Mock AMC 12 Solutions

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1. **Answer: (A)** The price of four candy bars is $4 \times \$2.50 = \$10$. If five candy bars are bought, the price is $5 \times \$2.50 \times 0.7 = \$8.75$. The amount saved by buying five candy bars instead of four is $\$10 - \$8.75 = \boxed{\textbf{(A) } \$1.25}$.

   Alternate solution: With the 30% discount, buying five candy bars is equivalent to buying $5 \times 0.7 = 3.5$ candy bars at full price. Then Nathan saves the cost of half a candy bar, or \$1.25.

2. **Answer: (A)** Suppose there are $3x$ boys and $4x$ girls in the class. Then $\frac{1}{4} \cdot 3x = \frac{3}{4}x$ boys have a pet, and $\frac{1}{3} \cdot 4x = \frac{4}{3}x$ girls have a pet. Then there are $\frac{3}{4}x + \frac{4}{3}x = \frac{25}{12}x$ sophomores who have a pet. The percentage of boys is

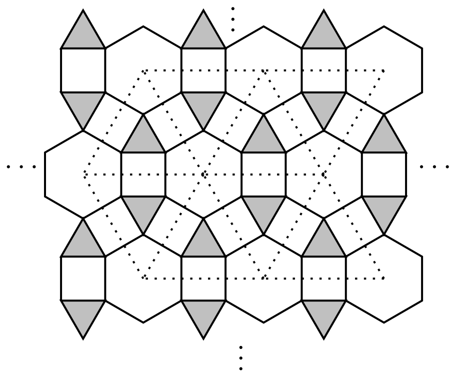
$$\frac{\frac{3}{4}x}{\frac{25}{12}x} = \frac{\frac{3}{4}}{\frac{25}{12}} = \frac{9}{25} = \boxed{\textbf{(A) } 36\%}$$

3. **Answer: (D)** We can simplify by dividing both numerator and denominator by 2017!, in which we obtain

$$\begin{aligned} \frac{2020! + 2019!}{2018! + 2017!} &= \frac{2020 \cdot 2019 \cdot 2018 + 2019 \cdot 2018}{2018 + 1} \\ &= 2020 \cdot 2018 + 2018 \\ &= 2021 \cdot 2018 \\ &= \boxed{\textbf{(D) } 4078378} \end{aligned}$$

   Note we can skip the multiplication of two 4-digit numbers as only choice **(D)** has a units digit of 8.

4. **Answer: (C)** We can tessellate the area as follows, using dotted triangles as indicated:



The fraction of area covered by the triangular tiles is approximately the fraction of area covered by one triangular tile within a dotted triangle. Without loss of generality, suppose each tile has a side length of 1. Then the side length of one of the dotted triangles is $1 + 2\left(\frac{\sqrt{3}}{2}\right) = 1 + \sqrt{3}$. The ratio of the area of the shaded triangle to the area of a dotted triangle is the square of the ratio of their side lengths, which is

$$\frac{1^2}{(1+\sqrt{3})^2} = \frac{1}{4+2\sqrt{3}} \approx \frac{1}{7.4} \approx \boxed{\textbf{(C) } 13\%}.$$

To be absolutely sure without use of a calculator, we note that $\frac{1}{4+2\sqrt{3}} > \frac{1}{8} = 12.5\%$, and $\frac{1}{4+2\sqrt{3}} < \frac{1}{7.4} < 14\%$.

5. **Answer: (C)** We do casework on whether $6 \in S$.

   If $6 \in S$, then we can choose any or all elements of $\{1, 2, 3, 4, 5\}$ to obtain a valid subset, giving $2^5 = 32$ possible subsets.

   If $6 \notin S$, then 3 must be in S, as well as either 2 or 4 (or both). There is 1 way to choose whether $3 \in S$, followed by $2^2 - 1 = 3$ ways to choose which of 2 and/or 4 are in S (either choose 2, 4, or both). We can choose any or all elements of $\{1, 5\}$ to determine S in 4 ways, giving $1 \times 3 \times 4 = 12$ possible subsets.

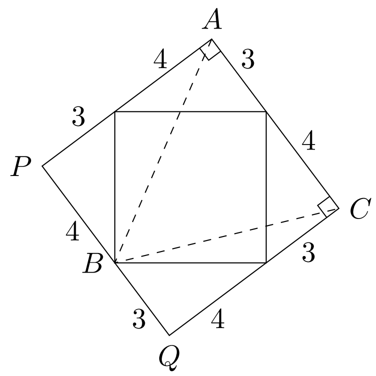
   Adding both cases, there are $32 + 12 = \boxed{\textbf{(C) } 44}$ subsets S whose product is divisible by 6.

6. **Answer: (B)** We have $f(g(x)) = |x^2 - 4|$ and $g(f(x)) = |x - 4|^2 = (x-4)^2$. Thus we wish to solve $|x^2 - 4| = (x-4)^2$ for real x.

   If $|x| \geq 2$, then $|x^2 - 4| = x^2 - 4$, and solving the equation $x^2 - 4 = (x-4)^2 = x^2 - 8x + 16$ gives $x = \frac{5}{2}$. If $|x| \leq 2$, then $|x^2 - 4| = 4 - x^2$, and the equation $4 - x^2 = x^2 - 8x + 16$ rearranges to the quadratic $2x^2 - 8x + 12 = 0 \iff x^2 - 4x + 6 = 0$. This has no real solutions as the discriminant is negative. Hence the number of real solutions is $\boxed{\textbf{(B) } 1}$.

7. **Answer: (A)** Rewrite the given equation as $a(bc + b + 1) = 64$. Clearly, a must be a power of 2 greater than 1, so $bc + b + 1$ must equal 2, 4, 8, 16, or 32. This implies $bc + b = b(c + 1)$ must equal 1, 3, 7, 15, or 31. As b and $c + 1$ are at least 2, the number $b(c + 1)$ must be composite, and only 15 satisfies this condition, giving $a = 4$. Then $b(c + 1) = 15$ giving solutions $(b, c) = (3, 4)$ or $(5, 2)$. There are two ordered triples (a, b, c) of integers greater than or equal to 2 which work, namely $(a, b, c) = (4, 3, 4)$ and $(4, 5, 2)$. In either case, $a+b+c = \boxed{\textbf{(A) } 11}$.

8. **Answer: (D)** Extend the side lengths of the triangles to form square $APQC$ of side length 7 as shown, where B is contained on $\overline{PQ}$:



Let $\angle ABC = \alpha$. Then $\tan\alpha = -\tan(\pi - \alpha) = -\tan(\angle ABP + \angle CBQ)$. Noting that $\tan\angle ABP = \frac{AP}{BP} = \frac{7}{4}$ and $\tan\angle CBQ = \frac{CQ}{QB} = \frac{7}{3}$, we can use the addition formula for tangent:

$$\begin{aligned}\tan(\angle ABP + \angle CBQ) &= \frac{\tan\angle ABP + \tan\angle CBQ}{1 - \tan\angle ABP\tan\angle CBQ}\\ &= \frac{\frac{7}{4} + \frac{7}{3}}{1 - \frac{7}{4}\cdot\frac{7}{3}}\\ &= \frac{\frac{49}{12}}{-\frac{37}{12}} = -\frac{49}{37}\end{aligned}$$

Then $\tan\alpha = -\left(-\frac{49}{37}\right) = \frac{49}{37}$, so $m + n = 49 + 37 = \boxed{\textbf{(D) } 86}$.

