Hnkevin42 Mock AMC 12 Answers

Question	Answer
1	B
2	C
3	E
4	A
5	E
6	A
7	D
8	A
9	D
10	A
11	E
12	A
13	B
14	B
15	C
16	E
17	B
18	D
19	B
20	D
21	C
22	A
23	E
24	D
25	C

Hnkevin42 Mock AMC 12 Solutions Manual

- This document deeply explains a solution for each problem on the contest and shows that all problems can be solved without the use of a calculator. If you managed to find an especially slick or creative solution to a problem, please feel free to mention it in the contest thread. I really hope the actual AMC 12 went well for you guys, and thank you immensely for the reception and attention.

1) **Answer: B**

We have that $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}=\frac{8+3+1}{24}+\frac{7+3}{105}=\frac{1}{2}+\frac{10}{105}=\frac{105+20}{210}=\frac{125}{210}=\frac{25}{42}$.

2) **Answer: C**

Clearly, Cal hits the wall 2 hours after they start. At this time, James is 8 miles from the starting point, and the two must meet inside of the 4-mile strip between them. Thus, we must find the time t such that Cal's distance after t hours is 4 minus James's distance after t hours. We have that $6t=4-4t \rightarrow t=\frac{2}{5}$. Adding this to the two hours elapsed yields that they will meet in $\frac{12}{5}$ hours.

3) **Answer: E**

The multiples of 3 or 4 attained by the two dice are 3, 4, 6, 8, 9, and 12. We can quickly make a table with the possible pairs of top faces that sum to these integers.

Multiple	Pairs
3	(2,1), (1,2)
4	(3, 1), (2,2), (1, 3)
6	(5, 1), (4, 2), (3, 3), (2, 4), (1, 5)
8	(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)
9	(6, 3), (5, 4), (4, 5), (3, 6)
12	(6, 6)

There are $2+3+5+5+4+1=20$ different pairs of top faces out of $6*6=36$ total. Thus, the desired probability is $\frac{20}{36}=\frac{5}{9}$.

4) **Answer: A**

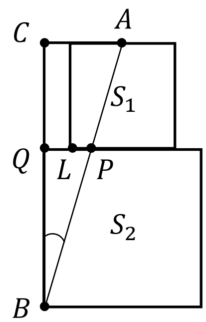
Suppose Mike has $x\%$ in a category that is worth $c\%$ of his total grade. That category will contribute $\frac{cx}{100}\%$ to his overall grade. We can set up an equation to find the minimum homework grade h needed for him to score at least 90%. This equation is

$$100(0.15)+90(0.4)+90(0.2)+25\left(\frac{h}{100}\right)=90.$$

Solving for h yields that $\frac{h}{4}=90-15-36-18=21 \rightarrow h=84$, and this is indeed the minimum since anything lower will make his grade less than 90%.

5) **Answer: E**

Call A the midpoint of the top side of S_1. Call B the bottom left vertex of S_2, and call Q the top left vertex of S_2. Extend the top side of S_1 and the left side of S_2 to meet at point C. It follows that triangles ABC and PBQ are similar, with ABC being scaled up from PBQ by a factor of $\frac{10}{6} = \frac{5}{3}$. The distance from S_1 to C equals the distance from S_2 to C, which is 3. So, segment QP has length $\frac{3}{5} * 3 = \frac{9}{5}$. To find the distance from the bottom left corner L of S_1 to P, we must subtract QL from QP. Clearly, $QL = 3 - 2 = 1$, and $QP = QL = \frac{9}{5} - 1 = \frac{4}{5}$.



6) **Answer: A**

If $a_1 = 1$, then the choices for a_2 are 3, 4, and 5. If $a_2 = 3$, a_3 must be 5. It follows that $a_4 = 2$ and $a_5 = 4$. If $a_2 = 4$, then a_3 must be 2. In this case $a_4 = 5$ and $a_5 = 3$. If $a_2 = 5$, then a_3 can be 2 or 3. $a_3 = 2$ yields no possibilities because the remaining integers 4 and 3 must then be next to each other. $a_3 = 3$ yields no possibilities because either 2 or 4 must then be next to 3. This totals up to 2 permutations.

If $a_1 = 2$, then the choices for a_2 are 4 and 5. If $a_2 = 4$, then $a_3 = 1$ must be true. Then, $a_4 = 5$ and $a_5 = 3$ or $a_4 = 3$ and $a_5 = 5$. If $a_2 = 5$, then $a_3 = 3$ because $a_3 \neq 4$ and if $a_3 = 1$, then 4 and 5 will be next to each other. So, if $a_2 = 5$ and $a_3 = 3$, then $a_4 = 1$ and $a_5 = 4$. This totals up to 3 permutations.

Now, note that $\{a_1, a_2, a_3, a_4, a_5\}$ being distant implies that $\{6 - a_1, 6 - a_2, 6 - a_3, 6 - a_4, 6 - a_5\}$ is distant – the magnitudes of the differences of consecutive terms is preserved, and each element is a distinct member of $\{1, 2, 3, 4, 5\}$. So, for every distinct distant permutation with $a_1 = 1$ and $a_1 = 2$, there is a unique distant permutation with $a_1 = 5$ and $a_1 = 4$ respectively. There are $2 * 5 = 10$ distant permutations with $a_1 \neq 3$.

Finally, we count all permutations with $a_1 = 3$. We have that the choices for a_2 are 1 and 5. If $a_2 = 1$, then the possibilities for the remaining terms are $a_3 = 4$, $a_4 = 2$, and $a_5 = 5$ or $a_3 = 5$ $a_4 = 2$, and $a_5 = 4$. If $a_2 = 5$, then the possibilities for the remaining terms are $a_3 = 2$, $a_4 = 4$, and $a_5 = 1$ or $a_3 = 1$ $a_4 = 4$, and $a_5 = 2$. This yields 4 permutations, so our grand total is $10 + 4 = 14$ distant permutations.

7) **Answer: D**

Notice that there are $9 + 8 + 7 + 6 + \cdots + 2 + 1 = 45$ games to be played. It is impossible for all 10 teams to have an odd number of wins because the number of games played is exactly the number of wins, the sum of 10 odd numbers is even, and 45 is odd.

