GOOD MODULI SPACES IN DERIVED ALGEBRAIC GEOMETRY

ERIC AHLQVIST, JEROEN HEKKING, MICHELE PERNICE, AND MICHAIL SAVVAS

Abstract. We develop a theory of good moduli spaces for derived Artin stacks, which naturally generalizes the classical theory of good moduli spaces introduced by Alper. As such, many of the fundamental results and properties regarding good moduli spaces for classical Artin stacks carry over to the derived context. In fact, under natural assumptions often satisfied in practice, we show that the derived theory essentially reduces to the classical theory. As applications, we establish derived versions of the étale slice theorem for stacks and the partial desingularization procedure of good moduli spaces.

Contents

1. Introduction

Some brief history. Moduli theory focuses on the understanding of geometric objects and how they can vary in families. The fundamental gadget which the modern foundations of the subject rest on is the notion of an algebraic stack, first introduced in [DM69] and later generalized in [Art74]. A parallel, important consideration, which is often useful in the study of the properties and behaviour of algebraic stacks, is the construction of a moduli space which keeps track of sufficient information about the stack, e.g., certain equivalence classes of the objects it parametrizes.

Mumford developed Geometric Invariant Theory (GIT) [Mum65] in order to be able to define moduli spaces for quotient stacks for reductive group actions on projective schemes equipped with appropriate extrinsic data (a linearization of the action). GIT has been wildly successful and used by Mumford to define moduli spaces of curves and abelian varieties, as well as in a plethora of other contexts, such as moduli spaces of sheaves (cf. [HL10]).

Still, many important examples of interest that arise in practice do not fall into the GIT regime. Keel–Mori [KM97] first partially lifted this restriction by showing that any stack with finite inertia admits a coarse moduli space.

Date: August 3, 2023.

Subsequently, Alper [Alp13] introduced and developed [Alp10, Alp12] the theory of good moduli spaces, which should be viewed as a complete generalization of GIT and allows for the construction of good moduli spaces in a wide variety of new contexts. Paired with recent results on existence criteria [AHLH19], Alper's theory has been extremely influential and allowed for important breakthroughs in moduli theory, including moduli of objects in abelian categories [AHLH19] and K-semistable Fano varieties in the minimal model program [ABHLX20].

The purpose of this paper is to initiate the study of good moduli spaces in derived algebraic geometry, generalizing the above theory of good moduli spaces for classical stacks. Derived algebraic stacks [TV04] can be thought of as (co)homological thickenings of classical stacks and materialize the hidden smoothness principle of Kontsevich [Kon95], with singular classical stacks being truncations of better behaved derived stacks. Even for classical applications, recent breakthroughs in derived algebraic geometry [Toë14] and shifted symplectic structures [PTVV13] have made derived stacks increasingly important. As such, the existence of derived good moduli spaces for derived stacks commonly used in practice is a desirable feature.

Statement of results. We now proceed to briefly state our main results. For purposes of simplicity in the introduction, we assume that all derived Artin stacks are finitely presented over $\mathbb{C}$ in what follows. We refer the reader to the main text for the precise assumptions made in each statement.

Here is our main definition.

Definition (Definition 3.1)**.** A morphism $X \to Y$ from a derived Artin stack to a derived algebraic space Y is a good moduli space if it satisfies the following conditions:

(i) q is universally of cohomological dimension 0.
(ii) The natural morphism $\mathcal{O}_Y \to q_*\mathcal{O}_X$ is an equivalence.

We comment that when X is classical, we show that Y must be classical as well (see Proposition 3.6). In fact, one of our main results says that the existence of a good moduli space can be detected at the level of the classical underlying stack.

Theorem (Theorem 3.8)**.** *(i) If X_{cl} admits a good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$, then X admits a good moduli space $q'\colon X \to Y$ such that $q'_{\mathrm{cl}} \simeq q_{\mathrm{cl}}$.*

(ii) If X admits a good moduli space $q\colon X \to Y$, then q_{cl} is a good moduli space for X_{cl}.

Interestingly, while condition (ii) of our definition is identical to the $\mathcal{O}$-connectedness condition imposed by Alper, condition (i) is in general stronger than requiring that q is cohomologically affine, as Alper does. The reason we need to impose a stronger condition is to ensure that the pushforward functor q_* is sufficiently well-behaved on derived stacks. In practice, this distinction is fairly benign, as it disappears when X has quasi-affine diagonal (see Proposition 3.4), so our moduli spaces are the same as the ones defined by Alper in that case.

Another main result of the paper is the universality of good moduli spaces.

Theorem (Theorem 3.11)**.** *Good moduli spaces are universal for maps to derived algebraic spaces.*

Following Alper's steps [Alp13] and largely bootstrapping from the classical case using the above results, we establish several properties of good moduli spaces, which mirror the classical theory and are listed in Proposition 3.7 and Lemma 3.14. Additionally, we prove results on good moduli spaces of closed substacks, gluing good moduli spaces and the descent of étale morphisms from stacks to their good moduli spaces.

We provide several applications of our theory. The first is a fully derived version of the étale slice theorem of [AHR20]. A simplified statement is below.

Theorem (Theorem 4.1)**.** *Suppose that X is an Artin stack with affine diagonal and good moduli space $q\colon X \to Y$. Let $x \in X_{\mathrm{cl}}$ be a closed point with stabilizer G_x.*

Then there exists an affine, étale morphism $\Phi\colon [U/G_x] \to X$, where U is an affine scheme with a G_x-action, $u \in U$ is fixed by G_x and maps to x via Φ, fitting in a cartesian square

$$\begin{tikzcd} {[U/G_x]} \arrow[r, "\Phi"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ U/\!\!/G_x \arrow[r] & Y, \end{tikzcd}$$

where $U/\!\!/G_x$ is the good moduli space of $[U/G_x]$.

A second application concerns the derived stabilizer reduction procedure for derived Artin stacks. Given an Artin stack X with affine diagonal such that X_{cl} admits a good moduli space, in [HRS22], the authors defined a canonical, Deligne–Mumford stack $\widetilde{X} \to X$, which resolves the positive dimensional stabilizers of X, via an iterated blow-up construction. While it was shown that $\widetilde{X}_{\mathrm{cl}}$ also admits a good moduli space, we now prove that the same holds for $\widetilde{X}$ itself, as expected. More precisely, the good moduli space of $\widetilde{X}$ is also obtained by an iterated blow-up of Y and should be viewed as a derived version of Kirwan's partial desingularization of GIT quotients [Kir85] and Edidin–Rydh's stabilizer reduction procedure [ER21].

Theorem (Theorem 4.8, Corollary 4.9)**.** *Suppose that X is a derived Artin stack with affine diagonal and a good moduli space $q\colon X \to Y$. Then the same holds for its stabilizer reduction $\widetilde{X}$.*

Finally, we also construct natural derived enhancements of fundamental examples of classical good moduli spaces, such as moduli spaces of stable maps and sheaves, and give a concrete description and examples of the good moduli space of affine quotient stacks $[\operatorname{Spec} A/G]$ for a linearly reductive group G.

Future directions. A natural direction for future inquiry would be to obtain positive and mixed characteristic as well as relative versions of the theory of derived moduli spaces.

More concretely, it would be desirable to have a theory of derived adequate moduli spaces [Alp14], which will cover the case of Artin stack over characteristic $p > 0$. Giving the correct condition, replacing being universally of cohomological dimension 0, looks like a subtler task, which we hope will prove to be within reach in the near future.

Furthermore, another extension of the present paper would be to prove an étale slice theorem [AHR19] and a partial desingularization result for stacks over *any* base scheme S and in particular over mixed characteristic.

Finally, it will be interesting to exhibit explicit, interesting, non-trivial examples of derived good moduli spaces which enjoy nice properties, such as being quasi-smooth.

Layout of the paper. In §2, we gather the necessary results and properties regarding morphisms which are universally of cohomogical dimension 0 and related notions. In §3, we define good moduli spaces and prove our main results and several properties. §4 is concerned with applications and examples, including an étale slice theorem, stabilizer reduction, an existence criterion and quotient stacks.

Acknowledgements. MS would like to thank Jarod Alper, Daniel Halpern-Leistner and David Rydh for discussions related to good moduli spaces and the étale slice theorem.

Notation and conventions. Throughout, everything is derived and ∞-categorical. Hence a stack is a space-valued sheaf $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{S}$.

An *Artin stack* is an 1-algebraic stack X. If moreover X_{cl} is a classical algebraic space, then X is an *algebraic space.*

For any stack X, the *category of quasi-coherent modules* $\mathsf{QCoh}(X)$ is the limit of the categories Mod_A, indexed over all $\operatorname{Spec} A \to X$. The category $\mathsf{QCoh}(X)$ is endowed with a canonical t-structure, where $M \in \mathsf{QCoh}(X)$ is connective if and only if $M_B := x^*M$ is connective for all $x\colon \operatorname{Spec} B \to X$.

A morphism $f\colon X \to Y$ of Artin stacks is called qcqs if f_{cl} is quasi-compact and quasi-separated. We assume throughout that all Artin stacks and morphisms of Artin stacks are qcqs.

Recall that an exact functor between stable categories is *right* t-exact if it preserves connective objects, and that the dual notion is *left* t-exact.

In a stable category, (co)fiber sequences are also called *exact* sequences.

A morphism of algebras, modules, ..., is *surjective* if the fiber is connective.

For an Artin stack X, we write $|X| := |X_{\mathrm{cl}}|$ for the topological space of points of X.

2. Connectivity and pushforward

The purpose of this section is to gather and develop the requisite background and results for a well-behaved pushforward functor in the context of derived algebraic geometry. This will be necessary towards a robust definition of good moduli spaces later on, which will rely on the properties of pushing forward. The appropriate notion of morphisms that we will consider is being universally of cohomological dimension zero, following [HLP19]. We will see that this guarantees the desired properties and bears close relation to affineness and t-exactness.

Let $f\colon X \to Y$ be a morphism of Artin stacks. Recall that we have an adjunction

$$(f^* \dashv f_*)\colon \mathsf{QCoh}(Y) \rightleftarrows \mathsf{QCoh}(X)$$

such that f^* is right t-exact and f_* is left t-exact.

For $y\colon \operatorname{Spec} A \to Y$, we write $f_A\colon X_A \to \operatorname{Spec} A$ for the pullback of f along y.

2.1. **Different notions of affineness.** We briefly discuss different notions of affineness and their relations.

Definition 2.1. Let $f\colon X \to Y$ be a morphism of Artin stacks.

If X and Y are classical, we say that f is *cohomologically affine* if the (underived) pushforward functor $\pi_0(f_*)\colon \mathsf{QCoh}(X)^\heartsuit \to \mathsf{QCoh}(Y)^\heartsuit$ is exact. If $X \to *$ is cohomologically affine, we call X *cohomologically affine.*

Suppose now that X and Y can be derived. We say that f is *affine* if for all morphisms $y\colon \operatorname{Spec} A \to Y$, the pullback X_A is an affine scheme.

Finally, f is *quasi-affine* if it factorizes as a quasi-compact open immersion $X \to X'$ followed by an affine morphism $X' \to Y$. If $X \to *$ is quasi-affine, we call X *quasi-affine.*

It is clear that affineness is stronger than cohomological affineness and quasi-affineness.

Lemma 2.2. *A morphism $f\colon X \to Y$ of Artin stacks is quasi-affine if and only if X_A is quasi-affine for all morphisms* $\operatorname{Spec} A \to Y$.

Proof. One direction is clear since open immersions and affine morphisms are stable under base-change. For the other direction, suppose that X_A is quasi-affine for all $\operatorname{Spec} A \to Y$. Consider the factorization

$$X \xrightarrow{h} \operatorname{Spec}_Y((f_*\mathcal{O}_X)_{\geq 0}) \xrightarrow{f'} Y$$

of f. It suffices to show that h is a quasi-compact open immersion. Since this is affine-local on the base and f' is affine, we reduce to $Y = *$. By assumption, we then have a quasi-compact open immersion $h'\colon X \to \operatorname{Spec} B$, which implies that X is a scheme, and moreover that f_* is t-exact. The map h' factorizes as

$$X \xrightarrow{h} \operatorname{Spec}(f_*\mathcal{O}_X) \to \operatorname{Spec} B$$

Now the proof of [Stcks, Tag 01P9] goes through verbatim to show that h is a quasi-compact open immersion. $\square$

Let $f\colon X \to Y$ still be a morphism of Artin stacks. Then f is *locally quasi-finite* if f_{cl} is locally quasi-finite. By [Stcks, Tag 0397], this definition coincides with the one given in [SAG, Def. 3.3.1.1] in case X, Y are algebraic spaces. Recall that f is *separated* if f_{cl} is, and that everything is assumed to be qcqs throughout.

