2022 OMC 10 Solutions

1. What is the value of $\sqrt{9}+\sqrt{16}+\sqrt{144}-\sqrt{9+16+144}$?

   (A) 0 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

   Answer (D): Compute as follows:

$$\sqrt{9}+\sqrt{16}+\sqrt{144}-\sqrt{9+16+144}=3+4+12-\sqrt{169}=19-13=\boxed{\textbf{(D) } 6}.$$

2. Ashley drives from San Jose to San Diego to visit some relatives. She takes off at 9:48 AM and arrives at her destination at 6:15 PM. Given that San Jose and San Diego were in the same time zone, how long was Ashley's drive, in minutes?

   (A) 327 **(B)** 333 **(C)** 507 **(D)** 513 **(E)** 693

   Answer (C): Note that 8 hours after 9:48 AM is 5:48 PM. Then, 27 minutes after 5:48 PM is 6:15 PM. Hence, the drive was 8 hours and 27 minutes. Converting this to minutes, the answer is $8\cdot 60+27=\boxed{\textbf{(C) } 507}$.

3. On a 20 question multiple choice exam, Sheldon gets exactly n questions correct. Each question on the exam was worth the same number of points. He notices that his score is exactly $n^2\%$. Given that Sheldon got at least one question right, what is the value of n?

   (A) 5 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 20

   Answer (A): Since Sheldon got at least one question right, n is nonzero. If Sheldon got exactly n questions correct, his score would be $\frac{n}{20}$. From his observation, this score equals $\frac{n^2}{100}$. Thus, $\frac{n}{20}=\frac{n^2}{100} \implies n=5$, where n can be divided out, since it is nonzero. Thus, the answer is $\boxed{\textbf{(A) } 5}$.

4. A circular table has 1001 seats some of which are occupied. If another person were to take a seat, he or she would have to sit next to another person that is already seated. What is the smallest possible number of seats that can be already occupied?

   (A) 201 **(B)** 333 **(C)** 334 **(D)** 500 **(E)** 501

   Answer (C): If there are 3 or more seats in a row that are unoccupied, the next person could sit in a seat not next to someone who is already seated. However, if there are always 2 or less seats in a row that are unoccupied, the next person would be forced to sit in a seat next to someone else. Thus, it suffices to place people in seats such that there are never 3 unoccupied seats in a row.

   Suppose there are n people already seated. Since there are n different gaps between consecutive seated people and each gap can have at most 2 seats, there are at most $3n$ seats around the table. For there to be 1001 seats, $3n \geq 1001$, implying the minimum n is $\boxed{\textbf{(C) } 334}$.

5. Three children, Ari, Rizzo, and Daeho, and their fathers are sitting in a row to watch a Minions movie. For safety concerns, both ends of the row should be occupied by adults. How many ways are there for the six individuals to be arranged?

   (A) 4 **(B)** 48 **(C)** 72 **(D)** 144 **(E)** 288

   Answer (D): There are 3 ways to choose the father sitting on the left end of the row, and then 2 ways to choose the father sitting on the right end. There are then 4 people left to rearrange, which can be done in any order. Thus, there are 4! ways to rearrange the remaining people. The number of arrangements is $3\cdot 2\cdot 4!=\boxed{\textbf{(D) } 144}$.

6. Suppose x and y are positive integers satisfying $x^y=2^{120}$. What is the smallest possible value of $x+y$?

   (A) 44 **(B)** 46 **(C)** 48 **(D)** 56 **(E)** 64

   Answer (B): Since x and y are positive integers and 2 is prime, x must be an integer power of 2. Let $x=2^a$ for some nonnegative integer a. Then, $2^{ay}=2^{120} \implies ay=120$. Since a and y are positive integers, they are positive divisors of 120. Then, $x+y=2^a+y$.

   If $(a,y)=(4,30)$, then $x+y=46$. To rigorously prove this is the minimum:

- If $a \le 2$, then $y \ge 60$ for $ay = 120$ to hold, but $x + y$ will clearly be too big
- If $a = 3$, then $y = 40$, resulting in $x + y = 48 > 46$
- If $a = 5$, then $y = 24$, resulting in $x + y = 56 > 46$
- If $a \ge 6$, then $x \ge 2^6$, where $x + y$ will clearly be too big

Hence, the minimum is $\boxed{\textbf{(B) } 46}$.

7. Let $ABCD$ be a rectangle in the Cartesian plane such that the midpoint of AB and the midpoint of BC are $(-22, 19)$ and $(14, 67)$ respectively. What is the length of BD?

   (A) 48 **(B)** 60 **(C)** 84 **(D)** 96 **(E)** 120

   Answer (E): Let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. By the distance formula,

$$MN = \sqrt{(14 - (-22))^2 + (67 - 19)^2} = \sqrt{36^2 + 48^2} = 12\sqrt{3^2 + 4^2} = 60.$$

   Since MN is a midsegment in $\triangle ABC$, it follows that $AC = 2MN = 120$. Furthermore, since $ABCD$ is a rectangle, $BD = AC = \boxed{\textbf{(E) } 120}$.

8. Call a positive integer *rhit* if all its digits are nonzero and it is divisible by each of its digits. How many two-digit positive integers are *rhit*?

   (A) 9 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 18

   Answer (B): Let $\overline{ab} = 10a + b$ be a two-digit rhit integer, where a and b are nonzero digits. Then, a and b must both divide $10a + b$. This implies a divides b and b divides $10a$. Consider cases on the value of a.

   Case 1: $a = 1$

   Then, the divisibility becomes b divides 10. There are 3 such digits b, namely 1, 2, and 5.

   Case 2: $a = 2$

   Then, the divisibility becomes 2 divides b and b divides 20. There are 2 such digits b, namely 2 and 4.

   Case 3: $a = 3$

   Then, the divisibility becomes 3 divides b and b divides 30. There are 2 such digits b, namely 3 and 6.

   Case 4: $a = 4$

   Then, the divisibility becomes 4 divides b and b divides 40. There are 2 such digits b, namely 4 and 8.

   Case 5: $5 \le a \le 9$.

