2020 Mock

AMC 12A

DO NOT OPEN UNTIL WEDNESDAY, JULY 1, 2020

****Administration On An Earlier Date Will Disqualify Your Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the imaginary TEACHER'S MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE JULY 1, 2020. Nothing is needed from inside this package until July 1.

2. YOU must verify on the AMC 12 CERTIFICATION FORM that you followed all rules associated with the conduct of this exam.

3. The Answer Form may be mailed First Class to the AMC office no later than 24 hours following the exam, but it may not be graded as this is not an official AMC exam.

4. *The publication, reproduction, or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, AoPS, telepathy, or media of an type is a violation of the competition rules. Consequences include, but are not limited to, bad karma and increased likelihood of silly errors on the official AMC exam.*

The problems were written by scrabbler94 and test-solved by djmathman, P_Groudon, fidgetboss_4000, nikenissan, and IMadeYouReadThis.

Questions and comments about problems or solutions within this exam should be communicated via private message (PM) to scrabbler94.

PUBLICATIONS

A complete listing of previous publications may be found at:
`https://artofproblemsolving.com/wiki/index.php/User:Scrabbler94`.

scrabbler94's Mathematics Competitions

2nd Biannual

Mock AMC 12A

scrabbler94's Mathematics Competitions 12A

Wednesday, July 1, 2020 -
Wednesday, July 15, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. **No copies.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the scrabbler94 Mathematics Competitions (i.e., myself) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CSMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this AMC 12 will be invited to take the 1^{st} Annual Interplanetary Mathematics Examination (AIME) on January 1, 9001. More details about the AIME are on the back page of this packet.

The publication, reproduction, or communication of the problems or solutions during the period when students are allowed to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, AoPS, telepathy, or media of any type is a violation of the competition rules.

1. A grocery store sells candy bars for \$2.50 each. There is a 30% discount if five or more candy bars are bought. How many dollars will Nathan save if he buys five candy bars instead of four?

   (A) \$1.25 **(B)** \$1.50 **(C)** \$1.75 **(D)** \$2.50 **(E)** \$3.75

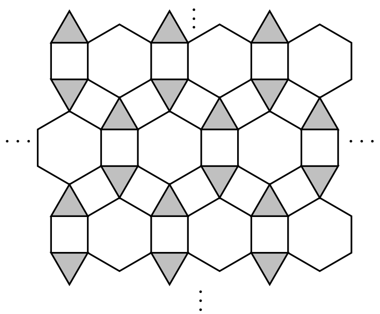
2. The ratio of boys to girls in a large sophomore class is $3:4$. When polled, it was found that exactly $\frac{1}{4}$ of the boys have a pet and $\frac{1}{3}$ of the girls have a pet. Out of all sophomores who have a pet, what percent of them are boys?

   (A) 36% **(B)** 40% **(C)** 50% **(D)** $56\frac{1}{4}\%$ **(E)** 64%

3. What is the value of $\dfrac{2020!+2019!}{2018!+2017!}$?

   (A) 4076360 **(B)** 4076361 **(C)** 4078374 **(D)** 4078378 **(E)** 4078380

4. An infinitely large floor is tessellated using the following pattern consisting of squares, regular hexagons, and equilateral triangles.



   Which of the following is closest to the percentage of the total floor area covered by the triangles?

   (A) 11% **(B)** 12% **(C)** 13% **(D)** 15% **(E)** 17%

24. Square $PQRS$ has coordinates $P=(0,0)$, $Q=(1,0)$, $R=(1,1)$, and $S=(0,1)$ in the Cartesian plane. For $i\in\{1,2,\ldots,2020\}$, points P_i, Q_i, R_i, and S_i are on segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$ respectively, with coordinates

$$P_i=\left(\frac{1}{2^i},0\right)$$
$$Q_i=\left(1,\frac{1}{2^i}\right)$$
$$R_i=\left(1-\frac{1}{2^i},1\right)$$
$$S_i=\left(0,1-\frac{1}{2^i}\right).$$

Let

$$K=\sum_{1\le p,q,r,s\le 2020}[P_pQ_qR_rS_s].$$

What is the remainder when $\lfloor K\rfloor$ is divided by 1000? ($[X]$ denotes the area of X, and $\lfloor K\rfloor$ denotes the greatest integer less than or equal to K)

