2019 CMC 10B Solutions Document

Christmas Math Competitions

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1. **Answer (D):** The first term can be computed as $\frac{20+19}{20-19} = 39$. The second term can be computed as $\frac{20 \cdot 19}{20/19} = \frac{19}{1/19} = 19^2 = 361$. Therefore, the expression is equal to $39 + 361 = 400$.

2. **Answer (D):** Clearly, if any of the absolute values of w, x, y, or z are greater than 2, the LHS will be too big. Therefore, $-1 \le w, x, y, z \le 1$, which means w^2, x^2, y^2, and z^2 are each either equal to 0 or 1. Clearly, among w^2, x^2, y^2, and z^2, two of them must be 1 and the other two must be 0. There are $\binom{4}{2} = 6$ ways to choose which ones are 1. For each of the two integers whose squares are equal to 1, we can choose the sign $\pm$ in 2 ways. Thus, there are $6 \cdot 2 \cdot 2 = 24$ quadruples.

3. **Answer (B):** The true value of the sum is 8.68. Fermat rounds each number as $3, 3, 1$, and 2, respectively, which gives him a sum of 9. Euler, on the other hand, rounds each number as $3.1, 2.7, 1.1$, and 1.7, respectively, which gives him a sum of 8.6. Since Euler is off by 0.08 and Fermat is off by 0.32, Euler is 4 times closer to the correct answer compared to Fermat.

4. **Answer (E):** Assume a standard bowl has 70 macaroni and 30 cheese. Since Aluminum's bowl has 50% more macaroni than normal, his bowl has 70 macaroni and 45 cheese.

   The total amount of macaroni amongst the group members is $3 \cdot 70 = 210$, while the total group order is $(70 + 45) + 2 \cdot (70 + 30) = 315$. Therefore, the macaroni makes up $100\% \cdot \frac{210}{315} = 66\frac{2}{3}\%$ of the group's order.

5. **Answer (E):** We consider what happens when we increase each variable in the expression. If we increase a, b, or c, the value of the expression must increase, but if we increase d, the value of the expression must decrease.

   For the maximum value of the expression, we want to minimize d, which means $d = 1$. Then, choosing c uniquely determines the sum $a + b$. By testing $c = 2, 3$, and 4, we find that the expression is maximized at $c = 4$, where its value is $(2 + 3)^4/1 = 625$.

   To minimize the expression, we maximize d, which means $d = 4$. Then, we test $c = 1, 2$, and 3. The expression is minimized at $c = 1$, where its value is $(2 + 3)^1/4 = \frac{5}{4}$.

   The ratio between the maximum and minimum is $625 \cdot \frac{4}{5} = 500$.

6. **Answer (B):** We factor the numerator as follows:

$$x^4 - 3x^2 + 2 = (x^2 - 1)(x^2 - 2) = (x + 1)(x - 1)(x + \sqrt{2})(x - \sqrt{2})$$

where the last step follows from difference of squares. We can also factor the denominator as $(x-1)(x-2)$. For the whole expression to be 0, the numerator must be 0, but to avoid the expression becoming undefined, the denominator cannot be 0.

The valid x are $-1, \sqrt{2}$, and $-\sqrt{2}$, while $x = 1$ does not work because it causes the denominator to be 0. The requested sum is $-1 + \sqrt{2} - \sqrt{2} = -1$.

7. **Answer (D):** There are 3 ways to choose which pair of people the boy will stand directly in between. WLOG, assume that the boy chooses to stand in between the two grandparents. There are 2 ways to choose which grandparent will be on the boy's left in which the other grandparent will be on the boy's right. Then, we treat the grandparents and boy as a single block, while the remaining four family members are four separate blocks. We can freely reorder these five blocks in 5! ways. Thus, there are $3 \cdot 2 \cdot 5! = 720$ possible arrangements.

8. **Answer (D):** At the beginning, cup A has $\frac{1}{2}$ cup of tea and cup B has $\frac{1}{2}$ cup of milk. During the first pouring, cup B transfers $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ cup of milk to cup A. Afterwards, cup A has $\frac{1}{6}$ cup of milk and $\frac{1}{2}$ cup of tea, while cup B has $\frac{1}{3}$ cup of milk.

