2023 Mock AMC 12 Solutions

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1. C
2. C
3. A
4. E
5. C
6. C
7. E
8. E
9. C
10. A
11. D
12. A
13. B
14. D
15. E
16. B
17. B
18. B
19. C
20. B
21. D
22. E
23. D
24. B
25. C

1. What is the value of

$$\frac{2^2 - 0^2 + 2^2 - 3^2}{2 - 0 + 2 - 3} - \frac{2 - 0 + 2 - 3}{2^2 - 0^2 + 2^2 - 3^2}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

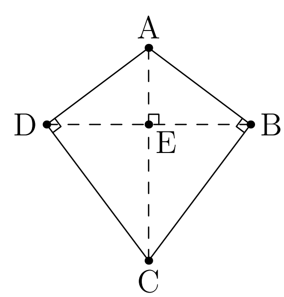
$$\frac{-1}{1} - \frac{1}{-1} = -1 - (-1) = \boxed{\textbf{(C) } 0}$$

2. Kite $ABCD$ has $AB = AD = 15$, $CB = CD = 20$, and $AC = 25$. What is BD?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 25 **(E)** 32

Solution 1: Similar triangles

Let E be the intersection of $\overline{AC}$ and $\overline{BD}$. From the converse of the Pythagorean theorem, we can tell $\angle ABC = \angle ADC = 90°$. Also note that the diagonals of all kites are perpendicular.



We can tell $\triangle AEB \cong \triangle AED \sim \triangle ABC \cong \triangle ADC$, so from similar triangles, $BE = DE = 15 \cdot \frac{20}{25} = 12$. Thus, $BD = BE + DE = 12 + 12 = \boxed{\textbf{(C) } 24}$.

Solution 2: Reverse kite area formula

First, we can find the kite's area as the sum of the areas of $\triangle ABC$ and $\triangle ADC$. From the converse of the Pythagorean theorem, we can tell $\angle ABC = \angle ADC = 90°$. So, the area of one triangle is $\frac{15 \cdot 20}{2} = 150$, and since the triangles are congruent, the area of the kite is $150 \cdot 2 = 300$.

Now we can use the kite area formula, $A = \frac{pq}{2}$, where p and q are the lengths of the kite's diagonals. Substituing $A = 300$ and $p = 25$, we find $q = \boxed{\textbf{(C) } 24}$.

3. Mindy wants to memorize her multiplication facts from 1×1 to 12×12 using a multiplication table that has rows and columns labeled with factors, such that the products form the body of the table. Of the 144 numbers in the body of the table, how many are less than 100?

(A) 133 **(B)** 134 **(C)** 135 **(D)** 136 **(E)** 137

We can tell for any $a \geq 10$ and $b \geq 10$, $ab \geq 100$. This rules out $3 \cdot 3 = 9$ numbers. Also, $9 \times 12 = 12 \times 9 \geq 100$, so this increases the number of ruled out numbers to 11. That makes the answer $144 - 11 = \boxed{\textbf{(A) } 133}$.

4. Five men and five women are standing in a line. Two men see a woman in the spot in front of them. w women see a man in the spot in front of them. What is the sum of all possible values of w?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

We can approach this problem by considering alternating blocks of arbitrary numbers of men and women. Notice these are the only possible configurations, where n_k for $k \geq 1$ represents an arbitrary positive integer that satisfies the premise that in total there are 5 men and 5 women, and a greater k implies a position closer to the front of the line.

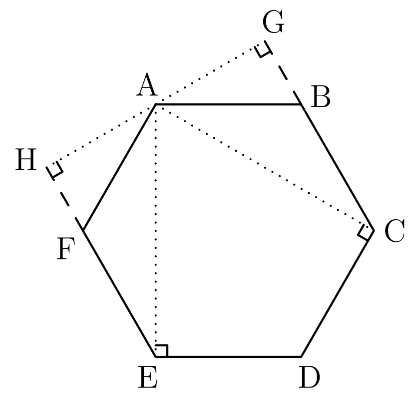
n_1 men — n_2 women — n_3 men — n_4 women $\implies w = 1$.
n_1 women — n_2 men — n_3 women — n_4 men — n_5 women $\implies w = 2$.
n_1 men — n_2 women — n_3 men — n_4 women — n_5 men $\implies w = 2$.
n_1 women — n_2 men — n_3 women — n_4 men — n_5 women — n_6 men $\implies w = 3$.

The answer is $1 + 2 + 3 = \boxed{\textbf{(E) } 6}$.

5. Let $ABCDEF$ be a regular hexagon. If the sum of the distances from A to $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{EF}$ (when all are extended) is $4\sqrt{3}$, what is the side length of the hexagon?

(A) 1 **(B)** $\frac{2\sqrt{3}}{3}$ **(C)** $\frac{4}{3}$ **(D)** $\frac{3}{2}$ **(E)** 2

The shortest distance from A to any of those lines is a perpendicular altitude, as shown. Let s denote the side length of the hexagon.



We can tell that the feet of the altitudes from A to $\overleftrightarrow{CD}$ and $\overleftrightarrow{DE}$ are C and E respectively. This can be proven because the sum of the angles in a triangle equals $180°$, and the angles opposite to equal sides in an isosceles triangle are equal. In order to satisfy both requirements, $\angle ACB = \angle AEF = 30°$, which implies $\angle ACD = \angle AED = 90°$. Based on the 30-60-90 triangles $\triangle ACD$ and $\triangle AED$, we can tell $AC = AE = \sqrt{3}s$.

Let G and H denote the feet of the altitudes from A to $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ respectively. Based on the 30-60-90 triangles $\triangle AGB$ and $\triangle AHF$, we can tell $AG = AH = \frac{\sqrt{3}}{2}s$.

Adding up the distances, the sum of the distances from A to $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{EF}$ is $3\sqrt{3}s$.

Solving $3\sqrt{3}s = 4\sqrt{3}$, $s = \boxed{\textbf{(C) } \frac{4}{3}}$.

6. A Martian analog clock has 3 minutes per hour and 3 seconds per minute. Its hour, minute, and second hands all rotate at constant rates of $\frac{1}{3}$ of a full circle per respective time unit. It is currently Martian noon. The next time two of the angles between the three hands are equal, how many Martian seconds will have elapsed? (An angle only counts if there are no hands between the two hands that form the angle.)

(A) $\frac{27}{16}$ **(B)** $\frac{9}{5}$ **(C)** $\frac{27}{14}$ **(D)** $\frac{9}{4}$ **(E)** $\frac{27}{10}$

Let s be seconds. The second hand moves $\frac{s}{3}$ rotations per second, the minute hand moves $\frac{s}{9}$ rotations per second, and the hour hand moves $\frac{s}{27}$ rotations per second.

