GENERALIZED DONALDSON-THOMAS INVARIANTS OF DERIVED OBJECTS VIA KIRWAN BLOWUPS

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Abstract. Let σ be a stability condition on the bounded derived category $D^b(\mathrm{Coh}W)$ of a Calabi-Yau threefold W and $\mathcal{M}$ a moduli stack of σ-semistable objects of fixed topological type. We define generalized Donaldson-Thomas invariants which act as virtual counts of objects in $\mathcal{M}$ by generalizing the approach introduced by Kiem, Li and the author in the case of semistable sheaves.

We construct an associated Deligne-Mumford stack $\widetilde{\mathcal{M}}$, called the $\mathbb{C}^*$-rigidified Kirwan partial desingularization of $\mathcal{M}$, with an induced semi-perfect obstruction theory of virtual dimension zero, and define the generalized Donaldson-Thomas invariant via Kirwan blowups to be the degree of the associated virtual cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$. This is invariant under deformations of the complex structure of W. Examples of applications include Bridgeland stability, polynomial stability, Gieseker and slope stability.

Contents

1. Introduction

1.1. **Background and history.** Donaldson-Thomas (abbreviated as DT from now on) invariants constitute one of the main enumerative theories of curves on Calabi-Yau threefolds. They naturally appear in many enumerative problems of interest in algebraic geometry and string theory and are conjecturally equivalent with other counting invariants, such as Gromov-Witten invariants, Stable Pair invariants [PT09] and Gopakumar-Vafa invariants [MT18]. These relations have now been proven in many cases (for example, in [MNOP06a, MNOP06b, Tod10]).

Let W be a smooth, projective Calabi-Yau threefold and $\gamma \in H^*(W, \mathbb{Q})$. Classical DT theory was introduced in [Tho00] in order to obtain virtual counts of stable sheaves on W of Chern character γ. More precisely, Thomas

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considered the moduli space $M := M_L^{ss}(\gamma)$ parameterizing Gieseker semistable sheaves on W of positive rank, fixed determinant L and Chern character γ. Assuming that every semistable sheaf is stable, M is proper and admits a perfect obstruction theory in the sense of Li-Tian [LT98] or Behrend-Fantechi [BF97] of virtual dimension zero. Thus, there exists a virtual fundamental cycle $[M]^{\mathrm{vir}} \in A_0(M)$ and the classical DT invariant is then defined as

$$\mathrm{DT}_\gamma := \deg\, [M]^{\mathrm{vir}}$$

and is shown to be invariant under deformation of the complex structure of W.

One important feature of DT invariants is their motivic nature. Behrend [Beh09] showed that the obstruction theory of M is symmetric and there is an equality

$$\mathrm{DT}_\gamma = \deg\, [M]^{\mathrm{vir}} = \chi(M, \nu_M) \tag{†}$$

where $\nu_M \colon M \to \mathbb{Z}$ is Behrend's canonical constructible function and the right-hand side a weighted Euler characteristic.

However, when stability and semistability of sheaves do not coincide, the above methods do not suffice and new techniques are required in order to define generalized DT invariants (or gDT for short) counting semistable sheaves. A main obstacle is that strictly semistable sheaves can have more automorphisms beyond scaling and, as a result, the stacks $\mathcal{M}^{ss}(\gamma)$ parameterizing semistable sheaves are generally Artin and no longer Deligne-Mumford (after rigidifying $\mathbb{C}^*$-scaling, cf. Subsection 5.3), which is necessary for the machinery of perfect obstruction theory and virtual cycles to apply.

In [JS12], Joyce and Song constructed gDT invariants of moduli stacks $\mathcal{M}^{ss}(\gamma)$ taking advantage of the above motivic behaviour and using motivic Hall algebras to generalize the right-hand side of (†). Their gDT invariant is easy to work with and amenable to computation. Their approach also works for moduli stacks of semistable complexes. However, the proof of the deformation invariance of the constructed invariants is indirect and proceeds via wall-crossing to a stable pairs theory where semistability and stability coincide and thus a virtual cycle exists. Kontsevich and Soibelman [KS10] have also defined motivic gDT invariants using similar ideas. We also mention the related work of Behrend and Ronagh [BR19, BR16].

In [KLS17], the authors develop a new direct approach towards defining gDT invariants of such a moduli stack $\mathcal{M}^{ss}(\gamma)$. Their method adapts Kirwan's partial desingularization procedure [Kir85] to define an invariant as the degree of a zero-dimensional virtual cycle in an associated Deligne-Mumford stack $\widetilde{\mathcal{M}}$. The constructed invariant is called the gDT invariant via Kirwan blowups (or DTK invariant for short) and is a direct generalization of the left-hand side of (†). By the usual properties of virtual cycles, they establish the deformation invariance of DTK invariants.

1.2. **Statement of results.** This paper serves as a sequel to [KLS17]. We generalize the construction of DTK invariants to moduli stacks $\mathcal{M} := \mathcal{M}^{\sigma-ss}(\gamma)$ of σ-semistable objects of Chern character γ in the derived category $D^b(\mathrm{Coh}\, W)$ of coherent sheaves on W, where σ is one of the following stability conditions:

(1) A Bridgeland stability condition [Bri07], as considered by Toda-Piyaratne [PT19] and [Li19].
(2) A polynomial stability condition [Bay09], as considered by Lo [Lo11, Lo13].
(3) Gieseker and slope stability of sheaves. These are examples of weak stability conditions on $\mathrm{Coh}\, W$ in the sense of Joyce-Song [JS12].

More generally, σ can be any nice stability condition (cf. Definition 5.1 and Definition 5.2).

Our main results are summarized in the following theorem, giving a definition of DTK invariants of σ-semistable complexes of Chern character γ.

Theorem. *Let W be a smooth, projective Calabi-Yau threefold, σ a stability condition on $D^b(\mathrm{Coh}\, W)$ as in Definition 5.2 and $\mathcal{M} := \mathcal{M}^{\sigma-ss}(\gamma)$ be the $\mathbb{C}^*$-rigidified moduli stack of σ-semistable complexes of Chern character γ. Then there exists an induced proper Deligne-Mumford stack $\widetilde{\mathcal{M}} \to \mathcal{M}$, called the $\mathbb{C}^*$-rigidified Kirwan partial desingularization of $\mathcal{M}$. $\widetilde{\mathcal{M}}$ is isomorphic to $\mathcal{M}$ over the stable locus $\mathcal{M}^{\sigma-s}(\gamma)$.*

$\widetilde{\mathcal{M}}$ admits an induced semi-perfect obstruction theory of virtual dimension zero, which extends the symmetric obstruction theory of the Deligne-Mumford stack $\mathcal{M}^{\sigma-s}(\gamma)$. It thus admits a virtual fundamental cycle

$$[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$$

and the generalized Donaldson-Thomas invariant via Kirwan blowups of $\mathcal{M}$ is defined as

$$\mathrm{DTK}(\mathcal{M}^{\sigma-ss}(\gamma)) := \deg\, [\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in \mathbb{Q}.$$

This is invariant under deformations of the complex structure of W.

1.3. **Brief review of the case of sheaves and sketch of construction.** We first give a brief account of the approach of [KLS17] and then explain the necessary adjustments in order to generalize their results.

Let $\mathcal{M}$ be an Artin stack. In [KLS17], it is assumed that $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack (cf. [PTVV13]), and moreover $\mathcal{M}$ is a global quotient stack

$$\mathcal{M} = [X/G], \tag{1.1}$$

obtained by Geometric Invariant Theory (GIT), meaning that G is reductive and X is an invariant closed subscheme $X \subset (\mathbb{P}^N)^{ss}$ so that G acts on $\mathbb{P}^N$ via a homomorphism $G \to GL(N+1, \mathbb{C})$.

By [BBBBJ15], it follows that $\mathcal{M}$ is a d-critical stack [Joy15]. In particular, for every closed point $x \in \mathcal{M}$ with (reductive) stabilizer H, there exists a smooth affine H-scheme V, an invariant function $f\colon V \to \mathbb{A}^1$ and an étale morphism

$$[U/H] \to \mathcal{M}, \tag{1.2}$$

where $U = (df = 0) \subset V$. Moreover for every two such local presentations, there exist appropriate comparison data.

We have the following H-equivariant 4-term complex

$$\mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_V|_U \xrightarrow{d(df)^\vee} F_V|_U = \Omega_V|_U \longrightarrow \mathfrak{h}^\vee. \tag{1.3}$$

For $u \in U$ with finite stabilizer, this is quasi-isomorphic to a 2-term complex which gives a symmetric perfect obstruction theory of $[U/H]$ and thus of $\mathcal{M}$ near u.

