2019 CMC 12A Solutions Document

Christmas Math Competitions

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Friday, December 21, 2018

1. **Answer (D):** We compute:

$$201 \times 9 + 20 \times 19 - 2 \times (0 + 1 + 9) = 1809 + 380 - 20 = 2169.$$

2. **Answer (D):** We compute $\frac{111{,}111{,}111}{9} = 12,345,679$. The missing digit is 8.

3. **Answer (D):** The number that is N% of 880 is $\frac{\text{N}}{100} \cdot 880 \implies \frac{44\text{N}}{5}$. Since this expression must be an integer, we let $\text{N} = 5k$ for some positive integer k. Then, our expression becomes $44k = 2^2 \cdot 11 \cdot k$. For this to be a perfect square, the exponent of each prime in the prime factorization must be even. Therefore, $k = 11$ and $\text{N} = 55$. We can check that 55% of 880 is $484 = 22^2$. The requested sum is $5 + 5 = 10$.

   To show that N is unique, we note that for the resulting number to be smaller than 880, we must have $\text{N} < 100$ or $k < 20$. Since the resulting number is a perfect square if $2^2 \cdot 11 \cdot k$ is a perfect square, k must be divisible by 11. But 11 is the only positive integer less than 20 that is divisible by 11, so N cannot equal another value less than 100. Thus, N is unique.

4. **Answer (B):** The ratio of the octagon's area to perimeter in inches is given by $\frac{5 \text{ in.}^2}{4 \text{ in.}} = \frac{5 \text{ in.}}{4}$.

   We note that 1 ft = 12 in.. Converting the unit of measurement to feet:

$$\frac{5 \text{ in.}}{4} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} = \frac{5 \text{ ft}}{48}$$

   Therefore, the requested ratio is $\frac{5}{48}$.

5. **Answer (A):** Let a positive integer have property P if it can be written as the sum of 18 consecutive integers. Once we establish the largest integer in the list, the rest of the list and the sum of the integers is uniquely determined. Ideally, we want the middle numbers of the list to be as close to 0 as possible so that the sum of the integers is a positive integer slightly above 0. If the 9^{th} and 10^{th} numbers on the list are -1 and 0, respectively, the sum is $(-9) + (-8) + (-7) + \cdots + (-1) + 0 + \cdots + 7 + 8 = -9$. However, if the 9^{th} and 10^{th} numbers on the list are 0 and 1, respectively, the sum is $(-8) + (-7) + (-6) \cdots + 0 + 1 + \cdots + 8 + 9 = 9$. Therefore, for the 1^{st} smallest positive integer with property P, 9 is the largest integer among the 18 integers. It then follows that we can increase the largest integer in the list by 1 and edit the rest of the list appropriately to produce the next smallest positive integer with property P.

For the 2^{nd} smallest positive integer with property P, 10 is the largest integer in the list, while for the 3^{nd} smallest, 11 is the largest integer in the list. Therefore, for the 65^{nd} smallest integer with property P, 73 is the largest integer in the list.

6. **Answer (D):** Objectively, to maximize the number of gangsters alive, one gangster should be shot by as many gangster as possible to prevent other gangsters from dying. Clearly, all 10 of them cannot remain alive.

   If 9 gangsters remain alive, then one gangster must be shot by every gangster, including himself. However, a gangster clearly cannot shoot himself because he is not the gangster furthest away from himself.

   However, 8 gangsters can remain alive. This can be done by placing 9 of the gangsters close together and placing the 10th gangster arbitrarily far from the other gangsters. Then, the 10th gangster will shoot one of the first 9 gangsters, while the first 9 gangsters will all shoot the 10th. Therefore, at most 8 gangsters can remain alive.

7. **Answer (D):** For $\sqrt{2} \cdot x$ to be an integer, x must be in the form $\frac{n}{\sqrt{2}}$ for some integer n. Then,
$$0 \leq x \leq 100 \implies 0 \leq \frac{n}{\sqrt{2}} \leq 100 \implies 0 \leq n \leq 100\sqrt{2}.$$
   The value of $\sqrt{2}$ is often approximated as 1.414, so $141 < 100\sqrt{2} < 142 \implies 0 \leq n \leq 141$. There are 142 possible values of n, which gives us 142 possible values of x.

