Shorter and simpler version of the paper: "Self-adjointness of a simplified Dirac interaction operator without any cutoffs"

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Abstract

This is a supplementary paper to (Damgaard, *Self-adjointness of a simplified Dirac interaction operator without any cutoffs*, arXiv:2311.12870 [quant-ph]), where we analyze a further simplified version of the same operator, namely one that does not leave the subspace of symmetric wave functions invariant. This allows us to reduce 18 pages of proof down to just about 12 pages, and furthermore with simpler formulas that are easier to read. This altered proof should thus generally be much easier to read and follow. And at the same time, it still keeps all the important parts of the proof which showcases the main ideas behind the solution. The structure of this paper is kept as close to the original version as possible, such that the latter can be read more easily upon having read this version.

1 Introduction

In Damgaard [7], we showed that the interaction operator given by

$$\hat{H}_{\mathrm{I}} = \int \frac{d\mathbf{k}\, d\mathbf{p}}{(2\pi)^6} \frac{1}{\sqrt{|\mathbf{k}|}} \big(\hat{a}(\mathbf{k}) + \hat{a}^\dagger(-\mathbf{k})\big) \hat{b}^\dagger(\mathbf{p}+\mathbf{k}) \hat{b}(\mathbf{p}) \tag{1}$$

is self-adjoint on some carefully constructed domain.[1] In that paper, we looked at a Hilbert space of a system containing only one fermion. This allowed us to only carry one fermion momentum vector $\mathbf{p}$ around in the formulas. And in this paper we will simplify matters even further by considering only a subspace of Dirac delta function-like wave functions where

$$\mathbf{p} + \mathbf{k}_1 + \ldots + \mathbf{k}_n = \text{const.} \tag{2}$$

Since the operator is momentum-conserving, it will leave each such subspace invariant anyway. And if we can therefore show that the operator is self-adjoint on each of these (infinitesimal) subspaces, it should also be self-adjoint on the full Hilbert space. (If this argument seems

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[1] Here the operators $\hat{a}^\dagger(\mathbf{k})$ and $\hat{a}(\mathbf{k})$ are the creation and annihilation operators for the photons of the system, and the operators $\hat{b}^\dagger(\mathbf{k})$ and $\hat{b}(\mathbf{k})$ are the creation and annihilation operators for the fermions (of which we assume there is only one in the system).

unsatisfactory, note that the original paper, Damgaard [7], carries out the proof for the full Hilbert space at once.) So instead of having

$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle + \ldots = \int d\mathbf{p}\, \psi_0(\mathbf{p})\, |\mathbf{p}\rangle + \int d\mathbf{k}_1\, d\mathbf{p}\, \psi_1(\mathbf{k}_1; \mathbf{p})\, |\mathbf{k}_1; \mathbf{p}\rangle$$
$$+ \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{p}\, \psi_2(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p})\, |\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}\rangle + \ldots \quad \text{(wrong in this version)} \tag{3}$$

as in the original paper, we now simply have

$$\begin{aligned} |\psi\rangle &= |\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle + \ldots \\ &= \psi_0\, |\,\rangle + \int d\mathbf{k}_1\, \psi_1(\mathbf{k}_1)\, |\mathbf{k}_1\rangle + \int d\mathbf{k}_1\, d\mathbf{k}_2\, \psi_2(\mathbf{k}_1, \mathbf{k}_2)\, |\mathbf{k}_1, \mathbf{k}_2\rangle + \ldots \end{aligned} \tag{4}$$

for the vectors of the Hilbert space, $\mathbf{H}$, as $\mathbf{p}$ will now always just be given by Eq. (2). Note also that, similarly to the original paper, we take each ψ (when there is no subscript) to be a member of the Hilbert space, $\mathbf{H'}$, of all infinite sequences of square-integrable functions,

$$\psi = (\psi_0, \psi_1, \psi_2, \ldots), \tag{5}$$

where $\psi_n : \mathbb{R}^{3n} \to \mathbb{C}$ for each $n \in \mathbb{N}_0$. We then take $\mathbf{H}$ to be isomorphic to $\mathbf{H'}$, and denote each vector in $\mathbf{H}$ using the bra-ket notation and letting $|\psi\rangle \in \mathbf{H}$ for any $\psi \in \mathbf{H'}$ (such that ψ labels the state $|\psi\rangle$ in $\mathbf{H}$). And as in the original paper, we also take $|\psi_n\rangle$ for all functions $\psi_n : \mathbb{R}^{3n} \to \mathbb{C}$ to denote

$$|\psi_n\rangle = |(0, \ldots, 0, \psi_n, 0, \ldots)\rangle \in \mathbf{H}. \tag{6}$$

The inner product between any two states, $|\psi\rangle$ and $|\phi\rangle$, is then given by

$$\langle\psi|\phi\rangle = \sum_{n=0}^{\infty} \int \psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)^* \phi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)\, d\mathbf{k}_1 \cdots d\mathbf{k}_n \tag{7}$$

and the norm of any state $|\psi\rangle$, which we will denote by $\|\psi\|$, is given by the relation

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \int |\psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)|^2 d\mathbf{k}_1 \cdots d\mathbf{k}_n \equiv \sum_{n=0}^{\infty} \|\psi_n\|^2. \tag{8}$$

Apart from this slight simplification of the Hilbert space, we also make another important modification in the version of we paper, which is that we define the operator $\hat{A}$ in a more simple way. As in Damgaard [7], we take

$$\hat{A}\psi = \hat{A}^+\psi + \hat{A}^-\psi = (\hat{A}_1^-\psi_1,\ \hat{A}_0^+\psi_0 + \hat{A}_2^-\psi_2,\ \hat{A}_1^+\psi_1 + \hat{A}_3^-\psi_3,\ \hat{A}_2^+\psi_2 + \hat{A}_4^-\psi_4,\ \ldots), \tag{9}$$

but in this version of the paper, we simply let the $\hat{A}^+$ and $\hat{A}^-$-operators be given by

$$\hat{A}_{n-1}^+\psi_{n-1}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \frac{1}{\sqrt{|\mathbf{k}_n|}} \psi_{n-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}), \tag{10}$$

$$\hat{A}_n^-\psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) = \int \frac{1}{\sqrt{|\mathbf{k}_n|}} \psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)\, d\mathbf{k}_n. \tag{11}$$

This operator no longer preserves the symmetry of the wave functions, which means that the photons in this system are not really bosons. Rather they are distinct particles with an order among themselves. When a photon is emitted, it always becomes the last (the nth) photon in the order, and at any point in time, it is only this last photon that can be absorbed again by the fermion. We can thus see the photons as constituting a 'stack,' where only the last photon on top of the stack can be absorbed at any given time and removed from the stack, leaving room for the second last photon to be absorbed.

This operator is of course not very realistic, but it makes the proof below a lot simpler compared to the original proof of Damgaard [7], without cutting away any of the most important ideas.

The difference first of all means that there are a lot of (lesser important) terms that we no longer need to handle in this version. Thus, we no longer get all the terms coming from '$\hat{A}^{\pm}_{n,l}\psi_n$,' $l \neq n$, in the original paper. Additionally, we can simplify a lot of the formulas by cutting out the almost-redundant instances of '$\times\mathbb{R}^3$' relating to $\mathbf{p}$. And we also do not need to juggle with all the combinations of terms containing instances of '$(\mathbf{k}_1, \ldots, \widehat{\mathbf{k}_j}, \ldots, \mathbf{k}_n)$' etc. Furthermore, we can define F_n below without a symmetrizing operator, '$\mathcal{S}$,' which generally makes the parts of the proof concerning F_n and $F_n^{\complement}$ much simpler.

To conclude this section in the same way as the original paper, let us also recall that we define $\hat{H}_{\mathrm{I}}$ more precisely[2] by

$$\hat{H}_{\mathrm{I}} \left|\psi\right\rangle = |\hat{A}\psi\rangle \tag{12}$$

for all $\left|\psi\right\rangle \in \mathrm{Dom}(\hat{H}_{\mathrm{I}})$. And unlike $\hat{A}$, we will not choose the domain of $\hat{H}_{\mathrm{I}}$ to be all of the Hilbert space. Rather we will let $\mathrm{Dom}(\hat{H}_{\mathrm{I}})$ be a proper subset of $\mathbf{H}$, which we will define in the following section.

2 A domain on which $\hat{H}_{\mathrm{I}}$ is self-adjoint

First of all, define $p_n : \mathbb{R}^{3(n-1)} \to \mathbb{R}$ for all $n \in \mathbb{N}$, $n \geq 2$, by

$$p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) = e^n \prod_{i=1}^{n-1} (\mathbf{k}_i^2 + 1)^2, \tag{13}$$

and define also E_n by

$$E_n = \big\{(\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbb{R}^{3n} \bigm| e^{p_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})/2} < |\mathbf{k}_n| < e^{p_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})}\big\}. \tag{14}$$

Then define F_n recursively for all $n \geq 2$ by

$$F_n = (F_{n-1}^{\complement} \times \mathbb{R}^3) \cap E_n, \quad F_1^{\complement} = \mathbb{R}^3. \tag{15}$$

Here we take $X^{\complement}$, when X is a set, to denote the complement of X.