Alternate solution: Assign coordinates $(0, 0)$, $(5, 0)$, $(5, 5)$, $(0, 5)$ to the square such that B has coordinates $(0, 0)$. We can then find coordinates $A(\frac{16}{5}, \frac{37}{5})$ and $C(\frac{37}{5}, \frac{9}{5})$. The slope of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are $\frac{37}{16}$ and $\frac{9}{37}$, respectively, so the angles formed by these lines and the positive

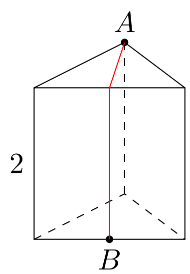
x-axis are $\tan^{-1}\frac{37}{16}$ and $\tan^{-1}\frac{9}{37}$. Thus, we have $\angle ABC = \tan^{-1}\frac{37}{16} - \tan^{-1}\frac{9}{37}$. To find $\tan\angle ABC$, we can compute $\tan(\tan^{-1}\frac{37}{16} - \tan^{-1}\frac{9}{37})$ using the subtraction formula for tangent, yielding

$$\begin{aligned}\tan\angle ABC &= \frac{\frac{37}{16} - \frac{9}{37}}{1 + \frac{37}{16}\cdot\frac{9}{37}} = \frac{\frac{37^2 - 9\cdot 16}{16\cdot 37}}{\frac{16\cdot 37 + 37\cdot 9}{16\cdot 37}} \\ &= \frac{1225}{37\cdot 25} \\ &= \frac{49}{37}\end{aligned}$$

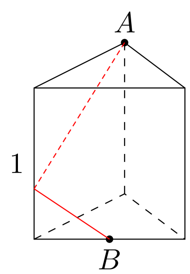
as above.

9. **Answer: (D)** It is easy to check that Joanna can make all monetary amounts of the form \$5k, where $k = 0, 1, \ldots, 74$, using denominations which are multiples of 5. Using one or two \$1 bills, she can also make the amounts \$5k+1 and \$5k+2. Thus, there are 75 possible values for k, each of which gives 3 different monetary amounts, giving $75 \times 3 = 225$ different amounts. However this includes \$0 which is invalid (since one or more bills is needed), so the answer is $225 - 1 = \boxed{\textbf{(D) } 224}$.

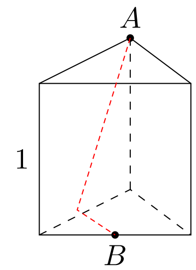
10. **Answer: (B)** If the ant crawls along the red path, as shown below, the ant travels $2 + \sqrt{3}$ units:



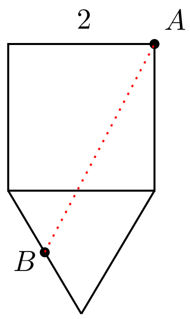
A better solution is for the ant to crawl along two square faces as shown:



Unfolding the two square faces, we see that the length of the path equals the hypotenuse of a right triangle with leg lengths 2 and 3; by the Pythagorean theorem, the distance crawled is $\sqrt{2^2+3^2} = \sqrt{13}$ units. However, the optimum solution is to crawl along one square face and one triangular face:



Unfolding the net, the path will appear as follows:



We see that the length of the dotted path is $\sqrt{\left(\frac{3}{2}\right)^2 + \left(2+\frac{\sqrt{3}}{2}\right)^2} = \boxed{\textbf{(B)}\ \sqrt{7+2\sqrt{3}}}$.

11. **Answer: (D)** By the change of base formula, we can rewrite $\log_8(a^3)$ as $\frac{\log_2(a^3)}{\log_2 8} = \frac{3\log_2 a}{3} = \log_2 a$. Thus, the given equation simplifies to a cubic in $\log_2 a$, namely $8(\log_2 a)^3 - 8\log_2 a - 1 = 0$.

We observe that the cubic polynomial $8x^3-8x-1$ has three distinct real roots. This can be shown by looking at the graph of $y = 8x^3 - 8x = 8(x^3 - x)$, which has roots $0, \pm 1$ as well as local minima/maxima at $x = \left(\pm\frac{\sqrt{3}}{3}, \mp\frac{16\sqrt{3}}{9}\right)$, in which shifting the graph down by 1 will still result in a graph which crosses the x-axis at exactly three distinct points (note that the computation of the local extrema is not needed; we can also observe that $y\left(\pm\frac{1}{2}\right) = \mp 3$, which yields the same conclusion). Thus, each real root of the cubic $8x^3-8x-1$ gives exactly one solution

a to the original equation. Suppose the roots of $8x^3 - 8x - 1$ are x_1, x_2, x_3, and suppose their corresponding solutions are a_1, a_2, a_3, where $x_i = \log_2 a_i$.

By Vieta's formulas, we have $x_1 + x_2 + x_3 = 0$ (as the x^2 coefficient is zero), which implies $\log_2 a_1 + \log_2 a_2 + \log_2 a_3 = 0 \iff \log_2 a_1 a_2 a_3 = 0 \iff a_1 a_2 a_3 = 1$, so the product of the real solutions is $\boxed{\textbf{(D) } 1}$.

12. **Answer: (D)** The numbers which are not multiples of 2, 3, or 5 eliminate children from the circle. Thus, if n children are originally in the circle, then the first $n-1$ such numbers eliminate children from the circle, in which the $(n-1)^{\text{th}}$ number is 121 by the conditions of the problem. That is, we want to find the number of positive integers less than or equal to 121 which are not multiples of 2, 3, or 5.

    Consider the system of modular congruences

    $$\begin{aligned} x &\equiv a \pmod{2} \\ x &\equiv b \pmod{3} \\ x &\equiv c \pmod{5} \end{aligned}$$

    where $a \in \{1\}$, $b \in \{1, 2\}$, and $c \in \{1, 2, 3, 4\}$. By the Chinese remainder theorem, any valid choice of a, b, c gives a unique solution for $x \pmod{30}$. As there are $1 \times 2 \times 4 = 8$ choices for (a, b, c), there are 8 numbers in $\{1, 2, \ldots, 30\}$ which are not multiples of 2, 3, or 5. Similarly, there are 8 numbers in $\{31, \ldots, 60\}$, $\{61, \ldots, 90\}$, $\{91, \ldots, 120\}$ which are not multiples of 2, 3, or 5. From this, we establish $n - 1 = 4 \times 8 + 1 = 33$, so the number of children is $n = \boxed{\textbf{(D) } 34}$.

13. **Answer: (D)** Observe that the prime factorization of 2021 is 43×47; this can be seen easily from the difference of squares factorization $2021 = 45^2 - 2^2 = (45 - 2)(45 + 2)$.