8. **Answer (C):** Note that for all real numbers k, $1 \circ k = 1 - \frac{1}{k} + \frac{1}{k} = 1$. Therefore, $1 \circ (2 \circ (\cdots (2018 \circ 2019))) = 1$.

9. **Answer (B):** The shaded area is given by the area of kite $NCMF$ minus the area of kite $NPMQ$. Since the area of a kite is half of the product of its diagonals, our desired area is $\frac{1}{2}(NM \cdot CF - NM \cdot PQ)$.

   Clearly, $NM = \sqrt{3}$ and $CF = 2$. In addition, $PQ = 1$ because $ABPQ$ is a square. Therefore, the desired area is $\frac{1}{2}(\sqrt{3} \cdot 2 - \sqrt{3} \cdot 1) = \frac{\sqrt{3}}{2}$.

10. **Answer (D):** Since $p(a) = 2$, $p(p(a)) = 17 \implies p(2) = 17$. Since $p(p(a)) = 17$, $p(p(p(a))) = 167 \implies p(17) = 167$.

    Let $p(x) = mx + n$ for real constant m and n. Plugging in $x = 2$ gives us $17 = 2m + n$. Plugging in $x = 17$ gives us $167 = 17m + n$. Solving these equations, $(m, n) = (10, -3)$.

    Since $p(a) = 2$, we have $2 = 10a - 3 \implies a = \frac{1}{2}$.

11. **Answer (B):** Let E denote the expected value of the expression, A denote the expected value of $\lfloor x \rfloor$, and B denote the expected value of $\{x\}$. The interval for x covers each value of $\{x\}$ the same number of times except for $\{x\} = 0$, but the probability that $\{x\} = 0$ is negligible. Therefore, each real value of $\{x\}$ is equally as likely to be chosen. It then follows that each possible value of $\lfloor x \rfloor$ is equally as likely to be chosen. As a result, the selection of the value of $\lfloor x \rfloor$ and the selection of the value of $\{x\}$ are independent events. Thus, $E = A \cdot B$.

    Since $0 \leq \lfloor x \rfloor \leq 99$, A is the average of the numbers $0, 1, 2, 3, ..., 98, 99$ or $\frac{99}{2}$. Since $0 \leq \{x\} < 1$, B is the average number in the interval for $\{x\}$ or $\frac{1}{2}$. Therefore, $E = \frac{99}{4}$ or 24.75.

12. **Answer (D):** Let R denote a rightwards move and U denote an upwards move. First, arrange all 9 of the Rs in a line: $RRRRRRRRR$. We have 10 spaces to insert a U, namely at the beginning of the string, at the end of the string, or in between two Rs. We can only insert up to 1 U in each of these 10 spaces, since each U must be followed

by an R. In addition, a U as the last character is allowed. There are $\binom{10}{5} = 252$ ways to choose the spaces for the U and hence, the number of possible sequences.

13. **Answer (E):** For a circle c of radius r to intersect $\mathcal{C}$, the distance between the center of c and a point on $\mathcal{C}$ must be less than or equal to r. Let $\mathcal{C}_r$ be the circle of radius $10-r$ with the same center as $\mathcal{C}$. Clearly, each point on $\mathcal{C}_r$ has a distance of r to some point on $\mathcal{C}$. It then follows that the region of acceptable points for a fixed r is the area between circles $\mathcal{C}$ and $\mathcal{C}_r$. The area of the region is given by $10^2\pi - (10-r)^2\pi = \pi(20r - r^2)$. In addition, the total area of the points we can choose from is $10^2\pi$ and each integer value of r has a $\frac{1}{9}$ chance of being picked. Therefore, the probability that we pick a fixed integer r and choose a point in the acceptable region is given by:

$$\frac{1}{9} \cdot \frac{20r - r^2}{100}$$

Our desired probability is the above expression summed from $r = 1$ to 9. We compute:

$$\sum_{r=1}^{9} \left(\frac{1}{9} \cdot \frac{20r - r^2}{100}\right) = \frac{1}{900} \cdot \sum_{r=1}^{9} (20r - r^2) = \frac{1}{900} \cdot \left(20 \sum_{r=1}^{9} r - \sum_{r=1}^{9} r^2\right)$$

$$\frac{1}{900} \cdot \left(20 \cdot \frac{9 \cdot 10}{2} - \frac{9 \cdot 10 \cdot 19}{6}\right) = \frac{1}{900} \cdot 615 = \frac{41}{60}$$

The requested sum is $41 + 60 = 101$.

14. **Answer (D):** Recall the identities $\log_x(y) = \frac{\log(y)}{\log(x)}$ and $\log_x(y) = \frac{1}{\log_y(x)}$.

Then, the first equation becomes:

$$\log(a)\left(\frac{1}{\log(4)} + \frac{1}{\log(9)}\right) = \frac{1}{\log(a)}\left(\log(4) + \log(9)\right)$$

We multiply both sides of the equation by $\log(a)$ and also use $\frac{1}{\log(4)} + \frac{1}{\log(9)} = \frac{\log(4)+\log(9)}{\log(4)\cdot\log(9)}$:

$$\log^2(a)\left(\frac{\log(4) + \log(9)}{\log(4) \cdot \log(9)}\right) = \log(4) + \log(9)$$

$$\log^2(a)\left(\frac{1}{\log(4) \cdot \log(9)}\right) = 1 \implies \log^2(a) = \log(4) \cdot \log(9)$$

Similarly, the second equation results in $\log^2(b) = \log(9) \cdot \log(16)$.

Because $\log_a(b) = \frac{\log(b)}{\log(a)}$, we divide the two equations to get:

$$\frac{\log^2(b)}{\log^2(a)} = \frac{\log(16)}{\log(4)} = \log_4(16) = 2$$

Therefore, $\frac{\log(b)}{\log(a)} = \sqrt{2}$.

15. **Answer (C):** Let E' be the rotation of E 180° around D. Then, $BECE'$ is a parallelogram because BC and EE' bisect each other. It then follows that $\angle BEC = \angle BE'C$.

Then, $\angle BEC + \angle BAC = \angle BE'C + \angle BAC = 180^\circ$. It then follows that $BE'CA$ is cyclic. By Power of a Point with respect to D, $DE' \cdot DA = BD \cdot CD$. Because $DE' = DE$, we have $DE \cdot DA = BD \cdot CD$. D is the midpoint of BC, so we have $BD = CD = \frac{\sqrt{7}}{2}$. Finally, $DE \cdot DA = \frac{7}{4}$. The requested sum is $7 + 4 = 11$.

16. **Answer (D):** Let k be the number of seconds before the expected sum of the outputted numbers is at least 280. Clearly, one output has no effect on the others, so the expected sum of the outputted numbers is the sum of the expected numbers from each second. At the nth second, the expected output is the average of the numbers $\{1, 2, 3, ..., n\}$ or $\frac{n+1}{2}$. Summing this expression from 1 to k, we have:

$$\frac{1+1}{2} + \frac{2+1}{2} + \frac{3+1}{2} + \cdots + \frac{k+1}{2} = \frac{k(k+3)}{4} \geq 280 \implies k^2 + 3k - 1120 \geq 0$$

By testing values of k or by factoring $k^2 + 3k - 1120 = (k - 32)(k + 35) \geq 0$, we find that $k = 32$ is the smallest value of k that satisfies the inequality.

17. **Answer (C):** Let l_1 be the line $y = 2x$ for $x \geq 0$ and l_2 be the line $y = -2x$ for $x \leq 0$. Clearly, A and C lie on l_1, while B lies on l_2. In addition, let l_3 be the line passing through M that is perpendicular to l_1. It follows that B is the intersection of l_2 and l_3.

