2021 CMC 12A Solutions Document

Christmas Math Competitions

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1. **Answer (E):** We compute the expression as follows:

$$\frac{21^2+\frac{21}{20}}{20^2+\frac{20}{21}}=\frac{21(21+\frac{1}{20})}{20(20+\frac{1}{21})}=\frac{21\cdot\frac{421}{20}}{20\cdot\frac{421}{21}}=\frac{\frac{21}{20}}{\frac{20}{21}}=\frac{441}{400}.$$

2. **Answer (C):** Label the two circles C_1 and C_2. If the total area covered by the two circles is 25π and the area of the overlap is 7π, the area covered by exactly one of C_1 or C_2 (but not both) is $25\pi - 7\pi = 18\pi$. Since both circles have the same radius, this means that the area covered by only C_1 is 9π. Adding back the overlap, the area of C_1 is $9\pi + 7\pi = 16\pi$. Since $\pi \cdot r^2 = 16\pi$, we have $r = 4$.

3. **Answer (C):** A regular k-gon only exists for integers $k \geq 3$, so $m, n \geq 3$.

   A pyramid with a regular n-gon as a base has $2n$ edges; there are n edges along the n-gon base and one edge from each vertex of the n-gon to the pyramid's apex.

   A prism with a regular m-gon as a base has $3m$ edges; there are m edges each along the two m-gon faces and one edge connecting each vertex of the m-gon face to some vertex of the other m-gon face.

   Thus, $2n = 3m$, where n and m are positive integers. We can see that n must be a multiple of 3. However, $n = 3$ causes $m = 2$, which is not allowed since $m \geq 3$. Testing $n = 6$ causes $m = 4$, which is allowed. Thus, the smallest possible value of n is 6.

4. **Answer (C):** Replace 1000 with x. Then, we can rewrite the given expression as

$$\frac{\frac{1}{x-1}+\frac{1}{x+1}}{2}-\frac{1}{x}=\frac{1}{N}.$$

   We simplify the left hand side:

$$\frac{\frac{1}{x-1}+\frac{1}{x+1}}{2}-\frac{1}{x}=\frac{\frac{2x}{x^2-1}}{2}-\frac{1}{x}=\frac{x}{x^2-1}-\frac{1}{x}=\frac{1}{x(x+1)(x-1)}$$

   Thus, $N = x(x+1)(x-1) = 1000 \cdot 999 \cdot 1001 = 1000 \cdot 999,999 = 999,999,000.$

   The requested answer is $9 \cdot 6 = 54$.

5. **Answer (A):** We have $n(1+2+3+\cdots m) = 2020$. Note that $1+2+3+...+m = \frac{m(m+1)}{2}$, the equation becomes $m(m+1)n = 4040$. Since m and $m+1$ are both integers, we need two consecutive factors of $4040 = 2^3 \cdot 5 \cdot 101$ to differ in absolute value by 1, and both factors must be greater than 1. The only such pair $(m, m+1)$ satisfying both requirements is $(4, 5)$, so $m = 4$ and $n = \frac{4040}{20} = 202$. The requested answer is $4 + 202 = 206$.

6. **Answer (B):** Let the parallelogram be $ABCD$.

   If $ABCD$ is not a square or rectangle, all 4 perpendicular bisectors clearly do not share a common point, so (i) is false.

   If $ABCD$ is a rectangle, then the perpendicular bisectors of AB and CD coincide, so (ii) is false.

   If all perpendicular bisectors of $ABCD$ meet at a common point, that does not necessarily imply $ABCD$ that is a square: all rectangles have this property too. Thus, also statement (iii) is false.

   If the perpendicular bisectors of AB and CD are distinct, then both bisectors are perpendicular to both AB and CD, which means the bisectors themselves are parallel to each other. Similarly, the perpendicular bisectors of AD and BC are also parallel to each other. Therefore, the four bisectors form a parallelogram and (iv) is true.

   All in all, only 1 of the 4 statements is true.

7. **Answer (A):** Obviously, $1 = 1; 2 = 1+1; 3 = 1+1+1; 4 = 4; 5 = 4+1; 6 = 4+1+1$.