   During the second pouring, cup A transfers $\frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$ cup of milk and $\frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$ cup of tea to cup B. Afterwards, cup A has $\frac{1}{10}$ cup of milk and $\frac{3}{10}$ cup of tea, which implies $m = 75$. In addition, cup B has $\frac{1}{15} + \frac{1}{3} = \frac{2}{5}$ cup of milk and $\frac{1}{5}$ cup of tea, which implies $n = 66.\overline{6}$.

   Therefore, $\frac{m}{n} = \frac{75}{200/3} = \frac{9}{8}$.

9. **Answer (B):** Let $A = (1,1), B = (13,-5), C = (1,4), D = (13,4)$, and $E = (x,4)$. Note that $\angle ACE = \angle BDE = 90°$. In addition, due to the reflection of the laser when it hits the mirror, $\angle AEC = \angle BED$. Therefore, $\triangle AEC \sim \triangle BED$. Since $AC = 3$ and $BD = 9$, it immediately follows that $CE : ED = 3 : 9$. Since, $CD = 12$, $CE = 3$ and $x = 4$.

10. **Answer (B):** First, $CF = 2 \cdot AB = 12$ by the properties of a regular hexagon. $ABCF$ is a trapezoid with AB parallel to FC. Since MQ is the midsegment of trapezoid $ABCF$, its length is the average of lengths AB and CF or $\frac{6+12}{2} = 9$.

    Consider $\triangle AFE$. Since $AF = FE = 6$ and $\angle AFE = 120°$, we have $AE = 6\sqrt{3}$. By the Midpoint Theorem, $PQ = \frac{1}{2} \cdot AE = 3\sqrt{3}$.

    Then, the area of rectangle $MNPQ$ is given by $MQ \cdot PQ = 27\sqrt{3}$.

11. **Answer (D):** We can factor $n^3 - n$ as $n(n^2-1)$ then $n(n-1)(n+1)$. This is the product of three consecutive positive integers. We note that $1001 = 7 \cdot 11 \cdot 13$. Since multiples of 13 are less common compared to multiples of 7 or 11, we will check sets where one of the three integers is a multiple of 13 and then locate the closest multiple of 11. Obviously, the difference between any two numbers in our set of three consecutive positive integers cannot be greater than 2.

    If the multiple of 13 is 13, the closest multiple of 11 is 11. However, $\{11, 12, 13\}$ does not contain a multiple of 7.

    If the multiple of 13 is 26, the closest multiple of 11 is 22. But $|26 - 22| > 2$, which is too far away. Similarly, if the multiple of 13 is 39 or 52, the closest multiple of 11 is too far away.

    If the multiple of 13 is 65, the closest multiple of 11 is 66. However, neither $\{64, 65, 66\}$ nor $\{65, 66, 67\}$ contains a multiple of 7.

If the multiple of 13 is 78, the closest multiple of 11 is 77, which is also divisible by 7. Both $\{76, 77, 78\}$ and $\{77, 78, 79\}$ work, but to minimize n, we choose $\{76, 77, 78\}$. The value of n is the middle number in the set, so $n = 77$. The requested sum is $7 + 7 = 14$.

12. **Answer (D):** Clearly, the graph G of the equation $x^{1000} + y^{1000} = 1$ is symmetric across the x-axis, y-axis, and the line $y = x$ (because x and y are interchangeable).

    Let S be the square centered at $(0, 0)$ in the coordinate plane with corners at $(\pm 1, \pm 1)$. Note that if $|x| > 1$, then $x^{1000} > 1$, but if $x \le 1$, $x^{1000} \le 1$. Therefore, $|x| \le 1$ and similarly, $|y| \le 1$. It follows that G is contained completely within S.

    We need x and y to be slightly under 1 in the graph G. If the values of $|x|$ and $|y|$ are not close enough to 1, then $x^{1000} + y^{1000}$ will be much less than 1 due to how quickly x^{1000} decreases. Therefore, intuitively G should be very close to the shape of S, which is has an area of 4.