 Since lines l_1 and l_3 are perpendicular, the slope of l_3 is the negative reciprocal of the slope of l_1. Therefore, the slope of l_3 is $-\frac{1}{2}$. Since, l_3 passes through $(3, 6)$, the equation of l_3 is given by $y = -\frac{1}{2}x + \frac{15}{2}$. Since l_2 has the equation $y = -2x$, $B = (-5, 10)$. Because $\triangle ABC$ is equilateral, $AC = BM \cdot \frac{2}{\sqrt{3}}$. Therefore, $[ABC] = \frac{BM^2}{\sqrt{3}}$. By the distance formula, $BM^2 = (3 - (-5))^2 + (6 - 10)^2 = 80$. Therefore, $[ABC] = \frac{80}{\sqrt{3}}$ or $[ABC] = \frac{80\sqrt{3}}{3}$.

18. **Answer (C):** Let the path have a southeast arrows, b horizontal arrows, and c southwest arrows. Then, $a + b + c = 8$. Since James must travel down 5 rows, we must have $a + c = 5$, as only the southeast and southwest arrows cause James to travel down a row. Lastly, to ensure that James ends up on AB among all the lines parallel to AB, we must have $b = c$. Solving these equations gives $(a, b, c) = (2, 3, 3)$.

 Let the point at the bottom left corner of the grid be C. We must make sure that our path does not go out of the bounds of $\triangle ABC$. Clearly, the directions James can travel in prevent us from ever crossing AC. For James to cross BC, he must travel down more than 5 rows. Our composition of directional arrows ensures that he will travel down exactly 5 rows. Therefore, James can't cross BC. However, if at any point in the path, James has traveled more times horizontally than he has in the southwest direction, he will cross AB.

 Consider a string of characters consisting of 3 bs and 3 cs. We will count how many strings there are such that at every point in the string, there are always at least as many cs as bs. There are 5 such strings, namely *cbcbcb*, *cbccbb*, *ccbcbb*, *ccbbcb*, and *cccbbb*. The 2 as can be inserted anywhere in these 5 strings to construct a possible path.

 Suppose we have 8 blank characters in a row. We have $\binom{8}{2} = 28$ ways to choose the locations of the 2 as. Then, we choose any of the 5 strings of bs and cs and insert the characters in order in the remaining 6 spaces. This gives $28 \cdot 5 = 140$ paths total.

19. **Answer (A):** Clearly, $p \neq q$. Otherwise, we would have $2p^2 = p^2r + 1 \Rightarrow p^2 \mid 1$, which is clearly impossible.

 Then, suppose WLOG $q > p \Rightarrow q \geq p + 1$. We will multiply by 2 at the end to take symmetry into account.

 We rearrange the given equation to $p^2 - 1 = pqr - q^2 = q(pr - q)$. It then follows that q divides $p^2 - 1 = (p + 1)(p - 1)$. By the definition of a prime number, this means than q divides either $p - 1$ or $p + 1$. In any case, $q \leq p + 1$. However, $q \geq p + 1$, so $q = p + 1$.

Hence the only pair of consecutive primes is $(p, q) = (2, 3)$, and solving for r gives $r = 2$; taking account for symmetry, we have 2 ordered triples (p, q, r).

20. **Answer (D):** Let $T = \triangle ABC$ be the equilateral triangle and P be the plane of projection. Additionally, let A' be the projection of A onto P, and define B' and C' similarly for B and C, respectively. Finally, let $T' = \triangle A'B'C'$.

    WLOG, assume $A = A'$ lies on P and $A'B' = 4$. We have right triangle $B'AB$ with hypotenuse $AB = 5$ and $A'B' = 4$. Thus, $BB' = 3$ and B is 3 units above plane P.

    Additionally, we have $A'C' = 4$. Right triangle $C'AC$ has hypotenuse $AC = 5$ and $A'C' = 3$. Thus, $CC' = 4$ and C is 4 units above the plane.

