thus a virtual cycle exists. Kontsevich and Soibelman [KS10] have also defined motivic generalized DT invariants using similar ideas. We also mention the related work of Behrend and Ronagh [BR19, BR16].

In [KLS17], the authors develop a new direct approach towards defining generalized DT invariants of such a moduli stack $\mathcal{M}^{ss}(\gamma)$. Their method adapts Kirwan's partial desingularization procedure [Kir85] to define an invariant as the degree of a zero-dimensional virtual cycle in an associated Deligne-Mumford stack $\widetilde{\mathcal{M}}$. The constructed invariant is called the generalized DT invariant via Kirwan blowups (or DTK invariant for short) and is a direct generalization of the left-hand side of ($\dagger$). By the usual properties of virtual cycles, they establish the deformation invariance of DTK invariants.

We make a final comment on the need for orientation data in the above approaches. Since motivic DT invariants are more refined, Kontsevich and

Soibelman do need to assume the existence of such data to define their invariants. This assumption has been recently proved to hold in [JU21]. Orientation data are not necessary for the definition of numerical generalized DT invariants given in [JS12] and [KLS17]. This will be the case for the DTK invariant defined in this paper as well.

1.2. **Statement of results.** This paper serves as a sequel to [KLS17]. We generalize the construction of DTK invariants to the case of moduli stacks $\mathcal{M} := \mathcal{M}^{\sigma-ss}(\gamma)$ parametrizing σ-semistable objects of Chern character γ in the derived category $D^b(\operatorname{Coh} W)$ of coherent sheaves on W, where σ is one of the following stability conditions:

(1) A Bridgeland stability condition [Bri07], as considered by Toda-Piyaratne [PT19] and [Li19].
(2) A polynomial stability condition [Bay09], as considered by Lo [Lo11, Lo13].
(3) Gieseker and slope stability of sheaves. These are examples of weak stability conditions on $\operatorname{Coh} W$ in the sense of Joyce-Song [JS12].

More generally, σ can be any nice stability condition (cf. Definition 6.1 and Definition 6.2).

Our main results are summarized in the following theorem, giving a definition of DTK invariants counting σ-semistable complexes.

Theorem. *Let W be a smooth, projective Calabi-Yau threefold, σ a stability condition on $D^b(\operatorname{Coh} W)$ as in Definition 6.2 and $\mathcal{M} := \mathcal{M}^{\sigma-ss}(\gamma)$ be the ($\mathbb{C}^*$-rigidified) moduli stack parametrizing σ-semistable complexes of Chern character γ. Then there exist:*

(1) *[Theorem-Construction 4.16, Theorem-Definition 6.8]*
A canonical proper Deligne-Mumford stack $\widetilde{\mathcal{M}}^{\mathbb{C}^*} \to \mathcal{M}$, called the $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction of $\mathcal{M}$. $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ is isomorphic to $\mathcal{M}$ over the open σ-stable locus $\mathcal{M}^{\sigma-s}(\gamma) \subseteq \mathcal{M}$.

(2) *[Theorem 5.18, Theorem-Definition 6.10]*
A natural semi-perfect obstruction theory of virtual dimension zero on $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$, which extends the symmetric obstruction theory of the Deligne-Mumford stack $\mathcal{M}^{\sigma-s}(\gamma)$.

We thus have a virtual fundamental cycle

$$[\widetilde{\mathcal{M}}^{\mathbb{C}^*}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}}^{\mathbb{C}^*})$$

and the generalized Donaldson-Thomas invariant via Kirwan blowups of $\mathcal{M}$ is defined as

$$\mathrm{DTK}(\mathcal{M}^{\sigma-ss}(\gamma)) := \deg\,[\widetilde{\mathcal{M}}^{\mathbb{C}^*}]^{\mathrm{vir}} \in \mathbb{Q}.$$

By Theorem 6.14, this is invariant under deforming the complex structure of W.

1.3. **Brief review of the case of sheaves and sketch of construction.**
We first give a brief account of the results of [KLS17] and then explain the necessary adjustments in order to generalize their approach.

Let $\mathcal{M}$ be an Artin stack parametrizing Gieseker semistable sheaves of a fixed Chern character on a smooth, projective Calabi-Yau threefold. By

construction (see [HL10]), $\mathcal{M}$ is obtained by Geometric Invariant Theory (GIT) (see [MFK94]), meaning that it is a quotient stack of the form

$$(1.1) \qquad \mathcal{M} = [X/G],$$

where G is a reductive group acting on a projective space $\mathbb{P}^N$ via a homomorphism $G \to GL(N+1, \mathbb{C})$ and X is an invariant closed subscheme of the GIT semistable locus $X \subseteq (\mathbb{P}^N)^{ss}$.

In order to define an invariant out of $\mathcal{M}$, the first step in [KLS17] is the construction of a proper DM quotient stack $\widetilde{\mathcal{M}} := [\widetilde{X}/G]$ over $\mathcal{M}$, called the intrinsic stabilizer reduction of $\mathcal{M}$. This is produced by an iterative blowup procedure, which successively resolves the loci of closed G-orbits in X with stabilizer groups of maximum dimension. This procedure is an adaptation of Kirwan partial desingularization construction [Kir85] to singular GIT quotient stacks, using the notion of intrinsic blowup introduced in [KL13] and suitably generalized in [KLS17].

The second step is to endow $\widetilde{\mathcal{M}}$ with a semi-perfect obstruction theory [CL11] of virtual dimension zero. This is a usual perfect obstruction theory in the sense of Behrend-Fantechi [BF97] defined on an étale cover of $\widetilde{\mathcal{M}}$ together with compatibility data which play the role of suitable descent data. It is at this step where the existence of a virtual structure on $\mathcal{M}$ plays a crucial role. More precisely, by [PTVV13], $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack and it then follows that $\mathcal{M}$ is a d-critical stack [Joy15]. In particular, applying Luna's étale slice theorem [Dré04] and using the theory of d-critical loci developed in [Joy15], we can obtain an explicit local description of $\mathcal{M}$: for every closed point $x \in \mathcal{M}$ with (reductive) stabilizer H, there exists a smooth affine scheme V with an H-action, an invariant function $f \colon V \to \mathbb{A}^1$ and an étale morphism

$$(1.2) \qquad [U/H] \to \mathcal{M},$$

containing x in its image, where $U = \{df = 0\} \subset V$ is the scheme-theoretic critical locus of f. Any two such local presentations can be compared as they give the same d-critical structure on $\mathcal{M}$ in the sense of [Joy15].

To obtain the semi-perfect obstruction theory of $\widetilde{\mathcal{M}}$, consider the following H-equivariant 4-term complex on U:

$$(1.3) \qquad \mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_V|_U \xrightarrow{d(df)} F_V|_U = \Omega_V|_U \longrightarrow \mathfrak{h}^\vee.$$

Around $u \in U$ with finite stabilizer group, this complex is quasi-isomorphic to a 2-term complex which gives a symmetric perfect obstruction theory of $[U/H]$ and thus of $\mathcal{M}$ near u.

The intrinsic stabilizer reduction algorithm produces lifts of the étale morphisms (1.2) to give an étale cover

$$(1.4) \qquad [T/H] \to \widetilde{\mathcal{M}},$$

where $T = \{\omega_S = 0\} \subseteq S$ for S a smooth affine H-scheme and $\omega_S \in H^0(S, F_S)$ an invariant section of an H-equivariant vector bundle F_S on S.

Moreover, there exists an effective invariant divisor D_S such that (1.3) lifts to a 4-term complex

$$(1.5) \qquad \mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_S|_T \xrightarrow{d\omega_S} F_S|_T \longrightarrow \mathfrak{h}^\vee(-D_S)$$

whose first arrow is injective with locally free cokernel and last arrow is surjective. Therefore, (1.5) is quasi-isomorphic to a 2-term complex, whose dual can be shown to be a perfect obstruction theory on $[T/H]$.

In [KLS17], it is shown that these perfect obstruction theories satisfy the axioms of a semi-perfect obstruction theory on $\widetilde{\mathcal{M}}$ of virtual dimension zero.

Since, as usual, semi-perfect obstruction theories produce virtual cycles, $\widetilde{\mathcal{M}}$ admits a virtual cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$ whose degree defines the DTK invariant counting Gieseker semistable sheaves parametrized by $\mathcal{M}$. A relative version of the above construction, using derived symplectic geometry, implies its deformation invariance.

In the present paper, we work with moduli stacks $\mathcal{M}$ of semistable perfect complexes. These are truncations of (-1)-shifted symplectic derived Artin stacks as before, however they are not global quotient stacks obtained by GIT of the form (1.1).

Our main contribution is to remove this assumption. To do this, we use several recent major technical results on the étale local structure of stacks [AHR19, AHR20], moduli spaces of objects in abelian categories [AHH18] and stability conditions in families [BLM$^+$21].

We generalize the requirement of a GIT presentation to the condition that $\mathcal{M}$ admit a good moduli space morphism [Alp13]. We show that such stacks (under additional reasonable assumptions) admit an intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$, which is a Deligne-Mumford stack and of independent interest in its own right (see Subsection 1.4 below).

We then proceed to show that if in addition $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack, then the above statements about local models of $\mathcal{M}$ and their comparison data carry over, using the main structural result of [AHR20] in place of Luna's étale slice theorem. This allows us to define a semi-perfect obstruction theory on $\widetilde{\mathcal{M}}$ and the associated DTK invariant.

Our approach thus works for a wide class of Artin stacks, which we call stacks of DT type (see Definition 3.7). Based on our discussion here, their two main characteristics can be summarized as the existence of a good moduli space and a derived enhancement that is (-1)-shifted symplectic.

Moduli stacks of semistable complexes fit into this context. This is the case by the recent results of [AHH18], which combine the theory of good moduli spaces with the notion of Θ-reductivity introduced in [Hal14], and apply to the stacks considered in the present paper.

Finally, the deformation invariance of the DTK invariant follows from the compatibility of the intrinsic stabilizer reduction with base change and the recent results on stability conditions in [BLM$^+$21].

1.4. **Relations to derived algebraic geometry.** Intrinsic blowups and the intrinsic stabilizer reduction procedure are expected to be the classical shadow of a corresponding construction in derived algebraic geometry. Namely, derived blowups were recently defined in full generality in the work of Hekking [Hek21].

We expect that the intrinsic blowup notion used in this paper coincides with the classical truncation of an appropriate derived blowup of a derived scheme with a G-action along its derived fixed locus by the group G. This suggests that the intrinsic stabilizer reduction construction is closely related to a derived version of the Kirwan partial desingularization in GIT [Kir85] and its generalization to stacks with good moduli spaces by Edidin-Rydh [ER21]. It should then be the case that it can be performed at the derived level directly to obtain a derived intrinsic stabilizer reduction.

If this is indeed so, the semi-perfect obstruction theory constructed in this paper is expected to be the one induced by the derived cotangent complex of the intrinsic stabilizer reduction of a (-1)-shifted symplectic derived stack.

We intend to explore these connections further in upcoming work.

1.5. **Layout of the paper.** In §2 we review background material on d-critical loci and (-1)-shifted symplectic structures. In §3, we define the notion of stacks of DT type and establish their main properties that will be used throughout. §4 reviews the intrinsic stabilizer reduction procedure for GIT quotient stacks and generalizes it to stacks with good moduli spaces. In §5, we explain how to construct a semi-perfect obstruction theory on stacks of DT type by following the arguments of [KLS17]. Finally, in §6, we construct DTK invariants of semistable complexes by combining the above with the results of [AHH18] and $\mathbb{G}_m$-rigidification for Artin stacks and discuss their deformation invariance using recent work of [BLM$^+$21].

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1.7. **Notation and conventions.** Here are the various notations and other conventions that we use throughout the paper:

- All schemes and stacks are defined over the field of complex numbers $\mathbb{C}$ or a smooth $\mathbb{C}$-scheme C, unless stated otherwise. For this reason, reductive group schemes will be linearly reductive automatically.
- $\mathcal{M}$ typically denotes an Artin stack, of finite type, with affine stabilizers and separated diagonal unless stated otherwise.
- W denotes a smooth, projective Calabi-Yau threefold over $\mathbb{C}$ and $D^b(\operatorname{Coh} W)$ its bounded derived category of coherent sheaves.
- C denotes a smooth quasi-projective scheme over $\mathbb{C}$.
- G, H denote complex reductive groups. Usually, H will be a subgroup of G. T denotes the torus $\mathbb{C}^*$.

– If $x \in \mathcal{M}$, G_x denotes the automorphism group/stabilizer of x. We will only consider stabilizers of closed points $x \in \mathcal{M}$. These will be reductive for most stacks of interest in this paper.
– If $U \hookrightarrow V$ is a closed embedding, $I_{U \subseteq V}$ or I_U (when V is clear from context) denotes the ideal sheaf of U in V.
– For a morphism $\rho \colon U \to V$ and a sheaf E on V, we systematically use $E|_U$ to denote $\rho^* E$, suppressing the pullback from the notation.
– If V is a G-scheme, V^G is used to denote the fixed point locus of G in V.
– If U is a scheme with a G-action, then $\widehat{U}$ is used to denote the Kirwan blowup of U with respect to G. $\widetilde{U}$ denotes the intrinsic stabilizer reduction of U.
– The abbreviations DT, DM, GIT, whenever used, stand for Donaldson-Thomas, Deligne-Mumford and Geometric Invariant Theory respectively.

2. D-Critical Loci and (-1)-Shifted Symplectic Derived Stacks

This section collects background material and terminology that will be used throughout the rest of the paper.

We first briefly recall Joyce's theory of d-critical loci, as developed in [Joy15], and establish some notation, and then proceed to quickly review shifted symplectic structures on derived stacks.

2.1. d-critical schemes. We begin by defining the notion of a d-critical chart.

Definition 2.1. (d-critical chart) *A d-critical chart for a scheme X is the data of (U, V, f, i) such that: $U \subseteq X$ is Zariski open, V is a smooth scheme, $f \colon V \to \mathbb{A}^1$ is a regular function on V and $U \xrightarrow{i} V$ is a closed embedding so that $U = \{df = 0\} = \mathrm{Crit}(f) \subseteq V$ is the scheme-theoretic vanishing locus of the derivative of f.*

If $x \in U$, then we say that the d-critical chart (U, V, f, i) is centered at x.