13. **Answer (D):** Distribute the pennies first, then the dimes, then the quarters. By stars and bars, there are $\binom{8}{2} = 28$ ways to distribute the pennies. Now, note that once we have distributed the pennies, we must distribute 6 of the dimes in the exact same way that we distributed the pennies. For example, if Andrew, Bob, and Charlie receive 2, 1, and 3 pennies, respectively, we will initially give them 2, 1, and 3 dimes, respectively. This way, each person will be forced to receive at least as many dimes as pennies, no matter how we distribute the remaining dimes. This will use up 6 of the dimes, leaving 2 dimes to distribute in $\binom{4}{2} = 6$ ways. Then, by the same process, we must use up 8 of the quarters to match each quarter with a dime. The remaining quarter can be distributed in 3 ways.

    Our total is $28 \cdot 6 \cdot 3 = 504$ ways.

14. **Answer (C):** Clearly, $P(a) = P(b) = P(c) = 0$. To find a polynomial with roots at a^2, b^2, and c^2, we must look at the function $P(\sqrt{x}) = 0$. We have $P(\sqrt{x}) = x\sqrt{x} - 3\sqrt{x} + 1 = 0$. However, this is not a polynomial because some of the terms are of a non-integer degree. We can group all the terms containing a square root and then square the expression so that each terms is of an integer degree:

    $$(x - 3)\sqrt{x} = -1$$

    $$(x^2 - 6x + 9)(x) = 1$$

    $$x^3 - 6x^2 + 9x - 1 = 0$$

    $$(x - 2)^3 - 3(x - 2) + 1 = 0$$

    Therefore, the polynomial with roots at a^2, b^2, and c^2 is given by $P(x - 2)$.

15. **Answer (C):** Take the equation in mod a. This tells us that $b \equiv 0 \pmod{a}$ or that a is a divisor of b. Let $b = am$ for some positive integer m. We have $a^2 + am = a(2019 - am) \implies a + m = 2019 - am$. We get $m = \frac{2019-a}{a+1}$ which is only an integer if $\frac{2019-a}{a+1} = \frac{2020}{a+1} - 1$ is an integer.

    Clearly, $a + 1$ must be a positive integer divisor of 2020. However, we must make sure that $a > 0$ and $m > 0$. The condition $a > 0$ is fulfilled if $a + 1 > 1$. The condition $m > 0$ tells us that $\frac{2020}{a+1} - 1 > 0$ or $2020 > a + 1$. Among the divisors of 2020, $a + 1$ may equal any of them except 1 or 2020. Since $2020 = 2^2 \cdot 5 \cdot 101$ has 12 divisors, our answer is $12 - 2 = 10$.

16. **Answer (D):** Let $R(x)$ be the polynomial defined by $P(x) - Q(x)$ for all real x. Then, we have $R(x) \leq 0$ if and only if $6 \leq x \leq 9$. Because polynomials are continuous functions, this means that $R(6) = R(9) = 0$, while $R(x) < 0$ when $6 < x < 9$. We seek $R(1)$ given that $R(0) = 243$.

    We are given that $P(x)$ and $Q(x)$ are both of degree 2, so $P(x) - Q(x) = R(x)$ has at most degree 2. If $R(x)$ has 1, then for every real number k, $R(x) = k$ must have exactly 1 solution in x. However, $R(6) = R(9) = 0$, so $R(x)$ cannot have a degree of 1. If $R(x)$ has a degree of 0, then since $R(6) = R(9) = 0$, $R(x) = 0$ for all real x. However, we are given that $R(0) = 243$, which is a contradiction. Therefore, $R(x)$ cannot have a degree of 0 and must have a degree of 2.

    We see that $R(x) = a(x-6)(x-9)$ for some real number a. Since $R(0) = 243$, $243 = a(-6)(-9) \implies a = \frac{9}{2}$. Then, $R(1) = \frac{9}{2} \cdot (1-6)(1-9) = 180$.

17. **Answer (D):** Let N be the midpoint of CD, and let E be the point in 3D space where A and B coincide after the folding. Now, we have tetrahedron $EMCD$, and we wish to find its height with respect to base MCD. To find this desired height, we simply need to find the altitude from E to MN in $\triangle EMN$.