One may then apply Kirwan's partial desingularization procedure, using the notion of intrinsic blowup introduced in [KL13], as adapted in [KLS17], to obtain the Kirwan partial desingularizations $\widetilde{X} \to X$ and $\widetilde{\mathcal{M}} := [\widetilde{X}/G] \to \mathcal{M}$, which is a proper DM stack. We may lift the étale cover (1.2) to an étale cover

$$[T/H] \to \widetilde{\mathcal{M}}, \tag{1.4}$$

where $T = (\omega_S = 0) \subset S$ for S a smooth affine H-scheme and $\omega_S \in H^0(S, F_S)$ an invariant section of an H-equivariant vector bundle F_S on S. Moreover, there exists an effective invariant divisor D_S such that (1.3) lifts to a 4-term complex

$$\mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_S|_T \longrightarrow F_S|_T \longrightarrow \mathfrak{h}^\vee(-D_S) \tag{1.5}$$

whose first arrow is injective and last arrow is surjective. Therefore, (1.5) is quasi-isomorphic to a 2-term complex

$$d(F_S^{\mathrm{red}}) : (d\omega_S^\vee)^\vee : T_{[S/H]}|_T \longrightarrow F_S^{\mathrm{red}}|_T, \tag{1.6}$$

where F_S^{red} is the kernel of the last arrow in (1.5). Dualizing and taking the quotient by H, we get

$$d\omega_S^\vee : F_{[S/H]}^{\mathrm{red}}|_{[T/H]}^\vee \longrightarrow \Omega_{[S/H]}|_{[T/H]}. \tag{1.7}$$

These perfect obstruction theories on $[T/H]$ are then shown to satisfy the axioms of a semi-perfect obstruction theory (cf. [CL11]) on $\widetilde{\mathcal{M}}$, giving rise to its virtual cycle.

The moduli stack of Gieseker semistable sheaves satisfies the above hypotheses and thus has a DTK invariant. A relative version of the above construction, using derived symplectic geometry, then implies its deformation invariance.

In the present paper, we work with moduli stacks $\mathcal{M}$ of semistable perfect complexes. These are truncations of (-1)-shifted symplectic derived Artin stacks, however they are not global quotient stacks obtained by GIT of the form (1.1).

Our main contribution is to remove this assumption. We use several recent technical results on the étale local structure of stacks [AHR20, AHR19], moduli spaces of objects in abelian categories [AHH18] and stability conditions in families [BLM+19].

Instead of requiring a GIT presentation, we assume in general that $\mathcal{M}$ admits a good moduli space [Alp13]. We then show that all of the above statements about the local presentations of $\mathcal{M}$ and their comparison data have identical analogues in this case, using the main structural result of [AHR20], and one may then carry out the Kirwan partial desingularization procedure as in [KLS17], also getting a semi-perfect obstruction theory of virtual dimension zero.

One needs to check that moduli stacks of semistable complexes fit into this context, admitting good moduli spaces, and their Kirwan partial desingularizations are proper DM stacks. This is the case by the recent results of [AHH18], which combine the theory of good moduli spaces with the notion of Θ-reductivity introduced in [Hal14], and apply to a wide range of stacks, including the ones appearing in the present paper.

Finally, the deformation invariance of the DTK invariant follows from the compatibility of the Kirwan partial desingularization with base change and the recent results on stability conditions in [BLM$^+$19].

1.4. **Layout of the paper.** In §2, we define the notion of stacks of DT type and establish their main properties that will be used throughout. §3 reviews the Kirwan partial desingularization procedure for GIT quotient stacks and generalizes it to stacks with good moduli spaces. In §4, we explain how to construct a semi-perfect obstruction theory on stacks of DT type by following the arguments of [KLS17]. Finally, in §5, we construct DTK invariants of semistable complexes by combining the above with the results of [AHH18] and $\mathbb{G}_m$-rigidification for Artin stacks and discuss their deformation invariance using recent work of [BLM$^+$19].

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1.6. **Notation and conventions.** Here are the various notations and other conventions that we use throughout the paper:

- All schemes and stacks are defined over $\mathbb{C}$, unless stated otherwise. $\mathcal{M}$ typically denotes an Artin stack, with affine stabilizers and separated diagonal unless stated otherwise.
- W denotes a smooth, projective Calabi-Yau threefold over $\mathbb{C}$ and $D^b(\operatorname{Coh} W)$ its bounded derived category of coherent sheaves.
- S denotes a smooth quasi-projective scheme over $\mathbb{C}$.
- G, H denote complex reductive groups. Usually, H will be a subgroup of G. T denotes the torus $\mathbb{C}^*$.
- If $x \in \mathcal{M}$, G_x denotes the automorphism group or stabilizer of x.
- If $U \hookrightarrow V$ is an embedding, $I_{U\subseteq V}$ or I_U (when V is clear from context) denotes the ideal sheaf of U in V.
- For a morphism $\rho\colon U \to V$ and a sheaf E on V, we often use $E|_U$ to denote ρ^*E.
- If V is a G-scheme, V^G and Z_G are both used to denote the fixed point locus of G in V.
- If U is a scheme with a G-action, then $\widehat{U}$ is used to denote the Kirwan blowup of U with respect to G. $\widetilde{U}$ denotes the Kirwan partial desingularization of U.
- The abbreviations DT, DM, GIT, whenever used, stand for Donaldson-Thomas, Deligne-Mumford and Geometric Invariant Theory respectively.

2. Stacks of DT Type

This section collects material that will be used throughout the rest of the paper.

We first briefly recall Joyce's theory of d-critical loci, as developed in [Joy15], and establish some notation. Then we move on to give an account of results regarding the étale local structure of Artin stacks and the theory of good moduli spaces. We conclude the section by defining stacks of DT type and stating some of their local properties.

2.1. **d-critical schemes.**

We begin by defining the notion of d-critical charts.

Definition 2.1. (d-critical chart) *A d-critical chart for a scheme M is the data of (U, V, f, i) such that: $U \subseteq M$ is Zariski open, V is a smooth scheme, $f \colon V \to \mathbb{A}^1$ is a regular function on V and $U \xrightarrow{i} V$ is an embedding so that $U = (df = 0) = \mathrm{Crit}(f) \subseteq V$.*

If $x \in U$, then we say that the d-critical chart (U, V, f, i) is centered at x.

Joyce defines a canonical sheaf $\mathcal{S}_M$ of $\mathbb{C}$-vector spaces with the property that for any Zariski open $U \subseteq M$ and an embedding $U \hookrightarrow V$ into a smooth scheme V with ideal I, $\mathcal{S}_M$ fits into an exact sequence

$$0 \longrightarrow \mathcal{S}_M|_U \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d} \Omega_V/I \cdot \Omega_V \tag{2.1}$$

For example, for a d-critical chart (U, V, f, i) the element $f + I^2 \in \Gamma(V, \mathcal{O}_V/I^2)$ gives a section of $\mathcal{S}_M|_U$.

Definition 2.2. (d-critical scheme) *A d-critical structure on a scheme M is a section $s \in \Gamma(M, \mathcal{S}_M)$ such that M admits a cover by d-critical charts (U, V, f, i) and $s|_U$ is given by $f + I^2$ as above on each such chart. We refer to the pair (M, s) as a d-critical scheme.*

2.2. **Equivariant d-critical loci.**

For our purposes, we need equivariant analogues of the results of Subsection 2.1. The theory works in parallel as before (cf. [Joy15, Section 2.6]).

Definition 2.3. (Good action) *Let G be an algebraic group acting on a scheme M. We say that the action is good if M has a cover $\{U_\alpha\}_{\alpha \in A}$ where every $U_\alpha \subseteq M$ is an invariant open affine.*

Remark 2.4. *If M is affine, then trivially every action of G on M is good.*

It is straightforward to extend Definitions 2.1, 2.2 to the equivariant setting (cf. [Joy15, Definition 2.40]).

Proposition 2.5. [Joy15, Remark 2.47] *Let G be a complex reductive group with a good action on a scheme M. Suppose that (M, s) is an invariant d-critical scheme. Then for any $x \in M$ fixed by G, there exists an invariant d-critical chart (U, V, f, i) centered at x, i.e. an invariant open affine $U \ni x$, a smooth scheme V with a G-action, an invariant regular function $f \colon V \to \mathbb{A}^1$ and an equivariant embedding $i \colon U \to V$ so that $U = \mathrm{Crit}(f) \subseteq V$.*

Remark 2.6. *If G is a torus $(\mathbb{C}^*)^k$, then Proposition 2.5 is true without the assumption that x is a fixed point of G.*

Remark 2.7. *One may replace Zariski open morphisms by étale morphisms without any difference to the theory. Another option is to work in the complex analytic topology.*

2.3. **d-critical Artin stacks.** The theory of d-critical loci extends naturally to Artin stacks. For more details, we point the interested reader to Section 2.8 of [Joy15]. We mention the following definition and basic properties which we will need in the form of remarks.

