July 2021 Mock AMC 12

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Instructions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. **No copies.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

1. Compute $2021 - 2^{0+1+2} + 10 \times 21$.

(A) 2217 **(B)** 2221 **(C)** 2223 **(D)** 2227 **(E)** 2233

2. How many ways are there to arrange the numbers $1, 2, 3, 4$ in a line so that the product of some two adjacent numbers in the line is odd?

(A) 8 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 20

3. How many positive integers $a \leq 50$ satisfy $\gcd(a, 50) = 5$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

4. Let $ABCDEF$ be a regular hexagon with side length 1. If M, N, and P are the midpoints of AB, BC, and CD, respectively, what is the area of $\triangle MNP$?

(A) $\frac{3\sqrt{3}}{16}$ **(B)** $\frac{\sqrt{3}}{4}$ **(C)** $\frac{3\sqrt{3}}{8}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** $\frac{9\sqrt{3}}{16}$

5. On a cube, Clark randomly writes the numbers $1, 2, 3, 4, 5, 6$ on the faces such that each number is used exactly once and each face contains exactly one number. What is the probability that the numbers on every pair of opposite faces add up to 7?

(A) $\frac{1}{24}$ **(B)** $\frac{1}{15}$ **(C)** $\frac{1}{14}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{2}{15}$

6. Let $x = 0.7111111111\ldots$ with an infinite number of ones trailing after the seven. The real number x can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 17 **(B)** 55 **(C)** 77 **(D)** 163 **(E)** 170

7. Let S be the set of complex numbers z in the complex plane that satisfy $|z| = |z + 6|$. Which of these best describes S?

(A) One point **(B)** Two points **(C)** A horizontal line **(D)** A vertical line **(E)** Two vertical lines

8. How many positive integers n less than or equal to 30 satisfy exactly two of the following statements?

- n is divisible by 2
- n is divisible by 3
- n is divisible by 5

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

9. Let $ABCD$ be a rhombus with $AC = 8$ and $BD = 6$. A circle that passes through A is tangent to both segment $\overline{CD}$ and segment $\overline{BC}$. What is the radius of this circle?

(A) $\frac{3}{2}$ **(B)** 2 **(C)** $\frac{7}{3}$ **(D)** $\frac{8}{3}$ **(E)** 3

10. Let T be the time between 3:00 pm and 4:00 pm on an analog clock such that the minute hand and hour hand coincide. Which of these times is closest to T?

(A) 3:15 pm **(B)** 3:16 pm **(C)** 3:17 pm **(D)** 3:18 pm **(E)** 3:19 pm

11. Triangle $\triangle ABC$ has side lengths $AC = 45$ and $BC = 53$ and $\angle A = 90°$. A point is randomly chosen inside the triangle. The probability that the point is closest to AB among the three sides of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 4 **(B)** 11 **(C)** 13 **(D)** 23 **(E)** 179

12. Suppose the interior angle measure of a regular m-gon is exactly 24 degrees greater than the interior angle measure of a regular n-gon. What is the sum of all possible values of n?

(A) 38 **(B)** 39 **(C)** 42 **(D)** 45 **(E)** 47

13. Let S be the set $\{1, 2, 3, 4, 5, 6, 7\}$. How many non-empty subsets of S have a median equal to 4?

(A) 20 **(B)** 23 **(C)** 24 **(D)** 26 **(E)** 29

14. Coins A and B are both unfair and flipping coin A and getting heads is more likely than flipping coin B and getting heads. When both coins are flipped simultaneously, the probability of obtaining heads on both coins is $\frac{8}{35}$ while the probability of obtaining tails on both coins is $\frac{1}{7}$. What is the probability that when both coins are flipped, coin A lands on heads and coin B lands on tails?