Before the second hand makes a full turn at $s = 3$, the angle between the second and minute hand is $\frac{s}{3} - \frac{s}{9}$ rotations, the angle between the minute and hour hand is $\frac{s}{9} - \frac{s}{27}$ rotations, and the angle between the second and hour hand is $1 - \frac{s}{3} + \frac{s}{27}$ rotations. Setting each pair of those expressions equal:

$$\frac{s}{3} - \frac{s}{9} = \frac{s}{9} - \frac{s}{27} \implies \text{no solution for } 0 < s < 3$$
$$\frac{s}{3} - \frac{s}{9} = 1 - \frac{s}{3} + \frac{s}{27} \implies s = \frac{27}{14}$$
$$\frac{s}{9} - \frac{s}{27} = 1 - \frac{s}{3} + \frac{s}{27} \implies s = \frac{27}{10}$$

The smallest of these is $\boxed{\textbf{(C) } \frac{27}{14}}$.

7. Let $P(x)$ be the probability that the sum of the values that show up when two fair 6-sided dice are rolled is equal to x. Let $Q(x)$ be the probability that the sum of the values that show up when a fair 5-sided die and a fair 7-sided die are rolled is equal to x. What is the smallest x for which $P(x) > Q(x)$? (Assume that for an n-sided die, the numbers on the die are integers from 1 through n.)

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Notice that for $x \leq 6$, there are an equal number of ways to roll a sum equal to x between the two 6-sided dice and combination of one 5-sided die and one 7-sided die. For $x = 2$, there's 1 way (1 & 1), for $x = 3$, there's 2 ways (2 & 1, 1 & 2), and so on until $x = 6$, where there's 5 ways (1 & 5, 2 & 4, 3 & 3, 4 & 2, 5 & 1). For the two 6-sided dice, there are 36 possibilities for the numbers on the two dice while for the combination of one 5-sided die and one 7-sided die, there are only 35. Thus, $P(x) < Q(x)$ for $x \leq 6$.

However, for $x = 7$, there are 6 ways to roll for the two 6-sided dice (1 & 6, 2 & 5, 3 & 4, 4 & 3, 5 & 2, or 6 & 1), but only 5 ways to roll the combination of one 5-sided die and one 7-sided die (1 & 6, 2 & 5, 3 & 4, 4 & 3, 5 & 2), where the 6 is rolled on the 7-sided die. Therefore, $P(7) = \frac{6}{36} > Q(7) = \frac{5}{35}$. The answer is $\boxed{\textbf{(E) } 7}$.

8. What is the greatest common divisor of all integers that can be represented as $k^6 - k^2$, where k is a positive integer?

(A) 10 **(B)** 12 **(C)** 20 **(D)** 30 **(E)** 60

Factor $k^6 - k^2$ as

$$k^2(k^4 - 1) = k^2(k^2 + 1)(k^2 - 1) = k^2(k^2 + 1)(k + 1)(k - 1)$$

First, we test divisibility by 3. One of k, $k + 1$, or $k - 1$ must be divisible by 3, so the expression is always divisible by 3.

Next, we test divisibility by 4. If k is even, k^2 is divisible by 4. If k is odd, $k + 1$ and $k - 1$ are both even. Thus, the expression is always divisible by 4.

Finally, we test divisibility by 5. If $k \equiv 0 \pmod 5$, the expression is clearly divisible by 5. If $k \equiv 1 \pmod 5$, $k - 1$ is divisible by 5. If $k \equiv 4 \pmod 5$, $k + 1$ is divisible by 5. Checking the units digit of $k^2 + 1$ for $k \equiv 2 \pmod 5$ and $k \equiv 3 \pmod 5$, we can tell it is always either 0 or 5. Thus the expression is always divisible by 5.

The expression must be divisible by 3, 4, and 5, and therefore must be divisible by 60. Additionally, we can prove this is the greatest common divisor because for $k = 2$, $2^6 - 2^2 = 60$. The answer is $\boxed{\textbf{(E)}\ 60}$.

9. Given a 5×5 grid composed of unit squares, how many ways are there to shade 12 unit squares such that no two shaded unit squares touch along a side? (Rotations and reflections of the same shading are considered separate.)

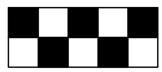
(A) 12 **(B)** 13 **(C)** 14 **(D)** 25 **(E)** 26

We can see that if you make a checkerboard with each corner shaded, 13 squares are shaded. This is 1 more than we need, so you can unshade any of the 13 squares, forming 13 ways.

You can also make a checkerboard with each corner unshaded, where there are 12 shaded squares. This is 1 more way.

It is quite clear there are no other ways, but to prove it, consider that each row can only have up to 3 shaded squares, and this is the only such configuration:

Also, if a row has 3 shaded squares, both rows adjacent to it can only have up to 2 shaded squares, and this is the only such configuration:



Logically, these are the only possibilities for the number of shaded squares in each row, and the number of ways in each case:

3, 2, 3, 2, 2 (3 ways)
3, 2, 2, 2, 3 (3 ways)
2, 2, 2, 2, 3 (3 ways)
3, 2, 3, 1, 3 (2 ways)
3, 1, 3, 2, 3 (2 ways)
2, 3, 2, 3, 2 (1 way)

The answer is $13 + 1$ or $3 + 3 + 3 + 2 + 2 + 1 = \boxed{\textbf{(C) } 14}$.

10. Which of the following numbers yields the smallest remainder when divided by 2023?

(A) 1978^2 **(B)** 1979^2 **(C)** 1980^2 **(D)** 1981^2 **(E)** 1982^2

Represent the answer choices as $(2023 - x)^2 = 2023^2 - 2 \cdot 2023 \cdot x + x^2$, where clearly, $x^2 \equiv (2023 - x)^2 \pmod{2023}$. So it suffices to examine which of 45^2, 44^2, 43^2, 42^2, and 41^2 have the smallest remainder when divided by 2023.

Observe that $2025 = 45^2 > 2023 > 44^2$. Thus, 45^2 yields a remainder of $2025 - 2023 = 2$, while the other 4 numbers leave the same remainder as their respective values. Clearly, $2 < 41^2$, so 45^2 has the smallest remainder. This corresponds to $\boxed{\textbf{(A) } 1978^2}$.

NOTE: Alternatively, you could use modular arithmetic to determine $n^2 \equiv (n - 2023)^2 \pmod{2023}$.

11. Hamza has 2023 cows and 119 of them are infected with the Cow-ronavirus. He tested a randomly chosen cow for the disease. The test is 90% accurate on cows who are infected and 95% percent accurate on cows who are healthy. If the test result shows healthy, what is the probability the cow is actually infected?

(A) $\frac{1}{170}$ **(B)** $\frac{1}{161}$ **(C)** $\frac{1}{160}$ **(D)** $\frac{1}{153}$ **(E)** $\frac{1}{152}$

There are two possible causes of a healthy test result: an accurate test on a healthy cow and an inaccurate test on an infected cow. Using the conditional probability formula, we wish to find the probability that it is an inaccurate test on an infected cow:

$$\frac{\text{P(inaccurate test on infected cow)}}{\text{P(accurate test on healthy cow)} + \text{P(inaccurate test on infected cow)}}$$

$$\frac{\frac{10}{100}\cdot\frac{119}{2023}}{\frac{95}{100}\cdot\frac{2023-119}{2023}+\frac{10}{100}\cdot\frac{119}{2023}} = \boxed{\textbf{(D)}\ \frac{1}{153}}$$

12. Let the base n representation of positive integer k be $\underline{a}\,\underline{b}_n$ for some positive integers $a \geq 2$, $b \geq 1$, and $n \leq 10$. Suppose the base is increased by 2, the digits are decreased by 1, and the numerical value remains k. How many possible k are there?