   Then, the only possible nonzero digit that is a multiple of a is a itself. Thus, $b = a$. Then, b is guaranteed to divide $10a$. Thus, there are 5 rhit integers here.

   The total number of two-digit rhit integers is $3 + 2 + 2 + 2 + 5 = \boxed{\textbf{(B) } 14}$.

9. A thirsty crow finds a large cylindrical container with radius 4 centimeters and height 20 centimeters that is filled halfway with water. If the crow drops three perfectly spherical rocks with radius 2 centimeters into the container, how many centimeters does the water level rise?

   (A) $\frac{2}{3}$ **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

   Answer (C): The volume of the three rocks is $3 \cdot \frac{4}{3}\pi \cdot 2^3 = 32\pi$. Thus, if a new cylinder has radius 4 (same as the container) and has height h and volume equal to the three rocks, the change in the water level is represented by h. The volume of this cylinder is $16\pi \cdot h$. Hence, $16\pi \cdot h = 32\pi \implies h = \boxed{\textbf{(C) } 2}$.

10. Let a, b, c be three nonzero integers satisfying $7a + 11b + 13c = 0$. What is the least possible value of $|a| + |b| + |c|$?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

    Answer (A): Observe that $(a, b, c) = (1, -3, 2)$ is a solution. It will be rigorously checked if $|a| + |b| + |c| = 6$ is minimal or not.

First, if (a, b, c) is a solution, so is $(-a, -b, -c)$ and vice versa. Thus, since a is nonzero, without loss of generality, assume a is positive. Since b and c are also nonzero, $|b|, |c| \geq 1$. Thus, a smaller value of $|a| + |b| + |c|$ cannot be achieved if $a \geq 4$.

Taking the equation modulo 11, $-4a + 2c \equiv 0 \pmod{11} \implies c \equiv 2a \pmod{11}$.

Case 1: $a = 1$

Then, $c \equiv 2 \pmod{11}$. Since $|b| \geq 1$, a smaller value of $|a| + |b| + |c|$ can only be achieved if $|c| \leq 3$. Thus, $c = 2$ and then $b = -3$, giving back $|a| + |b| + |c| = 6$.

Case 2: $a = 2$

Then, $c \equiv 4 \pmod{11}$. Since $|b| \geq 1$, a smaller value of $|a| + |b| + |c|$ can only be achieved if $|c| \leq 2$. No such c exists that also satisfies the congruence.

Case 3: $a = 3$

Then, $c \equiv 6 \pmod{11}$. Since $|b| \geq 1$, a smaller value of $|a| + |b| + |c|$ can only be achieved if $|c| \leq 1$. No such c exists that also satisfies the congruence.

Thus, a smaller value of $|a| + |b| + |c|$ cannot be achieved. The desired minimum is $\boxed{\textbf{(A) } 6}$.

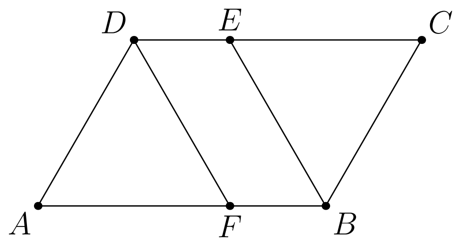
11. Let $ABCD$ be a parallelogram with $AB = 1$ and $\angle ABC = 120$. The angle bisectors of $\angle ABC$ and $\angle CDA$ trisect the quadrilateral into three regions of equal area. If $AB \geq BC$, what is the value of BC?

(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{3}}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** 1

Answer (C): Let the angle bisector of $\angle ABC$ intersect CD at E, and the angle bisector of $\angle CDA$ intersect AB at F.

By the properties of a parallelogram, $\angle CDA = \angle ABC = 120°$, implying that $\angle DAB = \angle BCD = 60°$. If $x = BC$, then $x = DA$ too.

By the angle bisectors, $\angle ADF = \angle CBE = 60°$. Thus, $\triangle ADF$ and $\triangle BCE$ are both equilateral. Since $CE = BC$ and $DC = AB \geq BC$, E will lie between C and D. Similarly, F lies between A and B. As shown in the diagram below, the three regions of the parallelogram are triangle ADF, quadrilateral $FDEB$, and triangle BCE.



Clearly, $\triangle ADF$ and $\triangle BCE$ are congruent because $AD = BC$, so they will have the same area. Thus, for the three regions to have equal area, it suffices for $\triangle ADF$ to be a third of the area of the parallelogram.

Let G be the foot of the altitude from A to line BC. Then, $\triangle ABG$ is a 30-60-90 triangle, implying $AG = \frac{\sqrt{3}}{2}$. Thus, $[ABCD] = BC \cdot AG = \frac{x\sqrt{3}}{2}$.

By the formula for the area of an equilateral triangle, $[ADF] = \frac{x^2\sqrt{3}}{4}$. Thus,

$$3[ADF] = [ABCD] \implies \frac{3x^2\sqrt{3}}{4} = \frac{x\sqrt{3}}{2} \implies x = \boxed{\textbf{(C) } \frac{2}{3}}$$

12. The three real roots of the polynomial $x^3 - 3x^2 + 2x - k$ are in geometric progression for some real number k. What is the value of k?

(A) $\frac{1}{8}$ **(B)** $\frac{8}{27}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{2}$ **(E)** $\frac{27}{8}$

Answer (B): Since the roots form a geometric progression, there are real numbers a and r such that the roots are a, ar, and ar^2.

By Vieta's formulas,

$$a + ar + ar^2 = 3 \implies a(1 + r + r^2) = 3$$

$$a \cdot ar + a \cdot ar^2 + ar \cdot ar^2 = 2 \implies a^2 r(1 + r + r^2) = 2.$$

Dividing the two equations implies $ar = \frac{2}{3}$. Then, by Vieta's formulas,

$$k = a \cdot ar \cdot ar^2 = (ar)^3 = \boxed{\textbf{(B)}\ \frac{8}{27}}$$

13. For everyday in 2022, Ethan has a 50% chance of playing video games. For any day Ethan plays video games, there is a 20% chance his parents catch him, in which case Ethan does not play video games for the rest of the year. The probability Ethan is playing video games on the 100^{th} day in 2022 can be expressed as $\frac{m}{n}$ for positive relatively prime integers m and n. What is the remainder when $m + n$ is divided by 100?

