Last-Minute Mock AMC 12

CountofMC

January 28, 2018

1 Problems

Good luck and have fun!

1. The Count of Monte Cristo has 32 coins in his pocket. Each coin is either a penny or a nickel, and the total monetary value of all the coins is \$1.08. How many nickels is The Count of Monte Cristo carrying in his pocket?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 19 **(E)** 24

2. What is the remainder when $\frac{2018!-2017!}{2016!+2015!}$ is divided by 1000?

(A) 240 **(B)** 256 **(C)** 272 **(D)** 289 **(E)** 306

3. Mr. Barnes tells his math class that they will have a party if the average of the scores on the next test is at least 90 or nobody misses his or her next homework assignment. Which of the following statements is always true?

(A) If the class did not have a party, then the average of the scores on the test was less than 90 and at least one person missed his or her homework assignment.

(B) If the class did not have a party, then the average of the scores on the test was less than 90 or at least one person missed his or her homework assignment.

(C) If the class has a party, then the average of the scores on the test was at least 90 and nobody missed his or her next homework assignment.

(D) If the class has a party, then the average of the scores on the test was less than 90 and nobody missed his or her next homework assignment.

(E) If nobody misses his or her next homework assignment and the average of the scores on the next test is at least 90, then the class will not have a party.

4. Two distinct positive integers between 1 and 100, inclusive, are chosen at random. What is the probability that their arithmetic mean will be an integer?

(A) $\frac{49}{100}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{49}{99}$ **(D)** $\frac{51}{99}$ **(E)** $\frac{51}{100}$

5. A right circular cone and right circular cylinder have the same volume. The radius of the base of the cone is increased by 40% while the height of the cylinder is decreased by 16%. What is the resulting ratio of the volume of the cylinder to the volume of the cone?

(A) $\frac{1}{7}$ **(B)** $\frac{3}{14}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{5}{14}$ **(E)** $\frac{3}{7}$

6. Aaron has some marbles he wishes to divide equally among a certain number of jars. When he places them into 3 jars, there is 1 marble left over. When he places them into 4 jars, there are 2 marbles left over. When he places them into 5 jars, there are 3 marbles left over. Let n be the least possible number of marbles Aaron could have. What is the sum of the digits of n?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

7. The angles of a convex dodecagon have integral values and are in arithmetic progression. What is the least possible value of the smallest angle, in degrees?

(A) 120 **(B)** 124 **(C)** 128 **(D)** 134 **(E)** 139

8. The sum of two positive two-digit integers a and b is another two-digit positive integer n. The positive difference of a and b is obtained by reversing the digits of n and is also a two-digit positive integer. How many ordered pairs (a, b) are possible?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

9. Point P is located outside a circle with center O such that $OP = 3$. Point Q is also outside the circle such that PQ is perpendicular to OP and $PQ = 4$. Let OP intersect the circle at R and OQ intersect the circle at S. Moreover, let T be the projection of R onto OQ, and let U be the point on the circle such that QU is tangent to the circle. If $QT = \frac{7}{2}$, then what is the length of QU?

(A) 2 **(B)** $2\sqrt{3}$ **(C)** $\frac{7\sqrt{3}}{4}$ **(D)** $\frac{5\sqrt{3}}{2}$ **(E)** $3\sqrt{2}$

10. Let $x = \sqrt{2 + \sqrt{6 + \sqrt{6 + \dots}}}$ and let $y = \sqrt{3 + \sqrt{12 + \sqrt{12 + \dots}}}$. Which of the following statements correctly expresses y in terms of x?

(A) $y = x+2$ **(B)** $y = x^2+2$ **(C)** $y = \sqrt{x+2}$ **(D)** $y = \sqrt{x^2+2}$ **(E)** $y = \sqrt{x^2-2}$

11. The positive integer N satisfies $N_8 + N_9 = 2017_{10}$, where the subscripts signify number bases. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

12. A primitive nth root of unity is a complex number z such that $z^n = 1$ and $z^k \neq 1$ for $k = 1, 2, 3, \ldots, n-1$. How many values of k are there less than 360 such that for some complex number ω that is a 360th root of unity, ω is also a primitive kth root of unity?

(A) 12 **(B)** 15 **(C)** 19 **(D)** 23 **(E)** 29

13. Let $S = \{P_1, P_2, P_3, \ldots, P_m\}$ be a set of lattice points in n-dimensional space; that is, each point P_i can be expressed as the n-tuple $(x_1, x_2, x_3, \ldots, x_n)$, where each of the x_j's are integers. In terms of n, what is the minimum number of points in S required to be able to find two points in S whose midpoint is also a lattice point?

