How the perturbed vacuum might be completely decoupled from all physical states, allowing for a well-defined Hamiltonian formulation of QED

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Abstract

We carry out a Dirac sea reinterpretation of a discretized version of the Hamiltonian of quantum electrodynamics (QED), and analyze the perturbed vacuum in the continuum limit. We argue that if certain operators can be shown to be the self-adjoint, the perturbed vacuum will have solutions that converge nicely in this limit to states of the infinitesimal subspace of the Hilbert space spanned by momentum eigenstates whose momenta sum to 0. This convergence allows us to analyze what effect the perturbed vacuum has on the physical states of the system. We then show that the interaction and the interference between the perturbed vacuum and the physical states actually vanish in the continuum limit, due to the fact that the states of the perturbed vacuum only live in this vanishingly small part of the momentum space. We furthermore show that this allows us to remove the terms that perturb the vacuum from the Hamiltonian, without changing the dynamics of the physical states, thus yielding us a well-defined Hamiltonian for the full theory of QED.

1 Motivation

It is conventional to use the path integral formulation of quantum mechanics when working with quantum electrodynamics (QED), and other theories of quantum field theory (QFT), rather than using the Hamiltonian formulation. Not much attention is therefore typically given to the Hamiltonian of such theories. However, since QFT assumes that the time evolution is unitary, a generator of that time evolution must exist. Any mathematically consistent QFT must therefore have an underlying Hamiltonian.

Before the negative-energy fermion solutions of QED are reinterpreted as antiparticles, in accordance with the principles of the Dirac sea, the Hamiltonian of the theory *can* indeed be derived.[1] But after this Dirac sea reinterpretation, we seem to get an infinite amount of vacuum fluctuations, as particles are created and annihilated again from the bare vacuum. This makes it, at least at first glance, seem like an almost impossible task to describe the theory using the Hamiltonian formulation.

It is therefore quite natural to want to abandon this approach, especially when the alternative one of the path integral formulation is available to us. However, finding the full Hamiltonian

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[1] See e.g. Weinberg [4], or see Damgaard [1].

of QED might lead to some new insights. And in particular, it might give us more insight into how the perturbed vacuum affects the dynamics of the *physical states* of the theory, i.e. the states which we might prepare and do experiments on in a laboratory.

In fact, the analysis of this paper into the Hamiltonian formulation of QED suggests that the perturbed vacuum actually has no effect at all on the physical states, and that the interaction terms that cause these vacuum fluctuations can simply be removed for the theory, without changing its dynamics.

2 The initial Hamiltonian

Before the Dirac sea reinterpretation, where the negative-energy fermion eigenstates are reinterpreted as antiparticles, the Hamiltonian of QED is formally given by[1]

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{D}} + \hat{H}_{\mathrm{C}} \tag{1}$$

on a Fock space, $\mathbf{H}$, where $\hat{H}_0$ is the free energy operator, and where $\hat{H}_{\mathrm{D}}$ and $\hat{H}_{\mathrm{C}}$ are the operators of the Dirac interaction and the Coulomb interaction, respectively. Note that while it is not typical to see the Coulomb interaction appear explicitly in the path integrals of QFT, this is not because it is not there: It is simply absorbed into the Feynman propagator of the photons (see e.g. Weinberg [4]).

The exact definitions of $\hat{H}_{\mathrm{D}}$ and $\hat{H}_{\mathrm{C}}$ are not important for the argument presented in this paper, but for the interested reader, they are elaborated upon in Appendix A. The important thing for our purposes is just to note that $\hat{H}_{\mathrm{D}}$ and $\hat{H}_{\mathrm{C}}$ have the general forms given by

$$\hat{H}_{\mathrm{D}} = \sum_{\lambda=1}^{2} \sum_{s,s'=1}^{4} \int \frac{d\mathbf{k}\, d\mathbf{p}}{(2\pi)^6} \frac{\alpha_{\lambda,s,s'}(\mathbf{k},\mathbf{p})}{\sqrt{|\mathbf{k}|}} \big(\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda^\dagger(-\mathbf{k})\big) \hat{b}_{s'}^\dagger(\mathbf{p}+\mathbf{k}) \hat{b}_s(\mathbf{p}), \tag{2}$$

$$\hat{H}_{\mathrm{C}} = \sum_{\lambda=1}^{2} \sum_{s,s',t,t'=1}^{4} \int \frac{d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^9} \frac{\beta_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)}{\mathbf{k}^2} \hat{b}_{s'}^\dagger(\mathbf{p}_1+\mathbf{k}) \hat{b}_{t'}^\dagger(\mathbf{p}_2-\mathbf{k}) \hat{b}_s(\mathbf{p}_1) \hat{b}_t(\mathbf{p}_2), \tag{3}$$

with $\alpha_{\lambda,s,s'}$ and $\beta_{\lambda,s,s',t,t'}$ being some bounded complex functions. Here $\hat{a}_\lambda^\dagger(\mathbf{k})$ and $\hat{a}_\lambda(\mathbf{k})$ are of course the creation and annihilation operators for the photons, and $\hat{b}_s^\dagger(\mathbf{p})$ and $\hat{b}_s(\mathbf{p})$ are the creation and annihilation operators for some fermions with 4 spin components.[2] And for the free energy operator, we have

$$\hat{H}_0 = \sum_{\lambda=1}^{2} \int \frac{d\mathbf{k}}{(2\pi)^3} |\mathbf{k}| \hat{a}_\lambda^\dagger(\mathbf{k}) \hat{a}_\lambda(\mathbf{k}) + \int \frac{d\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \Big(\sum_{s=1}^{2} \hat{b}_s^\dagger(\mathbf{p}) \hat{b}_s(\mathbf{p}) - \sum_{s=3}^{4} \hat{b}_s^\dagger(\mathbf{p}) \hat{b}_s(\mathbf{p}) \Big), \tag{4}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m_{\mathrm{F}}^2}$, with m_{F} being the mass of the fermions.

We will assume in this paper that $\hat{H}$ is self-adjoint on some dense domain of $\mathbf{H}$, and we will furthermore assume that we are free to make an ultraviolet cutoff, $|\mathbf{p}|, |\mathbf{k}| < \Lambda$, for $\hat{H}$, as well as for all other Hamiltonians appearing in this paper, and then expect the dynamics of these cut-off operators to converge to those of the original ones when $\Lambda \to \infty$.

[2] It is conventional to use the symbol d instead of b for the two spin indices of the negative-energy eigenstates, $s \in \{3, 4\}$, as is also seen in Appendix A. But here we use b for all spin indices in order to write these expressions more compactly.