Definition 2.8. [Joy15, Corollary 2.52, Definition 2.53] *Let $\mathcal{M}$ be an Artin stack. For every smooth morphism $\phi\colon U \to \mathcal{M}$, where U is a scheme, the assignment $\mathcal{S}(U,\phi) := \mathcal{S}_U$ defines a sheaf $\mathcal{S}_{\mathcal{M}}$ of $\mathbb{C}$-vector spaces in the lisse-étale topology of $\mathcal{M}$.*

A d-critical structure on $\mathcal{M}$ is a global section $s \in H^0(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$. We then say that $(\mathcal{M}, s)$ is a d-critical Artin stack.

Remark 2.9. [Joy15, Example 2.55] *A d-critical structure can be equivalently described in terms of a groupoid presentation of $\mathcal{M}$. It follows that d-critical structures on quotient stacks $[M/G]$ are in bijective correspondence with invariant d-critical structures on M.*

If $\mathcal{M}' \to \mathcal{M}$ is a smooth morphism of Artin stacks and $\mathcal{M}$ is d-critical, then one may pull back the d-critical structure on $\mathcal{M}$ making $\mathcal{M}'$ d-critical as well.

Remark 2.10. *In* [BBBBJ15]*, it is shown that if $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack, then $\mathcal{M}$ admits an induced d-critical structure.*

2.4. **Local structure of Artin stacks.** The following theorem is an étale slice theorem for stacks. It states that Artin stacks are étale locally quotient stacks.

Theorem 2.11. [AHR20, Theorem 1.2] *Let $\mathcal{M}$ be a quasi-separated Artin stack, locally of finite type over $\mathbb{C}$ with affine geometric stabilizers. Let $x \in \mathcal{M}$ and $H \subseteq G_x$ a maximal reductive subgroup of the stabilizer of x. Then there exists an affine scheme U with an action of H, a point $u \in U$ fixed by H, and a smooth morphism*

$$\Phi : [U/H] \to \mathcal{M}$$

which maps $u \mapsto x$ and induces the inclusion $H \hookrightarrow G_x$ of stabilizers at u. Moreover, if G_x is reductive, the morphism is étale. If $\mathcal{M}$ has affine diagonal, then Φ can be taken to be affine.

2.5. **Good moduli spaces.** In this section, we collect some useful results about the structure of a certain class of Artin stacks, namely those with affine diagonal admitting good moduli spaces, following the theory developed by Alper ét al. All the material of the section can be found in [Alp13] and [AHR20].

We have the following definition.

Definition 2.12. [Alp13, Definition 4.1] *A morphism $\pi\colon \mathcal{M} \to Y$, where $\mathcal{M}$ is an Artin stack and Y an algebraic space, is a good moduli space for $\mathcal{M}$ if the following hold:*

(1) *π is quasi-compact and* $\pi_*\colon \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(Y)$ *is exact.*
(2) *The natural map* $\mathcal{O}_Y \to \pi_*\mathcal{O}_{\mathcal{M}}$ *is an isomorphism.*

The intuition behind the introduction of the notion of good moduli space is that stacks $\mathcal{M}$ that admit good moduli spaces behave like quotient stacks $[X^{ss}/G]$ obtained from GIT with good moduli space given by the map $[X^{ss}/G] \to X/\!\!/G$. In this sense, it is a generalization of GIT quotients for stacks.

We state the following properties of stacks with good moduli spaces.

Proposition 2.13. [Alp13, Proposition 4.7, Lemma 4.14, Theorem 4.16, Proposition 9.1, Proposition 12.14] *Let $\mathcal{M}$ be locally noetherian and $\pi\colon \mathcal{M} \to Y$ be a good moduli space. Then:*

(1) *π is surjective.*
(2) *π is universally closed.*
(3) *Two geometric points $x_1, x_2 \in \mathcal{M}(k)$ are identified in Y if and only if their closures $\overline{\{x_1\}}$ and $\overline{\{x_2\}}$ in $\mathcal{M}$ intersect.*
(4) *Every closed point of $\mathcal{M}$ has reductive stabilizer.*
(5) *Let $y \in |Y|$ be a closed point. Then there exists a unique closed point $x \in |\pi^{-1}(y)|$.*
(6) *If $\mathcal{Z}$ is a closed substack of $\mathcal{M}$, then the morphism $\mathcal{Z} \to \pi(\mathcal{Z}) = \mathrm{im}(\mathcal{Z})$ is a good moduli space morphism.*
(7) *Let*

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of Artin stacks, such that Y, Y' are algebraic spaces.
   (a) *If $\mathcal{M} \to Y$ is a good moduli space, then $\mathcal{M}' \to Y'$ is a good moduli space.*
   (b) *If $Y' \to Y$ is fpqc and $\mathcal{M}' \to Y'$ is a good moduli space, then $\mathcal{M} \to Y$ is a good moduli space.*
(8) *If $\mathcal{M}$ is of finite type, then Y is of finite type.*

Regarding the étale local structure of good moduli space morphisms for stacks with affine diagonal, we have the following theorem.

Theorem-Definition 2.14. (Quotient chart) [AHR20, Theorem 4.12] *Let $\mathcal{M}$ be a locally noetherian Artin stack with a good moduli space $\pi\colon \mathcal{M} \to M$ such that π is of finite type with affine diagonal. If $x \in \mathcal{M}$ is a closed point, then there exists an affine scheme U with an action of G_x and a cartesian diagram*

$$\begin{tikzcd} {[U/G_x]} \arrow[r, "\Phi"] \arrow[d] & \mathcal{M} \arrow[d, "\pi"] \\ U/\!\!/G_x \arrow[r] & M \end{tikzcd} \tag{2.2}$$

such that Φ has the same properties as in Theorem 2.11, is affine and $U/\!\!/G_x$ is an étale neighbourhood of $\pi(x)$.

We refer to any choice of data (U, Φ) such that $\Phi\colon [U/G_x] \to \mathcal{M}$ is étale, affine and stabilizer preserving as a quotient chart for $\mathcal{M}$ centered at x.

2.6. **Stacks of DT type.** Combining the notions of the preceding subsections, we give the following definition.

Definition 2.15. (Stack of DT type) *Let $\mathcal{M}$ be an Artin stack. We say that $\mathcal{M}$ is of DT type if the following are true:*

(1) *$\mathcal{M}$ is quasi-separated and finite type over $\mathbb{C}$.*
(2) *There exists a good moduli space $\pi\colon \mathcal{M} \to M$, where π is of finite type and has affine diagonal.*
(3) *$\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack.*

Remark 2.16. *By Remark 2.10 a stack $\mathcal{M}$ of DT type admits a d-critical structure $s \in H^0(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$.*

Definition 2.17. (d-critical quotient chart) *Let $x \in \mathcal{M}$ be as in Theorem-Definition 2.14 and $\Phi\colon [U/G_x] \to \mathcal{M}$ as in Theorem 2.11, thus in particular affine and étale. We say that (U, V, f, Φ) is a d-critical quotient chart centered at x if there exists a G_x-invariant d-critical chart (U, V, f, i).*

A variant of the following proposition first appeared and was used in [Tod16].

Proposition 2.18. *Let $\mathcal{M}$ be a stack of DT type and $x \in \mathcal{M}$ a closed point. Then there exists a d-critical quotient chart for $\mathcal{M}$ centered at x.*

Proof. By Theorem 2.14, we have a quotient chart $\Phi\colon [U'/G_x] \to \mathcal{M}$. By Definition 2.8, Remark 2.9 and Proposition 2.5 there exists an invariant open affine $x \in U \subseteq U'$ and an invariant d-critical chart (U, V, f, i) and the conclusion follows. □

Using the fact that $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack and following verbatim the arguments of [KLS17, Section 8] and [BBBBJ15, Section 2.7] we obtain the following proposition, which gives a way to compare two choices of d-critical quotient charts.

Proposition 2.19. *Let $\mathcal{M}$ be a stack of DT type. Let*

$$(U_\alpha, V_\alpha, f_\alpha, \Phi_\alpha),\ (U_\beta, V_\beta, f_\beta, \Phi_\beta)$$

be two d-critical quotient charts with V_α, V_β affine and $z \in [U_\alpha/G_\alpha] \times_{\mathcal{M}} [U_\beta/G_\beta]$ with stabilizer $G := G_z$. Let $x_\alpha \in [U_\alpha/G_\alpha]$ and $x_\beta \in [U_\beta/G_\beta]$ be the two projections of z. Then the following hold:

(1) *There exists a d-critical G-invariant quotient chart $(U_{\alpha\beta}, S_{\alpha\beta}, f_{\alpha\beta}, \Phi_{\alpha\beta})$ for $\mathcal{M}$ with $t \in T_{\alpha\beta}$ fixed by G and $T_{\alpha\beta}, S_{\alpha\beta}$ affine.*
(2) *We have G-equivariant commutative diagrams for $\lambda = \alpha, \beta$*

$$\begin{tikzcd} T_{\alpha\beta} \arrow[r] \arrow[d, "i_\lambda"'] & S_{\alpha\beta} \arrow[d, "\theta_\lambda"] \\ U_\lambda \arrow[r] & V_\lambda, \end{tikzcd}$$

where the vertical arrows are unramified morphisms, the horizontal arrows are embeddings and $t \in T_{\alpha\beta}$ maps to $x_\lambda \in U_\lambda$.

