Last-Minute Mock AMC 12
January 28, 2018

Problem 1
The Count of Monte Cristo has 32 coins in his pocket. Each coin is either a penny or a nickel, and the total monetary value of all the coins is \$1.08. How many nickels is The Count of Monte Cristo carrying in his pocket?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 19 **(E)** 24

Answer: **D**

Solution: Let p and n denote the number of pennies and nickels in his pocket, respectively. Then, $p+n = 32$. Since there are a total of 108 cents in the pocket, $p + 5n = 108$. Subtracting these two equations and then dividing by 4, $n = 19$. Thus, there are 19 nickels in his pocket.

Problem 2
What is the remainder when $\frac{2018!-2017!}{2016!+2015!}$ is divided by 1000?

(A) 240 **(B)** 256 **(C)** 272 **(D)** 289 **(E)** 306

Answer: **C**

Solution: Observe that

$$\frac{2018! - 2017!}{2016! + 2015!} = \frac{2017! \cdot (2018 - 1)}{2015! \cdot (2016 + 1)} = \frac{2017! \cdot 2017}{2015! \cdot 2017} = 2016 \cdot 2017 = 4066272.$$

The requested remainder is 272.

Problem 3
Mr. Barnes tells his math class that they will have a party if the average of the scores on the next test is at least 90 or nobody misses his or her next homework assignment. Which of the following statements is always true?

(A) If the class did not have a party, then the average of the scores on the test was less than 90 and at least one person missed his or her homework assignment.

(B) If the class did not have a party, then the average of the scores on the test was less than 90 or at least one person missed his or her homework assignment.

(C) If the class has a party, then the average of the scores on the test was at least 90 and nobody missed his or her next homework assignment.

(D) If the class has a party, then the average of the scores on the test was less than 90 and nobody missed his or her next homework assignment.

(E) If nobody misses his or her next homework assignment and the average of the scores on the next test is at least 90, then the class will not have a party.

Answer: **A**

Solution: Let P denote the event that the average of the scores on the next test is at least 90, and let Q denote the event that nobody misses his or her next homework assignment. For the class to have party, at least one of P or Q must have occurred but not necessarily both.

- Statement A states that if the class did not have a party, both P and Q did not occur. This statement is true.
- Statement B states that if the class did not have a party, at least one of P or Q did not occur. However, this is not necessarily true because even if one of P or Q did not occur, the class would still have a party as long as the other event occurred.
- Statement C states that if the class had a party, both P and Q occurred. However, this is not necessarily true because only one of the two events could have occurred and the class would still have had a party.
- Statement D states that if the class had a party, P did not occur and Q occurred. However, this is not necessarily true because both events could have occurred and the class would have had a party.
- Statement E states that if both P and Q occurred, the class will not have a party. This is clearly false.

Thus, statement A is the only statement that is always true.

Problem 4
Two distinct positive integers between 1 and 100, inclusive, are chosen at random. What is the probability that their arithmetic mean will be an integer?

(A) $\frac{49}{100}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{49}{99}$ **(D)** $\frac{51}{99}$ **(E)** $\frac{51}{100}$

Answer: **C**

Solution: For the arithmetic mean of the two integers to also be an integer, the two integers must have the same parity.

After choosing the first integer, regardless of its parity, there are 99 integers left, 49 of which have the same parity as the first integer. Thus, the desired probability is $\frac{49}{99}$.

Problem 5
A right circular cone and right circular cylinder have the same volume. The radius of the base of the cone is increased by 40% while the height of the cylinder is decreased by 16%. What is the resulting ratio of the volume of the cylinder to the volume of the cone?

(A) $\frac{1}{7}$ **(B)** $\frac{3}{14}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{5}{14}$ **(E)** $\frac{3}{7}$

Answer: **E**

Solution: Since the height of the cylinder is modified by a factor of 0.84, this modifies the volume by a factor of 0.84.

Since the radius of the base of the cone is modified by a factor of 1.4, this modifies the volume by a factor of $(1.4)^2 = 1.96$.

Thus, the desired ratio is $\frac{0.84}{1.96} = \frac{3}{7}$.

Problem 6
Aaron has some marbles he wishes to divide equally among a certain number of jars. When he places them into 3 jars, there is 1 marble left over. When he places them into 4 jars, there are 2 marbles left over. When he places them into 5 jars, there are 3 marbles left over. Let n be the least possible number of marbles Aaron could have. What is the sum of the digits of n?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Answer: **D**

Solution: Since there is 1 marble left over when Aaron tries to equally divide the marbles into 3 jars, $n \equiv 1 \pmod 3$. Similarly, $n \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 5$.