3 Discretizing the operators

Before we can make the Dirac sea reinterpretation, where we want to turn $\hat{b}_3^\dagger(\mathbf{p}), \hat{b}_4^\dagger(\mathbf{p})$ into $\hat{b}_3(-\mathbf{p}), \hat{b}_4(-\mathbf{p})$ and vice versa, we will first need to discretize all the momenta. Let us therefore introduce the discretized creation and annihilation operators, $\hat{a}_{\lambda\mathbf{k}}^\dagger$, $\hat{a}_{\lambda\mathbf{k}}$, $\hat{b}_{s\mathbf{p}}^\dagger$, $\hat{b}_{s\mathbf{p}}$,[3] where $\mathbf{k}$ and $\mathbf{p}$ now range over a discrete and bounded set, $\mathbb{K}$, which we can define by

$$\mathbb{K} = \left\{\mathbf{k} \in \mathbb{R}^3 \,\middle|\, \mathbf{k} = \delta k \mathbf{z},\ \mathbf{z} \in \mathbb{Z}^3 \setminus \{(0,0,0)\},\ |\mathbf{k}| < \Lambda\right\} \tag{5}$$

for some lattice spacing δk, and for the ultraviolet cutoff of Λ.

With this discretization, we then need to replace the creation and annihilation operators of Eqs. (2–4) such that all transition probabilities converge when we let $\delta k \to 0$. Damgaard [1] (Section 8) shows a heuristic derivation of the following rules to ensure this:

$$\begin{gathered}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\, \hat{a}_\lambda^\dagger(\mathbf{k}) \to \sum_{\mathbf{k}\in\mathbb{K}} \delta k^{3/2} \hat{a}_{\lambda\mathbf{k}}^\dagger, \quad \int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\, \hat{a}_\lambda(\mathbf{k}) \to \sum_{\mathbf{k}\in\mathbb{K}} \delta k^{3/2} \hat{a}_{\lambda\mathbf{k}}, \\ \int \frac{d\mathbf{k}}{(2\pi)^3}\, \hat{a}_\lambda^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k}) \to \sum_{\mathbf{k}\in\mathbb{K}} \hat{a}_{\lambda\mathbf{k}}^\dagger \hat{a}_{\lambda\mathbf{k}}.\end{gathered} \tag{6}$$

And these results can be generalized by observing that for any transitions from an initial state, $|\mathbf{p}_1, \ldots, \mathbf{p}_n; \mathbf{k}_1, \ldots, \mathbf{k}_m\rangle$ to a final state, $|\mathbf{p}_1', \ldots, \mathbf{p}_{n'}'; \mathbf{k}_1', \ldots, \mathbf{k}_{m'}'\rangle$, we need to have

$$\langle \mathbf{p}_1', \ldots, \mathbf{p}_{n'}'; \mathbf{k}_1', \ldots, \mathbf{k}_{m'}' | \hat{A} | \mathbf{p}_1, \ldots, \mathbf{p}_n; \mathbf{k}_1, \ldots, \mathbf{k}_m\rangle \propto \delta k^{\Delta/2+d} \tag{7}$$

for the relevant matrix element of a given operator $\hat{A}$, where Δ, first of all, denotes the difference between the degrees of freedom of the final and the initial state. This will typically be equal to $n' - n + m' - m$, unless the initial or finial states have some Dirac delta function(s) in them. And d, second of all, denotes the degrees of freedom for the initial state in terms of how $(\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{k}_1, \ldots, \mathbf{k}_m)$ is allowed to vary given that $(\mathbf{p}_1', \ldots, \mathbf{p}_{n'}', \mathbf{k}_1', \ldots, \mathbf{k}_{m'}')$ is fixed for the final state.

The reasoning behind this general rule is that if the states in the neighborhood of the initial point $(\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{k}_1, \ldots, \mathbf{k}_m)$ have amplitudes that are proportional to $\delta k^{3a/2}$, then the states in the neighborhood of the final point $(\mathbf{p}_1', \ldots, \mathbf{p}_{n'}', \mathbf{k}_1', \ldots, \mathbf{k}_{m'}')$ need to get amplitudes that are proportional to $\delta k^{3(\Delta+a)/2}$ in order for probabilities to be preserved in the continuum limit. So the initial guess might be that the matrix elements should be proportional to simply $\delta k^{\Delta/2}$. However, when there are d degrees of freedom for the initial states going to the same, fixed final state, the sum over these degrees of freedom will yield a value that grows proportional to δk^{-d}. This is why we also need to add d to the exponent on the right-hand side of Eq. (7) in order to get the correct normalization.

[3] Such discretized operators can be constructed using step functions (in momentum space), which thus allows us to remain in the same Hilbert space.

So for the discretized versions of these operators, we can see that we need to have[4]

$$\hat{H}'_0 = \sum_{\lambda=1}^{2} \sum_{\mathbf{k}\in\mathbb{K}} |\mathbf{k}| \hat{a}^\dagger_{\lambda\mathbf{k}} \hat{a}_{\lambda\mathbf{k}} + \sum_{\mathbf{p}\in\mathbb{K}} E_{\mathbf{p}} \Big(\sum_{s=1}^{2} \hat{b}^\dagger_{s\mathbf{p}} \hat{b}_{s\mathbf{p}} - \sum_{s=3}^{4} \hat{b}^\dagger_{s\mathbf{p}} \hat{b}_{s\mathbf{p}} \Big), \tag{8}$$

$$\hat{H}'_{\mathrm{D}} \propto \sum_{\lambda=1}^{2} \sum_{s,s'=1}^{4} \sum_{\mathbf{k},\mathbf{p}\in\mathbb{K}} \delta k^{3/2} \frac{\alpha_{\lambda,s,s'}(\mathbf{k},\mathbf{p})}{\sqrt{|\mathbf{k}|}} (\hat{a}_{\lambda\mathbf{k}} + \hat{a}^\dagger_{\lambda(-\mathbf{k})}) \hat{b}^\dagger_{s'(\mathbf{p}+\mathbf{k})} \hat{b}_{s\mathbf{p}}, \tag{9}$$

$$\hat{H}'_{\mathrm{C}} \propto \sum_{\lambda=1}^{2} \sum_{s,s',t,t'=1}^{4} \sum_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2\in\mathbb{K}} \delta k^{3} \frac{\beta_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)}{\mathbf{k}^2} \hat{b}^\dagger_{s'(\mathbf{p}_1+\mathbf{k})} \hat{b}^\dagger_{t'(\mathbf{p}_2-\mathbf{k})} \hat{b}_{s\mathbf{p}_1} \hat{b}_{t\mathbf{p}_2}. \tag{10}$$

The factor of $\delta k^{3/2}$ here in $\hat{H}'_{\mathrm{D}}$ thus comes from having respectively $\Delta/2 + d = -3/2 + 3$ and $\Delta/2 + d = 3/2 + 0$ for the two terms. And the factor of δk^3 in $\hat{H}'_{\mathrm{C}}$ comes from having $\Delta/2 + d = 0 + 3$.

4 The Dirac sea reinterpretation

We will now try to turn $\hat{b}^\dagger_{3\mathbf{p}}, \hat{b}^\dagger_{4\mathbf{p}}$ into $\hat{b}_{3(-\mathbf{p})}, \hat{b}_{4(-\mathbf{p})}$ and vice versa in accordance with the principles of the Dirac sea reinterpretation.

