# Mock AMC 10 Solutions

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1. **Answer: (A)** The price of four candy bars is $4 \times \$2.50 = \$10$. If five candy bars are bought, the price is $5 \times \$2.50 \times 0.7 = \$8.75$. The amount saved by buying five candy bars instead of four is $\$10 - \$8.75 = \boxed{\textbf{(A)}\ \$1.25}$.

   *Alternate solution:* With the 30% discount, buying five candy bars is equivalent to buying $5 \times 0.7 = 3.5$ candy bars at full price. Then Nathan saves the cost of half a candy bar, or \$1.25.

2. **Answer: (A)** Suppose there are $3x$ boys and $4x$ girls in the class. Then $\frac{1}{4} \cdot 3x = \frac{3}{4}x$ boys have a pet, and $\frac{1}{3} \cdot 4x = \frac{4}{3}x$ girls have a pet. Then there are $\frac{3}{4}x + \frac{4}{3}x = \frac{25}{12}x$ sophomores who have a pet. The percentage of boys is

$$\frac{\frac{3}{4}x}{\frac{25}{12}x} = \frac{\frac{3}{4}}{\frac{25}{12}} = \frac{9}{25} = \boxed{\textbf{(A)}\ 36\%}$$

3. **Answer: (D)** Since the two trains pass each other halfway between Boston and New York, the southbound train must have traveled for $1\frac{1}{2}$ hours (as this is half of 3 hours, the travel time for the southbound train) and the northbound train must have traveled for $1\frac{1}{4}$ hours as it departed 15 minutes later. Therefore, the northbound train traveled 105 miles in $1\frac{1}{4} = \frac{5}{4}$ hours, so its average speed is $\frac{105 \text{ mi}}{\frac{5}{4} \text{ hr}} = \boxed{\textbf{(D)}\ 84}$ mph.

4. **Answer: (C)** The smallest possible sum of five different prime numbers is $2+3+5+7+11 = 28$, so the average of the primes must be at least 6. Notice that 2 cannot appear in Ellie's list, otherwise the sum $S$ of the five primes is even. In this case, if $\frac{S}{5}$ is an integer, then it is an even integer greater than 2, and cannot be prime.

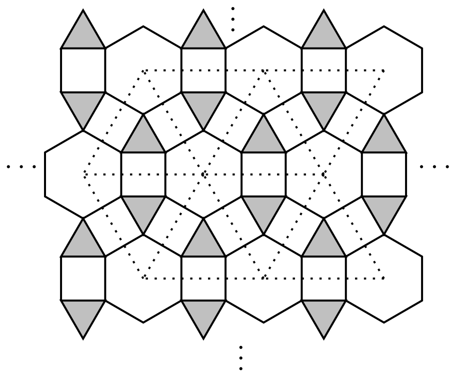
   Therefore the smallest possible sum is at least $3+5+7+11+13 = 39$, i.e., $S \geq 39$. The smallest prime $p$ such that $5p \geq 39$ is 11, so $S \geq 55$. We see that 55 is possible as $3+5+7+11+29 = 55$ and $\frac{55}{5} = 11$ is prime. Then the smallest possible value of Ellie's sum is $\boxed{\textbf{(C)}\ 55}$.

5. **Answer: (D)** We can simplify by dividing both numerator and denominator by 2017!, in which we obtain

$$\begin{aligned}\frac{2020! + 2019!}{2018! + 2017!} &= \frac{2020 \cdot 2019 \cdot 2018 + 2019 \cdot 2018}{2018 + 1} \\ &= 2020 \cdot 2018 + 2018 \\ &= 2021 \cdot 2018 \\ &= \boxed{\textbf{(D) } 4078378}\end{aligned}$$

Note we can skip the multiplication of two 4-digit numbers as only choice **(D)** has a units digit of 8.

6. **Answer: (C)** We can tessellate the area as follows, using dotted triangles as indicated:



The fraction of area covered by the triangular tiles is approximately the fraction of area covered by one triangular tile within a dotted triangle. Without loss of generality, suppose each tile has a side length of 1. Then the side length of one of the dotted triangles is $1 + 2\left(\frac{\sqrt{3}}{2}\right) = 1 + \sqrt{3}$. The ratio of the area of the shaded triangle to the area of a dotted triangle is the square of the ratio of their side lengths, which is

$$\frac{1^2}{(1+\sqrt{3})^2} = \frac{1}{4+2\sqrt{3}} \approx \frac{1}{7.4} \approx \boxed{\textbf{(C) } 13\%}.$$

To be absolutely sure without use of a calculator, we note that $\frac{1}{4+2\sqrt{3}} > \frac{1}{8} = 12.5\%$, and $\frac{1}{4+2\sqrt{3}} < \frac{1}{7.4} < 14\%$.

7. **Answer: (D)** Since the median score before and after dropping is 84, we know that the $4^{\text{th}}$ and $5^{\text{th}}$ quiz scores (in sorted order) are both 84. Further, we also know that the sum of Albert's seven quiz scores is

$84 \times 7 = 588$, and the sum of his highest five quiz scores is $85 \times 5 = 425$; thus, the sum of the two dropped scores is $588 - 425 = 163$.