(A) 18 **(B)** 20 **(C)** 21 **(D)** 24 **(E)** 25

Observe that $k = n \cdot a + b$. We can solve

$$n \cdot a + b = (n+2)(a-1) + (b-1)$$

arriving at $2a = n+3$. Also note the restrictions $a < n$ and $b < n$ because neither digit could be greater than the base.

If $n = 1$, $a = 2$ which is invalid because $a < n$ and $a \geq 2$.
If $n = 3$, $a = 3$ which is invalid because $a < n$.
If $n = 5$, $a = 4$ with 4 possibilities for b ($b = 0$ doesn't work.).
If $n = 7$, $a = 5$ with 6 possibilities for b.
If $n = 9$, $a = 6$ with 8 possibilities for b.

The total answer is $4 + 6 + 8 = \boxed{\textbf{(A)}\ 18}$.

13. Suppose positive integer k satisfies $\operatorname{lcm}(44100, k) = k \cdot \gcd(44100, k) \neq 44100$. What is the sum of the digits of the smallest possible value of k?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 15

Solution 1: Observations

Recall the prime factorization of $\operatorname{lcm}(44100, k)$ is the union of the prime factorizations of 44100 and k. Recall the prime factorization of $\gcd(44100, k)$ is the intersection of the prime factorizations of 44100 and k.

First, we find the prime factorization of 44100 as $2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$. We can tell that outside of satisfying the requirement that neither side of the equation equals 44100, $k = 2 \cdot 3 \cdot 5 \cdot 7 = 210$ works.

Secondly, if the prime factorization of k lacks any of $\{2, 3, 5, 7\}$, it would not work because $\operatorname{lcm}(44100, k)$ would be divisible by the missing prime factor(s) while $k \cdot \gcd(44100, k)$ would not.

Thirdly, the prime factorization of k cannot have any higher powers of $\{2, 3, 5, 7\}$. If it has n powers of prime factor $p \in \{2, 3, 5, 7\}$, where $n > 1$, $\operatorname{lcm}(44100, k)$ has n powers of p in its prime factorization, while $k \cdot \gcd(44100, k)$ has $n + 2$ powers.

Finally, we can try multiplying k by a prime other than $\{2, 3, 5, 7\}$. The smallest is 11, which satisfies our requirements as when $k = 210 \cdot 11 = 2310$, $\operatorname{lcm}(44100, k) = k \cdot \gcd(44100, k) = 44100 \cdot 11$.

The requested sum is $2 + 3 + 1 + 0 = \boxed{\textbf{(B) } 6}$.

Solution 2: LCM and GCD identity

The identity $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = ab$ holds for all positive integers a and b.

Using this identity and the given equation,

$$\operatorname{lcm}(44100, k)\gcd(44100, k) = 44100k \implies k(\gcd(44100, k))^2 = 44100k$$
$$\implies (\gcd(44100, k))^2 = 44100 \implies \gcd(44100, k) = 210$$

Thus, $k = 210m$ for some integer m. Since $\gcd(ac, bc) = c \cdot \gcd(a, b)$ for all positive integers a, b, and c,

$$\gcd(44100, 210m) = 210 \implies 210 \cdot \gcd(210, m) = 210 \implies \gcd(210, m) = 1.$$

Furthermore, since $\text{lcm}(ac, bc) = c \cdot \text{lcm}(a, b)$ for all positive integers a, b, and c,

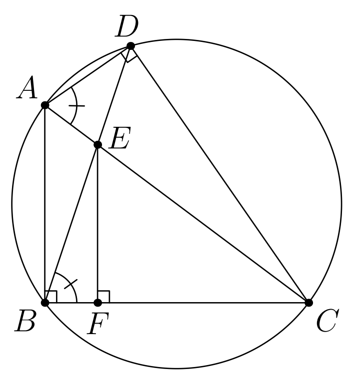
$$\text{lcm}(44100, k) \neq 44100 \implies \text{lcm}(44100, 210m) \neq 44100 \implies 210 \cdot \text{lcm}(210, m) \neq 44100$$
$$\implies \text{lcm}(210, m) \neq 210,$$

To minimize k, it suffices to minimize m. Since $\text{lcm}(210, m) \neq 210$, m cannot be 1. Because $\gcd(210, m) = 1$, m cannot contain a factor of 2, 3, 5, or 7. The smallest such m is the next prime after 7, which is 11. This satisfies all necessary conditions. So. $k = 210 \cdot 11 = 2310$, for which the requested sum is $2 + 3 + 1 + 0 = \boxed{\textbf{(B) } 6}$.

14. Quadrilateral $ABCD$ has $\angle ABC = \angle ADC = 90°$, $AB = 3$, and $BC = 4$. Let the point of intersection of $\overline{AC}$ and $\overline{BD}$ be E. If $AE = 1$, the area of quadrilateral $ABCD$ can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 19 **(B)** 29 **(C)** 32 **(D)** 43 **(E)** 54

By Pythagorean Theorem on $\triangle ABC$, $AC = 5$. Since $AE = 1$, $EC = 4$. Let F be the foot of the altitude from E to BC. Since $\angle ABC + \angle ADC = 180°$, quadrilateral $ABCD$ is cyclic. Thus, $\angle DAC = \angle DBC$.



Since EF and AB are both perpendicular to BC, the two lines are parallel. Hence, $\triangle ABC \sim \triangle EFC$. By the similarity, $FE = \frac{12}{5}$ and $FC = \frac{16}{5}$. This implies $FB = \frac{4}{5}$.

By AA similarity, $\triangle BFE \sim \triangle ADC$. Thus, $DA : DC = 1 : 3$. By Pythagorean theorem on $\triangle ADC$, $DA^2 + (3DA)^2 = 5 \implies DA = \frac{\sqrt{10}}{2}$ and $DC = \frac{3\sqrt{10}}{2}$.

Hence, the area of the quadrilateral is $\frac{3\cdot4}{2}+\frac{1}{2}\cdot\frac{\sqrt{10}}{2}\cdot\frac{3\sqrt{10}}{2}=\frac{39}{4}$. The requested sum is $39+4=\boxed{\textbf{(D) } 43}$.

15. What of the following values of x causes the following expression to evaluate to an integer?

$$6^{\log_6 12}12^{\log_6 12}18^{\log_6 12}24^{\log_6 12}x^{\log_6 12}$$

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 54

Due to the identity $x^{\log_b y}=y^{\log_b x}$, which we can prove as follows:

$$x^{\log_b y}=b^{\log_b x\log_b y}=y^{\log_b x}$$

We have:

$$12^{\log_6 6}12^{\log_6 12}12^{\log_6 18}12^{\log_6 24}12^{\log_6 x}$$

Then, we combine the exponents:

$$12^{\log_6 6+\log_6 12+\log_6 18+\log_6 24+\log_6 x}$$

$$12^{\log_6(6\cdot12\cdot18\cdot24\cdot x)}$$

From which we can see that if $6\cdot12\cdot18\cdot24\cdot x$ is a power of 6, the expression evaluates to an integer. The only answer choice that works is $x=\boxed{\textbf{(E) } 54}$.