(A) 0 **(B)** 1 **(C)** 11 **(D)** 89 **(E)** 99

Answer (D): For the Ethan to be playing video games on the 100^{th} day in 2022, he must not have been caught within the first 99 days.

On any given day,

- Ethan will not play video games with probability $\frac{1}{2}$. In this case, it is impossible for him to be caught.
- Ethan will play video games with probability $\frac{1}{2}$. He will not be caught with probability $\frac{4}{5}$.

Thus, the probability Ethan does not get caught on any given day (regardless of whether he was playing or not) is $\frac{1}{2} + \frac{1}{2} \cdot \frac{4}{5} = \frac{9}{10}$.

It follows that the probability Ethan does not get caught within the first 99 days is $(\frac{9}{10})^{99}$. Then, since he must play on the 100^{th} day, the desired probability is $\frac{1}{2} \cdot (\frac{9}{10})^{99}$. Thus, $m = 9^{99}$ and $n = 2 \cdot 10^{99}$. Clearly, $n \equiv 0 \pmod{100}$. Using the Binomial Theorem,

$$9^{99} = (10 - 1)^{99} \equiv \binom{99}{1} \cdot 10^1 - \binom{99}{0} \cdot 10^0 \equiv 89 \pmod{100}.$$

Alternatively, 9^{99} can be computed in modulo 4 and modulo 25 separately with the congruences merged at the end. The requested remainder is $\boxed{\textbf{(D)}\ 89}$.

14. How many positive integer ordered quintuplets (a, b, c, d, e) with $a, b, c, d, e \leq 10$ satisfy

$$3^a + 3^b + 3^c + 3^d + 3^e = 3^f$$

for some positive integer f?

(A) 0 **(B)** 9 **(C)** 90 **(D)** 100 **(E)** 180

Answer (C): Without loss of generality, assume $a \leq b \leq c \leq d \leq e$. Permutations will be taken into account later. Also, it is clear $3^e < 3^f$ must hold, implying $e < f$.

Dividing the given equation by 3^a,

$$1 + 3^{b-a} + 3^{c-a} + 3^{d-a} + 3^{e-a} = 3^{f-a}.$$

With the inequalities previously derived, $f - a$ is positive and the exponents of each term on the left hand side are nonnegative. Since $f - a$ is positive, the right hand side is divisible by 3. Thus, the left hand side must also be divisible by 3.

The left hand side consists of five integer powers of 3. Note that 1 is the only integer power of 3 that is not divisible by 3. Thus, the number of terms in the left hand side equal to 1 must be divisible by 3. Since there are 5 terms, the only possible way for this to happen, keeping in mind $a \le b \le c \le d \le e$, is for $a = b = c < d$. Then, the equation becomes

$$3 + 3^{d-a} + 3^{e-a} = 3^{f-a} \implies 1 + 3^{d-a-1} + 3^{e-a-1} = 3^{f-a-1}.$$

Since $a < d \le e < f$, the exponent $f - a - 1$ must be positive, implying the right hand side is divisible by 3. Thus, the left hand side is also divisible by 3. Like before, since 1 is the only integer power of 3 not divisible by 3, this is only possible when the two exponents on the left hand side are both 0. Thus, $d = e = a + 1$. Now,

$$3 = 3^{f-a-1} \implies f = a + 2.$$

Taking permutations into account, (a, b, c, d, e) is some permutation of $(k, k, k, k + 1, k + 1)$ for a positive integer k. Then, the corresponding value of f is $k + 2$.

There are $\binom{5}{3}$ ways to choose which variables in the quintuplet are equal to k. To ensure each variable is between 1 to 10, inclusive, there are 9 choices for k. Thus, the number of quintuplets is $\binom{5}{3} \cdot 9 = \boxed{\textbf{(C) } 90}$.

15. James has a total of $20! = 20 \cdot 19 \cdots 1$ marbles. Let N be the number of ways he selects 20 of them at random, where the order of the marbles selected is considered indistinguishable. What is the highest power of 20 that divides N?

(A) 20^3 **(B)** 20^4 **(C)** 20^5 **(D)** 20^6 **(E)** 20^7

Answer (A): By definition,

$$N = \binom{20!}{20} = \frac{20!(20! - 1)(20! - 2)(20! - 3)\dots(20! - 19)}{20!} = (20! - 1)(20! - 2)(20! - 3)\dots(20! - 19).$$

Recall that $20 = 2^2 \cdot 5$. Consider the number of powers of 5 that divide N separately from the number of powers of 2 that divide N.

If a is not divisible by 5, then $20! - a$ will clearly not be divisible by 5. Hence, it suffices to narrow down on $a = 5, 10,$ and 15. None of these values of a are divisible by 25, but 20! is divisible by 25, since $5 \cdot 10$ divides it. Hence, for $a = 5, 10,$ and 15, the number $20! - a$ will be divisible by 5 but not 25, implying that N has exactly 3 powers of 5 in its prime factorization.

Now, it is impossible for 20^b to divide N if $b \ge 4$, since N does not have a factor of 5^4.

Checking if 20^3 divides N is now equivalent to checking if 2^6 divides N. Since $20! - 16$ contains a factor of 16 and $20! - 8$ contains a factor of 8, N is indeed divisible by 2^6.

Hence, the largest power of 20 that divides N is $\boxed{\textbf{(A) } 20^3}$.

16. What is the number of positive integer values of $n \le 2025$ such that

$$x + \lfloor \sqrt{x} \rfloor + \lfloor \sqrt[3]{x} \rfloor = n$$

has no real solution for x?

(A) 54 **(B)** 55 **(C)** 56 **(D)** 57 **(E)** 58

Answer (C): Since the floor function only outputs integers and n is an integer, x must be an integer.