Joyce defines a canonical sheaf $\mathcal{S}_X$ of $\mathbb{C}$-vector spaces with the property that for any Zariski open $U \subseteq X$ and an embedding $U \hookrightarrow V$ into a smooth scheme V with ideal I, $\mathcal{S}_X$ fits into an exact sequence

$$(2.1) \qquad 0 \longrightarrow \mathcal{S}_X|_U \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d} \Omega_V/I \cdot \Omega_V$$

For example, for a d-critical chart (U, V, f, i) the element $f + I^2 \in \Gamma(V, \mathcal{O}_V/I^2)$ gives a section of $\mathcal{S}_X|_U$.

Definition 2.2. (d-critical scheme) *A d-critical structure on a scheme X is a section $s \in \Gamma(X, \mathcal{S}_X)$ such that X admits a cover by d-critical charts (U, V, f, i) and $s|_U$ is given by $f + I^2$ as above on each such chart. We refer to the pair (X, s) as a d-critical scheme.*

2.2. Equivariant d-critical loci. For our purposes, we need equivariant analogues of the results of Subsection 2.1. The theory works in parallel as before (cf. [Joy15, Section 2.6]).

Definition 2.3. (Good action) *Let G be an algebraic group acting on a scheme X. We say that the action is good if X has a cover $\{U_\alpha\}_{\alpha \in A}$ where every $U_\alpha \subseteq X$ is a G-invariant Zariski open affine subscheme.*

Remark 2.4. *If X is affine, then trivially every action of G on M is good.*

Suppose now that a scheme X admits a good action by a reductive group G. Definitions 2.1, 2.2 extend naturally to this equivariant setting (cf. [Joy15, Definition 2.40]) as follows: A G-invariant d-critical chart is given by the data (U, V, f, i) of Definition 2.1, where $U \subseteq X$ is a G-invariant Zariski open subscheme, V is a smooth scheme with a G-action, $i\colon U \to V$ an equivariant closed embedding and $f\colon V \to \mathbb{C}$ a G-invariant function. The sheaf $\mathcal{S}_X$ inherits a G-action by using a cover of X by G-invariant open subschemes $U \subseteq X$ in (2.1). Then we say that the d-critical scheme (X, s) is G-invariant if $s \in \Gamma(X, \mathcal{S}_X)^G$ is a G-invariant global section of the sheaf $\mathcal{S}_X$.

Proposition 2.5. [Joy15, Remark 2.47] *Let G be a complex reductive group with a good action on a scheme X. Suppose that (X, s) is an invariant d-critical scheme. Then for any $x \in X$ fixed by G, there exists an invariant d-critical chart (U, V, f, i) centered at x, i.e. an invariant open affine $U \ni x$, a smooth scheme V with a G-action, an invariant regular function $f\colon V \to \mathbb{A}^1$ and an equivariant embedding $i\colon U \to V$ so that $U = \mathrm{Crit}(f) \subseteq V$.*

Remark 2.6. *If G is a torus $(\mathbb{C}^*)^k$, then Proposition 2.5 is true without the assumption that x is a fixed point of G.*

Remark 2.7. *One may replace Zariski open morphisms by étale morphisms without any difference to the essence of the theory. Another option is to work in the complex analytic topology.*

2.3. **d-critical Artin stacks.** The theory of d-critical loci extends naturally to Artin stacks. For more details, we point the interested reader to Section 2.8 of [Joy15]. We mention the following definition and basic properties which we will need in the form of remarks.

Definition 2.8. [Joy15, Corollary 2.52, Definition 2.53] *Let $\mathcal{M}$ be an Artin stack. For every smooth morphism $\phi\colon U \to \mathcal{M}$, where U is a scheme, the assignment $\mathcal{S}(U, \phi) := \mathcal{S}_U$ defines a sheaf $\mathcal{S}_\mathcal{M}$ of $\mathbb{C}$-vector spaces in the lisse-étale topology of $\mathcal{M}$.*

A d-critical structure on $\mathcal{M}$ is a global section $s \in H^0(\mathcal{M}, \mathcal{S}_\mathcal{M})$. We then say that $(\mathcal{M}, s)$ is a d-critical Artin stack.

Remark 2.9. [Joy15, Example 2.55] *A d-critical structure can be equivalently described in terms of a groupoid presentation of $\mathcal{M}$. It follows that d-critical structures on quotient stacks $[X/G]$ are in bijective correspondence with invariant d-critical structures on X.*

If $\mathcal{M}' \to \mathcal{M}$ is a smooth morphism of Artin stacks and $\mathcal{M}$ is d-critical, then one may pull back the d-critical structure on $\mathcal{M}$ making $\mathcal{M}'$ d-critical as well.

2.4. **Shifted symplectic structures.** Let $\mathrm{cdga}_\mathbb{C}^{\leq 0}$ be the category of non-positively graded commutative differential graded $\mathbb{C}$-algebras.

There is a spectrum functor $\mathbf{Spec}\colon \mathrm{cdga}_\mathbb{C}^{\leq 0} \to \mathbf{dSt}_\mathbb{C}$ to the category of derived stacks (see [TV04, Definition 2.2.2.14] or [Toë09, Definition 4.2]). An

object of the form $\mathbf{Spec}\,A$ is called an *affine derived $\mathbb{C}$-scheme*. Such objects provide the Zariski local charts for general *derived $\mathbb{C}$-schemes*, see [Toë09, Section 4.2]. An object $\mathcal{M}$ in $\mathbf{dSt}_{\mathbb{C}}$ is called a *derived Artin stack* if it is m-geometric (cf. [Toë09, Definition 1.3.3.1]) for some m and its 'classical truncation' $t_0(\mathcal{M})$ is an Artin stack (and not a higher stack). A derived Artin stack $\mathcal{M}$ admits an *atlas*, i.e. a smooth surjective morphism $\boldsymbol{U} \to \mathcal{M}$ from a derived scheme. For a derived Artin stack $\mathcal{M}$, there exists a cotangent complex $\mathbb{L}_{\mathcal{M}}$ of finite cohomological amplitude in $[-m, 1]$ and a dual tangent complex $\mathbb{T}_{\mathcal{M}}$. Both are objects in a suitable stable ∞-category $L_{\mathsf{qcoh}}(\mathcal{M})$ (see [Toë09] or [TV04] for its definition).

Shifted symplectic structures on derived Artin stacks were introduced by Pantev–Toën–Vaquié–Vezzosi in [PTVV13]. The definition is given in the affine case first, and then generalized by showing the local notion satisfies smooth descent. We recall the local definition: let us set $\boldsymbol{M} = \mathbf{Spec}\,A$, so that $L_{\mathsf{qcoh}}(\boldsymbol{M}) \cong \mathbf{D}(\text{dg-Mod}_A)$. For all $p \geq 0$ one can define the exterior power complex $(\Lambda^p \mathbb{L}_{\boldsymbol{M}}, \mathrm{d}) \in L_{\mathsf{qcoh}}(\boldsymbol{M})$, where the differential d is induced by the differential of the algebra A. For a fixed $k \in \mathbb{Z}$, define a k-shifted p-form on $\boldsymbol{M}$ to be an element $\omega^0 \in (\Lambda^p \mathbb{L}_{\boldsymbol{M}})^k$ such that $\mathrm{d}\omega^0 = 0$. To define the notion of closedness, consider the de Rham differential $\mathrm{d}_{\mathrm{dR}} \colon \Lambda^p \mathbb{L}_{\boldsymbol{M}} \to \Lambda^{p+1} \mathbb{L}_{\boldsymbol{M}}$. A *$k$-shifted closed p-form* is a sequence $(\omega^0, \omega^1, \dots)$, with $\omega^i \in (\Lambda^{p+i} \mathbb{L}_{\boldsymbol{M}})^{k-i}$, such that $\mathrm{d}\omega^0 = 0$ and $\mathrm{d}_{\mathrm{dR}}\omega^i + \mathrm{d}\omega^{i+1} = 0$. When $p = 2$, any k-shifted 2-form $\omega^0 \in (\Lambda^2 \mathbb{L}_{\boldsymbol{M}})^k$ induces a morphism $\omega^0 \colon \mathbb{T}_{\boldsymbol{M}} \to \mathbb{L}_{\boldsymbol{M}}[k]$ in $L_{\mathsf{qcoh}}(\boldsymbol{M})$, and we say that ω^0 is *non-degenerate* if this morphism is an isomorphism in $L_{\mathsf{qcoh}}(\boldsymbol{M})$.

Definition 2.10 ([PTVV13, Definition 1.18])**.** *A k-shifted closed 2-form $\omega = (\omega^0, \omega^1, \dots)$ is called a k-shifted symplectic structure if ω^0 is non-degenerate. We say that $(\boldsymbol{M}, \omega)$ is a k-shifted symplectic (affine) derived scheme.*

When $k = -1$, it is shown in [BBBBJ15] that if $\mathcal{M}$ is the classical truncation of a (-1)-shifted symplectic derived Artin stack, then $\mathcal{M}$ admits an induced d-critical structure.

Theorem 2.11 ([BBBBJ15, Theorem 3.18])**.** *Let $(\mathcal{M}, \omega)$ be a (-1)-shifted symplectic derived Artin stack. Then the underlying classical Artin stack $M = t_0(\mathcal{M})$ extends in a canonical way to a d-critical Artin stack (M, s). This defines a 'truncation functor' τ from the ∞-category of (-1)-shifted symplectic derived Artin stacks to the 2-category of d-critical Artin stacks.*

3. Stacks of DT Type

In this section, we first give an account of results regarding the étale local structure of Artin stacks and the theory of good moduli spaces. We then proceed to define stacks of DT type, develop standard local models for them and describe how to compare these models.

3.1. **Local structure of Artin stacks.** The following theorem is an étale slice theorem for stacks, which generalizes Luna's étale slice theorem [Dré04]. It states that Artin stacks are étale locally quotient stacks.

Theorem 3.1. [AHR20, Theorem 1.2] *Let $\mathcal{M}$ be a quasi-separated Artin stack, locally of finite type over $\mathbb{C}$ with affine stabilizers. Let $x \in \mathcal{M}$ and $H \subseteq G_x$ a maximal reductive subgroup of the stabilizer of x. Then there exists an affine scheme U with an action of H, a point $u \in U$ fixed by H, and a smooth morphism*

$$\Phi_x : [U/H] \to \mathcal{M}$$

which maps $u \mapsto x$ and induces the inclusion $H \hookrightarrow G_x$ of stabilizers at u. Moreover, if G_x is reductive, the morphism is étale. If $\mathcal{M}$ has affine diagonal, then Φ_x can be taken to be affine.

3.2. **Good moduli spaces.** We now collect some useful results about the structure of a certain class of Artin stacks, namely those with affine diagonal admitting a good moduli space, following the theory developed by Alper ét al. All the material of the section can be found in [Alp13] and [AHR20].

We have the following definition of a good moduli space.

Definition 3.2. [Alp13, Definition 4.1] *A morphism $q\colon \mathcal{M} \to Y$, where $\mathcal{M}$ is an Artin stack and Y an algebraic space, is a good moduli space for $\mathcal{M}$ if the following hold:*

(1) *q is quasi-compact and $q_*\colon \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(Y)$ is exact.*
(2) *The natural map $\mathcal{O}_Y \to q_*\mathcal{O}_\mathcal{M}$ is an isomorphism.*

The intuition behind the introduction of the notion of good moduli space is that stacks $\mathcal{M}$ that admit good moduli spaces behave like quotient stacks $[X^{ss}/G]$ obtained from Geometric Invariant Theory (GIT) with good moduli space morphism given by the map $[X^{ss}/G] \to X/\!\!/G$. In this sense, it is a generalization of GIT quotients for stacks.

We state the following properties of stacks with good moduli space.

Proposition 3.3. [Alp13, Proposition 4.7, Lemma 4.14, Theorem 4.16, Proposition 9.1, Proposition 12.14] *Let $\mathcal{M}$ be locally noetherian over $\mathbb{C}$ and $q\colon \mathcal{M} \to Y$ be a good moduli space. Then:*

(1) *q is surjective.*
(2) *q is universally closed.*
(3) *Two geometric points $x_1, x_2 \in \mathcal{M}(k)$ are identified in Y if and only if their closures $\overline{\{x_1\}}$ and $\overline{\{x_2\}}$ in $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,k$ intersect.*
(4) *Every closed point of $\mathcal{M}$ has reductive stabilizer.*
(5) *Let $y \in |Y|$ be a closed point. Then there exists a unique closed point $x \in |q^{-1}(y)|$.*
(6) *If $\mathcal{Z}$ is a closed substack of $\mathcal{M}$, then the morphism $\mathcal{Z} \to q(\mathcal{Z}) = \mathrm{im}(\mathcal{Z})$ is a good moduli space morphism.*
(7) *Let*

$$
\begin{array}{ccc}
\mathcal{M}' & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

be a Cartesian diagram of Artin stacks, such that Y, Y' are algebraic spaces.

(a) If $\mathcal{M} \to Y$ is a good moduli space, then $\mathcal{M}' \to Y'$ is a good moduli space.

(b) If $Y' \to Y$ is fpqc and $\mathcal{M}' \to Y'$ is a good moduli space, then $\mathcal{M} \to Y$ is a good moduli space.

(8) If $\mathcal{M}$ is of finite type, then Y is of finite type.

The following notions will be useful for us.

Definition 3.4. [Alp13, Definition 6.1, Remark 6.2] *Let $\mathcal{M}$ be a locally noetherian Artin stack with a good moduli space $q \colon \mathcal{M} \to M$. A Zariski open substack $\mathcal{U} \subseteq \mathcal{M}$ is called saturated if $q^{-1}(q(\mathcal{U})) = \mathcal{U}$.*

If $\mathcal{U}$ is saturated, the morphism $\mathcal{U} \to U := q(\mathcal{U})$ gives a good moduli space for $\mathcal{U}$ fitting into a Cartesian diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow q \\ U & \longrightarrow & M, \end{array}$$

where $U \to M$ is an open embedding.

Definition 3.5. [AHR19, Definition 3.13] *Let $\mathcal{M}'$, $\mathcal{M}$ be Artin stacks with good moduli space morphisms $q' \colon \mathcal{M}' \to M'$, $q \colon \mathcal{M} \to M$. A morphism $\mathcal{M}' \to \mathcal{M}$ is called strongly étale if the induced morphism $M' \to M$ is étale and the diagram*

$$\begin{array}{ccc} \mathcal{M}' & \longrightarrow & \mathcal{M} \\ q' \downarrow & & \downarrow q \\ M' & \longrightarrow & M \end{array}$$

is Cartesian.