To show that 9 teams with odd wins is possible, draw a 10 x 10 grid that represents all the games being played. In the i^{th} row and j^{th} column with $i \neq j$, we place label "L," denoting the loss of the j^{th} team against the i^{th}, or the label "W," with a similar meaning for wins. The total number of W's in column j represents how many games team j won, and we want to find the maximum number of columns that can have an odd number of W's. Fill all unfilled spaces of columns 1, 2, 3, …, 10 in order. If a box in row i and column j has a label, then the box in row j and column i must have the opposite label – if one team wins, the team's opponent must lose and vice versa. If we do this, column j will have $10 - j$ unfilled spaces. However, we can fill all potential spaces such that either zero or one of them has a W, intentionally shifting the parity of W's in a column to odd when we need to. We can do this with all columns 1 to 9 so that 9 teams can have an odd number of wins.

8) **Answer: A**

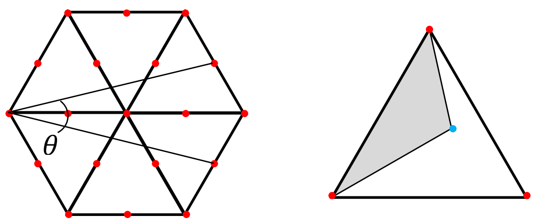
Set up a system of equations the formula for the sum of an arithmetic series, and call a the first term common to both series. For A_1, we have that the sum of the 30 terms is $15(2a+29d)=75 \rightarrow 2a+29d=5$. For A_2, we have that the sum of the 75 terms is $\frac{75}{2}(2a-74d)=30 \rightarrow 2a-74d=\frac{4}{5}$ using the formula for the sum of an arithmetic series. We solve for d by subtracting the two equations. We have $103d=\frac{21}{5} \rightarrow d=\frac{21}{515}$.

9) **Answer: D**

The given equation becomes $\log_{10} ab = 2016 \rightarrow ab = 10^{2016}$. Then, the number of distinct pairs (a,b) is exactly the number of distinct positive divisors of 10^{2016}. Breaking down 10^{2016} into its prime factorization yields that $10^{2016} = 2^{2016}5^{2016}$, and the number of divisors is equal to $(2016+1)(2016+1) = 2017^2$. For any pair (a,b) of integers satisfying the given equation, we see that ab immediately equals 2016. So, the logarithm to base 10 of the product of the values of ab across all distinct ordered pairs (a,b) in S is equal to 2016 times this number of pairs (a,b), which is $2016 \cdot 2017^2$.

10) **Answer: A**

Dissect the hexagon into 24 equilateral triangles. The area of the hexagon is $6 * \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$, so the area of each triangle will then be $\frac{3\sqrt{3}}{2*24} = \frac{\sqrt{3}}{16}$. We need $3+4+5+4+3=19$ points to capture all triangles' vertices.



Indeed, draw these 19 points. We claim that the shortest area of a triangle defined by any 3 non-collinear points of these 19 is $\frac{\sqrt{3}}{16}$, which in turn proves that more than 19 points are needed. If a segment between two of these points is parallel to one of the hexagon's sides, the smallest triangle can be obtained by choosing the third point to be on the nearest segment(s) parallel to the first, resulting in area at least $\frac{1}{2} * \frac{1}{2} * \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{16}$. Else no sides of the triangle are parallel to a side of the hexagon. The smallest side of this triangle is $\frac{\sqrt{3}}{2}$ and the smallest angle θ has $\sin\frac{\theta}{2} = \frac{\sqrt{3}}{4} \div \sqrt{\frac{3}{16}+\frac{49}{16}} = \frac{\sqrt{3}}{\sqrt{52}}$, so that $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2 * \frac{\sqrt{3}}{\sqrt{52}} * \frac{7}{\sqrt{52}} = \frac{7\sqrt{3}}{26}$. By the sine area formula, such a triangle's area is bounded below by $\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{7\sqrt{3}}{26}\right) > \left(\frac{3}{8}\right)\left(\frac{14}{26}\right) > \frac{1}{8} > \frac{\sqrt{3}}{16}$, proving the claim. The 20th point must be in or at the boundary of some equilateral triangle with area $\frac{\sqrt{3}}{16}$, so we can choose this point and two suitable vertices of this equilateral triangle for the resulting triangle to have area below $\frac{\sqrt{3}}{16}$.

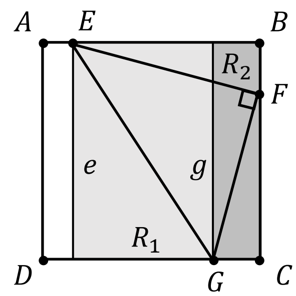
11) **Answer: E**

Draw a graph on the coordinate plane, with the x-axis being the train's arrival and the y-axis being Zach's arrival time. Each axis is labeled from 0 to 60 minutes to capture the entire sample space since the arrival times range from 2:00 to 3:00 PM, which spans 60 minutes. Since the train waits for 10 minutes, the departure time is modeled by the equation $y=x+10$. Then, Zach could arrive at any time such that the point lies under $y=x+10$. So, the probability that Zach catches the train is the area under $y=x+10$ divided by the total area of 60 x 60 square. Note that $y=x+10$ leaves a 50 x 50 right triangle above the graph but inside the square, so the area under the line is equal to the total area minus the area of that triangle. Thus, the probability is equal to $\frac{3600-\frac{1}{2}(50)(50)}{3600} = \frac{2350}{3600} = \frac{4700}{7200} = \frac{47}{72}$.

12) **Answer: A**

Suppose $F \neq B$. From E and G, draw segments parallel to BC (call them e and g respectively) to obtain three rectangles. Assuming the configuration below, E could be to the left or right of G, but EG cannot be vertical. In either case, the triangular portion of EFG inside the rectangle R_1 with sides e and g has less than half the area of R_1, as two of the portion's vertices coincide with R_1's vertices but not the third. The triangular portion of EFG inside the rectangle R_2 with side BC and the rightmost of e and g has area also less than half of R_2, because F is on BC, and only one of the remaining corners of this portion coincides with a vertex of R_2. Hence triangle EFG cannot have area more than $\frac{1}{2}$ for this case.

For $F = B$, the area is easily maximized when AB and BC hit their maximum of 1. Thus, the maximum area of triangle EFG is $\frac{1}{2}$.