   (A) 0 **(B)** 599 **(C)** 600 **(D)** 799 **(E)** 800

25. How many of the first 1000 positive integers can be written in the form

$$a!+b!+c!+d!+e!$$

for not necessarily distinct nonnegative integers a, b, c, d, and e? (Recall $0!=1$)

   (A) 187 **(B)** 188 **(C)** 189 **(D)** 190 **(E)** 191

19. Circles ω_1 and ω_2, each with radius 1, are drawn in the plane such that the distance between the centers of ω_1 and ω_2 is 1. Circles ω_1 and ω_2 intersect at two distinct points A and B. Points P and Q are on ω_1 and ω_2 respectively, such that P, A, and Q are collinear and A is between P and Q. Given that $QA = 2 \cdot AP$, what is the length of segment $\overline{PQ}$?

(A) $\frac{9}{7}$ **(B)** $\frac{3\sqrt{21}}{7}$ **(C)** $\frac{3\sqrt{2}}{2}$ **(D)** $\frac{3\sqrt{33}}{8}$ **(E)** $\frac{3\sqrt{26}}{7}$

20. Paige repeatedly rolls a fair six-sided die, and keeps a running total of the dice rolls she has obtained thus far. For example, if Paige's rolls are 2, 3, 1, 1, and 6 in that order, then her running total is 2, 5, 6, 7, then 13. If Paige rolls the die indefinitely, then the probability her running total equals 7 at some point in time equals $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the remainder when $m+n$ is divided by 1000?

(A) 7 **(B)** 241 **(C)** 585 **(D)** 733 **(E)** 929

21. Let $\mathcal{R}$ denote the set of all complex numbers z with the property that there exist real numbers $a, b \in [0,1]$ such that $z^2 + az + b = 0$. What is the area of $\mathcal{R}$, when graphed in the complex plane?

(A) $\frac{3\sqrt{3}+2\pi}{12}$ **(B)** $\frac{3\sqrt{3}+\pi}{6}$ **(C)** $\frac{\pi}{2}$ **(D)** $\frac{3\sqrt{3}+2\pi}{3}$ **(E)** 2π

22. Equiangular hexagon $ABCDEF$ has $AB = CD = EF = 28$ and $BC = DE = FA = 14$. Points M, N, O, P, Q, and R are the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$, respectively. There exists a unique circle ω which is tangent to line segments $\overline{MN}$, $\overline{OP}$, and $\overline{QR}$. What is the area of ω?

(A) $\frac{2023\pi}{12}$ **(B)** $\frac{833\pi}{4}$ **(C)** $\frac{700\pi}{3}$ **(D)** $\frac{1029\pi}{4}$ **(E)** 336π

23. How many ordered 11-tuples $(a_0, a_1, a_2, \ldots, a_{10})$ of integers satisfy the equation

$$a_0 + 2a_1 + 2^2a_2 + \ldots + 2^{10}a_{10} = 2020$$

where $0 \le a_i \le 2$ for all $0 \le i \le 10$?

(A) 27 **(B)** 33 **(C)** 39 **(D)** 45 **(E)** 51

5. How many nonempty subsets S of $\{1,2,3,4,5,6\}$ are there such that the product of the elements in S is divisible by 6?

(A) 32 **(B)** 40 **(C)** 44 **(D)** 46 **(E)** 48

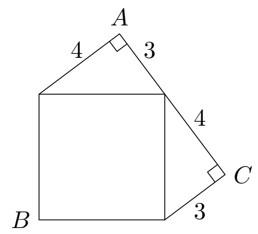
6. Let $f(x) = |x-4|$ and $g(x) = x^2$ for all real numbers x. How many real numbers x satisfy $f(g(x)) = g(f(x))$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** Infinitely many

7. Positive integers $a, b, c \ge 2$ satisfy the equation $abc + ab + a = 64$. What is $a+b+c$?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** Cannot be determined

8. The figure below contains two 3-4-5 right triangles attached to a square of side length 5. The value of $\tan \angle ABC$ equals $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

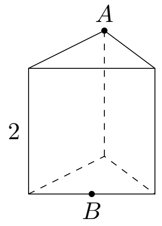


(A) 12 **(B)** 49 **(C)** 74 **(D)** 86 **(E)** 91

9. Joanna has \$372, consisting of two bills in each of the denominations \$1, \$5, \$10, \$20, \$50, and \$100. Using any combination of one or more of these bills, how many different monetary amounts can Joanna form? For example, she can form \$141 using one \$100 bill, two \$20 bills, and one \$1 bill.