We have the following theorem regarding the étale local structure of good moduli space morphisms for stacks with affine diagonal.

Theorem-Definition 3.6. (Quotient chart) [AHR20, Theorem 4.12] *Let $\mathcal{M}$ be a locally noetherian Artin stack with a good moduli space $q \colon \mathcal{M} \to M$ such that q is of finite type with affine diagonal. If $x \in \mathcal{M}$ is a closed point with (reductive) stabilizer G_x, then there exists an affine scheme U with an action of G_x and a Cartesian diagram*

$$(3.1) \qquad \begin{array}{ccc} [U_x/G_x] & \xrightarrow{\Phi_x} & \mathcal{M} \\ \downarrow & & \downarrow q \\ U_x /\!\!/ G_x & \longrightarrow & M \end{array}$$

such that Φ_x has the same properties as in Theorem 3.1, is affine and $U_x /\!\!/ G_x$ is an étale neighbourhood of $q(x)$.

We refer to any choice of data (U_x, Φ_x) such that $\Phi_x \colon [U_x/G_x] \to \mathcal{M}$ is étale, affine and stabilizer preserving as a quotient chart for $\mathcal{M}$ centered at x. We say that the quotient chart is strongly étale if the morphism Φ_x is strongly étale.

3.3. **Stacks of DT type.** We start with the following definition.

Definition 3.7. (Stack of DT type) *Let $\mathcal{M}$ be an Artin stack. We say that $\mathcal{M}$ is of DT type if the following are true:*

(1) *$\mathcal{M}$ is quasi-separated and finite type over $\mathbb{C}$.*
(2) *There exists a good moduli space $q\colon \mathcal{M} \to M$, where q is of finite type and has affine diagonal.*
(3) *$\mathcal{M}$ is the classical truncation of a (-1)-shifted symplectic derived Artin stack $\boldsymbol{\mathcal{M}}$.*

Remark 3.8. *By Theorem 2.11 a stack $\mathcal{M}$ of DT type admits a canonical d-critical structure $s \in H^0(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$.*

Definition 3.9. (d-critical quotient chart) *Let $x \in \mathcal{M}$ and $\Phi\colon [U_x/G_x] \to \mathcal{M}$ be as in Theorem-Definition 3.6. We say that (U_x, V, f, Φ_x) is a d-critical quotient chart centered at x if there exists a G_x-invariant d-critical locus determined by data (U_x, V, f, i).*

A variant of the following proposition first appeared and was used in [Tod16].

Proposition 3.10. *Let $\mathcal{M}$ be a stack of DT type and $x \in \mathcal{M}$ a closed point. Then there exists a d-critical quotient chart for $\mathcal{M}$ centered at x, which can be taken to be strongly étale.*

Proof. By Theorem-Definition 3.6, we have a quotient chart $\Phi_x\colon [U_x''/G_x] \to \mathcal{M}$. By Proposition 2.5, Definition 2.8 and Remark 2.9, there exists a G_x-invariant open affine $x \in U_x' \subseteq U_x''$ and an invariant d-critical chart (U_x', V', f, i). We obtain a commutative diagram

$$
\begin{array}{ccc}
[U_x'/G_x] & \xrightarrow{\Phi_x} & \mathcal{M} \\
{\scriptstyle q'}\downarrow & & \downarrow \\
U_x'/\!\!/G_x & \longrightarrow & M.
\end{array}
$$

Applying the Fundamental Lemma [Alp10, Theorem 6.10] at x, there exists a Zariski open subscheme $R \subseteq U_x'/\!\!/G_x$ such that the induced diagram

$$
\begin{array}{ccc}
(q')^{-1}(R) & \xrightarrow{\Phi_x} & \mathcal{M} \\
{\scriptstyle q'}\downarrow & & \downarrow \\
R & \longrightarrow & M
\end{array}
$$

is Cartesian and Φ_x is strongly étale.

Since $U'/\!\!/G_x$ is affine, we may take R to be affine as well. It is straightforward to check then that $(q')^{-1}(R) = [U_x/G_x]$, where U_x is a G_x-invariant Zariski open affine subscheme of U', containing x, and also $R = U_x/\!\!/G_x$.

Setting $V = V' \setminus (U' \setminus U_x)$, we have the d-critical chart $(U_x, V, f|_V, i)$. This concludes the proof. $\qquad\square$

We now obtain the following key lemma, which gives a way to compare two choices of d-critical quotient charts.

Lemma 3.11. *Let $\mathcal{M}$ be a stack of DT type. Let*

$$(U_\alpha, V_\alpha, f_\alpha, \Phi_\alpha),\ (U_\beta, V_\beta, f_\beta, \Phi_\beta)$$

be two d-critical quotient charts with V_α, V_β affine and $z \in [U_\alpha/G_\alpha] \times_{\mathcal{M}} [U_\beta/G_\beta]$ a closed point with stabilizer $G := G_z$. Let $x_\alpha \in [U_\alpha/G_\alpha]$ and $x_\beta \in [U_\beta/G_\beta]$ be the two projections of z. Then the following hold:

(1) *We have G-equivariant commutative diagrams for $\lambda = \alpha, \beta$*

(3.2)
$$\begin{array}{ccc} T_{\alpha\beta} & \longrightarrow & S_{\alpha\beta} \\ {\scriptstyle i_\lambda}\downarrow & & \downarrow{\scriptstyle \theta_\lambda} \\ U_\lambda & \longrightarrow & V_\lambda, \end{array}$$

where the vertical arrows are unramified morphisms, the horizontal arrows are embeddings, $t \in T_{\alpha\beta}$ maps to $x_\lambda \in U_\lambda$, and $T_{\alpha\beta}, S_{\alpha\beta}$ are affine.

(2) *We have an induced diagram with étale arrows:*

(3.3)
$$\begin{array}{ccc} & [T_{\alpha\beta}/G] & \\ \swarrow & & \searrow \\ [U_\alpha/G_\alpha] & & [U_\beta/G_\beta] \\ \searrow & & \swarrow \\ & \mathcal{M} & \end{array}$$

(3) *There exists a d-critical G-invariant quotient chart $(T_{\alpha\beta}, S_{\alpha\beta}, f_{\alpha\beta}, \Phi_{\alpha\beta})$ for $\mathcal{M}$ with $t \in T_{\alpha\beta}$ fixed by G.*

(4) *$I := (\theta_\lambda^* df_\lambda) = (df_{\alpha\beta})$ as ideal sheaves in $\mathcal{O}_{S_{\alpha\beta}}$ and $\theta_\lambda^* f_\lambda + I^2$, $f_{\alpha\beta} + I^2$ give the same invariant section $s|_{T_{\alpha\beta}} \in \Gamma(T_{\alpha\beta}, \mathcal{S}_{T_{\alpha\beta}})^G$, where θ_λ are the morphisms defined above in diagram (3.2).*

Proof. It follows by the second condition in Definition 3.7 that $\mathcal{M}$ must have affine diagonal. In particular, the Cartesian diagram

$$\begin{array}{ccc} U_\alpha \times_{\mathcal{M}} U_\beta & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{M}}} \\ U_\alpha \times U_\beta & \longrightarrow & \mathcal{M} \times \mathcal{M} \end{array}$$

implies that $U_{\alpha\beta} := U_\alpha \times_{\mathcal{M}} U_\beta$ is affine.

Let us write $G_{\alpha\beta} = G_\alpha \times G_\beta$. Since $U_{\alpha\beta}$ and $V_\alpha \times V_\beta$ are affine and the product embedding $U_\alpha \times U_\beta \hookrightarrow V_\alpha \times V_\beta$ is $G_{\alpha\beta}$-invariant, there exists a smooth affine scheme $V_{\alpha\beta}$ and a $G_{\alpha\beta}$-equivariant embedding fitting into a commutative diagram

$$\begin{array}{ccc} U_{\alpha\beta} & \longrightarrow & V_{\alpha\beta} \\ \downarrow & & \downarrow \\ U_\alpha \times U_\beta & \longrightarrow & V_\alpha \times V_\beta. \end{array}$$

Concretely, we may choose any $G_{\alpha\beta}$-equivariant embedding $U_{\alpha\beta} \hookrightarrow \mathbb{A}^N$ for big enough N and then take $V_{\alpha\beta} = V_\alpha \times V_\beta \times \mathbb{A}^N$ with the obvious choices of morphisms that make the above diagram commute.

Let $t \in U_{\alpha\beta}$ a representative of the point $z \in [U_\alpha/G_\alpha] \times_{\mathcal{M}} [U_\beta/G_\beta] \simeq [U_{\alpha\beta}/G_{\alpha\beta}]$ (we are liberally abusing notation at this point). Using Luna's étale slice theorem, take $S_{\alpha\beta} \subseteq V_{\alpha\beta}$ to be an affine étale slice for t in $V_{\alpha\beta}$ and $T_{\alpha\beta} := S_{\alpha\beta} \cap U_{\alpha\beta}$ the induced affine étale slice for $t \in U_{\alpha\beta}$.

It is clear that conditions (1) and (2) are now satisfied and remain so even after G-invariant shrinking of $S_{\alpha\beta}$ around t.

For conditions (3) and (4), notice that by diagrams (3.2) and (3.3) and the properties of d-critical structures, $[T_{\alpha\beta}/G]$ is a d-critical stack and thus, up to possibly shrinking $S_{\alpha\beta}$ G-invariantly around t, there exists an induced d-critical G-invariant quotient chart $(T_{\alpha\beta}, S_{\alpha\beta}, f_{\alpha\beta}, \Phi_{\alpha\beta})$ for $\mathcal{M}$ centered at $t \in T_{\alpha\beta}$. Finally, since by construction $S_{\alpha\beta}$, $T_{\alpha\beta}$ are étale slices of V_λ, U_λ at t, we obtain that $(df_{\alpha\beta}) = (\theta_\lambda^* df_\lambda)$ as ideals and $\theta_\lambda^* f_\lambda + I^2, f_{\alpha\beta} + I^2$ both represent the pullback of the d-critical structure of $\mathcal{M}$ to $[T_{\alpha\beta}/G]$, which completes the proof. □

Remark 3.12. *The above results only use the d-critical structure of $\mathcal{M}$ and not the existence of its (-1)-shifted symplectic derived enhancement. In fact, using the property that $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack $\mathbf{\mathcal{M}}$, Proposition 3.10 may be strengthened. Since we won't need this strengthening in this paper, we are content with just making this remark.*

Remark 3.13. *We finally make a remark on the conditions in Definition 3.7. The first condition is there to ensure boundedness. The second condition implies the existence of quotient charts as in diagram (3.1), which will be necessary in order to construct the intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ and its good moduli space. The third condition is necessary to induce a d-critical structure on $\mathcal{M}$ which is the crucial component in obtaining a semi-perfect obstruction theory for $\widetilde{\mathcal{M}}$.*

4. Intrinsic Stabilizer Reduction

In this section, we review the notions of intrinsic and Kirwan blowups and then generalize the construction of the intrinsic stabilizer reduction of a quotient stack obtained by GIT given in [KLS17] to the more general setting of stacks with good moduli space.

4.1. **Kirwan blowups for affine schemes.** We recall the notion of Kirwan blowup developed in [KLS17].

Suppose that U is an affine scheme with an action of a reductive group G. For now, let us assume that G is connected, as this will be the case when we take blowups throughout the paper.

Suppose that we have an equivariant closed embedding $U \to V$ into a smooth G-scheme V and let I be the ideal defining U. Since $U \subseteq V$ is G-invariant, G acts on I and we have a decomposition $I = I^{fix} \oplus I^{mv}$ into the fixed part of I and its complement as G-representations.

Let V^G be the fixed point locus of G inside V, defined by the ideal generated by I^{mv}, and $\pi\colon \mathrm{bl}_G(V) \to V$ the blowup of V along V^G. Let

$E \subseteq \mathrm{bl}_G(V)$ be its exceptional divisor and $\xi \in \Gamma(\mathcal{O}_{\mathrm{bl}_G(V)}(E))$ the tautologi-
cal defining equation of E.

G-equivariance implies that (cf. [KLS17, Section 2.2])

$$\pi^{-1}(I^{mv}) \subseteq \xi \cdot \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)},$$

and consequently, $\xi^{-1}\pi^{-1}(I^{mv}) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$, meaning that $\pi^{-1}(I^{mv})$ lies in
the image of the inclusion $\mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$, given by multiplication
by ξ.

We define $I^{intr} \subseteq \mathcal{O}_{\mathrm{bl}_G(V)}$ to be

(4.1) $I^{intr} = $ ideal sheaf generated by $\pi^{-1}(I^{fix})$ and $\xi^{-1}\pi^{-1}(I^{mv})$.

Definition 4.1. (Intrinsic blowup) *The G-intrinsic blowup of U is the sub-
scheme $U^{intr} \subseteq \mathrm{bl}_G(V)$ defined by the ideal I^{intr}.*

Lemma 4.2. [KLS17, Lemma 2.6] *The G-intrinsic blowup of U is inde-
pendent of the choice of G-equivariant embedding $U \subseteq V$, and hence is
canonical.*

Proof. We give a brief outline of the proof and refer the reader to the proof
of [KLS17, Lemma 2.6] and [KL13, Subsection 3.1] for details.

If $U^G = \emptyset$, then $\widehat{U} = U$ and there is nothing to show. So suppose this is
not the case and $u \in U^G$.

Let $U \to V_1$, $U \to V_2$ be two equivariant embeddings of U into smooth
G-schemes. By composing with the equivariant inclusions $V_1 \times \{u\} \subseteq V_1 \times V_2$
and $\{u\} \times V_2 \subseteq V_1 \times V_2$, we may reduce to the case of a sequence of equivariant
embeddings $U \to V \xrightarrow{\Phi} W$, where V, W are smooth G-schemes.

Applying the above construction using the two embeddings of U, we ob-
tain G-intrinsic blowups $\widehat{U}^V$, induced by the embedding $U \to V$, and $\widehat{U}^W$,
induced by the composite embedding $U \to W$.

Then an identical argument as in the proof of [KL13, Lemma 3.1], shows
that Φ gives rise to an isomorphism $\widehat{U}^V \simeq \widehat{U}^W$, which moreover is indepen-
dent of the choice of Φ interpolating between the two embeddings $U \to V$
and $U \to W$. This proves that $\widehat{U}$ is indeed canonical. $\square$

Suppose U is an affine G-scheme with an equivariant embedding into a
smooth affine G-scheme V as above. We can make sense of the notion of
semistability of points in U^{intr} without ambiguity as follows.