If we start by considering $\hat{H}'_0$, we see that we thus need to turn

$$\sum_{s=3}^{4} \hat{b}^\dagger_{s(-\mathbf{p})} \hat{b}_{s(-\mathbf{p})} \to \sum_{s=3}^{4} \hat{b}_{s\mathbf{p}} \hat{b}^\dagger_{s\mathbf{p}} = \sum_{s=3}^{4} (1 - \hat{b}^\dagger_{s\mathbf{p}} \hat{b}_{s\mathbf{p}}). \tag{11}$$

Due to our ultraviolet cutoff, $|\mathbf{p}| < \Lambda$, the first term of 1 within this sum on the right-hand side will only yield a constant energy, C. And since we are free to remove this constant energy at every step when we let $\delta k \to 0$, $\Lambda \to \infty$, without changing the dynamics of the system, we can thus write

$$\hat{H}''_0 = \sum_{\lambda=1}^{2} \sum_{\mathbf{k}\in\mathbb{K}} |\mathbf{k}| \hat{a}^\dagger_{\lambda\mathbf{k}} \hat{a}_{\lambda\mathbf{k}} + \sum_{s=1}^{4} \sum_{\mathbf{p}\in\mathbb{K}} E_{\mathbf{p}} \hat{b}^\dagger_{s\mathbf{p}} \hat{b}_{s\mathbf{p}} \tag{12}$$

for this discretized and reinterpreted $\hat{H}_0$.

If we then consider $\hat{H}'_{\mathrm{D}}$ and $\hat{H}'_{\mathrm{C}}$, we see that the creation and annihilation operators anti-commute almost everywhere, and we therefore do not need to add or subtract any constant energies here. Thus, for these operators, we can write similar formulas for the reinterpreted cases, namely

$$\hat{H}''_{\mathrm{D}} \propto \sum_{\lambda=1}^{2} \sum_{s,s'=1}^{4} \sum_{\mathbf{k},\mathbf{p}\in\mathbb{K}} \delta k^{3/2} \frac{\alpha_{\lambda,s,s'}(\mathbf{k},\mathbf{p})}{\sqrt{|\mathbf{k}|}} (\hat{a}_{\lambda\mathbf{k}} + \hat{a}^\dagger_{\lambda(-\mathbf{k})}) \hat{c}^\dagger_{s'(\mathbf{p}+\mathbf{k})} \hat{c}_{s\mathbf{p}}, \tag{13}$$

$$\hat{H}''_{\mathrm{C}} \propto \sum_{\lambda=1}^{2} \sum_{s,s',t,t'=1}^{4} \sum_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2\in\mathbb{K}} \delta k^{3} \frac{\beta_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)}{\mathbf{k}^2} \hat{c}^\dagger_{s'(\mathbf{p}_1+\mathbf{k})} \hat{c}^\dagger_{t'(\mathbf{p}_2-\mathbf{k})} \hat{c}_{s\mathbf{p}_1} \hat{c}_{t\mathbf{p}_2}, \tag{14}$$

but where $\hat{c}_{s\mathbf{p}} = \hat{b}_{s\mathbf{p}}$ when $s \in \{1,2\}$ and $\hat{c}_{s\mathbf{p}} = \hat{b}^\dagger_{s(-\mathbf{p})}$ when $s \in \{3,4\}$.

[4] We use proportional-to signs here instead of keeping track of the exact factors of $(2\pi)^a$.

5 Going back to the continuum limit

When we let $\delta k \to 0$, it would be nice if we could use the generalized rule of Eq. (6–7) in reverse and get a nice operator back with no divergences. However, this is not quite the case, as we will see in this section, since the terms that consist either exclusively of creation operators or of annihilation operators seem to cause us some trouble. These are in other words the terms that include either

$$\hat{a}_{\lambda\mathbf{k}}\hat{b}_{s'(-\mathbf{p}-\mathbf{k})}\hat{b}_{s\mathbf{p}}, \quad \hat{b}_{s'(-\mathbf{p}_1-\mathbf{k})}\hat{b}_{t'(-\mathbf{p}_2+\mathbf{k})}\hat{b}_{s\mathbf{p}_1}\hat{b}_{t\mathbf{p}_2}, \tag{15}$$

or their adjoint counterparts. Since these are the terms that perturb the vacuum, let us refer to such terms as the *vacuum-perturbing* terms in the remainder of this paper.

Before we analyze these troubling vacuum-perturbing terms further, let us discuss the other, less troubling terms first. We will refer to these as the *physical* terms of the operator for reasons that will be clear by the end of this analysis.

If we disregard the vacuum-perturbing terms of $\hat{H}''_{\mathrm{D}}$, we see that the rest of the terms either have two incoming particles and one outgoing particle, or have one incoming particle and two outgoing ones. We see that in all these cases, we have $\Delta/2 + d = 3/2$, which fits the factor of $\delta k^{3/2}$. So the sum of all these terms has a well-defined formal continuum limit, given by

$$\begin{aligned}\hat{H}'''_{\mathrm{D,phys}} = \sum_{\lambda=1}^{2}\sum_{s,s'=1}^{4}\int \frac{d\mathbf{k}\, d\mathbf{p}}{(2\pi)^6}\, \frac{\alpha_{\lambda,s,s'}(\mathbf{k},\mathbf{p})}{\sqrt{|\mathbf{k}|}}\big(\hat{a}_\lambda(\mathbf{k}) + \hat{a}^\dagger_\lambda(-\mathbf{k})\big)\hat{c}^\dagger_{s'}(\mathbf{p}+\mathbf{k})\hat{c}_s(\mathbf{p}) \\ - \text{ vacuum-perturbing terms.}\end{aligned} \tag{16}$$

(Here we take these continuous $\hat{c}_s$-operators to have the same relation to the $\hat{b}_s$-operators as in the discrete case above.)

If we then look at $\hat{H}''_{\mathrm{C}}$, we see that, apart from the vacuum-perturbing terms, we still have some terms with both two incoming and two outgoing particles, for which $\Delta/2 + d$ is thus still 3. Additionally, there are some terms with either three incoming and one outgoing particle, or with one incoming particle and three outgoing ones. Here we can see that we respectively have $\Delta/2+d = -6/2+6$ and $\Delta/2+d = 6/2+0$, which also gives us $\Delta/2+d = 3$ in both cases. This still fits the factor of δk^3, and the sum all of these terms therefore has a well-defined formal continuum limit as well, given by

$$\begin{aligned}\hat{H}'''_{\mathrm{C,phys}} = \sum_{\lambda=1}^{2}\sum_{s,s',t,t'=1}^{4}\int \frac{d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^9}\, \frac{\beta_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)}{\mathbf{k}^2}\hat{c}^\dagger_{s'}(\mathbf{p}_1+\mathbf{k})\hat{c}^\dagger_{t'}(\mathbf{p}_2-\mathbf{k})\hat{c}_s(\mathbf{p}_1)\hat{c}_t(\mathbf{p}_2) \\ - \text{ vacuum-perturbing terms.}\end{aligned} \tag{17}$$

Let us then look at the remaining, vacuum-perturbing terms, and see why we unfortunately cannot simply use the general rule of Eq. (6–7) in reverse for these. For the Dirac interaction, we here have either three incoming or outgoing particles with nothing on the other side. In particular, we can see that for the latter of these cases, due to momentum conservation, we have $\Delta/2 + d = (9-3)/2 + 0 = 3$. And this does not fit the factor of $\delta k^{3/2}$. Similarly, for the Coulomb interaction, we can see that we have $\Delta/2+d = (12-3)/2+0 = 9/2$ for the case when there are four outgoing particles and no incoming ones, which also does not fit the factor of δk^3. Thus, we cannot expect the vacuum-perturbing part of $\hat{H}''$ to have converging dynamics in the continuum limit, as its eigenvalues might very well diverge.