   To show that 7 cannot be the sum of fewer than 4 squares, note that the only squares less than 7 are 1 and 4. If we just use 1s, we will use seven 1s, which is not allowed. We can use at most one 4. If we use one 4, we need three more 1s for the sum to be 7, which is 4 squares total. Thus, 7 is the first integer with the given property.

   We can see that: $8 = 4+4; 9 = 9; 10 = 9+1; 11 = 9+1+1; 12 = 4+4+4; 13 = 9+4; 14 = 9+4+1$. Now, we will show that 15 cannot be expressed as the sum of fewer than 4 squares. The squares less than or equal to 15 are $1, 4$, and 9. If we use no 9s, the maximum sum of the squares is $4+4+4 = 12$, so we must use a 9. If we use one 9 and no 4s, the maximum sum of the squares is $9+1+1 = 11$, so we must use a 4. However, neither $9+4+1$ nor $9+4+4$ is 15, so 15 is the second integer with the given property.

   The requested sum is $7 + 15 = 22$.

8. **Answer (C):** Each ice cream flavor will be used up twice, once, or no times at all. Thus, to distribute the scoops of ice cream to 5 people, we establish two cases.

   Case 1: two scoops get used up twice and one scoop gets used up once

   We have $\binom{4}{2}$ ways to choose which scoops get used up twice and then 2 ways to choose which flavor gets used up once. We have $\binom{5}{2} \cdot \binom{3}{2}$ ways to distribute both flavors that get used up twice. Then, the remaining scoop goes to the last person. Thus, there are $\binom{4}{2} \cdot 2 \cdot \binom{5}{2} \cdot \binom{3}{2} = 360$ distributions in this case.

   Case 2: one scoop gets used up twice and every other scoop gets used up once

   We have 4 ways to choose which flavor gets used up twice in which we know every other flavor gets used up once. We have $\binom{5}{2}$ ways to determine who gets the flavor used up twice. Then, there are $3!$ ways to distribute the last 3 scoops. Thus, there are $4 \cdot \binom{5}{2} \cdot 3! = 240$ distributions in this case.

   In total, there are $360 + 240 = 600$ distributions.

9. **Answer (B):** Suppose the 6 points are on the number line with A at 0. WLOG, suppose B is at 1 (as opposed to -1). Then, since $BC = 4$, C is at the number 1 ± 4 (depending on whether C is to the left or right of B). Then, D is at the number $1 \pm 4 \pm 9$. Continuing this pattern, F is at $1 \pm 4 \pm 9 \pm 16 \pm 25$. Since $AF = 23$, we must have $1 \pm 4 \pm 9 \pm 16 \pm 25 = 23$ or -23.

   If we assume each $\pm$ is a $+$, the value of the expression equals 55. Thus, we have overshot by $55 - 23 = 32$ or $55 - (-23) = 78$. Note that for an integer k in the expression, if we change the sign from $+$ to $-$, the value of the expression decreases by $2k$. Immediately, we notice we can change the sign of 16 to yield 23. We can quickly check that we can't change the signs so that the expression equals -23. Thus, point F is at 23 in the number line and the expression is $1 + 4 + 9 - 16 + 25$. Then, C is at $1 + 4 = 5$, which means $CF = 23 - 5 = 18$.

10. **Answer (A):** If one of the teams wins in a game, the winning team earns 3 points, which contributes to a $+3$ increase in the sum of the points across all teams in the league. On the other hand, if the game ends in a draw, both teams earn 1 point each, which contributes to a $+2$ increase in the sum of the points across all teams.

    Suppose that x games ended up with one team winning and y games ended up in a tie. Since there were 500 points earned in total, $3x + 2y = 500$. In addition, since each possible pair of teams played each other once, there were $\binom{20}{2} = 190$ games. Thus, $x + y = 190$. Solving the system of equations, we have $x = 120$ and $y = 70$. Thus, 70 games ended in a draw.

