January 2021 Mock AMC 10 Solutions

Answer Key: EDBCA ECBCD DACED AAEBA CBCDA

1. Let $a \otimes b = \frac{1}{a} + \frac{1}{b} + \frac{ab+1}{ab}$ for nonzero a and b. What is $2020 \otimes 2021$?

   (A) $\frac{1010}{1011}$ **(B)** $\frac{2020}{2021}$ **(C)** $\frac{2022}{2021}$ **(D)** $\frac{2021}{2020}$ **(E)** $\frac{1011}{1010}$

   Answer (E): Note that $a \otimes b = \frac{1}{a} + \frac{1}{b} + \frac{ab+1}{ab} = \frac{ab+a+b+1}{ab} = \frac{(a+1)(b+1)}{ab}$ by Simon's Favorite Factoring Trick. Plugging in $a = 2020$ and $b = 2021$, we get that $2020 \otimes 2021 = \frac{2021 \times 2022}{2020 \times 2021} = \boxed{\frac{1011}{1010}}$.

2. The function $f(x) = x^2 + x - 2020$ has distinct roots a and b. What is $a^2 + b^2$?

   (A) 0 **(B)** 1 **(C)** 4040 **(D)** 4041 **(E)** 8081

   Answer (D): By the Vieta's Formulas, we have $a + b = -1$ and $ab = -2020$. We have $a^2 + b^2 = (a + b)^2 - 2ab = (-1)^2 - 2(-2020) = \boxed{4041}$.

3. Spiderman is delivering pizzas! He has to deliver pizzas in at most 5 minutes or he gets fired. He starts out on his moped, which travels at a rate of 1 block in 30 seconds, and then switches to using his spidey powers to travel 1 block in 10 seconds. If the pizza must be delivered to a place 20 blocks away, what is the maximum amount of blocks Spiderman can travel on his moped?

   (A) 1 **(B)** 5 **(C)** 6 **(D)** 9 **(E)** 10

   Answer (B): Let the x be the number of blocks Spiderman travelled on the moped and y the number of blocks travelled with his powers. Since he travels 20 blocks, $x + y = 20$. We also know that $30x + 10y \leq 5 \times 60$ if he is to deliver the pizzas on time. Combining the equations, we see that $20x \leq 100$. We see that x is maximized when $x = \boxed{5}$.

4. Andrew has 3 green socks, 2 blue socks, and 1 red sock in his drawer. If Andrew takes out socks one by one randomly without replacement, what is the probability Andrew can't choose a pair of socks of the same color from his drawer after taking out three socks?

   (A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{7}{10}$

   Answer (C): Andrew has to draw 2 green socks and 1 blue socks for no pairs of same color socks to remain. There are $\binom{3}{2} \times \binom{2}{1}$ ways he can choose socks for the condition to be met and $\binom{6}{3}$ ways he can choose socks regardless. Thus, our answer is $\frac{6}{20} = \boxed{\frac{3}{10}}$.

5. Harry's pet Hungarian Horntail (a flying dragon) is kept in a cubical cage. The Horntail is kept on a leash anchored at a corner of the floor of the cage. If the length of the

leash is the same as the length of a side of the cage, what fraction of the cage can the Horntail occupy?

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{2}$ **(D)** $\frac{2\pi}{3}$ **(E)** $\frac{3\pi}{4}$

Answer (A): Recognize that the space the Horntail can occupy is an eighth of a sphere with radius equal to the length of the cage. This is because the outer edge of this space is traced out by points that are equidistant from the point the leash is anchored. Thus, we see that

$$\frac{V_{\text{space}}}{V_{\text{cage}}} = \frac{\frac{4}{3}\pi r^3/8}{r^3} = \boxed{\frac{\pi}{6}}.$$

6. If the natural number n has exactly 8 positive divisors, the maximum number of divisors $27n^2$ can have is m. What is the sum of the digits of m?

   (A) 4 **(B)** 5 **(C)** 7 **(D)** 8 **(E)** 9

   Answer (E): Let $n = abc$ for prime numbers a, b, and c. Thus, n has 8 positive divisors. Then $27n^2 = 3^3a^2b^2c^2$. If $a, b, c \neq 3$, we have the maximum number of divisors $4 \times 3^3 = 108$, whose digits sum to $\boxed{9}$.

7. Let ω_1 and ω_2 be two concentric circles with radius r and $2r$ respectively. Given a fixed point P on the circumference of ω_2, a line ℓ is drawn at random going through P. What is the probability that ℓ does not intersect ω_1?

   (A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** $\frac{5}{6}$

   Answer (C): Drawing the tangents to circle ω_1 going through point P, we see that the lines between these tangents do not satisfy our condition. Taking one of the tangents as reference, we see that the lines between the tangents span $60°$ and all lines are within $180°$ of our reference tangent, so our probability is $1 - \frac{60}{180} = \boxed{\frac{2}{3}}$.

8. How many ordered triples of non-negative integers (x, y, z) exist such that

   $$2^x + 3^y = 4^z.$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Answer (B): Note that the triple $(0, 1, 1)$ works. A quick check shows us that no other triple involving 0 works. Note that any positive integer triple doesn't work because the parity of each side would be different (left side odd and right side even or vice versa). Thus, we only have $\boxed{1}$ triple that works.