Let $f(x) = x + \lfloor \sqrt{x} \rfloor + \lfloor \sqrt[3]{x} \rfloor$ for integers $x \ge 0$ (to prevent the number under the square root from being negative). If x is incremented by 1, x will always increase. While $\lfloor \sqrt{x} \rfloor$ or $\lfloor \sqrt[3]{x} \rfloor$ may not increase, they will never decrease. Thus, $f(x)$ is a strictly increasing function along the nonnegative integers.

Clearly, $f(0) = 0$ and $f(1) = 3$. It should be checked if there is an integer a such that $f(a) = 2025$ (if not, the integer a where $f(a)$ is as closest to 2025 as possible).

Clearly, $f(2025) > 2025$ and $f(44^2) = 1936 + 44 + 12 = 1992$. Since $f(x)$ is strictly increasing, $44^2 \le a < 45^2$ if a exists.

In this range, $\lfloor \sqrt{a} \rfloor$ can only be 44 and $\lfloor \sqrt[3]{a} \rfloor$ can only be 12.

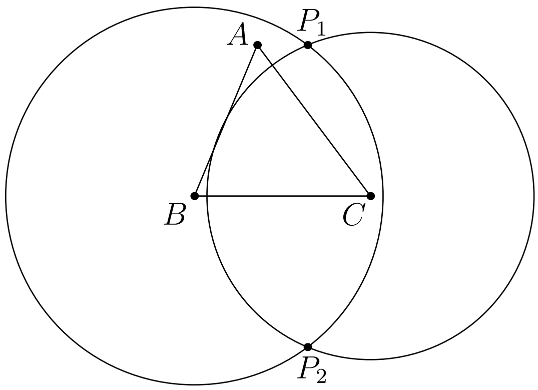
Then, $a = 2025 - 44 - 12 = 1969$, which lies in the range. Hence, $f(1969) = 2025$.

By the strictly increasing behavior of $f(x)$, the values $f(1), f(2), f(3), \cdots, f(1969)$ are distinct and the only values of f that can be one of the first 2025 positive integers. Thus, exactly 1969 of those integers are covered by f, so the number of integers n not covered by f is $2025 - 1969 = \boxed{\textbf{(C) } 56}$.

17. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. A point P is selected in the same plane such that triangles ABP and PCA are congruent. What is the sum of all possible values of the area of triangle ABP?

(A) 24 **(B)** 42 **(C)** 84 **(D)** 102 **(E)** 108

Answer (E): By congruency, corresponding sides of the triangle must be equal in length. Thus, $AP = PA$ (which always holds), $PC = AB = 13$, and $BP = CA = 15$. Thus, P is on the circle with radius 15 centered at B and on the circle with radius 13 centered at C. These two circles intersect at 2 points (P_1 and P_2) as shown:



By symmetry across the perpendicular bisector of $\overline{BC}$, $ABCP_1$ is an isosceles trapezoid. By rotational symmetry about the midpoint of $\overline{BC}$, ABP_2C is a parallelogram.

Let D and E be the feet of the altitudes to BC from A and P_1, respectively. By a property of a 13-14-15 triangle, $AD = 12$ and $BD = 5$. By symmetry, $EC = 5$. Thus, $DE = 4$. Clearly, $ADEP_1$ is a rectangle, so $AP_1 = 4$. Then, $[ABP_1] = \frac{1}{2} \cdot AP_1 \cdot AD = 24$.

In any parallelogram, the diagonals divide the parallelogram into four triangles of the same area. Since $\triangle ABP_2$ and $\triangle ABC$ comprise of half the area of ABP_2C, $[ABP_2] = [ABC] = \frac{1}{2} \cdot AD \cdot BC = 84$.

Thus, the sum of the values of $[ABP]$ is $24 + 84 = \boxed{\textbf{(E) } 108}$.

18. April has five identical fair coins. Every minute, she flips all coins and keeps only the ones that flip heads while discarding the ones that flipped tails. If April continues this process until she has no coins left, what is the probability that at some point in time, she had exactly three coins?

(A) $\frac{45}{128}$ **(B)** $\frac{34}{93}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{2}$ **(E)** $\frac{15}{31}$

Answer (B): Suppose April "wins" if and only if she has exactly three coins and "loses" if she can no longer do so. Let A and B be the probabilities that she wins given that the current number of coins is 5 and 4, respectively. Clearly, the number of coins she has can never increase. The desired value is A.

By binomial probabilities, if there are currently n coins, the probability that exactly k of them will be heads when all are flipped is $\frac{1}{2^n} \cdot \binom{n}{k}$.

Suppose there are currently 5 coins and she flips them all.

- All 5 coins will be heads with probability $\frac{1}{32}$. Now, her probability of winning is A.
- 4 of the coins will be heads with probability $\frac{5}{32}$. Now, her probability of winning is B.
- 3 of the coins will be heads with probability $\frac{10}{32}$. Now, she has won.
- Otherwise, less than 3 of the coins will be heads, in which she can no longer win.

Hence, $A = \frac{1}{32}A + \frac{5}{32}B + \frac{10}{32} \implies 31A = 5B + 10$.

Suppose there are currently 4 coins and she flips them all.

- All 4 coins will be heads with probability $\frac{1}{16}$. Now, her probability of winning is B.
- 3 of the coins will be heads with probability $\frac{4}{16}$. Now, she has won.
- Otherwise, less than 3 of the coins will be heads, in which she can no longer win.

Hence, $B = \frac{1}{16}B + \frac{4}{16} \implies B = \frac{4}{15}$.

Thus, $31A = 5B + 10 \implies A = \boxed{\textbf{(B) } \frac{34}{93}}$.

19. Ada is bored and decides to expand and simplify

$$f(x, y, z) = (x + y^2 + z^3)^K - (x^3 + y^2 + z)^K$$

for some positive integer K. If the number of monomials in Ada's expansion of $f(x, y, z)$ is greater than or equal to 300, what is the smallest possible value of K?

(Note: A monomial is a single nonzero term.)

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Answer (C): Any terms in the expansion of $(x + y^2 + z^3)^K$ will be in the form $x^a y^{2b} z^{3c}$, where a, b, and c are nonnegative integers such that $a + b + c = K$. Clearly, (a, b, c) uniquely determines a term because otherwise, one of the exponents of x, y, or z will be different, resulting in a different term. The number of terms in the expansion is equivalent to the number of solutions to $a + b + c = K$ in nonnegative integers, which is $\binom{K+2}{2}$ by stars and bars.

