2021 March MIMC 10 Solutions

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Video solution of all problems coming soon!

Please ask any questions related to this mock contest in the discussion forum.

1. What is the sum of $2^3 - (-3^4) - 3^4 + 1$?

(A) -155 **(B)** -153 **(C)** 7 **(D)** 9 **(E)** 171

By order of operations, we get that the expression is equal to $8-(-81)-81+1 = 8+81-81+1 = \boxed{\textbf{(B) } 9}$.

2. Okestima is reading a 150 page book. He reads a page every $\frac{2}{3}$ minutes, and he pauses 3 minutes when he reaches the end of page 90 to take a break. He does not read at all during the break. After, he comes back with food and this slows down his reading speed. He reads one page in 2 minutes. If he starts to read at $2:30$, when does he finish the book?

(A) $4:33$ **(B)** $5:30$ **(C)** $5:33$ **(D)** $6:30$ **(E)** $7:33$

To begin, we can start by find the amount of time Okestima takes to read 90 pages. $90 \cdot \frac{2}{3}$ gives us 60 minutes. After he comes back from break, he reads every page in two minutes. $60 \cdot 2 = 120$ minutes. Add the additional 3 minutes from break, he would finish the book at $\boxed{\textbf{(C) } 5:33}$.

3. Find the number of real solutions that satisfy the equation

$$(x^2+2x+2)^{3x+2} = 1$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

There are 2 cases when this expression can be equal to 1: $x^2+2x+2 = 1$ or $3x+2 = 0$. When $x^2+2x+2 = 1$, we can solve this quadratic to get $(x+1)^2 = 0$, or $x = -1$. We can solve the other solution by setting $3x+2 = 0$, or $x = -\frac{2}{3}$. However, we need to make sure that $x^2+2x+2 \neq 0$ because 0^0 is undefined. Therefore, our answer would be $\boxed{\textbf{(C) } 2}$.

4. Stiskwey wrote all the possible permutations of the letters $AABBCCCD$ ($AABBCCCD$ is different from $AABBCCDC$). How many such permutations are there?

(A) 420 **(B)** 630 **(C)** 840 **(D)** 1680 **(E)** 5040

Use the theorem of over-counting (When arrange n distinguishable items and m indistinguishable items, the total number of ways to arrange them is $\frac{(n+m)!}{m!}$.) Therefore, the number of permutations of AABBCCCD is $\frac{8!}{2!2!2!} = \boxed{\textbf{(D)}\ 1680}$.

5. Given $x : y = 5 : 3, z : w = 3 : 2, y : z = 2 : 1$, Find $x : w$.

(A) 3: 1 **(B)** 10: 3 **(C)** 5: 1 **(D)** 20: 3 **(E)** 10: 1

Clearly, w is the smallest variable here. We can set w to 2 (other variables also work, but consists more computations). Using our assumption, we can calculate $z = 3$, $y = 6$, and $x = 10$. Therefore, $x : w = 10 : 2 = \boxed{\textbf{(C)}\ 5{:}1}$.

6. A worker cuts a piece of wire into two pieces. The two pieces, A and B, enclose an equilateral triangle and a square with equal area, respectively. The ratio of the length of B to the length of A can be expressed as $a\sqrt[b]{c} : d$ in the simplest form. Find $a + b + c + d$.

(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 15

There are several different ways to solve this problems. For the sake of convenience, we can substitute a side length of the equilateral triangle. Let 2 be the side length, then the area of the equilateral triangle is $\sqrt{3}$. The side length of the square can be solved by computing $\sqrt[4]{3}$. However, the question is asking for the perimeter of triangle : the perimeter of the square. Therefore, the ratio is $4 \cdot \sqrt[4]{3} : 6 = 2 \cdot \sqrt[4]{3} : 3$. $2 + 4 + 3 + 3 = \boxed{\textbf{(C)}\ 12}$

7. Find the least integer k such that $838_k = 238_k + 1536$ where a_k denotes a in base-k.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

We can express $838_k = 238_k + 1536$ to be $8k^2 + 3k + 8 = 2k^2 + 3k + 8 + 1536$ in base 10. Notice that $3k + 8$ cancels out as it is on both sides of the equation, and we can move $2k^2$ to the left side of the equation. This results in $6k^2 = 1536$, $k^2 = 256$. Since k is a positive integer, $k = \boxed{\textbf{(E)}\ 16}$.