We first work on the ambient scheme V. As it is affine, we can think
of all points of V as being semistable (in the usual sense of GIT). For the
blowup $V^{intr} := \mathrm{bl}_G(V)$, G acts linearly on $E = \mathbb{P}N_{V^G/V}$ with respect to
the natural G-linearization of $\mathcal{O}_E(1) := \mathcal{O}_E(-E)$ and therefore we have a
GIT notion of stability of points on the exceptional divisor E.

Definition 4.3. *We say that $v \in V$ is **stable** if its G-orbit is closed in
V and its stabilizer finite. A point $\widetilde{v} \in \mathrm{bl}_G(V)$ is **unstable** if its G-orbit
closure meets the unstable locus of E. If $\widetilde{v}$ is not unstable, we say that it is
semistable.*

Thus for any smooth affine G-scheme V, we can define its Kirwan blowup
$\widehat{V} = (\mathrm{bl}_G(V))^{ss}$. By [Kir85], it satisfies $\widehat{V}^G = \emptyset$.

Now, if we have an equivariant embedding $V \to W$ between smooth G-schemes, then $(W^{intr})^{ss} \cap V^{intr} = (V^{intr})^{ss}$ based on our description. Hence we may define $(U^{intr})^{ss} := U^{intr} \cap (V^{intr})^{ss}$ for any equivariant embedding $U \to V$ into a smooth scheme V, which is independent of the choice of $U \to V$.

Definition 4.4. (Kirwan blowup) *Let U be a possibly singular affine G-scheme with non-empty G-fixed locus. The Kirwan blowup of U associated with G is the scheme $\widehat{U} = (U^{intr})^{ss}$. It satisfies $\widehat{U}^G = \emptyset$.*

The notion of semistability above is exactly motivated by the corresponding notion in GIT in Kirwan's original blowup procedure in [Kir85]. This can be seen by the following theorem of Reichstein, which asserts that the locus of unstable points on $\mathrm{bl}_G(V)$ is exactly the locus of unstable points in the sense of GIT for the ample line bundle $\mathcal{O}_{\mathrm{bl}_G(V)}(-E)$.

Theorem 4.5. [Rei89] *Let V as above and let $q \colon V \to V /\!\!/ G$ denote the GIT quotient map. The unstable locus of $\mathrm{bl}_G(V)$ with respect to the line bundle $\mathcal{O}_{\mathrm{bl}_G(V)}(-E)$ is the strict transform of the saturation $q^{-1}(q(V^G))$ of V^G inside V.*

We obtain the following corollary.

Corollary 4.6. *Let $\pi \colon \widehat{U} \to U$ be the Kirwan blowup of an affine G-scheme U with non-empty G-fixed locus. Then:*
  (1) *The quotient stack $[\widehat{U}/G]$ admits a good moduli space, denoted by $q \colon [\widehat{U}/G] \to \widehat{U} /\!\!/ G$.*
  (2) *The unstable locus of U^{intr} is the strict transform $\overline{\pi^{-1}(Z \setminus U^G)} \subseteq U^{intr}$ of the saturation $Z = q^{-1}(q(U^G))$ of U^G in U.*

Proof. Let $U \to V$ be a G-equivariant embedding into a smooth affine G-scheme V. By Theorem 4.5, $[\widehat{V}/G]$ admits a good moduli space $\widehat{V} /\!\!/ G$ and therefore the same is true for the closed substack $[\widehat{U}/G]$ by Proposition 3.3(6). (2) follows immediately from the construction of $\widehat{U}$, Definition 4.3 and Theorem 4.5. $\square$

Example 4.7. *Suppose that $G = \mathbb{C}^*$ is the one-dimensional torus acting on the affine plane $V = \mathbb{C}^2_{x,y}$ with weights 1 and -1 on the coordinates x and y respectively and $U \subseteq V$ is the closed G-invariant subscheme cut out by the ideal $I = (x^2 y, xy^2)$.*

V^{intr} is the blowup of V along the fixed locus $V^G = \{0\}$. The unstable points are the punctured x-axis and y-axis together with the points $0, \infty$ of the exceptional divisor $\mathbb{P}^1$. Thus we have that $\widehat{V} = \mathrm{Spec}[u, v, v^{-1}]$ where G acts on u, v with weights $1, -2$ respectively and the blowdown map $\widehat{V} \to V$ is given on coordinates by $x \mapsto u, y \mapsto uv$.

$\widehat{U} \subseteq \widehat{V}$ is the closed subscheme cut out by the ideal (u^2), so that

$$\widehat{U} = \mathrm{Spec}\left(\mathbb{C}[u, v, v^{-1}]/(u^2) \right).$$

We now explain how one can proceed if G is not connected.

Suppose that $U \to V$ is a G-equivariant embedding into a smooth G-scheme V. Let, as before, I be the ideal of U in V. Let G_0 be the connected

component of the identity. This is a normal, connected subgroup of G of finite index. Let $I = I^{fix} \oplus I^{mv}$ be the decomposition of I into fixed and moving parts with respect to the action of G_0. Using the normality of G_0, we see that the fixed locus V^{G_0} is a closed, smooth G-invariant subscheme of V and also I^{fix}, I^{mv} are G-invariant.

Let $\pi \colon \mathrm{bl}_{V^{G_0}} V \to V$ be the blowup of V along V^{G_0} with exceptional divisor E and local defining equation ξ. Then, as before, take I^{intr} to be the ideal generated by $\pi^{-1}(I^{fix})$ and $\xi^{-1}\pi^{-1}(I^{mv})$. Everything is G-equivariant and we define U^{intr} as the subscheme of $\mathrm{bl}_{V^{G_0}} V$ defined by the ideal I^{intr}.

Finally, we need to delete unstable points. By the Hilbert-Mumford criterion (cf. [MFK94, Theorem 2.1]) it follows that semistability on E with respect to the action of G is the same as semistability with respect to the action of G_0, since every 1-parameter subgroup of G factors through G_0, and hence we may delete unstable points exactly as before and define the Kirwan blowup $\widehat{U}$.

One may check in a straightforwardly analogous way that this has the same properties (and intrinsic nature). It is obvious that if G is connected we recover Definition 4.4.

4.2. **Kirwan blowups for GIT schemes.** So far, we have defined Kirwan blowups for affine G-schemes (with non-empty fixed locus of the G-action). We now generalize the construction to any G-scheme with a G-action coming from GIT (see Definition 4.8 below) and the associated quotient stack, as performed in [KLS17, Subsection 2.4].

We begin with an affine G-scheme whose G-fixed locus can now be possibly empty. Let $\mathcal{M} = [X/G]$ be a quotient Artin stack, where X is an affine G-scheme.

Fix a G-equivariant closed embedding $X \to Y$, where Y is a smooth affine G-scheme, and write $\mathcal{Y} = [Y/G]$. Let d be the maximum dimension of the stabilizers of closed points $x \in X$. By possibly equivariantly shrinking Y, we may assume that the maximum dimension of stabilizers of closed points of Y is also equal to d and such points lie in X.

Kirwan's partial desingularization, combined with Theorem 4.5, yields a smooth G-scheme $\widehat{Y}$ by blowing up the locus $Y_d \subseteq Y$ consisting of points satisfying $\dim G_y = d$ and deleting unstable points. By assumption, $X_d = Y_d$. $\widehat{Y}$ satisfies the crucial property that its maximum stabilizer dimension is strictly less than d.

Now, for any point $x \in X$ with closed G-orbit and $\dim G_x = d$, let $S_x \subseteq Y$ be an étale slice for the G-orbit of x in Y. The Kirwan blowups $\widehat{S}_x$ (cf. Definition 4.4) of the slices S_x associated with G_x provide natural strongly étale morphisms $\Psi_x \colon \widehat{S}_x \times_{G_x} G \to \widehat{Y}$. The joint morphism

$$\Psi := \coprod_{x \in X_d} \Psi_x \colon \coprod_{x \in X_d} \widehat{S}_x \times_{G_x} G \longrightarrow \widehat{Y}$$

is an étale cover of an open neighbourhood $\widehat{Y}'$ of the exceptional divisor E of $\widehat{Y}$ over the locus Y_d.

Let $T_x = X \cap S_x$. Then T_x is an étale slice for the G-orbit of x in X and we have the Kirwan blowup $\widehat{T}_x$ using the closed G_x-equivariant embedding

$T_x \subseteq S_x$. In [KLS17, Proposition 2.12], it is shown that the closed embedding

$$\coprod_{x \in X_d} \widehat{T}_x \times_{G_x} G \longrightarrow \coprod_{x \in X_d} \widehat{S}_x \times_{G_x} G$$

satisfies étale descent with respect to the morphism Ψ and thus defines a closed subscheme $\widehat{X}' \subseteq \widehat{Y}'$. $\widehat{X}'$ glues naturally with the pre-image of $X - X_d$ in $\widehat{Y}$ under the blowup morphism $\widehat{Y} \to Y$, defining a closed G-equivariant subscheme $\widehat{X}$ inside $\widehat{Y}$.

Lemma 4.2 and the above discussion on stability imply that these are canonical and independent of any choices made regarding the embedding $X \to Y$ and étale slices S_x.

If X is not affine, but is the GIT semistable locus of a G-scheme with a G-linearized ample line bundle, then it admits a cover by G-invariant Zariski open, saturated, affine subschemes $U_1, ..., U_n \subseteq X$. Here, as in Definition 3.4, saturated means that the U_i fit in Cartesian squares

$$
\begin{array}{ccc}
[U_i/G] & \longrightarrow & X \\
\downarrow & & \downarrow \\
U_i /\!\!/ G & \longrightarrow & X /\!\!/ G.
\end{array}
$$

We may thus apply the Kirwan blowup to each U_i. The Kirwan blowups $\widehat{U}_i$ glue to the Kirwan blowup $\widehat{X}$ of X and we are led to the following definition.

Definition 4.8. *Suppose that $[X/G]$ is a GIT quotient stack. Then we call $\widehat{X}$ and $\widehat{\mathcal{M}} = [\widehat{X}/G]$ the Kirwan blowups of X and $\mathcal{M} = [X/G]$ respectively.*

Remark 4.9. *When X is not affine, we could also argue globally as in the affine case using an equivariant closed embedding $X \to Y$ into a smooth G-scheme Y, afforded by GIT. However, the more local argument using a cover by saturated, affine, G-invariant open subschemes is closer in spirit to the proof of Theorem-Construction 4.16.*

4.3. **Intrinsic stabilizer reduction for GIT quotient stacks.** We quickly recall how the construction of the intrinsic stabilizer reduction works for GIT quotient stacks and some useful properties in that setting.

Let X be a G-scheme coming from GIT and $\mathcal{M} = [X/G]$ the associated quotient stack.

Write $\mathcal{M}_0 = \mathcal{M}$ and $X_0 = X$. Taking Kirwan blowups, we obtain $X_1 = \widehat{X}$ and $\mathcal{M}_1 = [\widehat{X}/G]$. The G-action on $\widehat{X}$ is still G-linearized, so $\mathcal{M}_1$ is a GIT quotient stack of lower maximum stabilizer dimension compared to $\mathcal{M}_0$. Thus we may iterate the procedure to produce a canonical sequence of GIT quotient stacks

$$\mathcal{M}_0 = [X/G], \ \mathcal{M}_1 = [X_1/G], \ \ldots, \ \mathcal{M}_\ell = [X_\ell/G],$$

where $\mathcal{M}_\ell$ is a Deligne-Mumford stack, since at each step the maximum stabilizer dimension strictly decreases.

Definition 4.10. *$\widetilde{\mathcal{M}} := \mathcal{M}_\ell$ is called the intrinsic stabilizer reduction of $\mathcal{M}$.*

4.4. **Intrinsic stabilizer reduction for stacks with good moduli spaces.**
We now generalize the procedure of intrinsic stabilizer reduction to the case
of stacks with good moduli spaces.

The following proposition about the properties of Kirwan blowups will be
useful, so we record it separately before our main construction.

Proposition 4.11. *Let $\varphi\colon G_1 \to G_2$ be a surjective homomorphism between
reductive groups, and X_i, $i = 1, 2$, affine G_i-schemes.*

*Suppose that $f\colon X_1 \to X_2$ is a smooth morphism which is equivariant with
respect to φ and such that the induced morphism $\underline{f}\colon [X_1/G_1] \to [X_2/G_2]$ on
quotient stacks is strongly étale (cf. Definition 3.5).*

*Then there is a canonically induced smooth, affine morphism $\widehat{f}\colon \widehat{X}_1 \to \widehat{X}_2$,
equivariant with respect to φ, such that the associated quotient morphism
$\underline{\widehat{f}}\colon [\widehat{X}_1/G_1] \to [\widehat{X}_2/G_2]$ is strongly étale.*

Proof. Let $x_1 \in X_1$ a point with closed G_1-orbit and of maximum stabilizer
dimension and $x_2 = f(x_1) \in X_2$. Since $\underline{f}$ is strongly étale, the G_2-orbit
of x_2 is also closed, by Proposition 3.3(5), and x_2 is of maximum stabilizer
dimension.

Up to G_1-equivariant shrinking around x_1, we can find smooth, affine G_i-
schemes Y_i, $i = 1, 2$, and a smooth morphism $g\colon Y_1 \to Y_2$, equivariant with
respect to φ, fitting in a commutative diagram

(4.2)
$$
\begin{array}{ccc}
X_1 & \longrightarrow & Y_1 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X_2 & \longrightarrow & Y_2
\end{array}
$$

where the horizontal arrows are equivariant closed embeddings. We may
additionally assume the induced morphism $\underline{g}\colon [Y_1/G_1] \to [Y_2/G_2]$ is strongly
étale. This is because we can initially take $\underline{g}$ to map x_1 to x_2, be stabilizer-
preserving at x_1 and $\underline{g}$ to be étale at x_1, so using the Fundamental Lemma
[Alp10, Theorem 6.10] at x_1, after possible further shrinking around x_1, we
may take $\underline{g}$ to be strongly étale.

By the intrinsic nature of intrinsic blowups, using an identical argument as
the one outlined in the proof of Lemma 4.2, one may consider diagram (4.2)
as x_1 varies to obtain a canonical morphism $f^{intr}\colon X_1^{intr} \to X_2^{intr}$.