By merging the first two congruences, $n \equiv 10 \pmod{12}$. By merging this congruence with the last congruence, $n \equiv 58 \pmod{60}$. Thus, $n = 58$. The sum of the digits of n is 13.

Problem 7
The angles of a convex dodecagon have integral values and are in arithmetic progression. What is the least possible value of the smallest angle, in degrees?

(A) 120 **(B)** 124 **(C)** 128 **(D)** 134 **(E)** 139

Answer: **C**

Solution: In a dodecagon, the average of the angles is $\frac{180^\circ \cdot (12-2)}{12} = 150^\circ$. Thus, since the angles of the dodecagon are in an arithmetic sequence, the sequence is centered around 150°. In particular, there exists a nonnegative real number x such that the angles of the dodecagon are

$$(150-11x)^\circ, (150-9x)^\circ, (150-7x)^\circ, \cdots, (150-x)^\circ,$$

$$(150+x)^\circ, (150+3x)^\circ, (150+5x)^\circ, \cdots, (150+11x)^\circ.$$

Since the angles are integers x must be an integer. In addition, since the dodecagon is convex, the largest angle must be less than 180°. Thus, $150 + 11x < 180 \implies x \leq 2$. To minimize the smallest angle, x must be maximized. Thus, $x = 2$. Then, the sequence of angles is

$$128^\circ, 132^\circ, 136^\circ, \cdots, 148^\circ, 152^\circ, \cdots, 172^\circ.$$

Thus, the least possible value of the smallest angle is 128°.

Problem 8
The sum of two positive two-digit integers a and b is another two-digit positive integer n. The positive difference of a and b is obtained by reversing the digits of n and is also a two-digit positive integer. How many ordered pairs (a, b) are possible?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

Answer: **A**

Solution: Since the positive difference between a and b is a two-digit positive integer, it is non-zero. Thus, $a \neq b$. There are 2 ways to choose which of a or b is greater. Without loss of generality, assume that $a > b$. If $n = \overline{xy}$ for digits x and y, then

$$a + b = \overline{xy} = 10x + y$$

$$a - b = \overline{yx} = 10y + x.$$

Since n has two digits and the number obtained by reversing the digits of n also has two digits, neither x nor y equals 0.

Adding the two equations and then dividing by 2, $a = \frac{11(x+y)}{2}$. Subtracting the two equations and then dividing by 2, $b = \frac{9(x-y)}{2}$. For a and b to be integers $x+y$ and $x-y$ must both be even. This occurs precisely when x and y have the same parity. In addition, for a and b to be two digit integers, $2 \leq x + y \leq 18$ and $4 \leq x - y \leq 22$.

- If $x = 1$, then $y \in \{5, 7, 9\}$.
- If $x = 2$, then $y \in \{6, 8\}$.
- If $x = 3$, then $y \in \{7, 9\}$.
- If $x = 4$, then $y = 8$.
- If $x = 5$, then $y = 9$.

Thus, there are 9 possible pairs (x, y), each of which corresponds to one pair (a, b). To account for the order of a and b, there are $2 \cdot 9 = 18$ pairs (a, b).

Problem 9
Point P is located outside a circle with center O such that $OP = 3$. Point Q is also outside the circle such that PQ is perpendicular to OP and $PQ = 4$. Let OP intersect the circle at R and OQ intersect the circle at S. Moreover, let T be the projection of R onto OQ, and let U be the point on the circle such that QU is tangent to the circle. If $QT = \frac{7}{2}$, then what is the length of QU?

(A) 2 **(B)** $2\sqrt{3}$ **(C)** $\frac{7\sqrt{3}}{4}$ **(D)** $\frac{5\sqrt{3}}{2}$ **(E)** $3\sqrt{2}$

Answer: **D**

Solution: Since $\triangle OPQ$ is a right triangle with legs of length 3 and 4, $OQ = 5$. Since $QT = \frac{7}{2}$, $OT = \frac{3}{2}$. Since $\triangle OPQ$ and $\triangle OTR$ are both right triangles that share $\angle POQ$, $\triangle OPQ \sim \triangle OTR$. Thus, $\frac{OR}{OT} = \frac{OQ}{OP} \implies OR = \frac{5}{2}$. Thus, the circle centered at O has radius $\frac{5}{2}$. Since QU is tangent to the circle, $\angle QUO = 90°$. Thus, by Pythagorean Theorem on $\triangle QUO$, $QU = \frac{5\sqrt{3}}{2}$.

Problem 10
Let $x = \sqrt{2 + \sqrt{6 + \sqrt{6 + \ldots}}}$ and let $y = \sqrt{3 + \sqrt{12 + \sqrt{12 + \ldots}}}$. Which of the following statements correctly expresses y in terms of x?