8. In the morning, Mr. Gavin always uses his alarm to wake him up. The alarm is special. It always rings in a cycle of ten rings. The first ring lasts 1 second, and each ring after lasts twice the time than the previous ring. Given that Mr. Gavin has an equal probability of waking up at any time, what is the probability that Mr. Gavin wakes up and end the alarm during the tenth ring?

(A) $\frac{511}{1023}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{512}{1023}$ **(D)** $\frac{257}{512}$ **(E)** $\frac{129}{256}$

First, we want to find the total length of a cycle. The length of a ten-rings cycle is equal to $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 = 2^{10} - 1 = 1023$. The probability is therefore $\frac{\text{length of the tenth ring}}{\text{Total time}} = \boxed{\textbf{(C)}\ \frac{512}{1023}}$

9. Find the largest number in the choices that divides $11^{11} + 13^2 + 126$.

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

We can look at the last digit of the expression first. $11^n \equiv 1 \pmod{10}$ and $13^2 \equiv 9 \pmod{10}$. Therefore, the expression $11^{11} + 13^2 + 126 \equiv 1 + 9 + 6 \equiv 6 \pmod{10}$. We know that it is divisible by 2 at this point. Then, we can look at the last two digits. $11^{11} \equiv 11 \pmod{100}$ and $13^2 \equiv 69 \pmod{100}$. The expression $11^{11} + 13^2 + 126 \equiv 11 + 69 + 6 \equiv 86 \pmod{100} \equiv 2 \pmod{4}$. Therefore, our answer is $\boxed{\textbf{(B)}\ 2}$.

10. If $x + \frac{1}{x} = -2$ and $y = \frac{1}{x^2}$, find $\frac{1}{x^4} + \frac{1}{y^4}$.

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

The simplest way to solve this problem is to solve x and y, respectively. To solve x, we can multiply both sides of $x + \frac{1}{x} = -2$ by x, and this gives us the quadratic $x^2 + 2x + 1 = 0$. Solve the quadratic, $x = -1$. Substitute x into the second equation to evaluate y, $y = 1$. $\frac{1}{1} + \frac{1}{1} = \boxed{\textbf{(E)}\ 2}$

11. How many factors of 16! is a perfect cube or a perfect square?

(A) 158 **(B)** 164 **(C)** 180 **(D)** 1280 **(E)** 3000

We need to calculate the number of perfect squares and the number of perfect cubes and then subtract the number of 6th power according to the principle of inclusion and exclusion. First of all, we need to factor $16! = 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$. Since we can choose even amount of each factor, there are a total of $8 \cdot 4 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 128$ perfect squares. Using the same logic, any number that is a cube must have multiple of 3 factors for each factor. Therefore, there are $6 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 36$ cubes. In addition, there are $3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 6$ numbers with 6th power. In total, there are $128 + 36 - 6 = \boxed{\textbf{(A)}\ 158}$ perfect square or perfect cube factors of 16!.

12. Given that $x^2 - \frac{1}{x^2} = 2$, what is $x^{16} - \frac{1}{x^8} + x^8 - \frac{1}{x^{16}}$?

(A) 1120 **(B)** 1188 **(C)** 3780 **(D)** $840\sqrt{2}$ **(E)** $1260\sqrt{2}$

We know that $2 = x^2 - \frac{1}{x^2} = (x + \frac{1}{x})(x - \frac{1}{x})$. Squaring both sides, we can get that $(x + \frac{1}{x})^2 \cdot (x + \frac{1}{x})^2 = 4$. Therefore, $(x^2 + 2 + \frac{1}{x^2})(x^2 - 2 + \frac{1}{x^2}) = 4$. Setting $a = x^2 + \frac{1}{x^2}$, we can get that $(a+2)(a-2) = 4$, and therefore $a^2 - 4 = 4$. So $x^2 + \frac{1}{x^2} = \sqrt{8} = 2\sqrt{2}$. In addition, $x^4 - \frac{1}{x^4} = (x^2 + \frac{1}{x^2})(x^2 - \frac{1}{x^2}) = 2 \cdot 2\sqrt{2} = 4\sqrt{2}$. $\frac{2}{x^2} = 2\sqrt{2} - 2$, so $\frac{1}{x^2} = \sqrt{2} - 1$ and $\frac{1}{x^4} = 3 - 2\sqrt{2}$. Therefore, $x^4 + \frac{1}{x^4} = 4\sqrt{2} + 2(3 - 2\sqrt{2}) = 6$. Using a similar approach, we can calculate $x^8 - \frac{1}{x^8} = 4\sqrt{2} \cdot 6 = 24\sqrt{2}$ and $\frac{1}{x^8} = (3 - 2\sqrt{2})^2 = 17 - 12\sqrt{2}$. Therefore, $x^8 + \frac{1}{x^8} = 24\sqrt{2} + 2(17 - 12\sqrt{2}) = 34$. Also, $x^{16} - \frac{1}{x^{16}} = 24\sqrt{2} \cdot 34 = 816\sqrt{2}$. As a result, the answer of $x^{16} - \frac{1}{x^8} + x^8 - \frac{1}{x^{16}}$ would be $816\sqrt{2} + 24\sqrt{2} = \boxed{\textbf{(D)}\ 840\sqrt{2}}$.