Before we abandon all hope of completing the Dirac sea reinterpretation, however, let us note that such diverging energies for the perturbed vacuum will only be a persisting problem if the states of the perturbed vacuum do not decouple from what we here refer to as the "physical states" in the continuum limit. If they do, on the other hand, we might be able to simply subtract this vacuum energy at every step going towards the continuum limit, leading us to converging dynamics for the physical states.

Motivated by this fact, let us consider the vacuum-perturbing terms separately. And since these terms can only cause transitions from the bare vacuum into other states with a combined momentum of 0, let us consider these operators on the subspace of $\mathbf{H}$, call it $\mathbf{H}_{\text{vac}}$, spanned by momentum eigenstates where all the momenta sum to 0.

On this subspace, we still have $\Delta/2 + d = (9-3)/2 + 0 = 3$ for the terms where there are three outgoing particles and no incoming particle, and $\Delta/2 + d = (12-3)/2 + 0 = 9/2$ for the terms where there are four outgoing particles. And for the adjoint of these terms, we then get $\Delta/2 + d = (-9+3)/2 + 6 = 3$ and $\Delta/2 + d = (-12+3)/2 + 9 = 9/2$, respectively. We thus see that all these terms are exactly a factor of $\delta k^{3/2}$ shy of fulfilling the rule of Eq. (7). So while $\hat{H}''_{\text{D,vac}}$ and $\hat{H}''_{\text{C,vac}}$ will not have converging formal continuum limits on $\mathbf{H}_{\text{vac}}$, we see that $\delta k^{3/2}\hat{H}''_{\text{D,vac}}$ and $\delta k^{3/2}\hat{H}''_{\text{C,vac}}$ *will*. These formal continuum limits of $\delta k^{3/2}\hat{H}''_{\text{D,vac}}$ and $\delta k^{3/2}\hat{H}''_{\text{C,vac}}$ on $\mathbf{H}_{\text{vac}}$, call them $\hat{H}'''_{\text{D,vac}}$ and $\hat{H}'''_{\text{C,vac}}$, will thus be given by

$$\hat{H}'''_{\text{D,vac}} = \sum_{\lambda=1}^{2} \sum_{s,s'=1}^{4} \int \frac{d\mathbf{k}\, d\mathbf{p}}{(2\pi)^6} \frac{\alpha'_{\lambda,s,s'}(\mathbf{k},\mathbf{p})}{\sqrt{|\mathbf{k}|}} \big(\hat{a}_\lambda(\mathbf{k})\hat{b}_{s'}(-\mathbf{p}-\mathbf{k})\hat{b}_s(\mathbf{p}) + \text{adjoint term}\big), \tag{18}$$

$$\begin{aligned}\hat{H}'''_{\text{C,vac}} = \sum_{\lambda=1}^{2} \sum_{s,s',t,t'=1}^{4} \int \frac{d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^9} \frac{\beta'_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)}{\mathbf{k}^2}& \\ \big(\hat{b}_{s'}(-\mathbf{p}_1-\mathbf{k})\hat{b}_{t'}(-\mathbf{p}_2+\mathbf{k})\hat{b}_s(\mathbf{p}_1)\hat{b}_t(\mathbf{p}_2) + \text{adjoint term}\big),&\end{aligned} \tag{19}$$

where all these creation and annihilation operators are then redefined as operators on $\mathbf{H}_{\text{vac}}$.[5] Note that $\alpha'_{\lambda,s,s'}(\mathbf{k},\mathbf{p})$ and $\beta'_{\lambda,s,s',t,t'}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)$ here are not the same coefficients as before. (In particular, they are both 0 whenever the relevant terms would otherwise break the law that the difference between the number of particles and antiparticles is conserved.)

Now, if the sum of these formal operators can be shown to be self-adjoint on some domain in $\mathbf{H}_{\text{vac}}$, the spectral theorem[6] will tell us that $\hat{H}'''_{\text{D,vac}} + \hat{H}'''_{\text{D,vac}}$ has ε-almost eigenstates[7] with real-valued eigenvalues. This would then suggest that, despite the divergences of $\hat{H}''_{\text{D,vac}} + \hat{H}''_{\text{C,vac}}$, the eigenstates of that operator, i.e. on the subspace of states whose momenta sum to 0, will still converge in the continuum limit, and that it is only the eigen*values* of these states which will diverge.

Of course, having diverging energies in a theory is still a problem, and therefore we still need for the perturbed vacuum to decouple from the physical states in the continuum limit in order for this Dirac sea reinterpretation to work. However, when we consider the perturbed vacuum, we see that it only lives on a very small part of the momentum space, whose volume vanishes when $\delta k \to 0$. Therefore it might indeed be the case that the interaction of the perturbed vacuum with the physical states of the system vanishes in the continuum limit. If this is true, and if the interference vanishes as well, we get that the perturbed vacuum *does* decouple from the physical states in the continuum limit, in which case the diverging energies will have no

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[5] One might note that the wave functions of this reduced Hilbert space should not be normalized according to $\int |\psi(\mathbf{p}_1,\ldots,\mathbf{p}_n;\mathbf{k}_1,\ldots,\mathbf{k}_m)|^2\, d\mathbf{p}_1\cdots d\mathbf{p}_n\, d\mathbf{k}_1\cdots d\mathbf{k}_m = P$, where P is some probability, but should have the following Dirac delta function in the integrand as well: $\delta^{3(n+m)}(\mathbf{p}_1+\ldots+\mathbf{p}_n+\mathbf{k}_1+\ldots+\mathbf{k}_m)$.

[6] See e.g. Hall [3].

[7] In accordance with Hall [3] (Definition 10.24), we take an *ε-almost eigenstate* of an operator, $\hat{A}$, with $\varepsilon > 0$, to refer to a state, ψ, for which $\|\hat{A}\psi - \lambda\psi\| < \varepsilon\|\psi\|$ for some *eigenvalue* $\lambda \in \mathbb{C}$.

effect on the dynamics of the latter. This is what we want to show, given a few mathematical assumptions, in the following section.