    To finish, let D be a point on CC' such that $C'D = 3$ and $CD = 1$. Note that $BD = B'C'$, so we just need to find BD. In right triangle BCD, we have $CD = 1$ and $BC = 5$, so $BD = B'C' = \sqrt{5^2 - 1^2} = 2\sqrt{6}$.

21. **Answer (B):** We claim that for positive integers a, b, x, and y, $a \star b = x \star y$ if and only if $a + b = x + y$. Our base case for $a + b = x + y = 2$ holds, as $(a, b) = (1, 1)$ is the only ordered pair of positive integers such that $a + b = 2$.

    Assume that our inductive hypothesis holds true for a, b, x, and y. Then, $a \star b = x \star y$ implies that $(a + 1) \star b = (x + 1) \star y$. By the second property of the operation, $(a + 1) \star b = (x + 1) \star y = a \star b + a + b = x \star y + x + y$. By our inductive hypothesis, $a \star b + a + b = x \star y + x + y$ implies that $a + b = x + y$. This completes the induction for our proposition. In addition, because our steps are reversible, the converse of our proposition holds true as well. Therefore, $a \star b = x \star y$ if and only if $a + b = x + y$.

    For a fixed value of c, $a \star b = c$ must be satisfied by exactly 25 ordered pairs of positive integers (a, b). Therefore, we must find a positive integer m such that $a + b = m$ has exactly 25 ordered pairs of positive integer solution. Clearly, $m = 26$ and our ordered pairs are $(25, 1), (24, 2), (23, 3), ..., (2, 24), (1, 25)$. Therefore, $c = 25 \star 1$.

    Let $f(n) = n \star 1$ for all $n \geq 1$. We are given that $f(1) = 2$, and we seek $f(25)$. By the second property of the function, $a \star b$, $(n+1) \star 1 = n \star 1 + n + 1$ or $f(n+1) = f(n) + (n+1)$.

    Therefore, $f(25) = 2 + (2 + 3 + 4 + ... + 24 + 25) = 326$.

22. **Answer (D):** Let O_1, O_2, O_3 be the centers of $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. By the tangency conditions, $O_1O_2 = 2 + 3 = 5$, $O_1O_3 = 6 - 2 = 4$ and $O_2O_3 = 6 - 3 = 3$, that is to say, O_3 lies on Γ_2 and since $3^2 + 4^2 = 5^2$, $\triangle O_1O_2O_3$ is right in O_3. Hence, since $O_3B = O_3C = 6$, $\triangle BO_3C$ is a right isosceles triangle.

    Let O be the circumcenter of $\triangle ABC$; by the definition of circumcenter, $OA = OB$, so $\triangle AOB$ is isosceles too. Since $\triangle ABO_1$ and $\triangle ACO_2$ are isosceles, $\angle BAO_1 + \angle CAO_2 = \angle ABO_1 + \angle ACO_2 = \frac{1}{2}\angle AO_1O_3 + \frac{1}{2}\angle AO_2O_3 = 45°$. Hence, $\angle BAC = 135°$.

    Finally, by the fact that the angle at the center is twice the angle at the circumference, considering arc BC not containing A, the concave $\angle BOC = 2\angle BAC = 270°$, hence $\angle BOC = 90°$ and BO_3CO is a square, that is to say, the circumradius of $\triangle ABC$ is $R = 6$.

23. **Answer (C):** All angles in the following solution expressed in degrees.

    Let $\mathcal{S}$ be the required sum. Since

$$\frac{1}{\tan x + \cot x} = \frac{1}{\tan x + \frac{1}{\tan x}} = \frac{\tan x}{\tan^2 x + 1} = \frac{\tan x}{\sec^2 x} = \sin x \cos x = \frac{1}{2}\sin(2x),$$

    we can see that $\mathcal{S} = \sum\limits_{n=1}^{44} \frac{1}{2}\sin(2n) = \sum\limits_{n=1}^{44} \frac{1}{2}\cos(2n)$.

