Graded Homework 1, exercise 4

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Number of Minimum Cuts (25 points)

In the class, it was shown that the number of minimum cuts of any unweighted graph is at most $\binom{n}{2}$. Here we want to discuss two generalizations of this result.

1. In the lecture we proved that the number of minimum cuts in any unweighted graph is at most $\binom{n}{2}$. Prove the same result for *weighted graphs.* All the weights are positive.
2. For a graph $G = (V, E)$, a subset of edges $E' \subseteq E$ is a k-cut if $G' = (V, E \setminus E')$ has at least k connected components. Prove that the number of minimum k-cuts in any unweighted graph is at most $2^{O(k)} n^{2k-2}$.

Solutions

1. Let $G = (V, E, w)$ be our weighted graph, $w : E \to \mathbb{R}^+$. Here $w(e)$ is the weight of the edge e. Let $n = \#V$. Let M be the minimum cut size.

   To prove that the number of minimum cut is at most $\binom{n}{2}$, we use a modified version of Karger's random contraction algorithm. When we sample edges, we don't do it uniformly, but the probability of sampling e from E' will be

   $$\Pr(\text{samplig } e \text{ from } E') = \frac{w(e)}{\sum_{e \in E'} w(e)}.$$

   We now prove the following lemma

   Lemma 1 (Karger's efficency for weighted graphs)**.** *Given a minimum cut $\{S, V \setminus S\}$ of size M. The probability that the modified Karger's algorithm selects it is at least* $\frac{1}{\binom{n}{2}}$

   Proof. Let $\tilde{E}$ be the set of edges that cut $\{S, V \setminus S\}$. We know that Krager's algorithm runs in no more than $n - 2$ steps. We say that the algorithm fails if at any timestep it samples an element from $\tilde{E}$. Note that if the algorithm doesn't fail it outputs $\{S, V \setminus S\}$. To have more insight about why this is true one can look into Karger's algorithm on the lecture notes chapter 13.

   Failing at step i: We now look at the probability of failing at step i assuming we didn't fail before.

   The probability of failing corresponds to the probability of sampling $e \in \tilde{E}$ from E_i. E_i is the set of edges in the contracted graph, V_i is the set of vertices which have not been contradiction. Note that $\#V_i = n - i + 1$. Note that $\sum_{e \in E_i} w(e) \geq \frac{(n-i+1)M}{2}$. To show this we rewrite it as

$$\begin{aligned}
\sum_{e \in E_i} w(e) &= \frac{1}{2} \sum_{v \in V_i} \sum_{e \text{ adj. to } v} w(e) \\
&\geq \frac{1}{2} \sum_{v \in V_i} M \\
&= \frac{1}{2} \#V_i M = \frac{1}{2}(n - i + 1)M
\end{aligned}$$

Where in the first equality we rewrite the sum looking vertex by vertex at its outgoing edges. In this way, however, we count every vertex twice, therefore we multiply by a $\frac{1}{2}$ factor to compensate. In the first line we use the fact that $\sum_{e \text{ adj. to } v} w(e)$ is just the size of the cut $\{\{e\}, V \setminus \{e\}\}$, therefore it must be larger than M.

Now, the probability of failing is

$$\begin{aligned}
\Pr\left(\text{sampling } e \in \tilde{E} \mid e \in E_i\right) &\leq \sum_{e \in \tilde{E}} \Pr\left(\text{sampling } e \mid e \in E_i\right) && \text{union bound} \\
&= \sum_{e \in \tilde{E}} \frac{w(e)}{\sum_{e \in E'} w(e)} && \text{modified algoritm} \\
&\leq \frac{\sum_{e \in \tilde{E}} w(e)}{\frac{(n-i+1)M}{2}} = \frac{M}{\frac{(n-i+1)M}{2}} = \frac{2}{n - i + 1}. && \text{using the result above}
\end{aligned}$$