13) **Answer: B**

Let the number F_n denote the n^{th} Fibonacci number. Note that the Fibonacci numbers are the sequence of numbers that satisfy $F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$. We claim that the zero of $f_n(x)$ is equal to $-\frac{F_n}{F_{n+1}}$. We approach using induction. First, we can easily see that the zero of $f_1(x)$ is equal to -1, which is $-\frac{F_1}{F_2} = -1$. Now assume this holds for some n. To get the zero of f_{n+1}, we must have that $1 + \frac{1}{x}$ equals the zero of $f_n(x)$. Thus, we need $1 + \frac{1}{x} = -\frac{F_n}{F_{n+1}} \to \frac{1}{x} = -\frac{F_n + F_{n+1}}{F_{n+1}} \to \frac{1}{x} = -\frac{F_{n+2}}{F_{n+1}} \to x = -\frac{F_{n+1}}{F_{n+2}}$, proving the claim. Then, we have that the zero of f_{12} is equal to $-\frac{F_{12}}{F_{13}}$. Computing F_{12} and F_{13} by hand yields that this zero is equal to $-\frac{144}{233}$.

14) **Answer: B**

Write down the prime factorization of 2016 as $2^5 3^2 7$. To complete the game in the minimum number of turns possible, Kevin and Felix need to finish each phase in the smallest amount of turns possible since the number of turns in any one phase doesn't depend on the number turns in another phase. Clearly, phase 3 can be completed in only 1 turn since choosing 672 will result in the number on the board being 3, which can't be divisible by any 3-digit prime factor.

Phase 2 cannot be completed in only one turn because every divisor d_2 except 12, 14, 16, and 18 exists such that $\frac{2016}{d_2}$ is also 2 digits, and 14 can be chosen after 12, 16, and 18 while 12 can be chosen after 14. However, it can be completed in two turns after choosing 63 and then 32, so 2 is the minimum number of turns that Kevin and Felix can complete phase 2 in.

Now we try to find the minimum number of turns that phase 1 can possibly be completed in. The one-digit divisors of 2016 excluding 1 are 2, 3, 4, 6, 7, 8, and 9. We show that phase 1 cannot be completed in three or less turns. Note that 7 must be erased. Additionally, one of the pairs (2, 4), (4, 8), and (2, 8) must be erased. But then, a power of 3 must be erased, contradicting that phase 1 can be completed in three or less turns. So, the minimum number of turns that phase 1 can be completed in is 4, which is done by erasing 7, 9, 4, and 8.

Thus, the least amount of turns that the game can possibly be completed in is $T = 4 + 2 + 1 + 1 = 8$.

15) **Answer: C**

We count the total number of possibilities of three slides. We say that a number "slides" if the square containing the number moves to an open space. If 5 slides first, then 7, 4, 1, or 5 can slide next. By symmetry, we have that if any of these numbers slide back to the middle space, we have three possibilities for the third slide since an open spot on the edge but not a corner has three adjacent squares. This will total to 12 groupings of three slides. If 8 slides first, then either 8 or 7 can slide. But for both of these numbers, we have three possibilities for the third slide. This totals to 6 groupings of three slides, and by symmetry, the case in which 2 slides first should also lead to 6 groupings of three slides. Thus, the total number of possibilities of three slides is $12 + 12 = 24$.

We count the number of successes. Say that a success happens if the numbers in the rightmost column sum to an odd integer. If 5 moves first and doesn't move second, then 5 will be on the rightmost column but can't move at all. Thus, there are already 9 possibilities for the third slide that will make a success happen. If 5 moves first and moves again, then 5 must move back to the right space for a success to happen. So, there will be 10 successes for the third slide if 5 moves first.

If 8 moves first and moves again, then 5 must move to the right side in order for a success to happen. If 8 moves first and 7 moves, then either 3 or 5 can move for a success to happen. We have 3 successes if 8 moves first, but now notice that the case in which 2 moves first is exactly the same as the case in which 8 moves first. This is because an odd number is to the left of 2, and the leftmost column doesn't have any effect on the parity of the rightmost column's sum. We have 3 more successes, and the desired probability is $\frac{10+3+3}{24} = \frac{16}{24} = \frac{2}{3}$.

16) **Answer: E**

We utilize the formula that, given the prime factorization of $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ the number of positive divisors of a positive integer n equals $(e_1 + 1)(e_2 + 1) \dots (e_k + 1)$. First evaluate $D(100)$, which, considering that the prime factorization of 100 is $2^2 5^2$, is $(2 + 1)(2 + 1) = 9$, and call an n that satisfies the given equation "good." Since no positive integer greater than 1 can have only 0 or 1 positive divisors, we have that the possible pairs $\left(D(n), D(n + 1)\right)$ are (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), and (7, 2). Note that $D(1) + D(2) = 1 + 2 = 3 \neq 9$.

Consider the pairs (7, 2) and (2, 7). Since 7 is a prime number, we must have that some n less than 100 is a 6th power of a prime. The only sixth power of a prime less than 100 is $2^6 = 64$, but clearly, neither 63 nor 65 have only two positive divisors. So, we don't have any good n for these pairs.

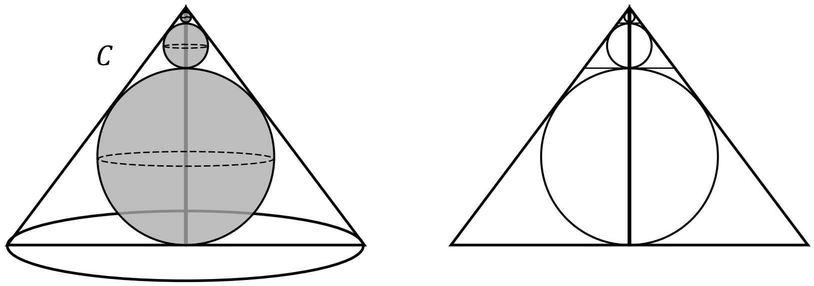
Consider the pairs (3, 6) and (6, 3). Since 3 is a prime number, we must have that some n less than 100 is a square of a prime. The only squares of primes less than 100 are 4, 9, 25, and 49. We rule out 4 because both 3 and 5 have only 2 positive divisors. We also rule out 9 because both 8 and 10 have only 4 positive divisors. Considering 25, we have that 24 has 8 divisors while 25 has 3, so we rule out 25. However, if we consider 49, then we have that $50 = 2 * 5^2$ has 6 positive divisors while $48 = 2^4 * 3$ has 10. So, 49 is a good n.

Lastly, consider the pairs (4, 5) and (5, 4). Since 5 is prime, we must have that some n less than 100 is a 4th power of a prime. The only fourth powers of primes less than 100 are 16 and 81. If we consider 16, we have that $15 = 3 * 5$ has exactly 4 positive divisors while 17 has only 2. If we consider 81, we have that $82 = 2 * 41$ has exactly 4 positive divisors while $80 = 2^4 * 5$ has 10. So, 15 and 81 are both good n.

