2022 WMC 10 Solutions and Results

WMC Committee

This manual contains the answers, solutions, projected cutoffs, statistics, and results for the 2022 WMC 10, which ran from October 9 to November 13.

Answers: ACBDE CBEDB CBDBA CDBCD EAABE

1. What is the value of
$$\frac{\left(\frac{\left(\frac{4}{3}\right)}{2}\right)}{1} \div \frac{4}{\left(\frac{3}{\left(\frac{2}{1}\right)}\right)}?$$

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

   Proposed by peace09

   Solution: We compute as follows:
$$\frac{\left(\frac{\left(\frac{4}{3}\right)}{2}\right)}{1} \div \frac{4}{\left(\frac{3}{\left(\frac{2}{1}\right)}\right)} = \frac{\left(\frac{2}{3}\right)}{1} \div \frac{4}{\left(\frac{3}{2}\right)} = \frac{2}{3} \div \frac{8}{3} = \boxed{\textbf{(A)}}\ \frac{1}{4}.$$

2. Two noncongruent rectangles with integer side lengths have the same area. What is the least possible difference between their perimeters?

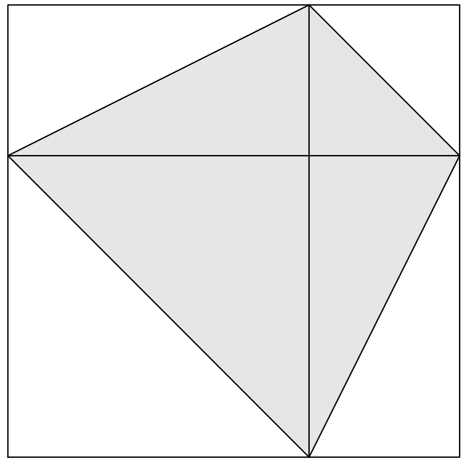
   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Proposed by peace09

   Solution: Intuitively, any two rectangles with the same area and perimeter must be congruent,[1] which eliminates option **(A)**. Also, any two rectangles with integer side lengths have even perimeters, meaning the difference between their perimeters is even as well. This eliminates option **(B)**. Thus, the least possible difference is $\boxed{\textbf{(C)}}$ 2, which is indeed attainable by 1×4 and 2×2 rectangles.

3. Two squares of side lengths 1 and 2 lie within a third square of side length 3, as shown below. What is the area of the shaded region?

[1]Rigorously, having the same area and perimeter implies that each pair of side lengths have the same sum s and the same product p. Consequently, both pairs comprise the roots of $x^2 - sx + p$ by Vieta's Formulas, so the two pairs are equal. It follows that the two rectangles are congruent.



(A) 4 **(B)** $4\frac{1}{2}$ **(C)** 5 **(D)** $5\frac{1}{2}$ **(E)** 6

Proposed by peace09

Solution: Each of the 4 shaded triangles can be paired with a congruent but unshaded triangle, and vice versa. Hence, the shaded region and the combined unshaded regions have the same area, so the shaded region has half the area of the entire square. The requested answer is therefore $\frac{3^2}{2} = \boxed{\textbf{(B)}}\ 4\frac{1}{2}$.

4. How many positive integer multiples of 42 have 42 divisors?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** infinitely many

Proposed by peace09

Solution: Recall that the number of divisors of a positive integer n is given by

$$d(n) = (e_2+1)(e_3+1)(e_5+1)(e_7+1)\ldots,$$

where $e_2, e_3, e_5, e_7, \ldots$ are the exponents of $2, 3, 5, 7, \ldots$ in its prime factorization. Any multiple of 42 has e_2, e_3, and e_7 positive, and thus e_2+1, e_3+1, and e_7+1 at least 2. As such, $d(n)$ has at least 3 factors that are ≥ 2, but if $d(n) = 42$, those 3 factors must be 2, 3, and 7 in some order, leaving no room for any other such factors. Now, it remains to assign 2, 3, and 7 to e_2+1, e_3+1, and e_7+1, which can be accomplished in $3! = \boxed{\textbf{(D)}}\ 6$ ways.

5. Consider the list of integers $1, 2, 2, 3, 4, 4, 5, \ldots, 999, 1000, 1000, 1001$, where every odd number between 1 and 1001 appears exactly once, and every even number between 1 and 1001 appears exactly twice. What is the mean of the list?

(A) 499 **(B)** $499\frac{1}{2}$ **(C)** 500 **(D)** $500\frac{1}{2}$ **(E)** 501

Proposed by A1001

Solution: Since the list is symmetric, its mean equals the average of the smallest and largest values, ergo $\frac{1+1001}{2} = \boxed{\textbf{(E)}}\ 501$.

6. Justin flips a fair coin 4 times and records the outcomes. What is the probability that there is a sequence of 2 or more consecutive heads?

   (A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{5}{8}$

   Proposed by peace09

   Solution: We employ complementary counting, by instead computing the probability that there are no 2 consecutive heads. In such a sequence of flips, there can either be no heads (1 way, $TTTT$), only 1 head (4 ways), or 2 heads that are nonconsecutive (3 ways: $HTHT$, $HTTH$, or $THTH$). Hence, the complement probability is $\frac{1+4+3}{2^4} = \frac{1}{2}$, which makes the requested answer $1 - \frac{1}{2} = \boxed{\textbf{(C)}}\ \frac{1}{2}$.

7. Abby is solving a system of equations in her homework, but the operations in the equations are missing, as shown below.

$$\begin{aligned} x \ \_\ y &= 55 \\ y \ \_\ z &= 89 \\ z \ \_\ x &= 34 \end{aligned}$$

   Not bothering to clarify with her teacher, she fills in each blank with a $+$ or a $-$ at random. What is the probability that there is a solution to the resulting system of equations?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{8}$ **(E)** 1

   Proposed by peace09

   Solution: We proceed with casework on the number of $+$s.

   - *Case 1:* 0 $+$s. Then, the operations are $-, -, -$, but adding all the equations yields $0 = 55 + 89 + 34 = 178$, impossible. So there are 0 ways here.
   - *Case 2:* 1 $+$. Consider when the operations are $+, -, -$.

$$\begin{aligned} x + y &= 55 \\ y - z &= 89 \\ z - x &= 34 \end{aligned}$$

     Adding the second and third equations gives $y - x = 89 + 34 = 123$, so we now have the values of $x + y$ and $y - x$. These uniquely determine x and y, and in turn z as well. It can similarly be derived that $-, +, -$ and $-, -, +$ each yield a unique solution, so there are 3 ways here.