Example 2.3. The diagonal Δ of an algebraic space is quasi-affine. To see this, by [SAG, Thm. 3.3.0.2], it suffices to show that Δ is separated and quasi-finite. Both reduce to the classical case, which can be found in [Stcks, Tag 02X4].

2.2. **Morphisms universally of cohomological dimension zero.** We now introduce the central definition of this section.

Definition 2.4. A morphism $f\colon X \to Y$ of Artin stacks is *of cohomological dimension zero* (cd_0 for short) if for all $M \in \mathsf{QCoh}(X)^\heartsuit$ it holds that f_*M is connective.

It is *universally of cohomological dimension zero* (ucd_0 for short) if for all morphisms $\operatorname{Spec} A \to Y$ and all $M \in \mathsf{QCoh}(X_A)^\heartsuit$ it holds that $(f_A)_*M$ is connective.

Recall that a morphism $f\colon X \to Y$ of stacks satisfies the *base-change formula* if for every cartesian diagram of stacks

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g_2"] & Y' \arrow[d, "g_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.1}$$

the natural transformation $g_1^* f_* \to f'_* g_2^*$ is an equivalence. Related to this, we say that f satisfies the *projection formula* if, for all $M \in \mathsf{QCoh}(X)$ and $N \in \mathsf{QCoh}(Y)$, the natural map $(f_*M)\otimes N \to f_*(M \otimes f^*N)$ is an equivalence.

Lemma 2.5. *Being* ucd_0 *is stable under composition and base-change. If f is* ucd_0*, then f_* preserves filtered colimits and satisfies the base-change and projection formulas.*

Proof. This is [HLP19, Prop. A.1.5; Prop. A.1.6]. □

Example 2.6. Affine morphisms are ucd_0.

Remark 2.7. Let $f\colon X \to Y$ be a morphism of Artin stacks. Since f_* is always left t-exact, it follows that f_* is t-exact if and only if $f_*(\mathsf{QCoh}(X)_{\geq 0}) \subseteq \mathsf{QCoh}(Y)_{\geq 0}$.

The proof of [HLP19, Lem. A.1.6] only uses that the categories of quasi-coherent modules in question are presentable and t-complete, which is true for all Artin stacks [GR17, Cor. 1.5.7]. It follows that $f_*(\mathsf{QCoh}(X)^\heartsuit) \subseteq \mathsf{QCoh}(Y)_{\geq 0}$ if and only if f is t-exact.

We have the following cancellation property (cf. [Alp13, Proposition 3.14]), which will be of use later.

Proposition 2.8. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of Artin stacks. Suppose that $g \circ f$ is* ucd_0 *and g has affine diagonal. Then f is* ucd_0.

Proof. Consider the diagram with cartesian squares

$$\begin{tikzcd} & X \arrow[r, "{(\mathrm{id},f)}"] \arrow[dl] & X \times_Z Y \arrow[r] \arrow[dl] \arrow[dr] & Y \arrow[dr] & \\ Y \arrow[r, "\Delta"'] & Y \times_Z Y & & X \arrow[r] & Z \end{tikzcd}$$

Since Δ is affine, it is ucd_0. Thus by the base-change property of being ucd_0, it follows that f is the composition of two ucd_0 morphisms and hence ucd_0. □

The following proposition shows that being ucd_0 is equivalent to being universally t-exact.

Proposition 2.9. *Let $f\colon X \to Y$ be a morphism of Artin stacks. Then the following are equivalent:*

(i) *f is ucd_0.*
(ii) *For all $Y' \to Y$ with Y' Artin, it holds that f'_* is t-exact, where $f'\colon X' \to Y'$ is the pullback of f along $Y' \to Y$.*
(iii) *For all $\operatorname{Spec} A \to Y$ it holds that $(f_A)_*$ is t-exact.*

Proof. Clearly we have that (ii) implies (iii) and that (iii) implies (i). By Lemma 2.5 and the definition of the t-structure on $\mathsf{QCoh}(Y')$, Remark 2.7 shows that (i) implies (ii). $\square$

Example 2.10. Quasi-affine morphisms are generally not ucd_0, since push-forwards along open immersions are in general not t-exact. For example, let $Y = \mathbb{P}^2$, $X \subseteq Y$ be the complement of the point $(0 : 0 : 1)$, and $f\colon X \to Y$ the natural inclusion morphism. By definition, f is an open immersion. There is a natural exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(1)^{\oplus 2} \longrightarrow \mathcal{O}_Y(2),$$

which is not surjective on the right. However, by Hartogs' lemma, we have $f_*f^*\mathcal{O}_Y(n) \simeq \mathcal{O}_Y(n)$ for all n. Since the restriction of the sequence to X becomes exact on the right, it follows that f is not t-exact.

When the target of a morphism has quasi-affine diagonal, universality is no longer a condition, as shown in the next proposition.

Proposition 2.11. *Let $f\colon X \to Y$ be a morphism of Artin stacks and assume that Y has quasi-affine diagonal. Then the following are equivalent:*

(i) *f is ucd_0.*
(ii) *f is cd_0.*
(iii) *f_* is t-exact.*

Proof. By definition, (i) implies (ii) and Remark 2.7 shows that (ii) is equivalent to (iii). It suffices to show that (ii) implies (i). To that end, let $g\colon \operatorname{Spec} A \to Y$ be a morphism. Since Y has quasi-affine diagonal, the morphism is quasi-affine. Now, as in the proof of [HLP19, Proposition A.1.9.], being cd_0 is preserved under affine base-change. We thus need to show that it is also preserved by base-change by open immersions, hence assume that g is an open immersion. Consider the cartesian diagram

$$\begin{tikzcd} X_A \arrow[r, "f_A"] \arrow[d, "g_A"'] & \operatorname{Spec} A \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Let $M \in \mathsf{QCoh}(X_A)^{\heartsuit}$. Since g_A is monic, the counit $g_A^*(g_A)_*M \to M$ is an equivalence, hence, using the fact that g_A^* is t-exact, so is

$$g_A^*\pi_0(g_{A*}M) \simeq \pi_0(g_A^*g_{A*}M) \to \pi_0 M \simeq M.$$

We may thus assume that M is of the form $(g_A)^*N$ for some $N \in \mathsf{QCoh}(X)^\heartsuit$. By [HLP19, Lemma A.1.3.], since g is flat and $M \in \mathsf{QCoh}(X_A)^\heartsuit$, we get an equivalence

$$(f_A)_*M \simeq (f_A)_*(g_A)^*N \simeq g^*f_*N \in \mathsf{QCoh}(\operatorname{Spec} A)^\heartsuit$$

since f is cd_0 and g^* is t-exact. Thus f_A is cd_0, as we want. □

Remark 2.12. Let $f\colon X \to Y$ be a morphism of Artin stacks. Define the following functors:

$$f_*^\heartsuit\colon \mathsf{QCoh}(X)^\heartsuit \xrightarrow{f_*} \mathsf{QCoh}(Y)_{\leq 0} \xrightarrow{\tau_{\geq 0}} \mathsf{QCoh}(Y)^\heartsuit$$

$$f_\heartsuit^*\colon \mathsf{QCoh}(Y)^\heartsuit \xrightarrow{f^*} \mathsf{QCoh}(X)_{\geq 0} \xrightarrow{\tau_{\leq 0}} \mathsf{QCoh}(X)^\heartsuit$$

Since $\tau_{\leq 0}$ is a left adjoint to the inclusion $\mathsf{QCoh}(X)_{\leq 0} \to \mathsf{QCoh}(X)$ and $\tau_{\geq 0}$ is a right adjoint to the inclusion $\mathsf{QCoh}(Y)_{\geq 0} \to \mathsf{QCoh}(Y)$, we get

$$\begin{aligned}\mathsf{QCoh}(X)^\heartsuit(f_\heartsuit^*M, N) &\simeq \mathsf{QCoh}(X)_{\leq 0}(\tau_{\leq 0}f^*M, N)\\ &\simeq \mathsf{QCoh}(X)(f^*M, N) \simeq \mathsf{QCoh}(Y)(M, f_*N)\\ &\simeq \mathsf{QCoh}(Y)_{\geq 0}(M, \tau_{\geq 0}f_*N) \simeq \mathsf{QCoh}(Y)^\heartsuit(M, f_*^\heartsuit N)\end{aligned}$$

It follows that $f^{*\heartsuit} \dashv f_*^\heartsuit$.

The following proposition gives a connection with cohomological affineness in the classical case, as defined in [Alp13]. Before giving the statement, recall that an Artin stack X is affine-pointed if every morphism $\operatorname{Spec} k \to X$, where k is a field, is affine. If X has quasi-affine or quasi-finite diagonal, then it follows that X is affine-pointed by [HR19, Lemma 4.5].

Proposition 2.13. *Let $f\colon X \to Y$ be a morphism of Artin stacks.*
(i) *If f is cd_0 then $(f_{\mathrm{cl}})_*^\heartsuit$ is exact.*
(ii) *Suppose that X and Y either are both affine-pointed with X Noetherian or both have affine diagonal. If $f_*^\heartsuit$ is exact, then f is cd_0.*
(iii) *Suppose that X and Y have quasi-affine diagonal with X Noetherian. Then, if $f_*^\heartsuit$ is exact, f is ucd_0.*

Proof. If f is cd_0, then f_* restricts to a functor $\mathsf{QCoh}(X)^\heartsuit \to \mathsf{QCoh}(Y)_{\geq 0}$, by Proposition 2.9. Since f_* is always left t-exact, f_* then in fact restricts to $\mathsf{QCoh}(X)^\heartsuit \to \mathsf{QCoh}(Y)^\heartsuit$. The latter restriction is equivalent to $(f_{\mathrm{cl}})_*^\heartsuit$, which is thus exact. This shows (i).

Note that conditions on the diagonal and affine-pointedness only depend on the underlying classical stack. Since $\mathsf{QCoh}(X)^\heartsuit$ is naturally equivalent to $\mathsf{QCoh}(X_{\mathrm{cl}})^\heartsuit$ via the pushforward of the natural embedding $X_{\mathrm{cl}} \to X$ and the same is true for Y, we may thus assume that X and Y are classical in parts (ii) and (iii).

Part (ii) is then a direct consequence of [HNR19, Lemma C.2, Lemma C.3], which imply that for any $M \in \mathsf{QCoh}(X)^\heartsuit$ and an injective resolution $M \to I^\bullet \in \mathsf{QCoh}(X)_{\leq 0}$, there is a natural equivalence between f_*M and $f_*^\heartsuit I^\bullet$. Thus, exactness of $f_*^\heartsuit$ implies that $f_*^\heartsuit I^\bullet \in \mathsf{QCoh}(Y)^\heartsuit$ and thus $f_*M \in \mathsf{QCoh}(Y)^\heartsuit$, so that f is cd_0.

Finally, part (iii) follows from part (ii) and Proposition 2.11, since having quasi-affine diagonal implies being affine-pointed. □

Example 2.14 (Hall–Rydh)**.** If the conditions in the preceding proposition are not satisfied, a morphism $f\colon X \to Y$ of classical Artin stacks with $f_*^\heartsuit\colon \mathsf{QCoh}(X)^\heartsuit \to \mathsf{QCoh}(Y)^\heartsuit$ exact need not be ucd_0. For instance, take $Y = BG$, with G a classical, non-affine group scheme, e.g. an elliptic curve, and f the universal torsor $* \to BG$. If f were ucd_0, then by base change so would $p\colon G \to *$. In particular, $p_*^\heartsuit$ would be exact by Proposition 2.13, hence p would be affine by Serre's criterion [Alp13, Prop. 3.3].