To maximize the highest score, we should minimize the remaining quiz scores, so we will set the scores optimally to be 81, 82, 82, 84, 84, 84, and $x$, where 81 and 82 are the two dropped scores. Solving, we get $x = \boxed{\textbf{(D) } 91}$.

8. **Answer: (C)** Let $A$, $B$, and $C$ be the following statements:

$$A: \text{Today is rainy.}$$
$$B: \text{Erin is wearing a raincoat.}$$
$$C: \text{Erin is wet outside.}$$

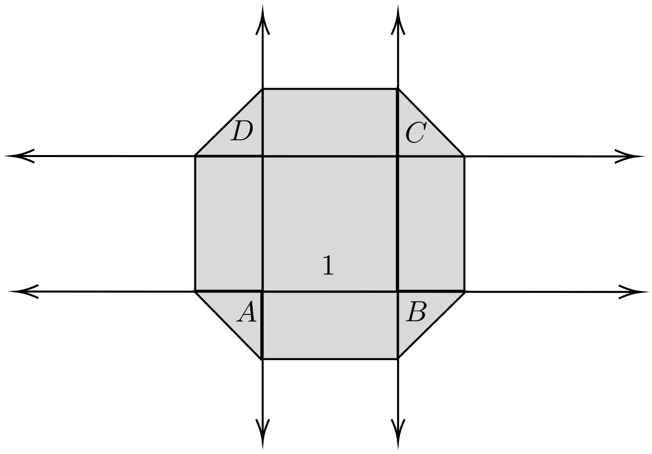
The given statements can be phrased as $A \implies B$ and $(A \wedge B) \implies C$. Observe that in the second statement, the phrase "is wearing a raincoat" is redundant as this is already implied by $A$; hence the second statement is equivalent to $A \implies C$.

Statement I is not implied as it is the converse of $A \implies C$ (perhaps, Erin got wet outside by playing in the sprinklers on a sunny day). Statement II is not implied as today could be rainy, in which Erin did not get wet outside. Statement III *is* implied as this is the contrapositive of $A \implies C$. Statement IV is not implied; note that if Erin is wet outside, then today is not rainy (III), but this does not imply she is not wearing a raincoat. Hence the only implied statement is $\boxed{\textbf{(C) } \text{III only}}$.

9. **Answer: (B)** Note that if point $P$ is inside the square, then the sum of the distances from $P$ to each of the lines is equal to 2. We must now consider when $P$ is outside the square.

If $P$ is between lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, then the sum of the distances from $P$ to these two lines equals 1. It follows that if $P$ is at most $\frac{1}{2}$ unit away from either line $\overleftrightarrow{BC}$ or line $\overleftrightarrow{CD}$, the sum of the distances from $P$ to these two lines is at most $\frac{1}{2} + \frac{3}{2} = 2$, and the sum of the distances from $P$ to all four lines is at most 3. A similar situation holds if $P$ is between lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{DA}$.

Lastly, suppose $P$ is not between lines $\{\overleftrightarrow{AB}, \overleftrightarrow{BC}\}$ or $\{\overleftrightarrow{BC}, \overleftrightarrow{DA}\}$. If $P$ is $x$ units away from the closer of the two vertical lines and $y$ units away from the closer of the two horizontal lines, then the sum of the distances is $x + (1 + x) + y + (1 + y) = 2 + 2x + 2y \leq 3$, so $x + y \leq \frac{1}{2}$. Using this information, $\mathcal{S}$ is the following octagon:




The octagon determined by $\mathcal{S}$ consists of a square with side length 1, four $1 \times \frac{1}{2}$ rectangles, and four isosceles right triangles of side length $\frac{1}{2}$. The area of region $\mathcal{S}$ is

$$1 + 4 \times \frac{1}{2} + 4 \times \frac{1}{8} = \boxed{\textbf{(B) } \frac{7}{2}}.$$

10. **Answer: (B)** We have $f(g(x)) = |x^2 - 4|$ and $g(f(x)) = |x - 4|^2 = (x-4)^2$. Thus we wish to solve $|x^2 - 4| = (x-4)^2$ for real $x$.

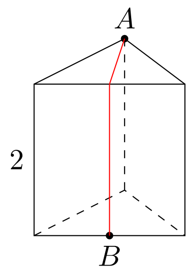
    If $|x| \geq 2$, then $|x^2 - 4| = x^2 - 4$, and solving the equation $x^2 - 4 = (x-4)^2 = x^2 - 8x + 16$ gives $x = \frac{5}{2}$. If $|x| \leq 2$, then $|x^2 - 4| = 4 - x^2$, and the equation $4 - x^2 = x^2 - 8x + 16$ rearranges to the quadratic $2x^2 - 8x + 12 = 0 \iff x^2 - 4x + 6 = 0$. This has no real solutions as the discriminant is negative. Hence the number of real solutions is $\boxed{\textbf{(B) } 1}$.

11. **Answer: (A)** Rewrite the given equation as $a(bc + b + 1) = 64$. Clearly, $a$ must be a power of 2 greater than 1, so $bc + b + 1$ must equal 2, 4, 8, 16, or 32. This implies $bc + b = b(c+1)$ must equal 1, 3, 7, 15, or 31. As $b$ and $c + 1$ are at least 2, the number $b(c+1)$ must be composite, and only 15 satisfies this condition, giving $a = 4$. Then $b(c+1) = 15$ giving solutions $(b, c) = (3, 4)$ or $(5, 2)$. There are two ordered triples $(a, b, c)$ of integers greater than or equal to 2 which work, namely $(a, b, c) = (4, 3, 4)$ and $(4, 5, 2)$. In either case, $a + b + c = \boxed{\textbf{(A) } 11}$.