    By inspection, $MN = 13$ and $ME = MA = 5$. After the folding, we have that $\triangle ECD$ is a triangle with lengths $DC = 10$ and $ED = EC = 13$. When we drop the altitude from E to DC in $\triangle ECD$, the foot is N. Using Pythagorean Theorem, we find that $EN = 12$, so $\triangle EMN$ is a $5 - 12 - 13$ right triangle. It then follows that the length of the altitude from E to MN in $\triangle EMN$ is $\frac{5 \cdot 12}{13} = \frac{60}{13}$.

18. **Answer (C):** In each valid way to shade the squares in the grid, there will always be the same number of shaded squares in each row and column. Therefore, if we were to choose the two squares in the first column to shade, we will always have the same number of ways to shade squares in the rest of the grid. WLOG, suppose we shade the two topmost squares in the first column. We will multiply by $\binom{4}{2} = 6$ at the end.

    Now, we will determine how to shade the second column. The cases will be dependent on what types of squares are immediately to the left of the shaded squares in the second column.

    Suppose both shaded squares in the second column have a shaded square immediately to their left. Then, we must shade all the squares in the 2×2 area in the bottom right corner of the grid. This gives us 1 valid way.

    Suppose one of the shaded squares in the second column has a shaded square immediately to its left, while the other shaded square has an unshaded square immediately to its left. WLOG, in the second column, shade the first an third squares. We will multiply by 4 later. Clearly, in the third and fourth columns, the two squares in the first row must be unshaded, while the two squares in the fourth row must be shaded. In the third column, we can choose either the second or third square to shade, which determines which square to shade in the fourth column. There are $4 \cdot 2 = 8$ valid ways in this case.

    Lastly, suppose both shaded squares in the second column have an unshaded square immediately to their left. Then, we can choose any two squares in the third column to shade and then the two shaded squares in the fourth column are uniquely determined. There are $\binom{4}{2} = 6$ valid ways in this case.

    There are $6 \cdot (1 + 8 + 6) = 90$ ways total.

19. **Answer (B):** Extend BP past P to intersect AC at E. We will proceed with area ratios.

By the Angle Bisector theorem, $\frac{AB}{AC} = \frac{DB}{CD} = 4$. Since $[ABC] = 120$, $[ACD] = 24$ and $[ADB] = 96$. If $[PCD] = a$, then $[PDB] = 4a$ by area ratios. In addition, because $\frac{[ADB]}{[ACD]} = \frac{[PDB]}{[PCD]} = 4$, then $\frac{[ADB]}{[ACD]} = \frac{[ADB]-[PDB]}{[ACD]-[PCD]} = \frac{[APC]}{[APB]} = 4$.

Let $[APC] = b$. Then, $[APB] = 4b$. Because M is the midpoint of AB, we know that $[AMP] = [PMB]$ and $[AMC] = [CMB]$. Then, $[APB] = 2[AMP] \implies [AMP] = [PMB] = 2b$. Using $[AMC] = [CMB]$, $[AMC] = [APC] + [CPD] = a + b$, and $[CMB] = [PCB] + [PMB] = 5a + b$, we get $b = 5a$. In addition, since $[APC] + [PCD] = [ACD] = 24$, we have $a + b = 24$. Solving these two equations tells us that $a = [PCD] = 4$.

20. **Answer (D):** We can check that $12^2 + 28^2 = 928 < 1024 = 32^2$, so $\angle B$ is obtuse. Therefore, segment BC does not enter the interior of the semicircle.

    Let D be the foot of the altitude from C to line AB. By Pythagorean Theorem, $BD^2 + CD^2 = 28^2$ and $(BD + 12)^2 + CD^2 = 32^2 \implies BD^2 + 24BD + CD^2 = 880$. Plugging in the first equation to second equation tells us that $24BD = 96 \implies BD = 4$. Furthermore, $AC = 2 \cdot AD = 32$ and $\angle ADC = 90°$. Therefore, $\angle CAD = \angle CAB = 60°$. Now, we will compute the desired area.

