2021 March MIMC 10 Solutions

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Video solution of all problems!

Please ask any questions related to this mock contest in the discussion forum.

1. What is the value of $\frac{1}{2} + 2^{-8}$?

By computing, you would get $\frac{128}{256} + \frac{1}{256} = \frac{129}{256}$ which is answer choice $\boxed{\text{(C) } \frac{129}{256}}$.

2. How many integers x are there such that $-2\pi < x \leq 3\pi$?

The least integer greater than -2π is -6, and the largest integer less than 3π is 9. Any x value larger or equal to -6 and less or equal to 9 fits in the solution. There are 16 solutions to x, so our answer would be $\boxed{\text{(C) } 16}$

3. There exist an integer x such that the twice of the reciprocal of x is 8 times the square of the reciprocal of x. Find $\frac{2x}{3}$.

Set up equation, $\frac{2}{x} = \frac{8}{x^2}$. After cross multiplication, we get $2x^2 - 8x = 0$. Solve the quadratic, we get the solution $x = 0, 4$. However, 0 does not have a inverse, so it is eliminated. Therefore, our answer is $\boxed{\text{(B) } \frac{8}{3}}$.

4. Find the area between the circumcircle and incircle of an equilateral triangle with area of $4\sqrt{3}$.

An equilateral triangle with area $4\sqrt{3}$ would have side length 4 and height of $2\sqrt{3}$. The distance from the center of the equilateral triangle to one side is the radius of the incircle r, or $\frac{1}{3}$ of the height, $\frac{2\sqrt{3}}{3}$. Therefore, the area of the incircle would be $\frac{4\pi}{3}$. The distance from the center of the equilateral triangle to one vertex is the radius of the circumcircle, or $\frac{2}{3}$ of the height, $\frac{4\sqrt{3}}{3}$. Therefore, the area of the circumcircle would be $\frac{16\pi}{3}$. The difference is $\frac{12\pi}{3}$, or $\boxed{\text{(B) } 4\pi}$.

5. Mr. James has 8 books on his shelf, which includes 3 math books, 4 English books and a Spanish book. He wants to rearrange them so that the math books are all adjacent and English books are all adjacent as well. Find the number of ways he can do that.

Since Mr. James wants all of same types of the books to be adjacent, the total way to arrange them would be $3! \cdot 4! \cdot 3! = 864$ which is answer choice $\boxed{\text{(C) } 864}$. Notice that $6 \cdot 24 \cdot 6$ does not contain 5 in its factor, so all of answer choices end with 0 are eliminated.

[Scroll down for #6-25 solutions]

6. What is the probability that a subset with more than 1 element randomly chosen in the set $\{1, 2, 3, 4, 5, 6, 7\}$ have no two elements that has $gcd > 1$, and that subset does not contain 1? The answer can be expressed in the form of $\frac{x}{y}$. What is $2x + y$?

There are a total of $2^7 = 128$ subsets in the set $\{1, 2, 3, 4, 5, 6, 7\}$ because each element can be included or excluded. However, there are 8 subsets with 0 or 1 elements, and we need to subtract them from the total number of subsets. Since 1 cannot be there, the total amount of subsets will be $2^6 = 64$ subsets. There are 1 subset of 0 elements and 6 subsets of 1 element. There are 4 subsets of 2 that we need to subtract: $(2, 4)$, $(2, 6)$, $(3, 6)$, $(4, 6)$. There are 12 subsets of 3 elements that we need to subtract: $(2, 3, 4)$, $(2, 4, 5)$, $(2, 4, 6)$, $(2, 4, 7)$, $(2, 3, 6)$, $(2, 5, 6)$, $(2, 6, 7)$, $(3, 4, 6)$, $(3, 5, 6)$, $(3, 6, 7)$, $(4, 5, 6)$, $(4, 6, 7)$. There are $\binom{6}{4} - 2 = 13$ subsets of 4 elements that we need to subtract. Here are the 2 that meet the requirements: $(2, 3, 5, 7)$, $(3, 4, 5, 7)$. There are $\binom{6}{5} = 6$ subsets of 5 elements and 1 subset of 6 elements that we need to subtract. The total number of subsets that meet the requirements is $64 - 1 - 6 - 4 - 12 - 13 - 6 - 1 = 21$. The probability would be $\frac{21}{120} = \frac{7}{40}$. Our answer is $2 \cdot 7 + 40 = \boxed{\text{(D) } 54}$.

