Mock American Mathematics Competitions (AMC 10) - Solutions

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1. **Answer: (D)** The six for \$5 is cheaper per bar, so she can buy six 6-packs and four additional bars, or $6 \times 6 + 4 = \boxed{40}$ candy bars.

2. **Answer: (B)** They save the cost of one fare, minus \$5.00 for the additional passenger. The cost of the 6.4-mile fare is \$3 for the first mile, plus the cost of 54 additional tenths of a mile, or $3 + 54 \times 0.2 = 13.80$. They pay 5 additional dollars, so their savings is $13.80 - 5 = \boxed{\$8.80}$.

3. **Answer: (D)** There are 16 side lengths on the perimeter, so the side length of one square is $\frac{48}{16} = 3$. The area of one square is $3^2 = 9$ square units, and the area of the figure is $9 \times 7 = \boxed{63}$ square units.

4. **Answer: (A)** Either Alice sits immediately to the left or to the right of Beth. In either case, the locations of the remaining four are uniquely determined:

	F				E	
E		D		C		F
C		B		B		D
	A				A	

However the problem states that Alice and David do not sit next to each other, ruling out the 2nd (right) seating. Then they are seated (in clockwise order) Alice, Charlie, Ellie, Frank, David, Beth, and $\boxed{\text{Alice}}$ sits across from Frank.

5. **Answer: (D)** Note that $3^{3^3} = 3^{27}$. The expression equals $\frac{3^{27}}{3^3} = \boxed{3^{24}}$.

6. **Answer: (C)** Let r denote the radius of the semicircle. Then we have
$$\frac{\pi r^2}{2} = \pi r + 2r$$
Dividing by r and multiplying by 2 gives $\pi r = 2\pi + 4$. Dividing by π, we obtain $r = \boxed{2 + \frac{4}{\pi}}$.

7. **Answer: (D)** Let p, q, r be the primes. We have $pqr = n^2 - 1 = (n-1)(n+1)$ for some integer n. If $n-1$ and $n+1$ are even, then 4 divides p, q, r, a contradiction since at most one of the primes is 2. Then $n-1$ and $n+1$ are odd, so one of them is prime and the other is a product of two different odd primes. The only possibility is $n - 1 = 13$ and $n + 1 = 15$ giving 3, 5, 13 as the primes, and their sum is $3 + 5 + 13 = \boxed{21}$.

8. **Answer: (C)** There are 36 possible outcomes. Out of these, there are seven for which the result is not composite ((1,2),(2,1),(1,3),(3,1),(1,5),(5,1) and (1,1)). The probability is $\boxed{\frac{29}{36}}$.

9. **Answer: (D)** Noting that $1 + 2 + 3 + \ldots + 63 = 2016 < 2018$, and $1 + 2 + 3 + \ldots + 64 > 2018$, the first 2016 characters end with a string of 63 C's, and the 2018th character is $\boxed{\text{D}}$.

10. **Answer: (E)** Note that the integer and the sum of the digits of the integer leave the same remainder upon division by 9. That is, $n \equiv s(n) \pmod 9$, and since $835 \equiv 7 \pmod 9$, we have $n + d(n) \equiv 2n \equiv 7 \pmod 9$, or $n \equiv 8 \pmod 9$. Hence the remainder is $\boxed{8}$.

    It turns out the only such value of n is $n = 818$.

11. **Answer: (C)** Since $\frac{m}{n}$ is necessarily a positive integer, we have $m = kn$ for some positive integer k. Then $kn+n+k = 19 \iff (k+1)(n+1) = 20$. The solutions are $(k + 1, n + 1) = (2, 10), (4, 5), (5, 4), (10, 2)$. Note that $(k + 1, n + 1) = (20, 1)$ is not valid as this gives $n = 0$, and $(1, 20)$ is not valid as this gives $k = 0 \Rightarrow m = 0$. The other pairs each

give 1 solution (m, n), which are $(9, 9), (12, 4), (12, 3), (9, 1)$. Altogether there are $\boxed{4}$ solutions.