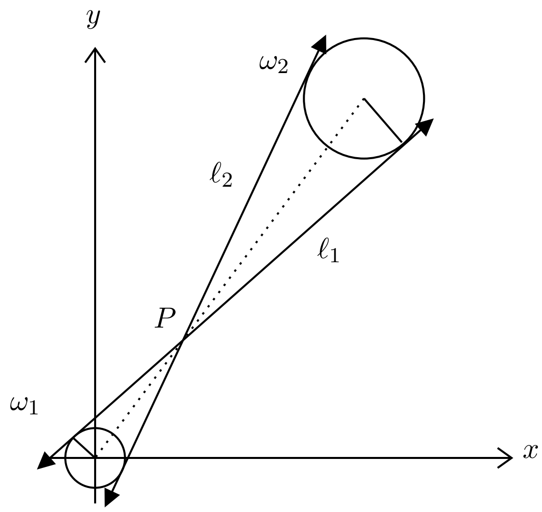
    Using Legendre's formula, we have $2021! = 43^{48} \times 47^{43} \times K$, where K is not divisible by 43 or 47. Also, we have $2020! = 43^{47} \times 47^{42} \times K$. In order for $\operatorname{lcm}(n, 2020!) = 2021!$, the smallest possible n is $n = 43^{48} \times 47^{43}$, and the number of factors of n is $(48 + 1)(43 + 1) = 49 \times 44 = \boxed{\textbf{(D) } 2156}$.

14. **Answer: (E)** The prime factorization of 216 is $2^3 \times 3^3$, so 216 has $(3+1)(3+1) = 16$ factors. There are $16^3 = 2^{12}$ equally likely outcomes for the triple of numbers Robert draws.

    Each divisor of 216 can be represented as an ordered pair (a, b) where $0 \le a \le 3$ and $0 \le b \le 3$, corresponding to the number $2^a \times 3^b$. Thus, we can consider the equivalent problem of choosing a random ordered pair $(a, b) \in \{0, \ldots, 3\} \times \{0, \ldots, 3\}$, and finding the probability that three such pairs sum to less than or equal to $(3,3)$ (where $(x,y) \le (3,3)$ if and only if $x \le 3$ and $y \le 3$).

Suppose these three pairs are (a_1,b_1), (a_2,b_2), (a_3,b_3). By stars and bars, the number of non-negative integer solutions to $a_1+a_2+a_3 \le 3$ equals the number of non-negative solutions to $a_1+a_2+a_3+s=3$, which is $\binom{6}{3} = 20$ (and similarly $b_1+b_2+b_3 \le 3$). Any such 6-tuple $(a_1,a_2,a_3,b_1,b_2,b_3)$ uniquely gives a sequence of three pairs or equivalently, a valid outcome whose product is a factor of 216. Thus, the number of valid outcomes is $20^2 = 400$, and the probability is $\frac{400}{2^{12}} = \boxed{\textbf{(E)}\ \frac{25}{256}}$.

15. **Answer: (E)** Note that there are four distinct lines which are tangent to both ω_1 and ω_2, but only one pair of lines intersects at a point on segment $\overline{O_1O_2}$, as shown. We will label the "steeper" line as ℓ_2, though this labeling is irrelevant.



We claim that P is the point $(3,4)$. Let A and B be the points where ℓ_1 is tangent to circles ω_1 and ω_2, respectively. By a simple AAA argument, $\triangle PAO_1 \sim \triangle PBO_2$; since the radius of ω_2 is twice that of ω_1, the ratio $PO_1 : PO_2$ is $1:2$. This implies ℓ_1 intersects $\overline{O_1O_2}$ at $(3,4)$. Similarly, ℓ_2 intersects $\overline{O_1O_2}$ at $(3,4)$, so $P=(3,4)$.

For simplicity, shift everything by $(-3,-4)$ so that P is now the origin. Thus, we have two lines passing through the origin which are tangent to the circle $(x+3)^2+(y+4)^2=1$, for which we can determine the possible slopes. Let $y=mx$ where m is the slope of ℓ_1 or ℓ_2. Then the following quadratic must have exactly one real solution x in order for $y=mx$ to be tangent:

$$\begin{aligned}(x+3)^2+(mx+4)^2&=1\\(m^2+1)x^2+(8m+6)x+24&=0\end{aligned}$$

The discriminant is $\Delta = (8m+6)^2 - 4(m^2+1)(24)$, which we set to zero in order to have exactly one real solution x. We obtain the resulting quadratic equation in m:

$$\begin{aligned}64m^2+96m+36-96m^2-96&=0\\-32m^2+96m-60&=0\\8m^2-24m+15&=0\end{aligned}$$

Let m_1 and m_2 be the two real solutions of the above quadratic, corresponding to the slopes of ℓ_1 and ℓ_2. By simple application of Vieta's formulas, we obtain $m_1+m_2=\frac{24}{8}=3$.

Back in the original setting (before shifting by $(-3,-4)$), we have that both lines intersect at $P=(3,4)$. Thus, if the equations of ℓ_1 and ℓ_2 are $y=m_1x+b_1$ and $y=m_2x+b_2$, we have

$$\begin{aligned}4&=3m_1+b_1\\4&=3m_2+b_2\end{aligned}$$

implying $b_1+b_2=8-3(m_1+m_2)=8-3(3)=\boxed{\textbf{(E)}\ -1}$.

16. **Answer: (D)** Let a_k denote the largest real solution to the equation $f^k(x)=0$, so that $x_1=a_{2020}$. We observe a pattern:

$$\begin{aligned}k=1:&\quad x^2-20=0 \implies a_1=\sqrt{20}\\k=2:&\quad (x^2-20)^2-20=0 \implies a_2=\sqrt{20+\sqrt{20}}\\k=3:&\quad ((x^2-20)^2-20)^2-20=0 \implies a_3=\sqrt{20+\sqrt{20+\sqrt{20}}}\\&\quad\vdots\end{aligned}$$

This pattern can be shown inductively. Suppose we know the value of a_k for some $k\ge 1$. Then $f^{k+1}(x)=(\ldots((x^2-20)^2-20)^2-\ldots)^2-20$. Here, the innermost x^2-20 must be a root of $f^k(x)$, so in order to maximize x, we set $x^2-20=a_k$ where $x>0$, giving $a_{k+1}=\sqrt{20+a_k}$.

We claim that the sequence $a_1,\ a_2,\ a_3,\ \ldots$ is strictly increasing and converges to the positive root of the polynomial a^2-a-20. To show the sequence is increasing, we note that $\sqrt{20}<a_k<5$ for all $k\ge 1$

(provable inductively), and that $\sqrt{20+a_k} > a_k$ iff $20 + a_k > a_k^2$ iff $(a_k - 5)(a_k + 4) < 0$. As $4 < a_k < 5$, this inequality holds, so (a_k) is strictly increasing. Let $a = \sqrt{20 + \sqrt{20 + \sqrt{20 + \ldots}}}$; squaring both sides gives $a^2 = 20 + a \implies a = -4, 5$ by the quadratic formula. Since $a > 0$, we take the positive root, or $a = 5$. However, a_{2020} is less than 5 (by an extremely small amount); that is, $x_1 = a_{2020} = 4.999\ldots < 5$. Here it is sufficient to bound $4.95 < x_1 < 5$, so that $24.5 < x_1^2 < 25$.

