Hnkevin42

Mock

AMC 12

American Mathematics Contest 12

Due: Sunday, January 31st, 2016

INSTRUCTIONS

1) You will have **75 minutes** to complete the test.
2) The problems will begin on the second page of this document. Do not look at that page or the following pages until you have started a timer for 75 minutes.
3) This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
4) Send me a PM of your answers when you're finished, clearly indicating which answer (including any omit) is of which problem. I'll PM you back your score as well the problems you got right and wrong. A running distribution of scores and item difficulty will be provided on the contest thread, but there will be no leaderboard.
5) SCORING: You will receive 6 points for each correct answer, 1.5 problems for each problem left unanswered, and 0 points for each incorrect answer.
6) No aids are permitted other than pencils, scratch paper, graph paper, rulers, compasses, protractors, and erasers. No calculators are allowed, and no problems on the test will *require* the use of a calculator.
7) IF YOU FIND A TYPO THAT INTERFERES WITH YOUR ABILITY TO ANSWER A PROBLEM CORRECTLY, please STOP your timer and PM me what the typo is. I will do my best to respond quickly and correct the typo in the main thread of the mock contest if the typo actually is detrimental to solving the problem.

All problems were written by me. Whether I will release other mock tests is up to popular demand, and when that will be depends on how busy I am for the rest of this academic year. In the case that I release a mock AIME, the mock AIME would be open to everybody that can see the corresponding thread on AoPS. Please do not discuss the problems in the time frame of the competition, as that seriously jeopardizes the integrity of the results as well as the preparation process of people who wish to score well on the actual AMC 12. Despite that, have fun, and best of luck! One last thing: **I'd like to thank Librian2000 for contributing to this test. If it wasn't for him, the test-making process would have been much harder.**

1) Compute $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ in simplest form.

(A) $\frac{4}{7}$ (B) $\frac{25}{42}$ (C) $\frac{13}{21}$ (D) $\frac{27}{42}$ (E) $\frac{2}{3}$

2) James and Cal start at the same point on a linear track that is 12 miles long. When they start running, James runs at a constant rate of 4 miles per hour while Cal runs at a constant rate of 6 miles per hour. However, at the end of the track is a wall, and when Cal reaches the wall, he turns around and runs back along the track to the starting point. After how many hours will Cal and James meet?

(A) 2 (B) $\frac{7}{3}$ (C) $\frac{12}{5}$ (D) $\frac{11}{4}$ (E) 3

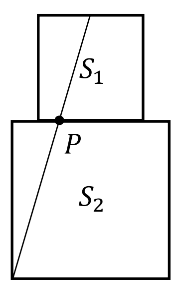
3) Janet rolls two fair dice labeled from 1 to 6. What is the probability that the two numbers on the top faces of the two dice sum to a multiple of 3 or 4?

(A) $\frac{7}{18}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{19}{36}$ (E) $\frac{5}{9}$

4) Mike's high school class weights grades as follows: tests are worth 40% of the overall grade, projects are worth 25% of the grade, quizzes are worth 20% of the grade, and homework is worth 15% of the grade. Mike knows his final homework grade is 100% and his final test grade is 90%. If Mike's quiz grade is also 90%, what is the lowest his project grade can be in order to ensure that his overall grade is at least 90%?

(A) 84 (B) 85 (C) 86 (D) 87 (E) 88

5) A square S_1 of side length 4 sits on top of a square S_2 of side length 6. The line segment connecting the centers of the two squares are vertical. Another segment connects the midpoint of the top side of S_1 to the bottom left corner of S_2 and hits the bottom side of S_1 at a point P. What is the distance from the bottom left corner of S_1 to P?



(A) $\frac{3}{5}$ (B) $\frac{13}{20}$ (C) $\frac{7}{10}$ (D) $\frac{3}{4}$ (E) $\frac{4}{5}$

6) A permutation $\{a_1, a_2, a_3, a_4, a_5\}$ of the integers 1, 2, 3, 4, 5 is called "distant" if it satisfies the rule $|a_{n+1} - a_n| > 1$ for all n from 1 to 4 inclusive. How many distant permutations are there?

(A) 14 (B) 16 (C) 18 (D) 20 (E) 22

7) 10 teams participate in a round-robin tournament, in which every team faces all nine other teams only once. Assuming that all games between two teams can only result in wins or losses, what is the maximum number of teams that can have an odd number of wins?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

8) 30 terms of a real-valued arithmetic sequence A_1 with common difference d sum to 75. Another arithmetic sequence A_2 has 75 terms and common difference $-d$, and the first term in A_2 is the same as the first term in A_1. If all terms in A_2 sum to 30, what is d?