13. Given that Giant want to put 12 green identical balls into 3 different boxes such that each box contains at least two balls, and that no box can contain 7 or more balls. Find the number of ways that Giant can accomplish this.

(A) 0 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 19

We can use stars and bars to calculate the total amount of ways without thinking about the condition "no box can contain 7 or more balls". Giving each box 1 ball in advance to meet "each box contains at least two balls" requirement since each box will get at least 1 extra, There are $\binom{8}{2}$ or 28 ways to accomplish this. We need to subtract the situations where a box contains 7 or more balls. There are two situations: $2, 2, 8$ in each box or $2, 3, 7$ in each box. In the first situation, since two boxes have the same amount, it will have $\frac{3!}{2} = 3$ ways of accomplishing it, and the second situation will have $3! = 6$ ways of accomplishing it. Therefore, the total amount of ways that meets the requirement is $28 - 3 - 6 = \boxed{\textbf{(E)}\ 19}$.

14. James randomly choose an ordered pair (x, y) which both x and y are elements in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, x and y are not necessarily distinct, and all of the equations:

$$x + y$$

$$x^2 + y^2$$

$$x^4 + y^4$$

are divisible by 5. Find the probability that James can do so.

(A) $\frac{1}{25}$ **(B)** $\frac{2}{45}$ **(C)** $\frac{11}{225}$ **(D)** $\frac{4}{75}$ **(E)** $\frac{13}{225}$

We can begin by converting all the elements in the set to Modular of 5. Then, we realize that all possible elements that can satisfy all the expressions to be divisible by 5 can only happen if x and y are both 0 (mod 5). Since x and y are not necessarily distinct, we have $3^2 = 9$ possible (x, y). There are total of $15 \cdot 15 = 225$ possible (x, y), therefore, the probability is $\frac{9}{25} = \boxed{\textbf{(A)}\ \frac{1}{25}}$.

15. Paul wrote all numbers that's less than 2021 and wrote their base 4 representation. He randomly choose a number out the list. Paul insist that he want to choose a number that had only 2 and 3 as its digits, otherwise he will be depressed and relinquishes to do homework. How many numbers can he choose so that he can finish his homework?

(A) 30 **(B)** 62 **(C)** 64 **(D)** 84 **(E)** 126

First, we can convert 2021 to base 4. $2021_10 = 133211_4$. Therefore, the total ways to obtain only 2 and 3 as its digits that are less than $2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 2^6 - 2 = \boxed{\textbf{(B)}\ 62}$.

16. Find the number of permutations of $AAABBC$ such that at exactly two As are adjacent, and the Bs are not adjacent.
(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

We can use casework counting to solve this problem.

The first case is $AA\_\_\_\_$. Since B cannot be adjacent, then there are three such cases. there are 2! for each of the case. However, A cannot be adjacent, therefore, there are 5 such arrangements.

The second case is $\_AA\_\_\_$. There are total of 4 possible cases for B to not be adjacent. There are $1+1+1+2=5$ total possible such arrangements. By symmetrical counting, the first case is the same as $\_\_\_\_AA$ and the second case is the same as $\_\_\_AA\_$.

The last we want to find is the number of arrangements of $\_\_AA\_\_$. For this case, there are total of 4 possible placement of two Bs to avoid adjacency. Each has 1 arrangement. Therefore, there are total of 4 such arrangements. $5+5+5+5+4=\boxed{\textbf{(D) } 24}$.