Similarly, the expansion of $(x^3 + y^2 + z)^K$ contains unique terms in the form $x^{3d} y^{2e} z^f$, where d, e, and f are nonnegative integers such that $d + e + f = K$. Similar to before, there are $\binom{K+2}{2}$ terms in the expansion.

However, when the two expressions are expanded and subtracting, some terms will be canceled out. It remains to search for what terms appear in both expansions.

This is equivalent to searching for (a, b, c, d, e, f) where $x^a y^{2b} z^{3c} = x^{3d} y^{2e} z^f$ and $a + b + c = d + e + f = K$. Comparing exponents, $a = 3d$, $b = e$, and $3c = f$. In turn with $a + c = d + f$, it follows that $a = 3c$, $c = d$, and $a = f$. Then, $b + 4c = K$.

If K is fixed, c can range from 0 to $\lfloor \frac{K}{4} \rfloor$ for b to be nonnegative. This leads to $\lfloor \frac{K}{4} \rfloor + 1$ solutions in (b, c), which determines each of a, d, e, and f. Hence, there are $\lfloor \frac{K}{4} \rfloor + 1$ terms in the first and second expansions each, that will eventually cancel out.

It follows that the number of terms $f(x, y, z)$ contains is

$$2 \cdot \left(\binom{K+2}{2} - \left(\left\lfloor \frac{K}{4} \right\rfloor + 1 \right) \right).$$

Since $\binom{K+2}{2}$ grows quadratically while $\frac{K}{4}$ grows linearly, $\binom{K+2}{2}$ will dominate $\frac{K}{4}$ for sufficiently large K. Hence, for the expression to be at least 300, the value of $\binom{K+2}{2}$ should be around 150.

If $K = 16$, the expression is 296, which is too small. However, if $K = 17$, the expression is 332. Hence, the smallest K is $\boxed{\textbf{(C) } 17}$.

20. For how many real numbers $0 \le x \le 10$ is $x^2 + \{x\}$ an integer?

(Note: $\{x\}$ is the fractional part of x. $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greater integer less than or equal to x.)

(A) 90 **(B)** 91 **(C)** 101 **(D)** 110 **(E)** 111

Answer (E): Using $x = \lfloor x \rfloor + \{x\}$, the given expression is

$$(\lfloor x \rfloor + \{x\})^2 + \{x\} = \lfloor x \rfloor^2 + 2\lfloor x \rfloor \{x\} + \{x\}^2 + \{x\} = \lfloor x \rfloor^2 + \{x\}(2\lfloor x \rfloor + 1 + \{x\}).$$

Since $\lfloor x \rfloor$ is always an integer by definition, it suffices for $f(x) = \{x\}(2\lfloor x \rfloor + 1 + \{x\})$ to be an integer.

Consider the number of solutions in the interval $[k, k+1)$ for some integer k. The interval $[k, k+1)$ is precisely the values of x such that $\lfloor x \rfloor = k$. For $x \geq 0$, the value $2k+1$ will be positive. Since both $\{x\}$ and $(2k+1+\{x\})$ will be nonnegative and strictly increasing along $[k, k+1)$, their product $f(x)$ will be strictly increasing along $[k, k+1)$. Plugging in the endpoints, $f(x)$ ranges along $[0, 2k+2)$. Because $f(x)$ is strictly increasing in the interval, each value is achieved exactly once, leading to $2k+2$ values of x that satisfy the condition when k is fixed.

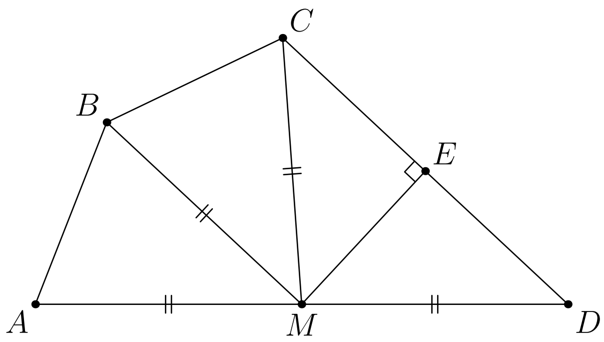
The range $0 \leq x \leq 10$ covers the intervals $[0,1), [1,2), [2,3), \cdots, [9,10)$ completely as well as $x = 10$. Since $x = 10$ is also a value that works, the total number of x is

$$(2+4+6+\cdots+20)+1 = \boxed{\textbf{(E) } 111}$$

21. Let $ABCD$ be a convex quadrilateral with $AB = BC = 1$ and $CD = 2$. If the perpendicular bisectors of AB, BC, and CD all intersect on the midpoint of DA, what is the length of side DA?

(A) $\sqrt{2}+1$ **(B)** $\sqrt{6}$ **(C)** $2\sqrt{3}-1$ **(D)** $\frac{5}{2}$ **(E)** $\sqrt{3}+1$

Answer (E): Let M be the midpoint of $\overline{DA}$. Since M lies on the perpendicular bisector of $\overline{AB}$, $MA = MB$. Similarly, $MB = MC$ and $MC = MD$. Let $x = MA = MB = MC = MD$. Let E be the foot of the altitude from M to CD, which is also the midpoint of $\overline{CD}$ because $MC = MD$.



Clearly, $\triangle MAB \cong \triangle MBC$ because the sides are equal. Let $\theta = \angle AMB = \angle BMC$. Thus, $\angle CMD = 180° - 2\theta$. Clearly, ME bisects $\angle CMD$, so $\angle CME = 90° - \theta$ and then $\angle MCE = \theta$.