9. What is the sum of the positive integers that are divisible by all perfect squares less than or equal to itself?

(A) 10 **(B)** 15 **(C)** 18 **(D)** 27 **(E)** 63

Answer (C):

10. Students at The Charter School of Wilmington have a total of 8 classes and alternate between taking 4 classes on *A day* and the other 4 classes on *B day*. If Nathan has 2 study halls and 6 other classes, and only has his study halls at either the beginning or end of the school day, how many possible class schedules could he have assuming classes (including study halls) and *days* are distinct.

(A) 4320 **(B)** 5640 **(C)** 7200 **(D)** 8640 **(E)**9280

Answer (D): There are 6 ways to choose places for study halls:

$$sccscccc$$

$$scccsccc$$

$$scccccccs$$

$$cccsscc c$$

$$cccsccc s$$

$$ccccsccs$$

with c representing a non study hall class and s representing a study hall. For each of these cases, there are 2 ways to arrange different study halls, and 6! ways to arrange the other classes. Thus, the number of possible schedules is $6 \times 2 \times 6! = \boxed{8640}$.

11. The Falcons are beating the Patriots in the Super Bowl by a score of 28-3 in the third quarter. In football, during a team's *possession*, it is possible for the team to score 0, 3, 6, 7, or 8 points. If the Patriots have exactly 4 *possessions* left, and the Falcons do not score any more points, how many ways can the Patriots take the game to overtime (tied score after their 4 *possessions*), where the order in which they score points matters?

(A) 7 **(B)** 14 **(C)** 24 **(D)** 28 **(E)** 32

Answer (D): Casework gets us that the Patriots can score the following unordered scores: 6 6 6 7, 3 6 8 8, and 3 7 7 8. They can be arranged 4, 12, and 12 ways respectively. Adding them gets $4 + 12 + 12 = \boxed{28}$.

12. Heran has \$10 in his bank account, and every day he withdraws some whole number amount of money from the account. He always withdraws at least \$1 and never withdraws more money than the amount left in the account. How many ways can he withdraw money so that his account reaches \$0 in exactly 5 days?

(A) 126 **(B)** 210 **(C)** 226 **(D)** 252 **(E)** 462

Answer (A): This question is equivalent to asking how many ordered groups of 5 positive integers are there that add up to 10. Thus, the answer is $\binom{9}{4} = \boxed{126}$ by stars and bars.

13. Functions $f(x) = x^2 + mx + 2021$ and $g(x) = x^2 - 91x + n$ are factors of $h(x) = x^3 - 138x^2 + kx - 47n$ for real numbers k, m, and n. What is the value of $h(50)$?

    (A) 21 **(B)** 37 **(C)** 42 **(D)** 47 **(E)** 55

    Answer (C): Notice that $2021 = 43 \times 47$. If the three roots of $h(x)$ are r_1, r_2, and r_3, with r_1 and r_2 being roots of $f(x)$ and r_2 and r_3 being roots of $g(x)$, we have $r_1r_2 = 2021$, $r_2r_3 = n$, and $r_1r_2r_3 = 47n$. Thus $r_2 = 43$, $r_1 = 47$, and since $r_1 + r_2 + r_3 = 138$, $r_3 = 48$. Thus, $h(x) = (x - r_1)(x - r_2)(x - r_3)$ since $h(x)$ is monic, and we get that

    $$h(50) = (50 - 47)(50 - 43)(50 - 48) = 3 \times 7 \times 2 = \boxed{42}.$$

14. Let Γ be the circle going through the points $(0, 0)$, $(4, 2)$, and $(-3, 1)$. Let A and B be the points on Γ diametrically opposite to $(4, 2)$, and $(-3, 1)$ respectively. Let A' and B' be the reflections of A and B over $(-3, 1)$ and $(4, 2)$ respectively. What is the area of quadrilateral $ABB'A'$?

    (A) 50 **(B)** $50\sqrt{2}$ **(C)** $50\sqrt{3}$ **(D)** $75\sqrt{2}$ **(E)** 100

    Answer (E): Let $(4, 2)$ be point C, and $(-3, 1)$ be point D. We know that quadrilateral $ABCD$ is a rectangle since its diagonals go through the center. And since A' and B' are just reflections over D and C, we know that $ABB'A'$ is twice the area of $ABCD$. Thus, we wish to find the area of $ABCD$ and multiply it by 2. Let the center of the circle be at point (a, b). Thus, we have the equations, $a^2 + b^2 = (a + 3)^2 + (b - 1)^2 = (a - 4)^2 + (b - 2)^2$ as the points on the triangle are equidistant from the center. Solving the equations gets $(a, b) = (5, 0)$. Thus, our circle has radius equal to 5. We notice that the length of BC is $\sqrt{(4 - (-3))^2 + (2 - 1)^2} = 5\sqrt{2}$, so $ABCD$ is actually a square with side length $5\sqrt{2}$. Thus the area of $ABCD$ is $(5\sqrt{2})^2 = 50$ and the area of $ABB'A' = 2 \times 50 = \boxed{100}$.

