2020 CMC 12B Solutions Document

Christmas Math Competitions

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1. **Answer (D):** We have $\frac{n}{100} \cdot n = 4$, which gives $n = 20$, and $(20^2)\% = 400\%$ of 20 equals 80.

2. **Answer (B):** To optimize the number of children, we give each child as few cards as necessary to satisfy the conditions. Therefore, we want one child to get 1 card, the next to get 2 cards, the next to get 3 cards, and so on until we have distributed all 52 cards.

   We notice that $1+2+3+4+...+9 = \frac{9 \cdot 10}{2} = 45$, but $1+2+3+4+...+10 = \frac{10 \cdot 11}{2} = 55 > 52$. With the sum $1+2+3+4+...+9$, we can adjust the 9 to a 16 in order for the sum to equal exactly 52, so the largest value of k is 9.

3. **Answer (B):** The expression $4(2n-5) = 8n-20$ is a multiple of $2n-5$, so we wish to locate n such that $(8n-20)+29$ is divisible by $2n-5$. It then follows that $2n-5$ must be a divisor of 29. To maximize n, we let $2n-5 = 29$, which gives $n = 17$. The answer is 8.

4. **Answer (D):** Pennies have an area of 1π, nickels have an area of 9π, dimes have an area of 25π, and quarters have an area of 64π.

   Note that if Josh has 5 pennies (which have a total area of 5π), he could exchange them for a nickel with an area of 9π. The latter is clearly more optimal and the price of cards is a multiple of 5, so we may completely disregard pennies. In addition, using two nickels as opposed to one dime is not optimal because $2 \cdot 9\pi < 25\pi$. However, it is possible to use only one nickel and still possibly be optimal; it is only unoptimal if we try to use two or more nickels.

   If we include at least one nickel, we have only one possibility where we never use two or more nickels, which is one nickel and one quarter for a total area of 73π.

   If we opt for no nickels, we must only use three dimes for a total area of 75π.

   The maximum value of A is 75.

5. **Answer (C):** We write the palindrome as $\overline{aba}$, where $a \neq 0$. Since the palindrome must be divisible by 6, a must be even.

   For divisibility by 3, we must have the sum of digits be divisible by 3, that is to say $2a + b \equiv 0 \pmod 3$ or $a \equiv b \pmod 3$.

   We do cases based on the value of a:

   $a = 2$ means that $b \in \{2, 5, 8\}$; 3 palindromes

$a = 4$ means that $b \in \{1, 4, 7\}$; 3 palindromes

$a = 6$ means that $b \in \{0, 3, 6, 9\}$; 4 palindromes

$a = 8$ means that $b \in \{2, 5, 8\}$; 3 palindromes

There are 13 palindromes total.

6. **Answer (D):** Let X be the intersection of P_1P_2 and P_5P_6. Define Y similarly for P_1P_2 and P_3P_4, and define Z similarly for P_3P_4 and P_5P_6. Because $\angle AP_1P_6 = \angle BP_1P_2 = \angle AP_1X = 30°$, $\angle P_6P_1X = 60°$ and by symmetry, $\angle XP_6P_1 = 60°$. It immediately follows that $\triangle XP_1P_6$ is equilateral and then $\triangle XYZ$ is equilateral.

   Clearly, $XY = 3P_1P_2$ by equilateral triangles $\triangle XP_1P_6$ and $\triangle YP_2P_3$. Since P_1 and P_2 are midpoints, $2P_1P_2 = AC$. Therefore, $\frac{XY}{AC} = \frac{3}{2}$. Because $\triangle XYZ$ and $\triangle ABC$ are both equilateral, the ratio of their areas is the square of the ratio of their sides: $(\frac{3}{2})^2 = \frac{9}{4}$.

7. **Answer (C):** Using the rule given by the operation, we re-write the given equation:

$$3n - \frac{n}{3} = n \cdot (3n - \frac{3}{n}) \implies \frac{8n}{3} = 3n \cdot \frac{n^2 - 1}{n}.$$

   Rearranging the equation, we have $9n^2 - 8n - 9 = 0$.