Let F be the foot of the altitude from B to CM. Since $\angle MCB = \angle CBM$, it is impossible for $\angle MCB$ to be obtuse by considering the angles of $\triangle BCM$. Thus, F lies on ray CM. Since the angles of $\triangle CEM$ and $\triangle MFB$ are the same and $CM = MB$, the two triangles are congruent. Thus, $MF = CE = \frac{1}{2}CD = 1$. Then, $FC = |MC - MF| = |x-1|$. Applying Pythagorean Theorem to both $\triangle BFM$ and $\triangle BFC$,

$$BF^2 = BM^2 - MF^2 = BC^2 - CF^2 \implies x^2 - 1 = 1 - (x-1)^2 \implies 2x^2 - 2x - 1 = 0 \implies x = \frac{1 \pm \sqrt{3}}{2}.$$

Discarding the negative solution, $x = \frac{1+\sqrt{3}}{2}$, implying that $DA = 2x = \boxed{\textbf{(E) } \sqrt{3}+1}$.

OR

As in the first solution, $x = MA = MB = MC = MD$. Thus, $ABCD$ is a cyclic quadrilateral with M as the center and AD being a diameter. The radius of the circle is clearly x. Since the sides of a cyclic quadrilateral can be rearranged without changing the radius of its circumcircle, rearrange the sides so that $AD = 2x$, $AB = CD = 1$, and $BC = 2$.

Since AD is a diameter, $\angle ABD = \angle ACD = 90°$. Thus, by Pythagorean Theorem, $AC = BD = \sqrt{4x^2 - 1}$. Then, by Ptolemy's Theorem on $ABCD$,

$$2 \cdot 2x + 1 \cdot 1 = 4x^2 - 1 \implies 2x^2 - 2x - 1 = 0,$$

which is the same quadratic found in the first solution.

22. How many positive three-digit integers $\overline{abc}$ with $a \neq 0$ satisfy

$$a^{b+c} = (a+b)^c?$$

(For example, 222 works as $2^{2+2} = (2+2)^2$.)

(A) 92 **(B)** 96 **(C)** 99 **(D)** 103 **(E)** 106

Answer (D):

Case 1: $b = 0$

Then, $a^{b+c} = a^c$ and $(a+b)^c = a^c$, implying all integers with $b = 0$ work. Since a and c are digits with a non-zero, there are 9 choices for a and then 10 choices for c. Thus, there are $9 \cdot 10 = 90$ integers in this case.

Case 2: $b \neq 0$

Then, $a \neq a + b$.

Case 2.1: $a = 1$

Then, it suffices for $(b+1)^c = 1$. Since b is non-zero, the equation can only be satisfied if $c = 0$ and then b can be any non-zero digit. There are 9 such integers here.

Case 2.2: $a \in \{2, 4, 8\}$

Then, a^{b+c} is a power of 2, implying it is necessary for $a + b$ to also be a power of 2 for $a^{b+c} = (a+b)^c$ to hold. The only pairs of digits (a, b) where both a and $a + b$ are powers of 2 are $(2, 2)$, $(2, 6)$, $(4, 4)$, $(8, 8)$. Plugging these pairs into the equation, the respective values of c are 2, 3, 8, and 24. Since 24 is not a digit, the last pair is discarded while the others are valid. There are 3 such integers here.

Case 2.3: $a \in \{3, 9\}$

Similar to Case 2.2, a^{b+c} is a power of 3, implying it is necessary for $a + b$ to also be a power of 3. The only pair of digits (a, b) when this occurs is $(3, 6)$. Plugging this pair into the equation yields $c = 6$. There is 1 such integer here.

Case 2.4: $a \in \{5, 7\}$

Then, $a + b$ must be a power of a, since 5 and 7 are prime. But this is impossible when $b \neq 0$, since the next power of a is too far when b is a digit.

Case 2.5: $a = 6$

Then, a^{b+c} is a power of 6, meaning the exponents of the 2 and 3 in the prime factorization of a^{b+c} are equal. Then, $a + b$ must also be a power of 6 for the exponents of the 2 and 3 in the prime factorization of $(a+b)^c$ to be equal. Similar to Case 2.4, this is impossible when $b \neq 0$, since the next power of 6 is too far when b is a digit.

Thus, the total number of integers is $90 + 9 + 3 + 1 = \boxed{\textbf{(D)}\ 103}$.

23. For a positive integer n, let $1 = d_1 < d_2 < \cdots < d_k = n$ be all of its positive divisors. How many positive integers $2 \leq n \leq 100$ satisfy the property that $\frac{d_{i+1} - d_i}{d_2 - d_1}$ is an integer for all $1 \leq i \leq k - 1$?

(A) 35 **(B)** 67 **(C)** 91 **(D)** 92 **(E)** 95

Answer (E): Let p be the smallest prime that divides n. Then, $d_2 = p$ and $d_{i+1} - d_i \equiv 0 \pmod{p-1}$ for all $1 \leq i \leq k - 1$. Since $d_2 \equiv 1 \pmod{p-1}$, it immediately follows that all divisors of n are $1 \pmod{p-1}$. Furthermore, any prime that divides n is also $1 \pmod{p-1}$.

Conversely, suppose all primes that divide n are $1 \pmod{p-1}$. Any divisor of n besides 1 is some product of (not necessarily distinct) primes that divide n. Since all of these primes are $1 \pmod{p-1}$, their product will be $1 \pmod{p-1}$, regardless of which primes or how many primes are in the product.

Thus, n satisfies the original condition if and only if every prime that divides n is $1 \pmod{p-1}$.

Consider the complement: the number of n such that $2 \leq n \leq 100$ and don't satisfy the condition. Eliminating the obvious n that satisfy the condition:

- All prime n satisfy the condition
- If $p = 2$, n will always satisfy the condition
- If $p = 3$, then n cannot be divisible by 2 by the minimality of p. Thus, all of n's divisors will be odd, meaning they are all $1 \pmod{p-1}$. Thus, if $p = 3$, n will always satisfy the condition

To locate an n that doesn't satisfy the condition, it is necessary (but not sufficient) for $p \geq 5$ and for n to have more than 1 prime in its prime factorization. By the definition of p, any other prime that divides n must be at least p. If n's prime factorization has 3 primes, then $n \geq p^3 \geq 125 > 100$. Thus, n's prime factorization has 2 primes. If $p \geq 11$, then $n \geq p^2 \geq 121 > 100$. Thus, either $p = 5$ or $p = 7$. Let q be the other prime that is at least p and divides n.