(A) $y = x + 2$ **(B)** $y = x^2 + 2$ **(C)** $y = \sqrt{x + 2}$ **(D)** $y = \sqrt{x^2 + 2}$ **(E)** $y = \sqrt{x^2 - 2}$

Answer: **D**

Solution: Let $a = \sqrt{6 + \sqrt{6 + \ldots}}$. Squaring both sides,

$$a^2 = 6 + \sqrt{6 + \sqrt{6 + \ldots}} \implies a^2 = 6 + a \implies (a - 3)(a + 2) = 0.$$

Since a is clearly positive, $a = 3$. Thus,

$$x = \sqrt{2 + a} = \sqrt{5}.$$

Let $b = \sqrt{12 + \sqrt{12 + \ldots}}$. Squaring both sides,

$$b^2 = 12 + \sqrt{12 + \sqrt{12 + \ldots}} \implies b^2 = 12 + b \implies (b - 4)(b + 3) = 0.$$

Since b is clearly positive, $b = 4$. Thus,

$$y = \sqrt{3 + b} = \sqrt{7}.$$

Among the given statements, the only one that is correct is $y = \sqrt{x^2 + 2}$.

Problem 11
The positive integer N satisfies $N_8 + N_9 = 2017_{10}$, where the subscripts signify number bases. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer: **E**

Solution: Let $f(N) = N_8 + N_9$. For N_8 to be well defined, no digit of N can be greater than 7. Since N_8 and N_9 both strictly increase as N increases, $f(N)$ is a strictly increasing function. It can be manually computed that $f(777) = 1148$. Thus, N has at least 4 digits. Suppose $N = \overline{abcd}$ for digits a, b, c, and d that are at most 7. Thus,

$$2017 = \overline{abcd}_8 + \overline{abcd}_9 = 1241a + 145b + 17c + 2d.$$

Clearly, a cannot equal 0, since it is the leading digit. If $a \geq 2$, then the right hand side is too big. Thus, $a = 1$, implying that

$$776 = 145b + 17c + 2d.$$

Observe that

$$776 = 145b + 17c + 2d \leq 145b + 17 \cdot 7 + 2 \cdot 7 \implies 643 \leq 145b \implies b \geq 5.$$

If $b = 5$, then $17c + 2d = 51$. Clearly, $(c, d) = (3, 0)$ is a solution to this equation. Since $f(N)$ is strictly increasing, the solution of $(a, b, c, d) = (1, 5, 3, 0)$ is unique. Thus, $N = 1530$ for which the sum of the digits is 9.

Problem 12
A primitive nth root of unity is a complex number z such that $z^n = 1$ and $z^k \neq 1$ for $k = 1, 2, 3, \ldots, n-1$. How many values of k are there less than 360 such that for some complex number ω that is a 360th root of unity, ω is also a primitive kth root of unity?

(A) 12 **(B)** 15 **(C)** 19 **(D)** 23 **(E)** 29

Answer: **D**

Solution: Let $\omega = \text{cis}(2\pi \cdot \frac{a}{b})$ for relatively prime positive integers a and b. Suppose ω is both a 360th root of unity and a primitive kth root of unity.

For ω to be a 360th root of unity, $\omega^{360} = 1$. By DeMoivre's Theorem, $\omega^{360} = \text{cis}(2\pi \cdot \frac{360a}{b})$. In particular, for $\text{cis}(2\pi \cdot \frac{360a}{b})$ to equal 1, $2\pi \cdot \frac{360a}{b}$ must be an integer multiple of 2π, implying that b must evenly divide $360a$. However, $\gcd(a, b) = 1$, so b must divide 360.

For ω to be a primitive kth root of unity, $\frac{ka}{b}$ must be an integer and $\frac{ja}{b}$ cannot be an integer for $j = 1, 2, ..., k-1$. In particular, b must divide k but not any integer smaller than k. Choosing $b = k$ is sufficient to satisfy this condition.

Thus, both conditions combined imply that k is a divisor of 360. Since it is given that $k < 360$, the desired quantity is the number of positive divisors of 360 except 360. Since $360 = 2^3 \cdot 3^2 \cdot 5$, the number of positive divisors is $4 \cdot 3 \cdot 2 = 24$. The desired quantity is 23.

Problem 13
Let $S = \{P_1, P_2, P_3, \ldots, P_m\}$ be a set of lattice points in n-dimensional space; that is, each point P_i can be expressed as the n-tuple $(x_1, x_2, x_3, \ldots, x_n)$, where each of the x_j's are integers. In terms of n, what is the minimum number of points in S required to be able to find two points in S whose midpoint

is also a lattice point?