- *Case 3:* 2 +s. Consider when the operations are $+, +, -$.

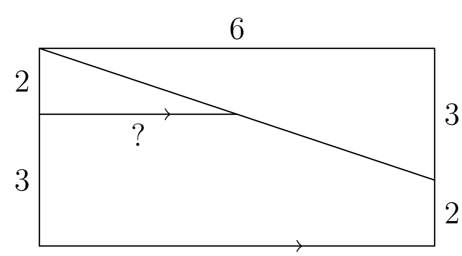
$$x + y = 55$$
$$y + z = 89$$
$$z - x = 34$$

  Adding the first and third equations gives $y+z = 55+34 = 89$, but we already have the value of $y + z$ as 89 from the second equation. So the system is dependent and has infinitely many solutions, because one of the equations is extraneous. Observe however that this particular system has infinitely many solutions only because $55 + 34 = 89$; in the systems for $+, -, +$ and $-, +, +$, similarly adding a two of the equations will yield results such as $34 + 89 = 55$ and $89 + 55 = 34$, which render the systems inconsistent (i.e. self-contradictory and having no solutions). Hence, there is 1 way here.

- *Case 4:* 3 +s. Then, the operations are $+, +, +$, and adding all the equations and dividing by 2 gives $x + y + z = \frac{55+89+34}{2} = 89$. Subtracting each of the original equations $x+y = 55$, $y+z = 89$, and $z+x = 34$ uniquely determines each of z, x, and y respectively. So there is 1 way here.

In all, the desired probability is $\frac{0+3+1+1}{2^3} = \boxed{\textbf{(B)}}\ \frac{5}{8}$.

8. What is the length of the segment marked with a question mark below?

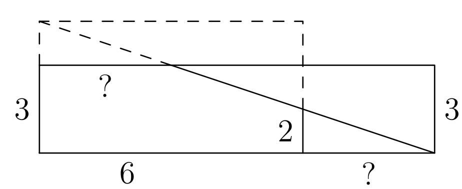


(A) 2 **(B)** $2\frac{1}{2}$ **(C)** 3 **(D)** $3\frac{1}{2}$ **(E)** 4

Proposed by peace09

Solution 1: The right triangle with legs 2 and ? is similar to that with legs 3 and 6 by AA similarity, so $\frac{2}{?} = \frac{3}{6} \implies ? = \boxed{\textbf{(E)}}\ 4$.

Solution 2: First, the area of the whole rectangle is $6(2+3) = 30$. Now, rearrange the pieces of the rectangle as follows:



The area of the rectangle is still 30, whence $3(6+?) = 30 \implies ? = \boxed{\textbf{(E)}}\ 4.$[2]

9. The three-term sequences A and G are arithmetic and geometric, respectively, and they share the same middle term of k, where $k > 0$. The common difference of A and the common ratio of G are both 4, and the product of the terms of A is equal to the sum of the terms of G. What is k?

   (A) $\frac{\sqrt{70}}{2}$ **(B)** $\frac{5\sqrt{3}}{2}$ **(C)** $2\sqrt{5}$ **(D)** $\frac{\sqrt{85}}{2}$ **(E)** $\frac{3\sqrt{10}}{4}$

   Proposed by peace09

   Solution: By the given we have that $A = k-4, k, k+4$ and $B = \frac{k}{4}, k, 4k$, whence

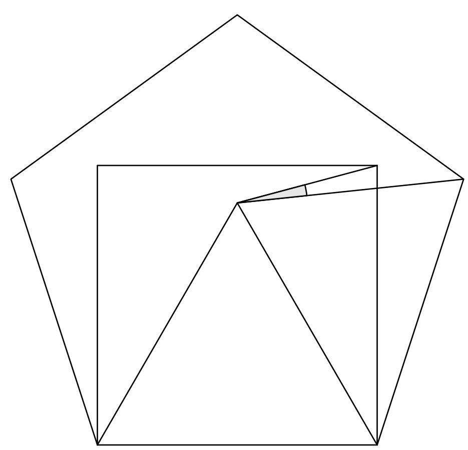
   $$\text{sum}(A) = (k-4)\cdot k\cdot(k+4) = \frac{k}{4} + k + 4k = \text{product}(B).$$

   Evidently $k \neq 0$, so dividing by k yields

   $$(k-4)\cdot(k+4) = k^2 - 16 = \frac{1}{4} + 1 + 4.$$

   Thus, $k^2 = \frac{1}{4} + 1 + 4 + 16 = \frac{85}{4} \implies k = \boxed{\textbf{(D)}}\ \frac{\sqrt{85}}{2}.$

10. An equilateral triangle lies inside a square that lies inside a regular pentagon, and all three polygons share a side, as shown below. What is the degree measure of the marked angle?

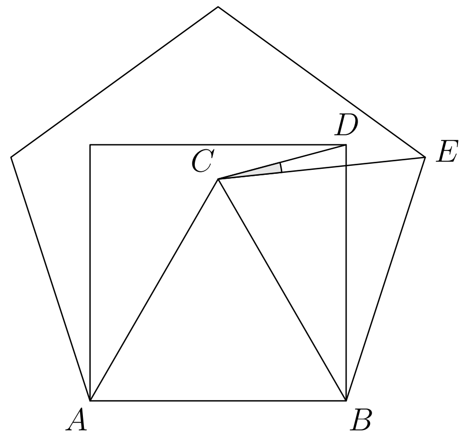


   (A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

   Proposed by peace09

[2]Albeit silly, this solution was the initial one, as the problem was written backwards around this rearrangement idea (cf. 2022 AMT Power 7a, which has the same author (yours truly!)). In that vein, the original problem statement provided the area of the rectangle instead of the length 2.

Label certain points in the diagram as follows.



Solution 1: We can compute $\angle DCE$ by taking $\angle BCD - \angle BCE$. One one hand, since $\angle CBD = \angle ABD - \angle ABC = 90° - 60° = 30°$ and $BC = BD$, we have that

$$\angle BCD = \frac{180° - \angle CBD}{2} = \frac{180° - 30°}{2} = 75°.$$

Similarly, because $\angle CBE = \angle ABE - \angle ABC = 108° - 60° = 48°$,

$$\angle BCE = \frac{180° - \angle CBE}{2} = \frac{180° - 48°}{2} = 66°.$$

Hence, $\angle DCE = \angle BCD - \angle BCE = 75° - 66° = \boxed{\textbf{(B)}}\ 9°$.