Let then T and $T^{\complement}$ denote the two sequences of sets given by

$$\begin{aligned} T &= (\emptyset,\, F_1,\, F_2,\, F_3,\, \ldots), \\ T^{\complement} &= (\mathbb{R}^3,\, F_1^{\complement},\, F_2^{\complement},\, F_3^{\complement},\, \ldots). \end{aligned} \tag{16}$$

And let furthermore $\hat{1}_T$ and $\hat{1}_{T^{\complement}}$ denote the two operators given for all $\psi \in \mathbf{H}'$ by

$$\begin{aligned} \hat{1}_T\psi &= (1_{\emptyset}\psi_0,\, 1_{F_1}\psi_1,\, 1_{F_2}\psi_2,\, 1_{F_3}\psi_3,\, \ldots), \\ \hat{1}_{T^{\complement}}\psi &= (1_{\mathbb{R}^3}\psi_0,\, 1_{F_1^{\complement}}\psi_1,\, 1_{F_2^{\complement}}\psi_2,\, 1_{F_3^{\complement}}\psi_3,\, \ldots), \end{aligned} \tag{17}$$

[2] Equation (1) is thus only there for introductory purposes.

where we for any set, X, we take 1_X to denote the indicator function of X. Note also that we take the product of any two functions, f and g (which could be e.g. 1_{F_n} and ψ_n in this case), to denote the function given by $x \mapsto f(x)g(x)$, $x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$.

With these definitions, we are now ready to define two subsets, V and U, of $\mathbf{H}$, the intersection of which we will choose as our domain for $\hat{H}_\mathrm{I}$. Let first V be the set of all $|\psi\rangle \in \mathbf{H}$ where

$$\|\hat{1}_T \hat{A}^+ \psi\| < \infty. \tag{18}$$

Before we move on to define U, note that in the original paper, we defined V by requiring that $\|\hat{1}_T \hat{A}^+ \hat{1}_{T^\complement} \psi\| < \infty$ instead. However, in this simplified version, it is easy to show that $F_{n-1} \times \mathbb{R}^3 \subset F_n^\complement$ (see Eq. (27) below). And because $\mathrm{supp}(\hat{1}_T \hat{A}^+ \hat{1}_T \psi_{n-1}) \subset F_n \cap (F_{n-1} \times \mathbb{R}^3)$ for all $n \geq 2$, it follows that $\mathrm{supp}(\hat{1}_T \hat{A}^+ \hat{1}_T \psi_{n-1}) \subset F_n \cap F_n^\complement \subset \emptyset$ for all $n \in \mathbb{N}_0$. Thus we get, due to linearity, that $\hat{1}_T \hat{A}^+ \psi = \hat{1}_T \hat{A}^+ \hat{1}_T \psi + \hat{1}_T \hat{A}^+ \hat{1}_{T^\complement} \psi = \hat{1}_T \hat{A}^+ \hat{1}_{T^\complement} \psi$. An equivalent definition of V is therefore that it the set of all $|\psi\rangle \in \mathbf{H}$ where

$$\|\hat{1}_T \hat{A}^+ \hat{1}_{T^\complement} \psi\| < \infty. \tag{19}$$

Let then U be the set of all $|\psi\rangle \in \mathbf{H}$ where

$$|\hat{A}\psi\rangle \in \mathbf{H}, \tag{20}$$

which is equivalent of saying that $\|\hat{H}_\mathrm{I} |\psi\rangle\| = \|\hat{A}\psi\| < \infty$. We will then define $\mathrm{Dom}(\hat{H}_\mathrm{I})$ as

$$\mathrm{Dom}(\hat{H}_\mathrm{I}) = V \cap U. \tag{21}$$

The proposition that we want to show in this paper then reads as follows.

Proposition 1 *Suppose that $\hat{H}_\mathrm{I}$ and $\mathrm{Dom}(\hat{H}_\mathrm{I})$ are defined as above. Then $\mathrm{Dom}(\hat{H}_\mathrm{I})$ is dense in $\mathbf{H}$ and $\hat{H}_\mathrm{I}$ is self-adjoint.*

In order to show this in the following sections, we will start by introducing a set of vectors, W, which we hope to show is a subset of both V and U. We furthermore hope to show that this W spans a dense subspace of $\mathbf{H}$, giving us the first part of Proposition 1. Then we want to show that $\hat{H}_\mathrm{I}$ is symmetric on its domain. And finally we want to show that the domain of its adjoint, $\mathrm{Dom}(\hat{H}_\mathrm{I}^*)$, is equal to $\mathrm{Dom}(\hat{H}_\mathrm{I})$, giving us the last part of Proposition 1.

3 Introducing a set $W \subset V$

We will now construct a certain set, $W \subset \mathbf{H}$, which we will then show is a subset of V. Let this W be the set of all $|\chi\rangle \in \mathbf{H}$ of the form

$$|\chi\rangle = |\chi_m\rangle + |\chi_{m+2}\rangle + |\chi_{m+4}\rangle + \ldots, \quad m \in \mathbb{N}_0, \tag{22}$$

for which we first of all require that for all $n \in \{m+2, m+4, m+6, \ldots\}$, we have

$$\chi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \frac{-1_{E_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) 1_{D_{n-1}^\complement}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}{2\pi p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) \sqrt{|\mathbf{k}_{n-1}|} \sqrt{|\mathbf{k}_n|^5}} \chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}), \tag{23}$$

where D_{m+1} is some superset of F_{m+1}, and where for all $n > m+2$, $D_{n-1}^\complement = \mathbb{R}^{3(n-1)}$, which means that $1_{D_{n-1}^\complement}$ is just 1 everywhere. Furthermore, we require that the functions $\hat{A}_m^- \chi_m$ and $1_{D_{m+1}} \hat{A}_m^+ \chi_m$, and of course also χ_m, are all square-integrable.

We want to show that $|\chi\rangle \in V$ for all $|\chi\rangle \in W$. In order to do this, we first seek to show that

$$\text{supp}(\chi_n) \subset F_n \tag{24}$$

for all $n \geq m+2$, where $\text{supp}(\chi_n)$ denotes the support of χ_n. We then first of all recall that

$$F_n = (F_{n-1}^{\complement} \times \mathbb{R}^3) \cap E_n, \quad F_1^{\complement} = \mathbb{R}^3. \tag{25}$$

for all $n \geq 2$. Due to the factor of $1_{E_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ in Eq. (23), we already have that $\text{supp}(\chi_n) \subset E_n$. Therefore we only need to show that

$$(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) \in F_{n-1}^{\complement} \tag{26}$$

for all $(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}, \mathbf{k}_n) \in \text{supp}(\chi_n)$ in order to obtain Eq. (24).

For $n = m+2$, we see that since we have $D_{m+1}^{\complement} \subset F_{n-1}^{\complement}$ by assumption, the factor of $1_{D_{n-1}^{\complement}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})$ in Eq. (23) already ensures that Eq. (26) holds. And it thus also follows that Eq. (24) holds for $n = m+2$.

Then for $n > m+2$, we want to make a recursive argument, namely by showing that if we have already shown that $\text{supp}(\chi_{n-2}) \subset F_{n-2}$, then Eq. (26) follows, and thereby also Eq. (24).

As part of this argument, we will first derive the following useful relation, which is that

$$F_{n-1} \times \mathbb{R}^3 \subset F_n^{\complement} \tag{27}$$

for all $n \geq 2$. This turns out to be an easy task in our simplified case. From Eq. (25), we get that

$$F_n \subset F_{n-1}^{\complement} \times \mathbb{R}^3 \tag{28}$$

for all $n \geq 2$, which means that

$$F_n^{\complement} \supset (F_{n-1}^{\complement} \times \mathbb{R}^3)^{\complement} = F_{n-1} \times \mathbb{R}^3. \tag{29}$$

We thus obtain Eq. (27).

Now, suppose that $\text{supp}(\chi_{n-2}) \subset F_{n-2}$. We see that Eq. (23) then tells us that for any $(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}, \mathbf{k}_{n-1}, \mathbf{k}_n) \in \text{supp}(\chi_n)$, we have

$$(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}, \mathbf{k}_{n-1}) \in \text{supp}(\chi_{n-2}) \times \mathbb{R}^3 \subset F_{n-2} \times \mathbb{R}^3 \subset F_{n-1}^{\complement}, \tag{30}$$

where we have used Eq. (27) to get the last relation. This gives us Eq. (26) as desired, and Eq. (24) thus follows for this recursive step. We therefore see, by recursion, that Eq. (24) indeed holds for all $n \geq m+2$.

We have thus obtained that χ_n is supported only on F_n for all $n > m$. We then want to use this result to show that $\|\hat{1}_T \hat{A}^+ \hat{1}_{T^{\complement}} \chi\| < \infty$ for all $|\chi\rangle \in W$, which, according to the definition of V in Eq. (19) will imply that $W \subset V$. We then first of all see that our result gives us that

$$|\hat{1}_{T^{\complement}} \chi\rangle = |1_{F_m^{\complement}} \chi_m\rangle. \tag{31}$$

And it follows, due to the fact that $D_{m+1} \supset F_{m+1}$, that

$$\|\hat{1}_T \hat{A}^+ \hat{1}_{T^{\complement}} \chi\| = \|1_{F_{m+1}} \hat{A}_m^+ (1_{F_m^{\complement}} \chi_m)\| \leq \|1_{D_{m+1}} \hat{A}_m^+ (1_{F_m^{\complement}} \chi_m)\| \leq \|1_{D_{m+1}} \hat{A}_m^+ \chi_m\|. \tag{32}$$

We thus, by our assumption that $1_{D_{m+1}} \hat{A}_m^+ \chi_m$ is square-integrable, get that $\|\hat{1}_T \hat{A}^+ \hat{1}_{T^{\complement}} \chi\| < \infty$ for all $|\chi\rangle \in W$. And we can therefore conclude that $W \subset V$.

