JANUARY 2021 MOCK

AMC 10

AMERICAN MATHEMATICS CONTEST 10

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU START YOUR TIMER.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the google form provided on the AoPS thread. Check to make sure that each question has a bubble marked, with blanked answers being marked as blank on the rightmost column. Otherwise, please PM **wertguk** your answers and make sure to include if you want to be on the leaderboard.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. On the google form, be sure to include your AoPS username in the correct spot, checking that spelling, capitalization, and symbols accurately match your username. Failure to do so might delay receiving your score.
8. You will have 75 minutes to complete the test, good luck!

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1. Let $a \otimes b = \frac{1}{a} + \frac{1}{b} + \frac{ab+1}{ab}$ for nonzero a and b. What is $2020 \otimes 2021$?

   (A) $\frac{1010}{1011}$ **(B)** $\frac{2020}{2021}$ **(C)** $\frac{2022}{2021}$ **(D)** $\frac{2021}{2020}$ **(E)** $\frac{1011}{1010}$

2. The function $f(x) = x^2 + x - 2020$ has distinct roots a and b. What is $a^2 + b^2$?

   (A) 0 **(B)** 1 **(C)** 4040 **(D)** 4041 **(E)** 8081

3. Spiderman is delivering pizzas! He has to deliver pizzas in at most 5 minutes or he gets fired. He starts out on his moped, which travels at a rate of 1 block in 30 seconds, and then switches to using his spidey powers to travel 1 block in 10 seconds. If the pizza must be delivered to a place 20 blocks away, what is the maximum amount of blocks Spiderman can travel on his moped?

   (A) 1 **(B)** 5 **(C)** 6 **(D)** 9 **(E)** 10

4. Andrew has 3 green socks, 2 blue socks, and 1 red sock in his drawer. If Andrew takes out socks one by one randomly without replacement, what is the probability Andrew can't choose a pair of socks of the same color from his drawer after taking out three socks?

   (A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{7}{10}$

5. Harry's pet Hungarian Horntail (a flying dragon) is kept in a cubical cage. The Horntail is kept on a leash anchored at a corner of the floor of the cage. If the length of the leash is the same as the length of a side of the cage, what fraction of the cage can the Horntail occupy?

   (A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{2}$ **(D)** $\frac{2\pi}{3}$ **(E)** $\frac{3\pi}{4}$

6. If the natural number n has exactly 8 positive divisors, the maximum number of divisors $27n^2$ can have is m. What is the sum of the digits of m?

   (A) 4 **(B)** 5 **(C)** 7 **(D)** 8 **(E)** 9

7. Let ω_1 and ω_2 be two concentric circles with radius r and $2r$ respectively. Given a fixed point P on the circumference of ω_2, a line ℓ is drawn at random going through P. What is the probability that ℓ does not intersect ω_1?

   (A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** $\frac{5}{6}$

8. How many ordered triples of non-negative integers (x, y, z) exist such that

   $$2^x + 3^y = 4^z.$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

9. What is the sum of the positive integers that are divisible by all perfect squares less than or equal to itself?

   (A) 10 **(B)** 15 **(C)** 18 **(D)** 27 **(E)** 63

10. Students at The Charter School of Wilmington have a total of 8 classes and alternate between taking 4 classes on *A day* and the other 4 classes on *B day*. If Nathan has 2 study halls and 6 other classes, and only has his study halls at either the beginning or end of the school day, how many possible class schedules could he have assuming classes (including study halls) and *days* are distinct.

    (A) 4320 **(B)** 5640 **(C)** 7200 **(D)** 8640 **(E)**9280

11. The Falcons are beating the Patriots in the Super Bowl by a score of 28-3 in the third quarter. In football, during a team's *possession*, it is possible for the team to score 0, 3, 6, 7, or 8 points. If the Patriots have exactly 4 *possessions* left, and the Falcons do not score any more points, how many ways can the Patriots take the game to overtime (tied score after their 4 *possessions*), where the order in which they score points matters?

    (A) 7 **(B)** 14 **(C)** 24 **(D)** 28 **(E)** 32

12. Heran has \$10 in his bank account, and every day he withdraws some whole number amount of money from the account. He always withdraws at least \$1 and never withdraws more money than the amount left in the account. How many ways can he withdraw money so that his account reaches \$0 in exactly 5 days?

    (A) 126 **(B)** 210 **(C)** 226 **(D)** 252 **(E)** 462

13. Functions $f(x) = x^2 + mx + 2021$ and $g(x) = x^2 - 91x + n$ are factors of $h(x) = x^3 - 138x^2 + kx - 47n$ for real numbers k, m, and n. What is the value of $h(50)$?

    (A) 21 **(B)** 37 **(C)** 42 **(D)** 47 **(E)** 55

14. Let Γ be the circle going through the points $(0,0)$, $(4,2)$, and $(-3,1)$. Let A and B be the points on Γ diametrically opposite to $(4,2)$, and $(-3,1)$ respectively. Let A' and B' be the reflections of A and B over $(-3,1)$ and $(4,2)$ respectively. What is the area of quadrilateral $ABB'A'$?

    (A) 50 **(B)** $50\sqrt{2}$ **(C)** $50\sqrt{3}$ **(D)** $75\sqrt{2}$ **(E)** 100

15. Andrew is playing a game where he writes numbers on a blackboard. He begins by writing a two-digit number n on the blackboard, and for each number k he writes, the next number he writes is $k+1$ if k is odd and $\frac{k}{2}$ if k is even. Andrew stops writing numbers once he writes the number 1. Let x be the maximum amount of numbers Andrew can write on the blackboard and let y be the amount of numbers n that Andrew can start with to write x numbers. What is $x + y$?