(3) *We have an induced diagram with étale arrows:*

$$\begin{tikzcd} & {[T_{\alpha\beta}/G]} \arrow[dl] \arrow[dr] & \\ {[U_\alpha/G_\alpha]} \arrow[dr] & & {[U_\beta/G_\beta]} \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

(4) $I := (\theta_\lambda^* df_\lambda) = (df_{\alpha\beta})$ *as ideals in* $\mathcal{O}_{S_{\alpha\beta}}$ *and* $\theta_\lambda^* f_\lambda + I^2$, $f_{\alpha\beta} + I^2$ *give the same invariant section* $s|_{T_{\alpha\beta}} \in H^0(T_{\alpha\beta}, \mathcal{S}_{T_{\alpha\beta}})$.

Remark 2.20. *We finally make a remark on the conditions in Definition 2.15. The first condition is there to ensure boundedness. The second condition implies the existence of quotient charts as in diagram* (2.2)*, which will be necessary in order to construct the Kirwan partial desingularization* $\widetilde{\mathcal{M}}$ *and its good moduli space. The third condition is necessary to induce a d-critical structure on* $\mathcal{M}$ *which is the crucial component in obtaining a semi-perfect obstruction theory for* $\widetilde{\mathcal{M}}$*.*

3. Kirwan Partial Desingularization

In this section, we briefly review Kirwan blowups and then generalize the construction of Kirwan partial desingularizations of quotient stacks obtained by GIT given in [KLS17] to the more general setting of stacks with good moduli spaces.

3.1. Kirwan blowups.

We recall the notion of Kirwan blowup as developed in [KLS17].

Suppose that U is an affine scheme with an action of a reductive group G. Let us assume that G is connected, as this will be the case when we take blowups throughout.

Suppose that we have an equivariant embedding $U \to V$ into a smooth G-scheme V and let I be the ideal defining U. Since $U \subset V$ is G-equivariant, G acts on I and we have a decomposition $I = I^{fix} \oplus I^{mv}$ into the fixed part of I and its complement as G-representations.

Let V^G be the fixed point locus of G inside V and $\pi\colon \mathrm{bl}_G(V) \to V$ the blowup of V along V^G. Let $E \subset \mathrm{bl}_G(V)$ be its exceptional divisor and $\xi \in \Gamma(\mathcal{O}_{\mathrm{bl}_G(V)}(E))$ the tautological defining equation of E. Then

$$\pi^{-1}(I^{mv}) \subset \xi \cdot \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}.$$

and consequently, $\xi^{-1}\pi^{-1}(I^{mv}) \subset \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}$. We define $I^{intr} \subset \mathcal{O}_{\mathrm{bl}_G(V)}$ to be

$$I^{intr} = \text{ideal generated by } \pi^{-1}(I^{fix}) \text{ and } \xi^{-1}\pi^{-1}(I^{mv}). \tag{3.1}$$

Definition 3.1. (Intrinsic blowup) *The G-intrinsic blowup of U is the subscheme $U^{intr} \subset \mathrm{bl}_G(V)$ defined by the ideal I^{intr}.*

Lemma 3.2. [KLS17] *The G-intrinsic blowup of U is independent of the choice of G-equivariant embedding $U \subset V$, and hence is canonical.*

Suupose U is an affine G-scheme with an equivariant embedding into a smooth affine G-scheme V as above. We can make sense of the notion of semistability of points in U^{intr} without ambiguity as follows.

We first work on the ambient scheme V. As it is affine, we can think of all points of V as being semistable. For the blowup $V^{intr} := \mathrm{bl}_G(V)$, G acts linearly on $E = \mathbb{P}N_{V^G/V}$ with respect to the natural G-linearization of $\mathcal{O}_E(1) := \mathcal{O}_E(-E)$ and therefore we have a GIT notion of stability of points on the exceptional divisor E.

Definition 3.3. *We say that $v \in V$ is **stable** if its G-orbit is closed in V and its stabilizer finite. A point $\widetilde{v} \in \mathrm{bl}_G(V)$ is **unstable** if its orbit closure meets the unstable locus of E. If $\widetilde{v}$ is not unstable, we say that it is **semistable**.*

Thus for any smooth affine G-scheme V, we can define its Kirwan blowup $\widehat{V} = (\mathrm{bl}_G(V))^{ss}$. By [Kir85], it satisfies $\widehat{V}^G = \emptyset$.

Now, if we have an equivariant embedding $V \to W$ between smooth G-schemes, then $(W^{intr})^{ss} \cap V^{intr} = (V^{intr})^{ss}$ based on our description. Hence we may define $(U^{intr})^{ss} := U^{intr} \cap (V^{intr})^{ss}$ for any equivariant embedding $U \to V$ into a smooth scheme V. This is independent of the choice of $U \to V$.

Definition 3.4. (Kirwan blowup) *The Kirwan blowup of a possibly singular affine G-scheme U associated with G is the scheme $\widehat{U} = (U^{intr})^{ss}$. It satisfies $\widehat{U}^G = \emptyset$.*

The notion of semistability above is exactly motivated by the corresponding notion in GIT in Kirwan's original blowup procedure in [Kir85]. This can be seen by the following theorem of Reichstein, which asserts that the locus of unstable points on $\mathrm{bl}_G(V)$ is exactly the locus of unstable points in the sense of GIT for the ample line bundle $\mathcal{O}_{\mathrm{bl}_G(V)}(-E)$.

Theorem 3.5. [Rei89] *Let V as above and denote $q \colon V \to V/\!\!/G$. The unstable locus of $\mathrm{bl}_G(V)$ with respect to the line bundle $\mathcal{O}_{\mathrm{bl}_G(V)}(-E)$ is the strict transform of the saturation $q^{-1}(q(V^G))$ of V^G.*

It follows that $[\widehat{V}/G]$ admits a good moduli space $\widehat{V}/\!\!/G$ and therefore the same is true for the closed substack $[\widehat{U}/G]$ by Proposition 2.13(6).

Remark 3.6. *Suppose that $U \to V$ is a G-equivariant embedding into a smooth G-scheme V. Let, as before, I be the ideal of U in V.*

We now explain how one can proceed if G is not connected. Let G_0 be the connected component of the identity. This is a normal, connected subgroup of G of finite index. Let $I = I^{fix} \oplus I^{mv}$ be the decomposition of I into fixed and moving parts with respect to the action of G_0. Using the normality of G_0, we see that the fixed locus V^{G_0} is a closed, smooth G-invariant subscheme of V and also I^{fix}, I^{mv} are G-invariant.

Let $\pi \colon \mathrm{bl}_{V^{G_0}} V \to V$ be the blowup of V along V^{G_0} with exceptional divisor E and local defining equation ξ. Then we take I^{intr} to be the ideal generated by $\pi^{-1}(I^{fix})$ and $\xi^{-1}\pi^{-1}(I^{mv})$. Everything is G-equivariant and we define U^{intr} as the subscheme of $\mathrm{bl}_{V^{G_0}} V$ defined by the ideal I^{intr}.

Finally, we need to delete unstable points. By the Hilbert-Mumford criterion (cf. [MFK94, Theorem 2.1]*) it follows that semistability on E with* *respect to the action of G is the same as semistability with respect to the action of G_0, since every 1-parameter subgroup of G factors through G_0, and hence we may delete unstable points exactly as before and define the Kirwan blowup $\widehat{U}$.*

One may check in a straightforwardly analogous way that this has the same properties (and intrinsic nature). It is obvious that if G is connected we obtain Definition 3.4.

3.2. **Kirwan partial desingularization for stacks with good moduli spaces.** We now generalize the construction of Kirwan partial desingularization to the case of stacks with good moduli spaces.