15. Andrew is playing a game where he writes numbers on a blackboard. He begins by writing a two-digit number n on the blackboard, and for each number k he writes, the next number he writes is $k + 1$ if k is odd and $\frac{k}{2}$ if k is even. Andrew stops writing numbers once he writes the number 1. Let x be the maximum amount of numbers Andrew can write on the blackboard and let y be the amount of numbers n that Andrew can start with to write x numbers. What is $x + y$?

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

    Answer (D): We work backwards. Start with the number 1, and after we write the number 2 we alternate between multiplying by 2 and subtracting 1. This gets us the sequence 1, 2, 4, 3, 6, 5, 10, 9, 18, 17, 34, 33, 66, 65, which has 14 terms. We see that no other set of moves gives the same or more terms since we can instead try multiplying by 2 twice in a row somewhere in the sequence. Checking all such sequences confirms that this is the only possible sequence to reach 14 terms. We don't bother to check multiplying more than twice in a row in the sequence since it is clear we want to minimize multiplying by 2 as the sequence reaches 100 faster and thus gives us less terms, and we can't subtract by 1 more than once in a row since we can only do that if the number is even. Thus $x + y = 14 + 1 = \boxed{15}$.

16. What is the largest possible value of

$$\frac{a^3 + a^2b + ab^2 + b^3}{1 + a + b}$$

such that a and b are the side lengths of a right triangle with hypotenuse equal to 1?

(A) $2 - \sqrt{2}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** 1

Answer (A): Notice that the top simplifies to

$$a^3 + a^2b + ab^2 + b^3 = a^2(a + b) + b^2(a + b) = (a^2 + b^2)(a + b).$$

Since a and b are side lengths of a right triangle with hypotenuse equal to 1, $a^2+b^2 = 1$. Thus, our expression equals $\frac{a+b}{1+a+b}$, which we can simplify to

$$\frac{1}{1 + \frac{1}{a+b}}.$$

Thus, we wish to maximize $a + b$, which is when $a = b = \frac{\sqrt{2}}{2}$. We can confirm this by Cauchy-Schwarz, as $(a^2 + b^2)(1^2 + 1^2) \geq (a + b)^2$, but this is also intuitive as the maximum perimeter occurs when the legs are equal length if the hypotenuse is fixed. Thus, our answer is

$$\frac{1}{1 + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} + 1} = \boxed{2 - \sqrt{2}}.$$

17. Let P be the set of all points inside equilateral triangle ABC such that the areas of triangles PBC, PCA, and PAB form an arithmetic sequence not necessarily in that order. The set of points in P splits triangle ABC into several distinct, non-overlapping regions. Let S be the set of the distinct areas of all the regions formed by P. What fraction of the area of triangle ABC is the total area of all regions in S?

(A) $\frac{1}{3}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{4}$ **(E)**1

Answer (A): Choosing a point P inside triangle ABC, let the altitude from P to BC, CA, and AB be h_1, h_2, and h_3 respectively. Note that $h_1 + h_2 + h_3$ is invariate, in other words equals a constant, because if the side length of ABC is a,

$$\text{Area} = \frac{ah_1}{2} + \frac{ah_2}{2} + \frac{ah_3}{2} = \frac{a(h_1 + h_2 + h_3)}{2}$$

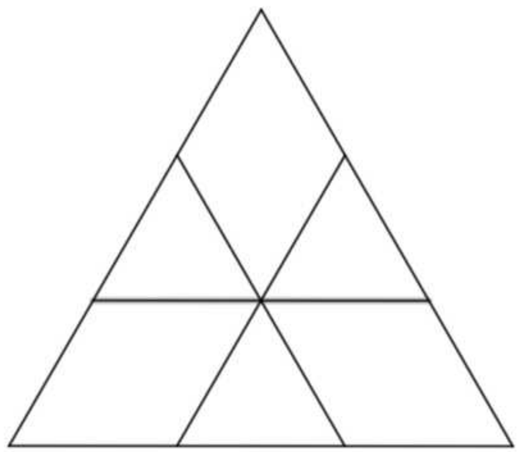
with the area of ABC and length a both being constants. Since the area of ABC is $\frac{a^2\sqrt{3}}{4}$, we see that

$$h_1 + h_2 + h_3 = \frac{a\sqrt{3}}{2},$$

which is the altitude of triangle ABC. WLOG, let $[PBC] < [PCA] < [PAB]$, which means that $2h_2 = h_1 + h_3$ by the arithmetic condition. Thus, substituting into the sum of heights formula gets

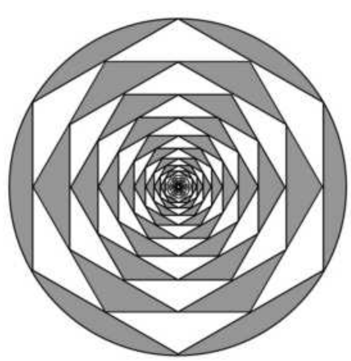
$$3h_2 = \frac{a\sqrt{3}}{2},$$

which implies that point P must lie on the line a third of the way from side BC to point A. Doing the same for the other three cases gives us that P can be the set of points on the lines through the centroid of ABC and parallel to either of the three sides of ABC, as shown below.



We see that there are two distinct areas that exist in the regions formed from these lines, which are $\frac{1}{9}$ and $\frac{2}{9}$ of the area of ABC. Thus, our answer is $\frac{1}{9} + \frac{2}{9} = \boxed{\frac{1}{3}}$.