(A) $\frac{21}{515}$ (B) $\frac{21}{340}$ (C) $\frac{21}{225}$ (D) $\frac{21}{71}$ (E) $\frac{21}{70}$

9) Call S the set of all ordered pairs of natural numbers (a, b) satisfying $\log_{10} a + \log_{10} b = 2016$. Let P be the product of the values of ab across all distinct ordered pairs (a, b) in S. Compute $\log_{10} P$. Note: this quantity is the same as $\log_{10}\left(\prod_{(a,b)\in S} ab\right)$.

(A) 2016 (B) 2017^2 (C) $2016 \cdot 2017$ (D) $2016 \cdot 2017^2$ (E) $2017!$

10) A regular hexagon has side length 1. What is the least number of distinct points we can put at the boundary or in the interior of this hexagon to ensure that the area of a nondegenerate triangle formed by some three of these points is less than $\frac{\sqrt{3}}{16}$?

(A) 20 (B) 21 (C) 23 (D) 24 (E) 25

11) Zach is trying to catch a train that will arrive at a random time between 2:00 PM and 3:00 PM, but whenever the train arrives, it waits 10 minutes before leaving, even if it has to leave after 3:00 PM. However, Zach doesn't know what the remaining traffic will be like to the train station, but he knows he will arrive at a random time between 2:00 PM and 3:00 PM and stay until 3:00 PM. What is the probability that Zach is successful in catching his train?

(A) $\frac{43}{72}$ (B) $\frac{11}{18}$ (C) $\frac{5}{8}$ (D) $\frac{23}{36}$ (E) $\frac{47}{72}$

12) In square $ABCD$, points E, F, and G lie on sides AB, BC, and CD, respectively. If triangle EFG has a right angle at F and $ABCD$ has side length 1, what is the maximum area of triangle EFG?

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{3}$ (C) $\frac{3}{5}$ (D) $\frac{2}{3}$ (E) $\frac{\sqrt{2}}{2}$

13) Define $f_1(x) = 1 + \frac{1}{x}$ and $f_n(x) = 1 + \frac{1}{f_{n-1}(x)}$ for all integers $n \geq 2$. Which of the following satisfies $f_{12}(x) = 0$?

(A) $-\frac{147}{231}$ (B) $-\frac{144}{233}$ (C) $-\frac{89}{144}$ (D) $-\frac{3}{5}$ (E) $-\frac{1}{2}$

14) Kevin and Felix are playing a co-op game on a board, which is split into phases. At the beginning of each phase, the number 2016 is written on the board. Either Kevin or Felix can play first, and the players start in phase 1. In each turn of phase k, for $1 \leq k \leq 4$, a player replaces the number N on the board by the integer $\frac{N}{d_k}$, where d_k is a k-digit positive divisor of 2016 excluding 1 and 2016. They cannot use a d_k chosen before in that phase, and when they cannot replace N by an integer, the phase ends and the players move on to phase $k+1$. The last turn happens in phase 4, where a player erases 1008 and replaces 2016 with 2. If Kevin and Felix want to complete the game in the least amount T of turns possible, what is T?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

15) A 3x3 slider puzzle has the following configuration with an open space on the right, where every other square has a number label.

3	7	8
4	5	
6	1	2

Define a "slide" to be a move of a square adjacent to an open space to that open space, with the formerly open space now being completely occupied by the first square. Note that there will now be an open space in the position the first square was last in before it moved. What is the probability that all numbers in the rightmost column will sum to an odd integer after three random slides?

(A) $\frac{7}{12}$ (B) $\frac{5}{8}$ (C) $\frac{2}{3}$ (D) $\frac{17}{24}$ (E) $\frac{3}{4}$

16) Let $D(n)$ be the number of positive divisors of an integer n. What is the sum of all positive integers n less than 100 that satisfy $D(n) + D(n+1) = D(100)$?

(A) 15 (B) 49 (C) 64 (D) 130 (E) 145

17) In the interior a right cone C of height 4 and radius 3, spheres $S_1, S_2, \ldots, S_n$ exist such that S_1 is tangent to the cone's circular base and has the largest possible volume that allows S_1 to fit inside C. For $n \geq 2$, S_n is externally tangent to S_{n-1}, is closer to the cone's vertex than S_{n-1}, and contains the largest possible volume that allows S_n to fit inside of C. Let $V(S_k)$ be the volume of sphere S_k. What is

$$\sum_{k=1}^{\infty} V(S_k)?$$

(A) 4π (B) $\frac{32\pi}{7}$ (C) $\frac{69\pi}{14}$ (D) $\frac{288\pi}{37}$ (E) $\frac{64\pi}{7}$

18) Triangle ABC has $BC = 6, \angle CAB = 60°$, and $\angle BCA = 80°$. Let F be the foot of the perpendicular from C to the angle bisector of $\angle CAB$ and let M be the midpoint of BC. What is the length of MF?