To find x_0, we observe that $f^k(x)$ is an even function for all $k \geq 1$. This can be shown inductively; $f^1(x)$ is even, and if $f^{k-1}(x)$ is even, then $f^k(-x) = f^1(f^{k-1}(-x)) = f^1(f^{k-1}(x)) = f^k(x)$. Thus for real x, we have $f^{2020}(x) = 0$ iff $f^{2020}(-x) = 0$, so $x_0 = -x_1 \approx -4.999\ldots$; that is, $|x_0| = |x_1| \approx 4.999\ldots$. Then $49 < x_0^2 + x_1^2 < 50$, so the largest integer less than or equal to $x_0^2 + x_1^2$ is $\boxed{\textbf{(D) } 49}$.

17. **Answer: (B)** Let us define a positive integer as *ascending* (resp. *descending*) if its digits are in strictly increasing (decreasing) order. Note that 1, 2, …, 9 are both ascending and descending. For now, we will count the number of ascending numbers which are $1 \pmod 3$.

Observe that there is a bijection between the set of ascending numbers which are $1 \pmod 3$ and the set of ascending numbers which are $2 \pmod 3$. The bijection is as follows: given an ascending number n, map it to the ascending number which contains all and only all digits 1-9 which are not contained in n. For example, 7 maps to 12345689, and 24578 maps to 1369. This is valid as $1 + 2 + 3 + \ldots + 9 = 45 \equiv 0 \pmod 3$.

There are $2^9 - 1 = 511$ ascending numbers; this is easily seen as any nonempty subset of $\{1, 2, \ldots, 9\}$ induces exactly one ascending number. It suffices to find the number of ascending numbers which are multiples of 3. Using the divisibility rule for 3, this is equivalent to finding the number of nonempty subsets of $\{1, 2, 3, \ldots, 9\}$ whose sum is divisible by 3. More generally, this can be done using the roots of unity filter (see Alternate Solution); a simple solution is to use casework by considering the digits modulo 3. Let x_1, x_2 represent the number of digits (elements) which are $1 \pmod 3$ and $2 \pmod 3$ respectively.

- **Case 1:** $(x_1, x_2) = (0, 0)$ (all digits are $0 \pmod 3$). This gives $2^3 - 1 = 7$ subsets.
- **Case 2:** $(x_1, x_2) = (1, 1)$. This gives $\binom{3}{1} \times \binom{3}{1} \times 2^3 = 72$ subsets.
- **Case 3:** $(x_1, x_2) = (2, 2)$. This gives $\binom{3}{2} \times \binom{3}{2} \times 2^3 = 72$ subsets.
- **Case 4:** $(x_1, x_2) = (3, 0)$. This gives $\binom{3}{3} \times 2^3 = 8$ subsets.
- **Case 5:** $(x_1, x_2) = (0, 3)$. Similarly this gives $\binom{3}{3} \times 2^3 = 8$ subsets.

- **Case 6:** $(x_1, x_2) = (3, 3)$. This gives 8 subsets.

Altogether we count $7+72+72+8+8+8 = 175$ nonempty subsets of $\{1, 2, 3, \ldots, 9\}$ (equivalently, ascending numbers) whose sum is divisible by 3. Then $511 - 175 = 336$ ascending numbers are not divisible by 3. By the above bijection, exactly half of these leave remainder 1 when divided by 3, so there are 168 such ascending numbers.

We can count the number of descending numbers similarly; the only difference is that a descending number may end in 0. To count this, consider any positive ascending number that is $1 \pmod 3$ and reverse its digits. We can optionally add a 0 at the end. The number of such descending numbers is $168 \times 2 = 336$. However we overcounted the numbers 1, 4, and 7 as they are both ascending and descending. Therefore the number of monotone numbers which leave remainder 1 when divided by 3 is $168 + 336 - 3 = \boxed{\textbf{(D) } 501}$.

Alternate solution: We find the number of ascending numbers which are $1 \pmod 3$ using the roots of unity filter. Let $\omega = e^{2i\frac{\pi}{3}}$ be a third root of unity, and let $\omega^2 = \overline{\omega}$ be its complex conjugate.

Let $f(x) = (1+x)(1+x^2)(1+x^3)\ldots(1+x^9) = a_{45}x^{45} + \ldots + a_1 x + a_0$. The coefficient of x^n represents the number of subsets of $\{1, 2, \ldots, 9\}$ whose sum is n. Now consider $g(x) = x^2 f(x) = a_{45}x^{47} + a_{44}x^{46} + \ldots + a_0 x^2$, so that we wish to find $a_{43} + a_{40} + \ldots + a_1$. We have $f(1) = 2^9 = 512$, and $f(\omega) = f(\omega^2) = 8$ (note that $(1+\omega)(1+\omega^2) = 1$). This gives $g(1) = 512$, $g(\omega) = 8\omega^2$, and $g(\omega^2) = 8\omega^4 = 8\omega$. Using the roots of unity filter:

$$\begin{aligned} g(1) + g(\omega) + g(\omega^2) &= 3(a_{43} + a_{40} + \ldots + a_1) \\ 512 + 8\omega^2 + 8\omega &= 3(a_{43} + a_{40} + \ldots + a_1) \end{aligned}$$

Since $\omega^2 + \omega = -1$:

$$\begin{aligned} 504 &= 3(a_{43} + a_{40} + \ldots + a_1) \\ 168 &= a_{43} + a_{40} + \ldots + a_1 \end{aligned}$$

Thus there are 168 ascending numbers which are $1 \pmod 3$, since the RHS represents this quantity. Proceed as above.

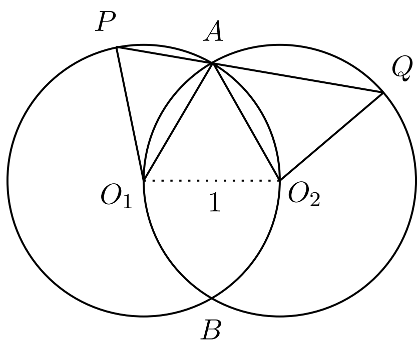
18. **Answer: (C)** Label the persons 1, $\ldots$, 6 with 1 the shortest and 6 the tallest. Note that 1 must stand next to 2 and 6 must stand next to 5. Thus we may treat 1 and 2 as one "pair" (similarly with 5 and 6). The remaining constraint is that 3 must stand next to 2 or 4, and 4 must stand next to 3 or 5.

We can do casework on whether 3 and 4 stand next to each other.