Before we move on to show that $W \subset U$ in the following section, note first that W might actually be empty for all we know at this point, since there might not exist any $|\chi\rangle$ of the above form for which $\|\chi\| < \infty$, which is required in order for $|\chi\rangle$ to be a member of $\mathbf{H}$. But as we will show in the Section 5, $\|\chi\|$ is in fact always finite whenever $\|\chi_m\|$ is.

4 Showing that W is a subset of U

In order to show that $W \subset U$, we need to show that $\|\hat{A}\chi\| < \infty$ for all $|\chi\rangle \in W$. Let us first recall that

$$\hat{A}^+_{n-1}\psi_{n-1}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \frac{1}{\sqrt{|\mathbf{k}_n|}}\psi_{n-1}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}), \tag{33}$$

$$\hat{A}^-_n\psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) = \int \frac{1}{\sqrt{|\mathbf{k}_n|}}\psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)\, d\mathbf{k}_n. \tag{34}$$

From Eq. (23), we therefore get that

$$\begin{aligned}\hat{A}^-_n\chi_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) &= \int \frac{1}{\sqrt{|\mathbf{k}_n|}}\chi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)\, d\mathbf{k}_n \\ &= -\int \frac{1_{E_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) 1_{D^\complement_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}{2\pi p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})\sqrt{|\mathbf{k}_{n-1}|}|\mathbf{k}_n|^3}\chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2})\, d\mathbf{k}_n\end{aligned} \tag{35}$$

Recalling the definition of E_n in Eq. (14), we can then evaluate the integral over $\mathbf{k}_n$ using polar coordinates, giving us

$$\int \frac{1_{E_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n)}{|\mathbf{k}_n|^3}\, d\mathbf{k}_n = \int_{e^{p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})/2}}^{e^{p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}} 4\pi k^{-1}\, dk = 2\pi p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}). \tag{36}$$

We thus get that

$$\hat{A}^-_n\chi_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) = \frac{-1_{D^\complement_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}{\sqrt{|\mathbf{k}_{n-1}|}}\chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}). \tag{37}$$

Let us then compare this to $\hat{A}^+_{n-2}\chi_{n-2}$, which we see is given by

$$\hat{A}^+_{n-2}\chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) = \frac{1}{\sqrt{|\mathbf{k}_{n-1}|}}\chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}). \tag{38}$$

Using the fact that $1 - 1_{D^\complement_{n-1}} = 1_{D_{n-1}}$, we therefore obtain that

$$\begin{aligned}(\hat{A}^-_n\chi_n + \hat{A}^+_{n-2}\chi_{n-2})(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}) &= \frac{1_{D^\complement_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}{\sqrt{|\mathbf{k}_{n-1}|}}\chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}) \\ &= (1_{D_{n-1}}\hat{A}^+_{n-2}\chi_{n-2})(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1}),\end{aligned} \tag{39}$$

where have used Eq. (38) again for the last equality to recognize the expression as simply the formula for $1_{D_{n-1}}\hat{A}^+_{n-2}\chi_{n-2}$. This shows us that

$$\hat{A}^-_n\chi_n + \hat{A}^+_{n-2}\chi_{n-2} = 1_{D_{n-1}}\hat{A}^+_{n-2}\chi_{n-2} \tag{40}$$

for all $n \geq m + 2$. And since D_{n-1} is just the empty set for all $n \geq m + 4$, we thus get that

$$\sum_{n=m+2}^{\infty} \|\hat{A}^-_{n,n}\chi_n + \hat{A}^+_{n-2}\chi_{n-2}\|^2 = \|1_{D_{m+1}}\hat{A}^+_m\chi_m\|^2, \tag{41}$$

which is finite by assumption.

For our simplified operator of this paper, we argument then ends here, since we can recognize the left-hand side of Eq. (41) as being equal to $\|\hat{A}\chi\|^2 - \|\hat{A}^-_m\chi_m\|^2$. And since $\|\hat{A}^-_m\chi_m\| < \infty$ by assumption, it follows that $\|\hat{A}\chi\|$ is also finite. This result applies for all $|\chi\rangle \in W$, and we thus get that $W \subset U$.

5 Computing the norms of the vectors in W

We have yet to analyze the norms of these χ-vectors in W, so let us do this now. From Eq. (23), we get, for $n \geq m+2$, that

$$\begin{aligned} \|\chi_n\|^2 &= \int \frac{1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n) 1_{D^\complement_{n-1}}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})}{4\pi^2 p_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})^2 |\mathbf{k}_{n-1}||\mathbf{k}_n|^5} \big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2 \, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &\leq \int \frac{1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n)}{p_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})^2 |\mathbf{k}_{n-1}||\mathbf{k}_n|^5} \big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2 \, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &= \int \frac{1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n)}{|\mathbf{k}_{n-1}||\mathbf{k}_n|^5 e^{2n} \prod_{i=1}^{n-1}(\mathbf{k}_i^2+1)^4} \big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2 \, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &= e^{-2n} \int \frac{1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n)}{|\mathbf{k}_n|^5} \, d\mathbf{k}_n \int \frac{1}{|\mathbf{k}_{n-1}|(\mathbf{k}_{n-1}^2+1)^4} \, d\mathbf{k}_{n-1} \\ &\qquad \int \frac{\big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2}{\prod_{i=1}^{n-2}(\mathbf{k}_i^2+1)^4} \, d\mathbf{k}_1 \cdots d\mathbf{k}_{n-2}, \end{aligned} \tag{42}$$

where we have substituted $p_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1}) = \exp(n) \prod_{i=1}^{n-1}(\mathbf{k}_i^2+1)^2$ to get the third line.

To get an upper bound for the first of these integrals, we can then use the fact that since $n \geq 2$, we have that $|\mathbf{k}_n| > 1$ for all $(\mathbf{k}_1,\ldots,\mathbf{k}_n) \in E_n$, which implies that $1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n) \leq 1_{B^\complement_{1,3}}(\mathbf{k}_n)$ everywhere. Here we take $B_{R,N}$ to denote the origin-centered ball in $\mathbb{R}^N$ with a radius of R, which means that $1_{B^\complement_{1,3}}(\mathbf{k}_n)$ is 0 whenever $|\mathbf{k}_n| \leq 1$, and is 1 everywhere else. We can then use polar coordinates to analyze the integral in question, giving us

$$\int \frac{1_{E_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n)}{|\mathbf{k}_n|^5} \, d\mathbf{k}_n \leq \int \frac{1_{B^\complement_{1,3}}(\mathbf{k}_n)}{|\mathbf{k}_n|^5} \, d\mathbf{k}_n = \int_1^\infty \frac{4\pi}{k^3} \, dk = \Big[\frac{-2\pi}{k^2}\Big]_1^\infty = 2\pi. \tag{43}$$

We can also integrate using polar coordinates to get an upper bound for the second integral:

$$\int \frac{1}{(\mathbf{k}^2+1)^4 |\mathbf{k}|} \, d\mathbf{k} = \int_0^\infty \frac{4\pi k}{(k^2+1)^4} \, dk = \Big[\frac{-2\pi}{3(k^2+1)^3}\Big]_0^\infty = \frac{2\pi}{3} < 2\pi. \tag{44}$$

And for the last integral, we can simply use the fact that both the numerator and the denominator are real, which means that if we decrease the denominator, the integral can only increase or remain constant. Since $\prod_{i=1}^{n-2}(\mathbf{k}_i^2+1)^4 \geq 1$ everywhere, we therefore get that

$$\int \frac{\big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2}{\prod_{i=1}^{n-2}(\mathbf{k}_i^2+1)^4} \, d\mathbf{k}_1 \cdots d\mathbf{k}_{n-2} \leq \int \big|\chi_{n-2}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-2})\big|^2 \, d\mathbf{k}_1 \cdots d\mathbf{k}_{n-2} = \|\chi_{n-2}\|^2. \tag{45}$$

With these upper bounds on the three integrals, we therefore get from Eq. (42) that

$$\|\chi_n\|^2 \leq (2\pi)^2 e^{-2n} \|\chi_{n-2}\|^2. \tag{46}$$

And from this recursive relation, we obtain that

$$\|\chi_n\|^2 \leq \prod_{j=2,4,6,\ldots}^{n-m} \Big((2\pi)^2 e^{-2(m+j)}\Big) \|\chi_m\|^2, \tag{47}$$

which can be seen to be finite due to our assumption that $\|\chi_m\|^2 < \infty$.

This result tells us that

$$\|\chi\|^2 \leq \sum_{n\in\{m,m+2,m+4\ldots\}}^{\infty} \prod_{j=2,4,6,\ldots}^{n-m} \Big((2\pi)^2 e^{-2(m+j)}\Big)\|\chi_m\|^2. \tag{48}$$

And it is easy to show that this expression is finite whenever $\|\chi_m\|$ is. This means that there exists a positive constant, C_2,[3] such that

$$\|\chi\|^2 \leq C_2 \|\chi_m\|^2 \tag{49}$$

for all $m \in \mathbb{N}_0$ and $|\chi\rangle = |\chi_m\rangle + |\chi_{m+2}\rangle + \ldots \in W$.