   By checking the discriminant, we see that $8^2 + 4 \cdot 9 \cdot 9 > 0$, so the quadratic has two distinct real solutions. We use Vieta's formulas to find that the product of the solutions to the quadratic is -1.

8. **Answer (D):** Using the quadratic formula. we get $x = \frac{2\sqrt{5} \pm 4\sqrt{2i}}{2} = \sqrt{5} \pm 2\sqrt{2i}$. Since $i = \text{cis}(\frac{\pi}{2})$, by DeMoivre's Theorem, we have $\sqrt{i} = \text{cis}(\frac{\pi}{4}) = \frac{1+i}{\sqrt{2}}$ so $x = \sqrt{5} \pm (2 + 2i)$. The sum of the real parts squared equals $(\sqrt{5} + 2)^2 + (\sqrt{5} - 2)^2 = 18$.

9. **Answer (A):** We know that $AD = 12$, $BD = 5$, and $DC = 9$. Let M be the midpoint of BC. Clearly, OM is perpendicular to BC and $DM = 2$. In addition, DM is the height of $\triangle AOD$ with respect to base AD. Then, $[AOD] = \frac{1}{2} \cdot 12 \cdot 2 = 12$.

10. **Answer (D):** Using the fact that $\triangle ABC$ is right isosceles and has an area of 1, $BC = \sqrt{2}$. It immediately follows that $AM = \frac{\sqrt{2}}{2}$ because M is the midpoint of AB and $AC = 2$. Because $\angle AXM = 90°$ and $\angle MAX = 45°$, $\triangle AXM$ is also right and isosceles. Therefore, $AX = \frac{1}{2}$. It then follows that $XC = \frac{3}{2}$. Because $\triangle CXY$ is also a right isosceles triangle with leg XC, its area is $\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{9}{8}$.

11. **Answer (A):** We can construct a sequence with 6 terms:

    We can let a_1 and a_2 be multiples of 55, a_3, a_4, and a_5 be multiples of 77, and a_6 be a multiple of 35. Then, we can distribute the factors of 2s and 3s to the terms in the sequence however we want. To show we cannot create a sequence with 7 terms, we will use the Greedy algorithm.

    We want every possible pair of terms in the sequence to share at least one prime factor. It is clearly not optimal for a term in the sequence to have more than one of the same prime in its prime factorization; we want to spread out each prime throughout the sequence.

    We let a_1, a_2, a_3, a_4, and a_5 each have one 11 in their prime factorization. Now, we need a_6 and a_7 to each share at least one prime with each of a_1, a_2, a_3, a_4, and a_5 in addition to a_6 sharing a prime with a_7. We will show this is not possible. Let a_6 have a 7 in its prime factorization. Then, we will distribute the remaining three 7s to a_1, a_2, and a_3.

Now, let a_6 have a 5 in its prime factorization. We can distribute the remaining two 5s to a_4 and a_5. Finally, we distribute the two 3s to a_6 and a_7. Therefore, a_6 shares a prime factor with every other term in the sequence. We only have a single 2 left to distribute. However, it will not matter where we distribute it because we do not have another 2. This then causes a_7 to be relatively prime to a_1, a_2, a_3, a_4, and a_5, so even if we distribute the primes optimally, we will never be able to construct a valid sequence of 7 integers. Therefore, the answer is 6.

12. **Answer (B):** Let $p = \frac{a}{5040}$ and $q = \frac{b}{3960}$ for some positive integers a and b such that $\gcd(a, 5040) = \gcd(b, 3960) = 1$. Since $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ and $3960 = 2^3 \cdot 3^2 \cdot 5 \cdot 11$, we have $p + q = \frac{11a+14b}{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11}$. However, we can still have factors from the denominator and numerator cancel, so we will consider each prime dividing the denominator separately.

