Proof exercises from SICP

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Exercise 1.13

> Prove that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction to prove that $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

The constants φ and ψ are the positive and negative solutions to the golden ratio equation for a rectangle with side lengths of 1 and x:

$$\frac{1}{x} = \frac{x}{1+x}.$$

Both φ and ψ satisfy two properties:

$$1 + x = x^2 \quad \text{and} \quad \frac{1}{x} + 1 = x, \quad \forall x \in \{\varphi, \psi\}.$$

The Fibonacci sequence begins with 0 and 1; each subsequent element is the sum of the two elements preceding it:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Using the Fib function, we can define the sequence recursively with

$$\begin{aligned}
\text{Fib}(0) &= 0, \\
\text{Fib}(1) &= 1, \\
\text{Fib}(n) &= \text{Fib}(n-1) + \text{Fib}(n-2).
\end{aligned}$$

Lemma 1. The exact value of $\text{Fib}(n)$ is given by

$$\text{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}. \tag{1}$$

Proof. First, we will demonstrate that equation 1 is true for the three base cases: Fib(0), Fib(1), and Fib(2). When $n = 0$, LS $=$ Fib$(0) = 0$ and

$$\text{RS} = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0.$$

When $n = 1$, LS $=$ Fib$(1) = 1$ and

$$\text{RS} = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1.$$

When $n = 2$, LS $=$ Fib$(2) = 1$ and

$$\begin{aligned}
\text{RS} &= \frac{\varphi^2 - \psi^2}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\frac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{4}}{\sqrt{5}} \\
&= \frac{\left((1+\sqrt{5}) - (1-\sqrt{5})\right)\left((1+\sqrt{5}) + (1-\sqrt{5})\right)}{4\sqrt{5}} \\
&= \frac{\left(2\sqrt{5}\right)(2)}{4\sqrt{5}} \\
&= 1.
\end{aligned}$$

Now comes the inductive step. For equation 1 to be true for the entire sequence, we must be able to prove that

$$\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2)$$

using equation 1 as the definiton of Fib. We know that LS $= (\varphi^n - \psi^n)/\sqrt{5}$. We can show that the right-hand side is equal:

$$
\begin{aligned}
\mathrm{RS} &= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\left(\varphi^{n-1} + \varphi^{n-2}\right) - \left(\psi^{n-1} + \psi^{n-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\left(\varphi^{-1} + \varphi^{-2}\right) - \psi^{n}\left(\psi^{-1} + \psi^{-2}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(1 + \varphi^{-1}\right) - \psi^{n}\psi^{-1}\left(1 + \psi^{-1}\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n}\varphi^{-1}\left(\varphi\right) - \psi^{n}\psi^{-1}\left(\psi\right)}{\sqrt{5}} \\
&= \frac{\varphi^{n} - \psi^{n}}{\sqrt{5}}.
\end{aligned}
$$

This completes the proof of equation 1. □

Theorem 1. $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$.

Proof. The absolute difference between these two values must remain less than or equal to one half for the latter to round to the former:

$$
\left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}}\right| \leq \frac{1}{2} \tag{2}
$$

$$
\begin{aligned}
\left|\frac{\varphi^n - \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|-\frac{\psi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\frac{|-\psi^n|}{\sqrt{5}} &\leq \frac{1}{2} \\
|\psi^n| &\leq \frac{\sqrt{5}}{2}.
\end{aligned}
$$

The value of ψ is about -0.618, therefore its absolute value is about 0.618. For all nonnegative values of n, $|\psi^n| \leq 1$. The value of $\sqrt{5}/2$ is about 1.12, which is greater than 1. Therefore, equation 2 is true and $\varphi^n/\sqrt{5}$ will always round to $\mathrm{Fib}(n)$. □

Exercise 1.35

> Show that the golden ratio φ (Section 1.2.2) is a fixed point of the transformation $x \mapsto 1 + 1/x$, and use this fact to compute φ by means of the `fixed-point` procedure.

The golden ratio φ is the positive real number that satisfies

$$\varphi = 1 + \frac{1}{\varphi}.$$

Multiplying by φ, we get another property of the golden ratio:

$$\varphi^2 = \varphi + 1.$$

For φ to be a fixed point, each repeated transformation $x \mapsto 1 + 1/x$ must cause the value of x to become closer to φ.

Theorem 2. Given any $x > 1$ and $y = 1/1 + x$, y is closer than x to φ, meaning x and y satisfy $|y - \varphi| < |x - \varphi|$.

Proof. To begin, we will simplify the left-hand side of the inequality:

$$\begin{aligned}
|y - \varphi| &= \left|1 + \frac{1}{x} - \varphi\right| \\
&= \left|\frac{x + 1 - \varphi x}{x}\right| \\
&= \left|\frac{x + 1 - (1 + 1/\varphi)x}{x}\right| \\
&= \left|\frac{x + 1 - x - x/\varphi}{x}\right| \\
&= \left|\frac{\varphi - x}{x\varphi}\right| \\
&= \frac{|x - \varphi|}{|x\varphi|}.
\end{aligned}$$

We already stipulated that $x > 1$, and we know the value of the golden ratio is $\varphi = (1 + \sqrt{5})/2 \approx 1.618 > 1$, therefore $x\varphi > 1$ and $|x\varphi| > 1$.