7. If $xy + 2x + 3y = 2$, What is the sum of all possible values of x if they are both integers?

Using Simon's Favorite Factoring Trick, we know that $xy + 2x + 3y + 6 = (x + 3)(y + 2)$, therefore, we add 6 to 2, and we turn this to the equation $(x + 3)(y + 2) = 8$. By solving the equation, there are total of 8 possible cases, which 4 of them are $positive \cdot positive$, and 4 of them are $negative \cdot negative$. All cases listed are $(1, 0)$, $(5, -1)$, $(-1, 2)$, $(-2, 6)$, $(-5, -6)$, $(-7, -4)$, $(-4, -10)$, $(-11, -3)$. The total of all possible values of x is therefore $\boxed{\text{(B) } -24}$.

8. Let x be the sum of the last four digits and y be the sum of all digits of the expression $99 + 9999 + 999999 + 99999999 + 9999999999 + ... + 999...99$(two hundred 9$s$). Find $x + y$.

Use the fact that $99 = 100-1, 9999 = 10000-1$, the sum of the total value would be 100+10000+1000000+100000000+10000000000+.......+100...00(two hundred 0s) $-$ 100, which is 10101010101010.........10100, (one hundred 1s and one hundred one 0s). Then, subtract 100 would result in 101010101010101010.......10000 with 99 1s, which the sum of all the digit is basically 99. Also, the last four digits are 0000 which sum in 0, so the answer for this problem is $\boxed{\text{(C) 99}}$.

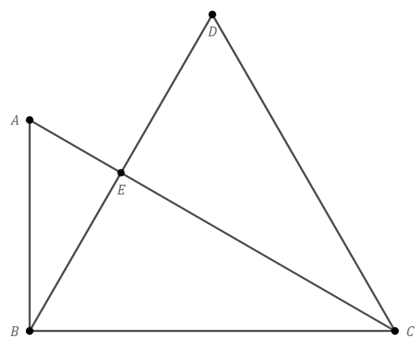
9. Define a function $f(x)$ such that $f(x) = \lfloor\sqrt{x}\rfloor$. What is $f(201600)$?

Using the answer choices, the answer would be around 450. Calculating $449^2 = 201601$. Therefore, $f(201600) = \lfloor\sqrt{201600}\rfloor = \boxed{\text{(D) 448}}$. Just in case, we can calculate 448^2 which results in 200704.

10. $(100!)^9 + 506$ is a multiple of what number?

Using Wilson's theorem, $100! \equiv -1(\text{mod } 101)$, and $-1^9 = -1$, and $506 \equiv 1(\text{mod } 101)$. Therefore, use the formula of addition in modular arithmetic, $(100!)^9 + 506 \equiv 0(\text{mod } 101)$, which is divisible by $\boxed{\text{(B) 101}}$.

11. A $30-60-90$ ABC triangle with the longer leg BC length 8 is overlapping with an equilateral triangle DBC with area $16\sqrt{3}$, and they shared a common base. The hypotenuse, AC, intersects the side DB, at E. The length of CE can be expressed as $x\sqrt{y}$. Find $xy+x+y$.



Rotate the equilateral triangle 60 degrees leftward to form a parallelogram, then the extended AC become the diagonal of the parallelogram. The height of the equilateral triangle is $4\sqrt{3}$ and the longer leg is therefore $8+4=12$. Use the pythagorean theorem, the length of the diagonal is $8\sqrt{3}$, which the length of CE is therefore $\frac{8\sqrt{3}}{2}$ which is $4\sqrt{3}$. Thus, by computing the answer is therefore $\boxed{\text{(C) } 19}$.