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

16. What is the largest possible value of

$$\frac{a^3 + a^2b + ab^2 + b^3}{1 + a + b}$$

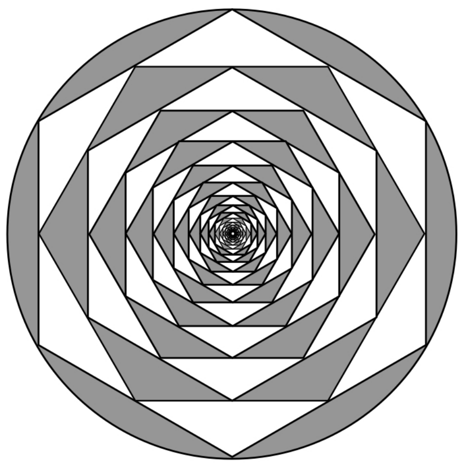
    such that a and b are the side lengths of a right triangle with hypotenuse equal to 1?

    (A) $2 - \sqrt{2}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** 1

17. Let P be the set of all points inside equilateral triangle ABC such that the areas of triangles PBC, PCA, and PAB form an arithmetic sequence not necessarily in that order. The set of points in P splits triangle ABC into several distinct, non-overlapping regions. Let S be the set of the distinct areas of all the regions formed by P. What fraction of the area of triangle ABC is the total area of all regions in S?

    (A) $\frac{1}{3}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{4}$ **(E)** 1

18. A regular hexagon H_1 is inscribed in a unit circle. A regular hexagon H_2 is inscribed in H_1 such that it has vertices at the midpoints of H_1. Define H_3, H_4, ..., similarly such that there are infinitely many hexagons. The area between the circle and H_1 is shaded, as well as between H_2 and H_3, and between all hexagons H_{2n} and H_{2n+1} for all positive integer n. What is the total shaded area?



    (A) $\pi - \frac{13\sqrt{3}}{11}$ **(B)** $\pi - \frac{8}{7}$ **(C)** $\pi - \frac{4\sqrt{3}}{7}$ **(D)** $\pi - \frac{3\sqrt{3}}{2}$ **(E)** $\pi - \frac{6\sqrt{3}}{7}$

19. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x (for example $\{1.5\} = 0.5$). If $\{n\} = \frac{1}{3}$ for postitive rational number n and

$$n^3 + 3n = 6\lfloor n \rfloor^2 + \frac{478}{27},$$

    what are the last two digits of $\lfloor n \rfloor^5$?

    (A) 01 **(B)** 25 **(C)** 32 **(D)** 43 **(E)** 68

20. Let ABC be a triangle with $AB = 3$, $BC = 5$, and $AC = 7$. The angle bisector of the angle at B intersects AC at point D and the circumcircle of triangle ABC at point E distinct from B. What is the area of triangle CBE?

    (A) $10\sqrt{3}$ **(B)** $\frac{15\sqrt{3}}{2}$ **(C)** 20 **(D)** $\frac{245\sqrt{3}}{32}$ **(E)** $\frac{235\sqrt{3}}{16}$

21. In a chess tournament, 10 players play matches against each other where no pair of players play more than once. Exactly 40 matches are played. The probability that each player played at least 6 matches is $\frac{m}{n}$ for relatively prime integers m and n. What is the sum of the distinct prime factors of n?

(A) 80 **(B)** 84 **(C)** 95 **(D)** 96 **(E)** 100

22. A circle ω_1 has radius 4 and is centered at the origin. A circle ω_2 passes through the origin such that ratio of the maximum length of a chord of ω_1 that is tangent to ω_2 to the minimum length of a chord of ω_1 that is tangent to ω_2 is equal to 2. What is the sum of all possible values of the radius of ω_2? Recall that a chord is a line segment and not a line.

(A) $\frac{5\sqrt{3}}{3}$ **(B)** $\frac{7\sqrt{3}}{3}$ **(C)** 3 **(D)** 4 **(E)** $\frac{6+\sqrt{7}}{2}$

23. There are 6 people sitting around a table. The seats are labeled $A, B, C, D, E,$ and F clockwise. They play a game where the person at A starts with a card. Every second, the person with the card passes it to a person next to them, either to the left or right. For example, the person at A can pass the card to either the person at B or the person at F. When the card reaches the person at D, the game ends. How many possible games are there if the game ends in at most 10 seconds?

(A) 64 **(B)** 72 **(C)** 80 **(D)** 84 **(E)** 96

24. Let $P(x)$ be a polynomial with real coefficients such that $P(x) = 4$ if and only if $x = 1, 3, 11, 29$ and $P(x)$ has at most 5 relative max/mins in the interval $[1, 29]$. Let N be the maximum number of solutions to $P(x) = ax + b$ for any real coefficients a and b, for real x in the interval $[1, 29]$. What is the sum of all possible values of N?

(A) 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 20

25. Let ABC be a right triangle with right angle at B, and $AB = \sqrt{3} - 1$ and $BC = \sqrt{3} + 1$. The angle bisector of $\angle ABC$ intersects $\overline{AC}$ at D. The angle bisector of $\angle ADB$ intersects $\overline{AB}$ at E and the angle bisector of $\angle CDB$ intersects BC at F. The circumcircle of triangle BEF intersects $\overline{AC}$ at two distinct points. The distance between these two points can be written as $\frac{a\sqrt{b}-c\sqrt{d}}{e}$ for positive integers a, b, c, d, and e where $\gcd(a, c, e) = 1$ and b and d do not contain a square divisor. What is $a + b + c + d + e$?

(A) 16 **(B)** 19 **(C)** 23 **(D)** 25 **(E)** 30