Theorem 3.7. *Let $\mathcal{M}$ be an Artin stack of finite type over $\mathbb{C}$ with affine diagonal. Moreover, suppose that $\pi\colon \mathcal{M} \to M$ is a good moduli space morphism with π of finite type and with affine diagonal. Then there exists a canonical DM stack $\widetilde{\mathcal{M}}$, called the Kirwan partial desingularization of $\mathcal{M}$, together with a morphism $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$. Moreover, $\widetilde{\mathcal{M}}$ admits a good moduli space $\widetilde{M}$ and the induced morphism $\widetilde{\mathcal{M}} \to \widetilde{M}$ is proper.*

Proof. Let $\mathcal{M}^{max}$ be the substack of $\mathcal{M}$ whose points have stabilizers of the maximum possible dimension. This is a closed substack of $\mathcal{M}$ with good moduli space M^{max} (cf. [ER17, Appendix B] for details on the algebraic structure of $\mathcal{M}^{max}$). For any closed point $x \in \mathcal{M}^{max}$, applying Theorem 2.14, we have a cartesian diagram

$$\begin{tikzcd} {[U_x/G_x]} \arrow[r, "\Phi_x"] \arrow[d, "\pi_x"'] & \mathcal{M} \arrow[d, "\pi"] \\ U_x /\!\!/ G_x \arrow[r] & M. \end{tikzcd} \tag{3.2}$$

The morphisms Φ_x cover the locus $\mathcal{M}^{max}$. We may take the Kirwan blowup of each quotient stack $[U_x/G_x]$ to obtain good moduli space morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x /\!\!/ G_x$.

We need to check that these glue to give a stack $\mathcal{M}_1$ with a universally closed projection $\mathcal{M}_1 \to \mathcal{M}$ and a good moduli space $\mathcal{M}_1 \to M_1$. By the properties of the Kirwan blowup, the maximum stabilizer dimension of $\mathcal{M}_1$ will be lower than that of $\mathcal{M}$ and we may then repeat the procedure.

Suppose x, y are two closed points of $\mathcal{M}$ such that G_x, G_y are of maximum dimension. We obtain a cartesian diagram of stacks

$$\begin{tikzcd} {[U_x \times_{\mathcal{M}} U_y/(G_x \times G_y)]} \arrow[r] \arrow[d] & {[U_y/G_y]} \arrow[d] \\ {[U_x/G_x]} \arrow[r] & \mathcal{M} \end{tikzcd} \tag{3.3}$$

where $U_{xy} := U_x \times_{\mathcal{M}} U_y$ is an affine scheme. This is due to the cartesian diagram

$$\begin{tikzcd} U_x \times_{\mathcal{M}} U_y \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta_{\mathcal{M}}"] \\ U_x \times U_y \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

and the fact that $\mathcal{M}$ has affine diagonal.

By the intrinsic nature of the Kirwan blowup, we obtain a diagram

$$\begin{tikzcd} & {[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[dl] \arrow[dr] & \\ {[\widehat{U}_x/G_x]} & & {[\widehat{U}_y/G_y]} \end{tikzcd} \tag{3.4}$$

with affine, étale arrows and moreover there are canonical isomorphisms between $[\widehat{U}_{xy}/(G_x \times G_y)]$ and $[\widehat{U}_x/G_x] \times_{\mathcal{M}} [U_y/G_y]$ and $[U_x/G_x] \times_{\mathcal{M}} [\widehat{U}_y/G_y]$.

Using the charts $[\widehat{U}_x/G_x]$ together with a cover of $\mathcal{M} \setminus \mathcal{M}^{max}$, we therefore obtain an atlas for a stack $\mathcal{M}_1$ with a map to $\mathcal{M}$. By the canonical isomorphisms of the previous paragraph, $\mathcal{M}_1$ is independent of the particular choices of charts for $\mathcal{M}$.

Note that the morphisms Φ_x, Φ_y are stabilizer preserving and induce étale morphisms at the level of good moduli spaces. It follows that all arrows in (3.3) are stabilizer preserving and thus both arrows in diagram (3.4) are stabilizer preserving and étale and therefore must be strongly étale (cf. for example [AHR19, Section 3.6] for more details on strongly étale morphisms). We thus obtain a corresponding diagram of étale arrows at the level of good moduli spaces of the Kirwan blowups

$$\begin{tikzcd} & {[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[dl] \arrow[dr] \arrow[dd] & \\ {[\widehat{U}_x/G_x]} \arrow[dd] & & {[\widehat{U}_y/G_y]} \arrow[dd] \\ & \widehat{U}_{xy}/\!\!/(G_x \times G_y) \arrow[dl] \arrow[dr] & \\ \widehat{U}_x/\!\!/G_x & & \widehat{U}_y/\!\!/G_y \end{tikzcd}$$

where both rhombi are cartesian.

By Proposition 2.13, $\mathcal{M} \setminus \pi^{-1}(M^{max}) \to M \setminus M^{max}$ is a good moduli space morphism. Hence the morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x/\!\!/G_x$ for all $x \in \mathcal{M}^{max}$ together with an atlas of $\mathcal{M} \setminus \pi^{-1}(M^{max})$ glue to give a morphism $\mathcal{M}_1 \to M_1$. By Proposition 2.13, this is a good moduli space morphism.

$\mathcal{M}_1$ has affine diagonal since we have a cartesian diagram

$$\begin{tikzcd}
{[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[d] \\
{[\widehat{U}_x/G_x] \times [\widehat{U}_y/G_y]} \arrow[r] & \mathcal{M}_1 \times \mathcal{M}_1
\end{tikzcd}$$

where the lower horizontal arrows give an étale cover of $\mathcal{M}_1 \times \mathcal{M}_1$ and the left vertical arrow is affine.

To see that $\mathcal{M}_1 \to M_1$ also has affine diagonal, we consider the diagram

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r] \arrow[dr] & \mathcal{M}_1 \times_{M_1} \mathcal{M}_1 \arrow[r] \arrow[d] & M_1 \arrow[d] \\
& \mathcal{M}_1 \times \mathcal{M}_1 \arrow[r] & M_1 \times M_1
\end{tikzcd}$$

where the right square is cartesian. The diagonal of M_1 is an immersion and hence in particular separated. Therefore, since the diagonal of $\mathcal{M}_1$ is affine, it follows by the usual cancellation property that $\mathcal{M}_1 \to M_1$ has affine diagonal.

$\mathcal{M}_1$, M_1 and the morphism $\mathcal{M}_1 \to M_1$ have the same properties as $\mathcal{M}$, M and $\mathcal{M} \to M$ and hence we may continue inductively to obtain the partial Kirwan desingularization $\widetilde{\mathcal{M}}$ and its good moduli space $\widetilde{\mathcal{M}} \to \widetilde{M}$.

Finally, $\widetilde{\mathcal{M}} \to \widetilde{M}$ admits an étale cover by morphisms of the form $[U/G] \to U/\!\!/G$ where G is finite, so its diagonal is finite (cf. [MFK94, Proposition 0.8]). Thus $\widetilde{\mathcal{M}} \to \widetilde{M}$ is separated and by Proposition 2.13 also universally closed, hence proper. □

Remark 3.8. *The fact that $\mathcal{M}$ and $\pi\colon \mathcal{M} \to M$ have affine diagonal is not crucial for the above proof to go through. In fact, by* [AHR19, Corollary 13.11] *and* [AHR19, Theorem 13.1]*, it is enough to assume that $\mathcal{M}$ has affine stabilizers and separated diagonal. Since the stacks we are interested in will have affine diagonal, we make this assumption for convenience of presentation.*

Remark 3.9. *Edidin-Rydh have also developed a desingularization procedure for stacks with good moduli spaces in* [ER17]*. For smooth stacks, our desingularization is the same as theirs. For singular stacks, Kirwan blowups can be phrased in their language of saturated blowups, however the desingularization they obtain is a closed substack of the one here. It would be interesting to investigate the reasons behind this discrepancy and whether they have to do with phenomena in derived algebraic geometry.*

4. Obstruction Theory

In this section, we give a description of the induced semi-perfect obstruction theory on the Kirwan partial desingularization of a stack of DT type. We begin by briefly recalling the definition of local models, their behaviour under Kirwan blowups and their reduced obstruction theory described in [KLS17]. We then observe that the results from [KLS17] apply to stacks of DT type and hence show that these glue to a semi-perfect obstruction theory of virtual dimension zero.

4.1. **Local models and standard forms.** Let V be a smooth affine G-scheme. The action of G on V induces a morphism $\mathfrak{g} \otimes \mathcal{O}_V \to T_V$ and its dual $\sigma_V : \Omega_V \to \mathfrak{g}^\vee \otimes \mathcal{O}_V$.

We consider the following data on V.

Setup-Definition 4.1. [KLS17] *The quadruple (V, F_V, ω_V, D_V), where F_V is a G-equivariant vector bundle on V, ω_V an invariant section with zero locus $U = (\omega_V = 0) \subset V$ and $D_V \subset V$ an effective invariant divisor, satisfying:*

(1) *$\sigma_V(-D_V) : \Omega_V(-D_V) \to \mathfrak{g}^\vee(-D_V)$ factors through a morphism ϕ_V as shown*

$$\Omega_V(-D_V) \longrightarrow F_V \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V). \tag{4.1}$$

(2) *The composition $\phi_V \circ \omega_V$ vanishes identically.*

(3) *Let R be the identity component of the stabilizer group of a closed point in V with closed orbit. Let V^R denote the fixed point locus of R. Then $\phi_V|_{V^R}$ composed with the projection $\mathfrak{g}^\vee(-D_V) \to \mathfrak{r}^\vee(-D_V)$ is zero, where $\mathfrak{r}$ is the Lie algebra of R.*

gives rise to data

$$\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$$

on V. We say that these data give a weak local model structure for V. We also say that U is in weak standard form.