By construction, f^{intr} is stabilizer-preserving and maps closed G_1-orbits
to closed G_2-orbits. It is moreover smooth and the associated quotient
morphism $\underline{f}^{intr}\colon [X_1^{intr}/G_1] \to [X_2^{intr}/G_2]$ is étale. This can be seen as
follows: Let T be an affine, étale slice for the G_1-orbit of x_1 in X_1, whose
stabilizer we denote by H. By the assumptions on the morphism f, T is
also an étale slice for the G_2-orbit of x_2 in X_2 and we have a commutative
diagram

$$
\begin{array}{ccc}
T \times_H G_1 & \longrightarrow & X_1 \\
{\scriptstyle \mathrm{id}\times\phi}\downarrow & & \downarrow{\scriptstyle f} \\
T \times_H G_2 & \longrightarrow & X_2
\end{array}
$$

and the corresponding commutative diagram

$$
\begin{array}{ccc}
[T/H] & \longrightarrow & [X_1/G_1] \\
\text{id} \downarrow & & \downarrow \underline{f} \\
[T/H] & \longrightarrow & [X_2/G_2]
\end{array}
$$

at the level of quotient stacks. All horizontal arrows are étale.

These give rise to identical diagrams with T, X_1, X_2 replaced by their intrinsic blowups. Again, varying x_1, this immediately implies the smoothness and affineness of f^{intr} and étaleness of $\underline{f}^{intr}$.

Now Corollary 4.6(1) lets us pass from intrinsic blowups to Kirwan blowups.

Finally, since $\widehat{f}$ is étale, stabilizer-preserving and maps closed points to closed points, so by another application of the Fundamental Lemma [Alp10, Theorem 6.10] it is strongly étale. This concludes the proof. $\qquad\square$

Before extending the notion of Kirwan blowup to stacks with good moduli space, it will be useful to generalize our definition of stability to this setting (cf. Definition 4.3).

Definition 4.12. *Let $\mathcal{M}$ be an Artin stack of finite type and $q\colon \mathcal{M} \to M$ a good moduli space morphism of finite type with affine diagonal. The prestable locus $\mathcal{M}^{ps} \subseteq \mathcal{M}$ of $\mathcal{M}$ is the (saturated) open substack whose closed points are the closed points $x \in \mathcal{M}$ satisfying $|q^{-1}(q(x))| = \{x\}$. The stable locus $\mathcal{M}^s \subseteq \mathcal{M}$ is the saturated, open substack consisting of prestable points with finite stabilizer group.*

This agrees with Definition 4.3 by the following proposition, which also explains why the stable locus is open and saturated, since this is a property that can be checked strongly étale locally.

Proposition 4.13. *With the same notation as above, for any strongly étale morphism $[U/G] \xrightarrow{\Phi} \mathcal{M}$, we have that $\Phi^{-1}(\mathcal{M}^s) = [U^s/G]$, where U^s is the open subscheme of $u \in U$ with closed G-orbit and finite stabilizer.*

Proof. Any strongly étale morphism

$$
\begin{array}{ccc}
[U/G] & \xrightarrow{\;\Phi\;} & \mathcal{M} \\
q_U \downarrow & & \downarrow q \\
U /\!\!/ G & \longrightarrow & M
\end{array}
$$

induces an isomorphism of fibers $q_U^{-1}(q_U(u)) \simeq q^{-1}(q(x))$ whenever $\Phi(u) = x$ and we identify $u \in U$ with its G-orbit as a point $u \in [U/G]$. The conclusion is immediate by Proposition 3.3(5) and the fact that Φ is stabilizer-preserving. $\qquad\square$

We can now construct the Kirwan blowup of an Artin stack with good moduli space.

Theorem-Construction 4.14. *Let $\mathcal{M}$ be an Artin stack of finite type over $\mathbb{C}$ with affine diagonal, which is not Deligne-Mumford. Moreover, suppose*

that $q\colon \mathcal{M} \to M$ *is a good moduli space morphism with q of finite type and with affine diagonal.*

Then there exists a canonical Artin stack $\widehat{\mathcal{M}}$, *called the Kirwan blowup of* $\mathcal{M}$, *together with a morphism* $\pi\colon \widehat{\mathcal{M}} \to \mathcal{M}$, *such that:*

(1) $\widehat{\mathcal{M}}$ *is of finite type over* $\mathbb{C}$, *has affine diagonal and admits a good moduli space morphism* $\widehat{q}\colon \widehat{\mathcal{M}} \to \widehat{M}$ *with affine diagonal.*

(2) *The maximum stabilizer dimension of closed points in* $\widehat{\mathcal{M}}$ *is strictly smaller than that of* $\mathcal{M}$.

(3) *For any strongly étale morphism* $[U/G] \to \mathcal{M}$, *the base change* $\widehat{\mathcal{M}} \times_{\mathcal{M}} [U/G]$ *is naturally isomorphic to the Kirwan blowup* $[\widehat{U}/G]$.

(4) $\pi|_{\pi^{-1}(\mathcal{M}^s)}$ *is an isomorphism.*

As is the case with Kirwan blowups of schemes, $\widehat{\mathcal{M}}$ *is a semistable locus* $\widehat{\mathcal{M}} = (\mathcal{M}^{intr})^{ss}$, *an open substack of a canonical Artin stack* $\pi^{intr}\colon \mathcal{M}^{intr} \to \mathcal{M}$, *called the intrinsic blowup of* $\mathcal{M}$.

Proof. Let $\mathcal{M}^{max}$ be the substack of $\mathcal{M}$ whose points have stabilizers of the maximum possible dimension. This is a closed substack of $\mathcal{M}$ with good moduli space M^{max} (cf. [ER21, Appendix B] for details on the algebraic structure of $\mathcal{M}^{max}$). For any closed point $x \in \mathcal{M}^{max}$, applying Theorem-Definition 3.6, we have a Cartesian diagram

(4.3)
$$
\begin{array}{ccc}
[U_x/G_x] & \xrightarrow{\ \Phi_x\ } & \mathcal{M} \\
\ {\scriptstyle q_x}\downarrow & & \downarrow{\scriptstyle q} \\
U_x /\!\!/ G_x & \longrightarrow & M.
\end{array}
$$

The morphisms Φ_x are affine, strongly étale and cover the locus $\mathcal{M}^{max}$. We may take the Kirwan blowup of each quotient stack $[U_x/G_x]$ to obtain good moduli space morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x /\!\!/ G_x$.

We need to check that these glue to give a stack $\widehat{\mathcal{M}}$ with a universally closed projection $\widehat{\mathcal{M}} \to \mathcal{M}$ and a good moduli space $\widehat{\mathcal{M}} \to \widehat{M}$ satisfying the same conditions as $\mathcal{M}$ and its good moduli space morphism. By the properties of the Kirwan blowup, the maximum stabilizer dimension of $\widehat{\mathcal{M}}$ will be lower than that of $\mathcal{M}$.

Suppose x, y are two closed points of $\mathcal{M}$ such that G_x, G_y are of maximum dimension. We obtain a Cartesian diagram of stacks

(4.4)
$$
\begin{array}{ccc}
[U_x \times_{\mathcal{M}} U_y/(G_x \times G_y)] & \longrightarrow & [U_y/G_y] \\
\downarrow & & \downarrow \\
[U_x/G_x] & \longrightarrow & \mathcal{M}
\end{array}
$$

where $U_{xy} := U_x \times_{\mathcal{M}} U_y$ is an affine scheme. This is due to the Cartesian diagram
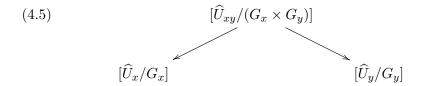
$$
\begin{array}{ccc}
U_x \times_{\mathcal{M}} U_y & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{M}}} \\
U_x \times U_y & \longrightarrow & \mathcal{M} \times \mathcal{M}
\end{array}
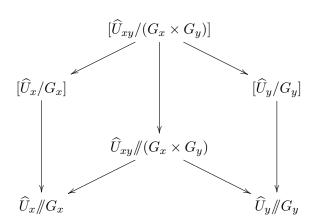$$

and the fact that $\mathcal{M}$ has affine diagonal.

By Proposition 4.11, we obtain a diagram

(4.5)
$$[\widehat{U}_{xy}/(G_x \times G_y)]$$

$$[\widehat{U}_x/G_x] \qquad\qquad [\widehat{U}_y/G_y]$$

with affine, strongly étale arrows. Moreover, there are canonical associated isomorphisms $[\widehat{U}_{xy}/(G_x \times G_y)] \to [\widehat{U}_x/G_x] \times_{\mathcal{M}} [U_y/G_y]$ and $[\widehat{U}_{xy}/(G_x \times G_y)] \to [U_x/G_x] \times_{\mathcal{M}} [\widehat{U}_y/G_y]$.

Using the charts $[\widehat{U}_x/G_x]$ together with a cover of $\mathcal{M} \setminus q^{-1}(M^{max})$, we therefore obtain an atlas for a stack $\widehat{\mathcal{M}}$ with a map to $\mathcal{M}$. By the canonical isomorphisms of the previous paragraph, $\widehat{\mathcal{M}}$ is independent of the particular choices of charts for $\mathcal{M}$.

Since the arrows in diagram (4.5) are strongly étale, we obtain a corresponding diagram of étale arrows at the level of good moduli spaces of the Kirwan blowups

$$[\widehat{U}_{xy}/(G_x \times G_y)]$$

$$[\widehat{U}_x/G_x] \qquad\qquad\qquad [\widehat{U}_y/G_y]$$

$$\widehat{U}_{xy} /\!/ (G_x \times G_y)$$

$$\widehat{U}_x /\!/ G_x \qquad\qquad\qquad \widehat{U}_y /\!/ G_y$$

where both rhombi are Cartesian.

By Proposition 3.3, $\mathcal{M} \setminus q^{-1}(M^{max}) \to M \setminus M^{max}$ is a good moduli space morphism. Hence the morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x /\!/ G_x$ for all $x \in \mathcal{M}^{max}$ together with an atlas of $\mathcal{M} \setminus q^{-1}(M^{max})$ glue to give a morphism $\widehat{\mathcal{M}} \to \widehat{M}$. By Proposition 3.3 again, this is a good moduli space morphism.

$\widehat{\mathcal{M}}$ has affine diagonal since we have a cartesian diagram

$$
\begin{array}{ccc}
[\widehat{U}_{xy}/(G_x \times G_y)] & \longrightarrow & \widehat{\mathcal{M}} \\
\downarrow & & \downarrow \\
[\widehat{U}_x/G_x] \times [\widehat{U}_y/G_y] & \longrightarrow & \widehat{\mathcal{M}} \times \widehat{\mathcal{M}}
\end{array}
$$

where the lower horizontal arrows give an étale cover of $\widehat{\mathcal{M}} \times \widehat{\mathcal{M}}$ and the left vertical arrow is affine.

To see that $\widehat{\mathcal{M}} \to \widehat{M}$ also has affine diagonal, we consider the diagram

$$
\begin{array}{ccc}
\widehat{\mathcal{M}} \longrightarrow \widehat{\mathcal{M}} \times_{\widehat{M}} \widehat{\mathcal{M}} \longrightarrow \widehat{M} \\
\searrow \quad\quad \downarrow \quad\quad\quad\quad \downarrow \\
\widehat{\mathcal{M}} \times \widehat{\mathcal{M}} \longrightarrow \widehat{M} \times \widehat{M}
\end{array}
$$

where the right square is cartesian. The diagonal of $\widehat{M}$ is an immersion and hence in particular separated. Therefore, since the diagonal of $\widehat{\mathcal{M}}$ is affine, it follows by the usual cancellation property that $\widehat{\mathcal{M}} \to \widehat{M}$ has affine diagonal.

$\widehat{\mathcal{M}}$, $\widehat{M}$ and the morphism $\widehat{\mathcal{M}} \to \widehat{M}$ thus have the same properties as $\mathcal{M}$, M and $\mathcal{M} \to M$, as desired. This establishes property (1) of the statement.

Property (3) is true by construction and properties (2) and (4) can be checked strongly étale locally. But by the definition of Kirwan blowups of quotient stacks, they are both true in that case.

Finally, the last assertion about the intrinsic blowup $\mathcal{M}^{intr}$ can be shown using the same arguments. Since we won't be needing $\mathcal{M}^{intr}$, we leave the details to the reader and only mention that

$$\widehat{\mathcal{M}} = (\mathcal{M}^{intr})^{ss} = \mathcal{M}^{intr} \setminus \overline{(\pi^{intr})^{-1}(q^{-1}(M^{max}) \setminus \mathcal{M}^{max})},$$

since the unstable locus of $\mathcal{M}^{intr}$ is the strict transform of the saturation of $\mathcal{M}^{max}$ inside $\mathcal{M}$, exactly as in Corollary 4.6(2). $\qquad\square$

Remark 4.15. *The fact that $\mathcal{M}$ and $q\colon \mathcal{M} \to M$ have affine diagonal is not crucial for the above proof to go through. In fact, by [AHR19, Theorem 13.1] and [AHR19, Corollary 13.11], it is enough to assume that $\mathcal{M}$ has affine stabilizers and separated diagonal. Since the stacks we are interested in will have affine diagonal, we make this assumption for convenience of presentation.*

As before, repeatedly applying the operation of Kirwan blowup we obtain the instrisic stabilizer reduction. By construction of the Kirwan blowup, the stable locus is preserved at each step and hence stays invariant under the whole procedure.

Theorem 4.16. *Let $\mathcal{M}$ be an Artin stack of finite type over $\mathbb{C}$ with affine diagonal, which is not Deligne-Mumford. Moreover, suppose that $q\colon \mathcal{M} \to M$ is a good moduli space morphism with q of finite type and with affine diagonal. Then there exists a canonical Deligne-Mumford stack $\widetilde{\mathcal{M}}$, called the intrinsic stabilizer reduction of $\mathcal{M}$, together with a morphism $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ which restricts to an isomorphism over the stable locus $\mathcal{M}^s \subseteq \mathcal{M}$. Moreover, $\widetilde{\mathcal{M}}$ admits a good moduli space $\widetilde{M}$ and the good moduli space morphism $\widetilde{q}\colon \widetilde{\mathcal{M}} \to \widetilde{M}$ is proper.*

Proof. We perform a sequence of Kirwan blowups by applying Theorem-Costruction 4.14 iteratively to get

$$\mathcal{M}_0 = \mathcal{M}, \ \mathcal{M}_1 = \widehat{\mathcal{M}}_1, \ \ldots, \ \mathcal{M}_\ell = \widehat{\mathcal{M}}_{\ell-1}.$$

At each step, the maximum stabilizer dimension strictly decreases, so we terminate as soon as $\mathcal{M}_\ell$ has finite stabilizers and is thus Deligne-Mumford. We define $\widetilde{\mathcal{M}} := \mathcal{M}_\ell$.