(A) $\dfrac{2}{35}$ **(B)** $\dfrac{2}{5}$ **(C)** $\dfrac{16}{35}$ **(D)** $\dfrac{4}{7}$ **(E)** $\dfrac{5}{7}$

15. What is the sum of all integers $30 \leq x \leq 50$ such that

$$\text{lcm}(1, 2, 3, ..., x-1) \neq \text{lcm}(1, 2, 3, ..., x)?$$

(A) 199 **(B)** 215 **(C)** 264 **(D)** 280 **(E)** 301

16. Let $ABCD$ be an isosceles trapezoid where AD is parallel to BC and $AD > BC$. Suppose the distance between lines BC and AD is 5 and $AB = CD = \sqrt{26}$. There exists a point E on line AB such that $\triangle CDE$ is equilateral. Then, BC can be written in the form $\frac{a\sqrt{b}}{c} - d$ for positive integers $a, b, c,$ and d, where a and c are relatively prime and b is not divisible by the square of any prime. What is $a+b+c+d$?

(A) 14 **(B)** 18 **(C)** 21 **(D)** 23 **(E)** 27

17. Let $\tau(n)$ denote the number of positive integer divisors of n. Call a positive integer n refactorable if $\tau(n)$ is a divisor of n. How many odd refactorable integers are there less than 500?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

18. How many real solutions x are there to $\sin(x^2) = \log_2(\log_2(x))$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

19. Let $(a_1, a_2, ..., a_6)$ be a permutation of $(1, 2, ..., 6)$. There are N permutations for which there is at least one ordered pair of integers (i, j) with $1 \leq i < j \leq 6$ that satisfy $a_i = j$ and $a_j = i$. What is the sum of the digits of N?

(A) 9 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 15

20. Let $\sigma(n)$ denote the sum of all the positive integer divisors of n, including 1 and n. How many positive integers n satisfy $\sigma(n) + \sigma(3n) = 954$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

21. Let $\triangle ABC$ have sides $AB = 6$ and $BC = 4$ and $\angle B = 90°$. Let G be the centroid of $\triangle ABC$. Line BG intersects the circumcircle of $\triangle AGC$ at two distinct points, G and E. Line CG intersects the circumcircle of $\triangle ABG$ at two distinct points, G and D. The area of $\triangle DGE$ equals $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 165 **(B)** 194 **(C)** 217 **(D)** 236 **(E)** 282

22. The polynomial $x^6 - 6x^5 + 15x^4 - 19x^3 + 18x^2 - 3x + 2$ has six distinct complex roots, only two of which can be written in the form $\frac{p \pm i\sqrt{q}}{r}$, where p, q, and r are positive integers and q is not divisible by the square of any prime. What is $p + q + r$?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

23. Three spheres, S_1, S_2, and S_3 are mutually externally tangent and each are tangent to distinct planes P_1 and P_2. Spheres S_1 and S_2 both have a radius of 6, and the radius of S_3 is less than 6. Suppose P_1 and P_2 form a $60°$ angle with the three spheres inside the $60°$ angle. The radius of S_3 can be written as $a - b\sqrt{c}$, where a, b, and c are all positive integers and c is squarefree. What is $a + b + c$?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

24. Matthew writes the number 2021^{2021} on a blackboard and chooses a positive integer $k \geq 2$. Each minute, he replaces the number m on the board with the sum of the digits when the base-10 number m is written in base-k. For example, if 125 is written on the board and $k = 7$, he replaces 125 with 11 because $125_{10} = 236_7$. However, he will stop when the number on the board is the same for two consecutive minutes. Let $f(k)$ be the number left on the board once he stops if he chooses the integer k. Suppose $A = f(7) \cdot f(8) \cdot f(9)$. Which of these intervals contains A?

(A) $[0, 50]$ **(B)** $[51, 100]$ **(C)** $[101, 150]$ **(D)** $[151, 200]$ **(E)** $[201, \infty)$

25. There are 13 chairs arranged in a circle and each chair will be painted red or blue. However, there is a rule stating that there must be exactly 2 sets of three adjacent chairs in the circle such that more than one chair in the set is painted red. How many distinct colorings are there that follow the rule? Two colorings are considered the same if one coloring can be rotated to obtain the other.

(A) 19 **(B)** 22 **(C)** 26 **(D)** 28 **(E)** 34