18. A regular hexagon H_1 is inscribed in a unit circle. A regular hexagon H_2 is inscribed in H_1 such that it has vertices at the midpoints of H_1. Define H_3, H_4, ..., similarly such that there are infinitely many hexagons. The area between the circle and H_1 is shaded, as well as between H_2 and H_3, and between all hexagons H_{2n} and H_{2n+1} for all positive integer n. What is the total shaded area?



(A) $\pi - \frac{13\sqrt{3}}{11}$ **(B)** $\pi - \frac{8}{7}$ **(C)** $\pi - \frac{4\sqrt{3}}{7}$ **(D)** $\pi - \frac{3\sqrt{3}}{2}$ **(E)** $\pi - \frac{6\sqrt{3}}{7}$

Answer (E): Recognize that we can construct the area by adding the area of the circle, subtracting the area of H_1, adding the area of H_2, subtracting the area of H_3, and so on. We also know that the area of H_1 is $\frac{3\sqrt{3}}{2}$, and the ratio of the side length of H_{n+1} to side length of H_n is $\frac{\sqrt{3}}{2}$. Thus, the ratio of the area of H_{n+1} to the area of H_n is $(\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$. Thus, the area of the hexagon is the geometric series

$$\pi - \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} \times \left(\frac{3}{4}\right) + \frac{3\sqrt{3}}{2} \times \left(\frac{3}{4}\right)^2 - ... = \pi - \frac{\frac{3\sqrt{3}}{2}}{1-(-\frac{3}{4})} = \boxed{\pi - \frac{6\sqrt{3}}{7}}$$

19. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x (for example $\{1.5\} = 0.5$). If $\{n\} = \frac{1}{3}$ for postitive rational number n and

$$n^3 + 3n = 6\lfloor n \rfloor^2 + \frac{478}{27},$$

what are the last two digits of $\lfloor n \rfloor^5$?

(A) 01 **(B)** 25 **(C)** 32 **(D)** 43 **(E)** 68

Answer (B): We can write $n = \lfloor n \rfloor + \{n\} = \lfloor n \rfloor + \frac{1}{3}$. We can substitute this into our equation to get

$$\left(\lfloor n \rfloor + \frac{1}{3}\right)^3 + 3\left(\lfloor n \rfloor + \frac{1}{3}\right) = 6\lfloor n \rfloor^2 + \frac{478}{27}.$$

Simple expansion gives us the cubic

$$3\lfloor n \rfloor^3 - 15\lfloor n \rfloor^2 + 10\lfloor n \rfloor - 5 = 0,$$

which simplifies to $(\lfloor n \rfloor - 5)(3\lfloor n \rfloor^2 + 10) = 0$ which gives us $\lfloor n \rfloor = 5$ since $\lfloor n \rfloor$ is an integer. Thus $\lfloor n \rfloor^5 = 5^5 = 3125$, whose last two digits are $\boxed{25}$.

20. Let ABC be a triangle with $AB = 3$, $BC = 5$, and $AC = 7$. The angle bisector of the angle at B intersects AC at point D and the circumcircle of triangle ABC at point E distinct from B. What is the area of triangle CBE?

(A) $10\sqrt{3}$ **(B)** $\frac{15\sqrt{3}}{2}$ **(C)** 20 **(D)** $\frac{245\sqrt{3}}{32}$ **(E)** $\frac{235\sqrt{3}}{16}$

Answer (A): By the Law of Cosines, $7^2 = 3^3 + 5^2 - 2 \cdot 3 \cdot 5\cos(B)$ and $\angle ABC = 120°$. Because quadrilateral $ABCE$ is cyclic, $\angle AEC = 60°$, so triangle AEC is an equilateral triangle with side length 7. We know $[CBE] = [CBP] + [CDE] = \frac{5}{8}([ABC] + [AEC])$. Now, we can find $[ABC] = \frac{1}{2} \cdot 3 \cdot 5 \cdot \sin(120) = \frac{15\sqrt{3}}{4}$ by using the Law of Sines and $[AEC] = \frac{49\sqrt{3}}{4}$. Using these values, we get

$$[CBE] = \frac{5}{8}\left(\frac{15\sqrt{3}}{4} + \frac{49\sqrt{3}}{4}\right) = \frac{5}{8}\left(16\sqrt{3}\right) = \boxed{10\sqrt{3}}.$$

Alternate Solution (no trig): Use Stewart's Theorem on triangle ABC, so

$$AB^2 \cdot DC + BC^2 \cdot AD = BD^2 \cdot AC + AC \cdot AD \cdot DC.$$

$$3^2 \cdot \left(\frac{5}{8} \cdot 7\right) + 5^2 \cdot \left(\frac{3}{8} \cdot 7\right) = BD^2 \cdot 7 + 7 \cdot \left(\frac{3}{8} \cdot 7\right) \cdot \left(\frac{5}{8} \cdot 7\right)$$

After solving the equation, we find that $BD = \frac{15}{8}$. Now, using Power of a Point, we see that $AD \cdot CD = BD \cdot DE$, or $(\frac{3}{8} \cdot 7) \cdot (\frac{5}{8} \cdot 7) = \frac{15}{8} \cdot DE$, so $DE = \frac{49}{8}$ and $BE = 8$. We observe that triangle CBE is a triangle with side lengths 5, 7, and 8 and proceed with Heron's Formula.

$$[CBE] = \sqrt{10 \cdot 5 \cdot 3 \cdot 2} = \boxed{10\sqrt{3}}$$

21. In a chess tournament, 10 players play matches against each other where no pair of players play more than once. Exactly 40 matches are played. The probability that each player played at least 6 matches is $\frac{m}{n}$ for relatively prime integers m and n. What is the sum of the distinct prime factors of n?