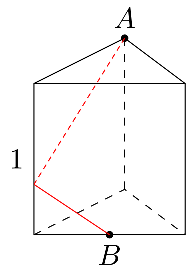
12. **Answer: (D)** It is easy to check that Joanna can make all monetary amounts of the form \$$5k$, where $k = 0, 1, \ldots, 74$, using denominations which are multiples of 5. Using one or two \$1 bills, she can also make the amounts \$$5k+1$ and \$$5k+2$. Thus, there are 75 possible values for

$k$, each of which gives 3 different monetary amounts, giving $75 \times 3 = 225$ different amounts. However this includes \$0 which is invalid (since one or more bills is needed), so the answer is $225 - 1 = \boxed{\textbf{(D) } 224}$.

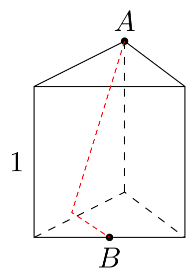
13. **Answer: (B)** If the ant crawls along the red path, as shown below, the ant travels $2 + \sqrt{3}$ units:



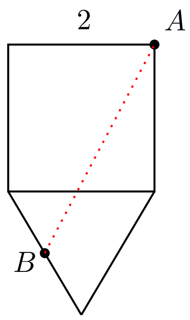

A better solution is for the ant to crawl along two square faces as shown:




Unfolding the two square faces, we see that the length of the path equals the hypotenuse of a right triangle with leg lengths 2 and 3; by the Pythagorean theorem, the distance crawled is $\sqrt{2^2 + 3^2} = \sqrt{13}$ units. However, the optimum solution is to crawl along one square face and one triangular face:




Unfolding the net, the path will appear as follows:




We see that the length of the dotted path is $\sqrt{\left(\frac{3}{2}\right)^2 + \left(2 + \frac{\sqrt{3}}{2}\right)^2} = \boxed{\textbf{(B) } \sqrt{7+2\sqrt{3}}}$.

14. **Answer: (B)** Because $M$ and $m$ have only odd digits and are divisible by 5, the units digit of $M$ and $m$ must both be 5. Let $M = \overline{abc5}$ and $m = \overline{def5}$, where $a, \ldots, f$ are digits with $a, d \neq 0$. In order for $M$ and $m$ to be divisible by 45, we must have $a+b+c+5 \equiv d+e+f+5 \equiv 0 \pmod{9}$, or equivalently, $a+b+c$ and $d+e+f$ leave remainder 4 when divided by 9.

    The only possible values for $a+b+c$ (or $d+e+f$) are 4, 13, and 22. However, since the digits are odd, the only possible candidate is $a+b+c = d+e+f = 13$. To find $M$, we want $a$ as large as possible, and setting $(a, b, c) = (9, 3, 1)$ accomplishes this. To find $m$, we want $a$ as small as possible, so set $(d, e, f) = (1, 3, 9)$. Then $M = 9315$ and $m = 1395$, and $\frac{M-m}{45} = \frac{9315-1395}{45} = \boxed{\textbf{(B) } 176}$.

15. **Answer: (D)** Note that $2020 \times \frac{1}{3} = 673\frac{1}{3}$, and $\frac{674}{2020} = \frac{337}{1010}$ appears after $\frac{1}{3}$. However $\frac{337}{1010}$ is *not* the fraction immediately after $\frac{1}{3}$, as we should consider fractions with other denominators. Ideally, the denominator should be as large as possible.

    We have $\frac{673}{2019} = \frac{1}{3}$, so $\frac{674}{2019}$ appears after $\frac{1}{3}$ in the Farey sequence; however, this appears further as $\frac{674}{2019} > \frac{337}{1010}$. We also consider $\frac{673}{2018}$; this turns out to be the optimum, in which $a + b = 673 + 2018 = \boxed{\textbf{(D) } 2691}$. To show this, we will consider fractions of the form $\frac{k+1}{3k+1}$ and $\frac{k+1}{3k+2}$ (where $k \in \mathbb{Z}_+$). These fractions can be written as $\frac{1}{3} + \frac{2/3}{3k+1}$ and $\frac{1}{3} + \frac{1/3}{3k+2}$ respectively, in which increasing $k$ results in a fraction which is closer to $\frac{1}{3}$.

16. **Answer: (D)** The numbers which are not multiples of 2, 3, or 5 eliminate children from the circle. Thus, if $n$ children are originally in the circle, then the first $n-1$ such numbers eliminate children from

the circle, in which the $(n-1)^{\text{th}}$ number is 121 by the conditions of the problem. That is, we want to find the number of positive integers less than or equal to 121 which are not multiples of 2, 3, or 5.