17. The following expression

$$\sum_{k=1}^{60}\binom{60}{k}+\sum_{k=1}^{59}\binom{59}{k}+\sum_{k=1}^{58}\binom{58}{k}+\sum_{k=1}^{57}\binom{57}{k}+\sum_{k=1}^{56}\binom{56}{k}+\sum_{k=1}^{55}\binom{55}{k}+\sum_{k=1}^{54}\binom{54}{k}+...+\sum_{k=1}^{3}\binom{3}{k}-2^{10}$$

can be expressed as $x^y - z$ which both x and y are relatively prime positive integers. Find $2^x(xy+2x+z)$.
(A) 4632 **(B)** 4844 **(C)** 4860 **(D)** 4864 **(E)** 8960

$$\sum_{k=0}^{60}\binom{60}{k}$$

can be expressed as 2^{60}, and $\binom{60}{0}$ is equal to 1. Therefore, we can simplify the original expression into $2^{60}-1+2^{59}-1+...+2^3-1-2^{10} = 2^{60}+2^{59}+...+2^3+2^3-58-1024 = 2^{61}-(8+58+1024) = 2^{61}-1090$. The expression that the answer wants would be $2^2\cdot(2\cdot 61+2\cdot 2+1090) = 4\cdot 1216 = \boxed{\textbf{(D) } 4864}$.

18. What can be a description of the set of solutions for this: $x^2+y^2 = |2x+|2y||$?
(A) Two overlapping circles with each area 2π.
(B) Four not overlapping circles with each area 4π.
(C) There are two overlapping circles on the right of the y-axis with each area 2π and the intersection area of two overlapping circles on the left of the y-axis with each area 2π.
(D) Four overlapping circles with each area 4π.
(E) There are two overlapping circles on the right of the y-axis with each area 4π and the intersection area of two overlapping circles on the left of the y-axis with each area 4π.

First, we want to graph this equation. use the technique of absolute value, there will be four cases of $x^2+y^2 = |2x+|2y||$. The four cases are all circles with radius of $\sqrt{2}$. However, we realize that $2x$ does not have an absolute value sign, so the left side is different from the right. Therefore, our answer would be $\boxed{\textbf{(C)}}$.

19. $(0.51515151...)_n$ can be expressed as $(\frac{6}{n})$ in base 10 which n is a positive integer. Find the sum of the digits of n^3.

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** Does Not Exist

We realize that when a decimal $0.abcdef$ is expressed in base n, the decimal would equal to the expression $\frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3} + \frac{d}{n^4} + \frac{e}{n^5} + \frac{f}{n^6} + ...$ Use this idea, $(0.515151515151.....)_n = \frac{5}{n} + \frac{1}{n^2} + \frac{5}{n^3} + \frac{1}{n^4} + \frac{5}{n^5} + \frac{1}{n^6} + ...$

This sum is basically the sum of two infinite geometric series. The first one has first term of $\frac{5}{n}$ and a common ratio of $\frac{1}{n^2}$. The second one has first term $\frac{1}{n^2}$ and a common ratio of $\frac{1}{n^2}$. The total sum is $\frac{5n+1}{n^2-1}$. This would result in $\frac{5n+1}{n^2-1} = \frac{6}{n}$. Turn this into a quadratic by cross-multiplication, we would get $n = 3$. HOWEVER, all numbers in base 3 can only have $0, 1, 2$ as its digits. Therefore, the answer will be $\boxed{\textbf{(E)}\text{ Does Not Exist}}$.

20. Given that $y = 24 \cdot 34 \cdot 67 \cdot 89$. Given that the product of the even divisors is a, and the product of the odd divisors is b. Find $a\colon b^4$.

(A) $512\colon 1$ **(B)** $1024\colon 1$ **(C)** $2^{64}\colon 1$ **(D)** $2^{80}\colon 1$ **(E)** $2^{160}\colon 1$

We can prime factorize the number first. $y = 24 \cdot 34 \cdot 67 \cdot 89 = 2^3 \cdot 3 \cdot 2 \cdot 17 \cdot 67 \cdot 89 = 2^4 \cdot 3 \cdot 17 \cdot 67 \cdot 89$. All of the odd factors of y would be factors of $3 \cdot 17 \cdot 67 \cdot 89$. Therefore, there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ odd factors of y. Let those factors form a set A, and all even factors would be $2A$ (all elements in A multiplied by 2), $4A$, $8A$, $16A$. Let the product of all odd factors in A be b, then the product of all even factors would be $a = 2^{16} \cdot b \cdot 4^{16} \cdot b \cdot 8^{16} \cdot b \cdot 16^{16} = 2^{16} \cdot 4^{16} \cdot 8^{16} \cdot 16^{16} \cdot b^4$. Therefore, the ratio of $a\colon b^4 = 2^{16} \cdot 4^{16} \cdot 8^{16} \cdot 16^{16}\colon 1 = \boxed{\textbf{(E)}\ 2^{160}\colon 1}$.