12. If there are x ordered six-ples (a,b,c,d,e,f) of positive integers such that all of them are multiples of 4 and $a+b+c+d+e+f=1956$. Find the remainder when x is divided by 11.

Since all of the integers are multiples of 4, we can divide both sides of the equation by 4, resulting $\frac{a+b+c+d+e+f}{4}=489$. Since all variables have to be positive integers, none of them can be 0. Using the principle of stars and bars, we can determine the total number of cases to be $\binom{488}{5}$ which is a multiple of 484. Since 484 is a multiple of 11, our final answer would be $\boxed{\text{(A) } 0}$.

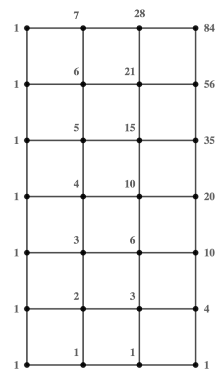
13. There is an equilateral triangle with area $16\sqrt{3}$, Let point X, Y, Z be on side AB, BC, AC, respectively, such that $AX : XB = 3 : 1$, $BY : YC = 3 : 1$, $AZ : ZC = 1 : 3$. The perimeter of triangle XYZ can be expressed as $m\sqrt{n}$ such that n does not have perfect square factors. Find mn.
By using the formula $A = \frac{a^2\sqrt{3}}{4}$, we can solve that the side length of equilateral triangle ABC is 8. Using coordinate geometry, we just have to find the length of XY. Set B as $(0, 0)$, then A would be $(4, 4\sqrt{3})$ and C would be $(8, 0)$. The coordinate of x is basically $(3, 3\sqrt{3})$ and y is (2,0). Use the distance formula, the length of XY is $2\sqrt{7}$. Therefore, the perimeter of equilateral triangle is $6\sqrt{7}$ and mn is $\boxed{\text{(B) } 42}$.

14. When 2579 is expressed in base b, the resulting number is $5n5$ which both b and n are integers. Find $b + n$.

$(5n5)_b = 5b^2 + nb + 5 = 2579$. We can get that b is a divisor of 2574 since factoring the equation above gives us $b(5b + n) = 2574$. Factoring 2574, we get that $2574 = 2 \cdot 3^2 \cdot 11 \cdot 13$. Some reasonable possible values for b are $11, 13, 18, 22, 26, 33, 39$ because it cannot be less than or equal to 10, and it is unlikely for it to exceed 40. We can start testing from the middle to eliminate incorrect values. After eliminations, $b = 22$ is the only case that makes $n = 7$ a single-digit integer. Therefore, $b + n = \boxed{\text{(C) } 29}$

15. Castoy, the frog, is at $(-3, 4)$. He want to reach $(0, 10)$ as his final destination. Given that he can only move 1 unit in the positive x or y direction. Find the number of ways he can accomplish this.

It takes Castoy 9 steps to reach $(0, 10)$. Of which 3 goes to the $+x$ direction, and 6 goes to the $+y$ direction. Therefore, the total number of ways that Castoy can accomplish this is $\binom{9}{3} = \boxed{\text{(E) } 84}$. Here is a visual representation of it, the number of ways each point can be accomplished is the sum of the bottom point and left point:



16. What is the area inside of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ but outside of $x^2 + y^2 = 4$.

The first equation is an ellipse with longer radius 3 and shorter radius 2. The second one is a circle with radius 2. The ellipse fully contains the circle. Set the longer radius of the ellipse x, and the shorter radius y, then the formula to calculate the area of ellipse is $x \cdot y \cdot \pi$ which is 6π, and the circle's area is $2^2\pi = 4\pi$. $6\pi - 4\pi = 2\pi$ which is $\boxed{\text{(B) } 2\pi}$.