    If we want a factor of 2 to cancel, we must have a be divisible by 2. However, this means that $\gcd(a, 5040)$ is divisible by 2, which is a contradiction. Similarly, if we want a factor of 7 to cancel, we need a to be divisible by 7, which causes $\gcd(a, 5040)$ to be divisible by 7, contradiction. If we want a factor of 11 to cancel, b must be divisible by 11, which causes $\gcd(b, 3960)$ to be divisible by 11, contradiction.

    However, since neither $11a$ nor $14b$ are necessarily divisible by 3 or 5, we can appropriately set a or b to cancel as many factors of 3 or 5 as desired. Therefore, the possible values of the content of $p + q$ are $2^4 \cdot 7 \cdot 11 \cdot d$, as d ranges across all positive integer factors of $3^2 \cdot 5$. The integer $3^2 \cdot 5$ has 6 positive factors.

13. **Answer (D):** Suppose $a = \overline{a_3a_2a_1}$ and $b = \overline{b_3b_2b_1}$, where the a_i and b_i represent digits and a_3 and b_3 may be equal to 0.

    Note that when Oliver computes his answer, it will be in the form $\overline{c_3c_2c_1}$, where the c_k are digits and c_k is the unique integer in the range $0 \le c_k \le 9$ such that $c_k \equiv a_k + b_k \pmod{10}$ for $1 \le k \le 3$. However, when the real answer is computed, it will be equal to $10^2 \cdot (a_3 + b_3) + 10 \cdot (a_2 + b_2) + (a_1 + b_1)$. Clearly, the difference between $(a_k + b_k)$ and c_k is nonzero if and only if $a_k + b_k \ge 10$ and the difference will be 10^{k+1}. Also, as Oliver randomly selects a and b across all integers from 0 to 999 inclusive, each a_k and b_k has an equal chance of assuming each of the 10 digits.

    Let p be the probability that the difference between $(a_k + b_k)$ and c_k is nonzero. In other words, p is the probability that if we choose a' and b' from the integers 0 to 9, uniformly and randomly, $a' + b' \ge 10$. Therefore, the expected value of D is $10^2 \cdot p + 10^3 \cdot p + 10^4 \cdot p = 1110p$.

    Now, we calculate p. If $a' = 0$, then no values of b' work. If $a' = 1$, then only $b' = 9$ works, so there is 1 case here. If $a' = 2$, then only $b' = 8, 9$ works, so there are 2 cases here. If $a' = 3$, then only $b' = 7, 8, 9$ works, so we have 3 cases. Continuing this until $a' = 9$, we have $1+2+3+4+5+6+7+8+9 = 45$ cases total. We can choose (a', b') in $10 \cdot 10 = 100$ different ways. So $p = \frac{9}{20}$ and $1110p = \frac{999}{2}$.

14. **Answer (A):** Note that segments MN, NP, and PM partition $\triangle ABC$ into four congruent triangles each with area $\frac{1}{4}$.

    There is a $\frac{1}{4}$ chance that X will be chosen inside $\triangle MNP$. In this case, the convex hull of the four points is simply $\triangle MNP$ with area $\frac{1}{4}$.

    There is a $\frac{3}{4}$ chance that X will be chosen outside $\triangle MNP$. WLOG, assume that X is chosen inside $\triangle ANP$. We know that $[ANX] + [NPX] + [PAX] = \frac{1}{4}$. Clearly, the expected values of $[ANX]$, $[NPX]$, and $[PAX]$ are all equal, so their common value is $\frac{1}{12}$. The convex hull of the four points is given by $[MNP] + [PNX]$, so the expected value of the convex hull in this case is $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$.

    Our answer is $\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{5}{16}$.

15. **Answer (A):** Let $f(x) = x^3 - 3x - 1$ and $g(x) = x^3 - 3x + 1$. Notice that $f(x) = -g(-x)$. Therefore, if r_i is a root of $f(x)$, then $-r_i$ is a root of $g(x)$. Since $r_1 < r_2 < r_3$, the roots of $g(x)$ are $-r_1 > -r_2 > -r_3$.

    Therefore, $s_1 = -r_3$, $s_2 = -r_2$, and $r_3 = -s_1$ and $|(r_1 + s_1)(r_2 + s_2)(r_3 + s_3)| = |(r_1 - r_3)(r_2 - r_2)(r_3 - r_1)| = 0$.