11. **Answer (A):** Let $PY = k$. By Pythagorean Theorem on $\triangle PAY$, $PA = \sqrt{PY^2 + YA^2} = \sqrt{k^2 + 4}$. By Pythagorean Theorem on $\triangle PBY$, $PB = \sqrt{PY^2 + YB^2} = \sqrt{k^2 + 25}$. Since the diagonals of any parallelogram bisect each other, $PD = PB = \sqrt{k^2 + 25}$.

    By Pythagorean Theorem on $\triangle PAX$, $PX = \sqrt{PA^2 - AX^2} = \sqrt{k^2 + 3}$. Finally, by Pythagorean Theorem on $\triangle PDX$, $DX = \sqrt{PD^2 - PX^2} = \sqrt{22}$.

12. **Answer (D):** Let $f(n) = \lceil \sqrt{n} \rceil - \lfloor \sqrt{n} \rfloor$. We arrange the given equation:

    $$\lceil \sqrt{a} \rceil - \lfloor \sqrt{a} \rfloor = \left\lceil \sqrt{b} \right\rceil - \left\lfloor \sqrt{b} \right\rfloor \implies f(a) = f(b)$$

    Now, we will analyze the behavior of $f(n)$.

    If $\sqrt{n}$ is not an integer (in other words, if n is not a perfect square), then $\lfloor \sqrt{n} \rfloor$ will strictly round $\sqrt{n}$ down to the nearest integer. On the other hand, $\lceil \sqrt{n} \rceil$ will strictly round $\sqrt{n}$ up to the nearest integer. Thus, in this case, $f(n) = 1$.

    If $\sqrt{n}$ is an integer (implying n is a perfect square), then $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$, implying $f(n) = 0$.

    Thus, for $f(a) = f(b)$, a and b must either both be non-perfect squares or both be perfect squares.

    There are 10 perfect squares less than or equal to 100 and 90 non-perfect squares less than or equal to 100. Thus, the desired probability is $\frac{10^2+90^2}{100^2} = \frac{41}{50}$.

13. **Answer (D):** Removing the logarithm, we have $a^d = \overline{b2c}$ for digits a, b, c, and d with $a \neq 1$ and $b \neq 0$.

    We do casework on the value of d.

    Clearly, if $d \in \{0, 1, 2\}$, a^d can never be 3 digits, since $a \leq 9$.

    If $d = 3$, then $5^3 = 125$ suffices.

If $d = 4$, then $5^4 = 625$ suffices.

If $d = 5$, we can check that $2^5 = 32$ is too small, $3^5 = 243$ does not have a tens digit of 2, and $4^5 = 1024$ is too big.

If $d = 6$, then $3^6 = 729$ suffices.

If $d = 7$, then $2^7 = 128$ suffices.

If $d = 8$, then $2^8 = 256$ doesn't work. Clearly, 3^8 is too big. If $d = 9$, then $2^9 = 512$ doesn't work.

Therefore, the sum of all possible values of d is $3 + 4 + 6 + 7 = 20$.

14. **Answer (C):** For complex numbers c_1 and c_2, define $\overline{(c_1)(c_2)}$ to be the line segment determined by c_1 and c_2 in the complex plane. Let $w = 1015 + 1516i$ be the midpoint of the $\overline{(z_1)(z_2)}$. There are two possible values of z_3 depending on which side of $\overline{(z_1)(z_2)}$ z_3 lies on. Let z_3' and z_3'' be the two values of z_3. We know that by symmetry, $\overline{(z_3')(z_3'')}$ has w as its midpoint. Thus, $z_1 + z_2 = z_3' + z_3'' = 2w = 2030 + 2032i$.

    If we let $z_3' = p + qi = az_1 + bz_2$ and $z_3'' = (2030 - p) + (2032 - q)i$, we have that

    $$z_3'' = 2030 + 2032i - z_3' = z_1 + z_2 - (az_1 + bz_2) = (1 - a)z_1 + (1 - b)z_2.$$

    Hence, the answer is $[a + b] + [(1 - a) + (1 - b)] = 2$.