3. Derived good moduli spaces

In this section, we give the main definition of this paper, that of a good moduli space for a (derived) Artin stack. We also establish the main properties of good moduli spaces. Our discussion and results closely follow Alper's original paper [Alp13] which introduced good moduli spaces for classical Artin stacks. Indeed, our main aim is to provide the appropriate generalization of these to the setting of derived algebraic geometry.

We show that, under certain natural assumptions, our definition is consistent with Alper's definition. Moreover, the vast majority of the classical results on good moduli spaces carry over to the derived world, as the property of having a good moduli space can be detected at the level of the classical underlying stack. We also establish fundamental properties, shared with the classical case, such as universality of good moduli spaces for maps to algebraic spaces, among others.

3.1. **Definitions: derived and classical.**

Definition 3.1. A morphism $q\colon X \to Y$ of Artin stacks is a *good moduli space morphism* if the following two conditions are satisfied:

(i) q is universally of cohomological dimension zero.
(ii) The natural morphism $\mathcal{O}_Y \to q_*\mathcal{O}_X$ is an equivalence.

If moreover Y is an algebraic space, then $q\colon X \to Y$ is called a *good moduli space* for X.

Remark 3.2. Since algebraic spaces have quasi-affine diagonal (see Example 2.3), we can also equivalently require, using Proposition 2.11, that q is cd_0 or that q_* is t-exact.

Before we explore properties of good moduli spaces, let us recall the classical definition given by Alper [Alp13] and compare it with the above definition for a classical Artin stack. This will often allow us to bootstrap from the classical case to the case of derived Artin stacks in what follows. We elect to use the term "$\mathcal{A}$-good" to eliminate possible confusion.

Definition 3.3. An *$\mathcal{A}$-good moduli space* for a classical Artin stack X is a morphism $q\colon X \to Y$ to a classical algebraic space Y such that $q_*^\heartsuit$ is exact and such that $\mathcal{O}_Y \simeq q_*^\heartsuit\mathcal{O}_X$ under the natural morphism.

Proposition 3.4. *Let X be a classical Artin stack and $q\colon X \to Y$ a morphism to a classical algebraic space Y.*

(i) *If q defines a good moduli space for X, then it defines an $\mathcal{A}$-good moduli space for X.*

(ii) *Suppose that X is Noetherian and has quasi-affine diagonal. Then if q defines an $\mathcal{A}$-good moduli space for X, it also defines a good moduli space for X.*

Proof. If q defines a good moduli space for X, then $q_*^\heartsuit$ is exact by Proposition 2.13(i) and $q_*^\heartsuit \mathcal{O}_X \simeq q_*\mathcal{O}_X \simeq \mathcal{O}_Y$ under the natural morphism. This shows part (i).

Conversely, if X is Noetherian with quasi-affine diagonal and $q_*^\heartsuit$ is exact, Proposition 2.13(iii) implies that q is ucd$_0$ and also t-exact by Proposition 2.11 (recall that Y has quasi-affine diagonal being an algebraic space). Thus, we also have $q_*^\heartsuit \simeq q_*$. It follows that q defines a good moduli space for X. □

Remark 3.5. We see that our definition of a good moduli space is in general stronger than Alper's definition for classical stacks. However, both definitions do coincide for most algebraic stacks encountered in practice, namely Noetherian stacks with quasi-affine diagonal. We thus consider this potential divergence a feature and not a bug.

The following proposition shows that the good moduli space of a classical Artin stack must be classical. In particular, under the conditions of part (ii) in the previous proposition, our definition of good moduli space is identical to the one given by Alper.

Proposition 3.6. *If X is a classical Artin stack with a good moduli space $q\colon X \to Y$, then Y is classical.*

Proof. Since q is t-exact and $\mathcal{O}_Y \simeq q_*\mathcal{O}_X$, it follows that $\pi_i(\mathcal{O}_Y) \simeq \pi_i(\mathcal{O}_X) = 0$ for all $i > 0$ and hence Y is classical. □

3.2. **Properties of good moduli spaces: base-change and passage between classical and derived.** We now lay the ground to show that most properties of good moduli spaces carry over from classical stacks to derived stacks. We first check that good moduli spaces behave well under base-change, which we then use to show that having a good moduli space is a property that only depends on the classical underlying stack, one of the essential features of our definition.

Proposition 3.7. *Good moduli space morphisms are stable under base-change.*

Proof. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "q'"] \arrow[d, "f"] & Y' \arrow[d, "g"] \\ X \arrow[r, "q"] & Y \end{tikzcd}$$

where q is a good moduli space and Y' an algebraic space. It is clear that q' is ucd$_0$. By base-change, Lemma 2.5, we have

$$q'_*\mathcal{O}_{X'} \simeq q'_* f^*\mathcal{O}_X \simeq g^* q_*\mathcal{O}_X \simeq g^*\mathcal{O}_Y \simeq \mathcal{O}_{Y'}.$$

□

We move on to one of the main results of the present paper.

Theorem 3.8. *Let X be an Artin stack.*

(i) *If X_{cl} admits a good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$, then X admits a good moduli space $q'\colon X \to Y$ such that $q'_{\mathrm{cl}} \simeq q_{\mathrm{cl}}$.*
(ii) *If X admits a good moduli space $q\colon X \to Y$, then q_{cl} is a good moduli space for X_{cl}.*

Proof. We first assume that X_{cl} admits a good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$. Note that by Proposition 3.6, Y_{cl} must be classical so the statement is well-defined. By using the Postnikov tower of X, which exhibits X as the colimit of a sequence of square-zero extensions

$$X_{\leq 0} = X_{\mathrm{cl}} \longrightarrow X_{\leq 1} \longrightarrow X_{\leq 2} \longrightarrow \dots \longrightarrow X,$$

it now suffices to prove that if X is a stack with good moduli space $q\colon X \to Y$, and X' is a square-zero extension of X by a quasi-coherent $\mathcal{O}_X$-module $M \in \mathsf{QCoh}(X)_{[d,d]}$, determined by a morphism

$$\alpha\colon \mathbb{L}_X \longrightarrow M \tag{3.1}$$

in $\mathsf{QCoh}(X)$, then X' admits a natural good moduli space $q'\colon X' \to Y'$, fitting in a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "q"'] & X' \arrow[d, "q'"] \\ Y \arrow[r, "j"'] & Y', \end{tikzcd} \tag{3.2}$$

where Y' is a square-zero extension of Y as well.

By definition and the universal property of $\mathbb{L}_X$, α is naturally equivalent to a morphism $\widetilde{\alpha}\colon X[M] \to X$ and X' is defined by the pushout diagram

$$\begin{tikzcd} X[M] \arrow[r, "\pi"] \arrow[d, "\widetilde{\alpha}"'] & X \arrow[d] \\ X \arrow[r] & X', \end{tikzcd} \tag{3.3}$$

where π denotes the projection map.

Since q is ucd$_0$, we have $q_*M \in \mathsf{QCoh}(Y)_{[d,d]}$ and we obtain a morphism

$$\beta\colon \mathbb{L}_Y \longrightarrow q_*\mathbb{L}_X \xrightarrow{q_*\alpha} q_*M \tag{3.4}$$

in $\mathsf{QCoh}(Y)$, which corresponds to a square-zero extension $Y \to Y'$.

Similarly to the above, β in turn corresponds to a morphism $\widetilde{\beta}\colon Y[q_*M] \to Y$ and Y' is defined by the pushout diagram

$$\begin{tikzcd} Y[q_*M] \arrow[r, "\pi"] \arrow[d, "\widetilde{\beta}"'] & Y \arrow[d] \\ Y \arrow[r] & Y', \end{tikzcd} \tag{3.5}$$

in which π again, by abuse of notation, denotes the projection map.

The counit $\epsilon\colon q^*q_*M \to M$ induces a morphism

$$\widetilde{q}\colon X[M] \to X[q^*q_*M] \simeq X \times_{q,Y,\pi} Y[q_*M] \to Y[q_*M]$$

compatible with the projection maps π.

Now $q \circ \widetilde{\alpha}$ is determined by the morphism

$$q^*\mathbb{L}_Y \to \mathbb{L}_X \xrightarrow{\alpha} M;$$

meanwhile $\widetilde{\beta} \circ \widetilde{q}$ is determined by the morphism

$$q^*\mathbb{L}_Y \to q^*q_*\mathbb{L}_X \xrightarrow{q^*q_*\alpha} q^*q_*M \xrightarrow{\epsilon} M.$$

By naturality of the counit, it follows that $q \circ \widetilde{\alpha} \simeq \widetilde{\beta} \circ \widetilde{q}$, hence that q induces a morphism from the pushout square (3.3) to the pushout square (3.5), and thus a morphism $q'\colon X' \to Y'$ fitting in a commutative diagram (3.2). We claim that q' is the desired good moduli space for X'.

Firstly, since i is affine, $i_*^\heartsuit$ is simply the restriction $i_*\colon \mathsf{QCoh}(X)^\heartsuit \to \mathsf{QCoh}(X')^\heartsuit$, which fits into the commutative square

$$\begin{tikzcd}
\mathsf{QCoh}(X_{\mathrm{cl}})^\heartsuit \arrow[r, "(i_{\mathrm{cl}})_*"] \arrow[d, "k_*"'] & \mathsf{QCoh}(X'_{\mathrm{cl}})^\heartsuit \arrow[d, "k'_*"] \\
\mathsf{QCoh}(X)^\heartsuit \arrow[r, "i_*"] & \mathsf{QCoh}(X')^\heartsuit
\end{tikzcd}$$

where $k\colon X_{\mathrm{cl}} \to X$, $k'\colon X'_{\mathrm{cl}} \to X'$ denote the natural closed embeddings. Since i_{cl} is an equivalence, and, by definition, the same holds for $k_*^\heartsuit$ and $k_*'^\heartsuit$, the same is true for $i_*^\heartsuit$. By the same argument, j_* is also an equivalence on the heart.

Since $j \circ q \simeq q' \circ i$ by diagram (3.2), we get $j_* \circ q_* \simeq q'_* \circ i_*$, which by Remark 2.7 implies that q' is t-exact, since q is t-exact. By Proposition 2.13 and Example 2.3, we conclude that q' is ucd$_0$.

Finally, we have a commutative diagram of distinguished triangles

$$\begin{tikzcd}
j_*q_*M[-1] \arrow[r] \arrow[d] & \mathcal{O}_{Y'} \arrow[r] \arrow[d] & j_*\mathcal{O}_Y \simeq j_*q_*\mathcal{O}_X \arrow[d] \\
q'_*i_*M[-1] \arrow[r] & q'_*\mathcal{O}_{X'} \arrow[r] & q'_*i_*\mathcal{O}_X.
\end{tikzcd}$$

Again, the equivalence $j \circ q \simeq q' \circ i$ by diagram (3.2) implies that the outer two vertical arrows are equivalences and hence the same is true for the middle arrow. This concludes (i).

Conversely, suppose that $q\colon X \to Y$ is a good moduli space morphism with Y an algebraic space. The t-exactness of q_* implies that $(q_{\mathrm{cl}})_*$ is also t-exact. Thus, to show that q_{cl} is a good moduli space morphism, we only need to check that the natural morphism $\mathcal{O}_{Y_{\mathrm{cl}}} \to (q_{\mathrm{cl}})_*^\heartsuit \mathcal{O}_{X_{\mathrm{cl}}}$ is an equivalence. Observe that we have a natural equivalence $(q_{\mathrm{cl}})_*^\heartsuit \mathcal{O}_{X_{\mathrm{cl}}} \simeq (q_{\mathrm{cl}})_*\mathcal{O}_{X_{\mathrm{cl}}}$ since q_{cl} is t-exact.