Solution 2: Observe that C, D, and E lie on a circle centered at B since $BC = BD = BE$, so by the Inscribed Angle Theorem, $\angle DCE = \frac{\angle DBE}{2}$. Substituting $\angle DBE = \angle ABE - \angle ABD = 108° - 90° = 18°$ yields $\frac{18°}{2} = \boxed{\textbf{(B)}}\ 9°$.

11. In how many ways can 18 indistinguishable adults and 6 indistinguishable children stand in a line such that any 2 children are separated by at least 3 adults?

    (A) 60 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

    Proposed by peace09

    Solution: Observe that for each such arrangement, we can remove 3 adults between each of the 5 pairs of adjacent children (i.e. pairs of children with no children between them) to obtain an arbitrary arrangement of $18 - 3 \cdot 5 = 3$ adults and 6 children. Conversely, starting with any arrangement of 3 adults and 6 children, adding 3 adults between each pair of adjacent children yields an arrangement of 18 adults and 6 children such that any 2 children are separated by at least 3 adults. Hence, the number of desired arrangements equals the number of arrangements of 3 adults and 6 children, ergo $\binom{9}{3} = \boxed{\textbf{(C)}}\ 84$.

12. The six-digit base-two integer $ABCDEF_{\text{two}}$ and the six-digit base-ten integer $ABCDEF_{\text{ten}}$ are both multiples of 6. What is the value of the six-digit base-six integer $ABCDEF_{\text{six}}$, expressed in base-ten?

(A) 7818 **(B)** 7998 **(C)** 8028 **(D)** 9078 **(E)** 9128

Proposed by peace09

Solution: First, $ABCDEF_{\text{two}}$ being defined implies that each of A, B, C, D, E, and F is either 0 or 1; we shall refer to this fact as $(*)$ passim. Additionally, since $ABCDEF$ is a *six-digit* integer (in either base), $A \neq 0$, so $(*)$ forces $A = 1$.

Consider the condition that $ABCDEF_{\text{ten}}$ is a multiple of 6. As a consequence:

- it is a multiple of 2, so its last digit F is even; $(*)$ forces it to be 0.
- it is a multiple of 3, so the sum of its digits $s = A + B + C + D + E + F$ is a multiple of 3. Now, $(*)$ forces $0 \leq s \leq 6$, with $s = 0$ and $s = 6$ impossible due to $A = 1$ and $F = 0$, respectively. Hence, the only possibility is $s = 3$.

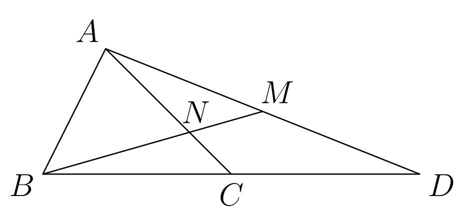
So in particular, $ABCDEF$ contains exactly three 0s and exactly three 1s. Then $ABCDEF_{\text{two}}$ can range from $100110_{\text{two}} = 38$ to $111000_{\text{ten}} = 56$, between which the only multiples of 6 are 42, 48, and 54. Of these, we find that only $42 = 101010_{\text{two}}$ works, which makes the requested answer $101010_{\text{six}} = \boxed{\textbf{(B)}}$ 7998.

13. In triangle ABC with area 60, side $\overline{BC}$ is extended past C to point D such that $BC = CD$. Let M be the midpoint of segment $\overline{AD}$, and let N be the intersection of segments $\overline{AC}$ and $\overline{BM}$. What is the area of triangle AMN?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 20 **(E)** 30

Proposed by A1001

Solution:



Observe that since $\overline{AC}$ and $\overline{BM}$ are medians of $\triangle ABD$, N is its centroid. It follows that $\triangle AMN$ has $\frac{1}{6}$ the area of $\triangle ABD$, which in turn has twice the area of $\triangle ABC$. So $[AMN] = \frac{2}{6} \cdot [ABC] = \frac{2}{6} \cdot 60 = \boxed{\textbf{(D)}}$ 20.

14. Let n be a randomly chosen divisor of 30^6. What is the probability that $2n$ has more divisors than $3n$ and $5n$?

(A) $\frac{12}{49}$ **(B)** $\frac{13}{49}$ **(C)** $\frac{14}{49}$ **(D)** $\frac{15}{49}$ **(E)** $\frac{16}{49}$

Proposed by peace09

Solution: Let $n = 2^a 3^b 5^c$ where $0 \le a, b, c \le 6$, and suppose $d(n)$ is the divisor function. Then $2n = 2^{a+1}3^b5^c$, $3n = 2^a3^{b+1}5^c$, and $5n = 2^a3^b5^{c+1}$, so

$$\begin{aligned} d(2n) &= (a+2)(b+1)(c+1) \\ d(3n) &= (a+1)(b+2)(c+1) \\ d(5n) &= (a+1)(b+1)(c+2). \end{aligned}$$

As a result, $d(2n) > d(3n)$ if and only if

$$(a+2)(b+1)(c+1) > (a+1)(b+2)(c+1) \iff \frac{a+2}{b+2} > \frac{a+1}{b+1} \iff a < b.$$

Analogously, $d(2n) > d(5n)$ if and only if $a < c$. Hence, given a fixed a, each of b and c has $6 - a$ possibilities in $a+1, a+2, \ldots, 6$, for $(6-a)^2$ feasible (a, b, c). Summing over $a = 0, 1, \ldots 6$ gives us

$$6^2 + 5^2 + \cdots + 0^2 = \frac{6 \cdot 7 \cdot 13}{6} = 91,$$

and dividing by $7^3 = 343$ yields $\boxed{\textbf{(B)}}$ $\frac{13}{49}$.[3]

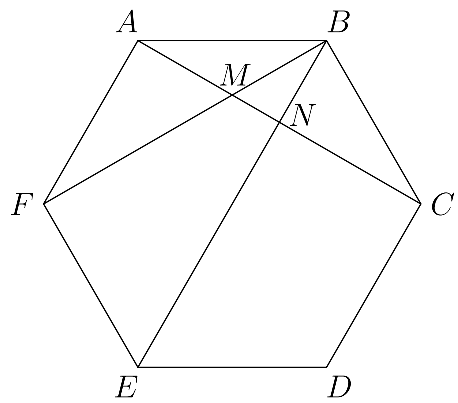
15. Three diagonals are drawn in a regular hexagon of side length 1, splitting the hexagon into several regions. What is the least possible area of such a region?