(A) 221 **(B)** 222 **(C)** 223 **(D)** 224 **(E)** 225

10. A right triangular prism has all edges of length 2. An ant crawls on the exterior of the prism from point A to point B, where B is the midpoint of the edge opposite A as shown. What is the shortest possible distance the ant crawls?



(A) $\sqrt{7}$ **(B)** $\sqrt{7+2\sqrt{3}}$ **(C)** $\sqrt{8+2\sqrt{3}}$ **(D)** $\sqrt{13}$ **(E)** $2+\sqrt{3}$

11. What is the product of all real solutions a to the equation $8(\log_2 a)^3 = 1 + 8\log_8(a^3)$?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 3

12. A large number of children sit in a circle. They count the numbers 1, 2, 3, ... in clockwise order, starting with the oldest child who counts the number "1." Once any child counts a number which is not divisible by 2, 3, or 5, that child leaves the circle and the next child continues with the next number. In particular, the oldest child leaves the circle after counting the number "1." They count until only one child remains, at which point they stop counting. Given that the number "121" was the last number counted, how many children were originally in the circle?

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

13. Let n be the smallest positive integer with the property that $\operatorname{lcm}(n, 2020!) = 2021!$, where $\operatorname{lcm}(a, b)$ denotes the least common multiple of a and b. How many positive factors does n have?

(A) 4 **(B)** 2021 **(C)** 2112 **(D)** 2156 **(E)** 2205

14. Robert writes all positive divisors of the number 216 on separate slips of paper, then places the slips into a hat. He randomly selects three slips from the hat, with replacement. What is the probability that the product of the numbers on the three slips Robert selects is a divisor of 216?

(A) $\frac{1}{64}$ **(B)** $\frac{81}{4096}$ **(C)** $\frac{25}{1024}$ **(D)** $\frac{225}{4096}$ **(E)** $\frac{25}{256}$

15. Circles ω_1 and ω_2 centered at $O_1 = (0, 0)$ and $O_2 = (9, 12)$ with radii 1 and 2, respectively, are drawn in the xy-plane. Two distinct lines ℓ_1 and ℓ_2 are tangent to both ω_1 and ω_2, and intersect at a point P on segment $\overline{O_1O_2}$. What is the sum of the y-intercepts of lines ℓ_1 and ℓ_2?

(A) -2 **(B)** $-\frac{3}{2}$ **(C)** $-\frac{4}{3}$ **(D)** $-\frac{5}{4}$ **(E)** -1

16. Let $f^1(x) = x^2 - 20$ for all real numbers x, and let $f^k(x) = f^1(f^{k-1}(x))$ for all integers $k \geq 2$. Let x_0 and x_1 be the smallest and largest real solutions to the equation $f^{2020}(x) = 0$, respectively. What is the largest integer less than or equal to $x_0^2 + x_1^2$?

(A) 32 **(B)** 40 **(C)** 41 **(D)** 49 **(E)** 50

17. A positive integer is *monotone* if its digits, when read left-to-right, are either in strictly increasing or strictly decreasing order. For example, 7, 540, and 24578 are monotone numbers, while 0 and 9986 are not. How many monotone numbers leave a remainder of 1 when divided by 3?

(A) 495 **(B)** 501 **(C)** 504 **(D)** 508 **(E)** 511

18. How many ways can six people of different heights stand in line such that for all $1 \leq k \leq 6$, the k^{th} tallest person must stand next to either the $(k+1)^{\text{th}}$ or $(k-1)^{\text{th}}$ tallest person (or both)? In particular, the tallest person must stand next to the second tallest person, and the shortest person must stand next to the second shortest person.

(A) 42 **(B)** 48 **(C)** 54 **(D)** 56 **(E)** 66