Let $k\colon X_{\mathrm{cl}} \to X$ and $\ell\colon Y_{\mathrm{cl}} \to Y$ denote the natural closed embeddings so that $q \circ k \simeq \ell \circ q_{\mathrm{cl}}$. Using the t-exactness of q_*, we have

$$\ell_*\mathcal{O}_{Y_{\mathrm{cl}}} \simeq \pi_0(\mathcal{O}_Y) \simeq \pi_0(q_*\mathcal{O}_X) \simeq q_*\pi_0(\mathcal{O}_X) \simeq q_*k_*\mathcal{O}_{X_{\mathrm{cl}}} \simeq \ell_*(q_{\mathrm{cl}})_*\mathcal{O}_{X_{\mathrm{cl}}}.$$

Since ℓ_* is an equivalence, this concludes the proof. $\square$

Remark 3.9. The preceding theorem is consistent with the usual analogy between the closed embedding $X_{\mathrm{cl}} \to X$ for a derived stack X and the closed embedding $X_{\mathrm{red}} \to X$ of the canonical reduced substack of a classical algebraic stack X. Namely, [Alp12, Corollary 5.7] asserts that a classical X admits an $\mathcal{A}$-good moduli space if and only if X_{red} does. Hence, it should not come as a surprise that the same holds for a derived stack X and its classical truncation X_{cl}.

3.3. **Properties of good moduli spaces: universality.** Before showing universality of good moduli spaces, we need the following lemma.

Lemma 3.10. *Let $q\colon X \to Y$ be a morphism of Artin stacks, and $M \in \mathsf{QCoh}(X)_{\geq 0}$. Then the commutative square*

$$\begin{tikzcd} X[M] \arrow[r] \arrow[d, "\pi"] & Y[q_*M] \arrow[d, "\pi"] \\ X \arrow[r] & Y \end{tikzcd}$$

where π denotes the projections, is a pushout.

Proof. Let $f\colon X \to T$ be given. A morphism $X[M] \to T$ factors through $\pi\colon X[M] \to X$ if and only if the corresponding map $f^*\mathbb{L}_T \to M$ is zero. Now the claim follows directly from the universal property of pushouts. ☐

Theorem 3.11. *Good moduli spaces are universal for maps to algebraic spaces.*

Proof. Let Z be an algebraic space and $q\colon X \to Y$ a good moduli space. We need to show that the natural morphism of mapping spaces

$$\alpha\colon \operatorname{Map}(Y, Z) \longrightarrow \operatorname{Map}(X, Z) \tag{3.6}$$

is an equivalence.

We will prove this by considering the Postnikov towers of X and Y. Thus, write $X = \varinjlim X_{\leq n}$ and $Y = \varinjlim Y_{\leq n}$. By construction and the proof of Theorem 3.8, we have good moduli spaces $q_{\leq n}\colon X_{\leq n} \to Y_{\leq n}$ which are compatible with the natural square-zero extensions $X_{\leq n} \to X_{\leq n+1}$ and $Y_{\leq n} \to Y_{\leq n+1}$. The strategy of proof will be to show that α has contractible fibers by induction on n, using [Toë14, §4.1]

For $n = 0$, we have $X_{\leq 0} = X_{\mathrm{cl}}$, $Y_{\leq 0} = Y_{\mathrm{cl}}$, and natural equivalences $\operatorname{Map}(X_{\mathrm{cl}}, Z) \simeq \operatorname{Map}(X_{\mathrm{cl}}, Z_{\mathrm{cl}})$ and $\operatorname{Map}(Y_{\mathrm{cl}}, Z) \simeq \operatorname{Map}(Y_{\mathrm{cl}}, Z_{\mathrm{cl}})$, so the equivalence $\operatorname{Map}(X_{\leq 0}, Z) \simeq \operatorname{Map}(Y_{\leq 0}, Z)$ follows from the fact that $\mathcal{A}$-good moduli spaces are universal for maps to algebraic spaces [AHR19, Theorem 3.12] and Proposition 3.4(i).

Let now $f\colon X \to Z$ be given. Write $f_{\leq m}$ for the restriction of f to $X_{\leq m}$, for any $m \geq 0$. By induction, we have a morphism $g_{\leq n}\colon Y_{\leq n} \to Z$ such that $g_{\leq n} q_{\leq n} \simeq f_{\leq n}$, and by assumption $f_{\leq n+1}$ extends $f_{\leq n}$. Now the obstruction for a lift of $g_{\leq n}$ to $Y_{\leq n+1}$ is an element

$$\begin{aligned} o(g, n) &\in \operatorname{Map}(g_{\leq n}^*\mathbb{L}_Z, (q_{\leq n})_* M[n+1]) \\ &\simeq \operatorname{Map}(q_{\leq n}^* g_{\leq n}^* \mathbb{L}_Z, M[n+1]) \simeq \operatorname{Map}(f_{\leq n}^*\mathbb{L}_Z, M[n+1]) \end{aligned} \tag{3.7}$$

where $M \in \mathsf{QCoh}(X_{\leq n+1})^{\heartsuit}$ is such that $X_{\leq n+1}$ is a square-zero extension of $X_{\leq n}$ by the module $M[n]$.

We claim that $o(g,n) \simeq 0$. To see this, observe that $o(g,n)$ corresponds to the morphism g'' obtained via the following diagram

$$\begin{tikzcd}
Y_{\leq n}[q_{\leq n*}M[n]] \arrow[r, "\alpha"] \arrow[d] & Y_{\leq n} \arrow[d] & \\
Y_{\leq n}[q_{\leq n*}(M[n] \oplus M[n+1])] \arrow[r] \arrow[d, "\pi"] & Y_{\leq n}[q_{\leq n*}M[n+1]] \arrow[d] \arrow[dl, dotted, "\pi"'] \arrow[dr, "g''"] & \\
Y_{\leq n} \arrow[r] \arrow[rr, bend right, "g_{\leq n}"'] & Y_{\leq n+1} \arrow[r, dotted, "g_{\leq n+1}"] & Z
\end{tikzcd}$$

where the squares are pushouts, and the dotted diagonal arrow is again the projection. To see that this diagram indeed induces g'', extend the top square with two more obvious pushout squares on the left. By construction, $o(g,n) = 0$ if and only if $g_{\leq n}\pi \simeq g''$. Now use that the square

$$\begin{tikzcd}
X_{\leq n}[M[n] \oplus M[n+1]] \arrow[r] \arrow[d] & X_{\leq n}[M[n+1]] \arrow[d] \\
Y_{\leq n}[q_{\leq n*}(M[n] \oplus M[n+1])] \arrow[r] & Y_{\leq n}[q_{\leq n*}M[n+1]]
\end{tikzcd}$$

is a pushout by Lemma 3.10 and that $f_{\leq n}\pi \simeq f''$ since $o(f,n) = 0$, where $f''\colon X_{\leq n}[M[n+1]] \to Z$ is similarly defined as is g'', to conclude that indeed $g_{\leq n}\pi \simeq g''$. We thus find an extension $g_{\leq n+1}\colon Y_{\leq n+1} \to Z$ of $g_{\leq n}$.

These data induce a diagram

$$\begin{tikzcd}
\mathrm{Map}(g_{\leq n}^*\mathbb{L}_Z, (q_{\leq n})_*M[n+1]) \arrow[r] \arrow[d] & \mathrm{Map}(Y_{\leq n+1}, Z) \arrow[r] \arrow[d] & \mathrm{Map}(Y_{\leq n}, Z) \arrow[d] \\
\mathrm{Map}(f_{\leq n}^*\mathbb{L}_Z, M[n+1]) \arrow[r] & \mathrm{Map}(X_{\leq n+1}, Z) \arrow[r] & \mathrm{Map}(X_{\leq n}, Z)
\end{tikzcd}$$

with (homotopy) exact rows. By induction, the vertical arrow on the right is an equivalence, and by construction so is the vertical arrow on the left. It follows that the middle arrow is an equivalence, since it is surjective on connected components, which concludes the argument. $\square$

The following two statements are now immediate.

Corollary 3.12. *For an Artin stack X, the space of good moduli spaces $X \to Y$ is contractible if X_{cl} admits a good moduli space, and else it is empty. In particular, any good moduli space of X is induced from a good moduli space of the classical truncation X_{cl} via the procedure in the proof of Theorem 3.8.*

Corollary 3.13. *For a good moduli space $q\colon X \to Y$, the derivedness of Y is bounded by the derivedness of X. That is, if $X_{\leq n} \to X$ is an equivalence, then so is $Y_{\leq n} \to Y$.*

3.4. **Properties of good moduli spaces continued.** We now list several properties that mirror the classical case (cf. [Alp13, Theorem 4.16]).

Lemma 3.14. *Let $q\colon X \to Y$ be a good moduli space and $f\colon X \to S$ a morphism from X to an algebraic space S.*

(i) *q is surjective and universally closed.*
(ii) *Being a good moduli space if fpqc-local on the target.*
(iii) *Let A be a sheaf of quasi-coherent $\mathcal{O}_X$-algebras. Then the morphism $\operatorname{Spec}_X A \to \operatorname{Spec}_Y q_*A$ is a good moduli space.*
(iv) *For $x_1, x_2 \in |X|$, the relation $\sim$ for which $x_1 \sim x_2$ if $\overline{\{x_1\}} \cap \overline{\{x_2\}} = \emptyset$ is an equivalence relation such that $|Y|$ is the quotient of $|X|$ by $\sim$.*
(v) *If X is flat over S, then the same is true for the induced morphism $Y \to S$.*

Proof. (i) and (iv) are a consequence of Theorem 3.8(ii) and the fact that a good moduli space for X_{cl} is an $\mathcal{A}$-good moduli space for X_{cl} by Proposition 3.4. Thus [Alp13, Theorem 4.16] applies.

For (ii), by Proposition 2.11, Lemma 3.7 and Theorem 3.8, we reduce to the case where we have a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q'"] & X \arrow[d, "q"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

of classical stacks—where q' is a good moduli space and f is faithfully flat—and we need to show that q is cd_0. Let $M \in \mathsf{QCoh}(X)^{\heartsuit}$ be given. By [HLP19, Lem. A.1.3], it holds that $f^*q_*M \simeq q'_*g^*M$, since f is flat and M is bounded above. By assumption and the flatness of g, it follows that $f^*q_*M \in \mathsf{QCoh}(Y')_{\geq 0}$. By essentially the same argument as given in [GR17, Prop. 1.5.4] applied to the fact that f is faithfully flat, it follows that $q_*M \in \mathsf{QCoh}(Y)_{\geq 0}$. This shows that q is ucd_0. And again by [HLP19, Lem. A.1.3], we have

$$f^*q_*\mathcal{O}_X \simeq q'_*g^*\mathcal{O}_X \simeq \mathcal{O}_{Y'}$$

since $q'_*\mathcal{O}_{X'} \simeq \mathcal{O}_{Y'}$. Since f is faithfully flat, f^* is conservative, hence $q_*\mathcal{O}_X \simeq \mathcal{O}_Y$.

For (iii), consider the commutative diagram

$$\begin{tikzcd} \operatorname{Spec}_X A \arrow[r, "i"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ \operatorname{Spec}_Y q_*A \arrow[r, "j"'] & Y. \end{tikzcd}$$

j is affine and thus has affine diagonal. Moreover $j \circ q' \simeq q \circ i$ is ucd_0, since i is ucd_0 as an affine morphism and q is ucd_0. It follows by Proposition 2.8 that q' is ucd_0.

Let F be the cofiber of the natural morphism $\mathcal{O}_{\operatorname{Spec}_Y q_*A} \to q'_*\mathcal{O}_{\operatorname{Spec}_X A}$. Since $j_*q'_*\mathcal{O}_{\operatorname{Spec}_X A} \simeq q_*i_*\mathcal{O}_{\operatorname{Spec}_X A} \simeq q_*A \simeq j_*\mathcal{O}_{\operatorname{Spec}_Y q_*A}$, we deduce that $j_*F \simeq 0$. But for any affine morphism $f\colon S \to T$, it holds that $\mathsf{QCoh}(S)$ is the category of $f_*\mathcal{O}_S$-modules in $\mathsf{QCoh}(T)$, and f_* is then the forgetful functor, which in particular is conservative. It follows that $F \simeq 0$ since $j_*F \simeq 0$, as desired.