Never failing: The probability of never failing is

$$\begin{aligned}
\Pr\left(\text{never failing}\right) &= \Pr\left(\text{not fail at step } n-2 \mid \text{it didn't fail before } n-2\right) \cdots \Pr\left(\text{not fail at step } 1\right) \\
&= \prod_{i=1}^{n-2} \Pr\left(\text{not fail at step } i \mid \text{didn't fail before } i\right) \\
&= \prod_{i=1}^{n-2} \left(1 - \Pr\left(\text{fail at step } i \mid \text{didn't fail before } i\right)\right) \\
&\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{n - i + 1}\right) \\
&= \prod_{i=1}^{n-2} \left(\frac{n - i - 1}{n - i + 1}\right) \\
&= \frac{\cancel{n-2}}{n} \frac{\cancel{n-3}}{n-1} \frac{\cancel{n-4}}{\cancel{n-2}} \cdots \frac{\cancel{3}}{\cancel{5}} \frac{2}{\cancel{4}} \frac{1}{\cancel{3}} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}.
\end{aligned}$$

Since the probability of not failing, is the probability to sample $\{S, V \setminus S\}$, we have concluded the proof. $\square$

Finally, it's very easy to prove that there are no more than $\binom{n}{2}$ minimum cuts. By contradiction, assume that there were more than $\binom{n}{2}$ minimum cuts. Each one of them will have a probability of at least $\frac{1}{\binom{n}{2}}$ to be the output of Karger's algorithm. This means, that the total probability of Karger's to output one of the minimum cuts is

$$\begin{aligned}
\Pr(\text{sample a min. cut}) &= \sum_{i=1}^{\text{num. of min. cuts}>\binom{n}{2}} \Pr(\text{the } i\text{-th min. cut is selected}) \\
&\geq \sum_{i=1}^{\text{num. of min. cuts}>\binom{n}{2}} \frac{1}{\binom{n}{2}} > 1
\end{aligned}$$

Where in the second line we used lemma 1. Of course having a probability to be more than 1 is a contradiction. This implies that the total number of minimum cuts cannot be more than $\binom{n}{2}$.

2. The proof of this part follows a very similar strategy to the one seen before. First we need and important lemma.

Lemma 2 (Bound number of edges)**.** *Let $G = (V, E)$ be a graph (or multigraph) where, $n = \#V$ and the minimum $k-$cut has size M. Then*

$$\#E \geq \frac{M}{1 - \left(1 - \frac{k-1}{n}\right)\left(1 - \frac{k-1}{n-1}\right)}.$$

Proof. Let $v_1, \ldots, v_{k-1}$ be $k-1$ different vertices in V, the order is not important. Then we define

$$\mathcal{E}_{v_1,\ldots,v_{k-1}} := \{e \in E \mid \text{ one of the extremes is in } \{v_1, \ldots, v_{k-1}\}\}.$$

One can notice that $\mathcal{E}_{v_1,\ldots,v_{k-1}}$ is just the set of edges in the k-cut

$$\mathcal{C}_{v_1,\ldots,v_{k-1}} = \left\{\{v_1\}, \ldots, \{v_{k-1}\}, V \setminus \bigcup_{i=1}^{k-1} \{v_i\}\right\}.$$

This is true since each edge which has at least one vertex on $v_1, \ldots, v_{k-1}$ is in the cut because v_1 only has 1 element, and therefore such edge cannot be fully inside $\{v_1\}$. Furthermore, each of the edges cutting $\mathcal{C}_{v_1,\ldots,v_{k-1}}$, must have one of the ending point in $v_1, \ldots, v_{k-1}$, since all but one component are just $\{v_1\}, \ldots, \{v_{k-1}\}$. Finally, we can conclude that