16. Given randomly chosen real numbers h and k within the interval $[-1,1]$, let P be the probability that the following system of equations has no real solutions. Which of the following is closest to P? (NOTE: $\sqrt{2}\approx1.414$ and $\sqrt{3}\approx1.732$)

$$x^2+y^2=4$$

$$|x-h|+|y-k|=\sqrt{2}$$

(A) 0.305 **(B)** 0.315 **(C)** 0.325 **(D)** 0.335 **(E)** 0.345

First, we can see that $x^2+y^2=4$ forms a circle with radius 2 centered at the origin while $|x-h|+|y-k|=\sqrt{2}$ forms a square centered at (h,k) with diagonals parallel to the x and y axes and a side length of 2.

As given in the problem, h and k must satisfy $\max(|h|,|k|) \leq 1$. We can rotate both the square $|x-h|+|y-k| = \sqrt{2}$ and the square region $\max(|h|,|k|) \leq 1$ by $45°$ to simplify the problem. This makes our new system of equations the following:

$$x^2 + y^2 = 4$$

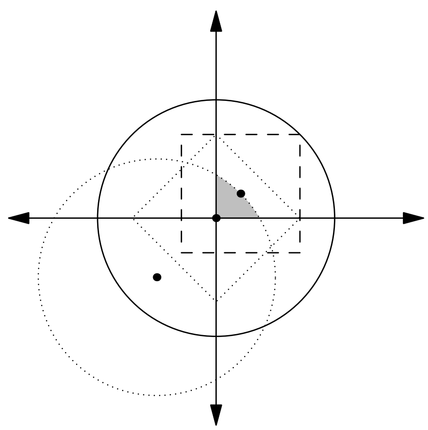
$$\max(|x|,|y|) = 1$$

And we are looking at randomly chosen x and y within the area $|x|+|y| \leq \sqrt{2}$.

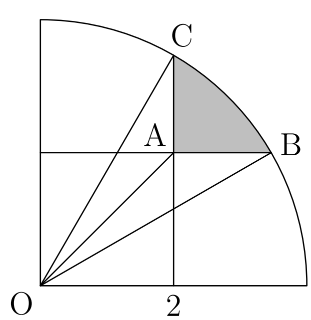
Now we can see that within Quadrant I, the area where there are no solutions consists of

$$(x+1)^2 + (y+1)^2 \leq 4$$

This is well within the bound of $|x|+|y| \leq \sqrt{2}$. The other quadrants are symmetrical, so we can simply multiply by 4 to obtain the final area. Here is a diagram, with the dashed square being an example of a square that is "right on the edge".



Now we need to examine the upper-right quadrant of the circle centered at $(-1,-1)$. Let the center of the circle be O, the origin be A, and the endpoints of the shaded area on the arc of the quadrant be B and C. Draw in $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$.



We can see that the shaded area is equal to the area of sector BOC minus two times the area of $\triangle OAB \cong \triangle OAC$. We can tell $\overset{\frown}{BC} = \arccos\left(\frac{1}{2}\right) - \arcsin\left(\frac{1}{2}\right) = 60° - 30° = 30°$. That means the area of sector BOC is ${}^2\pi \cdot \frac{30°}{360°} = \frac{\pi}{3}$. The area of $\triangle OAB \cong \triangle OAC$ is $\frac{\sqrt{3}-1}{2}$. This gives a total area of

$$\frac{\pi}{3} - 2\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{3} + 1 - \sqrt{3}$$

There are 4 separate quadrants so the total area is 4 times the calculated area. However, the area bounded by $|x| + |y| \leq \sqrt{2}$ is also 4, so the probability we're looking for is $\frac{\pi}{3} + 1 - \sqrt{3}$, which is approximately $\boxed{\textbf{(B)}\ 0.315}$.

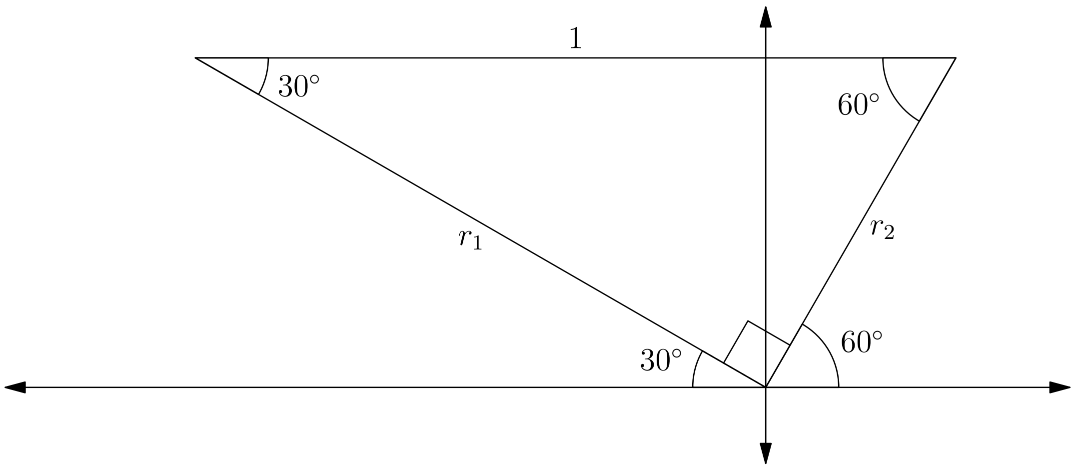
17. Let z be a complex number that satisfies $\text{Im}(z^6) = \text{Im}((z+1)^3) = 0$, $\text{Im}(z^m) \neq 0$ for all integers $0 < m < 6$, and $\text{Im}((z+1)^n) \neq 0$ for all integers $0 < n < 3$. What is the greatest possible value of z^6?

(A) -27 **(B)** $-\dfrac{27}{64}$ **(C)** $\dfrac{27}{64}$ **(D)** 1 **(E)** 27

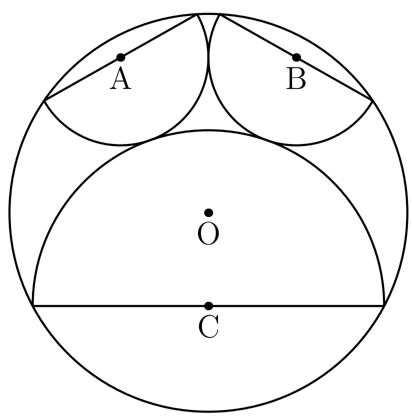
Consider z to have radius r_1 and angle θ_1 in polar coordinates. From DeMoivre's formula, $z^6 = r_1^6(\cos(6\theta_1) + i\sin(6\theta_1))$. Since the imaginary part of z^6 is 0, it follows that $6\theta_1$ is a multiple of $180°$, so θ_1 is a multiple of $30°$. However, if θ_1 is a multiple of $60°$, then $\text{Im}(z^3) = 0$, violating the given conditions. Similarly, if θ_1 is a multiple of $90°$, then $\text{Im}(z^2) = 0$, violating the given conditions. Hence, $\theta_1 \in \{\pm 30°, \pm 150°\}$.