12. **Answer: (A)** We express the lines using point-slope form:

$$y - 5 = m(x - 7)$$
$$y - 5 = -\frac{1}{m}(x - 7)$$

We have $m - \frac{1}{m} = 10$. The y-intercepts of the lines are $-7m + 5$ and $\frac{7}{m} + 5$ respectively. Their sum is $-7m + \frac{7}{m} + 10 = -7(10) + 10 = \boxed{-60}$.

13. **Answer: (E)** Since $365 \equiv 1 \pmod 7$, we can simply find the day 100 days after Monday. However there are an additional 24 leap days (not 25), so January 1, 2118 will be on the same day as 124 days from Monday, which is $\boxed{\text{Saturday}}$.

14. **Answer: (A)** Extend $\overline{BC}$ past C to point M such that $\angle CMD = 90°$. Then CMD is a 30-60-90 triangle with area $\frac{1}{2} \times 6 \times 6\sqrt{3} = 18\sqrt{3}$, and $ABMD$ is a trapezoid with area $\frac{1}{2} \times (3 + 6\sqrt{3}) \times 10 = 15 + 30\sqrt{3}$.

    We have $[ABCD] = [ABMD] - [CMD] = 15 + 30\sqrt{3} - 18\sqrt{3} = \boxed{15 + 12\sqrt{3}}$.

15. **Answer: (E)** Without loss of generality let the side length of the larger square be 1. Then the area of each rectangle and the smaller square is $1/5$, so the side length of the smaller square is $\frac{\sqrt{5}}{5}$. Let the shorter side length of each rectangle be x. We have $2x + \frac{\sqrt{5}}{5} = 1 \Rightarrow x = \frac{5-\sqrt{5}}{10}$. The longer side is $x + \frac{\sqrt{5}}{5} = \frac{5+\sqrt{5}}{10}$.

    The ratio of longer side to shorter side is $\frac{5+\sqrt{5}}{5-\sqrt{5}}$, which simplifies to $\boxed{\dfrac{3+\sqrt{5}}{2}}$.

16. **Answer: (B)** Similar to the area formula $A = rs$ for triangles, we have $V = \frac{1}{3}rS$ where S denotes the sum of the areas of the faces of the pyramid.

The volume of the pyramid is $\frac{1}{3} \times 100 \times 12 = 400$. Each triangular face has base 10 and height 13 (by Pythagorean thm.), so each triangular face has area 65. Then $S = 100 + 4 \times 65 = 360$, so $400 = \frac{1}{3} \times r \times 360 \Rightarrow r = \boxed{\dfrac{10}{3}}$.

Alternate solution: Consider a cross-section triangle of the pyramid containing the sphere's diameter. The cross section is a 10-13-13 isosceles triangle, and the radius of the sphere is equal to the inradius of the triangle, which is $\frac{10}{3}$.

17. **Answer: (A)** Rewrite as $(3^x)^2 = 9 \times 3^x - 1 \iff (3^x)^2 - 9 \times 3^x + 1 = 0$, which is quadratic in 3^x.

    The solutions are $3^x = \frac{9 \pm \sqrt{9^2-4}}{2}$, which are both positive (we do not care what the solutions are, just that they are positive). This gives two real solutions for x (say x_1 and x_2).

    To find their sum, we note that $3^{x_1} 3^{x_2} = 1$ by Vieta's formulas, or $3^{x_1+x_2} = 1 \Rightarrow x_1 + x_2 = \boxed{0}$.

18. **Answer: (B)** There are 24 ways to choose the first endpoint of such a space diagonal. For each vertex, there are 10 ways to choose the second endpoint (any vertex except the first vertex, or any vertex on either octagon containing the first vertex). Divide by 2 for overcounting each diagonal, giving $\frac{24 \times 10}{2} = \boxed{120}$ space diagonals.