Consider the system of modular congruences

$$\begin{aligned} x &\equiv a \pmod 2 \\ x &\equiv b \pmod 3 \\ x &\equiv c \pmod 5 \end{aligned}$$

where $a \in \{1\}$, $b \in \{1,2\}$, and $c \in \{1,2,3,4\}$. By the Chinese remainder theorem, any valid choice of $a$, $b$, $c$ gives a unique solution for $x \pmod{30}$. As there are $1 \times 2 \times 4 = 8$ choices for $(a,b,c)$, there are 8 numbers in $\{1,2,\dots,30\}$ which are not multiples of 2, 3, or 5. Similarly, there are 8 numbers in $\{31,\dots,60\}$, $\{61,\dots,90\}$, $\{91,\dots,120\}$ which are not multiples of 2, 3, or 5. From this, we establish $n-1 = 4\times 8+1 = 33$, so the number of children is $n = \boxed{\textbf{(D) } 34}$.

17. **Answer: (D)** Observe that the prime factorization of 2021 is $43 \times 47$; this can be seen easily from the difference of squares factorization $2021 = 45^2 - 2^2 = (45-2)(45+2)$.

    Using Legendre's formula, we have $2021! = 43^{48} \times 47^{43} \times K$, where $K$ is not divisible by 43 or 47. Also, we have $2020! = 43^{47} \times 47^{42} \times K$. In order for $\operatorname{lcm}(n, 2020!) = 2021!$, the smallest possible $n$ is $n = 43^{48} \times 47^{43}$, and the number of factors of $n$ is $(48+1)(43+1) = 49 \times 44 = \boxed{\textbf{(D) } 2156}$.

18. **Answer: (E)** The prime factorization of 216 is $2^3 \times 3^3$, so 216 has $(3+1)(3+1) = 16$ factors. There are $16^3 = 2^{12}$ equally likely outcomes for the triple of numbers Robert draws.

    Each divisor of 216 can be represented as an ordered pair $(a,b)$ where $0 \le a \le 3$ and $0 \le b \le 3$, corresponding to the number $2^a \times 3^b$. Thus, we can consider the equivalent problem of choosing a random ordered pair $(a,b) \in \{0,\dots,3\} \times \{0,\dots,3\}$, and finding the probability that three such pairs sum to less than or equal to $(3,3)$ (where $(x,y) \le (3,3)$ if and only if $x \le 3$ and $y \le 3$).

    Suppose these three pairs are $(a_1,b_1)$, $(a_2,b_2)$, $(a_3,b_3)$. By stars and bars, the number of non-negative integer solutions to $a_1+a_2+a_3 \le 3$ equals the number of non-negative solutions to $a_1+a_2+a_3+s = 3$, which is $\binom{6}{3} = 20$ (and similarly $b_1+b_2+b_3 \le 3$). Any such 6-tuple $(a_1,a_2,a_3,b_1,b_2,b_3)$ uniquely gives a sequence of three pairs or equivalently, a valid outcome whose product is a factor of 216. Thus, the number of valid outcomes is $20^2 = 400$, and the probability is $\frac{400}{2^{12}} = \boxed{\textbf{(E) } \frac{25}{256}}$.

19. **Answer: (D)** Let $a_k$ denote the largest real solution to the equation $f^k(x) = 0$, so that $x_1 = a_{2020}$. We observe a pattern:

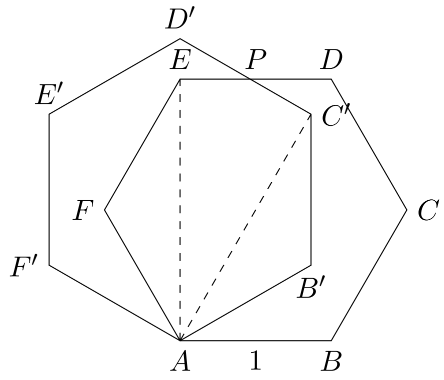
$$\begin{aligned}
k = 1 :&\quad x^2 - 20 = 0 \implies a_1 = \sqrt{20} \\
k = 2 :&\quad (x^2 - 20)^2 - 20 = 0 \implies a_2 = \sqrt{20 + \sqrt{20}} \\
k = 3 :&\quad ((x^2 - 20)^2 - 20)^2 - 20 = 0 \implies a_3 = \sqrt{20 + \sqrt{20 + \sqrt{20}}} \\
&\quad \vdots
\end{aligned}$$

This pattern can be shown inductively. Suppose we know the value of $a_k$ for some $k \geq 1$. Then $f^{k+1}(x) = (\ldots((x^2 - 20)^2 - 20)^2 - \ldots)^2 - 20$. Here, the innermost $x^2 - 20$ must be a root of $f^k(x)$, so in order to maximize $x$, we set $x^2 - 20 = a_k$ where $x > 0$, giving $a_{k+1} = \sqrt{20 + a_k}$.

We claim that the sequence $a_1$, $a_2$, $a_3$, $\ldots$ is strictly increasing and converges to the positive root of the polynomial $a^2 - a - 20$. To show the sequence is increasing, we note that $\sqrt{20} < a_k < 5$ for all $k \geq 1$ (provable inductively), and that $\sqrt{20 + a_k} > a_k$ iff $20 + a_k > a_k^2$ iff $(a_k - 5)(a_k + 4) < 0$. As $4 < a_k < 5$, this inequality holds, so $(a_k)$ is strictly increasing. Let $a = \sqrt{20 + \sqrt{20 + \sqrt{20 + \ldots}}}$; squaring both sides gives $a^2 = 20 + a \implies a = -4, 5$ by the quadratic formula. Since $a > 0$, we take the positive root, or $a = 5$. However, $a_{2020}$ is less than 5 (by an extremely small amount); that is, $x_1 = a_{2020} = 4.999\ldots < 5$. Here it is sufficient to bound $4.95 < x_1 < 5$, so that $24.5 < x_1^2 < 25$.