The sum of these good n is $49 + 15 + 81 = 145$.

17) **Answer: B**

Consider a cross-section of the cone through its diameter, which is an isosceles triangle T_1 of base 6 and height 4. Then images of the spheres in this cross section are a series of circles. The radii of all these circles are exactly the radii of the spheres $S_1, S_2, S_3, \dots$, and we find how these radii change to sum the volumes. Take the incircle of T_1 and draw a horizontal segment tangent to the top point of that incircle whose endpoints are on the sides of T_1. The apex of the triangle and these endpoints will form a triangle T_2. Repeat this process with T_2 to form T_3, and so on. Clearly, S_n must be above S_{n-1} assuming the cone is placed upright. The maximum volume of S_n is attained when its cross section is the incircle of T_n.



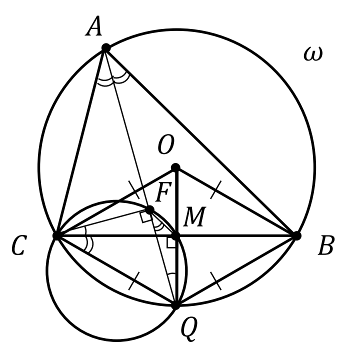
We show that $r_{n+1} = \frac{1}{4}r_n$. We first use the inradius area formula to compute r_1. We have that $\frac{6(4)}{2} = \frac{r_1(5+5+6)}{2} \rightarrow r_1 = \frac{3}{2}$, since the cone has slant height 5 (which is attained using the Pythagorean theorem). Then, it is immediate that T_2 is scaled down from T_1 by a factor of $\frac{1}{4}$ because the height of T_2 is $4 - 2\left(\frac{3}{2}\right) = 1$, the height of T_1 is 4, and the triangles are similar. However, T_3 is similar to T_2, T_4 is similar to T_3, and from our construction of T_{n+1} from T_n, all of $T_1, T_2, T_3 \ldots$ are similar. Hence, the height of T_{n+1} is scaled down from the height of T_n by a factor of $\frac{1}{4}$. Since $r_1, r_2, r_3, \ldots$ belong to $T_1, T_2, T_3, \ldots$, we have $r_{n+1} = \frac{1}{4}r_n$.

The sum of the volumes of spheres $S_1, S_2, S_3, \ldots$ turns out to be the sum of the infinite geometric series with first term $V(S_1) = \frac{4}{3}\pi\left(\frac{3}{2}\right)^3$ and common ratio $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$. This sum is equal to $\frac{\frac{4}{3}\pi\left(\frac{3}{2}\right)^3}{1-\frac{1}{64}} = \frac{\frac{9\pi}{2}}{\frac{63}{64}} = \frac{32\pi}{7}$ from the formula for the sum of an infinite geometric series.

18) **Answer: D**

Draw the circumcircle ω of triangle ABC. Call O the circumcenter of ABC. Notice that the bisector of $\angle CAB$ meets ω at the midpoint Q of minor arc $\widehat{BC}$. Also observe that $\angle COM = 60°$ and $\angle MCB = 30°$ (from the inscribed angles $\angle QAB \cong \angle QCB$). Then, $\angle AQC = 180° - 30° - 110° = 40°$. Then, $\angle AQM = \angle FQM = 20°$. However, $\angle CFQ = \angle CMQ = 90°$, which signals that $CFMQ$ is a cyclic quadrilateral. Then, $\angle MCQ = \angle MFQ = 30°$, and we can see that $MQ = OM = \frac{6/2}{\sqrt{3}} = \sqrt{3}$. Then by the Law of Sines, we have

$$\frac{\sin \angle MFQ}{MQ} = \frac{\sin \angle MQF}{MF} \rightarrow \frac{1}{2\sqrt{3}} = \frac{\sin 20°}{MF} \rightarrow MF = 2\sqrt{3}\sin 20°.$$



19) **Answer: B**

Note that $\sin x < \cos x$ for $0° \leq x \leq 30°$. Additionally, $\cos 2x \leq \cos x$ for these values of x, so the possible orders for our arithmetic sequence are $\sin x, \cos 2x, \cos x$ and $\cos 2x, \sin x, \cos x$. Let d be the common difference of this arithmetic sequence.

For the order $\cos 2x$, $\sin x$, $\cos x$, we have the system

$$\begin{cases} \cos 2x + d = \sin x \\ \sin x + d = \cos x \end{cases}.$$

Subtracting the two equations yields $\cos 2x - 2\sin x = -\cos x \rightarrow \cos^2 x + \cos x = \sin^2 x + 2\sin x \rightarrow$ $2\cos^2 x + \cos x - 1 = 2\sin x \rightarrow (2\cos x - 1)(\cos x + 1) = 2\sin x$ from the double angle formula. However, for $0° \leq x \leq 30°$, the left-hand side of the equation is at least $\left(\sqrt{3} - 1\right)\left(\frac{\sqrt{3}}{2} + 1\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} > 1$, whereas the right-hand side is at most 1. There is no solution for this case.

For the order $\sin x$, $\cos 2x$, $\cos x$, we have the system

$$\begin{cases} \sin x + d = \cos 2x \\ \cos 2x + d = \cos x \end{cases}.$$

Subtracting the two equations yields $2\cos 2x = \sin x + \cos x = 2(\cos x + \sin x)(\cos x - \sin x) = \cos x + \sin x$ using the double angle formula and factoring out the difference of squares. We can have $\cos x + \sin x = 0$, but this does not hold for $0° \leq x \leq 30°$. So, we can then divide both sides of the equation by $\cos x + \sin x$ to get that $2(\cos x - \sin x) = 1 \rightarrow 2\sqrt{1 - u^2} = 1 + 2u$ after setting $\sin x = u$ and $\cos x = \sqrt{1 - u^2}$. The equation then becomes the quadratic $8u^2 + 4u - 3 = 0$. Since sine is positive on $0° \leq x \leq 30°$, we only want the positive solution for u, which is $\frac{-1+\sqrt{7}}{4}$ after using the quadratic formula. Plugging this back into our first equation yields $d = 1 - \frac{8-2\sqrt{7}}{8} - \frac{-2+2\sqrt{7}}{8} = \frac{1}{4}$.

20) **Answer: D**

We note that there is no part of the hyperbola that satisfies $-1 < x < 1$. Thus, the equilateral triangle must lie on the right branch of the hyperbola, where $x > 1$. Now we show that the hyperbola is symmetric across the x-axis. Given some value x, we see that $y = \pm\sqrt{b(x^2 - 1)}$, and the plus or minus signals that both (x, y) and $(x, -y)$ lie on the hyperbola, implying that the hyperbola is symmetric across the x-axis.