$$\#\mathcal{E}_{v_1,\ldots,v_{k-1}} \geq M.$$

This is because $\mathcal{S}$ is the size of the minimum k-cut. Now, let's define $\mathcal{S}$ to be the sum of $\#\mathcal{E}_{v_1,\ldots,v_{k-1}}$ for all possible way we can take $v_1, \ldots, v_{k-1}$. Notice that there are $\binom{n}{k-1}$ ways for us to chose $v_1, \ldots, v_{k-1}$. Now, given an edge e, we want to count how many vertices choices $v_1, \ldots, v_{k-1}$ will be such that e is adjacent to at least one of $v_1, \ldots, v_{k-1}$. Let the number of such choices be $\mathcal{K}_e$. We can compute it by starting with the total number of choices, and subtracting all the ones which don't have any of the two vertices adjacent to e. Then we get:

$$\mathcal{K}_e = \underbrace{\binom{n}{k-1}}_{\text{total number of choices}} - \underbrace{\binom{n-2}{k-1}}_{\text{illigal choices}}.$$

The number of "illegal choices" is computed with $\binom{n-2}{k-1}$ since there are exactly $n-2$ vertices which are not adjacent o e. Furthermore, notice that $\mathcal{K}_e$ doesn't depend on e, therefore we rename it $\mathcal{K}$. Now we can rewrite $\mathcal{S}$ as

$$
\begin{aligned}
\mathcal{S} &= \sum_{\text{all possible permutations}} \#\mathcal{E}_{v_1,\dots,v_{k-1}} \\
&= \sum_{\text{all possible permutations}} \left(\sum_{e\text{adj. to one of } v_1,\dots,v_{k-1}} 1 \right) \\
&\stackrel{(i)}{=} \sum_{e\in E} \left(\sum_{\text{all possible perm. which have a vertex adj to } e} 1 \right) \\
&= \sum_{e\in E} \mathcal{K} = \#E\mathcal{K} = \#E\left(\binom{n}{k-1} - \binom{n-2}{k-1} \right). \\
&\Rightarrow \#E = \frac{\mathcal{S}}{\binom{n}{k-1} - \binom{n-2}{k-1}}
\end{aligned}
$$

Where in the sum inversion in (1) we just iterate first through edges, and then through permutations. Then we can also bound $\mathcal{S}$:

$$
\mathcal{S} = \sum_{\text{all possible permutations}} \#\mathcal{E}_{v_1,\dots,v_{k-1}} \geq \sum_{\text{all possible permutations}} M = \binom{n}{k-1} M.
$$

Putting every together, with some algebra, we get

$$
\begin{aligned}
\#E &\stackrel{(i)}{=} \frac{\mathcal{S}}{\binom{n}{k-1} - \binom{n-2}{k-1}} \\
&\stackrel{(ii)}{\geq} \frac{M\binom{n}{k-1}}{\binom{n}{k-1} - \binom{n-2}{k-1}} \\
&= \frac{M}{1 - \frac{\binom{n-2}{k-1}}{\binom{n}{k-1}}} \\
&= \frac{M}{1 - \frac{\frac{(n-2)!}{(n-2-(k-1))!\cancel{(k-1)!}}}{\frac{n!}{(n-(k-1))!\cancel{(k-1)!}}}} \\
&= \frac{M}{1 - \frac{\cancel{(n-2)!}(n-k+1)(n-k)\cancel{(n-k-1)!}}{n(n-1)\cancel{(n-2)!}\cancel{(n-k-1)!}}} \\
&= \frac{M}{1 - \frac{n-k+1}{n}\frac{n-k}{n-1}} \\
&= \frac{M}{1 - \left(1 - \frac{k-1}{n}\right)\left(1 - \frac{k-1}{n-1}\right)}
\end{aligned}
$$

Where in (i) we use the relationship we have found between $\#E$ and $\mathcal{S}$, and in (ii) we use the bound on $\mathcal{S}$. All the rest is simple algebra. This concludes the proof. $\square$

Now, we propose to use the Karger's algorithm but stopping after $n-k$ iterations. The outputs will be a graph with k nodes where each node correspond to a subset of V. The k subsets will represent the selected k-cut. Now we can prove the key theorem