(A) $\frac{\sqrt{3}}{24}$ **(B)** $\frac{\sqrt{3}}{18}$ **(C)** $\frac{\sqrt{3}}{16}$ **(D)** $\frac{\sqrt{3}}{12}$ **(E)** $\frac{\sqrt{3}}{8}$

Proposed by peace09

Solution: By intuition, the smallest possible region is produced when two diagonals stemming from a common vertex form the smallest angle possible, and the third diagonal tightly bounds that angle, as follows.

[3]Due to **brainfertilzer** (https://artofproblemsolving.com/community/c3038036h2938140p26294994). The author's original solution used the fact that given there is a unique maximum amongst $d(2n), d(3n), d(5n)$, the probability that $d(2n)$ is that unique maximum (the requested probability) is $\frac{1}{3}$ by symmetry, so it suffices to compute the probability that the former occurs and divide by 3.



Above, the two diagonals stemming from a common vertex are $\overline{BE}$ and $\overline{BF}$, which form the smallest angle possible of $\angle EBF = 30°$, and the third diagonal $\overline{AC}$ bounds $\angle EBF$ as tightly as possible.

The area we seek is $[BMN]$. First, triangles ABM and CFM are similar in ratio $\frac{AB}{CF} = \frac{1}{2}$, so $\frac{AM}{CM} = \frac{1}{2}$ or, equivalently, $\frac{AM}{AC} = \frac{1}{3}$. Additionally, $AN = CN$ by symmetry, which yields that $\frac{CN}{AC} = \frac{1}{2}$. Therefore,

$$\frac{MN}{AC} = 1 - \frac{AM}{AC} - \frac{CN}{AC} = 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}.$$

Now, triangles BMN and BAC share an altitude from B, implying $\frac{[BMN]}{[BAC]} = \frac{MN}{AC} = \frac{1}{6}$. Then, since ABC is a $30-30-120$ triangle with leg length $AB = 1$, it has area $\frac{\sqrt{3}}{4}$, giving us the requested answer of $[BMN] = \boxed{\textbf{(A)}}\ \frac{\sqrt{3}}{24}$.

16. Three rational fractions have denominators of 20, 21, and 30 when expressed in lowest terms. What is the least possible denominator that the sum of the three fractions, expressed in lowest terms, could have?

(A) 1 **(B)** 7 **(C)** 28 **(D)** 84 **(E)** 420

Proposed by peace09

Solution 1: Let the three fractions be $\frac{a}{20}$, $\frac{b}{21}$, and $\frac{c}{30}$, where a, b, and c are relatively prime to their respective denominators, so that their sum is

$$\frac{a}{20} + \frac{b}{21} + \frac{c}{30} = \frac{21a + 20b + 14c}{420}.$$

To obtain the least possible denominator, we wish to cancel as many factors of 420 from the numerator. Since $21a$ alone is odd, we cannot cancel any factors of 2; analogously, since $20a$ alone is not a multiple of 7, we cannot cancel the factor of 7. However, $21a$ is a multiple of 3, and $20b + 14c$ could possibly be a multiple of 3, so a cancellation in the 3's could occur. Similarly, since $21a + 14c$

can be a multiple of 5, the 5's can possibly cancel. Hence, the minimum possible denominator is $2^2 \cdot 7 = \boxed{\textbf{(C)}}\ 28$, attainable by $\frac{1}{20} + \frac{2}{21} + \frac{1}{30} = \frac{5}{28}$.

Solution 2: We approach the problem intuitively as follows. If, for example, $\frac{1}{2}$ were added to several fractions all with odd denominators, the denominator of the sum would contain a factor of 2, for the $\frac{1}{2}$ had no other fractions with even denominators to "cancel out the 2" with. As an example, the factor of 2 remains in summing $\frac{1}{2} + \frac{1}{3} + \frac{4}{5} = \frac{49}{30}$, but "cancels out" in $\frac{1}{2} + \frac{3}{2} = 2$. This logic can readily be extended to all primes other than 2.

Back to the original problem, we see that 3 can be cancelled in cooperation of 21 and 30, and 5 can be cancelled in cooperation of 20 and 30; but 7 divides 21 alone and thus cannot be cancelled. It seems as though 2 can be cancelled between 20 and 30; but since $20 = 2^2 \cdot 5$ contains "quarter components" while $30 = 2 \cdot 3 \cdot 5$ contains only "half components", we in fact cannot cancel any factors of 2. (For example, observe that $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ has no such cancellation.)

In all, we can cancel the 3's and 5's but not 2's and 7's, ergo $2^2 \cdot 7 = \boxed{\textbf{(C)}}\ 28$.

17. In rectangle $ABCD$ with $AB = 12$ and $BC = 9$, points W, X, Y, and Z lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that AW and CX are integers. If $WXYZ$ is a parallelogram with area 50, what is its perimeter?

(A) $11\sqrt{5}$ **(B)** $12\sqrt{5}$ **(C)** $13\sqrt{5}$ **(D)** $14\sqrt{5}$ **(E)** $15\sqrt{5}$

Proposed by peace09

Solution: Let $AW = m$ and $CX = n$, so $BW = 12 - m$ and $BX = 9 - n$. Instead of considering $[WXYZ]$, we look at its "complement":

$$[AZW] + [BWX] + [CXY] + [DYZ] = [ABCD] - [WXYZ] = 58.$$

Substituting $[AZW] = [CXY] = \frac{1}{2}mn$ and $[BWX] = [DYZ] = \frac{1}{2}(12-m)(9-n)$, we have that

$$\begin{aligned} mn + (12 - m)(9 - n) &= 2mn - 9m - 12n + 108 = 58 \\ \implies 2mn - 9m - 12n + 54 &= (6 - m)(9 - 2n) = 4. \end{aligned}$$

Since $9 - 2n$ is odd and a factor of 4, we must have $|9 - 2n| = 1$, so $n = 4$ or $n = 5$. It follows that $m = 2$ or $m = 10$ respectively; either yields the side lengths of the parallelogram to equal $\sqrt{2^2 + 4^2} = 2\sqrt{5}$ and $\sqrt{5^2 + 10^2} = 5\sqrt{5}$. Finally, the perimeter of parallelogram $WXYZ$ is

$$2\sqrt{5} + 5\sqrt{5} + 2\sqrt{5} + 5\sqrt{5} = \boxed{\textbf{(D)}}\ 14\sqrt{5}.$$

18. A field trip group has 3 adults, 3 boys, and 4 girls. How many (possibly empty) subsets of the 10 people can pose for a photo if at least one adult is left out to take the photo, one adult can't pose alone with one child, and each child refuses to pose with only children of the opposite gender? For example, there can be 1 adult, 1 boy, and 1 girl, but there can't be only 1 boy and 1 girl.