17. The numerical representation of 21! in base 10 has over 60000 positive integer divisors. Find the probability that a random chosen positive integer divisor is a power of 6. The probability can be written as $\frac{x}{y}$, find $x + y$.

Prime factorize 21! which we can get $2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1$. Multiply out, we get the total number of positive integer divisors of 21! is $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60800$. All the factors that are powers of 6 is basically $2^9 \cdot 3^9 = 6^9$, and the total number of factors that are powers of 6 would be $9 + 1 = 10$. The probability is $\frac{10}{60800} = \frac{1}{6080}$, and $x + y = \boxed{\text{(C) } 6081}$.

18. Find the largest real value of x that is a root of the polynomial $f(x) = x^4 - 14x^3 + 47x^2 + 14x - 15$. The largest real root can be expressed as $\frac{a+\sqrt{b}}{c}$. Find $2a + b + c$.

Using the idea of symmetrical polynomial, we can add 16 to both sides of the equation, which we get $x^4 - 14x^3 + 47x^2 + 14x + 1 = 16$. Divide x^2 by both sides, we get $x^2 - 14x + 47 + \frac{14}{x} - \frac{1}{x^2} = \frac{16}{x^2}$. Set $y = x - \frac{1}{x}$, then we can simplify the equation into $y^2 + 2 - 14y + 47 = (\frac{4}{x})^2$, and simplify further, the equation becomes $(y - 7)^2 = (\frac{4}{x})^2$, and $y - 7 = \frac{4}{x}$. Put $x - \frac{1}{x}$ back into the new equation, we get $x - \frac{5}{x} - 7 = 0$. Multiplying both side by x, we turned this into a quadratic $x^2 - 7x - 5 = 0$. The largest solution is therefore $\frac{7+\sqrt{69}}{2}$, and our answer is therefore $\boxed{\text{(E) } 85}$. Even though $y - 7 = -\frac{4}{x}$ can be another solution, but calculating the discriminant, $b^2 - 4ac$, shows that the solutions would be smaller.

19. Given $x + y = 2$ and $xy = -1$, what is the value of $\frac{(x^2+y^2)(x^4+y^4)(x^6+y^6)}{x^3y^3-3xy}$

First, we want to find $x^2 + y^2$ and we can evaluate by computing $(x + y)^2 - 2xy$ and this would result in 6. Then, we can find $x^4 + y^4 = (x^2 + y^2)^2 - 2(xy)^2 = 34$. Then, we want to find $x^3 + y^3 = (x + y)(x^2 + y^2 - xy) = 14$. $x^6 + y^6 = (x^3 + y^3)^2 - 2x^3y^3 = 198$. Next, we want to find $x^3y^3 - 3xy = -1 + 3 = 2$. Ultimately, computing the original equation would result in $\frac{(x^2+y^2)(x^4+y^4)(x^6+y^6)}{x^3y^3-3xy} = \frac{6\cdot34\cdot198}{2} = \boxed{\text{(B) } 20196}$.

[Scroll down for #20-25 solution]

20. Find the remainder when $1 + 11 + 11^2 + 11^3 + 11^4 + ... + 11^{1009}$ is divided by 1000.

Since it's asking for the remainder when this long sum is divided by 1000, we only have to find the last three digits. We can turn 11 to $10+1$ and use the expansion of binomial theorem. That is, $1 = (10+1)^0, 11 = (10+1)^1, 11^2 = (10+1)^2$, etc. According to binomial theorem,

$1 + 11 + 11^2 + 11^3 + 11^4 + ... + 11^{1009}$
$= 1 + (10 + 1) + [100+\binom{2}{1}10 + 1] + [1000+\binom{3}{1}100+\binom{3}{2}10 + 1] + ...$
$\equiv 1 \cdot 1010 + 10 \cdot [\binom{1}{0}+\binom{2}{1}+...+\binom{1009}{1008}] + 100 \cdot [\binom{2}{0}+\binom{3}{1}+...+\binom{1009}{1007}] \pmod{1000}$
$\equiv 1 \cdot 1010 + 10 \cdot (1 + 2 + ... + 1009) + 100 \cdot \frac{1\cdot2+2\cdot3+...+1008\cdot1009}{2} \pmod{1000}$
$\equiv 10 + 450 + 100 \cdot \frac{1\cdot2+2\cdot3+...+1008\cdot1009}{2} \pmod{1000}$.