Theorem 1 (Karger's efficency for k-cuts)**.** *Given a graph $G=(V,E)$ where $\#V=n$. Let*

$$
C = \left\{ A_1, A_2, \dots, A_{k-1}, V \setminus \left(\bigcup_{i=1}^{k-1} A_i \right) \right\}, \text{ where } A_i \cap A_j = \emptyset \; \forall i \neq j
$$

be a minimum k-cut. The probability that the early-stopped (after $n-k$ iterations) Karger's algorithm outputs C is

$$\Pr\left(\textit{Krager's outputs } C\right) \geq c\frac{(k-1)^{2(k-1)}}{(ne)^{2k-2}}$$

Proof. Let $\tilde{E}$ be the set of edges in the cut C. Let $M = \#\tilde{E}$. We say that the algorithm fails if at some point it picks an edge from $\tilde{E}$. If it does so, the algorithm will not output C. If it does not, then it must output C. We divide the proof in two parts:

Failing at step i: We now look at the probability of failing at step i given that it has not failed before. Let $G_i = (V_i, E_i)$ the multigraph output of the previous steps. Note that the minimum cut of such graph must be M. Furthermore, note that $\#V_i = n - i + 1$ (at step 1 $V_1 = V$, and then we take out one node per iteration).

Plugging G_i into lemma 2 we are guaranteed that

$$\#E_i \geq \frac{M}{1-\left(1-\frac{k-1}{\#V_i}\right)\left(1-\frac{k-1}{\#V_i-1}\right)} = \frac{M}{1-\left(1-\frac{k-1}{n-i+1}\right)\left(1-\frac{k-1}{n-i}\right)}.$$

Let $\hat{e}$ be the edge sampled at the i-th step. Since we sample uniformly at random from E_i, we get that

$$\Pr\left(\hat{e} \in \tilde{E}\right) \leq \frac{\#\tilde{E}}{\#E_i} = \frac{M}{\frac{M}{1-\left(1-\frac{k-1}{n-i+1}\right)\left(1-\frac{k-1}{n-i}\right)}} = 1-\left(1-\frac{k-1}{n-i+1}\right)\left(1-\frac{k-1}{n-i}\right).$$

Never failing: We now look at the probability that it never fails:

$$\begin{aligned}
\Pr\left(\text{never failing}\right) &= \Pr\left(\text{not fail at step } n-2 \mid \text{it didn't fail before } n-2\right)\cdots\Pr\left(\text{not fail at step } 1\right)\\
&= \prod_{i=1}^{n-k} \Pr\left(\text{not fail at step } i \mid \text{didn't fail before } i\right)\\
&= \prod_{i=1}^{n-k} \left(1-\Pr\left(\text{fail at step } i \mid \text{didn't fail before } i\right)\right)\\
&\geq \prod_{i=1}^{n-k} \left(\cancel{1-1}+\left(1-\frac{k-1}{n-i+1}\right)\left(1-\frac{k-1}{n-i}\right)\right)\\
&= \prod_{i=1}^{n-k} \left(\frac{n-i-k+2}{n-i+1}\right)\left(\frac{n-i-k+1}{n-i}\right)\\
&= \underbrace{\prod_{i=1}^{n-k} \left(\frac{n-i-k+2}{n-i+1}\right)}_{=A}\underbrace{\prod_{i=1}^{n-k} \left(\frac{n-i-k+1}{n-i}\right)}_{=B}
\end{aligned}$$