Case 1: $p = 5$

Then, $5q \leq 100 \implies 5 \leq q \leq 20$. For the condition to not be satisfied, $q \not\equiv 1 \pmod 4$. The only q in range are 7, 11, and 19. Thus, there are 3 values of n here that don't satisfy the condition.

Case 2: $p = 7$

Then, $7q \leq 100 \implies 7 \leq q \leq 14$. For the condition to not be satisfied, $q \not\equiv 1 \pmod 6$. The only q in range is 11. Thus, there is 1 values of n here that don't satisfy the condition.

Hence, 4 values of n don't satisfy the condition. Since this is the complement, the desired answer is $99 - 4 = \boxed{\textbf{(E)}\ 95}$.

24. Finn the hunter and a rabbit are at the points $(0, 0)$ and $(2002, 2000)$ respectively on the coordinate plane. Every minute, the rabbit randomly chooses to move one unit up, down, right, or left. Immediately after the rabbit moves, Finn moves one unit up, down, right, or left as to minimize the distance between the two. In the case where Finn has more than one path that would minimize the distance, he randomly selects one of them. The expected number of minutes until the rabbit and Finn are on the same point on the plane can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. What is the remainder when $m+n$ is divided by 1000?

(A) 11 **(B)** 47 **(C)** 191 **(D)** 675 **(E)** 731

Answer (A): Let a be the (possibly negative) difference in the x coordinates of the rabbit and Finn. Define b similarly for the y coordinates. The rabbit and Finn are on the same point precisely when $a = b = 0$.

Define $E(\mathcal{A}, \mathcal{B})$ to be the expected number of minutes until rabbit and Finn are on the same point, given that currently, $a = \mathcal{A}$ and $b = \mathcal{B}$. Clearly, $E(\mathcal{A}, \mathcal{B}) = E(\mathcal{B}, \mathcal{A})$ by symmetry. Also, changing the sign on either $\mathcal{A}$ or $\mathcal{B}$ will not change the value of $E(\mathcal{A}, \mathcal{B})$. The requested value is $E(2002, 2000)$.

Regardless of the values of a and b, the rabbit's four movements will increase or decrease either a or b by 1.

Suppose that after the rabbit moves, a and b are both positive. For optimization, Finn must make a movement that minimizes the distance between him and the rabbit. Clearly, increasing a or b will increase the distance, so Finn never wants to do that. Thus, using the distance formula and considering the possible distances depending on which variable he decreases, he needs to examine which of $\sqrt{(a-1)^2 + b^2}$ or $\sqrt{a^2 + (b-1)^2}$ is smaller. The first distance is smaller when $a > b$ is bigger, and the second distance is smaller when $a < b$. When $a = b$, both prospective distances are the same. While it is random which one he chooses, it does not matter because of symmetry.

Hence, given that a and b are positive after the rabbit moves, Finn will always react by decreasing the larger of the two variables. A similar argument applies when a or b are nonpositive, in which he will increase or decrease a or b that will result in the larger of $|a|$ or $|b|$ decreasing.

Clearly, $E(0, 0) = 0$. Consider the value of $E(k, k)$ for $k > 0$.

- The rabbit will increase either a or b by 1 with probability $\frac{1}{2}$. Finn will react by decreasing that variable back to the value k. Thus, the expected number of minutes in this case is given by $1 + E(k, k)$.
- The rabbit will decrease either a or b by 1 with probability $\frac{1}{2}$. Finn will react by decreasing the other variable to $k - 1$. Thus, the expected number of minutes in this case is given by $1 + E(k-1, k-1)$.

Hence,

$$E(k,k) = \frac{1}{2}(1 + E(k,k)) + \frac{1}{2}(1 + E(k-1,k-1)) \implies E(k,k) = 2 + E(k-1,k-1).$$

It becomes clear from this recursive formula that $E(k,k) = 2k$ for all $k \geq 0$.

Now, consider the value of $E(2,0)$:

- The rabbit will increase a by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a back to 2. Thus, the expected number of minutes in this case is given by $1 + E(2,0)$.
- The rabbit will decrease a by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a to 0, in which the two are on the same point. Thus, the expected number of minutes in this case is 1.
- The rabbit will change the value of b with probability $\frac{1}{2}$ (remember that the signs of the inputs of the function $E(a,b)$ do not matter). Finn will react by decreasing a by 1. Thus, the expected number of minutes in this case is given by $1 + E(1,1)$.

Thus,

$$E(2,0) = \frac{1}{2}(1 + E(1,1)) + \frac{1}{4}(1 + E(2,0)) + \frac{1}{4}(1) \implies E(2,0) = \frac{8}{3}.$$

Finally, consider the value of $E(k+2,k)$ for $k > 0$.

- The rabbit will increase a by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a back to $k+2$. Thus, the expected number of minutes in this case is given by $1 + E(k+2,k)$.
- The rabbit will decrease a by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a to k. Thus, the expected number of minutes in this case is given by $1 + E(k,k)$.
- The rabbit will increase b by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a to $k+1$. Thus, the expected number of minutes in this case is given by $1 + E(k+1,k+1)$.
- The rabbit will decrease b by 1 with probability $\frac{1}{4}$. Finn will react by decreasing a to $k+1$. Thus, the expected number of minutes in this case is given by $1 + E(k+1,k-1)$.

Thus, for $k > 0$,

$$E(k+2,k) = \frac{1}{4}(1 + E(k+2,k)) + \frac{1}{4}(1 + E(k,k)) + \frac{1}{4}(1 + E(k+1,k+1)) + \frac{1}{4}(1 + E(k+1,k-1))$$

$$\implies 3E(k+2,k) = 4 + (2k) + (2k+2) + E(k+1,k-1).$$

$$\implies E(k+2,k) = \frac{4k + 6 + E(k+1,k-1)}{3}. \quad (*)$$

Claim: $E(k+2,k) = 2k + 2 + \frac{2}{3^{k+1}}$ for all $k \geq 0$

This will be proven by induction on k. The claim holds for the base case of $k = 0$.