Remark 4.2. *Note that if $f \colon V \to \mathbb{A}^1$ is a G-invariant function on V, then $(U, V, \Omega_V, df, 0, \sigma_V)$ give a weak local model for V, being equivalent to an invariant d-critical chart (U, V, f, i) for U. Therefore, an invariant d-critical locus is a particular case of weak standard form.*

In the rest of the paper, we will omit the adjective "weak" for convenience.

4.2. **Kirwan blowup of local model.** Let $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ define a local model structure on V.

Since G is reductive, we have a splitting

$$F_V|_{V^G} = F_V|_{V^G}^{fix} \oplus F_V|_{V^G}^{mv}.$$

Let now $\pi \colon \widehat{V} \to V$ be the Kirwan blowup of V. We define $F_{\widehat{V}}$ as the kernel of the composite morphism $\pi^* F_V \to \pi^*(F_V|_{V^G}) \to \pi^*\left(F_V|_{V^G}^{mv}\right)$ so that we have an exact sequence

$$0 \longrightarrow F_{\widehat{V}} \longrightarrow \pi^* F_V \longrightarrow \pi^*\left(F_V|_{V^G}^{mv}\right) \longrightarrow 0$$

By equivariance, $\pi^*\omega_V$ maps to zero under the second map and hence induces an invariant section $\omega_{\widehat{V}}$ of $F_{\widehat{V}}$. A local computation shows the following.

Proposition 4.3. *The zero locus of $\omega_{\widehat{V}}$, denoted by $\widehat{U}$, is exactly the Kirwan blowup of U with respect to G.*

By the results of [KLS17], we have the following proposition.

Proposition 4.4. *Let $(U, V, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model. Then there exist induced data $(\widehat{U}, \widehat{V}, F_{\widehat{V}}, \omega_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}})$ of a local model on the Kirwan blowup $\widehat{V}$. The same is true for any étale slice S of a closed point of $\widehat{V}$ with closed G-orbit.*

4.3. **Obstruction theory of local model.** Let $(U, V, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model structure. Consider the sequence

$$\mathfrak{g} \to T_V|_U \xrightarrow{(d_V\omega_V^\vee)^\vee} F_V|_U \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V). \tag{4.2}$$

By [KLS17], this defines a complex. Let T_V^{red} be the cokernel of the first arrow and F_V^{red} the kernel of ϕ_V. By Definition 4.1, ω_V factors through a section ω_V^{red} of F_V^{red} and the stable locus U^s is the zero locus of ω_V^{red} inside V^s. After restricting to U^s, the two arrows become injective and surjective respectively, so we obtain a 2-term complex of vector bundles on U^s

$$T_{V^s}^{\mathrm{red}}|_{U^s} \xrightarrow{\left(d_V(\omega_V^{\mathrm{red}})^\vee\right)^\vee} F_{V^s}^{\mathrm{red}}|_{U^s}. \tag{4.3}$$

If $q_V\colon V \to [V/G]$ is the standard quotient map, then $T_{V^s}^{\mathrm{red}} \cong q_V^* T_{[V^s/G]}$ and the complex descends to a perfect two-term complex on $[U^s/G]$, whose dual $E_{U^s}^{\mathrm{red}}$ defines a perfect obstruction theory (in the sense of [BF97]) on $[U^s/G]$.

Definition 4.5. *Let $(U, V, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model structure. Then $E_{U^s}^{\mathrm{red}}$ is the induced reduced obstruction theory on $[U^s/G]$. Its virtual dimension is equal to* $\dim V - \mathrm{rk}\, F_V$.

4.4. **Semi-perfect obstruction theory of stacks of DT type.** Let $\mathcal{M}$ be an Artin stack of DT type.

Performing the Kirwan partial desingularization procedure as in the proof of Theorem 3.7, we see that beginning with a collection of d-critical quotient charts covering $\mathcal{M}$ we finally get a DM stack $\widetilde{\mathcal{M}}$ with an étale cover by quotient charts $[T_\alpha/H_\alpha]$ where each T_α is in standard form for a local model $\Lambda_\alpha = (T_\alpha, S_\alpha, F_\alpha, \omega_\alpha, D_\alpha, \phi_\alpha)$ and H_α acts on T_α with finite stabilizers. By taking further étale slices, we may assume that each H_α is finite. This is due to the fact that at each step of the construction we either perform a Kirwan blowup or take a slice and hence Proposition 4.4 implies that the local model structure is carried over at each step. Each quotient chart carries a reduced obstruction theory given by the 2-term complex

$$E_\alpha = [F_\alpha^\vee|_{T_\alpha} \xrightarrow{d\omega_\alpha^\vee} \Omega_{S_\alpha}|_{T_\alpha}]. \tag{4.4}$$

These perfect obstruction theories can be compared by combining Proposition 2.19 together with the notions and results on Ω-equivalence and Ω-compatibility from [KLS17, Section 5], as carried out in [KLS17, Section 6]. The arguments apply verbatim to our context as well.

More precisely, keeping track of the data of Proposition 2.19 when applying Kirwan blowups and taking étale slices of quotient charts, for any two indices α, β we obtain an étale cover $\{R_\gamma \to T_\alpha \times_{\widetilde{\mathcal{M}}} T_\beta\}$ where each R_γ is in standard form for a local model $(R_\gamma, P_\gamma, F_\gamma, \omega_\gamma, 0, 0)$ (with the group being the trivial group) and has an induced perfect obstruction theory

$$E_\gamma = [F_\gamma^\vee|_{R_\gamma} \xrightarrow{d\omega_\gamma^\vee} \Omega_{P_\gamma}|_{R_\gamma}]$$

such that:

(1) We have commutative diagrams for $\lambda = \alpha, \beta$

$$\begin{tikzcd} R_\gamma \arrow[r] \arrow[d] & P_\gamma \arrow[d, "\Psi_\lambda"] \\ T_\lambda \arrow[r] & S_\lambda \end{tikzcd}$$

where the horizontal arrows are closed embeddings and $\Psi_\lambda \colon P_\gamma \to S_\lambda$ is unramified.

(2) There exists a surjective morphism $\eta_{\Psi_\lambda} \colon F_\lambda|_{P_\gamma} \to F_\gamma$ such that $\omega'_\lambda := \eta_{\Psi_\lambda}(\omega_\lambda|_{P_\gamma})$ and ω_γ are Ω-equivalent (cf. [KLS17, Definition 5.9]). This implies in particular the existence of an isomorphism as perfect obstruction theories between E_γ and

$$E'_\lambda = [F_\gamma^\vee|_{R_\gamma} \xrightarrow{d(\omega'_\lambda)^\vee} \Omega_{P_\gamma}|_{R_\gamma}].$$

(3) η_{Ψ_λ} induces an isomorphism of obstruction sheaves

$$\eta_{\Psi_\lambda} \colon h^1(E_\lambda^\vee)|_{R_\gamma} \longrightarrow h^1(E_\gamma^\vee).$$

The results of [KLS17, Section 6] extend to show that, using the above, the perfect obstruction theories E_α on the given étale cover of $\widetilde{\mathcal{M}}$ satisfy the compatibility axioms of a semi-perfect obstruction theory. We thus have the following theorem.

Theorem 4.6. *The reduced obstruction theories on $[T_\alpha/H_\alpha]$ induce a semi-perfect obstruction theory on the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ of virtual dimension zero. Thererefore $\widetilde{\mathcal{M}}$ admits a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$.*

For details on semi-perfect obstruction theories, we point the reader to [CL11].

5. Donaldson-Thomas Invariants of Derived Objects

In this section, W denotes a smooth, projective Calabi-Yau threefold over $\mathbb{C}$. We first describe the stability conditions σ on $D^b(\mathrm{Coh}\, W)$ that we will be interested in. We then quote recent results in [AHH18] which imply that moduli stacks of σ-semistable complexes are stacks of DT type and explain how to rigidify the $\mathbb{C}^*$-automorphisms of stable objects. Finally, we define generalized DT invariants via Kirwan blowups and show their deformation invariance.

5.1. Stability conditions.

By [Lie06], there is an Artin stack $\mathcal{P} := \mathrm{Perf}(W)$ of (universally gluable) perfect complexes on W, which is locally of finite type and has separated diagonal. Following [AHH18], we will consider the following type of stability condition.

