Texas MATHCOUNTS AMC 10

1. This is a 75 minute, 25 question test.

2. There are five answer choices for each question. A correct answer is worth 6 points, a blank answer is worth 1.5 points, and an incorrect answer is worth 0 points.

3. No aids except for the test itself, paper, writing utensils, and erasers are allowed.

1. Find $14 + 34 \cdot 14 + 34$.
(A) 514 (B) 524 (C) 525 (D) 526 (E) 556

2. Let $ABCD$ be a square with side length 1. Let E be the midpoint of CD. Find the area of triangle AED.
(A) 1/8 (B) 1/4 (C) 1/2 (D) 5/8 (E) 7/8

3. Find the remainder when 14_{34} (14 in base thirty-four) is divided by 10_{10}.
(A) 1 (B) 3 (C) 4 (D) 8 (E) 9

4. Alice and Bob are running in a 100-meter race. Alice can run 100 meters in 15 seconds. Bob can run 100 meters in 12 seconds. Assuming that both people run at constant speeds, how far away from the finish line should Alice start so that she finishes three seconds before Bob?
(A) 40 (B) 50 (C) 60 (D) 80 (E) 100

5. Let $f(n)$ be $f(s(n))$ whenever $n \geq 10$, where $s(n)$ denotes the sum of the digits of n, and let $f(n) = n$ otherwise. Find $f(2023^{2024})$.
(A) 1 (B) 3 (C) 4 (D) 6 (E) 8

6. If x and y are positive integers such that $(1 + 3 + 5 + \cdots + 39)x = (2 + 4 + 6 + \cdots + 40)y$, find the smallest possible value of $x + y$.
(A) 10 (B) 11 (C) 20 (D) 21 (E) 41

7. Suppose that there are three unit circles that are mutually tangent to each other. Let T_1 be an equilateral triangle outside the circles that has each of its sides tangent to two of the circles. Let T_2 be the equilateral triangle that is contained in the triangle with vertices at the centers of the three circles such

that each of the sides is tangent to a circle. Find the value when the side length of T_2 is divided by the side length of T_1.

(A) $\frac{7-4\sqrt{3}}{2}$ (B) $\frac{2-\sqrt{3}}{4}$ (C) $\frac{3\sqrt{3}-5}{2}$ (D) $\frac{2-\sqrt{3}}{2}$ (E) $\frac{2\sqrt{3}-3}{2}$

8. Suppose that there are two rows of five seats, one front row and one back row. 10 people come and sit on those seats. The last person that comes must sit on the last seat of the second row. The back seat cannot be sat in if the corresponding seat in front is not sat in. Find the number of ways for the ten people to sit in the seats.

(A) 11340 (B) 20160 (C) 22680 (D) 40320 (E) 362880

9. Find, to the nearest minute, the time when the acute angle between the clock is 30° for the third time after 1 AM.

(A) 2:14 (B) 2:15 (C) 2:16 (D) 2:17 (E) 2:18

10. Let $ABCD$ be a unit square. Let E be the midpoint of AB and F be the midpoint of AD. The segments AC, BF, CE, BD split the square into 9 regions. One of the regions has no points in common with the perimeter of $ABCD$. Find the area of the region.

(A) 1/24 (B) 1/20 (C) 1/16 (D) 1/12 (E) 1/10

11. 64 people are randomly put into a single-elimination bracket, and they are numbered 1 through 64. The lower-numbered person always wins. If the probability that the semifinals will be 1 versus 3 and 2 versus 4 can be written as $\frac{2^a}{n^3-n}$ for positive integers a and n such that a is minimized, find $a+n$.

(A) 72 (B) 73 (C) 74 (D) 75 (E) 76

12. 64 people are randomly put into a single-elimination bracket with a third place match at the end, and they are numbered 1 through 64. The lower-numbered person always wins. What is the highest possible number of the third

place person?

(A) 17 (B) 32 (C) 33 (D) 47 (E) 48

13. Let $a \triangle b = \frac{a+2b}{a+4}$ for all $a \neq -4$. If the value of $a \triangle b \triangle c = (a \triangle b) \triangle c$, the value of $2024 \triangle 2023 \triangle \cdots \triangle 1$ can be expressed as $\frac{m}{n}$, where $\gcd(m, n) = 1$, find the remainder when $m + n$ is divided by 20.