    (A) 80 **(B)** 84 **(C)** 95 **(D)** 96 **(E)** 100

    Answer (C): Since there are only a total of $\binom{10}{2} = 45$ matches that could be played between two players, we can instead consider matches that aren't played between two players, since there are only $45 - 40 = 5$ of them. Thus, we have only two possible cases where a player doesn't play at least 6 matches.

    Case 1: We can have one player not play 5 of his possible matches, in other words only play 4 matches. Thus, there are $10 \times \binom{9}{4}$ ways to do this.

    Case 2: We can have one player not play 4 of his matches, in other words only play 5 matches, and then have two other players not play a match against each other. Thus there are $10 \times \binom{9}{5} \times \binom{9}{2}$ ways to do this.

    We have a total of $\binom{45}{5}$ ways to choose 5 matches out of 45 that aren't played, and thus our probability is

    $$\frac{10 \times \binom{9}{4} + 10 \times \binom{9}{5} \times \binom{9}{2}}{\binom{45}{5}} = \frac{2^2 \times 5 \times 37}{11 \times 41 \times 43}.$$

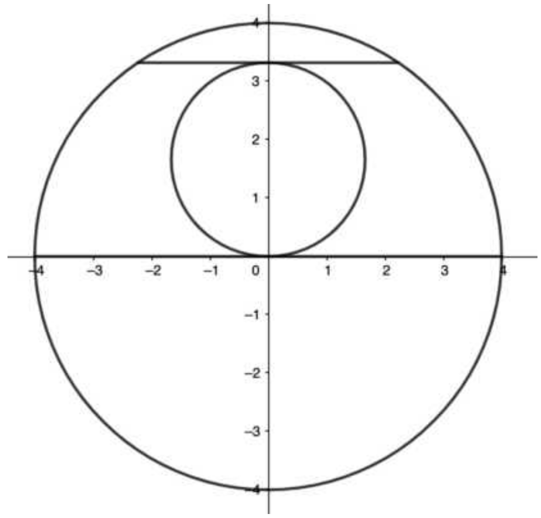
    Thus our answer is $11 + 41 + 43 = \boxed{95}$.

22. A circle ω_1 has radius 4 and is centered at the origin. A circle ω_2 passes through the origin such that ratio of the maximum length of a chord of ω_1 that is tangent to ω_2 to the minimum length of a chord of ω_1 that is tangent to ω_2 is equal to 2. What is the sum of all possible values of the radius of ω_2? Recall that a chord is a line segment and not a line.

(A) $\frac{5\sqrt{3}}{3}$ **(B)** $\frac{7\sqrt{3}}{3}$ **(C)** 3 **(D)** 4 **(E)** $\frac{6+\sqrt{7}}{2}$

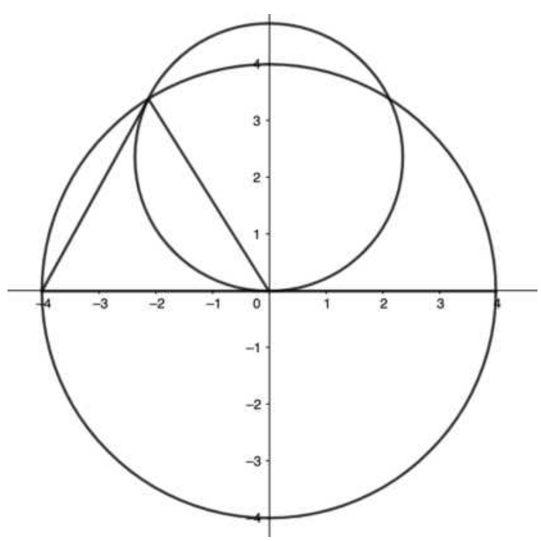
Answer (B): We have two cases to consider, one that ω_2 lies completely inside ω_1 and two that ω_2 intersects ω_1 at two points.

Case 1:



It is clear that the tangent with maximum length is the diameter and the tangent with minimum length is the chord opposite of the diameter. Take the center of ω_1 to be O, the intersection of the diametric tangent to be A, and the intersection of that tangent with ω_1 to be B. Thus, we have right triangle OAB with right angle at A. If the radius of ω_2 is r, we know that $AB = 2$, $OB = 4$, so $OA^2 + AB^2 = OB^2 \implies (2r)^2 + 2^2 = 4^2 \implies r = \sqrt{3}$.

Case 2:



The tangent with maximum length is again the diameter and the tangent with minimum length intersects the intersection point of the two circles. We observe that the tangent with minimum length has length 4, and the lengths between the ends of the tangent chord to the center of ω_1 are also r, and so the triangle formed by the two ends of the tangent chord and the center of ω_1 is equilateral. Thus we can find the radius of ω_2 by considering the triangle formed by the center of ω_2, the center of ω_1, and one of the intersections between ω_1 and ω_2. The triangle has side lengths R, R, and 4, with the angle between the equal side lengths being $120°$. Thus, we can obtain that $R = \frac{4}{\sqrt{3}}$.