(A) $n+1$ **(B)** $2n+1$ **(C)** 2^n+1 **(D)** 2^n+n+1**(E)** 2^n+2n+1

Answer: **C**

Solution: Two lattice points have a midpoint that is also a lattice point if and only if for each integer j such that $1 \le j \le n$, the jth coordinate of both points have the same parity.

For each lattice point in S, denote its binary string to be an n-character string of 0s and 1s, where the jth character of the string corresponds to the parity of the coordinate x_j. Thus, two lattice points have a midpoint that is also a lattice point if and only if they have the same binary string.

There are 2^n possible binary strings in terms of n. If S contains 2^n lattice points, they could all have different binary strings, which does not guarantee that the given condition is satisfied. However, if S contains 2^n+1 lattice points, by Pigeonhole Principle, some two lattice points in S must have the same binary string.

Thus, S must have at least 2^n+1 lattice points to guarantee that the given condition is satisfied.

Problem 14

Anna randomly chooses a positive divisor of the number 15! while Bob randomly chooses another positive divisor of the same number, not necessarily distinct from the one chosen by Anna. What is the probability that the sum of Anna's and Bob's divisors will be odd?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{11}{72}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{13}{72}$

Answer: **C**

Solution: By Legendre's formula, the exponent of the largest power of 2 that divides 15! is

$$\left\lfloor \frac{15}{2^1} \right\rfloor + \left\lfloor \frac{15}{2^2} \right\rfloor + \left\lfloor \frac{15}{2^3} \right\rfloor = 11.$$

Thus, $15! = 2^{11} \cdot m$ for some odd integer m. It then follows that when a divisor of 15! is chosen at random, the exponent of 2 in its prime factorization will be equal to an integer from 0 to 11, inclusive, each with equal probability. Thus, the probability the divisor is odd is $\frac{1}{12}$, while the divisor is even with probability $\frac{11}{12}$. For the sum of the Anna's and Bob's divisors to be odd, they must be of different parities.

There is a factor of 2 to account for who will pick an odd divisor so that the other person must pick an even divisor. Thus, the desired probability is $2 \cdot \frac{1}{12} \cdot \frac{11}{12} = \frac{11}{72}$.

Problem 15

In triangle ABC, $AB = 13$, $AC = 14$, and $BC = 15$. A point is randomly chosen inside the triangle. The probability that it will lie closer to A than to any other vertex can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the remainder when $m+n$ is divided by 1000?

(A) 219 **(B)** 297 **(C)** 342 **(D)** 388 **(E)** 435

Answer: **A**

Solution: Let O be the circumcenter of $\triangle ABC$.

Suppose ℓ_1 is the perpendicular bisector of $\overline{AB}$. If a point P lies on the side of ℓ_1 containing A, then $PA < PB$. Else, $PB \ge PA$. Thus, the randomly chosen point must lie on the side of ℓ_1 containing A.

Similarly, suppose ℓ_2 is the perpendicular bisector of $\overline{AC}$. For $PA < PC$ to hold, P must lie on the side of ℓ_2 containing A. Else, $PC \ge PA$.

Since the perpendicular bisectors of $\triangle ABC$ concur at O, the point must be chosen in the region bounded by AB, ℓ_1, ℓ_2, and AC to satisfy the given condition. Let M and N be the midpoints of AB and AC, respectively.

By Heron's formula, $[ABC] = 84$. Let R be the circumradius of $\triangle ABC$. Then, by the formula for the circumradius of a triangle, $R = \frac{65}{8}$. Then, since OM and ON are perpendicular to AB and AC, respectively, $OM = \frac{39}{8}$ and $ON = \frac{33}{8}$ by Pythagorean Theorem. Then,

$$[AMON] = [AMO] + [ANO] = \frac{1}{2} \cdot AM \cdot OM + \frac{1}{2} \cdot AN \cdot ON = \frac{969}{32}.$$

The desired probability is $\frac{[AMON]}{[ABC]} = \frac{969}{32 \cdot 84} = \frac{323}{896}$, for which $m + n = 1219$. The requested remainder is 219.

Problem 16
A positive integer is called happy if the sum of its digits equals the two-digit integer formed by its two leftmost digits. How many five-digit positive integers are happy?