- **Case 1:** 3 and 4 stand next to each other. Then we may treat 3 and 4 as one pair, in which we have three pairs (12), (34), (56). There are $3! = 6$ ways to order these pairs, followed by $2^3 = 8$ ways to order the two people within each pair, giving $6 \times 8 = 48$ ways.
- **Case 2:** 3 and 4 do not stand next to each other. Then 3 stands next to 2, and 4 stands next to 5. Thus, 1, 2, and 3 form a "triple" (denoted (123)), and so do 4, 5, 6. There are $2! \times 2 \times 2 = 8$ ways to arrange the two triples (123),(456). However, this overcounts the arrangements 123456 and 654321, as 3 and 4 are next to each other. Thus, the number of ways is $8 - 2 = 6$.

Adding, we obtain $48 + 6 = \boxed{\textbf{(C) } 54}$ ways.

19. **Answer: (B)** Let O_1 and O_2 be the centers of ω_1 and ω_2 respectively. Since $O_1O_2 = O_1A = O_2A$, triangle AO_1O_2 is equilateral.



Let $\angle O_1PA = \angle O_1AP = \alpha$. With a little angle chasing, we see that $\angle O_2AQ = \angle O_2QA = 120^\circ - \alpha$, $\angle AO_2Q = 2\alpha - 60^\circ$, and $\angle PO_1A = 180^\circ - 2\alpha$. Let $PA = x$ and $AQ = 2x$. Applying the law of cosines on $\triangle PO_1A$ and AO_2Q:

$$\begin{aligned} x^2 &= 2 - 2\cos(180^\circ - 2\alpha) \\ 4x^2 &= 2 - 2\cos(2\alpha - 60^\circ) \end{aligned}$$

Using basic trigonometric identities, we can write $\cos(180^\circ - 2\alpha)$ as $-\cos 2\alpha$, and $\cos(2\alpha - 60^\circ) = \cos 2\alpha \cos 60^\circ + \sin 2\alpha \sin 60^\circ = \frac{1}{2}\cos 2\alpha + \frac{\sqrt{3}}{2}\sin 2\alpha$. Now, observe that $\sin 2\alpha = \sqrt{1 - \cos^2 2\alpha}$; this follows from the Pythagorean identity and using the fact that $2\alpha \in (0, \pi)$ (so that $\sin 2\alpha$ is positive). Hence our system of equations implies

$$\begin{aligned} x^2 &= 2 + 2\cos 2\alpha \\ 4x^2 &= 2 - \cos 2\alpha - \sqrt{3}\sqrt{1 - \cos^2 2\alpha} \end{aligned}$$

Let $y = \cos 2\alpha$. Multiplying the first equation by 4, we can obtain an equation entirely in terms of y:

$$\begin{aligned} 8 + 8y &= 2 - y - \sqrt{3 - 3y^2} \\ \sqrt{3 - 3y^2} &= -6 - 9y \end{aligned}$$

Square both sides to obtain a quadratic in y:

$$\begin{aligned} 3 - 3y^2 &= 81y^2 + 108y + 36 \\ 84y^2 + 108y + 33 &= 0 \\ 28y^2 + 36y + 11 &= 0 \end{aligned}$$

Using the quadratic formula, we obtain $y = -\frac{11}{14}$ or $y = -\frac{1}{2}$, so the possible values of $\cos 2\alpha$ are $-\frac{11}{14}$ or $-\frac{1}{2}$. However, notice that $\frac{1}{2}$ is extraneous as it implies $x = 1 \implies PQ = 3$, which is impossible given the configuration of the problem. Hence $\cos 2\alpha = -\frac{11}{14} \implies x^2 = PA^2 = 2 - 2\left(-\frac{11}{14}\right) = \frac{3}{7}$, so $PA = \frac{\sqrt{21}}{7}$ and $PQ = 3x = \boxed{\textbf{(B) } \frac{3\sqrt{21}}{7}}$.

20. **Answer: (E)** For $n \geq 1$, let p_n denote the probability that Paige's first n rolls sum to 7. Note that $p_1 = 0$ since each roll is at most 6, and $p_n = 0$ for $n \geq 8$ since each roll is at least 1. The answer is $p_2 + p_3 + \ldots + p_7$.

    To find p_n for $2 \leq n \leq 7$, we can find the number of outcomes on n dice rolls which sum to 7, then divide by 6^n. The number of "good" outcomes equals the number of positive integer solutions to the equation $a_1 + \ldots + a_n = 7$ where $1 \leq a_i \leq 6$. By stars and bars, this yields $\binom{6}{n-1}$ outcomes.

The desired answer is

$$\begin{aligned}
p_2 + p_3 + \ldots + p_7 &= \sum_{n=2}^{7} \frac{\binom{6}{n-1}}{6^n} \\
&= \binom{6}{1}\frac{1}{6^2} + \binom{6}{2}\frac{1}{6^3} + \ldots + \binom{6}{6}\frac{1}{6^7} \\
&= \frac{1}{6}\left[\binom{6}{1}\frac{1}{6} + \binom{6}{2}\frac{1}{6^2} + \ldots + \binom{6}{6}\frac{1}{6^6}\right] \\
&= \frac{1}{6}\left[\left(1+\frac{1}{6}\right)^6 - 1\right] \\
&= \frac{7^6 - 6^6}{6^7}
\end{aligned}$$

Hence $m = 7^6 - 6^6$ and $n = 6^7$, in which $m+n = 7^6 - 6^6 + 6^7$. To find $m+n \pmod{1000}$, we can use $7^6 - 6^6 = 343^2 - 216^2 = 559 \times 127 \equiv 993 \pmod{1000}$, and $6^7 \equiv 216^2 \times 6 \equiv 936 \pmod{1000}$. Then $m + n \equiv 993 + 936 \equiv \boxed{\textbf{(E) } 929} \pmod{1000}$ (Note that $\frac{7^6-6^6}{6^7} \approx 25.4\%$).

Alternate solution: We can use recursion. For $n \geq 0$, let a_n denote the probability that Paige eventually gets a running sum of n, with $a_0 = 1$, $a_1 = \frac{1}{6}$, and $a_n = 0$ for $n < 0$. The recursion step is $a_n = \frac{1}{6}(a_{n-1} + a_{n-2} + \ldots + a_{n-6})$, by examining the last die roll before obtaining a running sum of n. We compute $a_n = \frac{7^{n-1}}{6^n}$ for $1 \leq n \leq 6$. This pattern breaks down at $n = 7$, since we only consider the last six terms of the sequence and not the entire sequence. We can compute $a_7 = \frac{7^6}{6^7} - \frac{1}{6} = \frac{7^6-6^6}{6^7}$, then proceed as above.

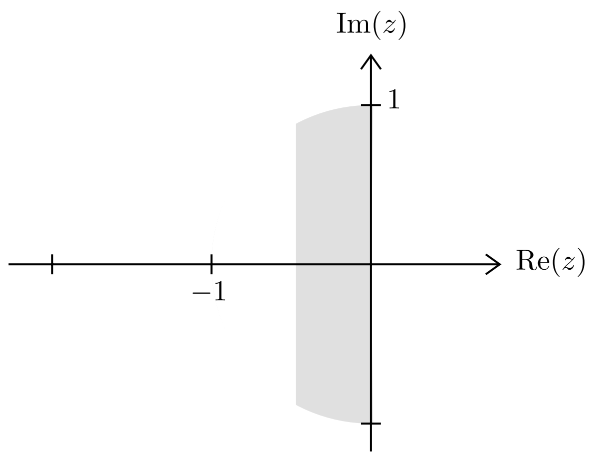
21. **Answer: (A)** By the quadratic formula, if $z^2 + az + b = 0$, then $z = \frac{-a\pm\sqrt{a^2-4b}}{2}$, so we wish to determine the set $\mathcal{R}$ of all complex numbers z which can be written in this form, for some $a, b \in [0, 1]$.