(A) 807 **(B)** 814 **(C)** 821 **(D)** 828 **(E)** 835

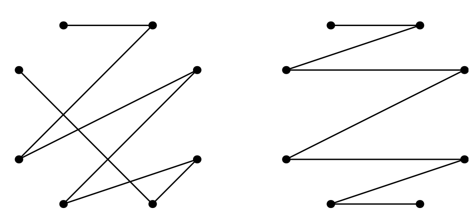
Proposed by peace09

Solution: Proceed by complementary counting, as follows: the total number of subsets without restrictions is $2^{10} = 1024$, and for each statement, we subtract any subsets contradicting it. Accordingly,

- for the first statement, all the adults are posing in $2^7 = 128$ subsets (the number of ways to choose the children).
- for the second statement, one adult poses alone with one child in $3 \cdot 7 = 21$ subsets (3 adults, 7 children).
- for the third statement, a boy (3 choices) poses only with girls ($2^4 - 1 = 15$ choices, excluding the empty set) in $3 \cdot 15 = 45$ subsets. Similarly, a girl poses only with boys in $4 \cdot (2^3 - 1) = 28$ subsets. However, we have doublecounted the subsets with only one boy and one girl, who can be chosen in $3 \cdot 4 = 12$ ways. Hence, there are $45 + 28 - 12 = 61$ invalid subsets here.[4]

Therefore, there are $1024 - 128 - 21 - 61 = \boxed{\textbf{(B)}}$ 814 valid subsets, as requested.

19. Ari the ant visits each vertex of a convex octagon exactly once in some order, crawling in straight lines from vertex to vertex. What is the probability that his path self-intersects? For example, the left path shown below is valid, but the right path is invalid.



(A) $\frac{103}{105}$ **(B)** $\frac{62}{63}$ **(C)** $\frac{311}{315}$ **(D)** $\frac{104}{105}$ **(E)** $\frac{313}{315}$

Proposed by peace09

[4] Alternatively, a valid subset with no adults either has $\neq 1$ boys and $\neq 1$ girls, for $(2^3 - 3)(2^4 - 4) = 60$ ways, or one child, for 7 ways. So there are $60 + 7 = 67$ valid subsets with no adults, and subtracting from the total $2^7 = 128$ renders 61 invalid, as before.

Solution: We consider the complementary probability. Starting from any given vertex, if Ari travels in a straight line ℓ to any nonadjacent vertex, the remaining 6 vertices are divided into two sets separated by ℓ, and he will at some point cross over from one set to the other, intersecting ℓ (bad!). Hence, at every vertex, Ari must next travel to one of the 2 adjacent vertices.

As a result, out of $7! = 5040$ total paths (where we fix the first vertex, without loss of generality), there are $2^6 = 64$ feasible paths, since there are 2 choices for the 2nd, 3rd, ..., 7th vertices as aforementioned (the last vertex is consequently forced). Therefore, the probability that Ari's path does *not* self-intersect is $\frac{64}{5040} = \frac{4}{315}$, which makes the requested answer $\boxed{\textbf{(C)}}\ \frac{311}{315}$.

20. A positive integer N is said to be *nearly-square* if $\sqrt{N}$ can be expressed in simplest radical form as $a\sqrt{b}$, where a and b are positive integers greater than 1 with $a > b$. How many of the first 1000 positive integers are nearly-square?

    (A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

    Proposed by peace09

    Solution: Setting $a\sqrt{b} = \sqrt{N}$ and squaring yields $a^2b = N$. Observe that if $b \geq 10$, we have that $N = a^2b > b^2b \geq 1000$, which is too large; and b cannot be divisible by the square of a prime. Hence, the only possibilities are $b = 2, 3, 5, 6, 7$.

    When $b = 2$, the least possible value of a is 3, and because $a^2b = N \leq 1000$, its greatest possible value is $\lfloor\sqrt{\frac{1000}{2}}\rfloor$, for $\lfloor\sqrt{\frac{1000}{2}}\rfloor - 2$ possibilities. Similarly, $b = 3$ has $\lfloor\sqrt{\frac{1000}{3}}\rfloor - 3$ possible choices for a, and continuing in this fashion gives

    $$\sum_{b=2,3,5,6,7}\left(\left\lfloor\sqrt{\frac{1000}{b}}\right\rfloor - b\right) = 20 + 15 + 9 + 6 + 4 = \boxed{\textbf{(D)}}\ 54,$$

    the requested answer.

21. The *Centinacci sequence* $\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$ is defined by $C_n = 0$ for $n < 0$, $C_0 = 1$, and $C_n = C_{n-1} + C_{n-2} + \cdots + C_{n-100}$ for $n > 0$. What is the largest positive integer k such that 2^k divides C_{123}?

    (A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

    Proposed by peace09

Solution: Experimenting, we compute $C_1 = 1$, $C_2 = 2$, $C_3 = 4$, and $C_4 = 8$, and it is apparent that $C_n = 2^{n-1}$ for $1 \le n \le 100$. Now:

$$\begin{aligned}
C_{101} &= 2^{99} + 2^{98} + 2^{97} + 2^{96} + \cdots + 1 = 2^{100} - 1 \\
C_{102} &= (2^{100} - 1) + 2^{99} + 2^{98} + 2^{97} + \cdots + 2 = 2^{101} - (1 + 2) \\
C_{103} &= (2^{101} - 1 - 2) + (2^{100} - 1) + 2^{99} + 2^{98} + \cdots + 4 = 2^{102} - (2 \cdot 1 + 1 \cdot 2 + 4) \\
C_{104} &= (2^{102} - 2 \cdot 1 - 2 - 4) + (2^{101} - 1 - 2) + (2^{100} - 1) + 2^{99} + \cdots + 8 \\
&= 2^{103} - (4 \cdot 1 + 2 \cdot 2 + 1 \cdot 4 + 8).
\end{aligned}$$

Continuing in this fashion (rigorizable by induction), we have that

$$C_{123} = 2^{122} - (2^{21} \cdot 1 + 2^{20} \cdot 2 + \cdots + 1 \cdot 2^{21} + 2^{22}) = 2^{122} - 24 \cdot 2^{21}.$$

Rewriting gives us $2^{122} - 3 \cdot 2^{24}$, which makes the requested answer $\boxed{\textbf{(E)}}$ 24.