(A) $n+1$ **(B)** $2n+1$ **(C)** 2^n+1 **(D)** 2^n+n+1**(E)** 2^n+2n+1

14. Anna randomly chooses a positive divisor of the number 15! while Bob randomly chooses another positive divisor of the same number, not necessarily distinct from the one chosen by Anna. What is the probability that the sum of Anna's and Bob's divisors will be odd?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{11}{72}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{13}{72}$

15. In triangle ABC, $AB = 13$, $AC = 14$, and $BC = 15$. A point is randomly chosen inside the triangle. The probability that it will lie closer to A than to any other vertex can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the remainder when $m + n$ is divided by 1000?

(A) 219 **(B)** 297 **(C)** 342 **(D)** 388 **(E)** 435

16. A positive integer is called *happy* if the sum of its digits equals the two-digit integer formed by its two leftmost digits. How many five-digit positive integers are happy?

(A) 880 **(B)** 950 **(C)** 1001 **(D)** 1075 **(E)** 1110

17. Let x, y, and z be positive real numbers with $1 < x < y < z$ such that

$$\log_x y + \log_y z + \log_z x = 8 \text{ and}$$

$$\log_x z + \log_z y + \log_y x = \frac{25}{2}.$$

The value of $\log_y z$ can then be written as $\frac{a+\sqrt{b}}{c}$ for positive integers a, b, and c such that b is not divisible by the square of any prime. What is the value of $a + b + c$?

(A) 31 **(B)** 42 **(C)** 47 **(D)** 53 **(E)** 64

18. A particle starts at the origin and takes a series of steps in the following manner: If the particle is at the point (x, y), it then moves randomly to one of $(x-1, y)$, $(x+1, y)$, $(x-1, y+1)$, $(x, y+1)$, or $(x+1, y+1)$. What is the expected number of steps the particle takes to reach the line $y = 3$?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

19. Let a, b, and c be positive real numbers such that

$$\frac{a^2}{2018-a} + \frac{b^2}{2018-b} + \frac{c^2}{2018-c} = \frac{1}{168}.$$

What is the largest possible value of $a + b + c$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

20. Let m and n be two odd positive integers less than 100, and let $k = 2^m 3^n$. Let N be the number of divisors of k^2 that are less than k but do not divide k. For how many ordered pairs (m, n) is it true that N is less than 1000?

(A) 729 **(B)** 822 **(C)** 872 **(D)** 916 **(E)** 987

21. Rectangle $ABCD$ has $AB = 15$ and $BC = 12$. Let E and F be the trisection points of AC, with E closer to A, and let G and H be the trisection points of BD, with G closer to B. Let AG intersect BC at P, PH intersect AD at Q, QE intersect AB at R, and RF intersect CD at S. What is the area of quadrilateral $DHFS$?

(A) 27 **(B)** 31 **(C)** 35 **(D)** 37 **(E)** 42

22. The sequence $1, 4, 5, 16, 17, \ldots$ consists of all of the positive integers that can be expressed as the sum of distinct nonnegative powers of 4, ordered starting from $4^0 = 1$, the smallest such integer. Let S be the sum of the first 31 elements of the sequence. What is the sum of the digits of S?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

23. Given that in triangle ABC, $\cos(nA) + \cos(nB) + \cos(nC) = 1$ for some positive integer n, which of the following statements is always true?

(A) There exists an angle that measures $\frac{360^\circ}{n}$ if and only if n is odd.

(B) There exists an angle that measures $\frac{360^\circ}{n}$ if and only if n is even.

(C) There exists an angle that measures $\frac{180^\circ}{n}$ if and only if n is odd.

(D) There exists an angle that measures $\frac{180^\circ}{n}$ if and only if n is even.

(E) More information is required in order to accurately determine the measure of at least one angle of triangle ABC.

24. In triangle $\triangle ABC$, $AB = 5$, $BC = 7$, and $CA = 8$. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively, and let M be the midpoint

of BC. The area of triangle MEF can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c such that the greatest common divisor of a and c is 1 and b is not divisible by the square of any prime. What is the value of $a+b+c$?

(A) 68 **(B)** 70 **(C)** 72 **(D)** 74 **(E)** 76

25. For every positive integer n, define S_n as the set of all permutations of the first n positive integers such that no pair of consecutive integers appears in that order; that is, 2 does not follow 1, 3 does not follow 2, and so on. For example, 2143 and 2431 are valid permutations in S_4. Denote by $f(n)$ the number of elements in S_n and $p(n)$ the probability that a randomly chosen permutation of the first n positive integers is contained in S_n. Let r be the units digit of $f(2018)$ and m be the positive integer formed by the first three digits after the decimal point in the expansion of $p(2018)$. What is the value of $r+m$?

(A) 372 **(B)** 373 **(C)** 374 **(D)** 375 **(E)** 376