For (v), by universality of good moduli spaces, we have a natural morphism $g\colon Y \to S$. As above, Y_{cl} is a good and hence an $\mathcal{A}$-good moduli space for X_{cl}, which must be flat over S by [Alp13, Theorem 4.16]. It remains to check that the natural morphism $g_{\mathrm{cl}}^*\pi_n(\mathcal{O}_S) \simeq \pi_n(\mathcal{O}_S) \otimes_{\pi_0(\mathcal{O}_S)} \pi_0(\mathcal{O}_Y) \to \pi_n(\mathcal{O}_Y)$ is an equivalence for all $n > 0$. By the flatness of X over S, this is the case for the morphism $\pi_n(\mathcal{O}_S) \otimes_{\pi_0(\mathcal{O}_S)} \pi_0(\mathcal{O}_X) \to \pi_n(\mathcal{O}_X)$. We then have by definition $\pi_n(\mathcal{O}_X) \simeq \pi_n(\mathcal{O}_S) \otimes_{\pi_0(\mathcal{O}_S)} \pi_0(\mathcal{O}_X) \simeq q_{\mathrm{cl}}^* g_{\mathrm{cl}}^* \pi_n(\mathcal{O}_S)$. Applying the pushforward and noting that $\pi_n(\mathcal{O}_Y) \simeq (q_{\mathrm{cl}})_*\pi_n(\mathcal{O}_X)$ for all $n \geq 0$ as $\pi_0(\mathcal{O}_Y)$-modules by the proof of Theorem 3.8, together with $(q_{\mathrm{cl}})_* q_{\mathrm{cl}}^* \simeq \mathrm{id}$ by the projection formula, we obtain

$$\pi_n(\mathcal{O}_Y) \simeq (q_{\mathrm{cl}})_*\pi_n(\mathcal{O}_X) \simeq (q_{\mathrm{cl}})_* q_{\mathrm{cl}}^* g_{\mathrm{cl}}^* \pi_n(\mathcal{O}_S) \simeq g_{\mathrm{cl}}^*\pi_n(\mathcal{O}_S),$$

which concludes the proof. □

Remark 3.15. Finite generation and other similar properties for good moduli spaces, as stated in [Alp13, Theorem 4.16], hold in the derived setting as well in the obvious way. Since the translation to our context is immediate for such properties, we omit them in the interest of space.

3.5. **Closed substacks.** Throughout, let $q\colon X \to Y$ be a good moduli space.

Definition 3.16. The *scheme-theoretic image* of a closed immersion $Z \to X$ is defined as $q(Z) := \mathrm{Spec}_Y\, q_*\mathcal{O}_Z$. For a closed immersion $Z' \to Y$, its *scheme-theoretic pre-image* is defined as $q^{-1}(Z') = Z' \times_Y X$. We say that a closed substack $Z \to X$ is *saturated* if the natural morphism $Z \to q^{-1}(q(Z))$ is an equivalence. $q^{-1}(q(Z)) = \mathrm{Spec}_X(q^*q_*\mathcal{O}_Z)$ is called the *saturation* of the closed substack Z and denoted by Z_{sat}.

Remark 3.17. Observe that $q(Z) \to Y$ is a closed immersion, and that $q(Z)_{\mathrm{cl}}$ is the classical scheme-theoretic image of $Z_{\mathrm{cl}} \to X_{\mathrm{cl}}$ under q_{cl}. By definition, it holds that $q(X) \simeq Y$.

Corollary 3.18. *Let $Z \to X$ be a closed immersion. The morphism $Z \to q(Z)$ defines a good moduli space for Z.*

Proof. This follows from Lemma 3.14(ii) applied to $A = \mathcal{O}_Z$. □

Lemma 3.19. *Let $Y' \to Y$ be a closed immersion, and let $q'\colon X' \to Y'$ be the pullback of q. Then the natural map $Y' \to q(X')$ is an equivalence.*

Proof. This is immediate from Lemma 3.7. □

The following proposition is an application of the projection formula for ucd_0 morphisms.

Proposition 3.20. *Let $Z_1 \to X$ and $Z_2 \to X$ be closed immersions. Let F_i denote the fiber of the morphism $\mathcal{O}_X \to \mathcal{O}_{Z_i}$ and C_i the cofiber of the morphism $q^*q_*F_i \to F_i$. Then the natural morphism*

$$q(Z_1 \times_X Z_2) \to q(Z_1) \times_Y q(Z_2) \tag{3.8}$$

is an equivalence if and only if $q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} C_2) = 0$. In particular, this is the case if Z_2 is saturated.

Proof. Consider the exact triangle $F_2 \to \mathcal{O}_X \to \mathcal{O}_{Z_2}$. Applying q_* and tensoring with $q_*\mathcal{O}_{Z_1}$ gives an exact triangle

$$q_*\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_Y} q_*F_2 \longrightarrow q_*\mathcal{O}_{Z_1} \longrightarrow q_*\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_Y} q_*\mathcal{O}_{Z_2}.$$

On the other hand, tensoring with $\mathcal{O}_{Z_1}$ and then applying q_* gives an exact triangle

$$q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} F_2) \longrightarrow q_*\mathcal{O}_{Z_1} \longrightarrow q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_2}).$$

It thus suffices to prove that the natural morphism

$$q_*\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_Y} q_*F_2 \longrightarrow q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} F_2)$$

is an equivalence.

Consider the exact triangle $q^*q_*F_2 \to F_2 \to C_2$. Applying the functor $q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} (-))$ and using the projection formula, the cofiber of this morphism is exactly $q_*(\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} C_2)$. This establishes the sufficiency of (ii).

If Z_2 is saturated, then by definition $\mathcal{O}_{Z_2} \simeq q^*q_*\mathcal{O}_{Z_2}$ and hence $C_2 \simeq 0$. Thus, the condition holds and we are done. □

Remark 3.21. The above proposition shows that there is a difference in behavior regarding scheme-theoretic images of closed substacks and their intersection between the classical and derived setting. Namely, the morphism (3.8) is always an isomorphism when everything is classical (including the closed immersions).

For a concrete example that shows that the condition of the proposition is not automatic, consider $X = B\mathbb{C}^*$ and $Z_1 = [\operatorname{Spec}\mathbb{C}[\epsilon_1]/\mathbb{C}^*]$, $Z_2 = [\operatorname{Spec}\mathbb{C}[\epsilon_2]/\mathbb{C}^*]$ be two trivial square-zero thickenings of X where the $\mathbb{C}^*$-weight of ϵ_i is equal to $(-1)^i$. Since X is classical Noetherian with affine diagonal, its good moduli space as a derived stack is the same as its $\mathcal{A}$-good moduli space and hence is given by the canonical morphism $q\colon X = B\mathbb{C}^* \to * = Y$. By Theorem 3.8, it follows that the good moduli spaces of Z_1 and Z_2 are also equivalent to the point $*$. In particular, we have that $q(Z_1) \times_Y q(Z_2) \simeq *$.

On the other hand, $Z_1 \times_X Z_2 = [\operatorname{Spec}\mathbb{C}[\epsilon_1, \epsilon_2]/\mathbb{C}^*]$. Considering $\mathbb{C}[\epsilon_1, \epsilon_2]$ as an $\mathcal{O}_{Z_1\times_X Z_2}$-module with the given $\mathbb{C}^*$-weights, we will see later on (Subsection 4.5) that $q_*\mathbb{C}[\epsilon_1, \epsilon_2] = \mathbb{C}[\epsilon_1, \epsilon_2]^{\mathbb{C}^*} = \mathbb{C} \oplus \mathbb{C}\epsilon_1\epsilon_2$, which is not equal to $\mathbb{C}$. Thus the morphism (3.8) cannot be an equivalence.

3.6. **Gluing good moduli spaces.** Suppose that X is an Artin stack with a good moduli space $q\colon X \to Y$. Let $U \subseteq X$ be an open substack. Consider the image $q_{\mathrm{cl}}(U_{\mathrm{cl}})$ inside Y_{cl}. If it is open, we get a corresponding open substack $q(U)$ of Y. In analogy with the case of a closed substack, we say that U is *saturated* if the natural morphism $U \to q^{-1}(q(U)) := q(U) \times_Y X$ is an equivalence.

Proposition 3.22. *Let $q\colon X \to Y$ be a good moduli space. If $Y' \subseteq Y$ is an open substack, then $q^{-1}(Y') := Y' \times_Y X$ is a saturated open substack of X.*

Proof. This is clear from the definition. □

The following proposition describes the conditions under which good moduli spaces on an open cover can be glued (cf. [Alp13, Proposition 7.9]).

Proposition 3.23. *Suppose that X is an Artin stack with an open cover $\{U_i\}_{i\in I}$ by substacks and that for each index i there exists a good moduli space $q_i\colon U_i \to Y_i$. Then there exists a good moduli space $q\colon X \to Y$ and open substacks $Y_i' \subseteq Y$ such that $Y_i' \simeq Y_i$ as Y-stacks and we have cartesian squares*

$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "q_i"'] & X \arrow[d, "q"] \\ Y_i \simeq Y_i' \arrow[r] & Y \end{tikzcd} \tag{3.9}$$

if and only if for any two indices $i, j \in I$, the intersection $U_{ij} := U_i \cap U_j = U_i \times_X U_j$ is a saturated open substack of U_i and U_j.

Proof. The only if direction follows from the preceding proposition.

For the converse, by Theorem 3.8, we obtain good moduli spaces for the classical truncations $(q_i)_{\mathrm{cl}}\colon (U_i)_{\mathrm{cl}} \to (Y_i)_{\mathrm{cl}}$. The proof of [Alp13, Proposition 7.9] now applies verbatim to show that X_{cl} admits a good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ with open substacks $(Y_i')_{\mathrm{cl}} \subseteq Y_{\mathrm{cl}}$ fitting in cartesian squares

$$\begin{tikzcd} (U_i)_{\mathrm{cl}} \arrow[r] \arrow[d, "(q_i)_{\mathrm{cl}}"'] & X_{\mathrm{cl}} \arrow[d, "q_{\mathrm{cl}}"] \\ (Y_i)_{\mathrm{cl}} \simeq (Y_i')_{\mathrm{cl}} \arrow[r] & Y_{\mathrm{cl}}. \end{tikzcd}$$

By Theorem 3.8, it follows that X admits a good moduli space $q\colon X \to Y$, whose classical truncation is the morphism q_{cl}. This concludes the proof, since the existence of the squares (3.9) can then be checked on classical truncations. □

3.7. **Descent of étale morphisms.** We proceed to describe conditions that ensure that étale morphisms between stacks descend to étale morphisms between their good moduli spaces.

Recall that a morphism $f_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ between classical Artin stacks is pointwise stabilizer-preserving if it preserves stabilizer groups of points. Moreover, when X_{cl} and Y_{cl} are of finite type, we say that f_{cl} is weakly saturated if it maps closed points to closed points.

Let $q\colon X \to Y$ be the good moduli space of a stack X. For each $n \geq 0$, write $q_{\leq n}\colon X_{\leq n} \to Y_{\leq n}$ for the corresponding truncation so that $X_{\leq n+1}$ and $Y_{\leq n+1}$ are square-zero extensions of $X_{\leq n}$ and $Y_{\leq n}$ by modules M_{n+1} and $(q_{\leq n})_* M_{n+1}$ respectively. We say that X is *invariantly generated* if for every $n \geq 0$ the natural morphism $(q_{\leq n})^*(q_{\leq n})_* M_{n+1} \to M_{n+1}$ is an equivalence.

Proposition 3.24. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & X' \arrow[d, "q'"] \\ Y \arrow[r, "g"'] & Y', \end{tikzcd}$$

where X, X' are locally Noetherian Artin stacks of finite type (over S), q, q' are good moduli spaces, and g is locally of finite type.

Then, if f is étale, X' is invariantly generated, and f_{cl} is pointwise stabilizer-preserving and weakly saturated, it follows that g is étale.