Note that if the discriminant $a^2 - 4b \geq 0$, then z is real, and the range of possible *real* numbers z is an interval which has zero area. Specifically, the set of all real numbers in $\mathcal{R}$ is the interval $[-1, 0]$. We can see this by observing that if the discriminant is non-negative, then $0 \leq a^2 - 4b \leq a^2$, so $-1 = \frac{-a-a}{2} \leq z \leq \frac{-a+a}{2} = 0$. Moreover, it is not hard to show that for every real $z \in [-1, 0]$, there exist $a, b \in [0, 1]$ such that $z^2 + az + b = 0$.

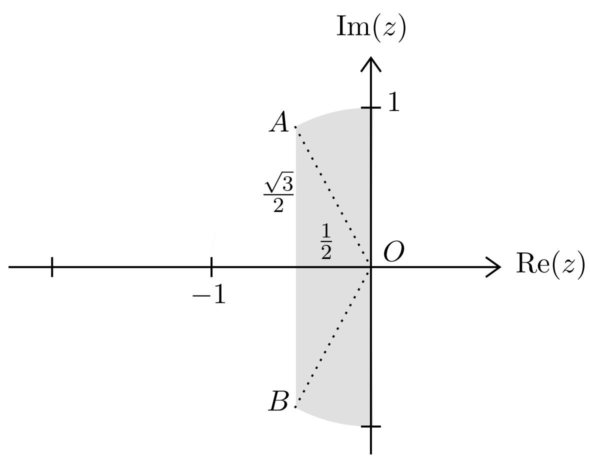
Now suppose the discriminant is non-positive; i.e., $a^2 - 4b \leq 0$, in which the roots of the quadratic z^2+az+b are $z = \frac{-a\pm\sqrt{a^2-4b}}{2} = -\frac{a}{2} \pm \frac{\sqrt{4b-a^2}}{2}i$. Geometrically, this represents the point $(-\frac{a}{2}, \pm\frac{\sqrt{4b-a^2}}{2})$.

Consider some $a \in [0, 1]$. The real part ("x-coordinate") of z is $-\frac{a}{2}$ and the imaginary part ("y-coordinate") is $\pm\frac{\sqrt{4b-a^2}}{2}$, whose absolute value varies continuously between 0 and $\frac{\sqrt{4-a^2}}{2}$, when $b = \frac{a^2}{4}$ and $b = 1$ respectively. Hence for a fixed $a \in [0, 1]$, the set of non-real numbers in $\mathcal{R}$ which are the root of some polynomial of the form $z^2 + az + b$ for some b is $-\frac{a}{2} + mi$ where $0 \leq |m| \leq \frac{\sqrt{4-a^2}}{2}$.

Consider points of the form $(-\frac{a}{2}, \frac{\sqrt{4-a^2}}{2})$, where $a \in [0, 1]$. Notice that these points lie on the unit circle; this is checked as $\left(-\frac{a}{2}\right)^2 + \left(\frac{\sqrt{4-a^2}}{2}\right)^2 = \frac{a^2}{4} + \frac{4-a^2}{4} = 1$. This gives us enough information to plot the region $\mathcal{R}$. Note that $\mathcal{R}$ also contains the interval $[-1, 0]$, but this has area zero.



The area of $\mathcal{R}$ can be calculated with basic geometry. For instance, we can divide R into an isosceles triangle $\triangle AOB$ and two $30°$ sectors, as shown:



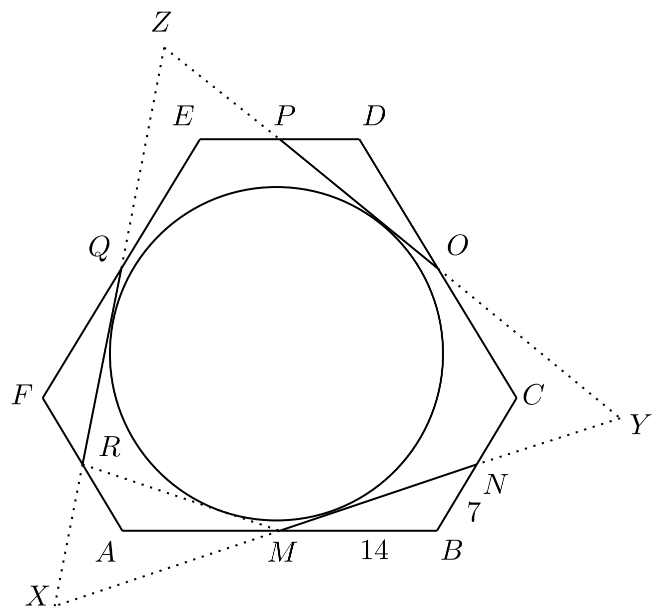
The area of isosceles triangle AOB is $\frac{1}{2} \cdot \sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4}$, and the combined area of the two $30°$ sectors is $\frac{\pi}{6}$. The total area of $\mathcal{R}$ is $\dfrac{\sqrt{3}}{4} + \dfrac{\pi}{6} = \boxed{\textbf{(A)}\ \dfrac{3\sqrt{3}+2\pi}{12}}$.

22. **Answer: (C)** First, we observe that $MNOPQR$ is an equilateral hexagon; note that MN is the side opposite the $120°$ angle in $\triangle BMN$. To find MN, we can either use the law of cosines on $\triangle BMN$, or drop an altitude from N onto BM and use the Pythagorean theorem. Though trigonometry is not required for this problem, we compute MN using the law of cosines:

$$\begin{aligned} MN^2 &= 7^2 + 14^2 - 2 \cdot 7 \cdot 14 \cos 120° \\ &= 7^2 + 14^2 + 7 \cdot 14 \\ &= 343 \end{aligned}$$

Then $MN = 7\sqrt{7}$. Similarly, $NO = OP = PQ = QR = RM = 7\sqrt{7}$, so hexagon $MNOPQR$ is equilateral. However, it is *not* regular as $\angle MNB = \angle ONC \neq 30°$.