6 The density of $\mathrm{Dom}(\hat{H}_\mathrm{I})$ in $\mathbf{H}$

In this section, we want to argue that the vectors in W span a dense subspace of $\mathbf{H}$, which means that we can get arbitrarily close to any vector in $\mathbf{H}$, i.e. in the norm topology of $\mathbf{H}$, by taking the sum of some finite set of vectors in W. And since $W \subset \mathrm{Dom}(\hat{H}_\mathrm{I})$, this sum will thus also be a member of $\mathrm{Dom}(\hat{H}_\mathrm{I})$, giving us that $\mathrm{Dom}(\hat{H}_\mathrm{I})$ is dense in $\mathbf{H}$.

To show that $\mathrm{span}(W)$ is dense in $\mathbf{H}$, let us consider all $|\chi\rangle = |\chi_m\rangle + |\chi_{m+2}\rangle + \ldots \in W$ where χ_m is smooth and has compact support bounded by some $R > 0$. Note that the set of all such χ_m functions spans $L^2(\mathbb{R}^{3m})$. For any such χ_m, it is then easy to see that for any $(\mathbf{k}_1, \ldots, \mathbf{k}_{m+1}) \in \mathrm{supp}(1_{F_{m+1}} \hat{A}^+_m \chi_m)$, $|\mathbf{k}_1|, \ldots, |\mathbf{k}_{m+1}|$ will all be bounded from above by $\exp[p_{m+1}(R, \ldots, R)] = \exp[\exp(m+1)\prod_{i=1}^{m}(R^2+1)^2]$. This means that there exists a sufficiently large R' such that $F_{m+1} \subset B^{m+1}_{R',3}$. And we can therefore let $D_{m+1} \supset F_{m+1}$ be equal to $B^{m+1}_{R',3}$.

Let us then first confirm that such a $|\chi\rangle$ is a member of W. We can first of all note that the fact that χ_m is smooth and with compact support means that both $\|\chi_m\|$ and $\|\hat{A}^-_m \chi_m\|$ are finite. And since $D_{m+1} = B^{m+1}_{R',3}$ for some finite R', we also get that $\|1_{D_{m+1}\times\mathbb{R}^3} \hat{A}^+_m \chi_m\| < \infty$. And since we get from Eq. (49) that $\|\chi\| < \infty$, we also have $|\chi\rangle \in \mathbf{H}$ in the first place, which is of course also a requirement. All the requirements are thus fulfilled, implying that $|\chi\rangle$ is indeed a member of W.

If we then look at Eq. (23), which we can recall states that

$$\chi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \frac{-1_{E_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) 1_{D^{\complement}_{n-1}}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})}{2\pi p_n(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})\sqrt{|\mathbf{k}_{n-1}|}\sqrt{|\mathbf{k}_n|^5}} \chi_{n-2}(\mathbf{k}_1, \ldots, \mathbf{k}_{n-2}), \tag{50}$$

we can see that when we expand this recursive formula, each χ_n with $n > m$ will include the factor of $1_{D^{\complement}_{m+1}}(\mathbf{k}'_1, \ldots, \mathbf{k}'_{m+1}) = 1_{\mathbb{R}^{3(m+1)}\setminus B^{m+1}_{R',3}}(\mathbf{k}'_1, \ldots, \mathbf{k}'_{m+1})$ in is formula. This means that if we let R' tend towards infinity, which we are allowed to do, then each χ_n, $n > m$, will tend to zero pointwise. This implies that each χ_n, $n > m$, as a vector will tend towards a zero vector in the norm topology of $L^2(\mathbb{R}^{3n})$. And we thus get that $|\chi\rangle \to |\chi_m\rangle$ for all such vectors when $R' \to \infty$.

Now, since the set of all such χ_m-functions spans $L^2(\mathbb{R}^{3m})$, we can therefore get arbitrarily close to any particular $|\psi_m\rangle$ this way, $\psi \in \mathbf{H}'$. And since we can do this for any $m \in \mathbb{N}_0$, we thus get that the vectors in W span a dense subspace of $\mathbf{H}$. Therefore, since $W \subset \mathrm{Dom}(\hat{H}_\mathrm{I})$, it follows that $\mathrm{Dom}(\hat{H}_\mathrm{I})$ is dense in $\mathbf{H}$.

[3] We call it C_2 here to be consistent with the original proof of Damgaard [7], even though we do not need the original C_1 in this simplified version.

7 The symmetry of $\hat{H}_\mathrm{I}$

In this section, we will show that the operator $\hat{H}_\mathrm{I}$, which we recall is given by $\hat{H}_\mathrm{I}\,|\psi\rangle = |\hat{A}\psi\rangle$ for all $|\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$, is *symmetric* in the sense that

$$\langle \hat{A}\phi|\psi\rangle = \langle \phi|\hat{A}\psi\rangle \tag{51}$$

for all $|\phi\rangle, |\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$.

Let us recall, first of all, that

$$\hat{A}\psi = (\hat{A}_1^-\psi_1,\ \hat{A}_0^+\psi_0 + \hat{A}_2^-\psi_2,\ \hat{A}_1^+\psi_1 + \hat{A}_3^-\psi_3,\ \ldots) \tag{52}$$

with

$$\hat{A}_{n-1}^+\psi_{n-1}(\mathbf{k}_1,\ldots,\mathbf{k}_n) = \frac{1}{\sqrt{|\mathbf{k}_n|}}\psi_{n-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1}), \tag{53}$$

$$\hat{A}_n^-\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1}) = \int \frac{1}{\sqrt{|\mathbf{k}_n|}}\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\,d\mathbf{k}_n. \tag{54}$$

To show that $\hat{H}_\mathrm{I}$ is symmetric, we thus want to show that the difference between the two series given by

$$\begin{aligned}\langle \hat{A}\phi|\psi\rangle &= \langle \hat{A}_1^-\phi_1|\psi_0\rangle + \langle \hat{A}_0^+\phi_0 + \hat{A}_2^-\phi_2|\psi_1\rangle + \langle \hat{A}_1^+\phi_1 + \hat{A}_3^-\phi_3|\psi_2\rangle + \ldots,\\ \langle \phi|\hat{A}\psi\rangle &= \langle \phi_0|\hat{A}_1^-\psi_1\rangle + \langle \phi_1|\hat{A}_0^+\psi_0 + \hat{A}_2^-\psi_2\rangle + \langle \phi_2|\hat{A}_1^+\psi_1 + \hat{A}_3^-\psi_3\rangle + \ldots\end{aligned} \tag{55}$$

is zero.

Let us then consider the following cut-off versions of $\hat{A}_n^+$ and $\hat{A}_n^-$, call them $\hat{A}_n^{R+}$ and $\hat{A}_n^{R-}$, defined for all $\psi \in \mathbf{H}'$ and $n \in \mathbb{N}_+$ by

$$\begin{aligned}\hat{A}_{n-1}^{R+}\psi_{n-1} &= 1_{(B_{R,3}\setminus B_{1/R,3})^n}\hat{A}_{n-1}^+\psi_{n-1},\\ \hat{A}_n^{R-}\psi_n &= 1_{(B_{R,3}\setminus B_{1/R,3})^{n-1}}\hat{A}_n^-(1_{(B_{R,3}\setminus B_{1/R,3})^n}\psi_n).\end{aligned} \tag{56}$$

We see that this gives us

$$\begin{aligned}\langle \hat{A}_{n-1}^{R+}\phi_{n-1}|\psi_n\rangle &= \int_{(B_{R,3}\setminus B_{1/R,3})^n} d\mathbf{k}_1\cdots d\mathbf{k}_n\,\frac{1}{\sqrt{|\mathbf{k}_n|}}\phi_{n-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})^*\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n),\\ \langle \phi_{n-1}|\hat{A}_n^{R-}\psi_n\rangle &= \int_{(B_{R,3}\setminus B_{1/R,3})^{n-1}} d\mathbf{k}_1\cdots d\mathbf{k}_{n-1}\\ &\qquad \int_{B_{R,3}\setminus B_{1/R,3}} d\mathbf{k}_n\,\frac{1}{\sqrt{|\mathbf{k}_n|}}\phi_{n-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})^*\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n).\end{aligned} \tag{57}$$