Proof. By [Alp12, Proposition 5.1], g_{cl} is étale. We now prove by induction on $n \geq 0$ that the truncation $g_{\leq n}$ is étale, i.e. the morphism $\mathbb{L}_{Y'_{\leq n}}|_{Y_{\leq n}} \to \mathbb{L}_{Y_{\leq n}}$ is an equivalence. We have a commutative diagram of distinguished triangles

$$\begin{tikzcd}
\mathbb{L}_{X'_{\leq n+1}}|_{X_{\leq n}} \arrow[r] \arrow[d] & \mathbb{L}_{X'_{\leq n}}|_{X_{\leq n}} \arrow[r] \arrow[d] & M'_{n+1}|_{X_{\leq n}} \arrow[d] \\
\mathbb{L}_{X_{\leq n+1}}|_{X_{\leq n}} \arrow[r] & \mathbb{L}_{X_{\leq n}} \arrow[r] & M_{n+1}
\end{tikzcd}$$

where the vertical arrows are equivalences, since f is étale.

The proof of Theorem 3.8 shows that there exists a corresponding commutative diagram of distinguished triangles

$$\begin{tikzcd}
\mathbb{L}_{Y'_{\leq n+1}}|_{Y_{\leq n}} \arrow[r] \arrow[d] & \mathbb{L}_{Y'_{\leq n}}|_{Y_{\leq n}} \arrow[r] \arrow[d] & g^*_{\leq n}(q'_{\leq n})_* M'_{n+1} \arrow[d] \\
\mathbb{L}_{Y_{\leq n+1}}|_{Y_{\leq n}} \arrow[r] & \mathbb{L}_{Y_{\leq n}} \arrow[r] & (q_{\leq n})_* M_{n+1}
\end{tikzcd}$$

where the middle arrow is an equivalence. Hence it suffices to verify that the rightmost arrow

$$g^*_{\leq n}(q'_{\leq n})_* M'_{n+1} \to (q_{\leq n})_* M_{n+1} \simeq (q_{\leq n})_* f^*_{\leq n} M'_{n+1}$$

is also an equivalence, where the last equivalence follows from the étaleness of $f_{\leq n}$.

Since X' is invariantly generated, it follows that M'_{n+1} is of the form $(q'_{\leq n})^* N$ for some module N. But then, since by the projection formula we have that $(q'_{\leq n})_*(q'_{\leq n})^* \simeq \mathrm{id}$,

$$g^*_{\leq n}(q'_{\leq n})_* M'_{n+1} \simeq g^*_{\leq n}(q'_{\leq n})_*(q'_{\leq n})^* N \simeq g^*_{\leq n} N.$$

On the other hand, we similarly have

$$(q_{\leq n})_* f^*_{\leq n} M'_{n+1} \simeq (q_{\leq n})_* f^*_{\leq n}(q'_{\leq n})^* N \simeq (q_{\leq n})_*(q_{\leq n})^* g^*_{\leq n} N \simeq g^*_{\leq n} N.$$

Combining the two last equivalences finishes the proof. □

Remark 3.25. Note that if a stack is classical, then the condition of being invariantly generated is vacuous. Therefore, the previous proposition does recover [Alp12, Proposition 5.1].

Being invariantly generated is not the only additional or weakest constraint we could impose to ensure that an étale morphism descends. It is however not far from that and simple to state, which are the reasons for our choice in exposition.

Proposition 3.26. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q"'] & X' \arrow[d, "q'"] \\
Y \arrow[r, "g"'] & Y',
\end{tikzcd}$$

where X, X' are locally Noetherian Artin stacks and q, q' are good moduli spaces. If f is étale, X' is invariantly generated, and f_{cl} is representable, separated, stabilizer-preserving and weakly saturated, then g is étale and the diagram is cartesian.

Proof. By [Alp12, Proposition 5.3], g_{cl} is étale and the underlying square of classical truncations is cartesian. By the previous proposition, g is also étale.

Consider the natural morphism $h\colon X \to Y \times_{Y'} X'$ and write $g'\colon X \times_{Y'} Y \to X'$ for the base-change of g. Clearly g' is étale.

We just saw that h_{cl} is an isomorphism, hence it remains to check that h is étale. But we have a factorization $g' \circ h \simeq f$. Since both g' and f are étale, the same holds for h and we are done. □

Finally, combining the proofs of the two previous propositions, we state and prove a stronger variant of descent, which seems better suited to applications in derived geometry. Note that we no longer impose the condition that X' is invariantly generated.

Proposition 3.27. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & X' \arrow[d, "q'"] \\ Y \arrow[r, "g"'] & Y', \end{tikzcd}$$

where X, X' are locally Noetherian Artin stacks and q, q' are good moduli spaces. If f is étale, and f_{cl} is representable, separated, stabilizer-preserving and weakly saturated, then g is étale and the diagram is cartesian.

Proof. By [Alp12, Proposition 5.3], g_{cl} is étale and the underlying square of classical truncations is cartesian. We modify the proof of Proposition 3.24.

We prove by induction on $n \geq 0$ that the truncation $g_{\leq n}$ is étale, i.e. the morphism $\mathbb{L}_{Y'_{\leq n}}|_{Y_{\leq n}} \to \mathbb{L}_{Y_{\leq n}}$ is an equivalence and that the truncated square

$$\begin{tikzcd} X_{\leq n} \arrow[r, "f_{\leq n}"] \arrow[d, "q_{\leq n}"'] & X'_{\leq n} \arrow[d, "q'_{\leq n}"] \\ Y_{\leq n} \arrow[r, "g_{\leq n}"'] & Y'_{\leq n}, \end{tikzcd} \tag{3.10}$$

is cartesian.

We have a commutative diagram of distinguished triangles

$$\begin{tikzcd} \mathbb{L}_{X'_{\leq n+1}}|_{X_{\leq n}} \arrow[r] \arrow[d] & \mathbb{L}_{X'_{\leq n}}|_{X_{\leq n}} \arrow[r] \arrow[d] & M'_{n+1}|_{X_{\leq n}} \arrow[d] \\ \mathbb{L}_{X_{\leq n+1}}|_{X_{\leq n}} \arrow[r] & \mathbb{L}_{X_{\leq n}} \arrow[r] & M_{n+1} \end{tikzcd}$$

where the vertical arrows are equivalences, since f is étale.

The proof of Theorem 3.8 shows that there exists a corresponding commutative diagram of distinguished triangles

$$\begin{tikzcd} \mathbb{L}_{Y'_{\le n+1}}|_{Y_{\le n}} \arrow[r] \arrow[d] & \mathbb{L}_{Y'_{\le n}}|_{Y_{\le n}} \arrow[r] \arrow[d] & g^*_{\le n}(q'_{\le n})_* M'_{n+1} \arrow[d] \\ \mathbb{L}_{Y_{\le n+1}}|_{Y_{\le n}} \arrow[r] & \mathbb{L}_{Y_{\le n}} \arrow[r] & (q_{\le n})_* M_{n+1} \end{tikzcd}$$

where the middle arrow is an equivalence. Hence it suffices to verify that the rightmost arrow

$$g^*_{\le n}(q'_{\le n})_* M'_{n+1} \to (q_{\le n})_* M_{n+1} \simeq (q_{\le n})_* f^*_{\le n} M'_{n+1}$$

is also an equivalence, where the last equivalence follows from the étaleness of $f_{\le n}$. But this is the case by base-change for the cartesian square (3.10).

It remains to check that the induced commutative square of the form (3.10) for $n+1$ is also cartesian.

Consider the natural morphism $h_{\le n+1} \colon X_{\le n+1} \to Y_{\le n+1} \times_{Y_{\le n+1}} X'_{\le n+1}$. We know that $(h_{\le n+1})_{\mathrm{cl}}$ is an isomorphism, so it suffices to check that $h_{\le n+1}$ is étale. But we have a factorization

$$f_{\le n+1} \colon X_{\le n+1} \xrightarrow{h_{\le n+1}} Y_{\le n+1} \times_{Y_{\le n+1}} X'_{\le n+1} \xrightarrow{g'_{\le n+1}} X'_{\le n+1},$$

where $g'_{\le n+1}$ is the base-change of $g_{\le n+1}$. Since $f_{\le n+1}, g'_{\le n+1}$ are both étale, the same holds for $h_{\le n+1}$ and we are done. □

4. Applications and examples

This section focuses on applications and examples of the theory developed thus far. These include a derived étale slice theorem, an upgrade of the stabilizer reduction algorithm for derived stacks [HRS22], derived geometric invariant theory, an existence criterion and quotients by group actions.

4.1. **A (fully) derived étale slice theorem.** Let X_{cl} be a classical Noetherian Artin stack with an $\mathcal{A}$-good moduli space $q_{\mathrm{cl}} \colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ such that X_{cl} has affine stabilizers and separated diagonal. Let $x \in X$ be a closed point with stabilizer G_x.

The classical étale slice theorem of [AHR20] asserts that there exists an affine, étale morphism $\Phi_{\mathrm{cl}} \colon [U_{\mathrm{cl}}/G_x] \to X_{\mathrm{cl}}$, where U_{cl} is an affine scheme with a G_x-action, $u \in U_{\mathrm{cl}}$ is fixed by G_x and maps to x via Φ_{cl}, fitting in a cartesian square

$$\begin{tikzcd} {[U_{\mathrm{cl}}/G_x]} \arrow[r, "\Phi_{\mathrm{cl}}"] \arrow[d, "q'"'] & X_{\mathrm{cl}} \arrow[d, "q"] \\ U /\!\!/ G_x \arrow[r] & Y. \end{tikzcd}$$

We can now obtain a fully derived version of this statement.

Theorem 4.1. *Let X be a Noetherian Artin stack with a good moduli space $q \colon X \to Y$ such that X_{cl} satisfies the conditions listed above. Let $x \in X$ be a closed point and write G_x for the classical stabilizer group of $x \in X_{\mathrm{cl}}$.*

Then there exists an affine, étale morphism $\Phi\colon [U/G_x] \to X$, where U is an affine scheme with a G_x-action, $u \in U$ is fixed by G_x and maps to x via Φ, fitting in a cartesian square

$$\begin{tikzcd} {[U/G_x]} \arrow[r, "\Phi"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ U /\!\!/ G_x \arrow[r] & Y. \end{tikzcd}$$

Proof. By the classical étale slice theorem [AHR20, Theorem 4.12] and [AHR19, Theorem 6.1], there exists a classical morphism $\Phi_{\mathrm{cl}}\colon [U_{\mathrm{cl}}/G_x] \to X_{\mathrm{cl}}$, which is affine, étale, stabilizer-preserving and fits in a cartesian square

$$\begin{tikzcd} {[U_{\mathrm{cl}}/G_x]} \arrow[r, "\Phi_{\mathrm{cl}}"] \arrow[d, "q'_{\mathrm{cl}}"'] & X_{\mathrm{cl}} \arrow[d, "q_{\mathrm{cl}}"] \\ U_{\mathrm{cl}} /\!\!/ G_x \arrow[r, "\phi_{\mathrm{cl}}"'] & Y_{\mathrm{cl}}, \end{tikzcd} \tag{4.1}$$

where the morphism $[U_{\mathrm{cl}}/G_x] \to U_{\mathrm{cl}}/\!\!/G_x$ is a good moduli space morphism and $U_{\mathrm{cl}}/\!\!/G_x \to Y_{\mathrm{cl}}$ is étale (here we have implicitly used Proposition 3.4(ii), since $[U_{\mathrm{cl}}/G_x]$ and $U_{\mathrm{cl}}/\!\!/G_x$ both have affine diagonals).

By topological invariance [HRS22, Lemma 4.11], $\phi_{\mathrm{cl}}\colon U_{\mathrm{cl}}/\!\!/G_x \to Y_{\mathrm{cl}}$ is the classical truncation of an étale morphism $\phi\colon V \to Y$. Since the classical truncation of the fiber product $V \times_Y X$ is the quotient stack $[U_{\mathrm{cl}}/G_x]$, by the argument used in the proof of [HRS22, Proposition 4.13], it follows that $V \times_Y X$ is a quotient stack of the form $[U/G_x]$ for some affine scheme U with a G_x-action, whose classical truncation is the G_x-scheme U_{cl}.

We thus have a cartesian square

$$\begin{tikzcd} {[U/G_x]} \arrow[r, "\Phi"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ V \arrow[r, "\phi"'] & Y, \end{tikzcd}$$

whose classical truncation is the square (4.1).