(A) $\sqrt{2}\sin 20°$ (B) $\sqrt{3}\sin 20°$ (C) $2\sin 20°$ (D) $2\sqrt{3}\sin 20°$ (E) $3\sqrt{3}\sin 20°$

19) For $0° \le x \le 30°$, the numbers $\cos x$, $\sin x$, and $\cos 2x$ form an arithmetic sequence, not necessarily in this order. What is the common difference of this arithmetic sequence?

(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt{7}}{8}$ (D) $\frac{\sqrt{2}}{2}$ (E) $\frac{1+\sqrt{7}}{4}$

20) An equilateral triangle exists such that all three of its vertices lie on the same branch of the hyperbola $x^2 - \frac{y^2}{b} = 1$, with one vertex on $(1,0)$. If this triangle's area can be any real number in the interval $[3\sqrt{3}, 12\sqrt{3}]$, then the interval that contains all of the possible values of b is $[x_1, x_2]$ for real numbers x_1, x_2. What is $x_1 + x_2$?

(A) $\frac{3}{10}$ (B) $\frac{1}{3}$ (C) $\frac{3}{7}$ (D) $\frac{9}{20}$ (E) $\frac{1}{2}$

21) A palindrome is an integer that reads the same forwards and backwards with no leading zeroes. How many positive base-10 palindromes less than 2016 are also palindromes in base-6?

(A) 10 (B) 14 (C) 15 (D) 16 (E) 18

22) A monic polynomial is a polynomial whose leading coefficient is 1. The monic quadratic polynomial $P(x)$ exists according to the following conditions.

(i) All coefficients of $P(x)$ are positive integers less than 10.
(ii) All roots of $P(x)$ are integers.
(iii) Two roots of $P(P(x))$ including multiplicities are integers.

Given polynomial $Q(x)$, all roots r_1, r_2, r_3, r_4 of the monic quartic polynomial $Q(Q(x))$ exist such that the roots of $P(P(x))$ are $r_k - n$, for $1 \le k \le 4$ and for some nonzero integer n. How many possible polynomials exist for $Q(x)$?

(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely Many

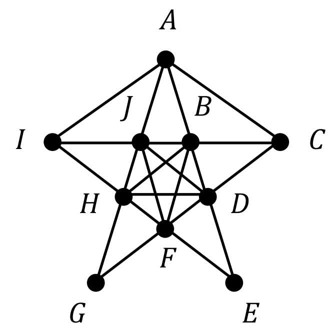
23) Let $f(n)$ be a function from the natural numbers to the natural numbers satisfying $f(1) = 1$ and

$$f(n) = \begin{cases} \prod_{i=1}^{n-1} f(i), \text{if } n \text{ is odd} \\ \sum_{i=1}^{n-1} f(i), \text{if } n \text{ is even.} \end{cases}$$

There is a value n such that $f(n) = 3^k \prod_{i=0}^{30} \left[2^{i+2} + \sum_{j=0}^{i+1} 2^j f(5 + 2i - 2j)\right]^{2^{30-i}}$ for some integer k. What is the remainder when $n + k$ is divided by 1000?

(A) 648 (B) 672 (C) 693 (D) 707 (E) 715

24) Students $A, B, C, D, E, F, G, H, I,$ and J stand on each circle in the network below with initial position shown below so that A is always on the topmost point. Each student must move to a neighboring circle so that after they move, no two students can stand on the same circle and no more than 1 pair of students swap positions. How many different formations of students are possible after they move?



(A) 22 (B) 52 (C) 62 (D) 70 (E) 94

25) Consider the set S of all points (x, y, z) in the three-dimensional coordinate system such that all of x, y, z are integers with absolute value at most 1. Certain groups of three points in S exist such that those points can be placed on a coordinate plane, and on this coordinate plane, a parabola of the form $y = ax^2 + bx + c$ with $a \neq 0$ can be translated and rotated to fit through those points. How many distinct groups of 4 points in S can exist in this manner?

(A) 720 (B) 860 (C) 960 (D) 1080 (E) 1200