Consider $z + 1$ to have radius r_2 and angle θ_2 in polar coordinates. From DeMoivre's formula again, we can tell $(z+1)^3 = r_2^3(\cos(3\theta_2) + i\sin(3\theta_2))$. We can tell $3\theta_2$ is a multiple of $180°$, so θ_2 is a multiple of $60°$. If θ_2 is a multiple of $180°$, $\text{Im}(z+1) = 0$, violating the given conditions. Hence, $\theta_2 \in \{\pm 60°, \pm 120°\}$.

If $\theta_1 = \pm 30°$, $z + 1$ would not have an valid angle. Therefore $\theta_1 = \pm 150°$. $6\theta_1 = \pm 900° = 180°$, and from DeMoivre's formula, $r_1^6(\cos(180°) + i\sin(180°)) = -r_1^6$. This means z^6 must be negative, so we are trying to find the least negative value, which is obtained from the lowest value of r_1. We can see that the smaller r_1 value is obtained when $\theta_2 = \pm 60°$, where $r_1 = \frac{\sqrt{3}}{2}$. $z^6 = -r_1^6 = \boxed{\textbf{(B)}\ -\dfrac{27}{64}}$.

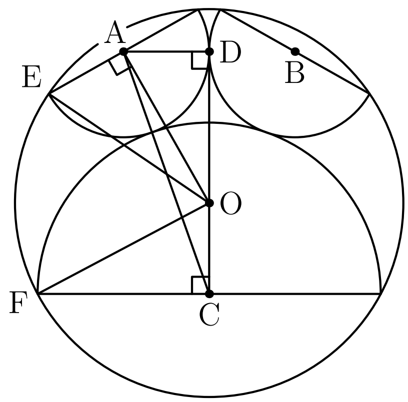


18. In the following diagram, semicircles A and B have radii of 1 while semicircle C has a radius of 2. The endpoints of the diameters of semicircles A, B, and C lie on circle O. In addition, semicircles A, B, and C are pairwise externally tangent to each other. The radius of circle O can be written in the form $\frac{\sqrt{x}}{y}$, where x and y are positive integers and y is as small as possible. What is $x+y$?



(A) 49 **(B)** 86 **(C)** 134 **(D)** 191 **(E)** 314

Let point D be the point of tangency between semicircles A and B. Let point E be one of the endpoints of the diameter of semicircle A. Let point F be one of the endpoints of the diameter of semicircle C. Draw in $\overline{DA}$, $\overline{DC}$, $\overline{AC}$, $\overline{OA}$, $\overline{OE}$, and $\overline{OF}$. Based on symmetry, we know that $\angle OCF = \angle ODA = \angle OAE = 90°$.



Let r be the radius of circle O. Then, $r = OF = OE$. From Pythagorean Theorem on $\triangle OCF$, it follows that $OC = \sqrt{r^2 - 4}$. From the tangency of semicircles A and C, $AC = 3$. Also, $AD = 1$. From Pythagorean Theorem on $\triangle ADC$, $DC = \sqrt{8}$. Therefore, $OD = \sqrt{8} - \sqrt{r^2 - 4}$. Applying Pythagorean theorem a third time to $\triangle ODA$ implies

$$OA = \sqrt{(\sqrt{8} - \sqrt{r^2 - 4})^2 + 1} = \sqrt{r^2 - 2\sqrt{8r^2 - 32} + 5}.$$

Applying it a fourth and final time to $\triangle OAE$, it follows that $OE = \sqrt{r^2 - 2\sqrt{8r^2 - 32} + 6}$.

But $OE = r$ from earlier. Thus,

$$\begin{aligned} \sqrt{r^2 - 2\sqrt{8r^2 - 32} + 6} = r &\implies r^2 - 2\sqrt{8r^2 - 32} + 6 = r^2 \\ &\implies 3 = \sqrt{8r^2 - 32} \\ &\implies 8r^2 = 41 \\ &\implies r = \frac{\sqrt{82}}{4} \end{aligned}$$

The requested answer is $\boxed{\textbf{(B) } 86}$.

19. Anita used a 4-digit combination lock to lock her bicycle, with each digit ranging from 0 to 9 and leading zeros permitted. However, she did not do a good job of scrambling the digits.

    A *turn* is defined as increasing a particular digit by 1, except that turning a 9 would make it a 0. Anita had the correct combination, and then turned each digit either $n-1$, n, or $n+1$ times, where n is a positive integer. If the lock reads “2023”, how many possible correct combinations are there? (For example, “2112” is a possible correct combination, where $n = 10$ and the digits are turned 10, 9, 11, and 11 times respectively.)

    (A) 630 **(B)** 640 **(C)** 650 **(D)** 660 **(E)** 670

    Let a be the number of times the first digit is turned. Define b, c, and d similarly for the second, third, and fourth digits. Let $m = \min(a, b, c, d)$. For a positive integer n to exist as defined, each of a, b, c, d must be equal to one of m, $m+1$, or $m+2$. Also, by the minimality of m, at least one of a, b, c, or d must be equal to m.

    Turning a digit 10 times will yield the exact same digit. Thus, if $m \geq 10$, then $(a-10, b-10, c-10, d-10)$ will correspond to the exact same combination as (a, b, c, d) and a positive integer n will still exist. Thus, it can be assumed that $0 \leq m \leq 9$. Let $M = \max(a, b, c, d)$.

    Case 1: $M = m$

    Then, $a = b = c = d = m$. There are 10 ways to choose m, corresponding to 10 different combinations.

    Case 2: $M = m + 1$

    Then, $m \leq a, b, c, d \leq m + 1$ with at least one variable being equal to m and at least one being equal to $m + 1$.

    There are $2^4 = 16$ ways to assign the value m or $m + 1$ to each of a, b, c, d. But this overcounts when each variable is equal to m and when each variable is equal to $m + 1$. So there are $16 - 2 = 14$ valid ways to assign values to each variable. Since there are 10 choices for m, there are $14 \cdot 10 = 140$ different combinations.

Case 3: $M = m + 2$

Then, $m \le a, b, c, d \le m + 2$ with at least one variable being equal to m and at least one being equal to $m + 2$.

There are $3^4 = 81$ ways to assign one of three values to each variable. But in $2^4 = 16$ of these ways, none of the variables are equal to m. Similarly, 16 of these ways have none of the variables equal to $m + 2$. In 1 way, none of the variables are equal to m or $m + 2$. Thus, by Principle of Inclusion/Exclusion, there are

$$81 - 16 - 16 + 1 = 50$$

ways to appropriately assign each variable a value.

Since there are 10 choices for m, there are $50 \cdot 10 = 500$ different combinations.

The total number of combinations is $10 + 140 + 500 = \boxed{\textbf{(C) } 650}$.