Since $|xp| > 1$ and $|y - \varphi| = |x - \varphi| / |xp|$, we have

$$|y - \varphi| < |x - \varphi|,$$

therefore y is closer than x to φ, and consequently φ is a fixed point of the transformation $x \mapsto 1 + 1/x$. □

Exercise 3.57

> How many additions are performed when we compute the nth Fibonacci number using the definition of `fibs` based on the `add-streams` procedure. Show that the number of additions would be exponentially greater if we had implemented `(delay` ⟨`exp`⟩`)` simply as `(lambda ()` ⟨`exp`⟩`)`, without using the optimization provided by the `memo-proc` procedure described in Section 3.5.1.

Let $A(n)$ represent the number of additions required to compute $\mathrm{Fib}(n)$ using `fibs`. The sequence begins with $\mathrm{Fib}(0) = 0$ and $\mathrm{Fib}(1) = 1$.

Theorem 3. Using `fibs` and the optimized implementation of `delay`, the number of additions grows as $\Theta(n)$; specifically,

$$A(0) = 0 \quad \text{and} \quad A(n) = n - 1,\ n > 1. \tag{3}$$

Proof. Since 0 and 1 form the base case of the procedure, they require no additions and therefore $A(0) = A(1) = 0$. To get the next element, we add the first element of the sequence, 0, to the first element of the `cdr` of the sequence, 1. We have $\mathrm{Fib}(2) = 0 + 1 = 1$ and $A(2) = 1$. In general,

$$A(n) = A(n-1) + 1, \tag{4}$$

because the next element is obtained by doing one more addition.

Equation 3 correctly produces zero for $n = 0$ and for $n = 1$. We can show that it is also true for all $n > 1$ by mathematical induction. If it is true, we should be able to verify equation 4 by substituting $n - 1$ for $A(n)$. We have $\mathrm{LS} = A(n) = n - 1$, and

$$\mathrm{RS} = A(n-1) + 1 = ((n-1) - 1) + 1 = n - 1 = \mathrm{LS},$$

therefore equation 3 is true for all $n \geq 0$. □

Theorem 4. Using `fibs` and the implementation of `delay` that does *not* use memoization, the number of additions grows as $\Theta(e^n)$; specifically,

$$A(n) = \frac{1}{4}\left(\left(1-\sqrt{2}\right)^n + \left(1+\sqrt{2}\right)^n - 2\right). \tag{5}$$

Proof. As before, $A(0) = A(1) = 0$, and equation 5 satisfies both of these. To get the next element, we add 0 to 1, so $A(2) = 1$. To compute $\mathrm{Fib}(3)$, we must repeat the one addition of $\mathrm{Fib}(2)$, repeat it again to extend the `cdr` of the sequence one more element, giving us $\mathrm{Fib}(3) = 1 + 1 = 2$ and $A(3) = 3$. In general,

$$A(n) = A(n-2) + 2A(n-1) + 1. \tag{6}$$

Here is why: to get the next element, we must repeat the work of getting the previous element in the stream: $A(n-1)$. We must do it again to extend the `cdr` of the sequence to that same element, because it will be the addend. We must then compute $\mathrm{Fib}(n-2)$ to extend the other stream going into `add-streams`, because this element will be the augend. Finally, we add the addend to the augend, and this is the last addition.

We can once again use induction to prove equation 5 for all $n > 1$. Let $a = 1 - \sqrt{2}$ and let $b = 1 + \sqrt{2}$. Substituing it into equation 6, we have

$$\mathrm{LS} = \frac{1}{4}\left(a^n + b^n - 2\right),$$

and on the other side, we have

$$\begin{aligned}
\mathrm{RS} &= A(n-2) + 2A(n-1) + 1 \\
&= \frac{1}{4}\left(a^{n-2} + b^{n-2} - 2\right) + 2 \cdot \frac{1}{4}\left(a^{n-1} + b^{n-1} - 2\right) + 1 \\
&= \frac{1}{4}a^{n-2} + \frac{1}{4}b^{n-2} - \frac{1}{2} + \frac{1}{2}a^{n-1} + \frac{1}{2}b^{n-1} - 1 + 1 \\
&= \frac{1}{4}\left(a^{n-2} + b^{n-2} + 2a^{n-1} + 2b^{n-1} - 2\right) \\
&= \frac{1}{4}\left(a^n a^{-2} + b^n b^{-2} + 2a^n a^{-1} + 2b^n b^{-1} - 2\right) \\
&= \frac{1}{4}\left(a^n\left(a^{-2} + 2a^{-1}\right) + b^n\left(b^{-2} + 2b^{-1}\right) - 2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left(a^n\left(\frac{1}{a^2}+\frac{2}{a}\right)+b^n\left(\frac{1}{b^2}+\frac{2}{b}\right)-2\right)\\
&= \frac{1}{4}\left(a^n\left(\frac{1+2a}{a^2}\right)+b^n\left(\frac{1+2b}{b^2}\right)-2\right)\\
&= \frac{1}{4}\left(a^n\left(\frac{1+2\left(1-\sqrt{2}\right)}{\left(1-\sqrt{2}\right)^2}\right)+b^n\left(\frac{1+2\left(1+\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^2}\right)-2\right)\\
&= \frac{1}{4}\left(a^n\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)+b^n\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)-2\right)\\
&= \frac{1}{4}\left(a^n+b^n-2\right)\\
&= \mathrm{LS},
\end{aligned}
$$

therefore equation 5 is true for all $n \geq 0$. $\square$