6 How the perturbed vacuum decouples from the physical states

Suppose that

$$\hat{H}'''_{\text{vac}}(\delta k) = \delta k^{3/2}\big(\hat{H}'''_0 + \hat{H}'''_{\text{D,phys}} + \hat{H}'''_{\text{C,phys}} + \delta k^{-3/2}(\hat{H}'''_{\text{D,vac}} + \hat{H}'''_{\text{C,vac}})\big) \tag{20}$$

is self-adjoint on some dense domain of $\mathbf{H}_{\text{vac}}$ for all $\delta k < 0$ and for $\delta k \to 0$, where $\hat{H}'''_0$ is the formal continuum limit of $\hat{H}''_0$. Let us also from now on refer to $\hat{H}''$ as $\hat{H}''(\delta k)$ instead to make its dependency on δk explicit. Then if we consider the part of $\hat{H}''(\delta k)$ that works only on states whose momenta sum to 0, we see that it approximates $\delta k^{-3/2}\hat{H}'''_{\text{vac}}(\delta k)$ arbitrarily well when δk tends towards 0. It is therefore not unreasonable to expect that for any generalized eigenstate, $|\phi\rangle$, of $\lim_{\delta k\to 0}\hat{H}'''_{\text{vac}}(\delta k) = \hat{H}'''_{\text{D,vac}} + \hat{H}'''_{\text{C,vac}}$ with eigenvalue E_ϕ, one can find a sequence of eigenstates[8] of $\hat{H}''(\delta k)$ such that the cross section of these eigenstates on $\mathbf{H}_{\text{vac}}$, when normalized at each step to preserve the overall norm, will converge to $|\phi\rangle$, and such that their eigenvalues, when divided by $\delta k^{-3/2}$ at each step, will converge to E_ϕ.

Let us assume this to be true for at least one such $|\phi\rangle$, and let $(|\phi_{\delta k}\rangle)_{\delta k^{-1}\in\mathbb{N}_+}$ be this sequence of eigenstates of $\hat{H}''(\delta k)$ that approximates $|\phi\rangle$. Let also $\delta k^{-3/2}E_{\phi_{\delta k}}$ be the eigenvalue of each $|\phi_{\delta k}\rangle$, with $E_{\phi_{\delta k}} \to E_\phi$ when $\delta k \to 0$.

Now, since $|\phi\rangle$ is a generalized state, it might contain Dirac delta functions in its formula. Let us, however, first assume that it does not, and analyze the interaction between $|\phi_{\delta k}\rangle$ and any physical states with this assumption. This then implies that the amplitude of $|\phi_{\delta k}\rangle$ over each point, $(\mathbf{p}_1,\ldots,\mathbf{p}_n,\mathbf{k}_1,\ldots,\mathbf{k}_m)$, in the discretized parameter space will go as $\delta k^{3(n+m-1)/2}$, where the -1 in the exponent comes from the fact that $\mathbf{p}_1+\ldots+\mathbf{p}_n+\mathbf{k}_1+\ldots+\mathbf{k}_m = 0$ for the support of $|\phi_{\delta k}\rangle$.

Let us then consider the ε-almost eigenstates[7] of the "physical" part of $\hat{H}''(\delta k)$. Here we will further make the assumption that

$$\hat{H}'''_{\text{phys}} = \hat{H}'''_0 + \hat{H}'''_{\text{D,phys}} + \hat{H}'''_{\text{C,phys}} \tag{21}$$

is self-adjoint on some dense domain of $\mathbf{H}$. We can then expect that for any ε-almost eigenstate, $|\psi\rangle$, of $\hat{H}'''_{\text{phys}}$ with eigenvalue E_ψ, the discretized version of this operator, i.e.

$$\hat{H}''_{\text{phys}}(\delta k) = \hat{H}''(\delta k) - \text{ vacuum-perturbing terms,} \tag{22}$$

will have a sequence of $\varepsilon_{\delta k}$-almost eigenstates, each with the same eigenvalue, E_ψ, that converges to $|\psi\rangle$ when $\delta k \to 0$, and where $\varepsilon_{\delta k} \to \varepsilon$ in the same limit as well. Let $(|\psi_{\delta k}\rangle)_{\delta k^{-1}\in\mathbb{N}}$ be such a sequence, and note furthermore that we are also free to assume that each $|\psi_{\delta k}\rangle$ includes no momentum eigenstates, $|\mathbf{p}_1,\ldots,\mathbf{p}_n;\mathbf{k}_1,\ldots,\mathbf{k}_m\rangle$, where there are three or more momentum vectors that sum to exactly 0. Since this only removes a combined state whose norm goes as $\delta k^{3/2}$, and since $\hat{H}''_{\text{phys}}(\delta k)$ approximates a nice, smooth operator, this will only add an $O(\delta k^{3/2})$ contribution to $\hat{H}''_{\text{phys}}(\delta k)\,|\psi_{\delta k}\rangle$. And therefore, it will not change the fact that $\varepsilon_{\delta k}$ can get arbitrarily close to ε when $\delta k \to 0$. With these assumptions, we see that since $|\psi\rangle$ is an

[8] Since $\hat{H}''(\delta k)$ is discrete and bounded, its generalized eigenstates will be actual eigenstates in $\mathbf{H}$.

actual state in $\mathbf{H}$, we get that the amplitude of $|\psi_{\delta k}\rangle$ over each point $(\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{k}_1, \ldots, \mathbf{k}_m)$ in the discretized parameter space will go as $\delta k^{3(n+m)/2}$.

Define then $\hat{A}^\dagger_{\phi_{\delta k}}$ and $\hat{A}^\dagger_{\psi_{\delta k}}$ as the combined creation operators that respectively create $|\phi_{\delta k}\rangle$ and $|\psi_{\delta k}\rangle$ from the bare vacuum. In other words, let

$$\hat{A}^\dagger_{\phi_{\delta k}} |\rangle = |\phi_{\delta k}\rangle, \quad \hat{A}^\dagger_{\psi'_{\delta k}} |\rangle = |\psi_{\delta k}\rangle. \tag{23}$$

And define further $|\chi_{\delta k}\rangle$ by

$$|\chi_{\delta k}\rangle = \hat{A}^\dagger_{\psi_{\delta k}} \hat{A}^\dagger_{\phi_{\delta k}} |\rangle. \tag{24}$$

We then hope to show that $|\chi_{\delta k}\rangle$ is an ε'-almost eigenstate of $\hat{H}''(\delta k)$, where ε' can get arbitrarily small when $\delta k \to 0$, $\varepsilon \to 0$.

