Golden Ratio Fixed Point

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Exercise 1.13 of *Structure and Interpretation of Computer Programs* asks us to

> Show that the golden ratio φ (Section 1.2.2) is a fixed point of the transformation $x \mapsto 1 + 1/x$, and use this fact to compute φ by means of the `fixed-point` procedure.

Definitions

The golden ratio φ is the positive real number that satisfies

$$\varphi = 1 + \frac{1}{\varphi}.$$

Multiplying by φ, we get another property of the golden ratio:

$$\varphi^2 = \varphi + 1.$$

Proof

We would like to prove that each transformation $x \mapsto 1 + 1/x$ causes the value of x to become closer to φ. Given an initial guess $x > 1$, let $y = 1 + 1/x$ be the improved guess. We must prove that $|y - \varphi| < |x - \varphi|$. To begin, we will simplify the left-hand side of the inequality:

$$\begin{aligned}
|y - \varphi| &= \left| 1 + \frac{1}{x} - \varphi \right| \\
&= \left| \frac{x + 1 - \varphi x}{x} \right| \\
&= \left| \frac{x + 1 - (1 + 1/\varphi)x}{x} \right| \\
&= \left| \frac{x + 1 - x - x/\varphi}{x} \right| \\
&= \left| \frac{\varphi - x}{x\varphi} \right|.
\end{aligned}$$

The error of the improved guess, $|y - \varphi|$, is related to the error of the orignal guess, $|x - \varphi|$. We can rearrange a bit more to make this clear:

$$|y - \varphi| = \left| \frac{\varphi - x}{x\varphi} \right| = \frac{|x - \varphi|}{|x\varphi|}.$$

To prove that $|y - \varphi| < |x - \varphi|$, we must show that $|x\varphi| > 1$, because dividing the original error by a number greater than one will produce a new error smaller than the original. We already stipulated that $x > 1$, and we know the value of the golden ratio is $\varphi = (1 + \sqrt{5})/2$, therefore $x\varphi > 1$ and $|x\varphi| > 1$.

Since $|xp| > 1$ and $|y - \varphi| = |x - \varphi| / |xp|$, we have

$$|y - \varphi| < |x - \varphi| ,$$

therefore each transformation $x \mapsto y$ brings x closer to φ, and consequently φ is a fixed point of the transformation $x \mapsto 1 + 1/x$. ■