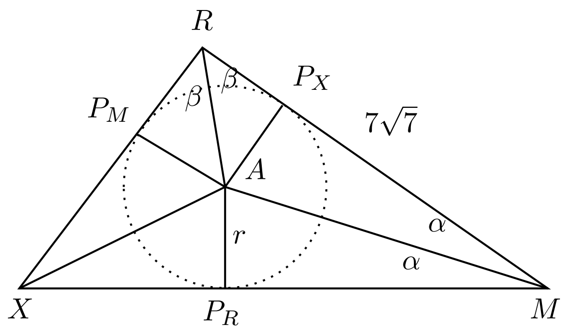
Extend $\overline{MN}$, $\overline{OP}$, and $\overline{QR}$ to form triangle XYZ as shown in the figure below. Then ω is the incircle of this triangle, as it is tangent to the sides of $\triangle XYZ$. Consider triangle RMX:



Let $\angle RMA = \alpha$ and $\angle MRA = \beta$, where $\alpha + \beta = 60^\circ$. Because $\triangle RMA \cong \triangle NMB$ by SSS congruence, we have $\angle AMX = \angle NMB = \alpha$, and similarly $\angle ARX = \angle FRQ = \beta$. Then A is the intersection of two angle bisectors in $\triangle RMX$, so A is the incenter of $\triangle RMX$. Using this, we establish $\angle RXM = 180^\circ - 2\alpha - 2\beta = 180^\circ - 2(60^\circ) = 60^\circ$, and similarly $\angle NYO = \angle PZQ = 60^\circ$, so $\triangle XYZ$ is equilateral. It suffices to find the side length of $\triangle XYZ$, since we can compute the inradius easily from there.

Recall that $MN = RM = 7\sqrt{7}$. Using the congruence $\triangle RXM \cong \triangle NYO$, we see that $NY = RX$, and that the side length of $\triangle XYZ$ equals the perimeter of $\triangle RMX$. Since $RM = 7\sqrt{7}$, it suffices to find $RX + XM$.

Let P_M, P_R, and P_X be the points where the incircle of $\triangle RMX$ is tangent to $\overline{RX}$, $\overline{XM}$, and $\overline{RM}$, respectively:



We first compute the inradius r of $\triangle RMX$. Fortunately this is not hard to find, as the inradius is simply the altitude from A to RM in $\triangle RAM$. We can use one of many methods to find $[\triangle RAM]$ (either Heron's formula, $\frac{1}{2}ab\sin C$, or dropping an altitude from M to AR), obtaining

$$[\triangle RAM] = \frac{49\sqrt{3}}{2} = \frac{1}{2}(7\sqrt{7})r.$$

Solving for r yields $r = \sqrt{21}$.

Notice that $P_M R = RP_X$ and $P_R M = MP_X$, so $P_M R + P_R M = 7\sqrt{7}$. Further, since $\triangle AP_R X$ and $AP_M X$ are 30-60-90, we have $P_M X = P_R X = r\sqrt{3} = 3\sqrt{7}$. It follows that the perimeter of $\triangle RMX$ is $7\sqrt{7} + 7\sqrt{7} + 2(3\sqrt{7}) = 20\sqrt{7}$. Therefore the side length of $\triangle XYZ$ is $20\sqrt{7}$.

Using 30-60-90 triangles, the inradius of an equilateral triangle with side length $20\sqrt{7}$ is $\frac{10\sqrt{21}}{3}$, so the area of ω is $\left(\frac{10\sqrt{21}}{3}\right)^2 \pi = \boxed{\textbf{(C) } \frac{700\pi}{3}}$.

Alternate solution: Extend $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ to form an equilateral triangle of side length $28 + 14 + 14 = 56$. By a rotational symmetry argument, the center of the circle is the incenter of the equilateral triangle.

Assign coordinates $M = (0, 0)$, $B = (14, 0)$. Using 30-60-90 triangles, the coordinates of the center are $\left(0, \frac{28\sqrt{3}}{3}\right)$. We can easily find the coordinates of N to be $\left(\frac{35}{2}, \frac{7\sqrt{3}}{2}\right)$. The equation of line $\overline{MN}$ is therefore $y = \frac{\sqrt{3}}{5}x$, or equivalently $\sqrt{3}x - 5y = 0$. Using the formula for the distance from a point to a line, we obtain

$$\text{radius of } \omega = \frac{|0 \cdot \sqrt{3} - 5 \cdot \frac{28\sqrt{3}}{3}|}{\sqrt{3 + 5^2}} = \frac{10\sqrt{21}}{3}.$$

Similarly as above, the area of ω is $\frac{700\pi}{3}$.

23. **Answer: (E)** We will consider the problem in binary. The binary representation of 2020 is 11111100100_2.

Let $f(n)$ denote the number of tuples of the form $(a_0, a_1, \ldots, a_k)$ such that $\sum_{i=0}^{k} a_i 2^i = n$ and $a_i \in \{0, 1, 2\}$. The desired answer is $f(2020)$ (note that $2^{11} > 2020$, so the maximum possible k is 10). To represent the tuples, we will express them in a way similar to binary, except that digits may be 0, 1, or 2 (for example, 102 corresponds to $a_2 = 1$, $a_1 = 0$, $a_0 = 4$):

n	$f(n)$	Solutions	n	$f(n)$	Solutions
1	1	1	9	3	1001, 201, 121
2	2	2, 10	10	5	1010, 1002, 210, 202, 122
3	1	11	11	2	1011, 211
4	3	100, 12, 20	12	5	1100, 1020, 220, 1012, 212
5	2	101, 21	13	3	1101, 1021, 221
6	3	110, 102, 22	14	4	1110, 1102, 1022, 222
7	1	111	15	1	1111
8	4	1000, 200, 120, 112	16	5	10000, 2000, 1200, 1120, 1112

We observe some patterns regarding f. First, we see that $f(2^k - 1) = 1$ (the only solution is $11\ldots1$), and $f(2^k) = k + 1$. We can state these claims more generally:

Lemma 1. *The following are true for any integer n:*

$$f(2n) = f(n) + f(n-1) \quad (1)$$
$$f(2n+1) = f(n) \quad (2)$$

Proof. Note that this "binary" representation of $2n$ can be obtained by appending a 0 to any representation of n, or by appending a 2 to any representation of $n-1$ (which proves (1)), and a "binary" representation of $2n+1$ must be obtained by appending a 1 to any representation of n. $\square$

Corollary 2. $f(2^k - 2) = k$.

This can be shown inductively with (1).

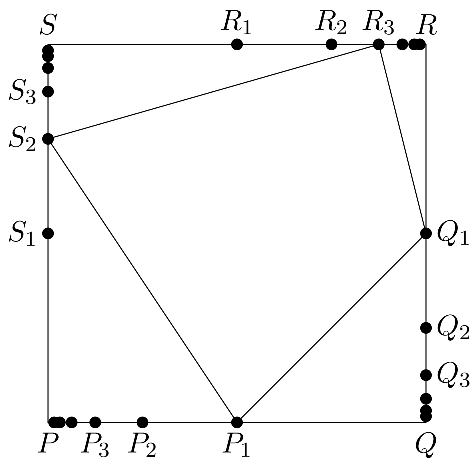
Corollary 3. $f(4n) = f(n) + 2f(n-1)$.

This can be shown by applying (1) then (2).