We can see that the right-hand sides of these two equations are equal, giving us

$$\langle \hat{A}_{n-1}^{R+}\phi_{n-1}|\psi_n\rangle = \langle \phi_{n-1}|\hat{A}_n^{R-}\psi_n\rangle \tag{58}$$

for all $\phi, \psi \in \mathbf{H}'$, $n \in \mathbb{N}_+$, and $R \geq 1$. Let us therefore define

$$\begin{aligned}\langle \hat{A}^R\phi|\psi\rangle_N &= \sum_{n=0}^{N}\big(\langle \hat{A}_{n-1}^{R+}\phi_{n-1}|\psi_n\rangle + \langle \hat{A}_{n+1}^{R-}\phi_{n+1}|\psi_n\rangle\big),\\ \langle \phi|\hat{A}^R\psi\rangle_N &= \sum_{n=0}^{N}\big(\langle \phi_n|\hat{A}_{n-1}^{R+}\psi_{n-1}\rangle + \langle \phi_n|\hat{A}_{n+1}^{R-}\psi_{n+1}\rangle\big),\end{aligned} \tag{59}$$

where we let $\hat{A}^R$ be defined similarly as in Eq. (9), and let $\hat{A}^{R+}_{-1} = \hat{A}^{R-}_0 = 0$. We then see that

$$\langle \hat{A}^R \phi | \psi \rangle_N - \langle \phi | \hat{A}^R \psi \rangle_N = \langle \hat{A}^{R-}_{N+1} \phi_{N+1} | \psi_N \rangle + \langle \phi_N | \hat{A}^{R-}_{N+1} \psi_{N+1} \rangle \,, \tag{60}$$

namely since all other terms cancel each other. Thus, since the left-hand side of this equation converges to $\langle \hat{A}\phi|\psi\rangle - \langle \phi|\hat{A}\psi\rangle$ when $N, R \to \infty$, we get that

$$\langle \hat{A}\phi|\psi\rangle - \langle \phi|\hat{A}\psi\rangle = \lim_{R\to\infty} \lim_{N\to\infty} \big(\langle \hat{A}^{R-}_{N+1} \phi_{N+1} | \psi_N \rangle + \langle \phi_N | \hat{A}^{R-}_{N+1} \psi_{N+1} \rangle \big). \tag{61}$$

So if we can only show that $\langle \phi_N | \hat{A}^{R-}_{N+1} \psi_{N+1} \rangle \to 0$ when $N, R \to \infty$ for any $|\phi\rangle, |\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$, we will get that both terms on the right-hand side of Eq. (61) will tend to zero in this limit, which will thus give us that $\hat{H}_\mathrm{I}$ is symmetric.

In Section 3 we showed Eq. (27), which states that

$$F_{n-1} \times \mathbb{R}^3 \subset F_n^\complement \tag{62}$$

for all $n \geq 2$. And since $\hat{A}^+_N 1_{F_N} \phi_N$ is only supported on $F_N \times \mathbb{R}^3$, we thus get that $\hat{A}^+_N 1_{F_N} \phi_N$ is zero everywhere on F_{N+1}. So if we split ϕ and ψ up into

$$\phi = \hat{1}_{T^\complement} \phi + \hat{1}_T \phi \equiv \phi' + \phi'', \quad \psi = \hat{1}_{T^\complement} \psi + \hat{1}_T \psi \equiv \psi' + \psi'', \tag{63}$$

we therefore see that

$$\langle \phi''_N | \hat{A}^{R-}_{N+1} \psi''_{N+1} \rangle = \langle \hat{A}^{R+}_N \phi''_N | \psi''_{N+1} \rangle = 0 \tag{64}$$

for all N and R, where we have used Eq. (58) to get the first equality.

This motivates us to analyze $\langle \hat{A}^{R+}_{N-1} \phi_{N-1} | \psi'_N \rangle$, $\psi' = \hat{1}_{T^\complement}\psi$, for all $|\psi\rangle \in U \cap V = \mathrm{Dom}(\hat{H}_\mathrm{I})$ and $|\phi\rangle \in \mathbf{H}$ in the hope to show that this inner product vanishes for all R when $N \to \infty$. In order to do this, let us first split up $\mathbb{R}^{3n}$ for all $n \in \mathbb{N}_+$ into a set of subsets, $\{X_{n,\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}_0^n}$, where $X_{n,\mathbf{i}} \subset \mathbb{R}^{3n}$ for all $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ is given by

$$X_{n,\mathbf{i}} = (B_{i_1+1,3} \setminus B_{i_1,3}) \times \cdots \times (B_{i_n+1,3} \setminus B_{i_n,3}). \tag{65}$$

Let us then split ψ'_N up into $\psi'_N = \sum_{\mathbf{i}\in\mathbb{N}_0^N} \psi'_{N,\mathbf{i}}$, where

$$\psi'_{N,\mathbf{i}} = 1_{X_{N,\mathbf{i}}} \psi'_N. \tag{66}$$

Below we will then show the following lemma: There exists a constant $C_3 > 0$ such that for any sufficiently large N, we have

$$\|1_{F_{N+1}} \hat{A}^+_N \psi'_N\|^2 \geq 2\pi C_3 e^{p'_{N+1}(i_1,\ldots,i_N)} \|\psi'_{N,\mathbf{i}}\|^2 \tag{67}$$

for all $\mathbf{i} \in \mathbb{N}_0^N$, where we define $p'_n(k_1, \ldots, k_{n-1}) = p_n(k_1\hat{\mathbf{e}}_1, \ldots, k_{n-1}\hat{\mathbf{e}}_1)$, $\hat{\mathbf{e}}_1 = (1, 0, 0)$, in order to write this exponent more compactly.

Assuming that this lemma is true, let us see how this gives us that $\hat{H}_\mathrm{I}$ is symmetric. First we note that

$$\langle \hat{A}^{R+}_{N-1} \phi_{N-1} | \psi'_{N,\mathbf{i}} \rangle = \langle 1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1} \phi_{N-1} | \psi'_{N,\mathbf{i}} \rangle \leq \|1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1} \phi_{N-1}\| \|\psi'_{N,\mathbf{i}}\|. \tag{68}$$

Now, because $|\psi\rangle \in V$, we know that $\sum_{n=0}^\infty \|1_{F_{n+1}} \hat{A}^+_n \psi'_n\|^2 = \sum_{n=0}^\infty \|1_{F_{n+1}} \hat{A}^+_n 1_{F_n^\complement} \psi_n\|^2$ will converge, which means that for any $\varepsilon_1 > 0$, we have

$$\|1_{F_{N+1}} \hat{A}^+_N \psi'_N\|^2 \leq \varepsilon_1 \tag{69}$$

for any sufficiently large N. Combining this with the assumed Eq. (67), we therefore get that

$$\|\psi'_{N,\mathbf{i}}\|^2 \leq \frac{\varepsilon_1}{2\pi C_3} e^{-p'_{N+1}(i_1,\ldots,i_N)} \tag{70}$$

for any $\mathbf{i} \in \mathbb{N}_0^N$

Let us then also derive an upper bound for $\|1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1}\phi_{N-1}\|$, such that Eq. (68) will yield us a bound on $\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi'_{N,\mathbf{i}}\rangle$. We first of all see that

$$\|1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1}\phi_{N-1}\|^2 = \int_{X_{N,\mathbf{i}}\cap(B_{R,3}\setminus B_{1/R,3})^N} d\mathbf{k}_1\cdots d\mathbf{k}_N \, \frac{1}{|\mathbf{k}_N|}\big|\phi_{N-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{N-1})\big|^2. \tag{71}$$

And since this integrand is non-negative everywhere, we thus get that

$$\begin{aligned} \|1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1}\phi_{N-1}\|^2 &\leq \int_{i_N\leq|\mathbf{k}_N|\leq i_N+1} d\mathbf{k}_1\cdots d\mathbf{k}_N \frac{1}{|\mathbf{k}_N|}\big|\phi_{N-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{N-1})\big|^2, \\ &= \|\phi_{N-1}\|^2 \int_{i_N}^{i_N+1} 4\pi k\, dk = 2\pi\big((i_N+1)^2 - i_N^2\big)\|\phi_{N-1}\|^2 \\ &= 2\pi(2i_N+1)\|\phi_{N-1}\|^2. \end{aligned} \tag{72}$$

Furthermore, since $\phi \in \mathbf{H}'$, we know that $\sum_{n=0}^{\infty}\|\phi_n\|^2$ converges, which means that for all $\varepsilon_2 > 0$, we have $\|\phi_{N-1}\|^2 \leq \varepsilon_2$ for any sufficiently large N (chosen for the particular ε_2). We therefore get that

$$\|1_{X_{N,\mathbf{i}}} \hat{A}^{R+}_{N-1}\phi_{N-1}\|^2 \leq 2\pi(2i_N+1)\varepsilon_2 \tag{73}$$

for large enough values of N.

So from Eq. (68), we now get that

$$\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi'_{N,\mathbf{i}}\rangle \leq \sqrt{\frac{\varepsilon_1\varepsilon_2}{C_3}}\sqrt{2i_N+1}e^{-p'_{N+1}(i_1,\ldots,i_N)/2} \tag{74}$$

for all $\mathbf{i} \in \mathbb{N}_0^N$. And it is not hard to see, when considering the formula for p_n of Eq. (13), that $\sum_{\mathbf{i}\in\mathbb{N}_0^N}\sqrt{2i_N+1}\exp(-p'_{N+1}(i_1,\ldots,i_N)/2) < \infty$, and that this expression is furthermore bounded when $N\to\infty$. This tells us that

$$\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi'_N\rangle = \sum_{\mathbf{i}\in\mathbb{N}_0^N}\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi'_{N,\mathbf{i}}\rangle \leq C_4\sqrt{\varepsilon_1\varepsilon_2} \tag{75}$$

for some constant $C_4 \geq 0$. Thus, since we can choose arbitrarily small ε_1 and ε_2 for a large enough N, we get that

$$\lim_{N\to\infty}\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi'_N\rangle = 0 \tag{76}$$

for all $R \geq 1$, $|\phi\rangle \in \mathbf{H}$, $|\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$, $\psi' = \hat{1}_{T^\complement}\psi$.