20. Buildings B_1, B_2, B_3, B_4, and B_5 have positive height. For $2 \le n \le 5$, building B_n is $4^{n-2}k$ times the height of building B_{n-1}, where k is some positive real constant. If building B_x has the same height as the average height of the five buildings, how many of those five buildings could B_x possibly be?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

For all integers $1 \le n \le 5$, let h_n be the height of building n. Without loss of generality, assume $h_1 = 1$. By the given information, $h_n = 4^{n-2}k \cdot h_{n-1}$ for $2 \le n \le 5$. Thus, $h_2 = k$, $h_3 = 4k^2$, $h_4 = 4^3k^3$, and $h_5 = 4^6k^4$. Setting the average height of the buildings equal to h_x implies

$$\frac{4^6k^4 + 4^3k^3 + 4k^2 + k + 1}{5} = h_x \implies 4^6k^4 + 4^3k^3 + 4k^2 + k + 1 - 5h_x = 0.$$

For the average to equal the height of one of the buildings, the left hand side as a polynomial in k must have at least one positive root.

If $x = 1$, then the polynomial is

$$4^6k^4 + 4^3k^3 + 4k^2 + k - 4.$$

Since this polynomial has 1 sign change, by Descartes' Rule of Signs, the polynomial has 1 positive root in k. Thus, the average can equal h_1.

Similarly, if $x = 5$, then the polynomial has 1 sign change:

$$-4^7k^4 + 4^3k^3 + 4k^2 + k + 1.$$

Thus, the average can equal h_5.

If $x = 2$, then the polynomial is

$$4^6k^4 + 4^3k^3 + 4k^2 - 4k + 1 = 4^6k^4 + 4^3k^3 + (2k-1)^2.$$

It becomes clear that whenever k is positive, the polynomial will always be positive. Thus, the polynomial will have no positive roots.

If $x = 3$, then the polynomial is

$$4^6k^4 + 4^3k^3 - 4^2k^2 + k + 1 = 4^6k^4 + 1 + k(4^3k^2 - 4^2k + 1) = 4^6k^4 + 1 + k(8k-1)^2.$$

Again, whenever k is positive, the polynomial will always be positive. Thus, the polynomial will have no positive roots.

If $x = 4$, then the polynomial is

$$4^6k^4 - 4^4k^3 + 4k^2 + k + 1 = k + 1 + 4k^2(4^5k^2 - 4^3k + 1) = k + 1 + 4k^2(32k-1)^2.$$

Again, whenever k is positive, the polynomial will always be positive. Thus, the polynomial will have no positive roots.

Only 2 of the values of x yielded a polynomial with a positive root in k. So the answer is $\boxed{\textbf{(B) } 2}$.

21. Define an *n-stretchable* integer as a positive, three-digit integer that is evenly divisible by n, and remains evenly divisible by n no matter how many times the middle digit is repeated. For example, 369 is 3-stretchable because 369, 3669, 36669, etc. are all evenly divisible by 3. How many 7-stretchable integers are there?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

Let $m = \underline{abc}$ for digits a, b, and c with $a \neq 0$. Define the sequence

$$(s_1, s_2, s_3, \cdots) = (\underline{abc}, \underline{abbc}, \underline{abbbc}, \cdots),$$

where in s_k, the middle digit appears k times.

For all integers $k \geq 1$,

$$s_k = \underline{a}\,\underbrace{\underline{bb...bb}}_{k\ bs}\,\underline{c} = \underline{a}\,\underbrace{\underline{bbb...bbb}}_{k+1\ bs}+\underline{c}-\underline{b} \implies 10s_k = \underline{a}\,\underbrace{\underline{bbb...bbb}}_{k+1\ bs}\,\underline{0}+10\underline{c}-10\underline{b} = \underline{a}\,\underbrace{\underline{bbb...bbb}}_{k+1\ bs}\,\underline{c}+9\underline{c}-10\underline{b}$$

$$\implies s_{k+1} = 10s_k + 10b - 9c.$$

For m to be 7-stretchable, it's clearly necessary for s_1 and s_2 to be divisible by 7. This implies $\underline{abc}$ is divisible by 7 and by the recurrence, $10b - 9c$ to be divisible by 7.

If those two quantities are divisible by 7, then every term in the sequence is divisible by 7 through inductively applying the recurrence. This implies sufficiency.

For $10b - 9c$ to be divisible by 7, it suffices for $3b - 2c$ to be divisible by 7. Manually listing out all values of $\underline{bc}$,

$$\underline{bc} \in \{00, 07, 70, 77, 15, 85, 23, 93, 31, 38, 46, 54, 62, 69\}.$$

It now remains for $\underline{abc} = 100 \cdot \underline{a} + \underline{bc}$ to be divisible by 7.

Recall that $a \neq 0$. There are 2 possible digits a that are 1 (mod 7) and 2 (mod 7), each. All other residues of $a \pmod 7$ occur only once.

Thus, if $\underline{bc}$ is 3 or 5 (mod 7), there are 2 choices for a so that $\underline{abc}$ is divisible by 7. Otherwise, there is only 1 choice for a.

Only 3 of the values of $\underline{bc}$, namely 31, 38, and 54, are 3 or 5 (mod 7). The remaining 11 values are not. Thus, the answer is $2 \cdot 3 + 11 = \boxed{\textbf{(D) } 17}$.

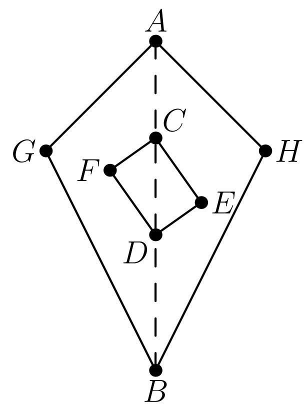
22. Right circular cone α with apex A has a base radius of 17 and a height of 17. Right circular cone β with apex B has a base radius of 17 and a height of 34. If α and β were joined at the base, the maximum possible side length of a cube inside the combined solid with one of its space diagonals on $\overline{AB}$ can be written in the form $\sqrt{m} - \sqrt{n}$ where m and n are positive integers. What is $m + n$?

(A) 441 **(B)** 459 **(C)** 477 **(D)** 495 **(E)** 513

Let $\mathcal{S}$ be the solid formed by joining α and β. Let C and D be vertices on the cube such that $\overline{CD}$ is the space diagonal of the cube lying on $\overline{AB}$. Furthermore, let E be a vertex of the cube such that $\overline{CE}$ is a face diagonal of the cube, and let F be the vertex such that $\overline{EF}$ is a space diagonal of the cube. It can be checked that $CEDF$ is a rectangle.

If s is the side length of the cube, then from the 3D application of the Pythagorean theorem, $CF = DE = s$, $CE = DF = s\sqrt{2}$, and $CD = s\sqrt{3}$. So, rectangle $CEDF$ is fixed up to similarity.

Orient $\mathcal{S}$ such that AB points straight up with A above B. Without loss of generality, orient the cube and label E and F such that these points are in this descending order: C, F, E, then D. Consider the 2D cross section of the cone and cube with the plane containing $CEDF$. $\mathcal{S}$ maps to a kite $AGBH$ as shown below.