Now note that, because of the hyperbola's symmetry, the side of the equilateral triangle in the hyperbola neither of whose endpoints are (1,0) must be vertical. This is because the perpendicular bisector of that side will not intersect (1,0) if the side isn't vertical, and we need the perpendicular bisector to hit (1,0) since the inscribed triangle must be equilateral.

Thus, we have the relation that $x - 1 = \sqrt{3b(x^2 - 1)}$ because $\sqrt{b(x^2 - 1)}$ is half the base of this triangle and $x - 1$ is the height. The area of this triangle is equal to $\frac{\sqrt{3}}{4}\left(2\sqrt{b(x^2 - 1)}\right)^2$ using the direct formula for the area of an equilateral triangle, but it's given that $3\sqrt{3} \leq \sqrt{3}\left(\sqrt{b(x^2 - 1)}\right)^2 \leq 12\sqrt{3} \rightarrow 3 \leq b(x^2 - 1) \leq 12$.

Now suppose that the equilateral triangle has area $a\sqrt{3}$. Note that, given this area, $x = 1 + \sqrt{3a}$. Then, we have

$$b = \frac{a}{3a + \sqrt{12a}} \rightarrow b = \frac{1}{3 + \sqrt{\frac{12}{a}}}.$$

As a increases, b increases. But a goes from 3 to 12, meaning that b goes from $\frac{1}{5}$ to $\frac{1}{4}$. Thus, our answer is $\frac{1}{5} + \frac{1}{4} = \frac{9}{20}$.

21) **Answer: C**

Note that the largest integer less than 2016 is 2015, and $2015_{10} = 13155_6$. We solve the problem by checking the palindromes in base-6 that are palindromes in base-10. We specifically perform casework based off of the k-digit palindromes in base-6, for $1 \leq k \leq 5$. We say that the first digit of a number is its leading digit, and the last digit of a number is its ones digit.

First consider the one-digit palindromes in base-6. These are 1, 2, 3, 4, and 5. Clearly, these integers are also palindromes in base-10, so we have 5 palindromes for this case.

Consider the two-digit palindromes in base-6. In base-10, these palindromes are expressed as $7a$, where a is an integer between 1 and 5. We get that $7a$ is a palindrome for $a = 1$ only, so we have 1 palindrome for this case.

Consider the three-digit palindromes in base-6. In base-10, these palindromes are expressed as $37a + 6b$, where a is an integer from 1 to 5 and b is an integer from 0 to 5. For $a = 1$, we need $37 + 6b$ to be a palindrome, and since the last digit is an even number plus an odd number, the first digit must be odd. In particular, $b = 3$ works and results in a base-10 palindrome. For $a = 2$, we need $74 + 6b$ to be a palindrome, and the leading digit must be even. Here, no values of b get us a palindrome. For $3 \leq a \leq 5$, the last digit of $37a + 6b$ must end in a 1 (For $n = 5$, we can't have the last digit be 2 because the last digit will always be odd, despite $185 + 6b$ being greater than 200 for $3 \leq b \leq 5$.). For $a = 3$, we have $b = 0$ or 5. For $a = 4$, we have no possibilities since the last digit will always be even. For $a = 5$, we have $b = 1$. Thus, there are 4 palindromes for this case.

Then, consider the four-digit palindromes in base-6. In base-10, these palindromes are expressed as $217a + 42b$, with a and b defined as they were last time. If a is odd, we need the first digit to be odd, and if a is even, we need the first digit to be even. For $a = 1$, we have that $b = 3$ works. For $a = 2$, we have that $b = 0$ works. For $a = 3$, we have that $b = 3$ works. For $a = 4$, we have that $b = 0$ works. For $a = 5$, we have that $b = 3$ works, but the second digit is not the same as the third in this representation. So, this doesn't get us a palindrome, and we have 4 palindromes for this case.

Finally, consider the five-digit palindromes in base-6. First note that the first and last digits of these palindromes must end in 1, and the second and fourth digits must be no greater than 3 for the palindrome to be less than 2016 in base-10. These palindromes can thus be expressed as $1297 + 222a + 36b$, where a goes from 0 to 3 and b goes from 0 to 3 unless $a = 3$, in which case b will go from 0 to 1. If $a = 0$, 1, or 2, we need the last digit to be a 1. For $a = 0$, we have that $b = 4$ works. For $a = 1$, we have that $b = 2$ works. For $a = 2$, we have that $b = 0$ and 5 work. For $a = 3$, we have that no b works. However, out of all these combinations of a and b, only $a = 0$ and $b = 4$ makes the second digit the same as the third. So, we have 1 palindrome for this case.

Thus, the number of positive integers less than 2016 that are palindromes in base-6 and base-10 is $5 + 1 + 4 + 4 + 1 = 15$.

22) **Answer: A**

We first try to locate those monic quadratic polynomials $P(x)$ that satisfy conditions (i) and (ii) only. We work with the prime factorization of the constant term c to find these polynomials. Clearly, all polynomials $P(x) = x^2 + (c + 1)x + c$ work, where this c goes from 1 to 9 inclusive. The other polynomials must exist such that the constant term is not prime. Since each of these c has only one pair other than $(c, 1)$ that multiplies to c, each of these c corresponds to only one polynomial. We find that the set of polynomials in question is

$$\{x^2 + 4x + 4, x^2 + 5x + 6, x^2 + 6x + 8, x^2 + 6x + 9, x^2 + 7x + 10\}.$$

Now, we utilize the fact that in order for $P\big(P(x)\big)$ to equal 0, $P(x) = r$ must be true, where r is a root of $P(x)$. This allows us to find the polynomials $P(x)$ that satisfy conditions (i), (ii), and (iii). Applying the aforementioned fact to the $P(x)$ that satisfy conditions (i) and (ii) and letting d_1 and d_2 be divisors that multiply to c, we have that $x^2 + (d_1 + d_2)x + d_1 d_2 = -d_1$ must have integer roots for some choice of d_1. More specifically, we want two positive integer divisors of $d_1 d_2 + d_1$ to sum to $d_1 + d_2$. Quickly checking, we have that this is only possible for $c = 3$ and $c = 10$, so the polynomials $P(x)$ that satisfy (i), (ii), and (iii) are $P_1(x) = x^2 + 4x + 3$ and $P_2(x) = x^2 + 7x + 10$.