19. **Answer: (E)** Note that $\lfloor \frac{1}{2} + x \rfloor$ simply rounds x to the nearest integer (rounding $\frac{1}{2}$ to 1). The following lists $\sqrt{x}$, rounded to the nearest integer:

$$\begin{aligned} \sqrt{1}, \sqrt{2} &\to 1 \quad \text{(2 values)} \\ \sqrt{3}, \ldots, \sqrt{6} &\to 2 \quad \text{(4 values)} \\ \sqrt{7}, \ldots, \sqrt{12} &\to 3 \quad \text{(6 values)} \\ &\vdots \\ \sqrt{1893}, \ldots, \sqrt{1980} &\to 44 \quad \text{(88 values)} \\ \sqrt{1981}, \ldots, \sqrt{2018} &\to 45 \quad \text{(38 values)} \end{aligned}$$

Hence the sum is $1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 + \ldots + 44 \cdot 88 + 45 \times 38 = 2(1^2 + 2^2 + \ldots + 44^2) + 45 \times 38 = \frac{44 \times 45 \times 89}{3} + 45 \times 38 = \boxed{60450}$.

Note: To make the computation slightly easier, rewrite as $44 \times 15 \times 89 + 45 \times 38$, then factor out 10 to get $10(66 \times 89 + 9 \times 19) = 10(6045) = 60450$.

20. **Answer: (B)** Extend AM past M and DC past C to meet at a point Q. By ASA congruence, we have $\triangle ABM \cong \triangle QCM$, so $CQ = 10$ and $QM = AM$.

    Similarly by ASA, we have $\triangle ABP \sim QDP$, and since $QD = 15$, the two triangles are in the ratio 3:2. It follows from these two facts that $AP = 4k$, $PM = k$, and $MQ = 5k$ for some k, and that $[BMP] = \frac{1}{5}[ABM] = \frac{1}{10}[ABC]$.

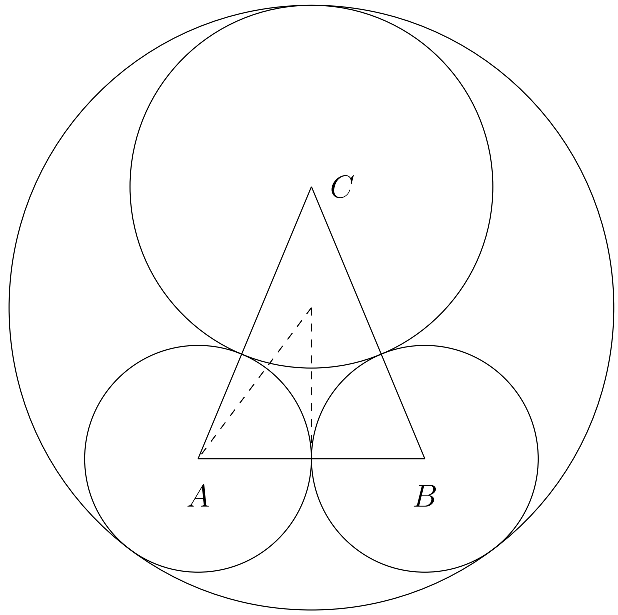
    If we extend AD and BC past D and C respectively to meet at a point R, we see that $\triangle ABR$ is isosceles with base 16 and equal sides of length 10. Since C is the midpoint of side BR, it follows that $\triangle ABC$ is a 6-8-10 triangle with area 24. Then $[BMP] = \frac{1}{10} \times 24 = \boxed{\frac{12}{5}}$.

21. **Answer: (B)** Note that 0 is not a root of $P(x)$. We have

$$\begin{aligned} Q(3) &= \prod_{i=1}^{3} \left(3 - (r_i + \frac{1}{r_i} + 1)\right) \\ &= \prod_{i=1}^{3} \frac{-r_i^2 + 2r_i - 1}{r_i} \\ &= \frac{\prod_{i=1}^{3} -(1-r_i)^2}{\prod_{i=1}^{3} r_i} \end{aligned}$$

    Note that $P(1) = 2(1-r_1)(1-r_2)(1-r_3)$, so $\frac{P(1)}{2} = \prod_{i=1}^{3}(1-r_i)$. Hence the numerator of the above expression is equal to $(-1)^3(P(1)/2)^2 = -(\frac{5}{2})^2 = -\frac{25}{4}$. The denominator is equal to $\frac{3}{2}$ by Vieta's. Then $Q(3) = \frac{-25/4}{3/2} = \boxed{-\frac{25}{6}}$.