16. **Answer (C):** Let E be the unique point on DC such that AE is parallel to CB. It immediately follows that $AE = EC = 5$ then $DE = 5$, so $\triangle ADE$ is equilateral. Furthermore, because $ABCD$ is an isosceles trapezoid and E is the midpoint of CD, $\triangle ABE$ and $\triangle BCE$ are also equilateral.

    Because $\angle EAT = \angle EAP = 90°$ and $\angle EAB = \angle EAD = 60°$, we have that $\angle BAT = \angle PAD = 30°$. In addition, because $\angle ADE = 60°$, $\angle ADP = 120°$. So $\triangle ADP$ is isosceles with $PD = AD$ and $\angle ADP = 120°$. Because $AD = 5$, $AP = 5\sqrt{3}$.

    We know that $BC = 5$, $\angle CBQ = 60°$, and $\angle BQC = 90°$, so $BQ = \frac{5}{2}$. In addition, we know that $AQ = \frac{15}{2}$, $\angle TAQ = 30°$, and $\angle AQT = 90°$, so $TQ = \frac{5\sqrt{3}}{2}$ and $AT = 5\sqrt{3}$. Then, $TP = AT + AP = 10\sqrt{3}$.

    Finally, $TP \cdot TQ = 10\sqrt{3} \cdot \frac{5\sqrt{3}}{2} = 75$.

17. **Answer (B):** Let there be k calls and let the lengths of the calls be $a_1, a_2, a_3, ..., a_k$. Now, we want to minimize $\sum_{i=1}^{k}(a_i^2 - 3a_i + 4.41)$ given that $\sum_{i=1}^{k} a_i = 17$.

    Intuitively, it seems that we should have $a_1 = a_2 = a_3 = ... = a_k$. Indeed, by the Cauchy-Schwarz Inequality, $289 = \left(\sum_{i=1}^{k} a_i\right)^2 \leq k \cdot \left(\sum_{i=1}^{k} a_i^2\right)$ with equality occurring when $a_1 = a_2 = a_3 = ... = a_k$. Therefore, let $a_1 = a_2 = a_3 = ... = a_k = \frac{17}{k}$. We must find the minimum value of $k((\frac{17}{k})^2 - 3 \cdot \frac{17}{k} + 4.41)$.

    Now, we want to minimize $\frac{289}{k} + 4.41k - 51$. By AM-GM, $\frac{289}{k} + 4.41k$ attains its minimum when $\frac{289}{k} = 4.41k$ or when $k = \frac{17}{2.1} \approx 8$. However, k must be an integer as there are an integer number of calls. Plugging in $k = 8$ and $k = 9$ shows that $k = 8$ gives a smaller value. We then compute $\frac{289}{8} + 4.41 \cdot 8 - 51 = \frac{4081}{200}$. The requested answer is $4 + 0 + 8 + 1 = 13$.

18. **Answer (D):** Suppose the cards have the integers 1 through 8, and we stack them so that card 1 is on top, card 2 is next, card 3 is next, and so on until card 8 is on the bottom. This will be our "original stack."

    If we use exactly n shufflings, that means at most n cards will be touched during the process and the remaining $8 - n$ cards will be untouched. We will also count the act of taking the topmost card and putting it back at the top as a shuffling. The $8 - n$ untouched cards will remain in increasing order and occupying the $8 - n$ bottom-most positions of the stack. Through this, we can manipulate which n cards will be the n topmost cards and in what order, while the remaining cards in the deck will be ordered in increasing order. In particular, for $n = 4$, have $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways to choose the 4 topmost cards and their order. However, not every stack of the 1680 can be achieved with a minimum of 4 shufflings. But since we know each of those stacks can be achieved in 4 shufflings, the minimum number of shufflings to create each stack must be 4 shufflings or less.