To find $x_0$, we observe that $f^k(x)$ is an even function for all $k \geq 1$. This can be shown inductively; $f^1(x)$ is even, and if $f^{k-1}(x)$ is even, then $f^k(-x) = f^1(f^{k-1}(-x)) = f^1(f^{k-1}(x)) = f^k(x)$. Thus for real $x$, we have $f^{2020}(x) = 0$ iff $f^{2020}(-x) = 0$, so $x_0 = -x_1 \approx -4.999\ldots$; that is, $|x_0| = |x_1| \approx 4.999\ldots$. Then $49 < x_0^2 + x_1^2 < 50$, so the largest integer less than or equal to $x_0^2 + x_1^2$ is $\boxed{\textbf{(D) } 49}$.

20. **Answer: (B)** The configuration will look like the figure below; note that a slighty different configuration arises if $ABCDEF$ is rotated clockwise, but the common region is the same.




We first find the area of $\triangle AEF$ and $\triangle AB'C'$. This is easy, as both triangles have the same area as an equilateral triangle of side length 1, which is $\frac{\sqrt{3}}{4}$. Let $P$ be the intersection of $\overline{DE}$ and $\overline{C'D'}$ as shown. Extend $AE$ past $E$ to meet at $D'$; it is not hard to show $A$, $E$, and $D'$ are collinear. Then $\triangle AC'D'$ is a 30-60-90 triangle with area $\frac{\sqrt{3}}{2}$. Further, $\triangle D'EP$ is also a 30-60-90 triangle with shorter side $D'E = 2 - \sqrt{3}$ and longer leg $EP = (2-\sqrt{3})\sqrt{3} = 2\sqrt{3} - 3$. Since $AE = AC' = \sqrt{3}$, we have

$$\begin{aligned}
[\triangle AEP] = [\triangle AC'P] &= \frac{1}{2}\sqrt{3}(2\sqrt{3}-3) \\
&= \frac{1}{2}\left(6 - 3\sqrt{3}\right) \\
[AC'PE] = 2[\triangle AEP] &= 6 - 3\sqrt{3}
\end{aligned}$$

Combining, we obtain $[AB'C'PEF] = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + (6 - 3\sqrt{3}) = 6 - \frac{5\sqrt{3}}{2} = \boxed{\textbf{(B) } \dfrac{12-5\sqrt{3}}{2}}$.

*Alternate solution:* Consider right triangles $\triangle AEP$ and $\triangle AC'P$. By HL congruence, they are congruent, so $\angle PAE = \angle PAC' = 15^\circ$, i.e., they are 15-75-90 triangles. Since $AE = \sqrt{3}$, we have $EP = PC' = \sqrt{3}\tan 15^\circ = \sqrt{3}(2-\sqrt{3}) = 2\sqrt{3} - 3$. Then $[AC'PE] = 2 \times \frac{1}{2} \times \sqrt{3}(2\sqrt{3} - 3) = 6 - 3\sqrt{3}$, and we obtain $[AB'C'PEF] = \frac{12-5\sqrt{3}}{2}$ as before.

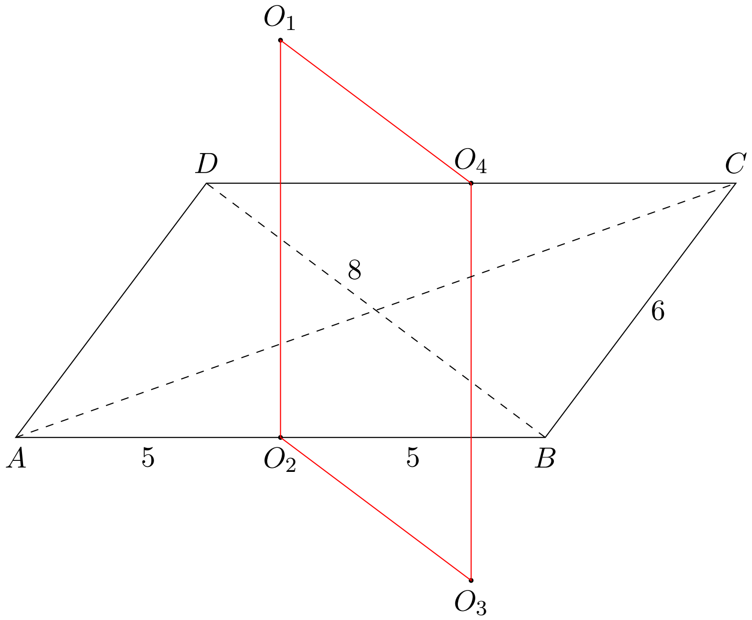
21. **Answer: (C)** Label the persons 1, …, 6 with 1 the shortest and 6 the tallest. Note that 1 must stand next to 2 and 6 must stand next to 5. Thus we may treat 1 and 2 as one “pair” (similarly with 5 and 6). The remaining constraint is that 3 must stand next to 2 or 4, and 4 must stand next to 3 or 5.