    Let segment AC intersect the semicircle at $E \neq A$. Since segment AC enters the interior of the semicircle but segment BC does not, the desired area is bounded by segment AE and minor arc AE on the semicircle. If O is the center of the semicircle, since $AO = OE = 6$ and $\angle EAO = 60°$, $\triangle AOE$ is equilateral. Hence, the desired area is clearly the area of minor sector AOE minus the area of equilateral triangle $\triangle AOE$. This can be computed as $\frac{6^2\pi}{6} - \frac{6^2\sqrt{3}}{4} = 6\pi - 9\sqrt{3}$.

21. **Answer (A):** Let $n = \overline{abcde}$. We will first consider n being a multiple of 8. Since 1000 is divisible by 8, only $\overline{cde}$ has an effect on n being a multiple of 8. Since $100 \equiv 4 \pmod 8$, if c is odd, then we need $\overline{de} \equiv 4 \pmod 8$. However, if c is even, then we need $\overline{de} \equiv 0 \pmod 8$.

    We need $\overline{de}$ to be divisible by 4, since 100 is divisible by 4. Overall, $\overline{de}$ has $\frac{100}{4} = 25$ possible values, 13 of which are $0 \pmod 8$ and 12 of which are $4 \pmod 8$. However, we need to subtract all the values where d or e is 0, namely $00, 04, 08, 20, 40, 60,$ and 80. From this list, 4 of these numbers are $0 \pmod 8$ and 3 of these are $4 \pmod 8$. So we have 18 acceptable values of $\overline{de}$, 9 of which are $0 \pmod 8$ and 9 of which are $4 \pmod 8$.

    We have 9 choices for c. No matter what the parity of c is, there will always be 9 possible values of $\overline{de}$ that will let $\overline{cde}$ be divisible by 8. Now, to let n be divisible by 3, we can randomly select b, and we choose a such that $a + b + c + d + e \equiv 0 \pmod 3$. There are always 3 such choices for a regardless of what the other digits are.

    Hence, we have a total of $3 \cdot 9 \cdot 9 \cdot 9 = 3^7$ values of n, which has 8 divisors.

22. **Answer (B):** For $\gcd(a, c)$ and $\gcd(b, c)$ to both greater than 1, c must share a prime factor with a and a prime factor with b. The shared prime factor between a and c is not necessarily the same as the shared prime factor between b and c.

    Let $g(a, b)$ denote the smallest prime that divides both a and b if $\gcd(a, b) > 1$ and undefined if $\gcd(a, b) = 1$. In addition, let $h(a)$ denote the smallest prime that divides a for $a > 1$.

    It follows that:

$$f(a,b) = \begin{cases} \min(g(a,b), h(a) \cdot h(b)), & \text{if } \gcd(a,b) > 1 \\ h(a) \cdot h(b), & \text{if } \gcd(a,b) = 1 \end{cases}$$

Since neither $55 = 5 \cdot 11$ nor $77 = 7 \cdot 11$ are prime, $f(x, y) = h(x) \cdot h(y) = 55$ and $f(y, z) = h(y) \cdot h(z) = 77$. Clearly, $(h(x), h(y), h(z)) = (5, 11, 7)$. However, we need to avoid changing the value of $f(x, y)$ and $f(y, z)$ through the min function. Therefore, we must make sure $g(x, y) > h(x) \cdot h(y) = 55$ when $\gcd(x, y) > 1$ and $g(y, z) > h(y) \cdot h(z) = 77$ when $\gcd(y, z) > 1$. The only prime that necessarily divides y with our current information is 11. Therefore, if x is divisible by 11, it will cause $g(x, y) = 11 > 55$, which is a contradiction. Similarly, if z is divisible by 11, it will cause $g(y, z) = 11 > 77$, which is a contradiction. Therefore, neither x nor z may be divisible by 11.

We will now find all possible values of $f(x, z)$. If we let $f(x, z) = h(x) \cdot h(z)$, then $f(x, z) = 35$. This is easily achievable with $(x, y, z) = (5, 11, 7)$.

However, if we let $g(x, z) < h(x) \cdot h(z) = 35$, then we consider $f(x, z) = p$ for some prime $p < 35$. Since $h(z) = 7$, we must have $p \geq 7$. However, recall earlier that x and z are not divisible by 11, so $p \neq 11$.

Checking the primes in the interval $7 \leq p < 35$ aside from 11, we have $p \in \{7, 13, 17, 19, 23, 29, 31$ Each of these values $f(x, z)$ can be achieved with $(x, y, z) = (5p, 11, 7p)$.