    However, for $n = 3$, we have $8 \cdot 7 \cdot 6 = 336$ stacks and for each of these stacks, the minimum number of shufflings to create the stack must be 3 shufflings or less. Thus, $1680 - 336 = 1344$ precisely describes the number of stacks that require a minimum of 4 shufflings to create.

19. **Answer (D):** We claim that if there exists a unique point P inside quadrilateral $ABCD$ such that $[ABP] = [BCP] = [CDP] = [DAP]$, then P is the midpoint of a diagonal of $ABCD$.

    Let M be the midpoint of AC. Clearly, $[ABM] = [BCM]$. In addition, any point Q that lies on segment BM will satisfy $[ABQ] = [BCQ]$ since $[APQ] = [CPQ]$. Similarly, any point R that lies on segment DM will satisfy $[CDR] = [DAR]$. For both $[ABQ] = [BCQ]$ and $[CDR] = [DAR]$ to hold, we must have $Q = R$ or $M = Q = R$. However, for $M = P$, we need $[ABM] = [BCM] = [CDM] = [DAM]$ to hold. Therefore, we must have $[ABM] = [ADM]$ or $[ABC] = [ACD]$. Since the length of AC is not uniquely determined by the side lengths of $ABCD$, we can adjust AC such that $[ABC] = [ACD]$ and then $M = P$.

    We can also use a similar process to show that if N is the midpoint of BD, we can adjust BD so that $N = P$. Therefore, if a quadrilateral $ABCD$ contains a unique point P such that $[ABP] = [BCP] = [CDP] = [DAP]$, P must be the midpoint of either AC or BD.

    WLOG, suppose P is the midpoint of AC. Let $PA = PC = d$. We seek $2d^2 + PB^2 + PD^2$. By Apollonius's Theorem on $\triangle ADC$, we have $\frac{CD^2+DA^2}{2} - d^2 = PD^2$. Similarly, by Apollonius's Theorem on $\triangle ABC$, we have $\frac{AB^2+BC^2}{2} - d^2 = PB^2$. Therefore, our answer is $\frac{AB^2+BC^2+CD^2+DA^2}{2}$ or 45. Note that the answer is the same for the quadrilateral where P is the midpoint of BD.

20. **Answer (B):** By the first equation, $b \geq 0$, since $\lceil a^4 \rceil$ and $\lceil b^2 \rceil^2$ are both positive. Similarly, by the second equation, $a \leq 0$. Therefore, let $c = -a$ so that $c \geq 0$. We substitute c into the equations, add the two equations, and then group the terms as follows:

$$0 = \lceil c^4 \rceil - 2\lfloor c^2 \rfloor b + \lceil b^2 \rceil^2 + \lceil c^2 \rceil - 2c\lfloor b^2 \rfloor + \lceil b \rceil^2$$

$$0 = (\lceil c^4 \rceil - 2\lfloor c^2 \rfloor b + \lceil b \rceil^2) + (\lceil b^2 \rceil^2 - 2c\lfloor b^2 \rfloor + \lceil c^2 \rceil).$$

   We note that for any real number m, $\lfloor m \rfloor \leq m$ and $\lceil m \rceil \geq m$. In addition, we can see that $-2c\lfloor b^2 \rfloor \geq -2c \cdot b^2$ and similarly, $-2b\lfloor c^2 \rfloor \geq -2b \cdot c^2$, since both b and c are nonnegative Using this, we can bound the terms on the RHS:

$$\lceil c^4 \rceil - 2\lfloor c^2 \rfloor b + \lceil b \rceil^2 \geq c^4 - 2c^2 b + b^2 = (c^2 - b)^2$$

$$\lceil b^2 \rceil^2 - 2c\lfloor b^2 \rfloor + \lceil c^2 \rceil \geq (b^2 - c)^2$$

   Adding these two inequalities together, we get that $0 \geq (c^2 - b)^2 + (b^2 - c)^2$. It immediately follows by the Trivial Inequality that $c^2 - b = 0$ and $b^2 - c = 0$. Solving these systems of equations, we find that $(c, b) \in \{(0,0), (1,1)\}$. We can verify that both of these pairs work, so the answer is 2.