21. How many solutions are there for the equation $\lfloor x \rfloor^2 - \lceil x \rceil = 0$. (Recall that $\lfloor x \rfloor$ is the largest integer less than x, and $\lceil x \rceil$ is the smallest integer larger than x.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** ∞

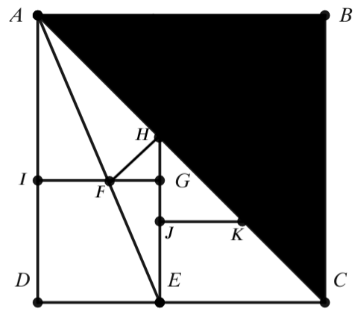
Let's split this question into different parts.

First case: $\lfloor x \rfloor = \lceil x \rceil$. In this case, both $\lfloor x \rfloor$ and $\lceil x \rceil$ has to be equal to x. Therefore, $\lfloor x \rfloor^2 - \lceil x \rceil = x^2 - x = 0$. Factoring this equation, we get that $x(x-1) = 0$ which provides two solutions, $x = 0$ and $x = 1$.

Second case: $\lfloor x \rfloor \neq \lceil x \rceil$. In this case, $\lfloor x \rfloor$ must be equal to $\lceil x \rceil - 1$. Setting $y = \lfloor x \rfloor$, we can get that $\lfloor x \rfloor^2 - \lceil x \rceil = y^2 - (y+1) = y^2 - y - 1 = 0$. Calculating the discriminant which is $1^2 - 4 \cdot 1 \cdot (-1) = 5$, we know that y is an irrational number. However, since y is a floor function of a number, it is always an integer. Therefore, there is no solution for this case.

In total, there are $\boxed{\textbf{(C)}\ 2}$ solutions for the equation.

22. In the diagram, $ABCD$ is a square with area $6+4\sqrt{2}$. AC is a diagonal of square $ABCD$. Square $IGED$ has area $11-6\sqrt{2}$. Given that point J bisects line segment HE, and AE is a line segment. Extend EG to meet diagonal AC and mark the intersection point H. In addition, K is drawn so that $JK//EC$. FH^2 can be represented as $\frac{a+b\sqrt{c}}{d}$ where a,b,c,d are not necessarily distinct integers. Given that $gcd(a,b,d)=1$, and c does not have a perfect square factor. Find $a+b+c+d$.



(A) 5 **(B)** 15 **(C)** 61 **(D)** 349 **(E)** 2009

To start this problem, we can first observe. Notice that FGH is a right triangle because angle FGH is supplementary to angle IGE which is a right angle. Therefore, we just have to solve for the length of side FG and HG.

Solve for FG: Triangles AIF and EGF are similar triangles, therefore, we can solve for length AI. $AI = AD - ID$. Use the technique of sum of squares and square root disintegration, $AD = 2+\sqrt{2}$. Using the same technique, $ID = 3-\sqrt{2}$. $AI = 2+\sqrt{2}-3+\sqrt{2} = 2\sqrt{2}-1$. Now, we can set up a ratio.

We can set $FG = x$, so $IF = 3-\sqrt{2}-x$. Using the similar triangle, $\frac{GE}{AI} = \frac{FG}{IF}$. Plugging the numbers into the ratio, we can get $\frac{3-\sqrt{2}}{2\sqrt{2}-1} = \frac{x}{3-\sqrt{2}-x}$.