22. Three right circular cones each with base radius 15 and height 20 share the same vertex, and their bases are pairwise tangent. The height of the smallest possible fourth cone that contains the other three cones in its interior can be expressed in the form $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?

(A) 316 **(B)** 317 **(C)** 318 **(D)** 319 **(E)** 320

Proposed by peace09

Solution: Let V be the common vertex, C_1, C_2, C_3 the centers of the bases, and T_1, T_2, T_3 the tangency points between the bases. Additionally, let O_C and O_T be the centers of $\triangle C_1C_2C_3$ and $\triangle T_1T_2T_3$, respectively, and orient $\overrightarrow{VO_CO_T}$ vertically.

Clearly, in order to minimize the volume of the fourth cone, it is optimal for it to have vertex V and vertical axis as well, so that its base lies on the plane determined by the (vertically) highest points on the other three bases. By inspection, these three highest points are the midpoints M_1, M_2, M_3 of arcs T_2T_3, T_3T_1, T_1T_2, and thus the length we seek is VO_M, where O_M is the center of $\triangle M_1M_2M_3$. (Refer to the following diagram: https://www.geogebra.org/calculator/rhtjzev2.)

We first compute VO_C, which is a height in tetrahedron $VC_1C_2C_3$, by letting P be the midpoint of $\overline{C_2C_3}$. Then, $C_2P = \frac{15 \cdot 20}{25} = 12$ from $\triangle VC_2T_1$, and $VP = \sqrt{20^2 - 12^2} = 16$ from $\triangle VC_2P$. Consequently, the $30-60-90$ triangle $\triangle C_2O_CP$ yields $O_CP = \frac{C_2P}{\sqrt{3}} = 4\sqrt{3}$, and Pythagoras on $\triangle VO_CP$ gives

$$VO_C = \sqrt{VP^2 - O_CP^2} = \sqrt{16^2 - (4\sqrt{3})^2} = 4\sqrt{13}.$$

As a result, since $\frac{VO_C}{VO_T} = \frac{VP}{VT_1} = \frac{16}{25}$ by similar triangles $\triangle VO_CP \sim \triangle VO_TT_1$, we have that $VO_T = \frac{25\sqrt{13}}{4}$ and $O_CO_T = VO_T - VO_C = \frac{9\sqrt{13}}{4}$. Furthermore, $O_TT_1 = \frac{25}{16}O_CP = \frac{25\sqrt{3}}{4}$, implying $T_1T_2 = T_2T_3 = T_3T_1 = \frac{75}{4}$.

Now, it suffices to compute the length O_MO_T. Observe that if Q is the midpoint of $\overline{T_2T_3}$, then $\frac{O_MO_T}{O_MO_C} = \frac{M_1Q}{M_1C_1}$ by similar triangles; equivalently, $\frac{O_CO_T}{O_CO_M} = \frac{C_1Q}{C_1M_1}$, where we already have that $O_CO_T = \frac{9\sqrt{13}}{4}$ and $C_1M_1 = 15$. Also, $C_1Q = \sqrt{15^2 - (\frac{75}{8})^2} = \frac{15\sqrt{39}}{8}$ by Pythagoras on $\triangle C_1QT_2$, so $O_CO_M = \frac{9\sqrt{13}}{4} \cdot \frac{15}{15\sqrt{39}/8} = 6\sqrt{3}$.

Hence, $VO_M = VO_C + O_CO_M = 4\sqrt{13} + 6\sqrt{3} = \sqrt{208} + \sqrt{108}$, ergo $\boxed{\textbf{(A)}}$ 316.[5,6]

23. For all nonnegative integers n, let $f(n)$ be the least number of pennies, dimes, and quarters needed to amount to n cents in total. What is the largest positive integer k such that $f(n) = k$ for some n between 0 and 99?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by peace09

Solution: At first, it seems as though it is optimal to maximize the number of quarters (i.e. use as many quarters as possible), then maximize the number of dimes, and then fill the remaining gap with pennies (according to the *greedy algorithm*). However, since pennies are so inefficient compared to dimes and quarters, it is actually optimal to minimize the number of pennies, and amount to the remaining cost with dimes and quarters (the "ungreedy algorithm").

For example, when $n = 43$, the greedy algorithm uses 1 quarter, 1 dime, and 8 pennies, for 10 coins, but the ungreedy algorithm only uses 4 dimes and 3 pennies (first fixing the number of pennies at the minimal 3, then filling what's left with dimes and quarters), for 7 coins. It follows that $f(43) = 7$.

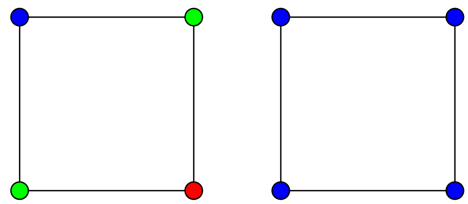
Now, observe that the value(s) of $n \in [0, 99]$ maximizing $f(n)$ end(s) in either 4 or 9, i.e., $n \equiv 4 \pmod 5$. This is because if n has a remainder of $m \not\equiv 4 \pmod 5$, then $n' = n - m + 4$ (the next largest $4 \pmod 5$ integer) requires $4 - m$ pennies more than n; there is no better way to amount to n' cents with respect to $f(n)$.

The rest is computation. As for n ending in 4, we see that $f(4)$, $f(14)$, $f(24)$, $f(34)$, and $f(44)$ equal 4, 5, 6, 7, and 8 using 0 quarters, and $f(54)$, $f(64)$, $f(74)$, $f(84)$, and $f(94)$ equal 6, 7, 8, 9, and 10 using 2 quarters, according to the aforementioned strategy. As for n ending in 9, $f(9), f(19) = 9, 10$ using 9 pennies, but for greater values we can avoid using ≥ 5 pennies: $f(29), f(39), f(49), f(59), f(69)$ $= 5, 6, 7, 8, 9$ using 1 quarter, and $f(79), f(89), f(99) = 7, 8, 9$ using 3 quarters.

All in all, the maximum of $f(n)$ in the domain $n \in [0, 99]$ is $\boxed{\textbf{(A)}}$ 10, attained at the values $n = 19$ and $n = 94$.

[5]Isn't it so amazing that $m \equiv n \pmod{100}$? The numbers for this problem virtually fell from the sky.
[6]For an even faster and more ingenious solution, see aops.com/community/p27693445.