$\widetilde{\mathcal{M}} \to \widetilde{M}$ admits a strongly étale cover by morphisms of the form $[U/G] \to U /\!/ G$ where G is finite, so its diagonal is finite (cf. [MFK94, Proposition 0.8]). Thus $\widetilde{\mathcal{M}} \to \widetilde{M}$ is separated and by Proposition 3.3 also universally closed, hence proper. □

Remark 4.17. *If $\mathcal{M}^s$ is non-empty, then we are guaranteed that the intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ is non-empty. While this is sufficient, it is not necessary, as can be seen for the quotient stack $\mathcal{M} = [U/G]$ in Example 4.7.*

In general, if at some step of the procedure the stack $\mathcal{M}_i$ is a gerbe over a Deligne-Mumford stack (i.e. $\mathcal{M}_i$ has constant stabilizer dimension and every point is prestable), then $\widehat{\mathcal{M}}_i$ is empty. We could elect to terminate the procedure at this step instead and define $\widetilde{\mathcal{M}} := \mathcal{M}_i$. We choose not to do so as for our purpose of defining Donaldson-Thomas invariants, setting $\widetilde{\mathcal{M}} = \emptyset$ in this case seems more geometrically motivated and better aligned with the BPS invariants considered in [JS12] and [DM20].

Remark 4.18. *Edidin-Rydh have also developed a blowup procedure for stacks with good moduli spaces in [ER21]. For smooth stacks, our stabilizer reduction is the same as theirs. For singular stacks, Kirwan blowups can be phrased in their language of saturated blowups, however the resolution they obtain is a closed substack of the one here. We expect that our construction is closely related to the Edidin-Rydh resolution of stabilizers performed at the level of derived stacks, using the recently developed notion of derived blowups in [Hek21].*

5. OBSTRUCTION THEORY

In this section, we first recall the basic principles of semi-perfect obstruction theories and then explain how to construct such a gadget on the intrinsic stabilizer reduction of a stack of DT type. Our discussion goes through the notion of a local model and its obstruction theory, as used in [KLS17].

5.1. **Semi-perfect obstruction theory.** This subsection contains background material about semi-perfect obstruction theories and their induced virtual cycles, as developed in [CL11].

Let $U \to C$ be a morphism, where U is a scheme of finite type and C denotes a smooth quasi-projective scheme, which will typically be either a point or a smooth quasi-projective curve. We first recall the definition of perfect obstruction theory [BF97, LT98].

Definition 5.1. (Perfect obstruction theory [BF97]) *A (truncated) perfect (relative) obstruction theory consists of a morphism $\phi \colon E \to L_{U/C}^{\geq -1}$ in $D^b(\operatorname{Coh} U)$ such that*

   (1) *E is of perfect amplitude, contained in $[-1, 0]$.*
   (2) *$h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.*

We refer to $\mathcal{O}b_\phi := H^1(E^\vee)$ as the obstruction sheaf of ϕ.

Definition 5.2. (Infinitesimal lifting problem) *Let $\iota \colon \Delta \to \bar{\Delta}$ be an embedding with $\bar{\Delta}$ local Artinian, such that $I \cdot \mathfrak{m} = 0$ where I is the ideal of Δ*

and $\mathfrak{m}$ the closed point of $\bar{\Delta}$. We call $(\Delta, \bar{\Delta}, \iota, \mathfrak{m})$ a small extension. Given a commutative square

(5.1)
$$\begin{array}{ccc} \Delta & \xrightarrow{\;g\;} & U \\ \downarrow{\scriptstyle\iota} & \nearrow{\scriptstyle\bar{g}} & \downarrow \\ \bar{\Delta} & \longrightarrow & C \end{array}$$

such that the image of g contains a point $p \in U$, the problem of finding $\bar{g} \colon \bar{\Delta} \to U$ making the diagram commutative is the "infinitesimal lifting problem of U/C at p".

Definition 5.3. (Obstruction space) *For a point $p \in U$, the intrinsic obstruction space to deforming p is $T^1_{p,U/C} := H^1\left((L^{\geq -1}_{U/C})^\vee|_p\right)$. The obstruction space with respect to a perfect obstruction theory ϕ is $\mathrm{Ob}(\phi, p) := H^1(E^\vee|_p)$.*

Given an infinitesimal lifting problem of U/C at a point p, there exists by the standard theory of the cotangent complex a canonical element

(5.2)
$$\omega\left(g, \Delta, \bar{\Delta}\right) \in \mathrm{Ext}^1\left(g^* L^{\geq -1}_{U/C}|_p, I\right) = T^1_{p,U/C} \otimes_{\mathbb{C}} I$$

whose vanishing is necessary and sufficient for the lift $\bar{g}$ to exist.

Definition 5.4. (Obstruction assignment) *For an infinitesimal lifting problem of U/C at p and a perfect obstruction theory ϕ the obstruction assignment at p is the element*

(5.3)
$$ob_U(\phi, g, \Delta, \bar{\Delta}) = h^1(\phi^\vee)\left(\omega\left(g, \Delta, \bar{\Delta}\right)\right) \in \mathrm{Ob}(\phi, p) \otimes_{\mathbb{C}} I.$$

Definition 5.5. *Let $\phi\colon E \to L^{\geq -1}_{U/C}$ and $\phi'\colon E' \to L^{\geq -1}_{U/C}$ be two perfect obstruction theories and $\psi\colon \mathcal{O}b_\phi \to \mathcal{O}b_{\phi'}$ be an isomorphism. We say that the obstruction theories give the same obstruction assignment via ψ if for any infinitesimal lifting problem of U/C at p*

(5.4)
$$\psi\left(ob_U(\phi, g, \Delta, \bar{\Delta})\right) = ob_U(\phi', g, \Delta, \bar{\Delta}) \in \mathrm{Ob}(\phi', p) \otimes_{\mathbb{C}} I.$$

We are now ready to give the definition of a semi-perfect obstruction theory.

Definition 5.6. (Semi-perfect obstruction theory [CL11, Definition 3.1]) *Let $\mathcal{M} \to C$ be a morphism, where $\mathcal{M}$ is a DM stack, proper over C, of finite presentation and C is a smooth quasi-projective scheme. A semi-perfect obstruction theory ϕ consists of an étale covering $\{U_\alpha\}_{\alpha \in A}$ of $\mathcal{M}$ and perfect obstruction theories $\phi_\alpha\colon E_\alpha \to L^{\geq -1}_{U_\alpha/C}$ such that*

(1) *For each pair of indices α, β, there exists an isomorphism*

$$\psi_{\alpha\beta}\colon \mathcal{O}b_{\phi_\alpha}|_{U_{\alpha\beta}} \longrightarrow \mathcal{O}b_{\phi_\beta}|_{U_{\alpha\beta}}$$

so that the collection $\{\mathcal{O}b_{\phi_\alpha}, \psi_{\alpha\beta}\}$ gives descent data of a coherent sheaf on $\mathcal{M}$.

(2) *For each pair of indices α, β, the obstruction theories $E_\alpha|_{U_{\alpha\beta}}$ and $E_\beta|_{U_{\alpha\beta}}$ give the same obstruction assignment via $\psi_{\alpha\beta}$ (as in Definition 5.5).*

Remark 5.7. *The obstruction sheaves $\{\mathcal{O}b_{\phi_\alpha}\}_{\alpha\in A}$ glue to define a sheaf $\mathcal{O}b_\phi$ on $\mathcal{M}$. This is the obstruction sheaf of the semi-perfect obstruction theory ϕ.*

Suppose now that $\mathcal{M} \to C$ is as above and admits a semi-perfect obstruction theory. Then, for each $\alpha \in A$, we have

$$\mathcal{C}_{U_\alpha/C} \subset N_{U_\alpha/C} = h^1/h^0((L^{\geq -1}_{U_\alpha/C})^\vee) \xrightarrow{h^1/h^0(\phi_\alpha^\vee)} h^1/h^0(E_\alpha^\vee) \longrightarrow h^1(E_\alpha^\vee),$$

where $\mathcal{C}_{U_\alpha/C}$ and $N_{U_\alpha/C}$ denote the intrinsic normal cone stack and intrinsic normal sheaf stack respectively, where by abuse of notation we identify a sheaf $\mathcal{F}$ on $\mathcal{M}$ with its sheaf stack.

We therefore obtain a cycle class $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ by taking the pushforward of the cycle $[\mathcal{C}_{U_\alpha/C}] \in Z_*N_{U_\alpha/C}$.

Theorem-Definition 5.8. [CL11, Definition-Theorem 3.7] *Let $\mathcal{M}$ be a DM stack, proper over C, of finite presentation and C a point or a smooth quasi-projective curve, such that $\mathcal{M} \to C$ admits a semi-perfect obstruction theory ϕ. The classes $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ glue to define an intrinsic normal cone cycle $[\mathfrak{c}_\phi] \in Z_*\mathcal{O}b_\phi$. Let s be the zero section of the sheaf stack $\mathcal{O}b_\phi$. The virtual cycle of $\mathcal{M}$ is defined to be*

$$[\mathcal{M}, \phi]^{\mathrm{vir}} := s^![\mathfrak{c}_\phi] \in A_*\mathcal{M},$$

where $s^! : Z_*\mathcal{O}b_\phi \to A_*\mathcal{M}$ is the Gysin map. This virtual cycle satisfies all the usual properties, such as deformation invariance.

Remark 5.9. *Observe that in Definition 5.6 it is not strictly necessary to take the U_α to be schemes. It is straightforward to generalize the definition and the construction of the associated virtual fundamental cycle to include étale covers by Deligne-Mumford stacks.*

5.2. **Local models, standard forms and their blowups.** Let V be a smooth affine G-scheme. The action of G on V induces a morphism $\mathfrak{g}\otimes\mathcal{O}_V \to T_V$ and its dual $\sigma_V : \Omega_V \to \mathfrak{g}^\vee \otimes \mathcal{O}_V$.

Setup-Definition 5.10. [KLS17, Setup-Definition 5.1] *Consider the quadruple (V, F_V, ω_V, D_V), where F_V is a G-equivariant vector bundle on V, ω_V a G-invariant section with scheme-theoretic zero locus $U = \{\omega_V = 0\} \subset V$ and $D_V \subset V$ an effective invariant divisor, satisfying:*

(1) *$\sigma_V(-D_V) : \Omega_V(-D_V) \to \mathfrak{g}^\vee(-D_V)$ factors through a morphism ϕ_V as shown*

(5.5) $$\Omega_V(-D_V) \longrightarrow F_V \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V);$$

(2) *The composition $\phi_V \circ \omega_V$ vanishes identically;*
(3) *Let R be the identity component of the stabilizer group of a closed point in V with closed orbit. Let V^R denote the fixed point locus of R. Then $\phi_V|_{V^R}$ composed with the projection $\mathfrak{g}^\vee(-D_V) \to \mathfrak{r}^\vee(-D_V)$ is zero, where $\mathfrak{r}$ is the Lie algebra of R.*

We say that the data

$$\Lambda_U = (U, V, G, F_V, \omega_V, D_V, \phi_V)$$

give a local model structure for U. Thinking of the quotient stack, we say that $[U/G]$ also has a local model structure and denote the data by $\Lambda_{[U/G]}$.

Remark 5.11. *Note that if* $f \colon V \to \mathbb{A}^1$ *is a G-invariant function on V, then $(U, V, G, \Omega_V, df, 0, \sigma_V)$ give a local model for U, being equivalent to an invariant d-critical chart (U, V, f, i) for U. Therefore, an invariant d-critical locus is a particular case of a local model.*

Now, let $\Lambda_U = (U, V, G, F_V, \omega_V, D_V, \phi_V)$ define a local model structure on U. Since G is reductive, we have a splitting

$$F_V|_{V^G} = F_V|_{V^G}^{fix} \oplus F_V|_{V^G}^{mv}.$$

Let now $\pi \colon \widehat{V} \to V$ be the Kirwan blowup of V. Define $F_{\widehat{V}}$ as the kernel of the composite morphism $\pi^* F_V \to \pi^* (F_V|_{V^G}) \to \pi^* (F_V|_{V^G}^{mv})$ so that we have an exact sequence

$$0 \longrightarrow F_{\widehat{V}} \longrightarrow \pi^* F_V \longrightarrow \pi^* (F_V|_{V^G}^{mv}) \longrightarrow 0.$$

By equivariance, $\pi^* \omega_V$ maps to zero under the second map and hence induces an invariant section $\omega_{\widehat{V}}$ of $F_{\widehat{V}}$. A local computation shows the following.

Proposition 5.12. *The zero locus of $\omega_{\widehat{V}}$ is the Kirwan blowup $\widehat{U}$ of U with respect to G.*

Proof. This is Proposition 2.11 in [KLS17]. $\qquad\qquad\qquad\qquad\qquad\square$

Local model structures are preserved under the operations of Kirwan blowups and taking étale slices.

Proposition 5.13. *Let $(U, V, G, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model. Then there exist induced data $(\widehat{U}, \widehat{V}, G, F_{\widehat{V}}, \omega_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}})$ of a local model on the Kirwan blowup $\widehat{U}$. The same is true for any étale slice T of a closed point of $\widehat{U}$ with closed G-orbit.*

Proof. This is [KLS17, Lemma 5.3]. The fact that $F_{\widehat{V}}$ is locally free is discussed shortly after Definition 2.10 in [KLS17]. Denoting the Kirwan blowup map $\widehat{V} \to V$ by π, the divisor $D_{\widehat{V}}$ is equal to $\pi^* D_V + 2E$, where E is the exceptional divisor of $\widehat{V}$. Finally, $\phi_{\widehat{V}}$ is induced by ϕ_V through a diagram chase and the definition of $F_{\widehat{V}}$. The reasoning for étale slices is similar. $\qquad\qquad\qquad\qquad\qquad\square$

5.3. **Obstruction theory of local model.** Let

$$\Lambda_U = (U, V, G, F_V, \omega_V, D_V, \phi_V)$$

be the data of a local model structure. Consider the sequence

$$(5.6) \qquad \mathcal{K}_U := [\mathfrak{g} \to T_V|_U \xrightarrow{(d_V \omega_V^\vee)^\vee} F_V|_U \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V)].$$

By [KLS17, Lemma 5.5], this defines a G-equivariant perfect complex on U that descends to a perfect complex on $[U/G]$, which we denote by $\mathcal{K}_{[U/G]}$.