By Lemma 3.7, q' is a good moduli space morphism, being a base-change of the good moduli space morphism q, and the proof is complete. □

Remark 4.2. By Proposition 3.4, if the diagonal of X is quasi-affine, it suffices to assume that X_{cl} admits an $\mathcal{A}$-good moduli space.

4.2. **Derived saturated blowups, stabilizer reduction and partial desingularization.** Let X be a derived Artin stack over $\mathbb{C}$ with a good moduli space $q\colon X \to Y$.

Stabilizer reduction refers to the process of producing a canonical stack $\widetilde{X}$ together with a canonical morphism $\pi\colon \widetilde{X} \to X$, which resolves the stackiness of X, meaning that $\widetilde{X}$ is Deligne–Mumford and π alters the original stack as little as possible. This has successfully been carried out within classical algebraic geometry originally for smooth quotient stacks obtained through

Geometric Invariant Theory (GIT) in [Kir85] and then more generally for singular classical stacks in the presence of a good moduli space in [ER21].

Subsequently, a derived stabilizer reduction procedure was developed in [HRS22]. However, the derived version only involves classical good moduli spaces in its statement and results. We can now give a unified, derived upgrade of the stabilizer reduction algorithm at the level of good moduli spaces as well.

We begin by recalling the definition of derived saturated blowups and setting up notation. These form the basic operation which is iteratively used in the stabilizer reduction process.

Definition 4.3. The *saturated projective spectrum* of a quasi-coherent, graded $\mathcal{O}_X$-algebra A is defined as the largest open substack $\mathrm{Proj}_X^q A \subseteq \mathrm{Proj}_X A$ for which the morphism $\mathrm{Proj}_X^q A \to \mathrm{Proj}_X q^*q_*A$ induced by $q^*q_*A \to A$ is well-defined. In particular, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Proj}_X^q A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Proj}_Y q_*A & \longrightarrow & Y. \end{array}$$

The following is then a direct generalization of [ER21, Proposition 3.4] to the derived setting.

Theorem 4.4. *Suppose that $q\colon X \to Y$ is a good moduli space and A a quasi-coherent, graded $\mathcal{O}_X$-algebra such that $\pi_0 A$ is finitely generated. Then $\mathrm{Proj}_X^q A \to \mathrm{Proj}_Y q_*A$ is a good moduli space and the induced morphism of good moduli spaces is the natural map $\mathrm{Proj}_Y q_*A \to Y$.*

Proof. Since we may work locally on Y by Proposition 3.14(i), we may assume that Y is affine. We then have an open cover of $\mathrm{Proj}_Y q_*A$ by affine schemes of the form $\mathrm{Spec}_Y (q_*A)_{(f)}$ as f runs through elements $f \in \pi_0(q_*A)_+$. Similarly, by definition, $\mathrm{Proj}_X^q A$ has an open cover by schemes of the form $\mathrm{Spec}_X A_{(f)}$ where again $f \in \pi_0(q_*A)_+$.

By Lemma 3.14(ii), we know that the morphism $\mathrm{Spec}_X A \to \mathrm{Spec}_Y q_*A$ is a good moduli space morphism, and thus, by Lemma 3.7, after localizing at $f \in \pi_0(q_*A)_+$, the morphism $q_f\colon \mathrm{Spec}_X A_f \to \mathrm{Spec}_Y (q_*A)_f$ is also a good moduli space morphism.

We now have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}_X A_f & \xrightarrow{\pi_X} & \mathrm{Spec}_X A_{(f)} \\ \downarrow{\scriptstyle q_f} & & \downarrow{\scriptstyle q_{(f)}} \\ \mathrm{Spec}_Y (q_*A)_f & \xrightarrow[\pi_Y]{} & \mathrm{Spec}_Y (q_*A)_{(f)} \end{array}$$

where π_X, π_Y are (coarse) quotients by the natural $\mathbb{G}_m$-action.

For any $M \in \mathsf{QCoh}(\mathrm{Spec}_X A_{(f)})$, we have that $M \simeq ((\pi_X)_*(\pi_X)^*M)_0$ and therefore, since $(q_{(f)})_*$ commutes with the $\mathbb{G}_m$-grading, we obtain $(q_{(f)})_*M = ((\pi_Y)_*(q_f)_*(\pi_X)^*M)_0$. This is a composition of right t-exact functors and hence t-exact. In particular, $q_{(f)}$ is a good moduli space morphism, as we want. $\square$

We may now define saturated blow-ups along closed substacks. By the preceding theorem and the properties of Rees algebras established in [Hek21], they automatically admit good moduli spaces.

Definition 4.5. Let $Z \to X$ be a closed immersion. Then the saturated blow-up of X along Z is defined by $\mathrm{Bl}^q_Z X := \mathrm{Proj}^q_X \mathcal{R}_{Z/X}$, where $\mathcal{R}_{Z/X}$ denotes the Rees algebra associated to $Z \to X$.

When the closed substack is saturated, the good moduli space of the saturated blow-up admits a nice description.

Proposition 4.6. *Let $Z \to X$ be a closed immersion.*

(i) *Suppose that Z is saturated. We then have a natural equivalence of graded algebras $q_*\mathcal{R}_{Z/X} \simeq \mathcal{R}_{q(Z)/Y}$.*

(ii) *If Z is not saturated, there exists $\delta \in \mathbb{N}$ and a closed immersion $Z'_{(\delta)} \to Y$ determined by Z and δ such that*

$$q_*\mathcal{R}_{Z_{(\delta)}/X} \simeq \mathcal{R}_{Z'_{(\delta)}/Y},$$

where $Z_{(\delta)}$ denotes the δ-th order infinitesimal neighborhood of Z in X defined in [HKR22, Example 5.26].

Proof. We first show part (i). In the following commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "q"] & D_{Z/X} \arrow[l, "p"'] \arrow[d, "q'"] \\ q(Z) \arrow[r] & Y & D_{q(Z)/Y} \arrow[l, "r"'] \end{tikzcd}$$

the square on the left is cartesian, hence so is the one on the right. It follows that q' is a good moduli space, and hence

$$q_*\mathcal{R}^{\mathrm{ext}}_{Z/X} \simeq q_*p_*\mathcal{O}_{D_{Z/X}} \simeq r_*q'_*\mathcal{O}_{D_{Z/X}} \simeq r_*\mathcal{O}_{D_{q(Z)/Y}} \simeq \mathcal{R}^{\mathrm{ext}}_{q(Z)/Y}.$$

Since pushforwards commute with the functor $(-)_{\geq 0}$ that sends a $\mathbb{Z}$-graded algebra to the $\mathbb{N}$-graded algebra obtained by discarding the negatively graded homogeneous pieces, the claim follows.

We move on to the more general part (ii). Consider the graded $\mathcal{O}_Y$-algebra $q_*\mathcal{R}^{\mathrm{ext}}_{Z/X}$. Since $\pi_0(q_*\mathcal{R}^{\mathrm{ext}}_{Z/X}) \simeq q_*\pi_0(\mathcal{R}^{\mathrm{ext}}_{Z/X})$ is finitely generated, there exists a positive integer δ such that $q_*(\mathcal{R}^{\mathrm{ext}}_{Z/X})^{(\delta)} \simeq q_*\mathcal{R}^{\mathrm{ext}}_{Z_{(\delta)}/X}$ is generated in degree 1. Taking $Z'_{(\delta)} = \mathrm{Spec}_Y(q_*\mathcal{R}^{\mathrm{ext}}_{Z_{(\delta)}/X}/(t^{-1}))_0$ concludes the proof using the properties of infinitesimal neighborhoods [HKR22, Example 5.26]. □

We deduce the following corollary.

Corollary 4.7. *Let $Z \to X$ define a closed substack of X. If Z is saturated, then the good moduli space of the saturated blow-up $\mathrm{Bl}^q_Z X$ is the derived blow-up of Y along $q(Z)$. More generally, the good moduli space of the saturated blow-up $\mathrm{Bl}^q_Z X$ takes the form of a derived blow-up of Y along a closed substack $Z' \to Y$ determined by $Z \to X$.*

Proof. This is a straightforward consequence of Theorem 4.4 and Proposition 4.6. □

Now, we consider the primary case of interest where $Z = X^{\max}$, defined in [HRS22]. $X^{\max}$ is a canonical, closed substack of X which parametrizes points with stabilizers of maximal dimension. As for classical stacks, $X^{\max}$ doesn't in general have to be saturated.

The saturated blow-up $\widehat{X} = \mathrm{Proj}_X^q \mathcal{R}_{X^{\max}/X}$ is called the Kirwan blow-up of X, defined in [HRS22]. We obtain the following result based on the above.

Theorem 4.8. *Let X be a quasi-compact Artin stack, locally of finite presentation with a good moduli space $q\colon X \to Y$. Then its Kirwan blow-up $\widehat{X} = \mathrm{Proj}_X^q \mathcal{R}_{X^{\max}/X}$ admits a good moduli space $\widehat{q}\colon \widehat{X} \to \widehat{Y}$, fitting in a commutative diagram*

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "\widehat{q}"'] & X \arrow[d, "q"] \\ \widehat{Y} \arrow[r] & Y, \end{tikzcd}$$

where $\widehat{Y} = \mathrm{Proj}_Y q_*\mathcal{R}_{X^{\max}/X} = \mathrm{Bl}_Z Y \to Y$ is the blow-up of Y along a closed substack Z determined by $X^{\max}$.

Proof. Since $A = \mathcal{R}_{X^{\max}/X}$ satisfies the condition that $\pi_0 A$ is finitely generated (and indeed in degree 1!), Theorem 4.4 applies and $\widehat{X} = \mathrm{Proj}_X^q \mathcal{R}_{X^{\max}/X}$ admits a good moduli space $\widehat{Y} = \mathrm{Proj}_Y q_*\mathcal{R}_{X^{\max}/X}$. Then Corollary 4.7 finishes the proof. $\square$

Using the fact that the operation of Kirwan blow-up reduces the maximal stabilizer dimension of a stack, by construction (cf. [HRS22]), the stabilizer reduction $\widetilde{X}$ of X is obtained by a finite sequence of iterated Kirwan blow-ups

$$X_0 := X,\ X_1 = \widehat{X}_0, \dots,\ \widetilde{X} = X_m = \widehat{X}_{m-1}.$$

In particular, Theorem 4.8 has the following implication.

Corollary 4.9. *Let X be a quasi-compact Artin stack, locally of finite presentation with a good moduli space $q\colon X \to Y$. Then the stabilizer reduction $\widetilde{X}$ admits a good moduli space $\widetilde{q}\colon \widetilde{X} \to \widetilde{Y}$, where the natural induced morphism $\widetilde{Y} \to Y$ is an iterated blow-up projection.*

4.3. **Geometric Invariant Theory (GIT).** We remark that an interesting special case of Theorem 3.8 and Proposition 3.4 covers derived, Noetherian Artin stacks X with quasi-affine diagonal whose classical truncation is obtained by GIT [MFK94], i.e. is of the form $X_{\mathrm{cl}} = [Q^{ss}/G]$, where $Q^{ss} \subseteq Q$ is the locus of semistable points for a linearized action of a complex reductive group G on a projective variety Q. This includes a plethora of stacks commonly encountered in the literature and provides natural derived enhancements of their corresponding GIT quotients. We mention the following example of this phenomenon.

Theorem 4.10. *Let W be a complex projective variety, $\beta \in H_2(W, \mathbb{Z})$ and $g, n \in \mathbb{N}$. The derived Deligne–Mumford stack $\mathbb{R}\mathcal{M}_{g,n}(X, \beta)$ of n-pointed stable maps of genus g and class β to X (see* [STV15, Definition 2.6]*) admits a good moduli space $\mathbb{R}M_{g,n}(X, \beta)$, which is a derived enhancement of the GIT quotient $M_{g,n}(X, \beta)$.*

4.4. **Existence Criteria.** In the same vein as in the previous subsection, let X be a Noetherian derived Artin stack with quasi-affine diagonal. By Theorem 3.8 and Proposition 3.4, the existence of a good moduli space for X is equivalent to the existence of an $\mathcal{A}$-good moduli space for X_{cl}.