22. **Answer: (C)** We have the following:



Let O denote the center of Ω and M denote the midpoint of AB, so that $AM = 5$. Because $\triangle ABC$ is isosceles, we have that AMC is a 5-12-13 triangle, so $MC = 12$.

Let the radius of Ω be r. We have that O, A, and the tangency point with Ω_A and Ω are concurrent, so $OA = r - 5$. We have $OC = r - 8$ and $MC = 12 - (r - 8) = 20 - r$. By Pythagorean theorem on triangle AMO:

$$5^2 + (20 - r)^2 = (r - 5)^2$$

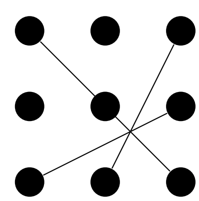
Expanding gives $425 - 40r + r^2 = r^2 - 10r + 25 \Rightarrow 400 = 30r \Rightarrow r = \boxed{\dfrac{40}{3}}$.

23. **Answer: (B)** There are 20 lines that pass through at least two of the points (3 vertical, 3 horizontal, 6 with slope ± 1, 4 with slope ± 2, 4 with slope $\pm \frac{1}{2}$), so there are $\binom{20}{2} = 190$ ways to choose two different lines, leading to potentially 190 intersection points. However this overcounts (i) pairs of parallel lines and (ii) 3 or more pairs of lines that intersect at the same point.

(i) There are 16 pairs of parallel lines, so subtract 16 for overcounting.

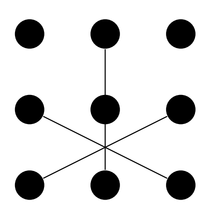
(ii) Note that the four corner points have 5 lines passing through; the four side points have 6 lines passing through, and the center point has 4 lines passing through it. Then there are $\binom{5}{2} = 10$ pairs of lines that intersect at a corner point, $\binom{6}{2} = 15$ pairs of lines that intersect at a side point, and $\binom{4}{2} = 6$ pairs of lines that intersect at the center. Hence we must subtract $4\times(\binom{5}{2}-1)+4\times(\binom{6}{2}-1)\times(\binom{4}{2}-1) = 36+56+5 = 97$ due to overcounting.

Additionally, we overcount four intersection points as follows:



We subtract $4 \times (\binom{3}{2} - 1) = 8$ for overcounting.

The other four intersection points that are overcounted are as follows:



Similarly we subtract 8 for overcounting.

The number of pairs of lines is $190 - 16 - 97 - 8 - 8 = \boxed{61}$.

24. **Answer: (D)** Let W and L denote the events "some team wins all 5 matches" and "some team loses all 5 matches" respectively. The complementary event is that some team wins all 5 or loses all 5, so the desired probability equals $1 - P(W \cup L)$.

We have $P(W) = 6 \times \frac{1}{32} = \frac{3}{16}$. This is because at most one team can win all 5 matches (i.e. the events "team i wins all 5" and "team j wins all 5" are disjoint for $i \neq j$), and the probability that any one team wins all five matches is $(\frac{1}{2})^5 = \frac{1}{32}$. Similarly $P(L) = \frac{3}{16}$.

We need to compute the intersection, which is $P(W \cap L)$. For any $1 \le i < j \le 6$, the probability that team i wins all 5 and team j loses all 5 is $(\frac{1}{2})^5(\frac{1}{2})^4 = \frac{1}{512}$ (team i wins all 5, and team j loses the remaining 4 matches). Then $P(W \cap L) = 6 \times 5 \times \frac{1}{512} = \frac{15}{256}$.