(A) 1 (B) 3 (C) 4 (D) 8 (E) 13

14. Let A be the least positive integer a such that there exists positive integers b and c where $2a^2 = 3b^3$ and $5a^5 = 7c^7$. If $A = 2^x 3^y 5^z 7^w$ for positive integers x, y, z, w, find $x + y + z + w$.

(A) 40 (B) 41 (C) 42 (D) 43 (E) 44

15. Find the remainder when the sum of the digits of $9 + 99 + 999 + \cdots + 999 \ldots 999$, where the last term has 2024 digits, is divided by 10.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

16. Let ABC be a right triangle with the right angle at C. Let F be a point on AB such that $AF = 2$, $BF = 6$, and $CF = 4$. Find the value of AC^2.

(A) 24 (B) 30 (C) 34 (D) 40 (E) 44

17. Channing has a probability of p of winning any countdown round question and a probability of $1 - p$ of losing. The first to gain a two point advantage wins the round. Find the probability that Channing wins the round in terms of p.

(A) $\frac{2p}{2p^2-p+2}$ (B) $\frac{p}{2p^2-2p+1}$ (C) $\frac{p^2-2p}{2p^2-2p-1}$ (D) $\frac{p^2+p}{2p^2-2p+2}$ (E) $\frac{p^2}{2p^2-2p+1}$

18. Let there be a quarter circle with center O and arc AB such that we can rotate A 90° counterclockwise around O to get B. Let A' be the point in the quarter circle such that $OA = AA'$ and $\angle OAA' = 20^\circ$. Let C be a point on arc AB such that $\angle AA'C = 20^\circ$. Find the measure of angle $A'CB$ in degrees.
(A) 20 (B) 25 (C) 30 (D) 35 (E) 40

19. The number $1800^2 + 1$ is the product of two primes. Find the sum of the two primes.
(A) 1802 (B) 1804 (C) 3600 (D) 3602 (E) 7202

20. Let ABC be an acute triangle. Let D, E, and F be the foot of the altitudes from A, B, and C to their corresponding sides. Let H be the orthocenter of triangle ABC. Let $f(\triangle ABC)$ be the number of pairs of distinct (as in not the same three vertices) similar (and possibly congruent) triangles that have their vertices in the set $\{A, B, C, D, E, F, H\}$. Find the sum of the minimum and maximum of $f(\triangle ABC)$ over all acute triangles ABC.
(A) 18 (B) 142 (C) 160 (D) 175 (E) 184

21. Find the number of non-congruent equiangular hexagons $ABCDEF$ with integer side lengths that have area $119999 \cdot \frac{\sqrt{3}}{2}$ and $AB = DE = 1$.
(A) 34 (B) 35 (C) 68 (D) 69 (E) 70

22. If a positive integer N leaves a remainder of 1 when divided by 99, 8 when divided by 98, and 27 when divided by 97, and N is minimized, find the sum of the digits of $\lfloor \frac{N}{99} \rfloor + \lfloor \frac{N}{98} \rfloor + \lfloor \frac{N}{97} \rfloor$.
(A) 12 (B) 13 (C) 14 (D) 15 (E) 16

23. Derek has two normal 12-sided dice. Eric has a fair 9-sided die and a fair 16-sided die with positive integers on each face (but not necessarily has the numbers $1-9$ and $1-16$). If the probability of rolling any sum on each pair of

dice is equal, find the smallest sum of numbers on the 25 faces on Eric's dice.

(A) 120 (B) 138 (C) 145 (D) 152 (E) 166

24. Find the sum of all not necessarily distinct prime factors of $1^1 + 4^4 + 8^8 - 2^{17} - 2^{13} - 2^5$.

(A) 200 (B) 203 (C) 207 (D) 208 (E) 213

25. Let $ABCDE$ be a convex pentagon satisfying $AB = BC = CD = DE$, $\angle ABC = \angle CDE$, $\angle EAB = \angle AED = \frac{1}{2}\angle BCD$. Let X be the intersection of lines AB and CD. If $\triangle BCX$ has a perimeter of 18 and an area of 11, find the area of $ABCDE$.

(A) 136 (B) 137 (C) 138 (D) 139 (E) 140