24. Each vertex of a cube is colored red, green, or blue. Then, for each face, define its *color-count* to be the unordered list of the frequencies of all colors present amongst its vertices. For example, the left face shown below has color-count $1, 1, 2$ (equivalently $1, 2, 1$ or $2, 1, 1$), while the right face has color-count 4.



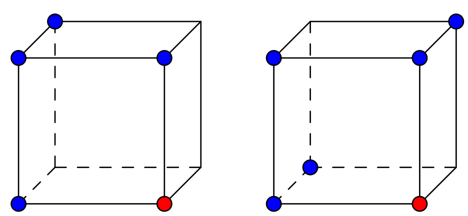
For how many such colorings is the color-count of each face the same? (Two colorings are considered identical if one can be rotated to match the other.)

(A) 30 **(B)** 33 **(C)** 36 **(D)** 39 **(E)** 42

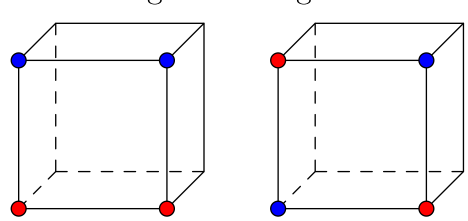
Proposed by peace09

Solution: We proceed with casework on the uniform color-count.

- *Case 1:* 4. Here, all the vertices must be colored uniformly, so 3 colorings.
- *Case 2:* $1, 3$. WLOG let the front face have 3 blue and 1 red vertex, as shown below. The left and top faces, already having 2 blue vertices, must consequently have 3 blue vertices. They can either share their third blue vertex (left below) or not (right below), yielding the following configurations.

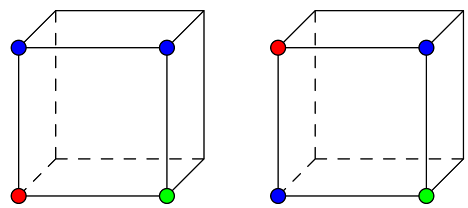


At left above, the remaining 3 vertices are forced to be red, giving $\binom{3}{2} = 3$ colorings in general (blue-red, red-green, or green-blue). At right above, the back bottom-right vertex must be blue, but the back top-left vertex can be either red or green. So in general, the "dominant" color (blue above) can be chosen in 3 ways, and the other 2 vertices can be chosen in 3 ways (red-red, green-green, or red-green), for $3 \cdot 3 = 9$ total. Ergo $3 + 9 = 12$ in this case.
- *Case 3:* $2, 2$. WLOG let the front face have 2 blue and 2 red vertices; these can be arranged in the following two configurations:



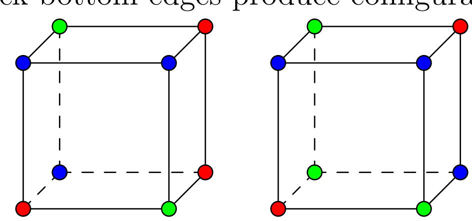
In both configurations, all the remaining vertices must be colored red or blue. At left above, the back face's top and bottom edges must be colored red and blue, respectively, for $\binom{3}{2} = 3$ colorings in general (BR, RG, or GB). At right above, each vertex of the back faces must be colored opposite to the vertex directly in front of it (blue-red and red-blue),[7] which gives us $\binom{3}{2} = 3$ colorings as before. Hence, the total for this case is $3 + 3 = 6$.

- *Case 4:* $1, 1, 2$. WLOG let the front face have 2 blue, 1 red, and 1 green vertices; these can be arranged in the following two configurations:



At the left, if the back face's top edge were colored RG, the back bottom edge would be colored GR according to the left and right faces, but then the bottom and back faces would have a color-count of $2, 2$, contradiction. As a result, the back top edge must be colored GR, and the back bottom edge can be either BB, BR, RB, BG, or GB.

  - The cube with BB as the bottom back edge yields 3 colorings in general, because choosing the color with 4 vertices (blue in our example) determines the rest of the cube.
  - The other four back bottom edges produce configurations as follows:



Careful mental cube-turning reveals that choosing the color with 2 vertices along a space diagonal (green and red above) gives 2 possibilities for the rest of the cube, yielding 6 colorings in general.

At the right, most colorings of the back face give configurations identical to those at the left; the only additional configurations are those with 2 colors each forming an equilateral triangle of side length $\sqrt{2}$, and the third color forming a space diagonal (e.g. BGRG in the back face, going clockwise). Consequently, we have 3 colorings here, and the aggregate number of colorings with $1, 1, 2$ as the color-count is $3 + 6 + 3 = 12$.

The cumulative number of such colorings is therefore $3 + 12 + 6 + 12 = \boxed{\textbf{(B)}}\ 33$.

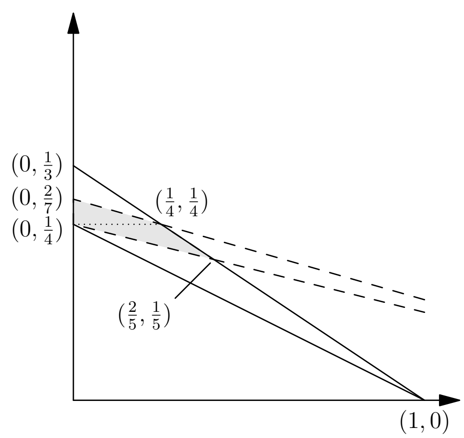
[7] The only other possibility is to color each such vertex identically to the vertex in front of it, but that would result in the same configuration as the left cube (by rotation).

25. A negligibly small grasshopper hops onto a random point on a horizontal sidewalk with meter-long tiles separated by negligibly thin grooves. It proceeds to hop in the horizontal direction with each hop of a constant length less than a meter, which it chooses uniformly at random. Exactly 4 of its hops (including the initial hop onto the sidewalk) land in the first tile it hops across. The probability that exactly 4 hops land in the second tile it hops across can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

Proposed by peace09

Solution: Let A be the event that 4 hops land in the first tile and B be the event that 4 hops land in the second tile, so we seek the conditional probability $P(B \text{ if } A) = \frac{P(A \text{ and } B)}{P(A)}$. To compute this, we proceed by geometric probability. Let x be the horizontal position (with respect to the first tile) of the first hop onto the sidewalk and y be the length of each hop, where $x, y \in [0, 1]$.



First, A occurs if and only if the fourth hop $x + 3y$ lands in the first tile and the fifth hop $x + 4y$ lands in the second tile, i.e., $x + 3y \le 1 \le x + 4y$. This region is bounded by the solid lines above. Next, B occurs if and only if the eighth hop $x + 7y$ lands in the second tile and the ninth hop $x + 8y$ lands in the third tile, i.e., $x + 7y \le 2 \le x + 8y$. This region is bounded by the dashed lines above. Accordingly, A and B both occur in the shaded region.