We can do casework on whether 3 and 4 stand next to each other.

- **Case 1:** 3 and 4 stand next to each other. Then we may treat 3 and 4 as one pair, in which we have three pairs (12), (34), (56). There are $3! = 6$ ways to order these pairs, followed by $2^3 = 8$ ways to order the two people within each pair, giving $6 \times 8 = 48$ ways.
- **Case 2:** 3 and 4 do not stand next to each other. Then 3 stands next to 2, and 4 stands next to 5. Thus, 1, 2, and 3 form a "triple" (denoted (123)), and so do 4, 5, 6. There are $2! \times 2 \times 2 = 8$ ways to arrange the two triples (123),(456). However, this overcounts the arrangements 123456 and 654321, as 3 and 4 are next to each other. Thus, the number of ways is $8 - 2 = 6$.

Adding, we obtain $48 + 6 = \boxed{\textbf{(C) } 54}$ ways.

22. **Answer: (D)** Note that $\triangle BCD$ and $\triangle ABD$ are 6-8-10 right triangles with $\angle DBC = \angle BDA = 90°$. Then the circumcenters $O_2$ and $O_4$ are the midpoints of their corresponding hypotenuses.




We observe that $O_1$ and $O_2$ are both circumcenters of two triangles containing side $AB$. As the circumcenter of a triangle is the intersection of its perpendicular bisectors, we have $\angle O_1O_2B = 90°$, and similarly $\angle O_3O_4D = 90°$, which implies $O_1O_2 \parallel O_4O_3$. Similarly, $O_1$ and $O_4$ are both circumcenters of two triangles containing $BC$; by similar reasoning we have $O_1O_4 \perp BC$ and $O_1O_4 \parallel O_2O_3$, so $O_1O_2O_3O_4$ is a parallelogram. Letting $P$ be the intersection of $O_1O_2$ and $CD$, a quick angle chase reveals that $\triangle O_1PO_4 \sim \triangle ADB$.

We will call $\overline{O_1O_2}$ and $\overline{O_3O_4}$ the bases of the parallelogram. The height can easily be found to be $PO_4 = 6 \times \frac{3}{5} = \frac{18}{5}$. To find $\overline{O_1O_2}$, we see that $O_2P = 6 \times \frac{4}{5} = \frac{24}{5}$, and $O_1P = \frac{3}{4} \times \frac{18}{5} = \frac{27}{10}$. Then $O_1O_2 = \frac{24}{5} + \frac{27}{10} = \frac{15}{2}$, and the area of $O_1O_2O_3O_4$ is therefore $\frac{18}{5} \times \frac{15}{2} = \boxed{\textbf{(D) } 27}$.

23. **Answer: (E)** For $n \geq 1$, let $p_n$ denote the probability that Paige's first $n$ rolls sum to 7. Note that $p_1 = 0$ since each roll is at most 6, and $p_n = 0$ for $n \geq 8$ since each roll is at least 1. The answer is $p_2 + p_3 + \ldots + p_7$.

    To find $p_n$ for $2 \leq n \leq 7$, we can find the number of outcomes on $n$ dice rolls which sum to 7, then divide by $6^n$. The number of "good" outcomes equals the number of positive integer solutions to the equation $a_1 + \ldots + a_n = 7$ where $1 \leq a_i \leq 6$. By stars and bars, this yields $\binom{6}{n-1}$ outcomes.

    The desired answer is

$$\begin{aligned} p_2 + p_3 + \ldots + p_7 &= \sum_{n=2}^{7} \frac{\binom{6}{n-1}}{6^n} \\ &= \binom{6}{1}\frac{1}{6^2} + \binom{6}{2}\frac{1}{6^3} + \ldots + \binom{6}{6}\frac{1}{6^7} \\ &= \frac{1}{6}\left[\binom{6}{1}\frac{1}{6} + \binom{6}{2}\frac{1}{6^2} + \ldots + \binom{6}{6}\frac{1}{6^6}\right] \\ &= \frac{1}{6}\left[\left(1 + \frac{1}{6}\right)^6 - 1\right] \\ &= \frac{7^6 - 6^6}{6^7} \end{aligned}$$

    Hence $m = 7^6 - 6^6$ and $n = 6^7$, in which $m + n = 7^6 - 6^6 + 6^7$. To find $m + n \pmod{1000}$, we can use $7^6 - 6^6 = 343^2 - 216^2 = 559 \times 127 \equiv 993 \pmod{1000}$, and $6^7 \equiv 216^2 \times 6 \equiv 936 \pmod{1000}$. Then $m + n \equiv 993 + 936 \equiv \boxed{\textbf{(E) } 929} \pmod{1000}$ (Note that $\frac{7^6 - 6^6}{6^7} \approx 25.4\%$).