(A) 880 **(B)** 950 **(C)** 1001 **(D)** 1075 **(E)** 1110

Answer: **E**

Solution: Let the positive integer be $\overline{abcde}$ for digits a, b, c, d, and e. The two-digit integer formed by its two leftmost digits is given by $10a + b$. Thus, for the positive integer to be happy,

$$10a + b = a + b + c + d + e \implies 9a = c + d + e. \quad (*)$$

Thus, the value of b does not affect the given condition. There are 10 choices for the value of b. Since a is the leading digit, $1 \leq a \leq 9$. Since $0 \leq c, d, e \leq 9$, this bound for a is satisfied as long as c, d, and e are not all equal to 0.

Consider the three digit positive integer $n = \overline{cde}$, where n may have leading zeroes. A positive integer is divisible by 9 if and only if the sum of the digits is divisible by 9. Since the sum of the digits of n is divisible by 9 by $(*)$, it follows there is a bijection between the solutions to $(*)$ and the positive multiples of 9 with at most 3 digits. Since 9 and 999 are the smallest and largest such multiples of 9, respectively, there are 111 choices for the quadruple (a, c, d, e). Since there are 10 choices for b, there are $111 \cdot 10 = 1110$ happy integers with five digits.

Problem 17
Let x, y, and z be positive real numbers with $1 < x < y < z$ such that

$$\log_x y + \log_y z + \log_z x = 8 \text{ and}$$

$$\log_x z + \log_z y + \log_y x = \frac{25}{2}.$$

The value of $\log_y z$ can then be written as $\frac{a+\sqrt{b}}{c}$ for positive integers a, b, and c such that b is not divisible by the square of any prime. What is the value of $a + b + c$?

(A) 31 **(B)** 42 **(C)** 47 **(D)** 53 **(E)** 64

Answer: **B**

Solution: Let $p = \log_x y$, $q = \log_y z$, and $r = \log_z x$. By the first given equation,

$$p + q + r = 8. \quad (1)$$

Observe that $pq = \frac{\log(y)}{\log(x)} \cdot \frac{\log(z)}{\log(y)} = \frac{\log(z)}{\log(x)} = \log_x z$. Similarly, $qr = \log_y x$ and $pr = \log_z y$. Thus, by the second given equation,

$$pq + qr + pr = \frac{25}{2}. \quad (2)$$

In addition,

$$pqr = \frac{\log(y)}{\log(x)} \cdot \frac{\log(z)}{\log(y)} \cdot \frac{\log(x)}{\log(z)} = 1. \quad (3)$$

By (1), (2), and (3) and Vieta's formulas, p, q, and r are the roots to the polynomial $2t^3 - 16t^2 + 25t - 2$. By Rational Theorem, the only possible rational roots are $t \in \{\pm 1, \pm 2, \pm \frac{1}{2}\}$. Since $t = 2$ is a root to the polynomial, it can be factored as follows,

$$(t-2)(2t^2 - 12t + 1) = 0.$$

By the quadratic formula on the second factor, $t = \frac{6 \pm \sqrt{34}}{2}$. Clearly, $2 < 3 < \frac{6+\sqrt{34}}{2}$. Since r is maximized when it equals the largest root of the polynomial, $r = \frac{6+\sqrt{34}}{2}$. The requested sum is 42.

Problem 18

A particle starts at the origin and takes a series of steps in the following manner: If the particle is at the point (x, y), it then moves randomly to one of $(x-1, y)$, $(x+1, y)$, $(x-1, y+1)$, $(x, y+1)$, or $(x+1, y+1)$. What is the expected number of steps the particle takes to reach the line $y = 3$?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer: **C**

Solution: In any given step, the particle's y-coordinate will increase by 1 with probability $\frac{3}{5}$ and stay the same with probability $\frac{2}{5}$. Thus, the particle's y-coordinate is expected to increase by $\frac{3}{5}$ every step. It then follows that for the particle to reach $y = 3$ from $y = 0$, it is expected to take $3 \div \frac{3}{5} = 5$ steps.

Problem 19

Let a, b, and c be positive real numbers such that

$$\frac{a^2}{2018-a} + \frac{b^2}{2018-b} + \frac{c^2}{2018-c} = \frac{1}{168}.$$

What is the largest possible value of $a + b + c$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer: **C**

Solution: By Titu's Lemma,

$$\frac{1}{168} = \frac{a^2}{2018-a} + \frac{b^2}{2018-b} + \frac{c^2}{2018-c} \geq \frac{(a+b+c)^2}{6054-(a+b+c)}.$$

Let $x = a + b + c$. Then,

$$\frac{1}{168} \geq \frac{x^2}{6054-x} \implies 6054 - x \geq 168x^2 \implies 168x^2 + x - 6054 \leq 0 \implies (x-6)(168x+1009) \leq 0.$$

The maximum value of x for which this inequality holds is 6. Indeed, $a + b + c = 6$ is attainable with the triple $(a, b, c) = (2, 2, 2)$. Thus, the maximum possible value of $a + b + c$ is 6.