Afterward, we want to find how many 100s are in the binomial expansion, and this is the sum of triangular numbers $1+3+6+10+15+21+28+...$ To compute this sum, we can use the formula $\frac{x(x+1)(x+2)}{6}$ which x denotes the number of triangular numbers in the sum. There are $1009-2+1 = 1008$ triangular numbers. After computing the formula, we can prove that the sum of 1008 triangular numbers ends in 0. When something ends with 0 and multiply by 100, the product would always end with 4 zeros. Since we only need to worry about the last three digits to find the remainder when $1+11+11^2+11^3+11^4+...+11^{1009}$ is divided by 1000. Thus, the answer is $\boxed{\text{(C) } 460}$.

21. Given that r, s, t are the three roots of polynomial $x^3 - 2x^2 + 4x - 8$, find $|sr^2 + rt^2 + ts^2 + st^2 + rs^2 + tr^2|$.

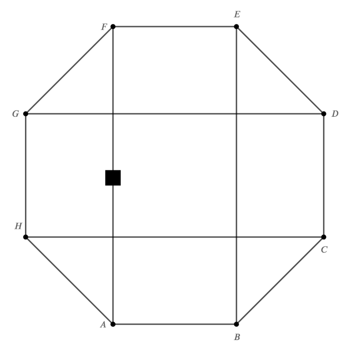
By Vieta's formula, $r + s + t = 2$, $rs + st + rt = 4$, $rst = 8$. We can simplify $sr^2 + rt^2 + ts^2 + st^2 + rs^2 + tr^2$ into $rst(\frac{r}{s} + \frac{r}{t} + \frac{s}{r} + \frac{s}{t} + \frac{t}{s} + \frac{t}{r})$. The value of $\frac{r}{s} + \frac{r}{t} + \frac{s}{t} + \frac{s}{r} + \frac{t}{r} + \frac{t}{s}$ can be simplified into $(r + s + t)(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}) - 3 = -2$. Since $(8)(-2) = -16$, our answer would be $|-16| = \boxed{\text{(D) } 16}$.

22. What is the remainder when $10001^{111} + 10001^{222} + 10001^{333} + 10001^{444} + 10001^{555} + 10001^{666} + 10001^{777} + 10001^{888} + 10001^{999} + 10001^{1100} + 9990$ is divided by 16?

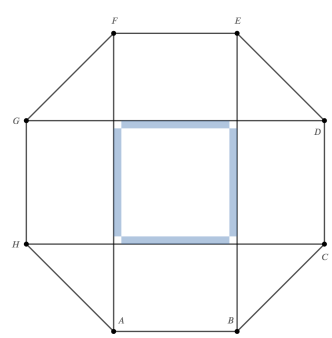
Using the binomial theorem, we can simplify 10001^{111} as $(10000+1)^{111}$. Since we want the remainder when divided by 16, we just need to know the last 4 digits. $(10000+1)^{111} \equiv 1(\text{mod } 1000)$. Using a similar logic, we can get that all terms except 9990 (mod 1000) would be 1. There are 10 of these similar terms, therefore, $10001^{111} + 10001^{222} + 10001^{333} + 10001^{444} + 10001^{555} + 10001^{666} + 10001^{777} + 10001^{888} + 10001^{999} + 10001^{1100} \equiv 10(\text{mod } 10000)$. Therefore, our answer would be $10 + 9990 \equiv 0(\text{mod } 10000)$. In conclusion, this long expansion is divisible by 16 which is the answer choice $\boxed{\text{(A) } 0}$.