Now we look at A and B independently. For A we have:

$$\begin{aligned}
A &= \prod_{i=1}^{n-k} \left(\frac{n-i-k+2}{n-i+1} \right) \\
&= \frac{n-k+1}{n} \frac{n-k}{n-1} \frac{n-k-1}{n-2} \cdots \frac{n-2k+2}{n-k+1} \cdots \frac{k-1}{2k-4} \cdots \frac{4}{k+3} \frac{3}{k+2} \frac{2}{k+1} \\
&= \frac{\cancel{n-k+1}}{n} \frac{\cancel{n-k}}{n-1} \frac{\cancel{n-k-1}}{n-2} \cdots \frac{\cancel{n-2k+2}}{\cancel{n-k+1}} \cdots \frac{\cancel{k+1}}{\cancel{2k}} \cdots \frac{4}{\cancel{k+3}} \frac{3}{\cancel{k+2}} \frac{2}{\cancel{k+1}} \\
&= \frac{k!}{\frac{n!}{(n-k+1)!}} = \frac{k(k-1)!}{\frac{n!}{(n-k+1)!}} = \frac{k}{\frac{n!}{(n-(k-1))!(k-1)!}} = \frac{k}{\binom{n}{k-1}}.
\end{aligned}$$

For B we have:

$$\begin{aligned}
B &= \prod_{i=1}^{n-k} \left(\frac{n-i-k+1}{n-i} \right) \\
&= \frac{n-k}{n-1} \frac{n-k-1}{n-2} \frac{n-k-2}{n-3} \cdots \frac{n-2k+1}{n-k} \cdots \frac{k}{2k-1} \cdots \frac{3}{k+2} \frac{2}{k+1} \frac{1}{k} \\
&= \frac{\cancel{n-k}}{n-1} \frac{\cancel{n-k-1}}{n-2} \frac{\cancel{n-k-2}}{n-3} \cdots \frac{\cancel{n-2k+1}}{\cancel{n-k}} \cdots \frac{\cancel{k}}{\cancel{2k-1}} \cdots \frac{3}{\cancel{k+2}} \frac{2}{\cancel{k+1}} \frac{1}{\cancel{k}} \\
&= \frac{(k-1)!}{\frac{(n-1)!}{(n-k)!}} = \frac{1}{\frac{(n-1)!}{(n-1-(k-1))!(k-1)!}} = \frac{1}{\binom{n-1}{k-1}}
\end{aligned}$$

Finally, we can bound the probability of never failing by using a well known bound on the binomial coefficient: $\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$.

$$\begin{aligned}
\Pr(\text{never failing}) \geq AB &= \frac{k}{\binom{n}{k-1}\binom{n-1}{k-1}} \\
&\geq \frac{k}{\left(\frac{ne}{k-1}\right)^{k-1} \left(\frac{(n-1)e}{k-1}\right)^{k-1}} \\
&\geq \frac{k}{\left(\frac{ne}{k-1}\right)^{2k-2}} \\
&\geq \frac{k(k-1)^{2k-2}}{(ne)^{2k-2}}
\end{aligned}$$

Since if the algorithm never fails, then it picks exactly C; we have concluded the proof. $\square$

Once we have theorem 1, then the result is easy. Indeed, we cannot have more than

$$\text{Number of minimum } k\text{-cuts} \leq \frac{(ne)^{2k-2}}{k(k-1)^{2k-2}}.$$

If, by contradiction we assume we had more, we get that the total probability of getting one of the minimum cuts is more than 1, which is impossible (For a similar argument more in detail look at the last part of the previous point). Now, rearranging some terms we get:

$$\begin{aligned}
\text{Number of minimum } k\text{-cuts} &\leq \frac{(ne)^{2k-2}}{k(k-1)^{2k-2}} \\
&= e^{O(k)} n^{2k-2} \frac{1}{k(k-1)^{2k-2}} \leq 2^{O(k)} n^{2k-2}.
\end{aligned}$$

Where $e^{O(k)} = 2^{O(k)\log_2 e} = 2^{O(k)}$, and $\frac{1}{k(k-1)^{2k-2}} \in O(1)$. This concludes the exercise. As a final note, one can notice that, $\frac{e^{2k-2}}{k(k-1)^{2k-2}}$ is bounded by a constant. A tighter bound could then be then obtained:

$$\text{Number of minimum } k\text{-cuts} \leq Cn^{2k-2} \in O(n^{2k-2})$$