Problem 20
Let m and n be two odd positive integers less than 100, and let $k = 2^m 3^n$. Let N be the number of divisors of k^2 that are less than k but do not divide k. For how many ordered pairs (m, n) is it true that N is less than 1000?

(A) 729 **(B)** 822 **(C)** 872 **(D)** 916 **(E)** 987

Answer: **B**

Solution: The integer k^2 has $(2m+1)(2n+1)$ positive divisors. Aside from the divisor k, the divisors of k^2 come in pairs that multiply to k^2, one of which is less than $\sqrt{k^2} = k$ and the other of which is greater than $\sqrt{k^2} = k$. Thus, there are $\frac{(2m+1)(2n+1)-1}{2}$ divisors of k^2 less than k. Any proper divisor of k will also be a factor of k^2 less than k, and there are $(m+1)(n+1)-1$ of these. Hence,

$$N = \frac{(2m+1)(2n+1)-1}{2} - ((m+1)(n+1)-1) = 2mn + m + n - mn - m - n = mn.$$

Thus, it suffices to find the number of ordered pairs of odd positive integers less than 100 such that $mn < 1000$.

<u>Case 1:</u> $m = n$

Then, $m^2 < 1000 \implies 1 \le m \le 31$. There are 16 pairs (m, n) in this case.

<u>Case 2:</u> $m < n$

- If $m = 1$, then $3 \le n \le 99$. There are 49 pairs here.
- If $m = 3$, then $5 \le n \le 99$. There are 48 pairs here.
- If $m = 5$, then $7 \le n \le 99$. There are 47 pairs here.
- If $m = 7$, then $9 \le n \le 99$. There are 46 pairs here.
- If $m = 9$, then $11 \le n \le 99$. There are 45 pairs here.
- If $m = 11$, then $13 \le n \le 89$. There are 39 pairs here.
- If $m = 13$, then $15 \le n \le 75$. There are 31 pairs here.
- If $m = 15$, then $17 \le n \le 65$. There are 25 pairs here.
- If $m = 17$, then $19 \le n \le 57$. There are 20 pairs here.
- If $m = 19$, then $21 \le n \le 51$. There are 16 pairs here.
- If $m = 21$, then $23 \le n \le 47$. There are 13 pairs here.
- If $m = 23$, then $25 \le n \le 43$. There are 10 pairs here.
- If $m = 25$, then $27 \le n \le 39$. There are 7 pairs here.
- If $m = 27$, then $29 \le n \le 37$. There are 5 pairs here.
- If $m = 29$, then $31 \le n \le 33$. There are 2 pairs here.

There are

$$49 + 48 + 47 + 46 + 45 + 39 + 31 + 25 + 20 + 16 + 13 + 10 + 7 + 5 + 2 = 403$$

pairs in this case.

<u>Case 3:</u> $m > n$

By symmetry of Case 2, there are 403 pairs in this case.

Thus, in total, there are $16 + 403 + 403 = 822$ pairs (m, n).

Problem 21
Rectangle $ABCD$ has $AB = 15$ and $BC = 12$. Let E and F be the trisection points of AC, with E closer to A, and let G and H be the trisection points of BD, with G closer to B. Let AG intersect BC at P, PH intersect AD at Q, QE intersect AB at R, and RF intersect CD at S. What is the area of quadrilateral $DHFS$?

(A) 27 **(B)** 31 **(C)** 35 **(D)** 37 **(E)** 42

Answer: **B**

Solution: By parallel lines, $\triangle ADG \sim \triangle PBG$. Since $AG : GP = 2 : 1$, $AD : PB = 2 : 1$, implying that $BP = 6$. Similarly, $\triangle BPH \sim \triangle DQH$ with a $2 : 1$ ratio, implying that $DQ = 3$.

Let rays RQ and CD intersect at T. Then, $\triangle QDT \sim \triangle QAR$. Let $TD = x$ so that $TC = x + 15$. Since $DQ : QA = 1 : 3$, $AR = 3x$. Since $AE : EC = 1 : 2$, $\triangle ARE \sim \triangle CTE$ with a $1 : 2$ ratio. Thus, $AR : TC = 1 : 2$, implying that $2 \cdot 3x = x + 15 \implies x = 3$. Thus, $AR = 9$.