In order to do this, let us first note that each term of $\hat{H}''(\delta k)$ includes an even number of $\hat{b}_{s,\mathbf{p}}$ or $\hat{b}^\dagger_{s,\mathbf{p}}$-operators. This means that if we want to commute the terms of $\hat{H}''(\delta k)$ with the terms of $\hat{A}^\dagger_{\psi_{\delta k}}$, the anticommutation relations of the $\hat{b}_{s,\mathbf{p}}$ and $\hat{b}^\dagger_{s,\mathbf{p}}$-operators will work exactly as if they were the corresponding *commutation* relations instead. This tells us that the difference between $\hat{H}''(\delta k)\hat{A}^\dagger_{\psi_{\delta k}}$ and $\hat{A}^\dagger_{\psi_{\delta k}}\hat{H}''(\delta k)$ is exactly equal to all the terms where one or more annihilation operator of $\hat{H}''(\delta k)$ hits a creation operator in $\hat{A}^\dagger_{\psi_{\delta k}}$ with the same indices and turns into a 1. Now, if we look at the terms where *all* (and at least one) of the annihilation operators in $\hat{H}''(\delta k)$ hit some corresponding creation operators in $\hat{A}^\dagger_{\psi_{\delta k}}$, we see that this is exactly the terms that make up

$$\hat{H}''_{\text{phys}}(\delta k)\hat{A}^\dagger_{\psi_{\delta k}} |\rangle = E_\psi \hat{A}^\dagger_{\psi_{\delta k}} |\rangle + O(\varepsilon_{\delta k}). \tag{25}$$

Here $O(\varepsilon_{\delta k})$ is a shorthand for some state with norm less than $C\varepsilon_{\delta k}$ for some constant $C > 0$. The reason why the vacuum-perturbing terms do not contribute to these terms is that $\hat{A}^\dagger_{\psi_{\delta k}} |\rangle$, by assumption, does not include any momentum eigenstates where three or more momentum vectors sum to 0. Therefore the terms that include only annihilation operators will not be able to have all these operators hit corresponding creation operators. And of course, since we need to have at least one annihilation operator hit a creation operator, we can also exclude the terms that include only creation operators. We therefore get that

$$\begin{aligned} \hat{H}''(\delta k) |\chi_{\delta k}\rangle &= \hat{A}^\dagger_{\psi_{\delta k}} \hat{H}''(\delta k) \hat{A}^\dagger_{\phi_{\delta k}} |\rangle + E_\psi \hat{A}^\dagger_{\psi_{\delta k}} \hat{A}^\dagger_{\phi_{\delta k}} |\rangle + \text{interaction terms} + O(\varepsilon_{\delta k}) \\ &= (\delta k^{-3/2} E_{\phi_{\delta k}} + E_\psi) |\chi_{\delta k}\rangle + \text{interaction terms} + O(\varepsilon_{\delta k}), \end{aligned} \tag{26}$$

where we by "interaction terms" refer to all the terms where there is at least one annihilation operator from $\hat{H}''(\delta k)$ that hits a creation operator in $\hat{A}^\dagger_{\psi_{\delta k}}$ and also at least one that hits a creation operator in $\hat{A}^\dagger_{\phi_{\delta k}}$. So if we can only show that the norm of these combined "interaction terms" goes to zero when δk does, we obtain the result that we are looking for.

This is fortunately not too hard to do, as it turns out. As per the assumptions above, a state, $|\mathbf{p}_1, \ldots, \mathbf{p}_n; \mathbf{k}_1, \ldots, \mathbf{k}_m\rangle$, created by $\hat{A}^\dagger_{\phi_{\delta k}}$ will have amplitudes that go as $\delta k^{3(n+m-1)/2}$. And a state state, $|\mathbf{p}_1, \ldots, \mathbf{p}_{n'}; \mathbf{k}_1, \ldots, \mathbf{k}_{m'}\rangle$, created by $\hat{A}^\dagger_{\phi_{\delta k}}$ will have amplitudes that go as $\delta k^{3(n'+m')/2}$. This means that the amplitudes of any state, $|\mathbf{p}_1, \ldots, \mathbf{p}_{n''}; \mathbf{k}_1, \ldots, \mathbf{k}_{m''}\rangle$, of $|\chi_{\delta k}\rangle$ will have amplitudes that go as $\delta k^{3(n''+m''-1)/2}$, matching the local degrees of freedom: $3(n''+m''-1)$. We can then analyze the norms of the "interaction terms" using the same rule of Eq. (7) once again, as long as we just remember that Δ will then always be plus 3 of what it would normally be since the initial amplitude is proportional to $\delta k^{3(n''+m''-1)/2}$ rather than to $\delta k^{3(n''+m'')/2}$.

For the Dirac interaction, the relevant terms here are those with two annihilated particles and one created particle, as well as those with three annihilated particles and zero created ones. In the first case, we see that we have $\Delta/2 + d = (-3+3)/2 + 0 = 0$. However, the matrix elements are proportional to $\delta k^{3/2}$ rather than δk^0, which means that the combined norm of these contributions to $\hat{H}''(\delta k)\,|\chi_{\delta k}\rangle$ will vanish when $\delta k \to 0$. And if we consider the terms where three particles annihilate each other with no particles created, we see that we have $\Delta/2 + d = (-9+3)/2 + 3 = 0$ as well, which means that these contributions also vanish in the same limit. And for the Coulomb interaction, we first of all have the terms with both two annihilated and two created particles, for which we get that $\Delta/2 + d = (0+3)/2 + 0 = 3/2$. Then there are also the terms with three annihilated particles and a single particle created, for which we have $\Delta/2 + d = (-6+3)/2 + 3 = 3/2$. And lastly, we have the terms where there are four annihilated particles and no particles created, in which case we have $\Delta/2 + d = (-12+3)/2 + 6 = 3/2$ yet again. Therefore, since the matrix elements here are proportional to δk^3 rather than $\delta k^{3/2}$, the combined norm of these contributions will also vanish when $\delta k \to 0$.

We thus obtain that

$$\hat{H}''(\delta k)\,|\chi_{\delta k}\rangle = (\delta k^{-3/2} E_{\phi_{\delta k}} + E_\psi)\,|\chi_{\delta k}\rangle + O(\delta k^{3/2}) + O(\varepsilon_{\delta k}). \tag{27}$$

We thus get that $|\chi_{\delta k}\rangle$ will indeed be an ε'-almost eigenstate of $\hat{H}''(\delta k)$, where ε' can get arbitrarily small when $\delta k \to 0$, $\varepsilon \to 0$.

Before we move on to discuss what this means, however, we should also look at what happens if $|\phi\rangle$ *does* include some Dirac delta functions. Now, we have just seen how the fact that the limit of $(|\phi_{\delta k}\rangle)_{\delta k^{-1} \in \mathbb{N}_+}$ includes a delta function when viewed in $\mathbf{H}$, namely the one that makes the momenta sum to 0, results in an interaction that vanishes in the continuum limit. And it is therefore natural to expect that if this limit also includes additional delta functions, it will only make the interaction vanish more rapidly, if anything. And indeed, it is not hard to convince oneself of this. We can see that for any degree of freedom that we remove from $|\phi_{\delta k}\rangle$ in the initial state, this either removes a degree of freedom from the final state, in which case Δ is invariant, or it will decrease d by 1. And if d decreases by 1, then even if Δ also increases by 1 at the same time, it will still only ever reduce $\Delta/2 + d$ in total. So having delta functions in $|\phi\rangle$ will thus indeed only make the interaction terms vanish more rapidly, if anything, and Eq. (27) will therefore still apply.