Definition 5.1. (Stability condition) *A stability condition σ on $D^b(\mathrm{Coh}\, W)$ consists of the following data:*

(1) *A heart $\mathcal{A} \subset D^b(\mathrm{Coh}\, W)$ of a t-structure on $D^b(\mathrm{Coh}\, W)$. Let $\mathcal{P}_\mathcal{A}$ denote the stack of perfect complexes in $\mathcal{A}$.*

(2) *A vector $\gamma \in H^*(W, \mathbb{Q})$. Let $\mathcal{P}_\mathcal{A}^\gamma$ denote the stack of perfect complexes in $\mathcal{A}$ with Chern character γ.*

(3) *A locally constant function*

$$p_\gamma \colon \pi_0(\mathcal{P}_{\mathcal{A}}) \longrightarrow V$$

where V is a totally ordered abelian group, $p_\gamma(E) = 0$ for $E \in \mathcal{P}_{\mathcal{A}}^{\gamma}$ and p_γ is additive so that $p_\gamma(E \oplus F) = p_\gamma(E) + p_\gamma(F)$.

We say that $E \in \mathcal{P}_{\mathcal{A}}^{\gamma}$ is semistable if for any subobject $F \subset E$ we have $p_\gamma(F) \leq 0$ and stable if $p_\gamma(F) < 0$. If E is not semistable, we say it is unstable.

In order for the stack of semistable objects to be of DT type, we will need to consider stability conditions satisfying certain properties.

Definition 5.2. (Nice stability condition) *Given a stability condition σ on $D^b(\operatorname{Coh} W)$, let $\mathcal{M}$ be the stack of σ-semistable objects in $\mathcal{P}_{\mathcal{A}}^{\gamma}$. We say that σ is nice if the following hold:*

(1) *$\mathcal{M}$ is an Artin stack of finite type.*
(2) *$\mathcal{M}$ is an open substack of* $\operatorname{Perf}(W)$.
(3) *$\mathcal{M}$ satisfies the existence part of the valuative criterion of properness. We then say that $\mathcal{M}$ is quasi-proper or universally closed.*

Remark 5.3. *The following are examples of nice stability conditions, by the results of the mentioned authors:*

(1) *A Bridgeland stability condition in the sense of Toda-Piyaratne* [PT19] *and Li* [Li19].
(2) *A polynomial stability condition in the sense of Lo* [Lo11, Lo13].
(3) *Gieseker and slope stability. These are examples of a weak stability condition in the sense of Joyce-Song* [JS12, Definition 3.5]*, where we take $\mathcal{A} = \operatorname{Coh} W$ and $K(A) = N(W)$.*

5.2. **Moduli stacks of semistable complexes are of DT type.** The following theorem is an application of [AHH18, Theorem 7.25] to our context.

Theorem 5.4. *Let σ be a nice stability condition on $D^b(\operatorname{Coh} W)$ and let $\mathcal{M}$ denote the stack of σ-semistable complexes. Then $\mathcal{M}$ is an Artin stack of DT type with proper good moduli space M and affine diagonal.*

Proof. By the niceness of σ, $\mathcal{M}$ is of finite type. By the results of [PTVV13], $\operatorname{Perf}(W)$ is the truncation of a (-1)-shifted symplectic derived Artin stack. Since $\mathcal{M}$ is an open substack of $\operatorname{Perf}(W)$, it is also such a truncation. By [AHH18, Theorem 7.25], $\mathcal{M}$ admits a good moduli space $\pi\colon \mathcal{M} \to M$ such that M is separated. By Proposition 2.13, π is universally closed and since $\mathcal{M}$ is universally closed as well, it follows that M is universally closed. Thus M must be proper.

$\mathcal{M}$ has affine diagonal by [AHH18, Lemma 7.19]. □

5.3. **$\mathbb{C}^*$-rigidified Kirwan partial desingularization.** Let $\mathcal{M}$ be as in Theorem 5.4. We denote $T = \mathbb{C}^*$.

To obtain a meaningful nonzero DT invariant, it will be necessary to rigidify the $\mathbb{C}^*$-automorphisms of objects in $\mathcal{M}$.

For each family of complexes $E_S \in \mathcal{M}(S)$ there exists an embedding

$$\mathbb{G}_m(S) \to \operatorname{Aut}(E_S)$$

which is compatible with pullbacks and moreover $\mathbb{G}_m(S)$ is central. In the terminology used in [AGV08], we say that $\mathcal{M}$ has a $\mathbb{G}_m$-2-structure.

Using the results of [AOV08] or [AGV08], we may take the $\mathbb{G}_m$-rigification $\mathcal{M}/\!\!/\mathbb{G}_m$ of $\mathcal{M}$. From the properties of rigification, for any point $x \in \mathcal{M}$, one has $\mathrm{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \mathrm{Aut}_{\mathcal{M}}(x)/T$. In particular, if $x \in \mathcal{M}^s$ is stable, then $\mathrm{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \{\mathrm{id}\}$. $\mathcal{M}/\!\!/\mathbb{G}_m$ has the same good moduli space M and, as in Section 3, an étale cover by cartesian diagrams of the form

$$\begin{tikzcd} {[U_\alpha/(G_\alpha/T)]} \arrow[r] \arrow[d] & \mathcal{M}/\!\!/\mathbb{G}_m \arrow[d] \\ U_\alpha/\!\!/(G_\alpha/T) = U_\alpha/\!\!/G_\alpha \arrow[r] & M. \end{tikzcd}$$

We may now replace $\mathcal{M}$ by $\mathcal{M}/\!\!/\mathbb{G}_m$. Then we can carry out the Kirwan partial desingularization procedure and Theorem 3.7 produces a DM stack $\widetilde{\mathcal{M}}$ with good moduli space $\pi\colon \widetilde{\mathcal{M}} \to \widetilde{M}$ such that π is proper. Since M is proper and $\widetilde{M}$ is proper over M, $\widetilde{\mathcal{M}}$ is proper too. One may check that the results of Section 4 go through by identical arguments to produce a semi-perfect obstruction theory on $\widetilde{\mathcal{M}}$. We immediately get the following theorem.

Theorem-Definition 5.5. *$\widetilde{\mathcal{M}}$ is called the $\mathbb{C}^*$-rigidified Kirwan partial desingularization of $\mathcal{M}$. It is a proper DM stack with a semi-perfect obstruction theory of virtual dimension zero.*

Remark 5.6. *In the case of semistable sheaves treated in* [KLS17]*, rigidification is much simpler, since the moduli stack is a global GIT quotient $\mathcal{M} = [X/G]$ where $G = \mathrm{GL}(N, \mathbb{C})$, and then one may work with $[X/\mathrm{PGL}(N, \mathbb{C})]$ as a $\mathbb{G}_m$-rigidication.*

We can also first take the Kirwan partial desingularization and then rigidify: We may follow the same steps as in Section 3 to obtain an Artin stack $\widetilde{\mathcal{M}}'$ whose stabilizers are all of dimension one, by resolving all higher dimensional stabilizers in order of decreasing dimension. Moreover, for any object $E_S \in \widetilde{\mathcal{M}}'(S)$ we get an induced embedding $\mathbb{G}_m(S) \to \mathrm{Aut}(E_S)$, compatible with pullbacks, such that $\mathbb{G}_m(S)$ is central.

By construction, $\widetilde{\mathcal{M}}'$ has a good moduli space $\widetilde{M}$ and is covered by étale morphisms

$$[U_\alpha/G_\alpha] \to \widetilde{\mathcal{M}}'$$

where each $[U_\alpha/G_\alpha]$ is in standard form for data $\Lambda_\alpha = (U_\alpha, V_\alpha, F_\alpha, \omega_\alpha, \phi_\alpha)$. It has a $\mathbb{G}_m$-2-structure, compatible with the one of $\widetilde{\mathcal{M}}'$, so that $T \subset G_\alpha$ acts trivially on U_α.