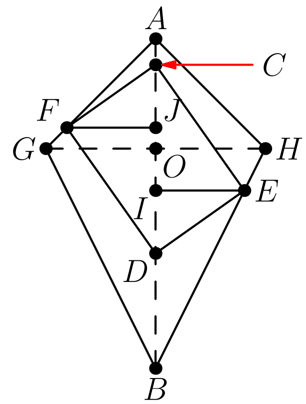


The vertices of the cube besides C and D are the points on the cube that are the furthest from $\overline{AB}$. So, as long as the rectangle is within the kite, the entire cube will be inside the confines of the $\mathcal{S}$. It remains to maximize the rectangle's size to maximize the cube's size, with $\overline{CD}$ on $\overline{AB}$ and the ratio $\frac{FC}{FD}$ fixed.

If the rectangle does not touch the perimeter of the kite, it can be expanded. So at least one of F or E must lie on the kite's perimeter for the size to be optimal.

If E lies on BH but F does not lie on AG, then the rectangle can be translated up by an arbitrarily small distance so that it is still in the confines of the kite but E will no longer be on BH. From there, the rectangle can be expanded. A similar argument applies if F is on AG but E does not lie on BH.

Thus, E must lie on BH and F must lie on AG, since if either is not true, the rectangle is non-optimal as worked out. This yields a unique configuration which must be optimal, since everything else is non-optimal.



O is defined to be the intersection of the diagonals of the kite, whereas I and J are the feet of the altitudes from E and F, respectively, to $\overline{AB}$.

Since the radius of the bases of cones α and β is 17, it follows that $OG = 17$. Also, $OA = 17$ and $OB = 34$ using the heights of the cones.

Using similar triangles $\triangle CJF$ and $\triangle CFD$, it can be deduced that $JF = IE = \frac{s\sqrt{2}}{\sqrt{3}}$ as well as $JC = ID = \frac{s}{\sqrt{3}}$, which implies $JI = \frac{s}{\sqrt{3}}$.

Similar triangles $\triangle AOG$ and $\triangle AJF$ imply $JA = JF = \frac{s\sqrt{2}}{\sqrt{3}}$. Similar triangles $\triangle BOH$ and $\triangle BIE$ imply $IB = 2IE = \frac{2s\sqrt{2}}{\sqrt{3}}$. Then,

$$AJ + JI + IB = AB \implies \frac{s\sqrt{2}}{\sqrt{3}} + \frac{s}{\sqrt{3}} + \frac{2s\sqrt{2}}{\sqrt{3}} = 51 \implies 3s\sqrt{2} + s = 51\sqrt{3}$$

$$\implies s = \sqrt{486} - \sqrt{27}.$$

The requested sum is $486 + 27 = \boxed{\textbf{(E)}\ 513}$.

23. Let $\mathbb{S}$ denote the set of all subsets of the set $\{x, y, z\}$. A function $f : \mathbb{S} \to \mathbb{S}$ satisfies

$$f(A \cup B) = f(A) \cup f(B)$$

for all A and B in $\mathbb{S}$. How many such functions f are possible?

(A) 343 **(B)** 512 **(C)** 585 **(D)** 729 **(E)** 4096

Solution 1: Considering the placement of x, y, and z in $\mathbb{S}$

First, a list of conditions will be made to ensure $f(A \cup B) = f(A) \cup f(B)$ holds for every choice of $A, B \subseteq \mathbb{S}$.

Clearly, it is necessary for the functional relation to hold for every pair of disjoint subsets. Under the assumption it holds for all pairs of disjoint subsets, let A and B be non-disjoint subsets. Then, let $C = A \cap B$, $A' = A \setminus C$ (the set difference), and $B' = B \setminus C$. Note that A', B', and C are pairwise disjoint, but their union is equal to $A \cup B$. Then, $A \cup B = (A' \cup C) \cup (B' \cup C) = (A' \cup B') \cup C$. Using the fact that we know the functional relation holds for all pairs of disjoint subsets,

$$f(A \cup B) = f((A' \cup B') \cup C) = f(A' \cup B') \cup f(C) = f(A') \cup f(B') \cup f(C)$$

$$= f(A') \cup f(C) \cup f(B') \cup f(C) = f(A' \cup C) \cup f(B' \cup C) = f(A) \cup f(B).$$

So the functional relation holding for every pair of disjoint subsets is sufficient to imply the relation also holds for pairs of non-disjoint subsets. It remains to ensure the relation holds for all pairs of disjoint subsets.

Plugging in $B = \emptyset$ into the relation, while letting $A \subseteq \mathbb{S}$ be arbitrary implies

$$f(A \cup \emptyset) = f(A) \cup f(\emptyset) \implies f(A) = f(A) \cup f(\emptyset).$$

This is equivalent to $f(\emptyset) \subseteq f(A)$.

There is no way to choose disjoint $A, B \neq \emptyset$ such that $A \cup B$ has 1 element.

Considering choices of disjoint $A, B \neq \emptyset$ such that $A \cup B$ has 2 elements implies

$$f(\{x, y\}) = f(\{x\}) \cup f(\{y\})$$

and similar relations for $f(\{y, z\})$ and $f(\{x, z\})$.

Finally, for choices of disjoint $A, B \neq \emptyset$ such that $A \cup B$ has 3 elements implies

$$f(\{x, y, z\}) = f(\{x, y\}) \cup f(\{z\}) = f(\{x, z\}) \cup f(\{y\}) = f(\{y, z\}) \cup f(\{x\}).$$

But since $f(\{x, y\}) = f(\{x\}) \cup f(\{y\})$ and similar equations from earlier, the last 3 terms in the equality chain above are necessarily equal.

Thus, the exhaustive list of necessary and sufficient conditions for f to be valid is:

- $f(\emptyset) \subseteq f(A)$ for all $A \subseteq \mathbb{S}$

- $f(\{x,y\}) = f(\{x\}) \cup f(\{y\})$ and similar statements
- $f(\{x,y,z\}) = f(\{x\}) \cup f(\{y\}) \cup f(\{z\})$

The last 2 properties in the list imply that once $f(\emptyset)$, $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$ are determined, all other values of f are uniquely built.

Furthermore, if $f(\emptyset) \subseteq f(\{x\}), f(\{y\}), f(\{z\})$ that is enough to imply $f(\emptyset) \subseteq f(A)$ for other choices of A, since the last 2 properties imply $f(\{x\}) \subseteq f(\{x,y\})$ (and similar statements) as well as $f(\{x\}) \subseteq f(\{x,y,z\})$.

All that remains now is to determine $f(\emptyset)$, $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$ so that $f(\emptyset) \subseteq f(\{x\}), f(\{y\}), f(\{z\})$.

We can have $x \in f(\varnothing)$, which means x is in all the other $f(\mathbb{S})$. That's 1 way.

Or we can have $x \notin f(\varnothing)$. Consider a subset of $\{f(\{x\}), f(\{y\}), f(\{z\})\}$, where we say elements in the subset has x in it and elements not in it don't. This is enough to construct x's placement in the rest of the $f(\mathbb{S})$. That is $2^3 = 8$ ways.

So, there are $1 + 8 = 9$ ways to determine x's placement among the $f(\mathbb{S})$. Same with y and z, which are independent of x's placement. Thus, the answer is $9^3 = \boxed{\textbf{(D) } 729}$.