This is a great result, since it implies that the dynamics of the physical states for $\hat{H}''(\delta k)$ converges to the dynamics of $\hat{H}'''_{\text{phys}}$ when $\delta k \to 0$. The way to see this is to note, first of all, that any state, $|\Psi\rangle$, of $\mathbf{H}$ can be approximated arbitrarily well by a sum of a finite set of ε-almost eigenstates, $\{|\psi_n\rangle\}_{n \in \{1,\ldots,N\}}$, such that $|\Psi\rangle \approx \sum_{n=1}^N \alpha_n\,|\psi_n\rangle$, where $\alpha_n \in \mathbb{C}$. And for each $|\psi_n\rangle$ with eigenvalue E_{ψ_n}, we can find a $|\chi_{n,\delta k}\rangle$, that is an ε'-almost eigenstate of the operator given by

$$\hat{H}''_{\phi_{\delta k}} = \hat{H}''(\delta k) - \delta k^{-3/2} E_{\phi_{\delta k}}, \tag{28}$$

with an eigenvalue of E_{ψ_n} as well. Thus, for any given final state, $|\Phi\rangle$, approximated by $|\Phi\rangle \approx \sum_{n=1}^N \beta_n\,|\psi_n\rangle$, we get that

$$\begin{aligned}
\langle\Phi|\, e^{-i\hat{H}'''_{\text{phys}}}\,|\Psi\rangle &\approx \sum_{m=1}^{N}\sum_{n=1}^{N} \beta_m^* \alpha_n\,\langle\psi_m|\, e^{-i\hat{H}'''_{\text{phys}}}\,|\psi_n\rangle \approx \sum_{n=1}^{N} \beta_n^* \alpha_n e^{-iE_{\psi_n}} \\
&\approx \sum_{m=1}^{N}\sum_{n=1}^{N} \beta_m^* \alpha_n\,\langle\chi_{m,\delta k}|\, e^{-i\hat{H}''_{\phi_{\delta k}}}\,|\chi_{n,\delta k}\rangle \\
&\approx \langle\,|\,\hat{A}_{\phi_{\delta k}} \hat{A}_\Psi e^{-i\hat{H}''_{\phi_{\delta k}}} \hat{A}_\Psi^\dagger \hat{A}_{\phi_{\delta k}}^\dagger\,|\,\rangle\,,
\end{aligned} \tag{29}$$

where $\hat{A}^{\dagger}_{\Psi}$ is the combined creation operator that creates $|\Psi\rangle$. When we let δk tend towards 0, and let $N \to \infty$ and $\varepsilon, \varepsilon_{\delta k} \to 0$ as well, we therefore see that any experiment in a system governed by $\hat{H}'''_{\text{phys}}$ can simulated to arbitrary precision using $\hat{H}''_{\phi_{\delta k}}$ instead, and vice versa!

The dynamics of the physical states in a system governed by $\hat{H}''(\delta k)$ thus converges to the dynamics of a system governed by $\hat{H}'''_{\text{phys}}$. And since $\hat{H}'''_{\text{phys}}$ includes no vacuum-perturbing terms, we thus get the the perturbed vacuum decouples from all the physical states in the continuum limit.

7 The Lorentz covariance of the resulting Hamiltonian

We have not yet argued why the procedure used in this paper for the Dirac sea reinterpretation is a correct approach. This is, however, not hard to justify, given our assumption above that in each of the steps of this procedure, going from $\hat{H} \to \hat{H}' \to \hat{H}'' \to \hat{H}'''_{\text{phys}}$, we can always impose an ultraviolet cutoff, $|\mathbf{p}|, |\mathbf{k}| < \Lambda$, and expect the dynamics to converge when we let $\Lambda \to \infty$. This assumption allows us to make Dyson expansions of the time evolution operators $\exp(-i\hat{H}')$ and $\exp(-i\hat{H}'')$, yielding us two scattering matrices that can both be written out purely as a linear combination of creation and annihilation operators. The point is then to note that all such terms of $\exp(-i\hat{H}'')$ will correspond to a similar term of $\exp(-i\hat{H}')$, with the same coefficients, since the Dirac sea reinterpretation does not change the commutation and anticommutation relations between the various creation and annihilation operators. This means that the scattering matrix of $\hat{H}''$ can be derived directly from that of $\hat{H}'$. So if the dynamics of $\hat{H}'$ become Lorentz-covariant in the continuum limit, and when we also let $\Lambda \to \infty$ at the same time, then the dynamics of $\hat{H}''$ must also become Lorentz-covariant in the same limit.[9]

And given that $\hat{H}$ of Eq. (1) is generally regarded as Lorentz-covariant to begin with,[10] i.e. with formulas of $\hat{H}_D$ and $\hat{H}_C$ as shown in Appendix A, we can therefore propose the following Hamiltonian of QED, call it $\hat{H}_{QED}$, given formally on $\mathbf{H}$ by

$$\hat{H}_{QED} = \hat{H}'''_{\text{phys}}. \tag{30}$$

Here we recall that $\hat{H}'''_{\text{phys}}$ is the operator obtained from $\hat{H}$ by first turning $\hat{b}^{\dagger}_3(\mathbf{p}), \hat{b}^{\dagger}_4(\mathbf{p})$ into $\hat{b}_3(-\mathbf{p}), \hat{b}_4(-\mathbf{p})$ and vice versa, and then removing all the vacuum-perturbing terms afterwards.

8 On confirming the mathematical assumptions

In order to get to $\hat{H}_{QED}$, we had to make some mathematical assumptions, in particular about the self-adjointness of $\hat{H}'''_{\text{vac}}(\delta k)$ and of $\hat{H}'''_{\text{phys}} \equiv \hat{H}_{QED}$, as well as their discretized versions. And we have also assumed that $\hat{H}$ is self-adjoint on some domain to begin with. These propositions are by no means trivial to show. However, Damgaard [2] does introduce a way that one might be able to show the self-adjointness of such operators. That paper thus proves the self-adjointness of a simplified Dirac interaction operator (with only one fermion and only one spin component), and the technique used for this proof might be extendable to more general operators as well. Thus, there is hope that these propositions can be shown in a near future.

[9] This argument can also be made even more compelling if one changes the basis of the creation and annihilation operators to a position basis.

[10] An overall argument for this can also be seen in Damgaard [1].

Additionally, there is also the matter of showing that we can always impose ultraviolet cutoffs for these operators, and then expect the ε-almost eigenstates, along with their corresponding eigenvalues, to converge to those of the original operators. This is not a trivial task either, but it might very well be possible. Appendix B briefly discusses a potential approach for doing this.