    To rigorously prove that the two values of $a + b$ obtained by $z_3' = az_1 + bz_2$ and $z_3'' = (1-a)z_1 + (1-b)z_2$ are distinct, it suffices to show $a + b \neq 1$. Assume for the sake of contradiction that $a + b = 1$.

    Then, subtracting the equations with z_3' and z_3'' and plugging in $b = 1 - a$, we have:

    $$z_3'' - z_3' = z_1 - z_2 - 2a(z_1 - z_2) = (z_1 - z_2)(1 - 2a).$$

    However, we know that $\overline{(z_1)(z_2)}$ and $\overline{(z_3')(z_3'')}$ are perpendicular lines in the complex plane by construction of the equilateral triangles. In particular, we can find that $\overline{(z_1)(z_2)}$ has a slope of -1, so $z_1 - z_2$ (and the right hand side of the equation) is a real multiple of $-1 + i$. Then, $\overline{(z_3')(z_3'')}$ has a slope of 1, which means $z_3'' - z_3'$ (and the left hand side of the equation) is a real multiple of $1 + i$. However, the only real multiple of both $1 + i$ and $-1 + i$ is 0, so both sides must be 0. However, this implies $z_3'' = z_3$, contradiction. Thus, $a + b \neq 1$ and the two possible values of $a + b$ are distinct.

15. **Answer (B):** Suppose the three roots of $P(x)$ are $r < s < t$. Because the three roots follow an arithmetic progression, $2s = r + t$, and by Vieta's formulas on $P(x)$ we know that $r + s + t = -a$.

    Using Vieta's formulas on $Q(x)$, we get that

    $$(r + 1) + (s + 1) + (t + 1) = 7 \implies r + s + t = 4.$$

    Therefore, $a = -4$. With the arithmetic progression condition,

    $$r + s + t = 4 \implies 3s = 4 \implies s = \frac{4}{3}.$$

    Thus, $\frac{4}{3}$ is a root of $P(x)$. Since $P(\frac{4}{3}) = 0$, we have:

    $$\left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)^2 + b\left(\frac{4}{3}\right) + 10 = 0 \implies b = -\frac{71}{18}.$$

    The requested sum is $71 + 18 = 89$.

16. **Answer (C):** By the equilateral triangle, $\angle QPC = 60^\circ$ and thus, $\angle APQ = 120^\circ$. The area of $\triangle APQ$ is given by $\frac{1}{2} \cdot \sin(120^\circ) \cdot PQ \cdot AP$ by the Law of Sines area formula. By equilateral triangle $\triangle CPQ$, $PQ = PC$. In addition, by a Power of a Point with respect to P and ω, $PA \cdot PC = PB \cdot PD = 28$. Thus, $[APQ] = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 28 = 7\sqrt{3}$.

17. **Answer (D):** Let $r_1 = 2 - \sqrt{3}$ and $r_2 = 2 + \sqrt{3}$ be the two roots of $P(x)$. By the definition of $P(x)$ and $Q(x)$ being friends, $Q(r_1) = r_i$ and $Q(r_2) = r_j$, where i and j are individually either 1 or 2. We will do casework on the values of i and j.

    Case 1: $Q(r_1) = Q(r_2) = k$, where $k \in \{r_1, r_2\}$

    Since $Q(x)$ is a quadratic polynomial with leading coefficient 1, we have $Q(x) = k + (x - r_1)(x - r_2)$. In this case, $Q(0) = k + r_1 \cdot r_2 = k + 1$. We choose $k = r_2 = 2 + \sqrt{3}$ in order to maximize $Q(0)$. Thus, the maximum value of $Q(0)$ in this case is $3 + \sqrt{3}$.

    Case 2: $Q(r_1) = r_1$ and $Q(r_2) = r_2$

    This means that the polynomial $Q(x) - x$ has roots r_1 and r_2. In addition, it has the same degree, leading coefficient, and roots as $P(x)$. Thus, $Q(x) - x = P(x)$. Plugging in $x = 0$ gives $Q(0) = P(0) = 1$.