We have a morphism

$$(5.7) \qquad\qquad E_{[U/G]} := \mathcal{K}_{[U/G]}^\vee \longrightarrow L_{[U/G]}^{\geq -1}$$

given by the diagram

$$(5.8) \qquad \mathfrak{g}(D_V) \xrightarrow{\phi_V^\vee} F_V^\vee|_U \xrightarrow{d_V \omega_V^\vee} \Omega_V|_U \longrightarrow \mathfrak{g}^\vee$$

with vertical map $\omega_V^\vee$ and bottom row $I/I^2 \xrightarrow{d_V} \Omega_V|_U \longrightarrow \mathfrak{g}^\vee,$

where I denotes the ideal sheaf of U in V. Observe that the diagram commutes, since by Setup-Definition 5.10(2), the composition $\phi_V \circ \omega_V$ is identically zero.

Let U^s, V^s denote the stable loci of U and V respectively (cf. Definition 4.3). We thus have that $\sigma_V|_{V^s} \colon \Omega_V|_{V^s} \to \mathfrak{g}^\vee$ is surjective. By Setup-Definition 5.10, it follows that $\phi_V|_{U^s}$ is surjective and hence its dual is injective with locally free cokernel.

Therefore, $\mathcal{K}_{[U^s/G]} := \mathcal{K}_{[U/G]}|_{[U^s/G]}$ is a two-term perfect complex, whose dual $E_{[U^s/G]}$ gives a perfect obstruction theory on the DM stack $[U^s/G]$ via the diagrams (5.7) and (5.8).

Definition 5.14. *Let $(U, V, G, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model structure. Then $E_{[U^s/G]}$ is the induced perfect obstruction theory on the stable locus $[U^s/G]$.*

5.4. Semi-perfect obstruction theory of the intrinsic stabilizer reduction of a stack of DT type.
Let $\mathcal{M}$ be an Artin stack of DT type. We now explain how to construct a semi-perfect obstruction theory on its intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$, one of the main results of this paper.

We begin by recording and abstracting some of the data associated to a stack of DT type.

Setup 5.15. *For any closed point $x \in \mathcal{M}$, by Proposition 3.10, there exists a strongly étale d-critical quotient chart for $\mathcal{M}$ centered at x. In particular, we obtain a strongly étale cover*

$$(5.9) \qquad \coprod_{x \in \mathcal{M}} [U_x/G_x] \longrightarrow \mathcal{M}$$

where, by Remark 5.11, each quotient stack $[U_x/G_x]$ is equipped with data of a local model

$$(5.10) \qquad \Lambda_{[U_x/G_x]} = (U_x, V_x, G_x, F_{V_x}, \omega_{V_x}, D_{V_x}, \phi_{V_x}),$$

with $F_{V_x} = \Omega_{V_x}$, $\omega_{V_x} = df_x$, $D_{V_x} = 0$ and $\phi_{V_x} = \sigma_{V_x}$.

By Proposition 3.11, for any two closed points $x, y \in \mathcal{M}$ we have an étale cover

$$(5.11) \qquad \coprod_z [T_z/G_z] \longrightarrow [U_x/G_x] \times_{\mathcal{M}} [U_y/G_y]$$

by d-critical quotient charts centered at closed points $z \in [U_x/G_x] \times_{\mathcal{M}} [U_y/G_y]$, so that we have local model structures on $[T_z/G_z]$ given by the data

$$(5.12) \qquad \Lambda_{[T_z/G_z]} = (T_z, S_z, G_z, F_{S_z}, \omega_{S_z}, D_{S_z}, \phi_{S_z}),$$

with $F_{S_z} = \Omega_{V_x}$, $\omega_{S_z} = df_z$, $D_{S_z} = 0$ and $\phi_{S_z} = \sigma_{S_z}$.

For any morphism $[T_z/G_z] \to [U_x/G_x]$ in (5.11), Proposition 3.11 also produces the following data:

(1) *A commutative G_z-equivariant diagram*

$$
\begin{array}{ccc}
T_z & \longrightarrow & S_z \\
\downarrow & & \downarrow{\scriptstyle \theta_{zx}} \\
U_x & \longrightarrow & V_x
\end{array}
$$

where the horizontal arrows are closed embeddings and θ_{zx} is unramified, inducing the étale morphism $[T_z/G_z] \to [U_x/G_x]$ in (5.11).

(2) *A surjective G_z-equivariant morphism $\eta_{zx} \colon F_{V_x}|_{S_z} \to F_{S_z}$, compatible with the morphisms $\phi_{V_x}|_{S_z}$ and ϕ_{S_z}, such that $\omega'_z := \eta_{zx}(\omega_{V_x}|_{S_z})$ and ω_{S_z} are Ω-equivalent (cf. [KLS17, Definition 5.9]).*

(3) *An induced isomorphism*

$$
\eta_{zx} \colon h^1(\mathcal{K}_{[U_x/G_x]})|_{[T_z/G_z]} \longrightarrow h^1(\mathcal{K}_{[T_z/G_z]}).
$$

(4) *Isomorphisms*

$$
\psi_{xy} \colon h^1(\mathcal{K}_{[U_x/G_x]})|_{[U_x/G_x] \times_{\mathcal{M}} [U_y/G_y]} \longrightarrow h^1(\mathcal{K}_{[U_y/G_y]})|_{[U_x/G_x] \times_{\mathcal{M}} [U_y/G_y]}
$$

satisfying the cocycle condition and $\psi_{xy}|_{[T_z/G_z]} = \eta_{zy}^{-1} \circ \eta_{zx}$.

*We abstract this situation by saying that the data of a strongly étale cover (5.9) equipped with a local model structure (5.10), an étale cover (5.11) with a local model structure (5.12) and compatibility data described by items (1)-(4) above constitute a **VC package** for an Artin stack $\mathcal{M}$ satisfying the conditions of Theorem-Construction 4.14. Note that if $\mathcal{M}$ is an Artin stack of DT type then, by our discussion, we may take the local model structures to correspond to invariant d-critical quotient charts, however in general we only require local models that satisfy the compatibilities (1)-(4).*

For a detailed account of Ω-equivalence, we refer the reader to [KLS17, Section 5]. For the purposes of our discussion, it suffices to note that ω'_z being Ω-equivalent to ω_{S_z} essentially means that we have another local model structure

$$
\Lambda'_{[T_z/G_z]} = (T_z, S_z, G_z, F_{S_z}, \omega'_z, D_{S_z}, \phi_{S_z})
$$

on $[T_z/G_z]$, such that the induced perfect obstruction theory $E'_{[T_z^s/G_z]}$ naturally has the same obstruction sheaf and gives the same obstruction assignment with $E_{[T_z^s/G_z]}$.

The data of Setup 5.15 are preserved under Kirwan blowups and hence lift to the intrinsic stabilizer reduction of a stack of DT type.

Theorem 5.16. *Let $\mathcal{M}$ be an Artin stack of DT type. Then its intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ admits a VC package.*

Proof. By the discussion in Setup 5.15, $\mathcal{M}$ admits a VC package where moreover all local models corresponds to d-critical quotient charts.

By the construction of $\widetilde{\mathcal{M}}$, it suffices to show that if a stack $\mathcal{N}$ (satisfying the conditions of Theorem-Construction 4.14) admits a VC package, then its Kirwan blowup $\widehat{\mathcal{N}}$ admits a canonically induced VC package.

By the construction in the proof of Proposition 3.11, Proposition 4.11 and Theorem-Construction 4.14, we may obtain new étale covers 5.9 and 5.11 for $\widehat{\mathcal{N}}$ by replacing every stack by its Kirwan blowup.

By Proposition 5.13, there are canonically induced local model structures on these étale covers. It is moreover clear that condition (1) is satisfied.

Conditions (2)-(4) follow from [KLS17, Lemma 6.1]. This concludes the proof. $\qquad\square$

Now, a VC package on a Deligne-Mumford stack naturally provides data of a semi-perfect obstruction theory.

Theorem 5.17. *Let $\mathcal{M}$ be a Deligne-Mumford stack equipped with a VC package. Then $\mathcal{M}$ admits an induced semi-perfect obstruction theory and thus a virtual fundamental cycle $[\mathcal{M}]^{\mathrm{vir}} \in A_*(\mathcal{M})$.*

Proof. Since $\mathcal{M}$ is Deligne-Mumford, all quotient stacks in (5.9) are also Deligne-Mumford and satisfy $[U_x/G_x] = [U_x^s/G_x]$, since all closed points are now stable.

Thus, by Definition 5.14, the local model structures $\Lambda_{[U_x/G_x]}$ induce perfect obstruction theories

$$E_x := E_{[U_x/G_x]} \xrightarrow{\phi_x} L_{[U_x/G_x]}^{\geq -1}$$

on each $[U_x/G_x]$.

Conditions (3) and (4) of a VC package give descent data for the obstruction sheaves $h^1(E_x^\vee)$, while conditions (1) and (2) and the properties of Ω-equivalence imply that these descent data are compatible with obstruction assignments of infinitesimal lifting problems.

Using Remark 5.9 and Theorem-Construction 5.8, we obtain a semi-perfect obstruction theory on $\mathcal{M}$ and a virtual fundamental cycle $[\mathcal{M}]^{\mathrm{vir}} \in A_*(\mathcal{M})$. $\qquad\square$

Combining the above two theorems, we arrive at the main result of this section.

Theorem 5.18. *Let $\mathcal{M}$ be an Artin stack of DT type. Then the intrinsic stabilizer reduction $\widetilde{\mathcal{M}}$ admits a semi-perfect obstruction theory of virtual dimension zero and an induced, canonical virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$.*

Proof. The existence of the semi-perfect obstruction theory follows immediately from Theorems 5.16 and 5.17.

The virtual dimension is zero, since for a d-critical chart the rank of $F_V = \Omega_V$ is equal to $\dim V$ and the virtual dimension $\dim V - \mathrm{rk}\, F_V$ is preserved under Kirwan blowups.

Finally, any two different étale covers (5.9) that are part of a VC package of $\mathcal{M}$ can be refined to a common étale cover. It is routine to check that the virtual fundamental cycle thus obtained is the same using this common refinement. $\qquad\square$

Remark 5.19. *The existence of the intrinsic stabilizer reduction of $\mathcal{M}$ and its obstruction theory only uses its (-1)-shifted symplectic derived enhancement to deduce the existence of a d-critical structure on $\mathcal{M}$. We could have*

thus used a weaker notion of stacks of DT type, replacing condition (3) in Definition 3.7 by the requirement that $\mathcal{M}$ admits a d-critical structure. However, we will be interested in replicating our arguments in the relative case where $\mathcal{M}$ is a stack over a base smooth scheme S. In that context, there is no well-developed theory of d-critical stacks to the author's knowledge and the existence of a derived enhancement will allow us to still perform our constructions.

6. Donaldson-Thomas Invariants of Derived Objects

In this section, W denotes a smooth, projective Calabi-Yau threefold over $\mathbb{C}$. We first describe the stability conditions σ on $D^b(\operatorname{Coh} W)$ that we will be interested in. We then quote results in [AHH18] which imply that moduli stacks of σ-semistable complexes are stacks of DT type and explain how to rigidify the $\mathbb{C}^*$-scaling automorphisms of objects. Finally, we define generalized DT invariants via Kirwan blowups and show their deformation invariance.

6.1. **Stability conditions.** By [Lie06], there is an Artin stack $\mathcal{P} := \operatorname{Perf}(W)$ of (universally gluable) perfect complexes on W, which is locally of finite type and has separated diagonal. Following [AHH18], we will consider the following type of stability condition.

Definition 6.1. (Stability condition) *A stability condition σ on $D^b(\operatorname{Coh} W)$ consists of the following data:*

(1) *A heart $\mathcal{A} \subset D^b(\operatorname{Coh} W)$ of a t-structure on $D^b(\operatorname{Coh} W)$. Let $\mathcal{P}_{\mathcal{A}}$ denote the stack of perfect complexes in $\mathcal{A}$.*

(2) *A vector $\gamma \in H^*(W, \mathbb{Q})$. Let $\mathcal{P}_{\mathcal{A}}^{\gamma}$ denote the stack of perfect complexes in $\mathcal{A}$ with Chern character γ.*

(3) *A locally constant function*

$$p_{\gamma} \colon \pi_0(\mathcal{P}_{\mathcal{A}}) \longrightarrow V$$

where V is a totally ordered abelian group, $p_{\gamma}(E) = 0$ for $E \in \mathcal{P}_{\mathcal{A}}^{\gamma}$ and p_{γ} is additive so that $p_{\gamma}(E \oplus F) = p_{\gamma}(E) + p_{\gamma}(F)$.

We say that $E \in \mathcal{P}_{\mathcal{A}}^{\gamma}$ is semistable if for any subobject $F \subseteq E$ we have $p_{\gamma}(F) \leq 0$ and stable if $p_{\gamma}(F) < 0$. If E is not semistable, we say it is unstable.

In order for the stack of semistable objects to be of DT type, we will need to consider stability conditions satisfying certain properties.

Definition 6.2. (Nice stability condition) *Given a stability condition σ on $D^b(\operatorname{Coh} W)$, let $\mathcal{M}$ be the stack of σ-semistable objects in $\mathcal{P}_{\mathcal{A}}^{\gamma}$. We say that σ is nice if the following hold:*

(1) *$\mathcal{M}$ is an Artin stack of finite type.*

(2) *$\mathcal{M}$ is an open substack of $\operatorname{Perf}(W)$.*

(3) *$\mathcal{M}$ satisfies the existence part of the valuative criterion of properness. We then say that $\mathcal{M}$ is quasi-proper or universally closed.*

Remark 6.3. *The following are examples of nice stability conditions, by the results of the mentioned authors:*

(1) *A Bridgeland stability condition in the sense of Toda-Piyaratne* [PT19] *and Li* [Li19].
(2) *A polynomial stability condition in the sense of Lo* [Lo11, Lo13].
(3) *Gieseker and slope stability. These are examples of a weak stability condition in the sense of Joyce-Song* [JS12, Definition 3.5], *where we take* $\mathcal{A} = \operatorname{Coh} W$ *and* $K(A) = N(W)$.

