Fibonacci and the Golden Ratio

Mitchell Kember

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Exercise 1.13 of *Structure and Interpretation of Computer Programs* asks us to

> Prove that $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction to prove that $\mathrm{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

Definitions

The constants φ and ψ are the positive and negative solutions to the golden ratio equation for a rectangle with side lengths of 1 and x:

$$\frac{1}{x} = \frac{x}{1+x}.$$

Both φ and ψ satisfy the following properties:

$$1 + x = x^2 \qquad \text{and} \qquad \frac{1}{x} + 1 = x.$$

The Fibonacci sequence begins with 0 and 1; each subsequent element is the sum of the two elements preceding it:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Using the Fib function, we can define the sequence recursively with

$$\begin{aligned}
\mathrm{Fib}(0) &= 0;\\
\mathrm{Fib}(1) &= 1;\\
\mathrm{Fib}(n) &= \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2).
\end{aligned}$$

Proof

We will begin by proving by induction that

$$\mathrm{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}. \tag{1}$$

First, we will demonstrate that equation (1) is true for the three base cases: Fib(0), Fib(1), and Fib(2). When $n = 0$, $\mathrm{LS} = \mathrm{Fib}(0) = 0$ and

$$\begin{aligned}
\mathrm{RS} &= \frac{\varphi^0 - \psi^0}{\sqrt{5}} \\
&= \frac{1 - 1}{\sqrt{5}} \\
&= 0.
\end{aligned}$$

When $n = 1$, $\mathrm{LS} = \mathrm{Fib}(1) = 1$ and

$$\begin{aligned}
\mathrm{RS} &= \frac{\varphi^1 - \psi^1}{\sqrt{5}} \\
&= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \\
&= 1.
\end{aligned}$$

When $n = 2$, $\mathrm{LS} = \mathrm{Fib}(2) = 1$ and

$$\begin{aligned}
\mathrm{RS} &= \frac{\varphi^2 - \psi^2}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\frac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{4}}{\sqrt{5}} \\
&= \frac{\left(\left(1+\sqrt{5}\right) - \left(1-\sqrt{5}\right)\right)\left(\left(1+\sqrt{5}\right) + \left(1-\sqrt{5}\right)\right)}{4\sqrt{5}} \\
&= \frac{\left(2\sqrt{5}\right)(2)}{4\sqrt{5}} \\
&= 1.
\end{aligned}$$

Now comes the inductive step. For equation (1) to be true for the entire sequence, we must be able to prove that

$$\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$$

using equation (1) as the definiton of Fib. We know that $\mathrm{LS} = (\varphi^n - \psi^n)/\sqrt{5}$.

We can show that the right-hand side is equal:

$$
\begin{aligned}
\mathrm{RS} &= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\left(\varphi^{n-1} + \varphi^{n-2}\right) - \left(\psi^{n-1} + \psi^{n-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\left(\varphi^{-1} + \varphi^{-2}\right) - \psi^{n}\left(\psi^{-1} + \psi^{-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(1 + \varphi^{-1}\right) - \psi^{n}\psi^{-1}\left(1 + \psi^{-1}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(\varphi\right) - \psi^{n}\psi^{-1}\left(\psi\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n} - \psi^{n}}{\sqrt{5}}.
\end{aligned}
$$

Now we must show that $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$. The absolute difference between these two values must remain less than or equal to one half for the latter to round to the former:

$$
\begin{aligned}
\left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|\frac{\varphi^n - \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|-\frac{\psi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\frac{\left|-\psi^n\right|}{\sqrt{5}} &\leq \frac{1}{2} \\
\left|\psi^n\right| &\leq \frac{\sqrt{5}}{2}.
\end{aligned}
$$

The value of ψ is about -0.618, therefore its absolute value is about 0.618. For all nonnegative values of n, $|\psi^n| \leq 1$. The value of $\sqrt{5}/2$ is about 1.12, which is greater than 1. Therefore, the equation above is true and $\varphi^n/\sqrt{5}$ will always round to $\mathrm{Fib}(n)$. ■