    Case 3: $Q(r_1) = r_2$ and $Q(r_2) = r_1$

    In this case, $Q(x) = (x - r_1)(x - r_2) + (r_1 + r_2 - x)$. Plugging in $x = 0$ gives $Q(0) = r_1 \cdot r_2 + r_1 + r_2 = 5$.

    Across all cases, 5 is the maximum value of $Q(0)$ and hence, the answer.

18. **Answer (C):** Suppose that the pyramid is folded up into its 3D form and all the edges are intact. We will do casework based on how many edges we cut among the edges of $BCDE$. Note that if we cut all the edges of $BCDE$, the square $BCDE$ will become a separate piece from the triangles, which is not allowed.

    - Three of the four edges of $BCDE$ are cut: WLOG, assume that BC is the edge that remains intact. Because rotations do not count as distinct nets, we do not multiply by 4. After the three edges are cut, we must cut one of the four edges emanating from A. If we cut more than one of the four edges, we will form more than one piece, which is not allowed. Each of the four possible cuts yields a distinct net, so there are 4 nets in this case.
    - Two opposite edges of $BCDE$ are cut: WLOG, suppose that CD and BE are cut. We must cut one of $\{AC, AD\}$ and one of $\{AB, AE\}$ in order to unfold the pyramid. There are $2 \cdot 2 = 4$ ways to make the cuts. However, if we cut AB and AC and unfold, it yields the same net as if we cut AD and AE and unfolded. All other nets are distinct. Thus, there are $4 - 1 = 3$ nets in this case.
    - Two adjacent edges of $BCDE$ are cut: WLOG, suppose that DE and BE are cut. Clearly, AC must be cut. Then, we have to cut one of the three remaining edges emanating from A, unfolding the rest of the pyramid. There are 3 nets in this case.
    - One edge of $BCDE$ is cut: WLOG, suppose that BE is cut. Then, one of AB or AE must be cut. Both cuts yield different nets. There are 2 nets in this case.
    - No edges of $BCDE$ are cut: This means that each edge emanating from A must be cut. There is 1 net in this case.

    In total, there are $4 + 3 + 3 + 2 + 1 = 13$ nets.

19. **Answer (C):** Let $g(x) = 2\sin x$ and let $g^n(x) = \underbrace{g \circ ... \circ g}_{n \text{ times}}(x)$, so that $f(x) = g^{200}(x)$. Notice that $0 \le g^n(x) \le 2$ for any $n \ge 1$. Indeed, since $0 \le 2\sin x \le 2$ for $x \in [0, \pi]$,

$$g^2([0,\pi]) = 2\sin(g([0,\pi])) = 2\sin([0,2]) = [0,2]$$

and the general case follows from induction. We claim that for any $n \ge 2$, the function $g^n(x)$ attains its maximum value 2 on $[0, \pi]$ in only two points. Indeed,

$$g^n(x) = 2\sin(g^{n-1}(x)) = 2 \iff g^{n-1}(x) = \frac{\pi}{2} + 2k\pi$$

for some integer k. However, since $0 \le g^{n-1}(x) \le 2$ as we just proved, the only possible value of k is 0, thus

$$g^{n-1}(x) = 2\sin(g^{n-2}(x)) = \frac{\pi}{2} \iff \sin(g^{n-2}(x)) = \frac{\pi}{4}.$$

Since $\sin t$ is increasing in $[0, \pi/2]$ and decreasing in $[\pi/2, 2]$, we know that as t ranges in the interval $[0, 2]$, $\sin t$ assumes each value in $[0, \sin 2) \cup \{1\}$ exactly once and each value in $[\sin 2, 1)$ exactly twice. Since $\pi/4 < \sin 2$ and $0 \le g^{n-2}(x) \le 2$, this proves that there exists a unique value of $g^{n-2}(x)$ satisfying $\sin(g^{n-2}(x)) = \pi/4$, which is

$$g^{n-2}(x) = \arcsin\left(\frac{\pi}{4}\right) \implies \sin(g^{n-3}(x)) = \frac{1}{2}\arcsin\left(\frac{\pi}{4}\right). \quad (\star)$$