Therefore, $f(x, z) \in \{7, 13, 17, 19, 23, 29, 31, 35\}$ for a total of 8 values.

23. **Answer (C):** Since $P(0) = 0$ and $P(x)$ is a monic cubic polynomial, $P(x)$ is in the form $x(x^2 + ax + b)$ for some real numbers a and b. For $P(1) = 1$ to hold, we must have $1 = 1(1 + a + b) \implies a = -b$.

    So $P(x) = x(x^2 - bx + b)$ for some real number b. By Vieta's formulas, $\mathcal{S} = b$. For $P(x)$ to have real roots, the discriminant of the quadratic $x^2 - bx + b$ must be nonnegative or $b^2 - 4b \geq 0 \implies b(b - 4) \geq 0$. The inequality $b(b - 4) \geq 0$ if and only if $b \leq 0$ or $b \geq 4$. So $b \notin \{1, 2, 3\}$ for a total of 3 values that cannot be attained by b.

24. **Answer (C):** Let d_n be the number of ways to pair n pairs of best friends such that no 2 best friends are together. We seek the value of d_5.

    Label the students as $A, A', B, B', C, C', ...$, where A is best friends with A', B is best friends with B', and so on.

    Since A cannot pair with A', A may choose to pair with any of the other $2(n - 1)$ students. WLOG, assume that A pairs with B. We will multiply by $2(n - 1)$ later. Next, we will do casework on whether B' pairs with A' or not.

    If B' pairs with A', then we have $n - 2$ pairs of best friends left to pair up. There are d_{n-2} ways to pair them up in this case.

    If B' does not pair with A', then A' and B' act as a virtual best friend pair and cannot be paired together. Since we have $n - 1$ pairs of best friends in this case, there are d_{n-1} ways to pair them up.

    Therefore, we have $d_n = 2(n - 1)(d_{n-1} + d_{n-2})$ for all $n \geq 2$. Since $d_0 = 1$ and $d_1 = 0$, we can compute $d_2 = 2, d_3 = 8, d_4 = 60$, and $d_5 = 544$.

25. **Answer (C):** Intuitively, the roots of the quadratic $x^2 + F_{100}x - F_{100}^2 - 1 = 0$ should be Fibonacci numbers. In particular, we claim that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all integers $n \geq 1$. We will prove this identity by induction. Checking the base case with $n = 2$, we have $F_3F_1 - F_2^2 = 1 \implies 2 \cdot 1 - 1^2 = 1$, which holds.

    Now, we will show that if we assume the induction hypothesis is true for $n = k$, it will also hold for $n = k + 1$:

$$\begin{aligned} F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - (F_k + F_{k-1})^2 \\ &= F_{k+1}F_k - 2F_kF_{k-1} - F_{k-1}^2 \end{aligned}$$

$$\begin{aligned}
&= (F_k + F_{k-1})F_k - 2F_kF_{k-1} - F_{k-1}^2 \\
&= F_k^2 - F_kF_{k-1} - F_{k-1}^2 \\
&= F_k^2 - F_{k-1}(F_k + F_{k-1}) \\
&= F_k^2 - F_{k+1}F_{k-1} \\
&= (-1) \cdot (F_{k+1}F_{k-1} - F_k^2) \\
&= (-1)^{k+1}
\end{aligned}$$

Since the induction hypothesis is true for $n = k + 1$ given that it is true for $n = k$, the induction is complete. (This identity is more commonly known as Cassini's identity)

By the identity, $F_{100}^2 + 1 = F_{99}F_{101}$. In addition, we can rewrite F_{100} as $F_{101} - F_{99}$.

The quadratic becomes $x^2 + (F_{101} - F_{99})x - F_{99}F_{101} = 0$, which can be factored as $(x + F_{101})(x - F_{99}) = 0$. Now, we wish to find $N \equiv F_{99} + F_{101} \pmod{11}$. Quickly checking, the Fibonacci numbers repeat with a period of 10 in modulo 11 and $F_1 \equiv F_9 \equiv 1 \pmod{11}$. Therefore, $N \equiv 2 \pmod{11}$.