Graded Homework 1, exercise 4

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Vertex Cover (15 points)

Consider the linear program that is used to 2-approximate the weighted vertex cover problem, that is we minimize $\sum_{u \in V(G)} w_u x_u$ given $x_u + x_v \geq 1$ for every edge uv and $x_u \geq 0$ for each node u. Prove that there is a polynomial time algorithm that finds an optimum solution to this linear program such that it has the additional property that for every node u we have $x_u \in \{0, 1/2, 1\}$.

Solution

We start by taking the valid solution that we get from linear programming, and then we transform it in a way that we get $x_u \in \{0, 1/2, 1\}$. To do so we create an iterative step that gets as input a valid solution, where some x_u are in $\{0, 1/2, 1\}$ and some are not; and outputs a solution where all the x_u that were inside $\{0, 1/2, 1\}$ in the input, will remain inside in the output; and at least one of those x_u which were not inside $\{0, 1/2, 1\}$ in the input, will be inside $\{0, 1/2, 1\}$ in the output. Furthermore, the output solution will have a cost no greater than the input one.

If we can devise an algorithm to run such step in polynomial time, then we are done. This is because we can iterate the algorithm n times, where n is the number of vertices. At each iteration one more x_u will be inside $\{0, 1/2, 1\}$. Since we do n iterations, in the end, we get a solution with the same cost as the original one (minimal), and with all the $x_u \in \{0, 1/2, 1\}$.

Iteration step: The iteration step gets as input $x_u, \forall u \in V$. Let's define

$$\mathcal{V}_+ = \{u \in V \mid 1/2 < x_u < 1\}$$
$$\mathcal{V}_- = \{u \in V \mid 0 < x_u < 1/2\}$$

Here we assume that either $\mathcal{V}_+$ or $\mathcal{V}_-$ are not empty. If they are empty, then we are in a trivial situation. This is because $x_u \notin \{0, 1/2, 1\} \Rightarrow x_u > 1$. But if that is the case we can just update $\hat{x}_u = 1 \forall u$ such that $x_u > 1$. This will still yield a valid solution and decrease the cost (notice that this cannot happen in practice, because this would imply that our LP solution is not optimal).

Now we create a new solution:

$$\hat{x}_u(\delta) = \begin{cases} x_u - \delta & u \in \mathcal{V}_+ \\ x_u + \delta & u \in \mathcal{V}_- \\ x_u & \text{otherwise} \end{cases}$$

We investigate for which value of δ, $\hat{x}_u(\delta)$ is still a solution. The conditions to be respected are:

1. $\hat{x}_u(\delta) \geq 0$: therefore $\delta \leq \min_{u \in \mathcal{V}_+} x_u, \delta \geq -\min_{u \in \mathcal{V}_-} x_u$

2. Given an edge uv, assuming $u \in \mathcal{V}_+$ and $v \in \mathcal{V}_-$, $\hat{x}_u(\delta) + \hat{x}_v(\delta) \geq 1$. This is always true since

$$\hat{x}_u(\delta) + \hat{x}_v(\delta) = x_u - \cancel{\delta} + x_v + \cancel{\delta} = x_u + x_v \geq 1.$$

   The same holds if $u \in \mathcal{V}_-$ and $v \in \mathcal{V}_+$.

3. Given an edge uv, assuming $u \in \mathcal{V}_+$ and $x_v = 1/2$. Then we have

$$\hat{x}_u(\delta) + \hat{x}_v(\delta) = x_u - \delta + 1/2 \geq 1$$

   Therefore we have that $\delta \leq x_u - 1/2$.

If we think about, all the other scenarios are not problematic. If we have one of the vertices is 0, then the other one must be at least 1, therefore it will not be inside $\mathcal{V}_+$ or $\mathcal{V}_-$, and it will not be modified. If we have one of the vertices is 1, then the only condition for the other vertex is that it cannot drop below 0, but this has already been encoded in condition 1. Finally, if one of vertices is in $\mathcal{V}_-$, then the other one must either be in $\mathcal{V}_+$ (condition 2) or it's ≥ 1. It cannot be less than one, otherwise the input solution would not be valid.

Now we notice that δ can be taken to be > 0 or < 0. This is because:

- $\min_{u \in \mathcal{V}_+} x_u \geq .5 > 0$
- $x_u - 1/2 > 0$ since $x_u \in \mathcal{V}_+$, and $x_u > 1/2$
- $-\min_{u \in \mathcal{V}_-} x_u < 0$

From this we can look at

$$D := \partial_\delta \left(\sum_{u \in V} w_u x_u \right)$$

But, since the solution is optimal, we know that $D = 0$. If, by contradiction, $D \neq 0$, we would have that picking a small δ, either positive or negative, because the cost is linear in δ, we improves the result. But this is impossible by optimality. Hence $D = 0$, hence any δ which is valid, yields the same cost as the original problem.

Now, we take $\hat{\delta}$ to be the largest δ that is still valid. Since it's valid, of course $\hat{x}_u(\hat{\delta})$ is still a solution. This would be our output.

The last thing to show is that $\hat{x}_u(\hat{\delta})$ has one more variable in $\{0, 1/2, 1\}$. But this is not hard: assume that the dominating condition for $\hat{\delta}$ is condition 1, in this scenario, there would be x_u that goes to 0. In the case where the dominating condition is condition 3, then we have that $\hat{\delta} = x_u - 1/2$. This would yield that:

$$\hat{x}_u(\hat{\delta}) = x_u - (x_u - 1/2) = 1/2.$$

And therefore, we still have one more variable in $\{0, 1/2, 1\}$. Finally, it's easy to see that all the variables which where already in $\{0, 1/2, 1\}$ have not been changed.