Since $\arcsin(x) \le 2x$, we also have that

$$\frac{1}{2}\arcsin\left(\frac{\pi}{4}\right) \le \frac{\pi}{4} < \sin 2$$

thus we can apply the same reasoning as above to conclude that there exists a unique value of $g^{n-3}(x)$ satisfying $(\star)$. Thus, by induction $g^n(x) = 2$ iff $g(x) = 2\sin(x) = \xi$ for some $0 < \xi < 2$ which is uniquely determined applying the reasoning above. Since this equation has clearly 2 distinct solutions for $x \in [0, \pi]$, we conclude that $M = N = 2$, thus the required answer is $M + N = 4$.

20. **Answer (E):** Let $v_p(x)$ denote the exponent of p in the prime factorization of x. In particular, suppose that for an arbitrary prime p that divides n, $v_p(n) = e$.

    For brevity, let $v_p(d_1) = a$ and $v_p(d_2) = b$. For d_1 and d_2 to be relatively prime, one of a and b must be 0. Otherwise, if both a and b were at least 1, that implies p divides both d_1 and d_2, which prevents them from being relatively prime. As d_1 and d_2 are divisors of n, a and b can equal any integer in the interval $[0, e]$. There are $e+1$ ordered pairs (a, b) with $a = 0$ and similarly, $e + 1$ ordered pairs with $b = 0$. However, we must subtract 1 to avoid overcounting $(a, b) = (0, 0)$.

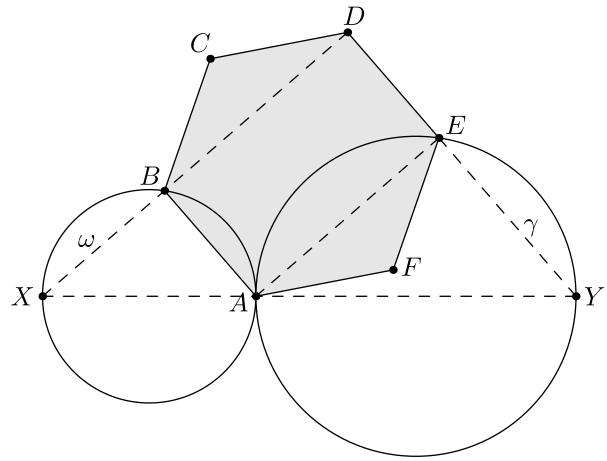
    To summarize, if $v_p(n) = e$, there are $2(e + 1) - 1 = 2e + 1$ valid ways to decide the exponents of p in the prime factorizations of d_1 and d_2. However, the decision of the exponents of d_1 and d_2 regarding one prime has no effect on the decisions of the exponents of other primes in d_1 and d_2 with respect to the condition. Thus, if the prime factorization of n is $p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot ... \cdot p_k^{e_k}$, the number of valid pairs (d_1, d_2) is $(2e_1 + 1)(2e_2 + 1)(2e_3 + 1)...(2e_k + 1) = 2021$. Clearly, each $2e_m + 1$ term is an integer, so we consider ways to write 2021 as the product of positive integers greater than 1. However, the only two ways to do so are 2021 and $43 \cdot 47$.

    If $2e_1 + 1 = 2021$, then $e_1 = 1010$. Therefore, n is in the form q^{1010} for some prime q. Then, n has $1010 + 1 = 1011$ divisors in this case.

    If WLOG $2e_1 + 1 = 43$ and $2e_2 + 1 = 47$, then $(e_1, e_2) = (21, 23)$. Therefore, $n = q^{21} \cdot r^{23}$ for distinct primes q and r. Then, n has $(21 + 1)(23 + 1) = 528$ divisors in this case.

    Thus, the sum of the possible values of the number of divisors of n is $1011 + 528 = 1539$.