    *Alternate solution:* We can use recursion. For $n \geq 0$, let $a_n$ denote the probability that Paige eventually gets a running sum of $n$, with $a_0 = 1$, $a_1 = \frac{1}{6}$, and $a_n = 0$ for $n < 0$. The recursion step is $a_n = \frac{1}{6}(a_{n-1} + a_{n-2} + \ldots + a_{n-6})$, by examining the last die roll before obtaining a running sum of $n$. We compute $a_n = \frac{7^{n-1}}{6^n}$ for $1 \leq n \leq 6$. This pattern breaks down at $n = 7$, since we only consider the last six terms of the sequence and not the entire sequence. We can compute $a_7 = \frac{7^6}{6^7} - \frac{1}{6} = \frac{7^6 - 6^6}{6^7}$, then proceed as above.

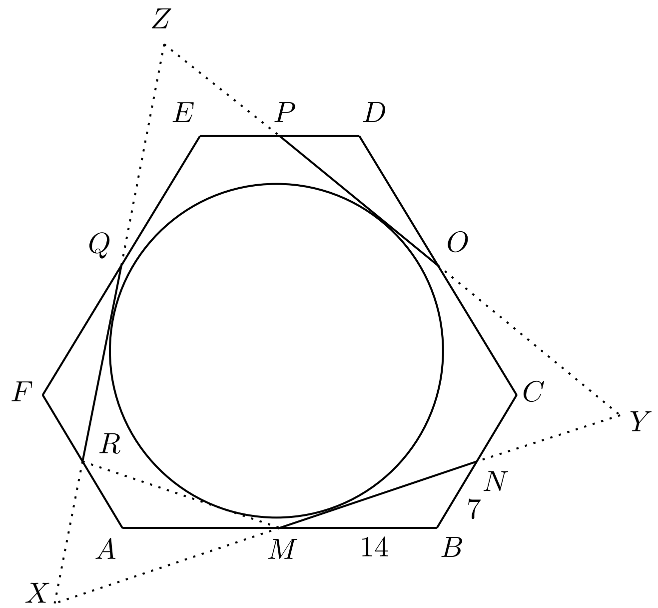
24. **Answer: (C)** First, we observe that $MNOPQR$ is an equilateral hexagon; note that $MN$ is the side opposite the $120^\circ$ angle in $\triangle BMN$.

To find $MN$, we can either use the law of cosines on $\triangle BMN$, or drop an altitude from $N$ onto $BM$ and use the Pythagorean theorem. Though trigonometry is not required for this problem, we compute $MN$ using the law of cosines:

$$\begin{aligned} MN^2 &= 7^2 + 14^2 - 2 \cdot 7 \cdot 14 \cos 120^\circ \\ &= 7^2 + 14^2 + 7 \cdot 14 \\ &= 343 \end{aligned}$$

Then $MN = 7\sqrt{7}$. Similarly, $NO = OP = PQ = QR = RM = 7\sqrt{7}$, so hexagon $MNOPQR$ is equilateral. However, it is *not* regular as $\angle MNB = \angle ONC \neq 30^\circ$.

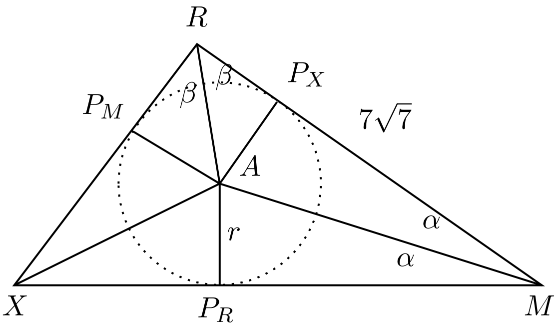
Extend $\overline{MN}$, $\overline{OP}$, and $\overline{QR}$ to form triangle $XYZ$ as shown in the figure below. Then $\omega$ is the incircle of this triangle, as it is tangent to the sides of $\triangle XYZ$. Consider triangle $RMX$:




Let $\angle RMA = \alpha$ and $\angle MRA = \beta$, where $\alpha + \beta = 60^\circ$. Because $\triangle RMA \cong \triangle NMB$ by SSS congruence, we have $\angle AMX = \angle NMB = \alpha$, and similarly $\angle ARX = \angle FRQ = \beta$. Then $A$ is the intersection of two angle bisectors in $\triangle RMX$, so $A$ is the incenter of $\triangle RMX$. Using this, we establish $\angle RXM = 180^\circ - 2\alpha - 2\beta = 180^\circ - 2(60^\circ) = 60^\circ$, and similarly $\angle NYO = \angle PZQ = 60^\circ$, so $\triangle XYZ$ is equilateral. It suffices to find the side length of $\triangle XYZ$, since we can compute the inradius easily from there.

Recall that $MN = RM = 7\sqrt{7}$. Using the congruence $\triangle RXM \cong \triangle NYO$, we see that $NY = RX$, and that the side length of $\triangle XYZ$ equals the perimeter of $\triangle RMX$. Since $RM = 7\sqrt{7}$, it suffices to find $RX + XM$.

Let $P_M$, $P_R$, and $P_X$ be the points where the incircle of $\triangle RMX$ is tangent to $\overline{RX}$, $\overline{XM}$, and $\overline{RM}$, respectively:




We first compute the inradius $r$ of $\triangle RMX$. Fortunately this is not hard to find, as the inradius is simply the altitude from $A$ to $RM$ in $\triangle RAM$. We can use one of many methods to find $[\triangle RAM]$ (either Heron's formula, $\frac{1}{2}ab\sin C$, or dropping an altitude from $M$ to $AR$), obtaining

$$[\triangle RAM] = \frac{49\sqrt{3}}{2} = \frac{1}{2}(7\sqrt{7})r.$$

Solving for $r$ yields $r = \sqrt{21}$.