$$\frac{3-\sqrt{2}}{2\sqrt{2}-1} = \frac{x}{3-\sqrt{2}-x}$$

$$(3-\sqrt{2})(3-\sqrt{2}-x) = x(2\sqrt{2}-1)$$

$$11-6\sqrt{2}-(3-\sqrt{2})x = x(2\sqrt{2}-1)$$

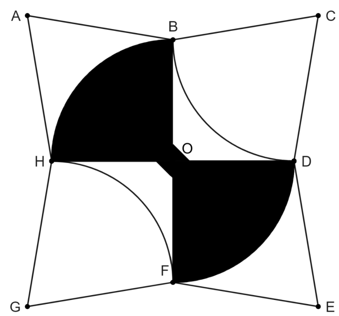
$$(2+\sqrt{2})x = 11-6\sqrt{2}$$

$$x = \frac{11-6\sqrt{2}}{2+\sqrt{2}}$$

$$x = \frac{34-23\sqrt{2}}{2}$$

Solve for HG: Since angle HEC is 90 and angle HCE is 45, $EC = HE = 2\sqrt{2}-1$. Since $GE = 3-\sqrt{2}$, $HG = 2\sqrt{2}-1-(3-\sqrt{2}) = 3\sqrt{2}-4$. Finally, we can solve for FH^2, that is, $FG^2+HG^2 = (\frac{34-23\sqrt{2}}{2})^2+(3\sqrt{2}-4)^2 = \frac{1175-830\sqrt{2}}{2}$. Therefore, our answer would be $1175-830+2+2 = \boxed{\textbf{(D) } 349}$.

23. On a coordinate plane, point O denotes the origin which is the center of the diamond shape in the middle of the figure. Point A has coordinate $(-12, 12)$, and point C, E, and G are formed through 90, 180, and 270 rotation about the origin O, respectively. Quarter circle BOH (formed by the arc BH and line segments BO and GH) has area 25π. Furthermore, another quarter circle DOF formed by arc DF and line segments OF, OD is formed through a reflection of sector BOH across the line $y = x$. The small diamond centered at O is a square, and the area of the little square is 2. Let x denote the area of the shaded region, and y denote the sum of the area of the regions ABH (formed by side AB, arc BH, and side HA), DFE (formed by side ED, arc DF, and side FE) and sectors FGH and BCD. Find $\frac{x}{y}$ in the simplest radical form.



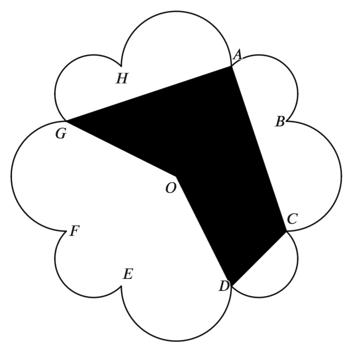
(A) $\frac{50\pi+1}{280}$ **(B)** $\frac{50\pi\sqrt{2}+\sqrt{2}}{560}$ **(C)** $\frac{50\pi+1}{140+100\pi}$ **(D)** $\frac{50\pi+1}{280+100\pi}$ **(E)** $\frac{50\pi^2+700\pi\sqrt{2}+3001\pi-70\sqrt{2}+60}{2\pi^2+240\pi+6920}$

First of all, we know that $BO = OH$. Since the area of the quarter circle is 25π, we can get that $BO = OH = \sqrt{\frac{25\pi \cdot 4}{\pi}} = 10$ Then, we can calculate the area of shaded region. It is made of two quarter circles and two right triangles. The total area would be $2 \cdot 25\pi + 2 \cdot \frac{2}{4} = 50\pi + 1$.

The sum of the area of the regions ABH (formed by side AB, arc BH, and side HA), DFE (formed by side ED, arc DF, and side FE) and sectors FGH and BCD can be calculated by turning $AHBO$ into a square, and subtract the extra areas. Since BO has length 10, we know that the height of the two right triangles are 2 and the based are 12. $144 - 24 = 120$. We want to also subtract the shaded quarter circle. The area is $120 - 25\pi$. The region enclosed by arc DF and length DE, FE is the reflection of the previous area. The area $HGF = 120 - 100 + 25\pi = 20 + 25\pi$. The region BCD is also the reflection. Therefore, the total area is $120 - 25\pi + 120 - 25\pi + 20 + 25\pi + 20 + 25\pi = 280$. As a result, the ratio is $\boxed{\textbf{(A)}\ \frac{50\pi+1}{280}}$.

24. One semicircle is constructed with diameter $AH = 4$ and let the midpoint of AH be M. Construct a point O on the side of segment AH (closer to segment AH than arc AH) such that the distance from A to O is $2\sqrt{5}$, and that OM is perpendicular to the diameter AH. Three more such congruent semicircles are formed through multiple 90 rotations around the point O. Name the 6 endpoints of the diameters B, C, D, E, F, G in a circular direction from A to H. Another four congruent semicircles are constructed with diameters AB, CD, EF, GH, and that the distance from the diameters to the point O are less than the distance from the arcs to the point O. Connect AC, CD, DO, OG, and GA. Find the ratio of the area of the pentagon $ACDOG$ to the total area of the shape formed by arcs AB, BC, CD, DE, EF, FG, GH, HA.