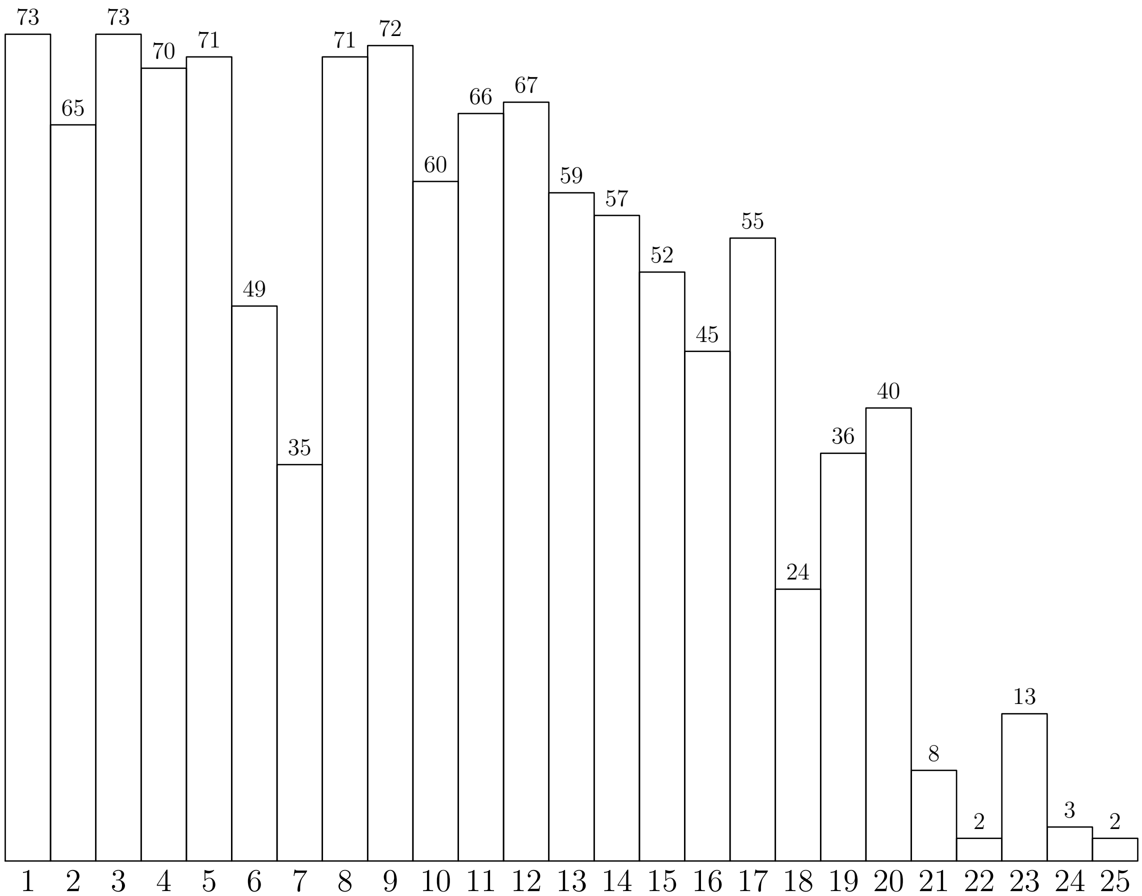
Now, the region bounded by the solid lines can be interpreted as having base $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ and height 1, so it has area $P(A) = \frac{1/12 \cdot 1}{2} = \frac{1}{24}$. Furthermore, by drawing the dotted line shown, the shaded region can be interpreted as having base $\frac{1}{4}$ and height $\frac{2}{7} - \frac{1}{5} = \frac{3}{35}$, so it has area $P(A \text{ and } B) = \frac{1/4 \cdot 3/35}{2} = \frac{3}{280}$. Hence, $P(B \text{ if } A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{3/280}{1/24} = \frac{9}{35}$, which makes the requested answer $\boxed{\textbf{(E)}}$ 44.

Contest Summary: Thank you to all 74 people who submitted; we received nearly 50 more submissions than last year's contest! Congratulations to the following top scorers:

1. Anonymous (141/150)
2. john0512 (135/150)
3. GrantStar (130.5/150)

Comprehensive results can be accessed on the pages that follow.

Solves per Problem:



Cutoffs:[8]

- Average Score: 60
- AIME Cutoff: 96
- Distinction: 108
- Distinguished Honor Roll: 120

[8]Projected and based on personal opinion and feedback. Consider them with a grain of salt.

Username	Score
Anonymous	141
john0512	135
GrantStar	130.5
bobthegod78	126
mathtiger6	126
KevinYang2.71	126
Anonymous	124.5
Anonymous	124.5
channing421	124.5
Anonymous	123
awesomehuman	120
HamstPan38825	120
Anonymous	118.5
Anonymous	117
Nishanth1234	117
Anonymous	115.5
pi271828	114
Anonymous	114
Anonymous	114
SouradipClash_03	114
JustAnotherNoob28	112.5
binderclips1	112.5
Anonymous	112.5
Anonymous	112.5
ftwmaster65	112.5
sidchukkayapally	111
Math4Life7	111
Anonymous	111
ambiguous	111
brainfertilzer	109.5
mannshah1211	108
gladIasked	108
Anonymous	106.5
Anonymous	106.5
Anonymous	106.5
Taco12	105
miguel00	105
Anonymous	105
Anonymous	103.5
Gumball	103.5
Significant	103.5
Sky9313	103.5
hansenhe	103.5

Username	Score
Geometry285	102
Spectator	102
Anonymous	102
pieMax2713	102
Anonymous	100.5
ABCSD	100.5
zap4pan	99
samrocksnature	99
Anonymous	97.5
akliu	97.5
stonkpotato	96
RedFireTruck	96
idunknowhowaboutyou	94.5
S308663	94.5
Anonymous	94.5
RithwikGupta	93
ilikemath40	91.5
Anonymous	91.5
Anonymous	90
resources	88.5
YBSuburbanTea	87
Anonymous	85.5
Anonymous	85.5
metricpaper	85.5
BrainyBYH	82.5
Dogman63	76.5
Turtle09	75
Anonymous	72
Anonymous	72
programmeruser	64.5
Anonymous	46.5