From this result, we see that for any $|\phi\rangle, |\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$, we have that $\langle \hat{A}^{R+}_N\phi''_N|\psi'_{N+1}\rangle$, $\langle \hat{A}^{R+}_N\phi'_N|\psi''_{N+1}\rangle$, and $\langle \hat{A}^{R+}_N\phi'_N|\psi'_{N+1}\rangle$ all tend towards zero when $N\to\infty$, where $\phi', \phi'', \psi', \psi''$ are defined according to Eq. (63). And we have already shown that $\langle \hat{A}^{R+}_N\phi''_N|\psi''_{N+1}\rangle = 0$. We therefore get that

$$\lim_{N\to\infty}\langle \hat{A}^{R+}_{N-1}\phi_{N-1}|\psi_N\rangle = \lim_{N\to\infty}\langle \phi_{N-1}|\hat{A}^{R-}_N\psi_N\rangle = 0 \tag{77}$$

for all $|\phi\rangle, |\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$. And this tells us that both terms on the right-hand side of Eq. (61) vanishes in the limit when $N \to \infty$. This means that the left-hand side also vanishes, and we therefore get that

$$\langle \hat{A}\phi|\psi\rangle - \langle\phi|\hat{A}\psi\rangle = 0, \tag{78}$$

which is exactly what we want to show in order to conclude that $\hat{H}_\mathrm{I}$ is symmetric.

So all that is left in order to show that $\hat{H}_\mathrm{I}$ is symmetric is to show the lemma of Eq. (67), which stated that for all $|\psi\rangle \in U \cap V$ and $|\phi\rangle \in \mathbf{H}$, there exists a positive constant C_3 such that for any $N \in \mathbb{N}$ that is sufficiently large,

$$\|1_{F_{N+1}} \hat{A}_N^+ \psi'_N\|^2 \geq 2\pi C_3 e^{p'_{N+1}(i_1,\ldots,i_N)} \|\psi'_{N,\mathbf{i}}\|^2 \tag{79}$$

for all $\mathbf{i} \in \mathbb{N}_0^N$, where $\psi' = \hat{1}_{T^\complement}\psi$.

This is quite easy to do for our simplified operator of this supplementary paper. Since $\mathrm{supp}(\psi'_N) \subset F_N^\complement$ by definition, we have

$$\mathrm{supp}(\hat{A}_N^+ \psi'_N) \subset F_N^\complement \times \mathbb{R}^3. \tag{80}$$

And because $F_{N+1} = (F_N^\complement \times \mathbb{R}^3) \cap E_{N+1}$, we thus have that

$$1_{F_{N+1}} \hat{A}_N^+ \psi'_N = 1_{E_{N+1}} \hat{A}_N^+ \psi'_N. \tag{81}$$

By the same argument, this also applies for each $\psi'_{N,\mathbf{i}}$, and we therefore get that

$$1_{F_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{i}} = 1_{E_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{i}}. \tag{82}$$

This then allows us to write

$$\begin{aligned}
\|1_{F_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{i}}\|^2 &= \int d\mathbf{k}_1 \cdots d\mathbf{k}_N \int_{(\mathbf{k}_1,\ldots,\mathbf{k}_{N+1}) \in E_{N+1}} d\mathbf{k}_{N+1} \, \frac{1}{|\mathbf{k}_{N+1}|} \left|\psi'_{N,\mathbf{i}}(\mathbf{k}_1,\ldots,\mathbf{k}_N)\right|^2 \\
&= \int_{X_{N,\mathbf{i}}} d\mathbf{k}_1 \cdots d\mathbf{k}_N \int_{(\mathbf{k}_1,\ldots,\mathbf{k}_{N+1}) \in E_{N+1}} d\mathbf{k}_{N+1} \, \frac{1}{|\mathbf{k}_{N+1}|} \left|\psi'_{N,\mathbf{i}}(\mathbf{k}_1,\ldots,\mathbf{k}_N)\right|^2,
\end{aligned} \tag{83}$$

where we for the last equation have used the fact that $\psi'_{N,\mathbf{i}}$ is only supported on $X_{N,\mathbf{i}}$ to limit the first integral to $(\mathbf{k}_1,\ldots,\mathbf{k}_N) \in X_{N,\mathbf{i}}$. Then for a sufficiently large N, we see that

$$\begin{aligned}
\int_{(\mathbf{k}_1,\ldots,\mathbf{k}_{N+1}) \in E_{N+1}} \frac{1}{|\mathbf{k}_{N+1}|} \, d\mathbf{k}_{N+1} &= \int_{e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)/2}}^{e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)}} 4\pi k \, dk \\
&= \Big[2\pi k^2\Big]_{e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)/2}}^{e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)}} \\
&\geq 2\pi e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)},
\end{aligned} \tag{84}$$

where we for the last inequality have used the fact that for a sufficiently large x, we have $\exp(2x) - \exp(x) = \exp(x)(\exp(x) - 1) \geq \exp(x)$. We thus obtain that

$$\begin{aligned}
\|1_{F_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{i}}\|^2 &\geq 2\pi \int_{X_{N,\mathbf{i}}} d\mathbf{k}_1 \cdots d\mathbf{k}_N \, e^{p_{N+1}(\mathbf{k}_1,\ldots,\mathbf{k}_N)} \left|\psi'_{N,\mathbf{i}}(\mathbf{k}_1,\ldots,\mathbf{k}_N)\right|^2 \\
&\geq 2\pi e^{p'_{N+1}(i_1,\ldots,i_N)} \|\psi'_{N,\mathbf{i}}\|^2.
\end{aligned} \tag{85}$$

Next we want to argue that any two functions $1_{F_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{i}}$ and $1_{F_{N+1}} \hat{A}_N^+ \psi'_{N,\mathbf{j}}$ will have no overlap in terms of their support when $\mathbf{j} \neq \mathbf{i}$. This is easy to do, since for any $(\mathbf{k}_1,\ldots,\mathbf{k}_{N+1}) \in$

$\mathrm{supp}(1_{F_{N+1}}\hat{A}_N^+\psi'_{N,\mathbf{i}}) \cap \mathrm{supp}(1_{F_{N+1}}\hat{A}_N^+\psi'_{N,\mathbf{j}})$, we must have $(\mathbf{k}_1,\ldots,\mathbf{k}_N) \in X_{N,\mathbf{i}} \cap X_{N,\mathbf{j}}$, which is only possible when $\mathbf{j} = \mathbf{i}$. This result then tells us that

$$\|1_{F_{N+1}}\hat{A}_N^+\psi'_N\|^2 = \sum_{\mathbf{j}\in\mathbb{N}_0^N} \|1_{F_{N+1}}\hat{A}_N^+\psi'_{N,\mathbf{j}}\|^2 \geq \|1_{F_{N+1}}\hat{A}_N^+\psi'_{N,\mathbf{i}}\|^2 \tag{86}$$

for any $\mathbf{i} \in \mathbb{N}_0^N$. And from Eq. (85), we therefore get that

$$\|1_{F_{N+1}}\hat{A}_N^+\psi'_N\|^2 \geq 2\pi e^{p'_{N+1}(i_1,\ldots,i_N)}\|\psi'_{N,\mathbf{i}}\|^2. \tag{87}$$

We thus obtain the lemma of Eqs. (67/79), where we can choose C_3 simply as 1. This was the last piece that we needed, and we therefore get that $\hat{H}_\mathrm{I}$ is symmetric.

8 The self-adjointness of $\hat{H}_\mathrm{I}$

We now want to finally show that $\hat{H}_\mathrm{I}$ is self-adjoint on its domain, which will thus give us Proposition 1.

We have the following definitions regarding self-adjointness according to Hall [1]. First of all, an *unbounded operator*, $\hat{T}$, on $\mathbf{H}$ is any linear operator where $\mathrm{Dom}(\hat{T})$ is a dense subspace of $\mathbf{H}$ and $\hat{T}\psi \in \mathbf{H}$ for all $\psi \in \mathrm{Dom}(\hat{T})$. And the *adjoint*, $\hat{T}^*$, of such an operator is defined as the operator for which $\mathrm{Dom}(\hat{T}^*)$ is equal to the subspace of all $\phi \in \mathbf{H}$ where

$$\psi \mapsto \frac{\langle\phi|\hat{T}\psi\rangle}{\|\psi\|}, \quad \psi \in \mathrm{Dom}(\hat{T}), \quad \|\psi\| > 0, \tag{88}$$

is bounded as a function,[4] and where $\langle\phi|\hat{T}\psi\rangle = \langle\hat{T}^*\phi|\psi\rangle$ for all $\phi \in \mathrm{Dom}(\hat{T}^*)$ and $\psi \in \mathrm{Dom}(\hat{T})$. And, as the name suggests, a *self-adjoint* operator $\hat{T}$ is one for which $\mathrm{Dom}(\hat{T}^*) = \mathrm{Dom}(\hat{T})$ and $\hat{T}^*\psi = \hat{T}\psi$ for all $\psi \in \mathrm{Dom}(\hat{T})$.