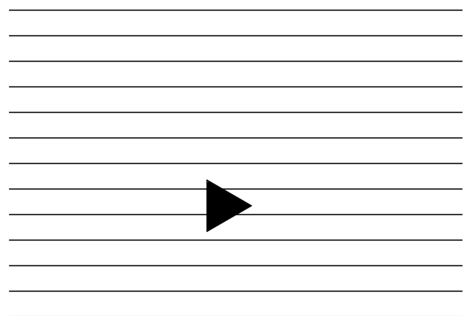
(A) $\frac{14+10\pi}{17}$ **(B)** $\frac{13+\sqrt{2}}{28}$ **(C)** $\frac{4+\sqrt{2}}{7+3\pi}$ **(D)** $\frac{13}{28+6\pi}$ **(E)** $\frac{13}{30\pi}$



We can first draw the diagram with the instructions. Since $AH = 4$ and $AO = 2\sqrt{5}$, we know that $OM = 4$. We know that $GO = OC = 2\sqrt{5}$, we can calculate the area of triangle AGC. $[AGC] = \frac{1}{2} \cdot 4\sqrt{5} \cdot 2\sqrt{5} = 20$. The final hard step for this problem is to solve for CD, or the diameter of the small circle. We can use coordinate geometry to finish this problem. Set E as the origin point and ED as the positive x direction. We then can get that $D = (4, 0)$ since $DE = AH = 4$. We can then solve for O, and it is easy as well because we can split it into two parts. $O = (2, 4)$. Since OC is perpendicular to OE, their slopes must be opposite reciprocal. Therefore, OC is 4 to the right and 2 down. Thus, the coordinate of C would be $(6, 2)$. We can then use the distance formula on CD. $CD = \sqrt{(2-0)^2 + (6-4)^2} = 2\sqrt{2}$. Then, we can draw a height from O to CD and call that N. $ON = \sqrt{(2\sqrt{5})^2 - (\sqrt{2})^2} = 3\sqrt{2}$. Thus, the area of the remaining shape of the shaded part would be $\frac{1}{2} \cdot 2\sqrt{2} \cdot 3\sqrt{2} = 6$. The whole shaded area would have a area of $20 + 6 = 26$. We can connect the diameters to get a octagon. It is formed by four big triangles like DOE and four small triangles like COD. Therefore, $A_{octagon} = 4 \cdot \frac{1}{2} \cdot 4 \cdot 4 + 4 \cdot 6 = 56$ The area of the outer semicircles would form two big circles and two small circles. $A_{semicircles} = 2 \cdot (\sqrt{2})^2 \cdot \pi + 2 \cdot (2)^2 \cdot \pi = 12\pi$. Thus, the ratio would be $\frac{26}{56+12\pi} = \boxed{\textbf{(E)}\ \frac{13}{28+6\pi}}$.

25. Suppose that a researcher hosts an experiment. He tosses an equilateral triangle with area $\sqrt{3}$ cm^2 onto a plane that has a strip every 1 cm horizontally. Find the expected number of intersections of the strips and the sides of the equilateral triangle.

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ \frac{12}{\pi} \qquad \textbf{(C)}\ \frac{2+3\sqrt{3}}{2} \qquad \textbf{(D)}\ \frac{4+\sqrt{3}}{2} \qquad \textbf{(E)}\ \frac{12+4\sqrt{2}-2\sqrt{3}}{4}$



Clearly, we can set up an equation about the side length of the triangle. $\frac{\sqrt{3}}{4}s^2 = \sqrt{3}$. By solving that, we can get that $s = 2$. Therefore, the perimeter of the triangle is 6.

This problem can be solved using calculus. Since calculus is not allowed in AMC 10, there is obviously another alternative.

Buffon's Needle Problem has the same concept as this problem. Although a rigorous proof does need calculus, but one can think of a circle needle with a 1 cm diameter. Thus, the perimeter of that circle would be π cm. A circle always intersect the plane at two points, so a triangle with perimeter 6 would have an expected intersection of $\frac{6}{\pi} \cdot 2 = \boxed{\textbf{(B)}\ \frac{12}{\pi}}$.