We have computed the two possible values of the radius of ω_2, which are $r = \sqrt{3}$ and $R = \frac{4}{\sqrt{3}}$. Their sum is $\sqrt{3} + \frac{4}{\sqrt{3}} = \frac{3}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \boxed{\dfrac{7}{\sqrt{3}}}$.

23. There are 6 people sitting around a table. The seats are labeled A, B, C, D, E, and F clockwise. They play a game where the person at A starts with a card. Every second, the person with the card passes it to a person next to them, either to the left or right. For example, the person at A can pass the card to either the person at B or the person at F. When the card reaches the person at D, the game ends. How many possible games are there if the game ends in at most 10 seconds?

(A) 64 **(B)** 72 **(C)** 80 **(D)** 84 **(E)** 96

Answer (C): We can set up a table to arrange the possible paths of the card each second.

	A	***B***	***C***	***D***	***E***	***F***
0	1	0	0	**0**	0	0
1	0	1	0	**0**	0	1
2	2	0	1	**0**	1	0
3	0	3	0	**2**	0	3
4	6	0	3	**0**	3	0
5	0	9	0	**6**	0	9
6	18	0	9	**0**	9	0
7	0	27	0	**18**	0	27
8	54	0	27	**0**	27	0
9	0	81	0	**54**	0	81
10	162	0	135	**0**	135	0

In this table, the top row represents the seats and the left column represents the time elapsed. Let $P_{(N,k)}$ denote the number of paths for the card to reach seat N in k seconds. At 0 seconds, the card must be at A, so there is only one path and $P_{(A,0)} = 1$. We can write 1 in cell $(A, 0)$. At 1 second, the card has one path leading to seat B and one path leading to seat F, so $P_{(B,1)} = P_{(F,1)} = 1$. We can write 1 in cells $(B, 1)$

and $(F, 1)$. In general we can write,

$$P_{(A,k)} = P_{(F,k-1)} + P_{(B,k-1)}$$

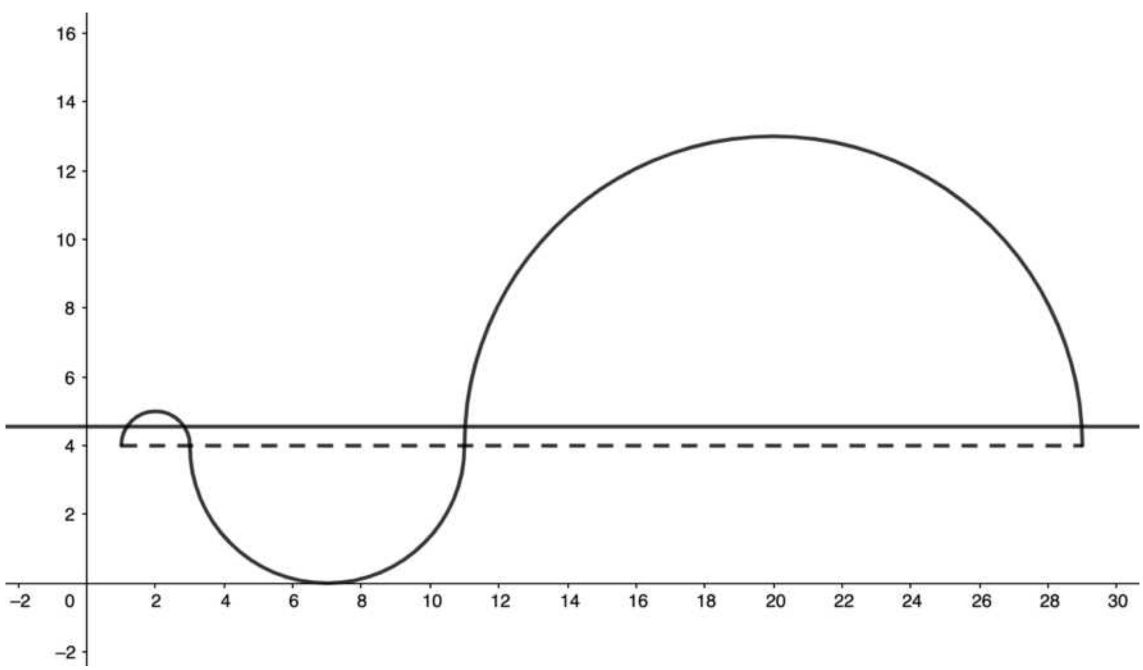
$$P_{(B,k)} = P_{(A,k-1)} + P_{(C,k-1)}$$
$$P_{(D,k)} = P_{(C,k-1)} + P_{(E,k-1)}$$
$$P_{(F,k)} = P_{(E,k-1)} + P_{(A,k-1)}$$
$$P_{(C,k)} = P_{(B,k-1)}$$
$$P_{(E,k)} = P_{(F,k-1)}$$