Then we have the $\mathbb{G}_m$-rigification $\widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m$ of $\widetilde{\mathcal{M}}'$. From the properties of rigidification, for any point $x \in \widetilde{\mathcal{M}}$, one has $\mathrm{Aut}_{\widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m}(x) = \mathrm{Aut}_{\widetilde{\mathcal{M}}'}(x)/T$, and this is now zero-dimensional, hence finite. In particular, G_α/T is finite and $\widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m$ is then a DM stack, which has the same good moduli space $\widetilde{M}$. It has an étale cover of the form

$$[U_\alpha/(G_\alpha/T)] \to \widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m$$

Each $[U_\alpha/G_\alpha]$ comes with a 4-term complex

$$\mathfrak{g}_\alpha \to T_{V_\alpha}|_{U_\alpha} \xrightarrow{(d\omega_{V_\alpha}^\vee)^\vee} F_\alpha|_{U_\alpha} \xrightarrow{\phi_V} \mathfrak{g}_\alpha^\vee(-D_\alpha). \tag{5.1}$$

Setting $\mathfrak{t} = \mathrm{Lie}(T)$, we see that the compositions $\mathfrak{t} \to \mathfrak{g}_\alpha \to T_{V_\alpha}|_{U_\alpha}$ and $F_\alpha|_{U_\alpha} \to \mathfrak{g}_\alpha^\vee(-D_\alpha) \to \mathfrak{t}^\vee(-D_\alpha)$ are zero, since the T-action on V_α is trivial and using property (3) of Setup 4.1. It follows that we have an induced 4-term complex

$$\mathfrak{g}_\alpha/\mathfrak{t} \to T_{V_\alpha}|_{U_\alpha} \xrightarrow{(d\omega_{V_\alpha}^\vee)^\vee} F_\alpha|_{U_\alpha} \xrightarrow{\phi_V} (\mathfrak{g}_\alpha/\mathfrak{t})^\vee(-D_\alpha), \tag{5.2}$$

with injective first arrow and surjective last arrow, since $[U_\alpha/(G_\alpha/T)]$ is DM. As in Section 4, by taking the kernel and cokernel of these two arrows, this yields a reduced 2-term complex

$$E_\alpha = [(F_\alpha^{\mathrm{red}})^\vee|_{U_\alpha} \xrightarrow{d(\omega_\alpha^{\mathrm{red}})^\vee} \Omega_{V_\alpha}|_{U_\alpha}] \tag{5.3}$$

giving a reduced perfect obstruction theory on $[U_\alpha/(G_\alpha/T)]$. These glue to a semi-perfect obstruction theory on $\widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$.

It is easy to observe that the following proposition is true.

Proposition 5.7. *We have an isomorphism $\widetilde{\mathcal{M}} \simeq \widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$ and a commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathcal{M}}' \arrow[r] \arrow[d] & \widetilde{\mathcal{M}} \simeq \widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m \arrow[r] \arrow[d] & \widetilde{M} \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{M} /\!\!/ \mathbb{G}_m \arrow[r] & M,
\end{tikzcd}$$

where the vertical arrows are induced by the Kirwan desingularization procedure.

Remark 5.8. *By identical reasoning, all of the above hold in greater generality when $\mathcal{M}$ is an Artin stack of DT type with a $\mathbb{G}_m$-2-structure.*

5.4. **Generalized DT invariants via Kirwan blowups.** Suppose as before that $\mathcal{M}$ is an Artin stack of DT type parametrizing σ-semistable objects in a heart $\mathcal{A}$ of a t-structure on $D^b(\mathrm{Coh}\, W)$ with Chern character γ, for a nice stability condition σ. Then, by Theorem 3.7 and the previous subsection, there is an induced $\mathbb{C}^*$-rigidified Kirwan partial desingularization $\widetilde{\mathcal{M}}$ with a good moduli space $\widetilde{M}$ and, by Section 4, an induced semi-perfect obstruction theory and virtual cycle of dimension zero. We can now state our main theorem.

Theorem-Definition 5.9. *Let W be a smooth, projective Calabi-Yau threefold, σ a nice stability condition on a heart $\mathcal{A} \subset D^b(\mathrm{Coh}\, W)$ of a t-structure, as in Definition 5.1, $\gamma \in H^*(W)$ and let $\mathcal{M}$ denote the stack of σ-semistable complexes with Chern character γ. Then we may define the associated generalized Donaldson-Thomas invariant via Kirwan blowups as*

$$\mathrm{DTK}\,(\mathcal{M}) := \deg\ [\widetilde{\mathcal{M}}]^{\mathrm{vir}}.$$

Remark 5.10. *In* [KS20]*, the results of* [KLS17] *and the present paper are refined to define a virtual structure sheaf* $[\mathcal{O}^{\mathrm{vir}}_{\widetilde{\mathcal{M}}}] \in K_0(\widetilde{\mathcal{M}})$ *and a corresponding K-theoretic generalized DTK invariant.*

5.5. **Relative theory and deformation invariance.** Let S be a smooth quasi-projective scheme over $\mathbb{C}$ and $W \to S$ a smooth, projective family of Calabi-Yau threefolds. Without loss of generality, we assume that $H^*(W_t, \mathbb{Q})$ stays constant for $t \in S$ and identify it with $H^*(W_0, \mathbb{Q})$ where W_0 is the fiber of the family over a point $0 \in S$. Let $\gamma \in H^*(W_0, \mathbb{Q})$. Moreover, let $\mathrm{Perf}(W/S)$ denote the stack of (universally gluable) perfect complexes on the morphism $W \to S$ as in [Lie06].

We consider families σ of stability conditions σ_t, where, for each $t \in S$, σ_t is a stability condition on $D^b(\mathrm{Coh}\, W_t)$ as in Definition 5.2 with Chern character $\gamma \in H^*(W_t, \mathbb{Q}) = H^*(W_0, \mathbb{Q})$. Let $\mathcal{M} \to S$ be the stack parametrizing relatively semistable objects in $D^b(\mathrm{Coh}\, W)$, i.e. perfect complexes E such that the derived restriction $E_t := E|_{W_t}$ is σ_t-semistable for all $t \in S$. We require that the conditions characterizing a nice stability condition hold relative to the base S as follows.

Setup 5.11. *We say that the family of stability conditions σ_t is nice if the following hold.*

(1) $\mathcal{M}$ *is an Artin stack of finite type.*
(2) $\mathcal{M}$ *is an open substack of* $\mathrm{Perf}(W/S)$*.*
(3) $\mathcal{M} \to S$ *admits a good moduli space $M \to S$, proper over S.*

Remark 5.12. *When we have a GIT description of $\mathcal{M} \to S$, then the above conditions are satisfied. This is the case for Gieseker and slope stability of coherent sheaves.*

When we are in the situation of the above setup, all of our results extend to the relative case.

By (1) and (3), Theorem 3.7 and Subsection 5.3 yield a $\mathbb{C}^*$-rigidified Kirwan partial desingularization $\widetilde{\mathcal{M}} \to \mathcal{M}$ with a good moduli space $\widetilde{\mathcal{M}} \to \widetilde{M}$ such that $\widetilde{\mathcal{M}}$ and $\widetilde{M}$ are both proper over S.

By (2), the morphism $\mathcal{M} \to S$ is (-1)-shifted symplectic.

By [KLS17, Subsection 7.1], the Kirwan partial desingularization construction is compatible with base change, so that over $t \in S$ the fiber $\widetilde{\mathcal{M}}_t$ of $\widetilde{\mathcal{M}}$ is the ($\mathbb{C}^*$-rigidified) Kirwan partial desingularization of $\mathcal{M}_t$. Following the arguments of [KLS17, Section 7] verbatim, using the shifted symplectic structure of the morphism $\mathcal{M} \to S$ and a suitable notion of quotient charts relative to S, it follows that $\widetilde{\mathcal{M}} \to S$ admits a semi-perfect obstruction theory, which fiberwise over $t \in S$ induces the semi-perfect obstruction theory of $\widetilde{\mathcal{M}}_t$ constructed in Section 4. We thus obtain the following theorem.

Theorem 5.13. *Let $W \to S$ be a smooth, projective family of Calabi-Yau threefolds over a smooth, quasi-projective scheme S, $\{\sigma_t\}_{t\in S}$ a nice family of stability conditions on $D^b(\mathrm{Coh}\, W)$, $\gamma \in H^*(W_0, \mathbb{Q})$ and $\mathcal{M} \to S$ the stack of fiberwise σ_t-semistable objects of Chern character γ in $D^b(\mathrm{Coh}\, W)$.*

Then there exists an induced $\mathbb{C}^*$-rigidified Kirwan partial desingularization $\widetilde{\mathcal{M}} \to S$, a proper DM stack over S, endowed with a semi-perfect obstruction theory and a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$. For $t \in S$, $\widetilde{\mathcal{M}}_t$ is *the $\mathbb{C}^*$-rigidified Kirwan partial desingularization of $\mathcal{M}_t$ and the obstruction theory pulls back to the one constructed in the absolute case.*

In the case of Bridgeland stability conditions constructed in [PT19] and [Li19], the results of [BLM+19] imply that we get nice families of Bridgeland stability conditions.

As an immediate corollary, we have the following theorem.

Theorem 5.14. *The generalized DT invariant via Kirwan blowups for σ-semistable objects on Calabi-Yau threefolds, where σ is a Bridgeland stability condition as in* [PT19] *and* [Li19]*, is invariant under deformations of the complex structure of the Calabi-Yau threefold.*

Remark 5.15. *In the case of Gieseker stability and slope stability of coherent sheaves, deformation invariance follows from* [KLS17]*.*

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