The roots of $P_1(P_1(x))$ consist of the solutions of $x^2+4x+3=-1 \to x^2+4x+4=0$ and the solutions of $x^2+4x+3=-3 \to x^2+4x+6=0$. These roots are -2 and $\frac{-4\pm\sqrt{-8}}{2}=-2\pm i\sqrt{2}$. From the problem statement, we want $Q(Q(x))$ to have the roots $-2+n$ and $-2+n\pm i\sqrt{2}$ for a nonzero integer n. We want to find whether coefficients for $Q(x)$ exist. If we let $Q(x)$ have roots a and b, we need to find solutions to

$$\begin{cases} (-2+n)^2-(-2+n)(a+b)+ab=a \\ \left(-2+n+i\sqrt{2}\right)^2-\left(-2+n+i\sqrt{2}\right)(a+b)+ab=b. \\ \left(-2+n-i\sqrt{2}\right)^2-\left(-2+n-i\sqrt{2}\right)(a+b)+ab=b \end{cases}$$

This system particularly comes from Vieta's formulas. We know that b must be the right-hand side of the second and third equations because if a root to a real polynomial is complex, that solution's complex conjugate must also solve the equation. Subtracting the second equation from the third reveals that $a+b=2(-2+n)$. Substituting this into the first and second equations and subtracting reveals that $a-b=2$. Then, we have that $a=n-1$ and $b=n-3$ after solving the previous equations for a and b in terms of n. Plugging this into our second equation yields

$$(-2+n)^2+2i\sqrt{2}(-2+n)-2-2(-2+n)^2-2i\sqrt{2}(-2+n)+(n-1)(n-3)=n-3 \to -(-2+n)^2$$
$$=-n^2+5n-4 \to -n^2+4n-4=-n^2+5n-4 \to n=0.$$

But we were given that n cannot be zero, so there are 0 polynomials $Q(x)$ for this case.

The roots of $P_2(P_2(x))$ consist of the solutions of $x^2+7x+10=-2 \to x^2+7x+12=0$ and the solutions of $x^2+7x+10=-5 \to x^2+7x+15=0$. These roots are $-3, -4$, and $\frac{-7\pm i\sqrt{11}}{2}$. Here, we want $Q(Q(x))$ to have the roots $-3+n$, $-4+n$, and $\frac{-7+2n}{2}\pm\frac{i\sqrt{11}}{2}$ for a nonzero integer n. We need to find solutions to

$$\begin{cases} (-3+n)^2-(-3+n)(a+b)+ab=a \\ (-4+n)^2-(-4+n)(a+b)+ab=a \\ \left(\frac{-7+2n}{2}+\frac{i\sqrt{11}}{2}\right)^2-\left(\frac{-7+2n}{2}+\frac{i\sqrt{11}}{2}\right)(a+b)+ab=b. \\ \left(\frac{-7+2n}{2}-\frac{i\sqrt{11}}{2}\right)^2-\left(\frac{-7+2n}{2}-\frac{i\sqrt{11}}{2}\right)(a+b)+ab=b \end{cases}$$

Proceeding similarly to how we did with $P_1(x)$, we have that $a+b=-7+2n$ and $a-b=3$. We then have that $a=n-2$ and $b=n-5$. We plug this into our first equation to get

$$(-3+n)^2-(-3+n)(-7+2n)+n^2-7n+10=n-2$$
$$\to 9-6n+n^2-21+13n-2n^2+n^2-7n+10=n-2 \to n=0.$$

Again, we can't have $n=0$, so we have 0 polynomials for this case.

Thus, there are 0 possible polynomials for $Q(x)$ that satisfy the problem's conditions.

23) **Answer: E**

Note the following:

$$f(n)=\prod_{i=1}^{n-1} f(i)=f(n-1)f(n-2)\prod_{i=1}^{n-3} f(i)=f(n-1)f(n-2)^2, \text{for } n \text{ odd, (I)}$$
$$f(n)=\sum_{i=1}^{n-1} f(i)=f(n-1)+f(n-2)+\sum_{i=1}^{n-3} f(i)=f(n-1)+2f(n-2), \text{for } n \text{ even. (II)}$$

We see that the value n mentioned in the problem is in the form of a large product, which indicates that n is most likely odd. Combining (I) and (II) and using the fact that $f(3) = 1$, we have that for n odd,

$$f(n) = f(n-1)f(n-2)^2 = f(n-1)f(n-3)^2f(n-4)^4$$

$$= f(n-1)f(n-3)^2f(n-5)^4f(n-6)^8 = f(n-1)f(n-3)^2f(n-5)^4 \dots f(4)^{2^{\frac{n-5}{2}}}.$$

For n even, after again combining (I) and (II) and using the fact that $f(1) = 1$, we have

$$f(n) = f(n-1) + 2f(n-2) = f(n-1) + 2f(n-3) + 4f(n-4)$$

$$= f(n-1) + 2f(n-3) + 4f(n-5) + 8f(n-6) = f(n-1) + 2f(n-3) + 4f(n-5) + \cdots + 2^{\frac{n-2}{2}}$$

We simplify the equation given to us.

$$f(n) = 3^k \prod_{i=0}^{30} \left[2^{i+2} + \sum_{j=0}^{i+1} 2^j f(5+2i-2j)\right]^{2^{30-i}}$$

$$= 3^k \prod_{i=0}^{30} \left[f(5+2i) + 2f(3+2i) + \cdots + 2^i f(5) + 2^{i+1} f(3) + 2^{i+2}\right]^{2^{30-i}}$$

$$= 3^k (f(65) + 2f(63) + \cdots + 2^{31}) \dots (f(7) + 2f(5) + 4f(3) + 8)^{2^{29}} (f(5) + 2f(3) + 4)^{2^{30}}$$

$$= f(66)f(64)^2f(62)^4 \dots f(8)^{2^{29}} f(6)^{2^{30}} 3^k.$$

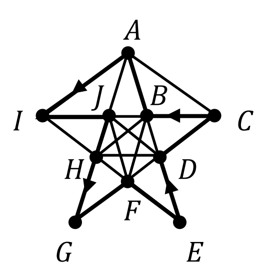
However, we're missing $f(4)$ in the representation of $f(n)$ above, but we know that the representation needs an $f(4)^{2^{31}} = 3^{2^{31}}$. So, we have found the exact k that we want: $k = 2^{31}$. This also is in the exact same form for $f(n)$ with n odd. So, we have $n - 1 = 66 \rightarrow n = 67$. The last three digits of $k + n = 2^{31} + 67$ will be our answer, which is

$$(2^{31} + 67) \bmod 1000 = (2(2^{10})^3 + 67) \bmod 1000 = (2(1024)^3 + 67) \bmod 1000 = (2(24)^3 + 67) \bmod 1000$$

$$= (2(576)(24) + 67) \bmod 1000 = 648 + 67 = 715.$$

24) **Answer: D**

Represent the moves of students by drawing a directed, contiguous path that starts from a student S and connects neighboring students. For the students to move successfully, every student must be part of some closed loop composed of a certain group of neighboring students.