21. **Answer (C):**



Let X be the point diametrically opposite A on ω and let Y be the point diametrically opposite A on γ so that X, A, Y are collinear. Note that $\angle XBA = \angle DBA = 90^\circ$ and $\angle YEA = \angle DEA = 90^\circ$, whence $D = XB \cap YE$. Set a as the side-length of the hexagon so that $AE = a\sqrt{3}$ and $BX = \sqrt{AX^2 - AB^2} = \sqrt{16 - a^2}$. From $DB \parallel AE$ we obtain $\triangle ABX \sim \triangle YEA$ thus

$$\left(\frac{BX}{AX}\right)^2 = \left(\frac{AE}{AY}\right)^2 \implies \frac{16 - a^2}{16} = \frac{3a^2}{36}$$

so we can solve for $a^2 = \frac{48}{7}$. The area of $ABCDEF$ can be computed as $\frac{3a^2\sqrt{3}}{2} = \frac{72\sqrt{3}}{7}$ giving an answer of $72 + 3 + 7 = 82$.

22. **Answer (E):** Let the prime factorization of n be $p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k}$.

In general, for a positive integer x to be neither a multiple nor divisor of n, we must fulfill two conditions:

- one of the exponents of the primes in n must be greater than the exponent of the said prime in x (in this way, x is not a multiple of n);
- one of the exponents of the primes in x must be greater than the exponent of the said prime in n (in this way, x is not a divisor of n).

Suppose n has at least 3 distinct prime divisors. Picking the divisors $p_1^{e_1}$ and $p_1 \cdot p_2$ violates the condition. Therefore, n has either 1 or 2 prime divisors.

Now consider numbers of the form $p_1^{e_1} \cdot p_2^{e_2}$ for $e_1 \geq 2$. Picking the divisors $p_2^{e_2}$ and $p_1 \cdot p_2$ violates the condition. Therefore, the exponent of any of the primes in n cannot be greater than 1 if n has 2 prime divisors.

Thus, the only possibilities left are integers in the form p^k for any prime p and any positive integer k and integers in the form pq for distinct primes p and q. It is easy to verify that all integers in these forms work.

For the first case, note that all primes within range are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47, for a total of 15 possible values of n, and all prime powers less than or equal to 50 are $4, 8, 16, 32, 9, 27, 25$ and 49, yielding 8 more values of n.

For the second case, we must count the number of integers of the form pq, with $p < q$.

If $p = 2$, then $q \in \{3, 5, 7, 11, 13, 17, 19, 23\}$, yielding 8 valid n.

If $p = 3$, then $q \in \{5, 7, 11, 13\}$, yielding 4 valid n.

If $p = 5$, then $q = 7$, yielding 1 valid n.

Therefore, in total there are $15 + 8 + 8 + 4 + 1 = 36$ possible values for n.

23. **Answer (C):** The equation $\sum_{k=1}^{n}(-1)^k a_k = 0$ implies $(a_2 + a_4 + a_6 + ...) = (a_1 + a_3 + a_5 + ...)$. In conjunction with $\sum_{k=1}^{n} a_k = 12$, we have $(a_2 + a_4 + a_6 + ...) = (a_1 + a_3 + a_5 + ...) = 6$. Now, we must have $\sum_{k=1}^{\ell}(-1)^k a_k \leq 0$ for all integers ℓ in the interval $[1, n-1]$.

    Consider a particle in the coordinate plane that starts at $(0, 0)$. Now, suppose the particle moves a_1 units to the right, a_2 units up, a_3 units to the right, a_4 units up, and so on (and in that order). The sum of the odd index terms at some point in the sequence corresponds to the x-coordinate of the particle at that point in the path. For, example, if $a_1 = 3, a_3 = 4$, and $a_5 = 1$, then the particle's x-coordinate is $3, 7$, and 8 after the first, third, and fifth moves, respectively. Similarly, the sum of the even index terms at some point in the sequence corresponds to the y-coordinate of the particle at that point in the path. Since $(a_2+a_4+a_6+...) = (a_1+a_3+a_5+...) = 6$, the coordinate's path must end at $(6, 6)$. In addition, since $\sum_{k=1}^{\ell}(-1)^k a_k \leq 0$ for all integers ℓ in the interval $[1, n-1]$, this means the y-coordinate can never be strictly greater than the x-coordinate at any point in the path.