[Scroll down for #23-25 solution]

23. In the regular octagon $ABCDEFGH$ with side length 8, diagonals BG, CF, AD, EH are drew to form 4 isoceles right triangle at the four corners, 4 rectangles on sides, and a giant square in center. A piece of unit square landed in the region of octagon. The sides of the unit square must be parallel to the diagonals. Find the area of region that the center of the unit square can locate in the octagon given that the landed unit square must have at least half of the unit square in the middle giant square and the unit square must intersect or touch the side of the giant square.



The center of the shaded unit square can land on the bigger square (made up by the diagonals of the octagon) or 0.5 units away from the inner region of the bigger square. However, the unit square cannot land on the corners of the square because more than half of the square will land outside of the expected region. The possible area is shaded in blue:



There are four rectangles with width 0.5 and length 7. The total area is $0.5 \cdot 7 \cdot 4 = \boxed{\text{(B) } 14}$.

24. Given that $(x-2\sqrt{2})^2+(y+2\sqrt{2})^2=8$ and the two axises x and y on the standard Cartesian plane intersects the equation at A and B, respectively. Also, a line with slope -1 both intersects the equation at A and intersects y axis at C, and another line with slope -1 both intersects the equation at B and intersect the x axis at point D. Also given that $(-2\sqrt{2}, 2\sqrt{2})$ is point E, the enclosed area of the curve AB and line segments BD, DE, EC, and AC can be expressed as $x \cdot (y - z\pi)$, where x, y, z are distinct integers such that $gcd(y, z) = 1$. Find $2x + y - z$.

We know that $(x-2\sqrt{2})^2+(y+2\sqrt{2})^2 = 8$ is a circle equation with radius $2\sqrt{2}$ and center lies in the fourth quadrant $(2\sqrt{2}, -2\sqrt{2})$. Also, set $(0, 0)$ as F, we can draw it out and figure out that triangles CFA and DFB are both isoceles right triangles with side length $2\sqrt{2}$, and $ECFD$ is a square with side length $2\sqrt{2}$. Also, let G denote the center of the circle enclosed by the equation $(x-2\sqrt{2})^2+(y+2\sqrt{2})^2 = 8$, the area enclosed by line FA, FB and the curve AB is basically square $FAGB$ subtract the quarter circle ABG. Square $FAGB$ has side length $2\sqrt{2}$ which is area 8, and the area of the quarter circle is 2π. Therefore, the total area is $8+4+4+8-2\pi = 2(4+2+2+4-\pi) = 2(12-1\pi)$. Thus, $2a + b - c = \boxed{\text{(E) 15}}$.

[Scroll down for #25 solution]

25. Note that the all possible values for the first 3-digits for the 6-digit number for $abcd, bcde, edcf$ to be a geometric sequence are: 942, 931, 842, 521, and 421. **This means that there are** 5000 **or less ways which eliminates choice (E).**

From here, let's try to figure out a possible set of the next digits. *abcd* cannot be from 9311 to 9319. If it starts with 931 then it starts with 9310. However, 9310 is not divisible by 3, or the common ratio, so starting with 931 is impossible.

abcd cannot be from 9320 to 9329 because the next digit won't meet the requirements.

abcd cannot be from 8420 and 8422 to 8429. If it starts with 842, then it starts with 8421. However, 8421 is not divisible by 2, or the common ratio, so starting with 842 is impossible.

abcd cannot be from 5210 to 5219 because the next digit won't meet the requirements.

At this point, Since 4000 **options were eliminated, this means that there are** 1000 **or less options which makes** B, C **and** D **all impossible.**

Just in case, let's look at 421.

abcd cannot be from 4211 to 4219. If it starts with 421 so it must start with 4210. We can continue with 42105. However now we are stuck because 2105 is not divisible by 2, or the common ratio. Therefore, this option impossible.

All possibilities have been eliminated. Thus, our answer is $\boxed{\text{(A) } 0}$.

Solution for #25 is created by jupiter314.