Solution 2: Casework

As proven in Solution 1, the function f is uniquely determined by $f(\varnothing)$, $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$. That means we can determine the values of every other $f(A)$ from just $f(\varnothing)$, $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$. Also, as proven in Solution 1, $f(\varnothing) \subseteq f(A)$ for all A in $\mathbb{S}$. Now we simply do casework.

Case 1: $f(\varnothing) = \varnothing$

Since $\varnothing \subseteq A$ for all sets A, $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$ can each be chosen independently with 8 choices for each. This gives $8^3 = 512$ possible f for this case.

Case 2: $|f(\varnothing)| = 1$

For this case, the number of possible f are the same for $f(\varnothing) = \{x\}$, $f(\varnothing) = \{y\}$, and $f(\varnothing) = \{z\}$, so we will just consider $f(\varnothing) = \{x\}$ and multiply by 3 at the end.

In this subcase, we must have $\{x\}$ be a subset of $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$. There are 4 sets in $\mathbb{S}$ which are supersets of $\{x\}$ so there are $4^3 = 64$ possibilities for this subcase.

We multiply this by 3 to get $3 \cdot 64 = 192$ total possible f for this case.

Case 3: $|f(\varnothing)| = 2$

Similarly to the last case, we can just consider $f(\varnothing) = \{x, y\}$ and multiply by 3 at the end.

In this subcase, we must have $\{x, y\}$ be a subset of $f(\{x\})$, $f(\{y\})$, and $f(\{z\})$. There are 2 sets in $\mathbb{S}$ which are supersets of $\{x, y\}$ so there are $2^3 = 8$ possibilities for this subcase.

We multiply by 3 to get $3 \cdot 8 = 24$ total possibilities for this case.

Case 4: $f(\varnothing) = \{x, y, z\}$

Obviously, this case only has 1 possibility, because the only set in $\mathbb{S}$ that is a superset of $\{x, y, z\}$ is itself.

Total: $512 + 192 + 24 + 1 = \boxed{\textbf{(D) } 729}$.

24. Let $P(x)$ be a polynomial of degree 5 which satisfies

$$P(2^i) = i$$

for all integers $0 \le i \le 5$. The coefficient of x^4 in $P(x)$ can be written in the form $-\frac{a}{b}$, where a and b are relatively prime positive integers. What is the sum of the digits of b?

(A) 7 **(B)** 15 **(C)** 17 **(D)** 19 **(E)** 20

Define $Q(x) = P(2x) - P(x) - 1$. Then $Q(x) = 0$ for all 2^i where $0 \le i \le 4$. Therefore we can write

$$Q(x) = k(x - 2^0)(x - 2^1) \cdots (x - 2^4)$$

for some constant k.

The constant term of $Q(x)$ is then $-k(2^0)(2^1)\cdots(2^4)$. But since $Q(x) = P(2x) - P(x) - 1$, we know that the two constant terms from $P(2x)$ and $P(x)$ cancel out and so the constant term of $Q(x)$ is -1. This sets up the following equation:

$$-1 = -k(2^0)(2^1)\cdots(2^4) = -a(2^{10})$$

so we know that $k = \dfrac{1}{2^{10}}$.

Now we can find the coefficient of x^4 in $Q(x)$. By Vieta's Formulas, this is

$$-k(2^0 + 2^1 + \cdots + 2^4) = -k(2^5 - 1) = -\frac{2^5 - 1}{2^{10}}$$

Observe that the coefficient of x^4 in $P(2x)$ is $2^4 = 16$ times the coefficient of x^4 in $P(x)$. Thus, the coefficient of x^4 in $Q(x)$ is $16 - 1 = 15$ times more than the coefficient of x^4 in $P(x)$, implying the coefficient of x^4 in $P(x)$ to be $-\dfrac{2^5-1}{2^{10}\cdot 15} = -\dfrac{31}{15360}$, leading to the answer $\boxed{\textbf{(B) } 15}$.

25. Suppose that real number x satisfies

$$5(\tan^6 x + \tan^4 x + \tan^2 x) + 2^{60} \cdot \sin^{10} x + \cos^2 x = 2^{60}$$

What is the sum of the digits of $\tan^2 x$?

(A) 18 **(B)** 19 **(C)** 25 **(D)** 26 **(E)** 31

Using the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$ and the Binomial Theorem, it follows that $\sec^8 x = (\tan^2 x + 1)^4 = \tan^8 x + 4\tan^6 x + 6\tan^4 x + 4\tan^2 x + 1$.

Subtract $\tan^8 x + 4\tan^6 x + 6\tan^4 x + 4\tan^2 x + 1$ and add $\sec^8 x$ to the left hand side of the equation to get the following:

$$\sec^8 x - \tan^8 x + \tan^6 x - \tan^4 x + \tan^2 x - 1 + 2^{60} \cdot \sin^{10} x + \cos^2 x = 2^{60}$$

Combine $\cos^2 x - 1$ to $-\sin^2 x$ using the Pythagorean identity:

$$\sec^8 x - \tan^8 x + \tan^6 x - \tan^4 x + \tan^2 x - \sin^2 x + 2^{60} \cdot \sin^{10} x = 2^{60} \quad (1)$$

Using the Pythagorean identity,

$$\tan^2 x - \sin^2 x = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x = \sin^2 x(\sec^2 x - 1) = \sin^2 x \tan^2 x.$$

Substituting this in to (1) implies:

$$\sec^8 x - \tan^8 x + \tan^6 x - \tan^4 x + \sin^2 x \tan^2 x + 2^{60} \cdot \sin^{10} x = 2^{60} \quad (2)$$

Using $\sin^2 x - \tan^2 x = -\sin^2 x \tan^2 x$ from earlier,

$$-\tan^4 x + \sin^2 x \tan^2 x = \tan^2 x(\sin^2 x - \tan^2 x) = \tan^2 x(-\sin^2 x \tan^2 x) = -\tan^4 x \sin^2 x$$

which can be substituted in to (2):

$$\sec^8 x - \tan^8 x + \tan^6 x - \tan^4 x \sin^2 x + 2^{60} \cdot \sin^{10} x = 2^{60}$$

Now, the same process can be repeated two more times to further simplify the equation into this form:

$$\sec^8 x - \tan^8 x \sin^2 x + 2^{60} \cdot \sin^{10} x = 2^{60}$$

But $\tan^8 x \sin^2 x = \dfrac{\sin^{10} x}{\cos^8 x} = \sin^{10} x \sec^8 x$. Hence, the equation becomes

$$\sec^8 x - \sin^{10} x \sec^8 x + 2^{60} \cdot \sin^{10} x = 2^{60}$$

Rearranging the terms and factoring:

$$(\sec^8 x - 2^{60})(\sin^{10} x - 1) = 0$$

If $\sin^{10} x - 1 = 0$, then $\sin x = \pm 1$, implying $x = \dfrac{\pi}{2} + n\pi$ for some integer n. However, the domain of the original equation does not contain those values because $\tan x$ is asymptotic at those values.

Thus, $\sec^8 x - 2^{60} = 0$ or $\sec^2 x = 32768$. From the Pythagorean identity, $\tan^2 x = \sec^2 x - 1 = 32767$, so our answer is $\boxed{\textbf{(C)}\ 25}$.