9 Conclusion

We went through the process of making the Dirac sea reinterpretation for the Hamiltonian of QED, and argued that while the energies of the perturbed vacuum seem to diverge in the continuum limit, there likely exist sequences of eigenstates of the perturbed vacuum which converge, at least given some mathematical assumptions about the self-adjointness of certain operators. We then showed that the interaction and interference between such perturbed vacuum states and the "physical states" of the system, i.e. the states which we might prepare and investigate in any realistic experiment, vanish in the continuum limit, namely due to the fact that the perturbed vacuum states only occupy a vanishingly small part of the momentum space. We discussed how this means that the physical states of the discretized and reinterpreted Hamiltonian, $\hat{H}''$, can then be simulated to arbitrary precision by an operator, $\hat{H}'''_{\text{phys}} \equiv \hat{H}_{QED}$, with all of the vacuum-perturbing terms removed. Finally, we argued that this operator then ought to be Lorentz-covariant, at least if we also make the assumption that the Hamiltonians in this paper can all be simulated to arbitrary precision by ones with ultraviolet cutoffs imposed on them. We can therefore propose this $\hat{H}_{QED}$ as a promising candidate for the Hamiltonian of the full theory of QED.

Appendices

A The Hamiltonian of QED before the Dirac sea reinterpretation

According to Damgaard [1], the continuous Hamiltonian of QED before the Dirac sea reinterpretation, on a Fock space of photons and some Dirac spinor fermions with a charge of q_{F}, is formally given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{D}} + \hat{H}_{\text{C}}, \tag{31}$$

with

$$\hat{H}_0 = \sum_{\lambda=1}^{2} \int \frac{d\mathbf{k}}{(2\pi)^3} \, |\mathbf{k}| \hat{a}^\dagger_\lambda(\mathbf{k}) \hat{a}_\lambda(\mathbf{k}) + \int \frac{d\mathbf{p}}{(2\pi)^3} \, E_{\mathbf{p}} \Big(\sum_{s=1}^{2} \hat{b}^\dagger_s(\mathbf{p}) \hat{b}_s(\mathbf{p}) - \sum_{s=3}^{4} \hat{d}^\dagger_s(\mathbf{p}) \hat{d}_s(\mathbf{p}) \Big) \tag{32}$$

$$\hat{H}_{\text{D}} = -q_{\text{F}} \sum_{\lambda=1}^{2} \int \frac{d\mathbf{k}\, d\mathbf{p}}{(2\pi)^6} \, \frac{1}{\sqrt{2|\mathbf{k}|}} \big(\hat{a}_\lambda(\mathbf{k}) + \hat{a}^\dagger_\lambda(-\mathbf{k}) \big) \hat{\bar{\psi}}(\mathbf{p}+\mathbf{k}) \gamma^\mu \epsilon_{\mu\lambda}(\mathbf{k}) \hat{\psi}(\mathbf{p}), \tag{33}$$

$$\hat{H}_{\text{C}} = \int \frac{d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^9} \, \frac{q_{\text{F}}^2}{2\mathbf{k}^2} \big(\hat{\bar{\psi}}(\mathbf{p}_1+\mathbf{k}) \otimes \hat{\bar{\psi}}(\mathbf{p}_2-\mathbf{k}) \big) (\gamma^0 \otimes \gamma^0) \big(\hat{\psi}(\mathbf{p}_1) \otimes \hat{\psi}(\mathbf{p}_2) \big), \tag{34}$$

and with

$$\begin{aligned} \hat{\psi}(\mathbf{p}) &= \frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big(\sum_{s=1}^{2} u_s(\mathbf{p})\hat{b}_s(\mathbf{p}) + \sum_{s=3}^{4} v_s(\mathbf{p})\hat{d}_s(\mathbf{p})\Big), \\ \hat{\bar{\psi}}(\mathbf{p}) &= \frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big(\sum_{s=1}^{2} \hat{b}_s^\dagger(\mathbf{p})\bar{u}_s(\mathbf{p}) + \sum_{s=3}^{4} \hat{d}_s^\dagger(\mathbf{p})\bar{v}_s(\mathbf{p})\Big). \end{aligned} \tag{35}$$

Most of the symbols here are either explained above or are well-known to any readers familiar with QFT. As mentioned in Footnote 2, we have simply used the symbols $\hat{b}_3$ and $\hat{b}_4$ instead of $\hat{d}_3$ and $\hat{d}_4$ in the body of this paper in order to write some of the expressions more compactly. Apart from this, the most important thing to note is just that the $\otimes$ symbols in Eq. (34) represent the matrix direct product. Since u_s, v_s are 4-by-1 column vectors (namely the spinor solutions to the Dirac equation) and $\bar{u}_s, \bar{v}_s$ are 1-by-4 row vectors, the integrand in this formula thus consists of a 1-by-16 row vector multiplied by a 16-by-16 matrix, $\gamma^0 \otimes \gamma^0$, multiplied by a 16-by-1 column vector.

For further elaboration on these formulas, see Damgaard [1].

B On showing that the ultraviolet cutoffs are valid

In this appendix, we will briefly discuss a potential approach for showing that the ultraviolet cutoffs used in this paper are valid, in the sense that the ε-almost eigenstates and eigenvalues converge when the cutoffs are lifted again.

Damgaard [2] uses an approach for constructing states that are mapped into normalizable states by the Hamiltonian in which transitions that create a particle (specifically a photon) are canceled by transitions that annihilate a particle. And in that paper, the momenta of the annihilated particles in these latter transitions take values in a set, E, whose radii grow very rapidly with increasing momenta and increasing photon number. However, one might also be able to use transitions where the momenta of the annihilated particles take values from an E set with constant radii instead.

Now, the trouble that we are facing for this task is that when we make an ultraviolet cutoff, we might ruin an ε-almost eigenstate's ability to cancel its own "creation transitions," as we might call them, since this might remove the "annihilation transitions" that the state otherwise used for this. So if we let the cut-off version of the Hamiltonian work on a given ε-almost eigenstate, it might actually (somewhat counterintuitively) result in state that is no longer normalizable.

But if the created particles can indeed also be canceled by using an E set with constant radii instead, the idea is then that for any ε-almost eigenstate, $|\psi\rangle$, of the original operator, one might be able to construct an ε'-almost eigenstate, $|\psi_\Lambda\rangle$, of the cut-off version by taking $|\psi\rangle$ and then adding a small state (a small "tail," so to speak) that conforms to this E. The job of this small "tail" is then to cancel all the "creation transitions" that are no longer canceled by the "annihilation transitions" due to the cutoff. One might then be able to show that this "tail" can be made arbitrarily small compared to the norm of $|\psi\rangle$ when $\Lambda \to \infty$, and thus that the sequence of $|\psi_\Lambda\rangle$'s will converge to $|\psi\rangle$, with $\varepsilon' \to \varepsilon$ and with eigenvalues of the sequence converging to that of $|\psi\rangle$ as well.

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