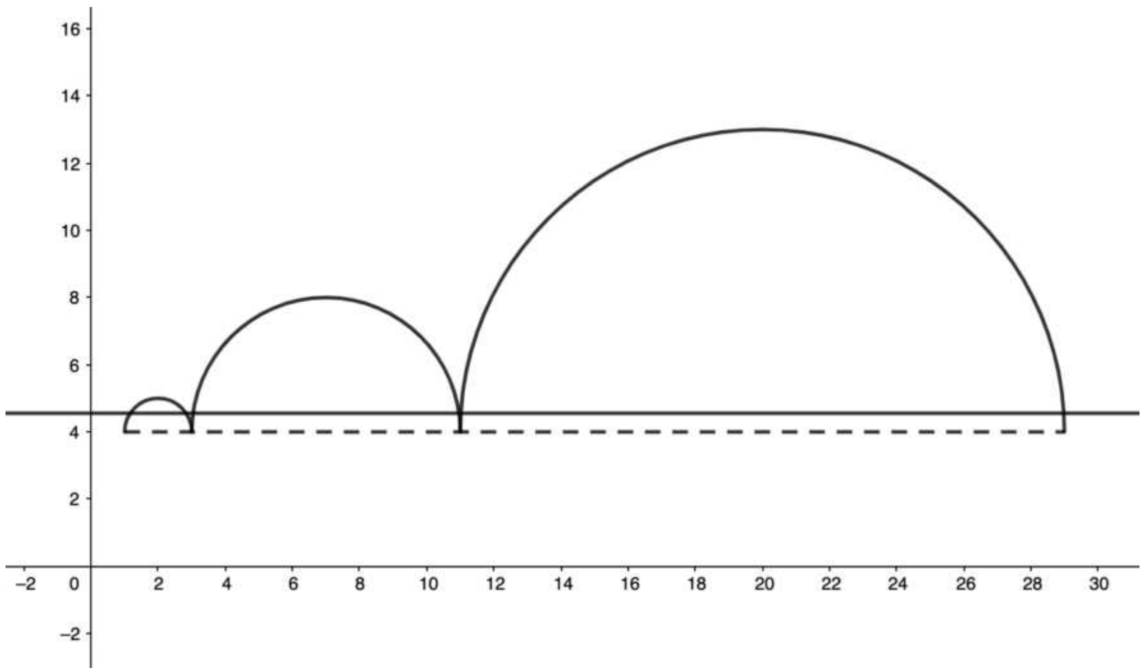
Note that $P_{(C,k)} \neq P_{(B,k-1)} + P_{(D,k-1)}$ and $P_{(E,k)} \neq P_{(D,k-1)} + P_{(F,k-1)}$ because the game ends and the card stops moving when it reaches seat D. We can now fill out the chart and add the values in the D column, which is $0+0+0+2+0+6+0+18+0+54+0 = \boxed{80}$

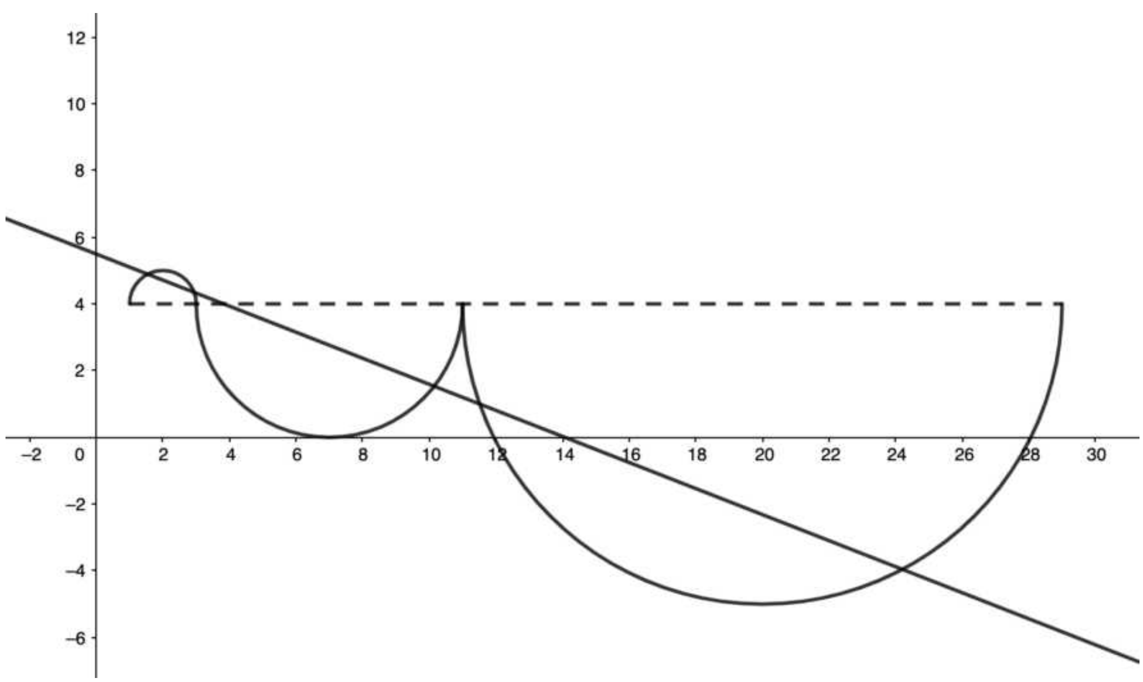
24. Let $P(x)$ be a polynomial with real coefficients such that $P(x) = 4$ if and only if $x = 1, 3, 11, 29$ and $P(x)$ has at most 5 relative max/mins in the interval $[1, 29]$. Let N be the maximum number of solutions to $P(x) = ax + b$ for any real coefficients a and b, for real x in the interval $[1, 29]$. What is the sum of all possible values of N?