6.2. **Moduli stacks of semistable complexes are of DT type.** The following theorem is an application of [AHH18, Theorem 7.25] to our context.

Theorem 6.4. *Let* σ *be a nice stability condition on* $D^b(\operatorname{Coh} W)$ *and let* $\mathcal{M}$ *denote the stack of* σ*-semistable complexes. Then* $\mathcal{M}$ *is an Artin stack of DT type with proper good moduli space* M *and affine diagonal.*

Proof. By the niceness of σ, $\mathcal{M}$ is of finite type. By the results of [PTVV13], $\operatorname{Perf}(W)$ is the truncation of a (-1)-shifted symplectic derived Artin stack. Since $\mathcal{M}$ is an open substack of $\operatorname{Perf}(W)$, it is also such a truncation. By [AHH18, Theorem 7.25], $\mathcal{M}$ admits a good moduli space $\pi \colon \mathcal{M} \to M$ such that M is separated. By Proposition 3.3, π is universally closed and since $\mathcal{M}$ is universally closed as well, it follows that M is universally closed. Thus M must be proper.

$\mathcal{M}$ has affine diagonal by [AHH18, Lemma 7.19]. $\square$

6.3. $\mathbb{C}^*$**-rigidified intrinsic stabilizer reduction.** Let $\mathcal{M}$ be as in Theorem 6.4. We denote $T = \mathbb{C}^*$.

To obtain a meaningful nonzero DT invariant, it will be necessary to rigidify the $\mathbb{C}^*$-scaling automorphisms of objects in $\mathcal{M}$.

For each family of complexes $E_S \in \mathcal{M}(S)$ there exists an embedding

$$\mathbb{G}_m(S) \to \operatorname{Aut}(E_S)$$

which is compatible with pullbacks and moreover $\mathbb{G}_m(S)$ is central. In the terminology used in [AGV08], we say that $\mathcal{M}$ has a $\mathbb{G}_m$-2-structure.

Using the results of [AOV08] or [AGV08], we may take the $\mathbb{G}_m$-rigification $\mathcal{M} /\!\!/ \mathbb{G}_m$ of $\mathcal{M}$. From the properties of rigification, for any point $x \in \mathcal{M}$, one has $\operatorname{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \operatorname{Aut}_{\mathcal{M}}(x)/T$. In particular, if $x \in \mathcal{M}^s$ is stable, then $\operatorname{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \{\operatorname{id}\}$.

Even though $\mathcal{M} /\!\!/ \mathbb{G}_m$ is not a stack of DT type, we show in the next two propositions that we can construct its intrinsic stabilizer reduction and equip it with a semi-perfect obstruction theory of virtual dimension zero.

Proposition 6.5. $\mathcal{M} /\!\!/ \mathbb{G}_m$ *satisfies the conditions of Theorem 4.16. It thus admits an intrinsic stabilizer reduction, which we denote by* $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$.

Proof. By [Rom05, Theorem 5.1] or [AOV08, Theorem C.1.1], $\mathcal{M} /\!\!/ \mathbb{G}_m$ has the same good moduli space M. Since the morphism $\mathcal{M} \to \mathcal{M} /\!\!/ \mathbb{G}_m$ is a $\mathbb{G}_m$-gerbe, it follows that $\mathcal{M} /\!\!/ \mathbb{G}_m$ and the good moduli space morphism $\mathcal{M} /\!\!/ \mathbb{G}_m \to M$ have affine diagonal. $\square$

Proposition 6.6. $\mathcal{M} /\!\!/ \mathbb{G}_m$ *is equipped with a VC package.*

Proof. $\mathcal{M}$ is a stack of DT type and hence admits a VC package by Setup 5.15. By the existence of the $\mathbb{G}_m$-2-structure, T naturally embeds in each stabilizer group G_x. Replacing all such groups by their quotients G_x/T, we obtain a VC package for $\mathcal{M} /\!\!/ \mathbb{G}_m$. $\square$

Remark 6.7. *In the case of semistable sheaves treated in* [KLS17], *rigidification is much simpler, since the moduli stack is a global GIT quotient stack* $\mathcal{M} = [X/G]$ *where* $G = \mathrm{GL}(N, \mathbb{C})$, *and then one may work with* $[X/\mathrm{PGL}(N, \mathbb{C})]$ *as the* $\mathbb{G}_m$-*rigidication.*

We thus obtain the following.

Theorem-Definition 6.8. $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ *is called the* $\mathbb{C}^*$-*rigidified intrinsic stabilizer reduction of* $\mathcal{M}$. *It is a proper DM stack with a semi-perfect obstruction theory of virtual dimension zero.*

Proof. For properness, the good moduli space $q\colon \widetilde{\mathcal{M}}^{\mathbb{C}^*} \to \widetilde{M}^{\mathbb{C}^*}$ is proper. Since M is proper and $\widetilde{M}^{\mathbb{C}^*}$ is proper over M, $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ is proper.

Everything else follows from Theorems 5.16, 5.17 as applied in the proof of Theorem 5.18. $\square$

Remark 6.9. *By identical reasoning, all of the above hold in greater generality when* $\mathcal{M}$ *is an Artin stack of DT type with a* $\mathbb{G}_m$-*2-structure.*

6.4. **Generalized DT invariants via Kirwan blowups.** Suppose as before that $\mathcal{M}$ is an Artin stack of DT type parametrizing σ-semistable objects in a heart $\mathcal{A}$ of a t-structure on $D^b(\mathrm{Coh}\, W)$ with Chern character γ, for a nice stability condition σ. Then, by Theorem-Definition 6.8, there is an induced $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ with a good moduli space $\widetilde{M}^{\mathbb{C}^*}$ and an induced semi-perfect obstruction theory and virtual cycle of dimension zero. We can now state the main theorem of this paper.

Theorem-Definition 6.10. *Let* W *be a smooth, projective Calabi-Yau threefold,* σ *a nice stability condition on a heart* $\mathcal{A} \subset D^b(\mathrm{Coh}\, W)$ *of a* t-*structure, as in Definition 6.1,* $\gamma \in H^*(W)$ *and let* $\mathcal{M}$ *denote the stack parametrizing* σ-*semistable complexes with Chern character* γ. *Then we may define the associated generalized Donaldson-Thomas invariant via Kirwan blowups (DTK invariant) as*

$$\mathrm{DTK}\,(\mathcal{M}) := \deg\,[\widetilde{\mathcal{M}}^{\mathbb{C}^*}]^{\mathrm{vir}}.$$

Remark 6.11. *In* [KS21], *the results of* [KLS17] *and the present paper are refined to define a virtual structure sheaf* $[\mathcal{O}_{\widetilde{\mathcal{M}}}^{\mathrm{vir}}] \in K_0(\widetilde{\mathcal{M}}^{\mathbb{C}^*})$ *and a corresponding* K-*theoretic generalized DTK invariant.*

6.5. **Relative theory and deformation invariance.** Let C be a smooth quasi-projective scheme over $\mathbb{C}$ and $W \to C$ a smooth, projective family of Calabi-Yau threefolds. Without loss of generality, we assume that $H^*(W_t, \mathbb{Q})$ stays constant for $t \in C$ and identify it with $H^*(W_0, \mathbb{Q})$ where W_0 is the fiber of the family over a point $0 \in C$. Let $\gamma \in H^*(W_0, \mathbb{Q})$. Moreover, let $\mathrm{Perf}(W/S)$ denote the stack of (universally gluable) perfect complexes on the morphism $W \to C$ as in [Lie06].

We consider families σ of stability conditions σ_t, where, for each $t \in C$, σ_t is a stability condition on $D^b(\mathrm{Coh}\, W_t)$ as in Definition 6.2 with Chern character $\gamma \in H^*(W_t, \mathbb{Q}) = H^*(W_0, \mathbb{Q})$. Let $\mathcal{M} \to C$ be the stack parametrizing relatively semistable objects in $D^b(\mathrm{Coh}\, W)$, i.e. perfect complexes E such that the derived restriction $E_t := E|_{W_t}$ is σ_t-semistable for all $t \in C$. We

require that the conditions characterizing a nice stability condition hold relative to the base C as follows.

Setup 6.12. *We say that the family of stability conditions σ_t is nice if the following hold.*

(1) $\mathcal{M}$ *is an Artin stack of finite type.*
(2) $\mathcal{M}$ *is an open substack of* $\mathrm{Perf}(W/C)$.
(3) $\mathcal{M} \to C$ *admits a good moduli space* $M \to C$, *proper over* C.

Remark 6.13. *When we have a GIT description of $\mathcal{M} \to C$, then the above conditions are satisfied. This is the case for Gieseker and slope stability of coherent sheaves.*

When we are in the situation of the above setup, all of our results extend to the relative case.

Theorem 6.14. *Let $W \to C$ be a smooth, projective family of Calabi-Yau threefolds over a smooth, quasi-projective scheme C, $\{\sigma_t\}_{t \in C}$ a nice family of stability conditions on $D^b(\mathrm{Coh}\, W)$, $\gamma \in H^*(W_0, \mathbb{Q})$ and $\mathcal{M} \to C$ the stack of fiberwise σ_t-semistable objects of Chern character γ in $D^b(\mathrm{Coh}\, W)$.*

Then there exists an induced $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction $\widetilde{\mathcal{M}} \to C$, a DM stack proper over C, endowed with a semi-perfect obstruction theory and a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$.

Moreover, the fiber $\widetilde{\mathcal{M}}_t$ over $t \in C$ is the $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction of $\mathcal{M}_t$ and the obstruction theory pulls back to the one constructed in the absolute case, so that if $i_t \colon \widetilde{\mathcal{M}}_t \to \widetilde{\mathcal{M}}$ is the inclusion, we have

$$i_t^![\widetilde{\mathcal{M}}]^{\mathrm{vir}} = [\widetilde{\mathcal{M}}_t]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}}_t).$$

Proof. This is the generalization of [KLS17, Theorem 7.17] adapted to our context. We briefly explain the steps and necessary changes.

$\mathcal{M}$ admits a $\mathbb{G}_m$-2-structure. By conditions (1) and (3) of Setup 6.12, $\mathcal{M} /\!\!/ \mathbb{G}_m$ satisfies the conditions of Theorem 4.16, so using Proposition 6.5 we obtain the $\mathbb{C}^*$-rigidified intrinsic stabilizer reduction $\widetilde{\mathcal{M}}^{\mathbb{C}^*} \to \mathcal{M} \to C$.

To see that the fiber over $t \in C$ is $\widetilde{\mathcal{M}}_t^{\mathbb{C}^*}$, observe that by definition $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ is obtained by taking iterated Kirwan blowups of a cover by strongly étale quotient charts (where $T = \mathbb{C}^*$)

$$
\begin{array}{ccc}
\mathcal{N}_x := [U_x/(G_x/T)] & \longrightarrow & \mathcal{M} /\!\!/ \mathbb{G}_m \\
\downarrow & & \downarrow \\
N_x := U_x /\!\!/ G_x & \longrightarrow & M \\
& & \downarrow \\
& & C.
\end{array}
$$

Taking fibers over $t \in C$, we obtain a cover of $\mathcal{M}_t /\!\!/ \mathbb{G}_m$ by strongly étale quotient charts, so it suffices to show that $(\widehat{\mathcal{N}_x})_t = \widehat{(\mathcal{N}_x)_t}$. But this is true by the same argument used in the proof of [KLS17, Proposition 7.4].

Constructing the obstruction theory of $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ is slightly more subtle. First, we observe that there is an obvious generalization of Setup-Definition 5.2 to

define local model structures on G-schemes U over the base scheme C, cf. [KLS17, Definition 7.11].

Using the fact that the morphism $\mathcal{M} \to C$ is the truncation of a (-1)-shifted symplectic derived stack over C, we can show that $\mathcal{M} /\!\!/ \mathbb{G}_m$ admits a VC package, as in Setup 5.15. The only difference is that now in (5.10), we have $F_{V_x} = \Omega_{V_x/C}$ and ω_{V_x} is a G_x-invariant 1-form, and similarly in (5.12), $F_{S_z} = \Omega_{S_z/C}$ and ω_{S_z} is a G_z-invariant 1-form. The construction of the VC package follows verbatim the reasoning of Subsections 7.2 and 7.3 in [KLS17]. We refer the reader there for details.

Finally, to see that the restriction of the semi-perfect obstruction theory of $\widetilde{\mathcal{M}}^{\mathbb{C}^*}$ to $\widetilde{\mathcal{M}}_t^{\mathbb{C}^*}$ agrees with the absolute semi-perfect obstruction theory constructed using the d-critical structure of $\mathcal{M}_t /\!\!/ \mathbb{G}_m$, we use the fact that for any choice of relative local models in the VC package of $\mathcal{M} /\!\!/ \mathbb{G}_m$, the sections ω_{V_x} can be taken so that $\omega_{V_x}|_{(V_x)_t}$ is Ω-equivalent to an exact 1-form df_x induced by the d-critical structure of $\mathcal{M}_t /\!\!/ \mathbb{G}_m$. As a consequence of properties of Ω-equivalence (cf. [KLS17, Lemma 5.14]), the construction of the semi-perfect obstruction theory of $\widetilde{\mathcal{M}}_t^{\mathbb{C}^*}$ can be performed equivalently using $\omega_{V_x}|_{(V_x)_t}$ instead of df_x in the VC package of $\mathcal{M}_t /\!\!/ \mathbb{G}_m$. The necessary arguments are carried out in detail in [KLS17, Subsection 7.4]. $\square$

In the case of Bridgeland stability conditions constructed in [PT19] and [Li19], the results of [BLM+21] imply that we get nice families of Bridgeland stability conditions.

As an immediate corollary, we have the following theorem.

Theorem 6.15. *The generalized DT invariant via Kirwan blowups for σ-semistable objects on Calabi-Yau threefolds, where σ is a Bridgeland stability condition as in [PT19] and [Li19], is invariant under deformations of the complex structure of the Calabi-Yau threefold.*

Remark 6.16. *In the case of Gieseker stability and slope stability of coherent sheaves, deformation invariance follows directly from the results of [KLS17].*

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