Proof exercises from SICP

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Exercise 1.13

> Prove that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction to prove that $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

The constants φ and ψ are the positive and negative solutions to the golden ratio equation for a rectangle with side lengths of 1 and x:

$$\frac{1}{x} = \frac{x}{1+x}.$$

Both φ and ψ satisfy two properties:

$$1 + x = x^2 \quad \text{and} \quad \frac{1}{x} + 1 = x, \quad \forall x \in \{\varphi, \psi\}.$$

The Fibonacci sequence begins with 0 and 1; each subsequent element is the sum of the two elements preceding it:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Using the Fib function, we can define the sequence recursively with

$$\begin{aligned}\text{Fib}(0) &= 0;\\ \text{Fib}(1) &= 1;\\ \text{Fib}(n) &= \text{Fib}(n-1) + \text{Fib}(n-2).\end{aligned}$$

Proof. We will begin by proving by induction that

$$\text{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}. \tag{1}$$

First, we will demonstrate that equation 1 is true for the three base cases: Fib(0), Fib(1), and Fib(2). When $n = 0$, LS = Fib(0) = 0 and

$$\mathrm{RS} = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0.$$

When $n = 1$, LS = Fib(1) = 1 and

$$\mathrm{RS} = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1.$$

When $n = 2$, LS = Fib(2) = 1 and

$$\begin{aligned}
\mathrm{RS} &= \frac{\varphi^2 - \psi^2}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\frac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{4}}{\sqrt{5}} \\
&= \frac{\left((1+\sqrt{5}) - (1-\sqrt{5})\right)\left((1+\sqrt{5}) + (1-\sqrt{5})\right)}{4\sqrt{5}} \\
&= \frac{\left(2\sqrt{5}\right)(2)}{4\sqrt{5}} \\
&= 1.
\end{aligned}$$

Now comes the inductive step. For equation 1 to be true for the entire sequence, we must be able to prove that

$$\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$$

using equation 1 as the definiton of Fib. We know that LS $= (\varphi^n - \psi^n)/\sqrt{5}$. We can show that the right-hand side is equal:

$$
\begin{aligned}
\mathrm{RS} &= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\left(\varphi^{n-1} + \varphi^{n-2}\right) - \left(\psi^{n-1} + \psi^{n-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\left(\varphi^{-1} + \varphi^{-2}\right) - \psi^{n}\left(\psi^{-1} + \psi^{-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(1 + \varphi^{-1}\right) - \psi^{n}\psi^{-1}\left(1 + \psi^{-1}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(\varphi\right) - \psi^{n}\psi^{-1}\left(\psi\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n} - \psi^{n}}{\sqrt{5}}.
\end{aligned}
$$

This completes the proof of equation 1.

Now we must show that $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$. The absolute difference between these two values must remain less than or equal to one half for the latter to round to the former:

$$
\begin{aligned}
\left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \qquad (2) \\
\left|\frac{\varphi^n - \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|-\frac{\psi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\frac{\left|-\psi^n\right|}{\sqrt{5}} &\leq \frac{1}{2} \\
\left|\psi^n\right| &\leq \frac{\sqrt{5}}{2}.
\end{aligned}
$$

The value of ψ is about -0.618, therefore its absolute value is about 0.618. For all nonnegative values of n, $|\psi^n| \leq 1$. The value of $\sqrt{5}/2$ is about 1.12, which is greater than 1. Therefore, equation 2 is true and $\varphi^n/\sqrt{5}$ will always round to $\mathrm{Fib}(n)$. $\square$

Exercise 1.35

> Show that the golden ratio φ (Section 1.2.2) is a fixed point of the transformation $x \mapsto 1 + 1/x$, and use this fact to compute φ by means of the `fixed-point` procedure.

The golden ratio φ is the positive real number that satisfies

$$\varphi = 1 + \frac{1}{\varphi}.$$

Multiplying by φ, we get another property of the golden ratio:

$$\varphi^2 = \varphi + 1.$$

Proof. We would like to prove that each transformation $x \mapsto 1 + 1/x$ causes the value of x to become closer to φ. Given an initial guess $x > 1$, let $y = 1 + 1/x$ be the improved guess. We must prove that $|y - \varphi| < |x - \varphi|$. To begin, we will simplify the left-hand side of the inequality:

$$\begin{aligned}
|y - \varphi| &= \left|1 + \frac{1}{x} - \varphi\right| \\
&= \left|\frac{x + 1 - \varphi x}{x}\right| \\
&= \left|\frac{x + 1 - (1 + 1/\varphi)x}{x}\right| \\
&= \left|\frac{x + 1 - x - x/\varphi}{x}\right| \\
&= \left|\frac{\varphi - x}{x\varphi}\right| \\
&= \frac{|x - \varphi|}{|x\varphi|}.
\end{aligned}$$

We now have a relation between $|y - \varphi|$ and $|x - \varphi|$. To prove that $|y - \varphi| < |x - \varphi|$, we must show that $|x\varphi| > 1$, because dividing the original error by a number greater than one will produce a smaller value as the new error. We already stipulated that $x > 1$, and we know the value of the golden ratio is $\varphi = (1 + \sqrt{5})/2 \approx 1.618 > 1$, therefore $x\varphi > 1$ and $|x\varphi| > 1$.

Since $|xp| > 1$ and $|y - \varphi| = |x - \varphi| / |xp|$, we have

$$|y - \varphi| < |x - \varphi|,$$

therefore each transformation $x \mapsto y$ brings x closer to φ, and consequently φ is a fixed point of the transformation $x \mapsto 1 + 1/x$. $\square$