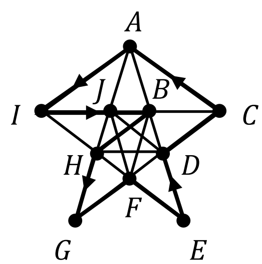
We are given that we can draw at most 1 loop of two students (which is basically a line segment connecting two students that represents them switching positions). So, to count the number of positions possible, we first partition the 10 students into multiple groups with at most one group of 2. We have the following partitions:

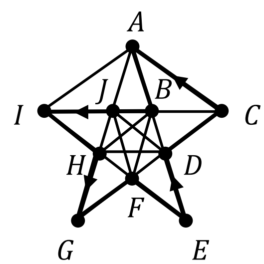
$$\{10\}, \{8,2\}, \{7,3\}, \{6,4\}, \{5,5\}, \{5,3,2\}, \{4,4,2\}, \{4,3,3\}.$$

Now note that the loops are directed, meaning that if more than 2 neighboring students are in the loop, then traveling the loop clockwise will result in a different combination of students than if the loop was traveled counterclockwise. So, for each set of integers above, the number of formations of students is equal to 2^c times the number of configurations of loops, where c is the number of elements greater than 2. Additionally, no two different configurations of loops will result in the same configuration of students since some student must move to a different position in the two configurations.

We then count the number of loop configurations. We proceed with casework where, without loss of generality, all loops go counterclockwise and multiply the powers of 2 for each case afterward. When orders of students are reversed, it is because the configuration of the loop is different, *not* because the loop was travelled clockwise instead of counterclockwise.

$\{10\}$: Here, H, G, F, D, and E must be connected in the loop. Assume student J is not connected to B in the loop. Then, we have two paths corresponding to two orders (AIJ/AJI and BCA/CBA), leading to four loop configurations. If J is connected to B, we have four more configurations from the orders IJB, BJI, JBC, and CBJ (in which the former pair is shown below for clarification). For this case we have $8 * 2^1 = 16$ formations.





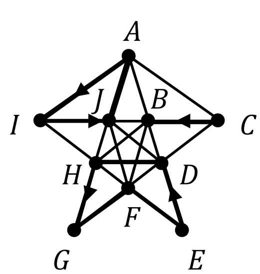
$\{8, 2\}$: We count the number of segments of two neighboring students that allow for a loop of 8 students. Then, we must count the possible number of loops of 8 students as there could be multiple loops of 8 students allowed by a segment of two neighboring students. These segments are BJ, AI, CA, JI, CB, HG, and ED. All segments allow only one loop of 8 students except HG and ED, which allow two (recalling the orders AIJ/AJI and BCA/CBA). So, we have 9 configurations of loops. The number of formations here is $9 * 2^1 = 18$.

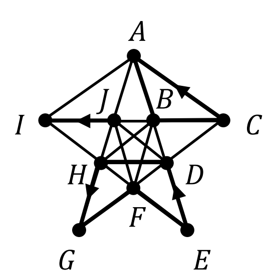
$\{7, 3\}$: We count the number of loops of 3 neighboring students that allow for a loop of 7 students. The only possible loops are AIJ and ABC, which both allow for only one loop of 7 students. There are two configurations for this case, so we have $2 * 2^2 = 8$ formations.

$\{6, 4\}$: Here, note that no possible loops of 4 neighboring students allow for a loop of 6 students, so we have 0 formations for this case.

$\{5, 5\}$: Here, there is only one configuration of loops: the large triangle passing through points $AIJBC$ and the concave pentagon $HGFED$. We have $1 * 2^2 = 4$ formations of students for this case.

$\{5, 3, 2\}$: We count the number of loops of 5 students that allow for a loop of 3 students and a segment of 2 students. The possible loops of 5 students include concave pentagons $HGFED$, $FEDCB$, and $JIHGF$, as well as the triangle passing through $AIJBC$. Loops $HGFED$ and $AIJBC$ allow for two different groupings of a loop of 3 students and a loop of two students, while the other loops allow for only one grouping. So, we have 6 total configurations of loops, leading to $6 * 2^2 = 24$ formations of students.





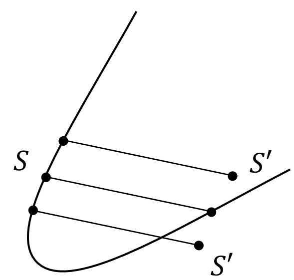
{4, 4, 2}: With this case, observe that no loop of 4 neighboring students can allow for another loop of 4 neighboring students, so there are 0 formations for this case. We can see this by observing that students G, F, and E cannot be included in a loop of 4 students and a loop of 2 students and have another loop of 4 students existing.

{4, 3, 3}: Like the last case, no loop of 4 students can allow for two loops of three students, so there are 0 formations for this case. Similarly, we can see this by observing that students G, F, and E cannot be included in two loops of three students.

Thus, there are $16 + 18 + 8 + 0 + 4 + 24 + 0 + 0 = 70$ formations of students possible after they move.

25) **Answer: C**

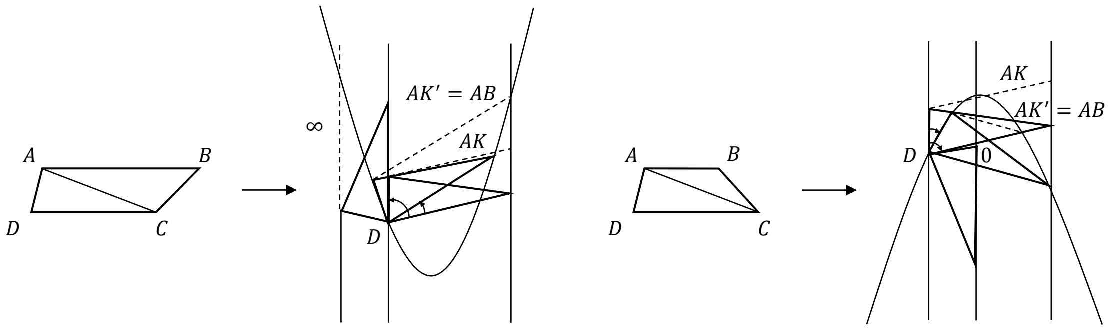
First, we claim that no parallelogram can exist such that all four of its vertices are distinct points on a parabola. Take any segment S with endpoints on parabola P and move S to S', where S' has an endpoint on P and has the same slope as S. As S' moves closer to the vertex of P, the points in common with P and S' move closer together, and the opposite happens as S' moves further away. Hence, no other segment of the same slope and length as S can have both endpoints on P, proving this claim.