Note that $2020 = 11111100100_2$. Using the above Lemma and corollaries, we compute $f(2020)$ successively:

$$\begin{aligned}
f(63) &= f(111111_2) = 1 \\
f(252) &= f(11111100_2) = f(63) + 2f(62) \\
&= 1 + 12 = 13 \\
f(505) &= f(111111001_2) = f(252) = 13 \\
f(2020) &= f(11111100100_2) = f(505) + 2f(504) \\
&= 13 + 2(f(252) + f(251)) \\
&= 13 + 2(13 + f(62)) && ((2),\ f(251) = f(125) = f(62)) \\
&= 13 + 2(13 + 6) = \boxed{\textbf{(E)}\ 51}
\end{aligned}$$

24. **Answer: (E)** Note that points (P_i), (Q_i), (R_i), (S_i) are positioned on the sides of the square roughly as follows, with a sample quadrilateral $P_1Q_1R_3S_2$ shown.



We use an expected value argument to find K. Note that $[P_pQ_qR_rS_s] = 1 - [\triangle PS_sP_p] - [\triangle QP_pQ_q] - [\triangle RQ_qR_r] - [\triangle SR_rS_s]$. First, we will find $\mathbb{E}([\triangle PS_sP_p])$, where $p, \ldots, s$ are randomly and independently selected from $\{1, 2, \ldots, 2020\}$. Using Area $= \frac{1}{2}bh$, this equals $\frac{1}{2}\mathbb{E}(PS_s \cdot PP_p) = \frac{1}{2}\mathbb{E}(PS_s)\mathbb{E}(PP_p)$ using the fact that the expected value of a product of two uncorrelated random variables equals the product of the expected values of those random variables.

By definition of expectation, we have

$$\mathbb{E}(PP_p) = \frac{1}{2020}\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{2020}}\right) = \frac{1}{2020}\left(1 - \frac{1}{2^{2020}}\right) \approx \frac{1}{2020}$$

$$\mathbb{E}(PS_s) = 1 - \mathbb{E}(PP_p) \approx \frac{2019}{2020}$$

More precisely, $\mathbb{E}(PP_p) = \frac{1}{2020} - \varepsilon$ and $\mathbb{E}(PS_s) = \frac{2019}{2020} + \varepsilon$ where $\varepsilon = \frac{1}{2020 \cdot 2^{2020}}$. Then we have

$$\begin{aligned}
\mathbb{E}([\triangle PS_sP_p]) &= \frac{1}{2}\mathbb{E}(PS_s)\mathbb{E}(PP_p) \approx \frac{2019}{2 \cdot 2020^2} \\
&= \frac{1}{2}\left(\frac{2019}{2020} + \varepsilon\right)\left(\frac{1}{2020} - \varepsilon\right) \\
&= \frac{2019}{2 \cdot 2020^2} - \varepsilon'
\end{aligned}$$

for some small $\varepsilon' > 0$ (note that ε and ε' are on the order of 2^{-2020}, which can be treated as a negligibly small number). By symmetry, the quantities $\mathbb{E}([\triangle QP_pQ_q])$, $\mathbb{E}([\triangle RQ_qR_r])$, and $\mathbb{E}([\triangle SR_rS_s])$ are also equal. Thus, by linearity of expectation, we have

$$\begin{aligned}
\mathbb{E}([P_pQ_qR_rS_s]) &= \mathbb{E}(1 - [\triangle PS_sP_p] - [\triangle QP_pQ_q] - [\triangle RQ_qR_r] - [\triangle SR_rS_s]) \\
&= 1 - \mathbb{E}([\triangle PS_sP_p]) - \mathbb{E}([\triangle QP_pQ_q]) \\
&- \mathbb{E}([\triangle RQ_qR_r]) - \mathbb{E}([\triangle SR_rS_s]) \\
&= 1 - 4 \cdot \frac{1}{2}\left(\mathbb{E}(PS_s)\mathbb{E}(PP_p)\right) \\
&= 1 - \frac{2 \cdot 2019}{2020^2} + 4\varepsilon'
\end{aligned}$$

Since K is the sum over all possible areas $[P_pQ_qR_rS_s]$, and there are 2020^4 equally likely choices for (p, q, r, s), then we have

$$K = 2020^4\left(1 - \frac{2 \cdot 2019}{2020^2} + 4\varepsilon'\right)$$

It follows that $\lfloor K \rfloor = 2020^4 - 2 \cdot 2019 \cdot 2020^2 \equiv -2 \cdot 2019 \cdot 2020^2 \equiv \boxed{\textbf{(E) } 800} \pmod{1000}$.

25. **Answer: (B)** Note that $0! = 1! = 1$, so we may assume WLOG that $a, b, c, d, e \geq 1$, since we are only looking for the integers expressible as a sum of five factorials, not the 5-tuples (a, b, c, d, e) themselves. Further, since $7! = 5040 > 1000$, the greatest possible value for each of the variables $a, \ldots, e$ is 6.

For $i = 1, \ldots, 6$, let x_i denote the number of variables $a, \ldots, e$ which are equal to i. In particular, $x_6 \leq 1$ since $2 \times 6! > 1000$. Consider the following instance of stars and bars:

$$x_1 + x_2 + \ldots + x_6 = 5$$

where $x_6 \in \{0, 1\}$. Each non-negative solution $(x_1, x_2, \ldots, x_6)$ to the equation which satisfies this constraint gives us a positive integer expressible as the sum of five factorials (we do overcount a bit, which we will mention later).

- If $x_6 = 0$, then the number of non-negative solutions to $x_1 + x_2 + \ldots + x_5 = 5$ is $\binom{9}{4} = 126$.
- If $x_6 = 1$, we observe that $720 + 120 + 120 + 120 > 1000$ so $x_5 \leq 2$. Further, $720 + 120 + 120 + 24 + 24 > 1000$ and $720 + 120 + 24 + 24 + 24 \leq 1000$, so if $x_6 = 1$ and $x_5 = 2$, then $x_4 \leq 1$.
  - If $x_5 = 0$, then the number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 4$ is $\binom{7}{3} = 35$.
  - If $x_5 = 1$, then the number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 3$ is $\binom{6}{3} = 20$.
  - If $x_5 = 2$, then the number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 2$ is $\binom{5}{3} = 10$. However this counts $(x_1, x_2, x_3, x_4) = (0, 0, 0, 2)$ which is invalid as this results in a number greater than 1000, so there are $10 - 1 = 9$ solutions.

Therefore there are $126 + 35 + 20 + 9 = 190$ 6-tuples of non-negative integers $(x_1, \ldots, x_6)$ for which $x_1 + \ldots + x_6 = 5$ and the sum $x_1 \cdot 1! + x_2 \cdot 2! + \ldots x_6 \cdot 6!$ is at most 1000. However, this overcounts as some positive integers can be represented as the sum of five factorials of positive integers in two or more different ways (e.g., $10 = 3! + 1! + 1! + 1! + 1! = 2! + 2! + 2! + 2! + 2!$). It can be checked that the only integers less than or equal to 1000 that can be represented as the sum of five factorials in two different ways are 10 (as illustrated above), and $30 = 4! + 2! + 2! + 1! + 1! = 3! + 3! + 3! + 3! + 3!$. Thus we have overcounted by 2, so the answer is $190 - 2 = \boxed{\textbf{(B)}\ 188}$.