(A) 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 20

Answer (D): Since the polynomial only equals 4 at $x = 1, 3, 11, 29$, we know that $P(x) = k(x-1)^a(x-3)^b(x-11)^c(x-29)^d + 4$ for real k and positive integers a, b, c, d. We can find the maximum number of solutions to $P(x) = ax+b$ by graphing $y = P(x)$ and an arbitrary line $y = ax + b$ and finding the maximum number of intersections for different $P(x)$.





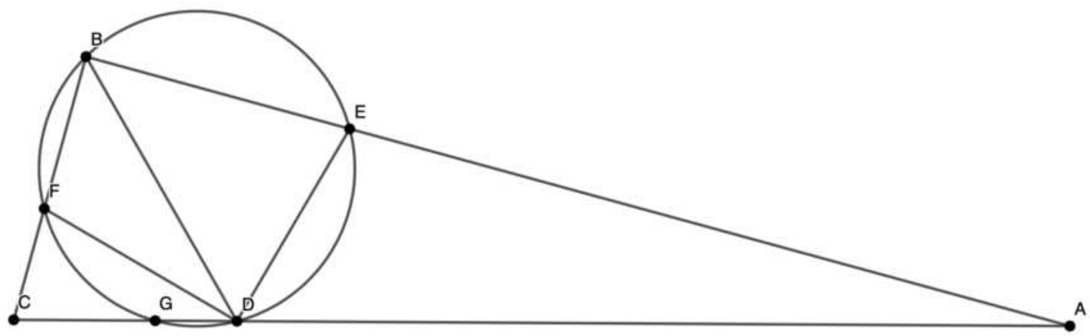


The above diagrams show the three possible ways to get different numbers of maximum intersections, simplified with semicircles rather than a continuous polynomial function. Note that we can choose each interval of $(1, 3), (3, 11), (11, 29)$ to have either a local maximum or minimum since we know that we only have at most 5 local maximums and minimums on $x = [1, 29]$, so we cannot have more than one local maximum or minimum in each of the three intervals. Thus, we add up the total number of intersections that we see, which is $4 + 6 + 5 = \boxed{15}$.

25. Let ABC be a right triangle with right angle at B, and $AB = \sqrt{3} - 1$ and $BC = \sqrt{3} + 1$. The angle bisector of $\angle ABC$ intersects $\overline{AC}$ at D. The angle bisector of $\angle ADB$ intersects $\overline{AB}$ at E and the angle bisector of $\angle CDB$ intersects BC at F. The circumcircle of triangle BEF intersects $\overline{AC}$ at two distinct points. The distance between these two points can be written as $\frac{a\sqrt{b}-c\sqrt{d}}{e}$ for positive integers a, b, c, d, and e where $\gcd(a, c, e) = 1$ and b and d do not contain a square divisor. What is $a + b + c + d + e$?

(A) 16 **(B)** 19 **(C)** 23 **(D)** 25 **(E)** 30

Answer (A):



We first recognize that $\angle FDE = 90$, since $\angle FDE = \angle FDB + \angle EDB = \frac{1}{2}(\angle CDB + \angle EDB) = \frac{1}{2} \times 180$. Thus, we know that quadrilateral $BFDE$ is cyclic since $\angle B$ and $\angle FDE$ are both right. Let the G be the point other than D such that the circumcircle of BEF intersects line AC. We desire the distance GD, which we can get by applying power of a point on point C, which gives $CF \times BC = CG \times DC$.

We want to obtain the length CF. We obtain the lengths of CD and DA by angle bisector theorem, and thus we have $BD^2 = BC \times AB - CD \times DA$ giving us

$$BD = \sqrt{(\sqrt{3}-1)(\sqrt{3}+1) - \left(\frac{(\sqrt{3}-1)\sqrt{2}}{\sqrt{3}}\right)\left(\frac{(\sqrt{3}+1)\sqrt{2}}{\sqrt{3}}\right)} = \sqrt{\frac{2}{3}}.$$

Now we can apply angle bisector theorem again on triangle BDC, from which we obtain

$$CF = \left(\frac{\frac{(\sqrt{3}-1)\sqrt{2}}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}} + \frac{(\sqrt{3}-1)\sqrt{2}}{\sqrt{3}}}\right)(\sqrt{3}-1) = \frac{4-2\sqrt{3}}{\sqrt{3}}.$$

Now we can substitute into our power of a point formula, and if we let the desired length $GD = x$, we get

$$\frac{4-2\sqrt{3}}{\sqrt{3}} \times (\sqrt{3}-1) = \left(\frac{(\sqrt{3}-1)\sqrt{2}}{\sqrt{3}} - x\right) \times \frac{(\sqrt{3}-1)\sqrt{2}}{\sqrt{3}}.$$

We simplify this to get $x = \frac{2\sqrt{6}-3\sqrt{2}}{3}$, giving us an answer of $2+6+3+2+3 = \boxed{16}$.