Remember the sum to product identity $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$. In order for the sum to telescope, we can multiply $\mathcal{S}$ by $\sin 1$ so that we get

$$\mathcal{S}\sin 1 = \sum_{n=1}^{44}\frac{1}{2}\cos(2n)\sin 1 = \sum_{n=1}^{44}\frac{1}{4}[\sin(2n+1) - \sin(2n-1)] = \frac{1}{4}[\sin 89 - \sin 1] = \frac{\cos 1 - \sin 1}{4}.$$

Therefore,

$$\mathcal{S} = \frac{\cos 1 - \sin 1}{4\sin 1} = \frac{\cot 1 - 1}{4}.$$

The requested answer is $|1-4| = 3$.

24. **Answer (A):** Let E be the desired expected value. Because there are so many arrangements of the initial ten lighted candles, we will bound E instead of calculating its exact value.

The extreme case to consider for an upper bound on E is when the ten initial lighted candles are all consecutive.

Consider a string of k unlit candles with one lighted light at either end of the string. We claim that on average, it will take k seconds for the string of k candles to be lit.

Proof: Let e_k be the expected number of seconds for the string of k candles to become lit. Clearly, $e_0 = 0$ and $e_1 = 1$.

We will show by induction that $e_k = k$ for all $k \geq 0$. Our base cases are $k = 0$ and $k = 1$, which clearly both hold. Now, suppose we have a string of k unlit candles. Consider the two unlit candles on either end of the string. After the first second, there is a $\frac{1}{4}$ chance that both of these unlit candles will become lit, a $\frac{1}{2}$ chance that one of the two unlit candles will become lit, and a $\frac{1}{4}$ chance that neither unlit candle will become lit. Therefore, $e_k = \frac{1}{4}(e_k+1) + \frac{1}{2}(e_{k-1}+1) + \frac{1}{4}(e_{k-2}+1)$. If we assume our inductive hypothesis holds true for $k-1$ and $k-2$, then $e_k = \frac{1}{4}(e_k+1) + \frac{1}{2}((k-1)+1) + \frac{1}{4}((k-2)+1) \implies e_k = k$. Our induction is complete.

We got back to our extreme case where we have 10 initial lighted candles in a row and 90 unlighted candles in a row. By our result, the expected number of seconds for the 90 unlighted candles to become lit is 90. However, since this is the extreme case for the upper bound of E, it follows that $E < 90$. Among the answer choices, 90 is the closest to E.

25. **Answer (B):** Note that

$$\angle AXB = 90^\circ + \frac{\angle APB}{2} = 90^\circ + \frac{\angle ACB}{2} = \angle AIB,$$

so quadrilateral $AIXB$ is cyclic, which implies $IX = XB$ (as AX bisects $\angle BAI$). Similarly, $AIYC$ is cyclic, and $IY = YC$. In fact, $\triangle IXB \sim \triangle IYC$, since both vertex angles have measure equal to $180^\circ - \frac{A}{2}$.

Now further angle chasing yields

$$\angle XIY = \angle XIP + \angle YIP = \angle ABX + \angle ACY = \frac{\angle ABP + \angle ACP}{2} = 90^\circ.$$

Thus, letting $x = IX = XB$ and $y = IY = IC$, we have

$$\begin{aligned} IB^2 + IC^2 &= 2x^2(1-\cos\angle IXB) + 2y^2(1-\cos\angle IYC) \\ &= 2(x^2+y^2)(1+\cos\tfrac{A}{2}) = 8(1+\cos\tfrac{A}{2}). \end{aligned}$$

The first equation comes from Law of Cosines on $\triangle IXB$ and $\triangle IYC$, while $x^2+y^2 = 4$ comes from Pythagorean Theorem on $\triangle IXY$. Setting this equal to 15 yields $\cos\frac{A}{2} = \frac{7}{8}$, so $\cos A = 2\cos^2\frac{A}{2} - 1 = 2\cdot\frac{49}{64} - 1 = \frac{17}{32}$. The requested sum is $17 + 32 = 49$.