Assume the inductive hypothesis holds for $k = t-1$. It will be shown that this implies it will also hold for $k = t$. Using $(*)$,

$$E(t+2,t) = \frac{4t + 6 + \left(2(t-1) + 2 + \frac{2}{3^t}\right)}{3} = \frac{6t + 6 + \frac{2}{3^t}}{3} = 2t + 2 + \frac{2}{3^{t+1}}.$$

Thus, if the hypothesis holds for $k = t-1$, it will hold for $k = t$, completing the induction. ■

Plugging in $k = 2000$ gives

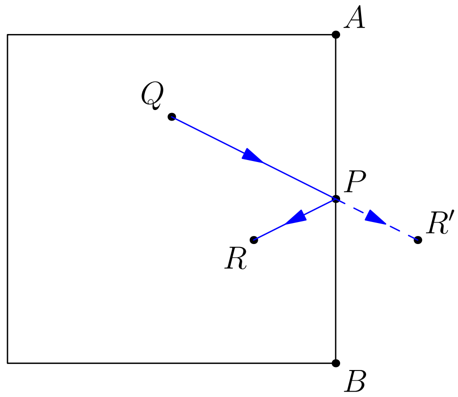
$$E(2002, 2000) = 4002 + \frac{2}{3^{2001}} = \frac{4002 \cdot 3^{2001} + 2}{3^{2001}}.$$

Thus, $m + n = 4003 \cdot 3^{2001} + 2$. By Euler's Theorem, $3^{400} \equiv 1 \pmod{1000}$, so $3^{2000} \equiv 1 \pmod{1000}$. Hence, $m + n \equiv 3 \cdot 3 + 2 \equiv 11 \pmod{1000}$. The requested remainder is $\boxed{\textbf{(A) } 11}$.

25. A bouncy ball with negligible size is shot from the top left hand corner of a unit square. The angle the ball is shot at with respect to the top of the unit square is θ. Whenever the ball hits a side of the square it rebounds so that the angle that it makes with the side it hit stays the same, but it does not go along the same path again. If the ball rebounds off the sides 2022 times before reaching a corner of the square, what is the number of different possible values for θ?

(A) 672 **(B)** 880 **(C)** 1080 **(D)** 1632 **(E)** 1932

Answer (B): Let the square be a unit square, and place the ball in the coordinate plane with its starting position at the origin. Tessellate the coordinate plane into unit squares with sides parallel to the coordinate axes and vertices at lattice points. Suppose the ball hits some side of a square at a point P as shown in the diagram:

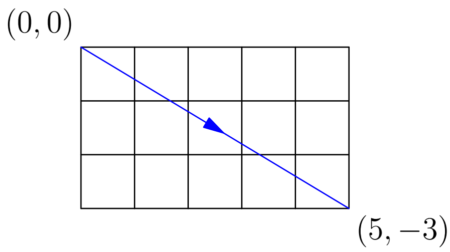


$\overline{AB}$ is the side of the square the ball hits. Q is an arbitrary point on the ball's path before it hits AB at P with no point where it hits a side of the square between Q and P. R is defined similarly as Q but is a point after the ball hits AB at P. Furthermore, R' is the reflection of R across AB.

By the reflection, $\angle BPR = \angle BPR'$. Since the ball rebounds at P with the same angle it hit the side of the square, $\angle APQ = \angle BPR$. Thus, $\angle APQ = \angle BPR'$, so Q, P, and R' are collinear.

Henceforth, whenever the ball hits a side of the square, assume it continues into the next square in a straight line instead of rebounding off the side of the square.

The ball reaching a corner of the square with rebounds is equivalent to the ball reaching a lattice point in the coordinate plane without rebounds. Clearly, the slope of the line representing the ball's path will be some negative real number. Suppose the next time the ball reaches a lattice point after $(0,0)$ is $(m,-n)$ for positive integers m and n. For example, if $m=5$ and $n=3$, the ball's path can be modelled in a grid of unit squares as follows:



Clearly, every angle θ where the ball eventually hits a lattice point can be modelled by such a grid of tiles. Thus, instead of finding the number of values of θ, the number of pairs (m,n) will be found.

It hasn't been shown yet that $(m,-n)$ is actually the next time the ball reaches a lattice point. The ball's path is given by the equation $y = -\frac{n}{m}x$. Let $d = \gcd(m,n)$.

If $d>1$, then $(\frac{m}{d}, -\frac{n}{d})$ is a lattice point on the path strictly between $(0,0)$ and $(m,-n)$, contradicting that $(m,-n)$ is the next time the ball reaches a lattice point.

Let $d = 1$ and suppose there existed a lattice point $(m', -n')$ on the path strictly between $(0,0)$ and $(m, -n)$. Then,

$$\frac{n}{m} = \frac{n'}{m'} \implies m'n = mn'$$

so that the point is on the path. In addition, $0 < m' < m$ and $0 < n' < n$ so that the point is strictly between the endpoints. The right hand side of the equation above must be divisible by n, but since $\gcd(m, n) = 1$ by assumption, n would have to divide n'. Thus, $n' \geq n$, since n' is positive. However, this contradicts $0 < n' < n$. It follows that $\gcd(m, n) = 1$ always allows $(m, -n)$ to be the next lattice point in the ball's path.

The ball hits a side whenever it hits $x = 1, 2, 3, \ldots, (m-1)$ or $y = -1, -2, -3, \ldots, -(n-1)$, corresponding to a total of $(m-1) + (n-1)$ rebounds. There is no overcount, since the ball does not hit a lattice point between the two endpoints of the path, and thus, cannot hit two sides simultaneously. Since the total number of rebounds is 2022, it follows that $m + n = 2024$.

With $\gcd(m, n) = 1$, $\gcd(m, m+n) = \gcd(m, 2024)$ must also be 1. Thus, the number of positive integers m is given by $\phi(2024)$ (where ϕ is Euler's Totient Function). Since $2024 = 2^3 \cdot 11 \cdot 23$, it follows that $\phi(2024) = 1 \cdot 2^2 \cdot 10 \cdot 22 = \boxed{\textbf{(B) } 880}$.