    It is well-known that the mth Catalan number is equal to the number of North-East lattice paths that start at $(0, 0)$ and end at (m, m) and never go above the line $y = x$. In addition, the mth Catalan number is given by $\frac{1}{m+1}\binom{2m}{m}$. Thus, we seek the 6th Catalan number, which is $\frac{1}{7}\binom{12}{6} = 132$.

24. **Answer (B):** Let M be the midpoint of BC (and center of ω). Note that $\angle EPF = \frac{\widehat{BC}+\widehat{EF}}{2}$, where $\widehat{BC}$ and $\widehat{EF}$ are measured with respect to ω. Since BC is the diameter of ω, $\widehat{BC} = 180^\circ$. In addition, $\widehat{EF} = \angle EMF$. Since $AEMF$ is quadrilateral with $\angle AEM = \angle AFM = 90^\circ$, $\angle EMF = 180^\circ - \angle EAF$. Therefore, $\angle EPF = 180^\circ - \frac{1}{2}\angle EAF$.

    Let γ be the circle with center A and radius AF. Major arc $\widehat{FE}$ in γ has measure $360^\circ - \angle EAF$, which is twice the measure of $\angle EPF$. Thus, P lies on γ and $AP = AF = AE$. We will compute AF via Pythagorean Theorem on $\triangle AFM$.

    Clearly, FM is equal to the radius of ω, which is 3. Using Stewart's Theorem on $\triangle ABC$ with cevian AM, $AM = 2\sqrt{7}$. Thus, $AP = AF = \sqrt{19}$.

25. **Answer (B):** For a set of points, define its convex hull to be the smallest convex polygon containing each of the points. In particular, for a set of 4 points, no three of which are collinear, the convex hull is either a non-degenerate quadrilateral or it is a triangle (in which case, one of the points lies in the interior of the triangle determined by the other 3 points).

    Among the $\binom{15}{4}$ sets of 4 points out of the 15, suppose that there are X instances in which the convex hull is a quadrilateral and Y instances in which it is a triangle. Clearly, $X + Y = \binom{15}{4} = 1365$. We seek the maximum value of X, as it is also the number of convex quadrilaterals that can be formed by the 15 points.

    Suppose that the convex hull of the 4 points is a quadrilateral. Since we are told no 4 points are cyclic, the quadrilateral has one pair of opposite angles whose sum is more than 180° and another pair whose sum is less than 180°. Let $ABCD$ be an arbitrary cyclic quadrilateral. Clearly, $\angle A + \angle C = 180^\circ$. Fix points A, B, and D. Recall that if $\angle BCD$ is fixed, then the locus of C forms the arc of a circle. As we increase the size of $\angle C$, $\angle A + \angle C$ increases and the point C will move closer to BD, while staying within the boundaries of the circumcircle of $\triangle ABD$. However, if we decrease the size of $\angle C$, $\angle A + \angle C$ decreases and the point C will leave the circumcircle of $\triangle ABD$. Thus, in a non-cyclic convex quadrilateral formed by 4 of the points, label the vertices such that $\angle A + \angle C > \angle B + \angle D$. Then, $(\odot(BCD), A)$ and $(\odot(ABD), C)$ are two valid pairs, while the pairs involving B and D are not valid. Thus, every set of 4 points in this case produces 2 valid pairs $(\mathcal{C}, P)$.

Now, suppose that the convex hull of the 4 points is a triangle. Let A be the point such that if the other three points are labelled B, C, and D, A lies strictly inside $\triangle BCD$. Clearly, $(\odot(BCD), A)$ is valid. Due to A lying inside $\triangle BCD$, $\angle CAD > \angle CBD$ and similarly, $\angle BAD > \angle BCD$ and $\angle BAC > \angle BDC$, which prevents any additional valid pairs. Thus, every set of 4 points in this case produces 1 valid pair $(\mathcal{C}, P)$.

From our findings about how many valid pairs each case produces, we have $2X + Y = 2021$. Subtracting $X + Y = 1365$ from the equation, $X = 656$ and hence, the maximum (and only) value of X.