By inclusion-exclusion, we have

$$P(W \cup L) = \frac{3}{16} + \frac{3}{16} - \frac{15}{256} = \frac{81}{256}$$

The desired probability is

$$1 - P(W \cup L) = 1 - \frac{81}{256} = \boxed{\frac{175}{256}}.$$

25. **Answer: (A)** N is interesting iff there is a sequence of length between 2 and 10 (inclusive), starting with 1, such that each successive term is obtained by doubling the previous term or adding 1, and whose last term is N. The largest interesting number is $2^9 = 512$. This suggests using binary.

Given positive integer N, let $f(N)$ denote the length of the shortest such sequence beginning with 1 and ending with N. For example, $f(12) = 5$ since the sequence $1, 2, 3, 6, 12$ has length 5, and no shorter sequence exists.

Let $d_2(N)$ denote the number of digits of N when expressed in binary, and let $s_2(N)$ denote the sum of these digits.

Lemma 1. *For positive integer N, we have $f(N) = d_2(N) + s_2(N) - 1$.*

The proof is at the end of the solution.

Using the lemma, we have $2 \le f(N) \le 10 \iff 3 \le d_2(N) + s_2(N) \le 11$, i.e. N is interesting iff the number of digits of N in binary, added with their sum (or the number of 1's) is between 3 and 11 inclusive.

We can put the set of interesting integers in bijection with sequences of 1's and 2's whose sum is between 1 and 9 inclusive. The bijection is as follows: Given such an integer N, convert it to binary. Add 1 to each digit to obtain a sequence of 1's and 2's, then drop the leading 2. For example, $N = 81 = 1010001_2$ maps to 121112.

Now we've reduced the problem to a simpler counting exercise of counting the number of sequences of 1's and 2's whose sum is 1, 2, 3, ..., 9. The number of sequences whose sum is m (between 1 and 9 inclusive) is solvable recursively and is equal to F_{m+1}, where F_n is the n^{th} Fibonacci number. Summing from $m = 1$ to $m = 9$, the number of interesting integers is

$$\begin{aligned}\sum_{m=1}^{9} F_{m+1} &= F_2 + F_3 + \ldots + F_{10} \\ &= F_{12} - 1 - F_1 \\ &= 144 - 1 - 1 \\ &= \boxed{142}.\end{aligned}$$

Proof of Lemma. This can be shown by strong induction on N. The base case $N = 1$ holds, as $f(1) = 1$ and $d_2(1)+s_2(N)-1 = 1+1-1 = 1$. Suppose the statement is true for all $n < N$. We wish to show the equation holds for N. We have two cases:

Case 1: N is odd. Then $f(N) = f(N-1)+1 = d_2(N-1)+s_2(N-1)$. Note that $d_2(N) = d_2(N-1)$, and $s_2(N) = s_2(N-1)+1$. Substituting gives $f(N) = d_2(N) + s_2(N) - 1$.

Case 2: N is even. Then $f(N) = \min(f(N/2) + 1, f(2N - 2) + 2)$ (either double to get N, or add 1 twice). By induction hypothesis:

$$f(N) = \min(d_2(N/2) + s_2(N/2), d_2(N-2) + s_2(N-2) + 1$$

For even N, we have $d_2(N/2) = d_2(N) - 1$ and $s_2(N/2) = s_2(N)$ trivially. Also, $d_2(N) \leq d_2(N-2) + 1$ and $s_2(N) \leq s_2(N-2) + 1$ (to see the last inequality, note that adding 1 increases the digit sum by 1, but adding 1 again either leaves the digit sum unchanged or decreases it due to carrying). Using these facts we have that

$$\begin{aligned} f(N) &= \min(d_2(N) + s_2(N) - 1, d_2(N) + s_2(N) - 1 + C) \\ &= d_2(N) + s_2(N) - 1 \end{aligned}$$

completing the induction. It is not too difficult to give an algorithm constructing such a sequence, given N in binary. $\square$