We also have the following proposition, which is important for our purposes. If $\hat{T}$ is symmetric, then $\hat{T}^*$ is an extension of $\hat{T}$, which means that $\mathrm{Dom}(\hat{T}^*) \supset \mathrm{Dom}(\hat{T})$ and $\hat{T}^*\psi = \hat{T}\psi$ for all $\psi \in \mathrm{Dom}(\hat{T})$. (See e.g. Proposition 9.4 of Hall [1].) So since we have already shown that $\mathrm{Dom}(\hat{H}_\mathrm{I})$ is dense in $\mathbf{H}$ and that $\hat{H}_\mathrm{I}$ is symmetric, we have therefore only left to show that $\mathrm{Dom}(\hat{H}_\mathrm{I}^*) \subset \mathrm{Dom}(\hat{H}_\mathrm{I})$ in order to show that $\hat{H}_\mathrm{I}$ is self-adjoint, as this will imply that $\mathrm{Dom}(\hat{H}_\mathrm{I}^*) = \mathrm{Dom}(\hat{H}_\mathrm{I})$.

In order to do this, let us consider the set, call it W_Σ, of all $|\psi\rangle \in \mathrm{Dom}(\hat{H}_\mathrm{I})$ of the form

$$|\psi\rangle = \sum_{m=0}^{M} |\chi^{(m)}\rangle = \sum_{m=0}^{M}\sum_{n=m}^{\infty} |\chi_n^{(m)}\rangle \tag{89}$$

for some $M \in \mathbb{N}$, where $|\chi^{(m)}\rangle = |\chi_m^{(m)}\rangle + |\chi_{m+2}^{(m)}\rangle + |\chi_{m+4}^{(m)}\rangle + \ldots \in W$ for all $m \in \{0,\ldots,M\}$. Let us also for each m let $D_{m+1}^{(m)}$ be the corresponding D_{m+1} set for $|\chi^{(m)}\rangle$ that is part of its free parameters. We then have

$$\begin{aligned} |\hat{A}\chi^{(m)}\rangle &= |\hat{A}_m^-\chi_m^{(m)}\rangle + \sum_{n=m+2}^{\infty} |\hat{A}_n^-\chi_n^{(m)} + \hat{A}_{n-2}^+\chi_{n-2}^{(m)}\rangle \\ &= |\hat{A}_m^-\chi_m^{(m)}\rangle + |1_{D_{m+1}^{(m)}}\hat{A}_m^+\chi_m^{(m)}\rangle\,, \end{aligned} \tag{90}$$

[4] Hall [1] uses the terminology of a *linear functional* with which the division by $\|\psi\|$ is implicit.

for each of these $|\chi^{(m)}\rangle$, where we have also used Eq. (41) to get the second equality. And from Eq. (49), we also get that

$$\|\psi\|^2 \leq C_2 \sum_{m=0}^{\infty} \|\chi_m^{(m)}\|^2 = C_2 \Big\| \sum_{m=0}^{\infty} |\chi_m^{(m)}\rangle \Big\|^2 \tag{91}$$

for some $C_2 > 0$.

In order to then first of all show that $\mathrm{Dom}(\hat{H}_{\mathrm{I}}^*) \subset U$, let us pick an M and a $|\psi\rangle \in W_\Sigma$ for which

$$\chi_m^{(m)} = \hat{A}_{m-1}^{R+}\phi_{m-1} + \hat{A}_{m+1}^{R-}\phi_{m+1} \tag{92}$$

for all $m \in \{0, \ldots, M\}$, where we take $\hat{A}_{-1}^{R+}\phi_{-1}$ to be 0. Let us here use similar definitions for A^{R+} and A^{R-} as in Eq. (56), but let us use $1/R^a$, $a \geq 1$, instead of $1/R$ for the lower cutoff, such that

$$\begin{aligned} \hat{A}_{n-1}^{R+}\psi_{n-1} &= 1_{(B_{R,3}\setminus B_{1/R^a,3})^n} \hat{A}_{n-1}^{+}\psi_{n-1}, \\ \hat{A}_{n}^{R-}\psi_{n} &= 1_{(B_{R,3}\setminus B_{1/R^a,3})^{n-1}} \hat{A}_{n}^{-}(1_{(B_{R,3}\setminus B_{1/R^a,3})^n}\psi_n). \end{aligned} \tag{93}$$

Since the support of each $\chi_m^{(m)}$ is therefore bounded by R with respect to the **k**-parameters, we can for each m choose $D_{m+1}^{(m)}$ to be equal to $B_{R_m,3}^{m+1} \supset F_{m+1}$ for some sufficiently large $R_m > R$, similarly to what we did in Section 6. This gives us that $\|1_{D_{m+1}^{(m)}} \hat{A}_m^{+}\chi_m^{(m)}\| < \infty$. And since A^{R-} and A^{R+} are bounded, we also have that $\|\chi_m^{(m)}\| < \infty$. We can furthermore see that

$$\hat{A}_m^{-}\chi_m^{(m)} = \hat{A}_m^{R_m-}(\hat{A}_{m-1}^{R+}\phi_{m-1} + \hat{A}_{m+1}^{R-}\phi_{m+1}) = \hat{A}_m^{R_m-}\chi_m^{(m)}, \tag{94}$$

where we have thus used the fact that $\hat{A}_m^{-}\hat{A}_{m-1}^{R+} = \hat{A}_m^{R_m-}\hat{A}_{m-1}^{R+}$ and $\hat{A}_m^{-}\hat{A}_{m+1}^{R-} = \hat{A}_m^{R_m-}\hat{A}_{m+1}^{R-}$ when $R_m \geq R$. (We can always put a cutoff on an operator if it stands to the left of an operator that already has a smaller cutoff.) Thus, we also get that $\|\hat{A}_m^{-}\chi_m^{(m)}\| < \infty$, which means that each $\chi_m^{(m)}$ fulfills its requirements, confirming that $|\psi\rangle$ exists as a member of W_Σ.

Next we note that since

$$B_{R_m,3}^{m+1} \setminus (B_{R_m,3} \setminus B_{1/R_m^a,3})^{m+1} = B_{R_m,3}^{m+1} \setminus (\mathbb{R}^3 \setminus B_{1/R_m^a,3})^{m+1} = B_{R_m,3}^{m+1} \setminus (B_{1/R_m^a,3}^{\complement})^{m+1}, \tag{95}$$

we have

$$\begin{aligned} 1_{D_{m+1}^{(m)}} \hat{A}_m^{+}\chi_m^{(m)} &= 1_{B_{R_m,3}^{m+1}} \hat{A}_m^{+}\chi_m^{(m)} \\ &= 1_{(B_{R_m,3}\setminus B_{1/R_m^a,3})^{m+1}} \hat{A}_m^{+}\chi_m^{(m)} + 1_{B_{R_m,3}^{m+1}\setminus(B_{R_m,3}\setminus B_{1/R_m^a,3})^{m+1}} \hat{A}_m^{+}\chi_m^{(m)} \\ &= \hat{A}_m^{R_m+}\chi_m^{(m)} + 1_{B_{R_m,3}^{m+1}\setminus(B_{1/R_m^a,3}^{\complement})^{m+1}} \hat{A}_m^{+}\chi_m^{(m)}. \end{aligned} \tag{96}$$

We can see that this last term on the right-hand side tends to a zero vector when $a \to \infty$, namely since $B_{1/R_m^a,3}^{\complement}$ tends towards $\mathbb{R}^3$. Let us therefore define[5]

$$|\psi''\rangle = |\hat{A}\psi\rangle - \sum_{m=0}^{M}\sum_{l=1}^{n-1} |1_{B_{R_m,3}^{m+1}\setminus(B_{1/R_m^a,3}^{\complement})^{m+1}} \hat{A}_m^{+}\chi_m^{(m)}\rangle \tag{97}$$

[5]We also call it ψ'' here to be consistent with the original proof of Damgaard [7], even though we have not defined ψ' here in this version.

to remove this term for now and analyze $\langle\phi|\psi''\rangle$ instead of $\langle\phi|\hat{A}\psi\rangle$. For when we let $a \to \infty$ at the end of this argument, the contributions from this term will vanish anyway. Rewriting this $|\psi''\rangle$, we thus have that

$$\begin{aligned}
|\psi''\rangle &= \sum_{m=0}^{M} |\hat{A}\chi^{(m)}\rangle - \sum_{m=0}^{M}\sum_{l=1}^{n-1} |1_{B^{m+1}_{R_m,3}\setminus(B^{\complement}_{1/R^a_m,3})^{m+1}}\hat{A}^+_m\chi^{(m)}_m\rangle \\
&= \sum_{m=0}^{M} |\hat{A}^-_m\chi^{(m)}_m\rangle + \sum_{m=0}^{M} |1_{D^{(m)}_{m+1}}\hat{A}^+_m\chi^{(m)}_m\rangle - \sum_{m=0}^{M}\sum_{l=1}^{n-1} |1_{B^{m+1}_{R_m,3}\setminus(B^{\complement}_{1/R^a_m,3})^{m+1}}\hat{A}^+_m\chi^{(m)}_m\rangle \\
&= \sum_{m=0}^{M} |\hat{A}^-_m\chi^{(m)}_m\rangle + \sum_{m=0}^{M} |\hat{A}^{R_m+}_m\chi^{(m)}_m\rangle\,,
\end{aligned} \tag{98}$$

where we have used Eq. (90) to get the second equation, and Eq. (96) to get the last. Then for $\langle\phi|\psi''\rangle$, we see that

$$\begin{aligned}
\langle\phi|\psi''\rangle &= \sum_{m=1}^{M} \langle\phi_{m-1}|\hat{A}^{R_m-}_m\chi^{(m)}_m\rangle + \sum_{m=0}^{M} \langle\phi_{m+1}|\hat{A}^{R_m+}_m\chi^{(m)}_m\rangle \\
&= \sum_{m=1}^{M} \langle\hat{A}^{R_m+}_{m-1}\phi_{m-1}|\chi^{(m)}_m\rangle + \sum_{m=0}^{M} \langle\hat{A}^{R_m-}_{m+1}\phi_{m+1}|\chi^{(m)}_m\rangle\,,
\end{aligned} \tag{99}$$

where we for the last equality here have used the result of Eq. (58). And if we take $\hat{A}^{R_m+}_{-1}\phi_{-1}$ to be 0, and use Eq. (92), we thus get that

$$\frac{\langle\phi|\psi''\rangle}{\|\psi\|} = \frac{1}{\|\psi\|}\sum_{m=0}^{M} \langle\hat{A}^{R_m+}_{m-1}\phi_{m-1} + \hat{A}^{R_m-}_{m+1}\phi_{m+1}|\hat{A}^{R+}_{m-1}\phi_{m-1} + \hat{A}^{R-}_{m+1}\phi_{m+1}\rangle\,. \tag{100}$$