Finally, $\triangle ARF \sim \triangle CSF$ with a $2 : 1$ ratio, implying that $SC = \frac{9}{2}$. Thus, $DS = \frac{21}{2}$. Clearly, both H and F lie a third of the way from line DC to line AB. Thus, $DHFS$ is a trapezoid with parallel lines DS and HF and a height of $\frac{1}{3} \cdot AD = 4$. Since H lies a third of the way from AD to BC, while F lies two thirds of the way from AD to BC, $HF = \frac{1}{3} \cdot 15 = 5$. Thus,

$$[DHFS] = 4 \cdot \frac{HF + DS}{2} = 31.$$

Problem 22
The sequence $1, 4, 5, 16, 17, \ldots$ consists of all of the positive integers that can be expressed as the sum of distinct nonnegative powers of 4, ordered starting from $4^0 = 1$, the smallest such integer. Let S be the sum of the first 31 elements of the sequence. What is the sum of the digits of S?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Answer: **D**

Solution: Writing each term in the sequence in base-4, the sequence is

$$1_4, 10_4, 11_4, 100_4, 101_4, \cdots.$$

The base-4 representations of the terms is reminiscent of a binary sequence. Indeed, when each term of the sequence is written as the sum of distinct powers of 4, each power of 4 will appear either 0 or 1 time in the sum. Thus, the base-4 representation of each term in the sequence can only contain 0s and 1s as digits. It then follows that the base-4 representation of the nth term in the sequence is the same as the binary representation of the nth positive integer.

Since $31 = 11111_2$, the 31st term of the sequence will be 11111_4. Thus, the first 31 terms of the sequence covers all possible 5 digit binary strings (possibly with leading zeroes) except 00000. Let T be the set of binary strings representing the first 32 nonnegative integers (0 is included in T for simplicity sake and because it does not affect the total sum).

Suppose the kth digit from the right is 1. Then, there are $2^4 = 16$ possible ways to determine the rest of the string. Clearly, each of these possible strings are in T. Every instance of the kth digit from the right being equal to 1 contributes 4^{k-1} to the total sum for each integer k with $1 \leq k \leq 5$. Because there are 16 instances of each possible power of 4,

$$S = 16 \cdot (4^4 + 4^3 + 4^2 + 4^1 + 4^0) = 5456.$$

The sum of the digits of S is 20.

Problem 23
Given that in triangle ABC, $\cos(nA) + \cos(nB) + \cos(nC) = 1$ for some positive integer n, which of the following statements is always true?

(A) There exists an angle that measures a multiple $\frac{360^\circ}{n}$ if and only if n is odd.

(B) There exists an angle that measures a multiple $\frac{360^\circ}{n}$ if and only if n is even.

(C) There exists an angle that measures a multiple $\frac{180^\circ}{n}$ if and only if n is odd.

(D) There exists an angle that measures a multiple $\frac{180^\circ}{n}$ if and only if n is even.

(E) None of the above.

Answer: **A**

Solution: Let $a = \text{cis}(A)$, $b = \text{cis}(B)$, and $c = \text{cis}(C)$. If $\text{cis}(\theta) = x$, recall the identity

$$\cos(\theta) = \frac{x + x^{-1}}{2}.$$

Then,

$$\frac{a^n + a^{-n}}{2} + \frac{b^n + b^{-n}}{2} + \frac{c^n + c^{-n}}{2} = 1$$
$$\implies a^n b^n c^n (a^n + b^n + c^n) + a^n b^n + b^n c^n + a^n c^n = 2a^n b^n c^n \quad (*)$$

after denominators are cleared. Since $A + B + C = 180^\circ$,

$$\text{cis}(A) \cdot \text{cis}(B) \cdot \text{cis}(C) = \text{cis}(180^\circ) = -1.$$

Thus, $abc = -1$.

Case 1: n is odd. Since $abc = -1$, $a^n b^n c^n = -1$. Then, $(*)$ becomes

$$-a^n + b^n c^n - b^n + a^n c^n - c^n + a^n b^n = -2.$$

By adding $-a^n b^n c^n + 1$ to the left hand side and 2 to the right hand side, the left hand side can be factored:

$$(a^n - 1)(b^n - 1)(c^n - 1) = 0.$$

Without loss of generality, assume $a^n = 1$. Then, the angle An is a multiple of 360°, implying that the angle A is a multiple of $\frac{360^\circ}{n}$.

Case 2: n is even.

Then, $a^n b^n c^n = 1$. Thus, $(*)$ becomes

$$a^n + b^n + c^n + a^n b^n + b^n c^n + a^n c^n = 2.$$

By adding $a^n b^n c^n + 1$ to the left hand side and 2 to the right hand side, the left hand side can be factored:

$$(a^n + 1)(b^n + 1)(c^n + 1) = 4.$$

Even in combination with $a^n b^n c^n = 1$, these equations alone do not imply that one of a^n, b^n, or c^n must necessarily equal -1 or 1. Thus, the second and fourth statements do not always hold.