Notice that $P_MR = RP_X$ and $P_RM = MP_X$, so $P_MR + P_RM = 7\sqrt{7}$. Further, since $\triangle AP_RX$ and $AP_MX$ are 30-60-90, we have $P_MX = P_RX = r\sqrt{3} = 3\sqrt{7}$. It follows that the perimeter of $\triangle RMX$ is $7\sqrt{7} + 7\sqrt{7} + 2(3\sqrt{7}) = 20\sqrt{7}$. Therefore the side length of $\triangle XYZ$ is $20\sqrt{7}$.

Using 30-60-90 triangles, the inradius of an equilateral triangle with side length $20\sqrt{7}$ is $\frac{10\sqrt{21}}{3}$, so the area of $\omega$ is $\left(\frac{10\sqrt{21}}{3}\right)^2 \pi = \boxed{\textbf{(C) } \frac{700\pi}{3}}$.

*Alternate solution:* Extend $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ to form an equilateral triangle of side length $28 + 14 + 14 = 56$. By a rotational symmetry argument, the center of the circle is the incenter of the equilateral triangle.

Assign coordinates $M = (0, 0)$, $B = (14, 0)$. Using 30-60-90 triangles, the coordinates of the center are $\left(0, \frac{28\sqrt{3}}{3}\right)$. We can easily find the coordinates of $N$ to be $\left(\frac{35}{2}, \frac{7\sqrt{3}}{2}\right)$. The equation of line $\overline{MN}$ is therefore

$y = \frac{\sqrt{3}}{5}x$, or equivalently $\sqrt{3}x - 5y = 0$. Using the formula for the distance from a point to a line, we obtain

$$\text{radius of } \omega = \frac{|0 \cdot \sqrt{3} - 5 \cdot \frac{28\sqrt{3}}{3}|}{\sqrt{3 + 5^2}} = \frac{10\sqrt{21}}{3}.$$

Similarly as above, the area of $\omega$ is $\frac{700\pi}{3}$.

25. **Answer: (E)** We will consider the problem in binary. The binary representation of 2020 is $11111100100_2$.

Let $f(n)$ denote the number of tuples of the form $(a_0, a_1, \dots, a_k)$ such that $\sum_{i=0}^{k} a_i 2^i = n$ and $a_i \in \{0, 1, 2\}$. The desired answer is $f(2020)$ (note that $2^{11} > 2020$, so the maximum possible $k$ is 10). To represent the tuples, we will express them in a way similar to binary, except that digits may be 0, 1, or 2 (for example, 102 corresponds to $a_2 = 1$, $a_1 = 0$, $a_0 = 4$):

| $n$ | $f(n)$ | Solutions | $n$ | $f(n)$ | Solutions |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 9 | 3 | 1001, 201, 121 |
| 2 | 2 | 2, 10 | 10 | 5 | 1010, 1002, 210, 202, 122 |
| 3 | 1 | 11 | 11 | 2 | 1011, 211 |
| 4 | 3 | 100, 12, 20 | 12 | 5 | 1100, 1020, 220, 1012, 212 |
| 5 | 2 | 101, 21 | 13 | 3 | 1101, 1021, 221 |
| 6 | 3 | 110, 102, 22 | 14 | 4 | 1110, 1102, 1022, 222 |
| 7 | 1 | 111 | 15 | 1 | 1111 |
| 8 | 4 | 1000, 200, 120, 112 | 16 | 5 | 10000, 2000, 1200, 1120, 1112 |

We observe some patterns regarding $f$. First, we see that $f(2^k - 1) = 1$ (the only solution is $11\dots1$), and $f(2^k) = k + 1$. We can state these claims more generally:

**Lemma 1.** *The following are true for any integer $n$:*

$$f(2n) = f(n) + f(n-1) \tag{1}$$
$$f(2n+1) = f(n) \tag{2}$$

*Proof.* Note that this "binary" representation of $2n$ can be obtained by appending a 0 to any representation of $n$, or by appending a 2 to any representation of $n - 1$ (which proves (1)), and a "binary" representation of $2n + 1$ must be obtained by appending a 1 to any representation of $n$. □

**Corollary 2.** $f(2^k - 2) = k$.

This can be shown inductively with (1).

**Corollary 3.** $f(4n) = f(n) + 2f(n-1)$.

This can be shown by applying (1) then (2).

Note that $2020 = 11111100100_2$. Using the above Lemma and corollaries, we compute $f(2020)$ successively:

$$\begin{aligned}
f(63) &= f(111111_2) = 1 \\
f(252) &= f(11111100_2) = f(63) + 2f(62) \\
&= 1 + 12 = 13 \\
f(505) &= f(111111001_2) = f(252) = 13 \\
f(2020) &= f(11111100100_2) = f(505) + 2f(504) \\
&= 13 + 2(f(252) + f(251)) \\
&= 13 + 2(13 + f(62)) && ((2),\ f(251) = f(125) = f(62)) \\
&= 13 + 2(13 + 6) = \boxed{\textbf{(E) } 51}
\end{aligned}$$