We may thus promote the existence criteria for $\mathcal{A}$-good moduli spaces of classical Artin stacks developed in [AHLH19] to the case of derived Artin stacks. Here is the translation of [AHLH19, Theorem A] to our context. Of course, one may obtain analogues of similar results, e.g. [AHLH19, Theorem 4.1].

Theorem 4.11. *Let X be a finitely presented derived Artin stack over $\mathbb{C}$ with quasi-affine diagonal and affine stabilizers. Then X admits a separated good moduli space $q\colon X \to Y$ if and only if X_{cl} is Θ-reductive and S-complete.*

Moreover, Y is proper if and only if X_{cl} satisfies the valuative criterion for properness.

As an immediate application of this theorem, we obtain derived enhancements for good moduli spaces of moduli stacks parametrizing objects in abelian categories, when these stacks admit derived enhancements.

For a concrete example, let W be a projective scheme over $\mathbb{C}$ and X be a quasi-compact component of the derived moduli stack of coherent sheaves on W. By [AHLH19, Lemma 7.20], X has affine diagonal and, by [AHLH19, Theorem 7.23], X_{cl} admits a proper good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$.

Theorem 4.12. *X admits a proper good moduli space $q\colon X \to Y$, which is a derived enhancment of the good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$.*

4.5. **Group actions and quotients.** Suppose that G is a linearly reductive classical group acting on an affine, finitely presented derived scheme $U = \operatorname{Spec} A$. Then the quotient stack $X = [U/G]$ has affine diagonal and admits a good moduli space since its classical truncation $X_{\mathrm{cl}} = [U_{\mathrm{cl}}/G]$ admits an $\mathcal{A}$-good and hence a good moduli space $q_{\mathrm{cl}}\colon X_{\mathrm{cl}} \to U_{\mathrm{cl}}/\!\!/G = \operatorname{Spec}(\pi_0 A)^G$.

To describe the good moduli space, we may assume that the G-action on U is strict, after possibly replacing $\operatorname{Spec} A$ by an equivalent affine G-scheme, meaning that it is induced by a morphism $\mathcal{O}_U \to \mathcal{O}_U \otimes_{\mathbb{C}} \mathcal{O}_G$ where the axioms for a group action are satisfied not up to homotopy but up to equality. This can be achieved by a standard argument, using that G is reductive and starting with a G-invariant closed embedding $U_{\mathrm{cl}} \to \mathbb{A}^N$, where G acts on $\mathbb{A}^N$ linearly. For each positive degree, we then add generators iteratively by adjoining G-representations. For more details, the reader can consult [HRS22, Section 3.7].

It now follows that G also acts on the ring A and thus we can consider the ring of invariants A^G and the G-invariant inclusion $A^G \subseteq A$, which induces a natural morphism $q\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. We claim that this is the good moduli space of $X = [U/G]$ and write $U/\!\!/G$ for $\operatorname{Spec} A^G$.

To see this, it suffices to check that q_* is t-exact and preserves the structure sheaf $\mathcal{O}$. For any object of $\mathsf{QCoh}([\operatorname{Spec} A/G])$, by a similar argument as before, we may assume, up to equivalence, that it is given by a G-equivariant A-module M and the same is true for morphisms between sheaves. We now have $q_* M = M^G$ as an A^G-module. Since G is reductive, taking invariants is exact, and $q_* \mathcal{O}_{[\operatorname{Spec} A/G]} = A^G$, as required.

Example 4.13. Here is a concrete example of the above. Consider the matrices

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \quad R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

let $\mathbb{C}[X, Y]$ be the free $\mathbb{C}$-algebra generated by the entries of X and Y and let $\mathbb{C}[R]$ be the free $\mathbb{C}$-algebra generated by the entries of R. Now let $\mathbb{C}[R] \to \mathbb{C}[X, Y]$ be the map sending R to the commutator matrix $[X, Y] = XY - YX$. For example, r_{12} maps to $[X, Y]_{12} = x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22}$. Let A be the simplicial ring which is defined by the pushout diagram

$$\begin{tikzcd} \mathbb{C}[R] \arrow[r] \arrow[d] & \mathbb{C}[X, Y] \arrow[d] \\ \mathbb{C} \arrow[r] & A. \end{tikzcd}$$

Then $G = \mathrm{GL}_{2,\mathbb{C}}$ acts by simultaneous conjugation on X, Y, R and hence on the derived affine scheme $\operatorname{Spec} A$. We thus have a derived good moduli space $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. It follows that the ring A^G is the simplicial ring which is the pushout of the diagram

$$\begin{tikzcd} \mathbb{C}[r] \arrow[r] \arrow[d] & (\mathbb{C}[x, y] \otimes_{\mathbb{C}} \mathbb{C}[x, y])^{S_2} \arrow[d] \\ \mathbb{C} \arrow[r] & A^G, \end{tikzcd}$$

where r is sent to zero via both maps and the symmetric group S_2 acts in the obvious way. This means that $\operatorname{Spec} A^G$ is a derived square zero thickening of the classical affine scheme $\mathrm{Sym}^2(\mathbb{A}^2_{\mathbb{C}})$.

This should not come as a surprise: $[\operatorname{Spec} A/G]_{\mathrm{cl}}$ is the classical moduli stack $\mathcal{M}_2$ of length 2 sheaves on $\mathbb{A}^2_{\mathbb{C}}$, whose good moduli space is given by the Hilbert-Chow morphism $\mathcal{M}_2 \to \mathrm{Sym}^2(\mathbb{A}^2_{\mathbb{C}})$.

Example 4.14. Another interesting example of derived nature arises when $X = BG$, where G is a derived group scheme over $\mathbb{C}$, whose classical truncation G_{cl} is linearly reductive. For a definition of a group object in any ∞-topos and its classifying space, we encourage the reader to consult [NSS15].

We have $X_{\mathrm{cl}} = BG_{\mathrm{cl}}$ and hence X is a derived Artin stack. We write $\mathfrak{g}_{\mathrm{cl}} = \mathbb{T}_{BG_{\mathrm{cl}}}[-1]$ for the Lie algebra of G_{cl} and $\mathfrak{g} = \mathbb{T}_{BG}[-1]$, which has a natural structure of a dg-Lie-algebra as a shifted tangent complex of a derived Artin stack (cf. [Hen18]). Moreover, write $\mathfrak{g}/\mathfrak{g}_{\mathrm{cl}}$ for the cofiber of the derivative morphism $\mathfrak{g}_{\mathrm{cl}} \to \mathfrak{g}|_{BG_{\mathrm{cl}}}$.

We thus have an exact triangle $\mathfrak{g}_{\mathrm{cl}} \to \mathfrak{g}|_{BG_{\mathrm{cl}}} \to \mathfrak{g}/\mathfrak{g}_{\mathrm{cl}}$ where each term is naturally G_{cl}-equivariant. It is easy to see that the triangle is split and we have a G_{cl}-invariant splitting $\mathfrak{g}|_{BG_{\mathrm{cl}}} = \mathfrak{g}_{\mathrm{cl}} \oplus \mathfrak{g}/\mathfrak{g}_{\mathrm{cl}}$, where $\mathfrak{g}/\mathfrak{g}_{\mathrm{cl}} \simeq \mathfrak{g}|^{\geq 0}_{BG_{\mathrm{cl}}}$.

Since $|BG| \simeq |BG_{\mathrm{cl}}| \simeq *$, by Koszul duality, we obtain an equivalence $[\operatorname{Spec}(\mathbb{C} \oplus \mathrm{Sym}(\mathfrak{g}/\mathfrak{g}_{\mathrm{cl}})^\vee[1])/G_{\mathrm{cl}}] \simeq BG$. By our previous discussion, we conclude that the good moduli space of BG is given by the morphism

$$BG \simeq [\operatorname{Spec}\left(\mathbb{C} \oplus \mathrm{Sym}(\mathfrak{g}/\mathfrak{g}_{\mathrm{cl}})^\vee[1]\right)/G_{\mathrm{cl}}] \to \operatorname{Spec}\left(\mathbb{C} \oplus \mathrm{Sym}(\mathfrak{g}/\mathfrak{g}_{\mathrm{cl}})^\vee[1]\right)^{G_{\mathrm{cl}}}.$$

In particular, we see that the good moduli space of BG is in general not the point, unless $G = G_{\mathrm{cl}}$. Interestingly, even when $G_{\mathrm{cl}} = *$, the good moduli space admits a non-trivial derived structure when G is not classical.

References

[ABHLX20] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, *Reductivity of the automorphism group of K-polystable Fano varieties*, Invent. Math. **222** (2020), no. 3, 995–1032.

[AHLH19] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth, *Existence of moduli spaces for algebraic stacks*, 2019, arXiv:1812.01128.

[AHR19] Jarod Alper, Jack Hall, and David Rydh, *The étale local structure of algebraic stacks*, 2019, arXiv:1912.06162.

[AHR20] Jarod Alper, Jack Hall, and David Rydh, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) **191** (2020), no. 3, 675–738.

[Alp10] Jarod Alper, *On the local quotient structure of Artin stacks*, J. Pure Appl. Algebra **214** (2010), no. 9, 1576–1591.

[Alp12] Jarod Alper, *Local properties of good moduli spaces*, Tohoku Math. J. (2) **64** (2012), no. 1, 105–123.

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[Alp14] Jarod Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531.

[Art74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[ER21] Dan Edidin and David Rydh, *Canonical reduction of stabilizers for Artin stacks with good moduli spaces*, Duke Math. J. **170** (2021), no. 5, 827–880.

[GR17] Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry: Correspondences and duality*, vol. 1, American Mathematical Soc., 2017.

[Hek21] Jeroen Hekking, *Graded algebras, projective spectra and blow-ups in derived algebraic geometry*, arXiv e-prints (2021), arXiv:2106.01270, arXiv:2106.01270.

[Hen18] Benjamin Hennion, *Tangent Lie algebra of derived Artin stacks*, J. Reine Angew. Math. **741** (2018), 1–45.

[HKR22] Jeroen Hekking, Adeel Khan, and David Rydh, *Deformation to the normal cone and blow-ups via derived Weil restrictions*, Forthcoming, 2022.

[HL10] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010.

[HLP19] Daniel Halpern-Leistner and Anatoly Preygel, *Mapping stacks and categorical notions of properness*, 2019, arXiv:1402.3204.

[HNR19] Jack Hall, Amnon Neeman, and David Rydh, *One positive and two negative results for derived categories of algebraic stacks*, J. Inst. Math. Jussieu **18** (2019), no. 5, 1087–1111.

[HR19] Jack Hall and David Rydh, *Coherent Tannaka duality and algebraicity of Hom-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675.

[HRS22] Jeroen Hekking, David Rydh, and Michail Savvas, *Stabilizer reduction for derived stacks and applications to sheaf-theoretic invariants*, arXiv e-prints (2022), arXiv:2209.15039, arXiv:2209.15039.

[Kir85] Frances Clare Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. (2) **122** (1985), no. 1, 41–85.

[KM97] Seán Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213.

[Kon95] Maxim Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368.

[MFK94] David Mumford, James Fogarty, and Frances Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[Mum65] David Mumford, *Geometric invariant theory*, vol. Band 34, Springer-Verlag, Berlin-New York, 1965.

[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson, *Principal ∞-bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), no. 4, 749–801.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted symplectic structures*, Publications mathématiques de l'IHÉS **117** (2013), no. 1, 271–328 (English).

[SAG] Jacob Lurie, *Spectral Algebraic Geometry*, Preprint, available at math.ias.edu/~lurie (2016).

[Stcks] The Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu`, 2018.

[STV15] Timo Schürg, Bertrand Toën, and Gabriele Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. Reine Angew. Math. **702** (2015), 1–40.

[Toë14] Bertrand Toën, *Derived algebraic geometry*, EMS Surv. Math. Sci. **1** (2014), no. 2, 153–240.

[TV04] Bertrand Toën and Gabriele Vezzosi, *From HAG to DAG: derived moduli stacks*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 173–216.

Department of Mathematics, KTH Royal Institute of Technology, Stockholm, 114 28, Sweden
Email address: `hekking@kth.se`

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA
Email address: `msavvas@utexas.edu`