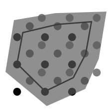
Second, we claim that every other convex quadrilateral can have all four of its vertices on distinct points of a parabola.

Suppose one pair of opposite sides of the quadrilateral is parallel. Call this quadrilateral $ABCD$ and let $AB \parallel CD$ without loss of generality. Notice that we can split this quadrilateral into two triangles, where one of the quadrilateral's diagonals acts as the common side. Let CA be this diagonal. Take triangle CDA and rotate it so that DA is vertical – and call this orientation "default" for brevity. Suppose we rotate triangle CDA by some angle about point D from default orientation. Let P' be the parabola passing through C, D, and A. If we draw segment AK parallel to CD, where $K \neq A$ lies on P', we find that the length of AK will increase as we rotate triangle CDA counterclockwise from default (P' opening upwards), with upper limit at infinity when CD is vertical. Conversely the length of AK will decrease the more we rotate triangle CDA clockwise (P' opening downwards), with lower limit 0 when CA is vertical. Also, by rotating by some arbitrarily small angle, we can get the length of AK as close as we want to the length of CD. Thus, any length in $(0, CD) \cup (CD, \infty)$ can be the length of AB, where $AB \parallel CD$ and all of A, B, C, and D lie on some parabola. Since ACD has arbitrary shape, any convex quadrilateral with only one pair of parallel opposite sides can have its vertices on a parabola.

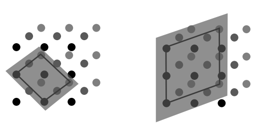
The second case is that no pair of opposite sides of the quadrilateral are parallel. Call this quadrilateral $ABCD$ again. Note that there must exist a diagonal d on $ABCD$ such that the foot F of the altitude from a vertex on $ABCD$ outside d, but to d lies inside $ABCD$. We can split $ABCD$ into two triangles. Without loss of generality let this diagonal be CA, and let B be the vertex in which the foot of its altitude to CA doesn't lie outside $ABCD$. Here angle CAD is not obtuse. Call F the foot from B onto CA. We can repeat the argument indicated in our previous case, first taking triangle CDA and rotating it so that DA is vertical (calling this orientation default). We move on by rotating ACD about D and again calling P' the parabola passing through A, C, and D. Draw segment FL perpendicular to CA, where L lies on P'. If we rotate clockwise from default, FL decreases in length until it reaches 0, when CA is vertical, and if we rotate counterclockwise from default, FL increases in length until infinity, when CA is horizontal. So, FB can have any length except that of FL if $ALCD$ is a parallelogram. But F can be any arbitrary point on CA, and triangle ACD has arbitrary shape, so any quadrilateral that does not have a pair of parallel opposite sides can have its vertices on a parabola.



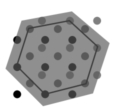
Then, we note that no parabola described in the problem can pass through the vertices of a concave quadrilateral. From what we have proved earlier, in order to solve the problem, we just need to count the number of coplanar groupings of 4 points in S that form a convex quadrilateral that is not a parallelogram. We proceed with cases, basing those cases on the distinct types of planes that can contain at least four points in S. Here, we say that a grouping of 4 points "works" if it satisfies the problem's conditions.



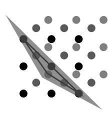
This type of plane only contains 4 points, and these points form a rectangle. Thus, any of these types of planes will result in 0 groupings of 4 points that work.



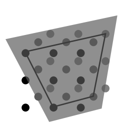
These types of planes contain a rectangle passing through 6 points in S, two of which are the midpoints of opposite sides of the rectangle. Out of these 6 points, we can form 4 convex quadrilaterals that aren't parallelograms. There are 12 planes like the one on the left and 24 planes like the one on the right, so we have $4 * 36 = 144$ groupings of 4 points that work.



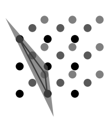
This type of plane contains a regular hexagon containing 7 points in S (the center lies within the hexagon). Out of these 7 points, 6 distinct isosceles trapezoids can be drawn, as well as 6 kites, from 4 points. There are 4 distinct planes existing in this manner, meaning we have $4 * 12 = 48$ groupings of 4 points that work.



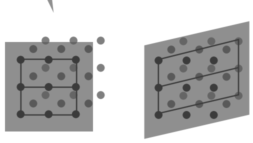
This type of plane contains an equilateral triangle of 6 points in S. Out of these 6 points, we can draw 3 isosceles trapezoids from 4 points. There are 8 distinct planes like this, meaning that there are $3 * 8 = 24$ groupings of 4 points that work.



This type of plane contains an isosceles trapezoid passing through 5 points in S. Out of these 5 points, only one grouping of 4 points can form a convex quadrilateral that isn't a parallelogram: the four that are the vertices of said isosceles trapezoid. There are 24 planes like this, meaning that we have 24 groupings of 4 points that work.



This type of plane contains a triangle passing through 4 points in S. However, three of these points are collinear, and no three collinear points can lie on a parabola. Thus, these types of planes will result in 0 groupings of 4 points that work.



These types of planes contain a rectangle that contains 9 points in S: the rectangle's vertices, the midpoints of its sides, and its center. There are 9 planes like the one on the left and 6 planes like the one on the right. The number of convex quadrilaterals that aren't parallelograms that can be made remain the same for both these types of planes, so we find the number of those quadrilaterals on the left plane and multiply it by 15. We can draw 4 kites, 24 trapezoids with two right angles, 8 quadrilaterals with one right angle, and 12 quadrilaterals with no right angles (4 of which are isosceles trapezoids). This amounts to 48 quadrilaterals, leading to $48 * 15 = 720$ groupings of 4 points that work.

Now note that all other planes either pass through only 3 points in S or have been covered already. Thus, our final answer is $144 + 48 + 24 + 24 + 720 = 960$ groupings of 4 points.