Thus, from the given statements, the only one that is always true is the first one.

Problem 24

In triangle $\triangle ABC$, $AB = 5$, $BC = 7$, and $CA = 8$. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively, and let M be the midpoint of BC. The area of triangle MEF can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c such that the greatest common divisor of a and c is 1 and b is not divisible by the square of any prime. What is the value of $a + b + c$?

(A) 68 **(B)** 70 **(C)** 72 **(D)** 74 **(E)** 76

Answer: **A**

Solution: Since $\angle BEC = \angle BFC = 90°$, quadrilateral $BFEC$ is cyclic. Let ω denote the circumcircle of $BFEC$. Then, BC is a diameter of ω. Furthermore, M is the center of ω, since it is the midpoint of BC. Thus, $MF = ME$.

By Law of Cosines on $\triangle ABC$,

$$\cos(A) = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC} = \frac{1}{2}.$$

Thus, $A = 60°$. Furthermore, $B + C = 120°$.

Since $BM = MF$, $\angle BFM = B \implies \angle AFM = 180° - B$. Similarly, since $CM = ME$, $\angle CEM = C \implies \angle AEM = 180° - C$. By considering the angles of quadrilateral $AFME$,

$$A + (180° - B) + (180° - C) + \angle EFM = 360° \implies \angle EFM = (B + C) - A = 60°.$$

Since $MF = ME$, $\triangle MEF$ is equilateral. The side length of $\triangle MEF$ is equal to the radius of ω, which is half the length of BC. Thus, the side length is $\frac{7}{2}$. By the formula for the area of an equilateral triangle, $[MEF] = (\frac{7}{2})^2 \cdot \frac{\sqrt{3}}{4} = \frac{49\sqrt{3}}{16}$. The requested sum is 68.

Problem 25

For every positive integer n, define S_n as the set of all permutations of the first n positive integers such that no pair of consecutive integers appears in that order; that is, 2 does not immediately follow 1, 3 does not immediately follow 2, and so on. For example, 2143 and 2431 are valid permutations in S_4. Denote by $f(n)$ the number of elements in S_n. What is the units digit of $f(2018)$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Answer: **D**

Solution: Call a permutation of $\{1, 2, ..., n\}$ "valid" if it is in S_n. Consider removing n from any valid permutation of length n. There are two cases:

Case 1: Removing n does not cause two consecutive integers to be adjacent.

Then, this is a valid permutation of length $n - 1$. Since n cannot appear after $n - 1$, there are $n - 1$ possible locations for n to be inserted in the permutation of length $n-1$. This accounts for $(n-1)f(n-1)$ permutations.

Case 2: Removing n does cause two consecutive integers to be adjacent.

Suppose these two integers are k and $k + 1$, where $k \neq n - 1$. Since the permutation was valid before n was removed, this is the only instance where two adjacent integers in the new permutation are consecutive integers. Thus, $k + 2$ will not appear directly after $k + 1$. Consider removing $k + 1$ from the permutation after removing n. Then, decrease every element in the permutation greater or equal to $k + 2$ by 1. Since $k+2$ is not after $k+1$ in the original permutation, $k+1$ will not be after k in the new permutation. Thus, the new permutation is a valid permutation of length $n - 2$.

Indeed, this process is reversible. Consider a valid permutation of length $n-2$ and choose any of the $n-2$ elements. If k is the chosen element, then increase every element in the permutation that is greater than

k by 1. Then, insert n and $k+1$ in that order directly after k. The element after $k+1$ will not be $k+2$, since that would imply $k+1$ appears after k in the permutation of length $n-2$. Thus, this accounts for $(n-2)f(n-2)$ permutations.

It follows that $f(n) = (n-1)f(n-1) + (n-2)f(n-2)$ for all positive integers $n \geq 3$. The only permutation of length 1 in S_1 is 1, while the only permutation of length 2 in S_2 is 21. Thus, the base cases of the recursion are $f(1) = 1$ and $f(2) = 1$. By using the recursion to compute $f(n)$ in mod 10,

n	$f(n) \pmod{10}$
1	1
2	1
3	3
4	1
5	3
6	9
7	9
8	7
9	9
10	7
11	1
12	1

Since $f(1) = f(11)$ and $f(2) = f(12)$ and the value of $f(n)$ is solely dependent on the previous two terms, $f(n)$ in modulo 10 is periodic with a period of 10. Thus $f(2018) \equiv f(8) \equiv 7 \pmod{10}$.