

Proof exercises from SICP

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16 August 2014

Exercise 1.13

Prove that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1 + \sqrt{5})/2$. Hint: Let $\psi = (1 - \sqrt{5})/2$. Use induction to prove that $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

The constants φ and ψ are the positive and negative solutions to the golden ratio equation for a rectangle with side lengths of 1 and x :

$$\frac{1}{x} = \frac{x}{1+x}.$$

Both φ and ψ satisfy two properties:

$$1+x = x^2 \quad \text{and} \quad \frac{1}{x} + 1 = x, \quad \forall x \in \{\varphi, \psi\}.$$

The Fibonacci sequence begins with 0 and 1; each subsequent element is the sum of the two elements preceding it:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

Using the Fib function, we can define the sequence recursively with

$$\text{Fib}(0) = 0;$$

$$\text{Fib}(1) = 1;$$

$$\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2).$$

Proof. We will begin by proving by induction that

$$\text{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}. \tag{1}$$

First, we will demonstrate that equation 1 is true for the three base cases: Fib(0), Fib(1), and Fib(2). When $n = 0$, $LS = \text{Fib}(0) = 0$ and

$$RS = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0.$$

When $n = 1$, $LS = \text{Fib}(1) = 1$ and

$$RS = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1.$$

When $n = 2$, $LS = \text{Fib}(2) = 1$ and

$$\begin{aligned} RS &= \frac{\varphi^2 - \psi^2}{\sqrt{5}} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\ &= \frac{\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4}}{\sqrt{5}} \\ &= \frac{((1 + \sqrt{5}) - (1 - \sqrt{5}))((1 + \sqrt{5}) + (1 - \sqrt{5}))}{4\sqrt{5}} \\ &= \frac{(2\sqrt{5})(2)}{4\sqrt{5}} \\ &= 1. \end{aligned}$$

Now comes the inductive step. For equation 1 to be true for the entire sequence, we must be able to prove that

$$\text{Fib}(n) = \text{Fib}(n - 1) + \text{Fib}(n - 2)$$

using equation 1 as the definition of Fib. We know that $LS = (\varphi^n - \psi^n)/\sqrt{5}$. We can show that the right-hand side is equal:

$$\begin{aligned}
\text{RS} &= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{(\varphi^{n-1} + \varphi^{n-2}) - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}} \\
&= \frac{\varphi^n (\varphi^{-1} + \varphi^{-2}) - \psi^n (\psi^{-1} + \psi^{-2})}{\sqrt{5}} \\
&= \frac{\varphi^n \varphi^{-1} (1 + \varphi^{-1}) - \psi^n \psi^{-1} (1 + \psi^{-1})}{\sqrt{5}} \\
&= \frac{\varphi^n \varphi^{-1} (\varphi) - \psi^n \psi^{-1} (\psi)}{\sqrt{5}} \\
&= \frac{\varphi^n - \psi^n}{\sqrt{5}}.
\end{aligned}$$

This completes the proof of equation 1.

Now we must show that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$. The absolute difference between these two values must remain less than or equal to one half for the latter to round to the former:

$$\begin{aligned}
\left| \text{Fib}(n) - \frac{\varphi^n}{\sqrt{5}} \right| &\leq \frac{1}{2} \\
\left| \frac{\varphi^n - \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}} \right| &\leq \frac{1}{2} \\
\left| -\frac{\psi^n}{\sqrt{5}} \right| &\leq \frac{1}{2} \\
\frac{|-\psi^n|}{\sqrt{5}} &\leq \frac{1}{2} \\
|\psi^n| &\leq \frac{\sqrt{5}}{2}.
\end{aligned} \tag{2}$$

The value of ψ is about -0.618 , therefore its absolute value is about 0.618 . For all nonnegative values of n , $|\psi^n| \leq 1$. The value of $\sqrt{5}/2$ is about 1.12 , which is greater than 1 . Therefore, equation 2 is true and $\varphi^n/\sqrt{5}$ will always round to $\text{Fib}(n)$. \square

Exercise 1.35

Show that the golden ratio φ (Section 1.2.2) is a fixed point of the transformation $x \mapsto 1 + 1/x$, and use this fact to compute φ by means of the **fixed-point** procedure.

The golden ratio φ is the positive real number that satisfies

$$\varphi = 1 + \frac{1}{\varphi}.$$

Multiplying by φ , we get another property of the golden ratio:

$$\varphi^2 = \varphi + 1.$$

Proof. We would like to prove that each transformation $x \mapsto 1 + 1/x$ causes the value of x to become closer to φ . Given an initial guess $x > 1$, let $y = 1 + 1/x$ be the improved guess. We must prove that $|y - \varphi| < |x - \varphi|$. To begin, we will simplify the left-hand side of the inequality:

$$\begin{aligned} |y - \varphi| &= \left| 1 + \frac{1}{x} - \varphi \right| \\ &= \left| \frac{x + 1 - \varphi x}{x} \right| \\ &= \left| \frac{x + 1 - (1 + 1/\varphi)x}{x} \right| \\ &= \left| \frac{x + 1 - x - x/\varphi}{x} \right| \\ &= \left| \frac{\varphi - x}{x\varphi} \right| \\ &= \frac{|x - \varphi|}{|x\varphi|}. \end{aligned}$$

We now have a relation between $|y - \varphi|$ and $|x - \varphi|$. To prove that $|y - \varphi| < |x - \varphi|$, we must show that $|x\varphi| > 1$, because dividing the original error by a number greater than one will produce a smaller value as the new error. We already stipulated that $x > 1$, and we know the value of the golden ratio is $\varphi = (1 + \sqrt{5})/2 \approx 1.618 > 1$, therefore $x\varphi > 1$ and $|x\varphi| > 1$.

Since $|xp| > 1$ and $|y - \varphi| = |x - \varphi| / |xp|$, we have

$$|y - \varphi| < |x - \varphi|,$$

therefore each transformation $x \mapsto y$ brings x closer to φ , and consequently φ is a fixed point of the transformation $x \mapsto 1 + 1/x$. \square