We can then note that in order for ϕ to be a member of $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$, the right-hand side of Eq. (100) has to be bounded when we let a and each R_m tend to infinity, as $\langle\phi|\psi''\rangle$ converges to $\langle\phi|\hat{A}\psi\rangle$ in this limit. And it also has to be bounded if we then subsequently let $R, M \to \infty$ as well. We must therefore have

$$\infty > \lim_{R,M\to\infty}\lim_{R_0,\ldots,R_M,a\to\infty}\frac{\langle\phi|\psi''\rangle}{\|\psi\|} \geq \lim_{R\to\infty}\frac{\sum_{m=0}^{\infty}\langle(\hat{A}\phi)_m|(\hat{A}^R\phi)_m\rangle}{\sqrt{C_2}\|\sum_{m=0}^{\infty}|\chi^{(m)}_m\rangle\|} = \frac{\|\hat{A}\phi\|^2}{\sqrt{C_2}\|\hat{A}\phi\|} = \frac{\|\hat{A}\phi\|}{\sqrt{C_2}} \tag{101}$$

for all $\phi \in \mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$, where we have also used Eq. (92) to recognize $\chi^{(m)}_m$ as $(\hat{A}^R\phi)_m$ to get the third equality. This shows us that $\|\hat{A}\phi\| < \infty$ for all $\phi \in \mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$, and we therefore get that $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}}) \subset U$.

To then show that $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}}) \subset V$, we will use a similar strategy, but choose $D^{(m)}_{m+1} = F_{m+1}$ and

$$\chi^{(m)}_m = 1_{F_m}\hat{A}^{R+}_{m-1}\phi_{m-1} \tag{102}$$

instead for all $m \in \{2, \ldots, M\}$. We will also choose $\chi^{(0)}_0$ and $\chi^{(1)}_1$ to be zero everywhere. (It is easy to see that we also have $\|\chi^{(m)}_m\|, \|1_{D^{(m)}_{m+1}\times\mathbb{R}^3}\hat{A}^+_m\chi^{(m)}_m\|, \|\hat{A}^-_m\chi^{(m)}_m\| < \infty$ in this case, by similar arguments as before.) We then first of all note that $\mathrm{supp}(\chi^{(m)}_m) \subset F_m$, which means that $\mathrm{supp}(\hat{A}^+_m\chi^{(m)}_m) \subset F_m \times \mathbb{R}^3$. And since we have from Eq. (27/62) that $F_m \times \mathbb{R}^3 \subset F^{\complement}_{m+1}$, we thus get that

$$\mathrm{supp}(\hat{A}^+_m\chi^{(m)}_m) \subset F^{\complement}_{m+1}. \tag{103}$$

So for our $1_{D^{(m)}_{m+1}} \hat{A}^+_m \chi^{(m)}_m$ function with $D^{(m)}_{m+1} = F_{m+1}$, we therefore get that

$$\mathrm{supp}(1_{D^{(m)}_{m+1}} \hat{A}^+_m \chi^{(m)}_m) \subset (F_{m+1} \cap F^{\complement}_{m+1}) \times \mathbb{R}^3 = \emptyset. \tag{104}$$

This means that $|1_{D^{(m)}_{m+1}} \hat{A}^+_m \chi^{(m)}_m\rangle$ is just a zero vector for each m. And for $\langle \phi | \hat{A} \psi \rangle$, we thus get that

$$\begin{aligned}
\langle \phi | \hat{A} \psi \rangle &= \sum_{m=2}^{M} \langle \phi_{m-1} | \hat{A}^-_m \chi^{(m)}_m \rangle + \sum_{m=2}^{M} \langle \phi_{m+1} | 1_{D^{(m)}_{m+1}} \hat{A}^+_m \chi^{(m)}_m \rangle \\
&= \sum_{m=2}^{M} \langle \phi_{m-1} | \hat{A}^-_m \chi^{(m)}_m \rangle \\
&= \sum_{m=2}^{M} \langle \phi_{m-1} | \hat{A}^-_m 1_{F_m} \hat{A}^{R+}_{m-1} \phi_{m-1} \rangle \,,
\end{aligned} \tag{105}$$

We can then use the fact that $\hat{A}^-_m 1_{F_m} \hat{A}^{R+}_{m-1} = \hat{A}^{R_{m-1}-}_m 1_{F_m} \hat{A}^{R+}_{m-1}$ for sufficiently large and fast-growing values of $R_0, \ldots, R_M$, to further obtain

$$\begin{aligned}
\langle \phi | \hat{A} \psi \rangle &= \sum_{m=2}^{M} \langle \phi_{m-1} | \hat{A}^{R_{m-1}-}_m 1_{F_m} \hat{A}^{R+}_{m-1} \phi_{m-1} \rangle \\
&= \sum_{m=2}^{M} \langle \hat{A}^{R_{m-1}+}_{m-1} \phi_{m-1} | 1_{F_m} \hat{A}^{R+}_{m-1} \phi_{m-1} \rangle \\
&= \sum_{m=2}^{M} \langle 1_{F_m} \hat{A}^{R_{m-1}+}_{m-1} \phi_{m-1} | 1_{F_m} \hat{A}^{R+}_{m-1} \phi_{m-1} \rangle \,.
\end{aligned} \tag{106}$$

We can then repeat the process of letting each $R_m \to \infty$, followed by $R, M \to \infty$. (We can keep a constant in this case.) This then gives us that

$$\infty > \lim_{R,M\to\infty} \lim_{R_0,\ldots,R_M\to\infty} \frac{\langle \phi | \hat{A} \psi \rangle}{\|\psi\|} \geq \frac{\sum_{m=2}^{\infty} \|1_{F_m} \hat{A}^+_{m-1} \phi_{m-1}\|^2}{\sqrt{C_2}\,\left\| \sum_{m=2}^{\infty} |1_{F_m} \hat{A}^+_{m-1} \phi_{m-1}\rangle \right\|} = \frac{\|\hat{1}_T \hat{A}^+ \phi\|}{\sqrt{C_2}} \tag{107}$$

for all $\phi \in \mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$, where we have used Eq. (17) to get the last equality. We thus obtain that $\|\hat{1}_T \hat{A}^+ \phi\| < \infty$ for all $\phi \in \mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$, and we can therefore also conclude that $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}}) \subset V$, i.e. from Eq. (18).

This completes the proof as we have now shown that $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}}) \subset U \cap V = \mathrm{Dom}(\hat{H}_{\mathrm{I}})$. And since we already have that $\mathrm{Dom}(\hat{H}_{\mathrm{I}}) \subset \mathrm{Dom}(\hat{H}^*_{\mathrm{I}})$ due to $\hat{H}_{\mathrm{I}}$ being symmetric, we thus get that $\mathrm{Dom}(\hat{H}^*_{\mathrm{I}}) = \mathrm{Dom}(\hat{H}_{\mathrm{I}})$. We also already have that $\hat{H}^*_{\mathrm{I}} \psi = \hat{H}_{\mathrm{I}} \psi$ for all $\psi \in \mathrm{Dom}(\hat{H}_{\mathrm{I}})$, and we can therefore conclude that $\hat{H}_{\mathrm{I}}$ is self-adjoint. And since we have thus shown both that $\mathrm{Dom}(\hat{H}_{\mathrm{I}})$ is dense in $\mathbf{H}$ and that $\hat{H}_{\mathrm{I}}$ is self-adjoint, we hereby obtain Proposition 1.

9 Conclusion

We have defined a further simplified version of the Dirac interaction operator, $\hat{H}_{\mathrm{I}}$, with no cutoffs on a certain domain. We have then shown, in Sections 3–6, that this domain is dense in the Hilbert space. In Section 7, we have furthermore shown that $\hat{H}_{\mathrm{I}}$ is symmetric on the domain. And finally, in Section 8, we have shown that $\hat{H}_{\mathrm{I}}$ is also self-adjoint on the domain, thus obtaining Proposition 1.

Appendices

A Symmetrizing the Hilbert space

See Damgaard [7], Appendix A.

B Adding a free energy to the operator

See Damgaard [7], Appendix B.

C Handling the perturbed vacuum

See Damgaard [7], Appendix C, or better yet, see Damgaard [8].

D On the motivation for proving Proposition 1

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