21. **Answer (C):** Let the common intersection point of lines XE, FI, and BC be Z. Note that $\triangle FZB \cong \triangle DAB$, so $BZ = 8$.

    Next, note that $\angle XEA = 90 - \angle XEI = 90 - \left(\frac{180 - \angle XIE}{2}\right) = \frac{1}{2}\angle XIE$. Furthermore, $\angle XIE = \angle AIE = \angle C$. Thus, $\angle CZE = 180 - \angle ECZ - \angle ZEC = 180 - (180 - \angle C) - (\angle XEA) = \angle C - \frac{1}{2}\angle C = \frac{1}{2}\angle C = \angle XEA = \angle ZEC$ or $\angle CZE = \angle ZEC$.

    This implies $\triangle ECZ$ is isosceles. Since $\triangle ABC$ is also isosceles, we see that $BD = DC = CE = CZ$. Thus, $\frac{3}{2}BC = BZ = 8 \implies BC = \frac{16}{3}$. The requested sum is $16 + 3 = 19$.

22. **Answer (D):** We will use the following well-known fact: if p is a prime and a is a positive integer such that $p-1 \nmid a$, then

$$1^a + 2^a + ... + (p-1)^a \equiv 0 \pmod{p}.$$

Proof: Let ξ be a primitive root modulo p, which exists as p is a prime. Then,

$$1^a + 2^a + ... + (p-1)^a = 1 + \xi^a + \xi^{2a} + ... + \xi^{(p-2)a}$$

and the RHS is a geometric series with sum

$$\frac{\xi^{(p-1)a} - 1}{\xi^a - 1}.$$

By the definition of a primitive root, $p-1$ is the minimum positive integer value of k such that $\xi^k \equiv 1 \pmod{p}$. Thus, $\xi^m \equiv 1 \pmod{p} \implies (p-1) \mid m$. Therefore, numerator is always $0 \pmod{p}$. As long as the denominator is not $0 \pmod{p}$ or as long as $p-1 \nmid a$, the expression will be $0 \pmod{p}$.

Let $S = 1^{2020} + 2^{2020} + \cdots + 2019^{2020}$. We know that $2020 = 2^2 \cdot 5 \cdot 101$. Applying the fact above with $p = 101$, we have that $S \equiv 0 \pmod{101}$. Then, we can directly compute the remainders modulo 4 and 5: Since the square of every odd number is 1 modulo 4 and the square of every even number is 0 modulo 4,

$$S \equiv 1 + 0 + ... + 1 = 1010 \equiv 2 \pmod{4} \Rightarrow S^{2020} \equiv 0 \pmod{4}$$

Since $1^4 \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1 \pmod{5}$ by either Fermat's Little Theorem or direct computation,

$$S \equiv 1 + 1 + 1 + 1 + 0 + ... + 1 = 4 \cdot 404 = 1616 \equiv 1 \pmod{5} \Rightarrow S^{2020} \equiv 1 \pmod{5}$$

Thus, by the Chinese Remainder Theorem the answer is $S \equiv 1616 \pmod{2020}$.

23. **Answer (D):** We note that $\triangle BAC \sim \triangle BDE$ by cyclic quadrilateral $AEDC$, so $\frac{DE}{AC} = \frac{BD}{AB} = \cos B$. So from $\frac{DE}{AC} = \cos B$, we have $AC = \frac{2}{\cos B}$. But by Law of Cosines on $\triangle ABC$, $AC^2 = 25 - 24\cos B$. Thus,

$$\frac{4}{\cos^2 B} = 25 - 24\cos B \implies 24\cos^3 B - 25\cos^2 B + 4 = 0$$

$$(3\cos B - 2)(8\cos^2 B - 3\cos B - 2) = 0 \implies \cos B \in \{\tfrac{2}{3}, \tfrac{3 \pm \sqrt{73}}{16}\}$$

Since $\triangle ABC$ is acute, we know that $\cos B \neq \frac{3-\sqrt{73}}{16}$, which is negative. Thus, $\cos B \in \{\frac{2}{3}, \frac{3+\sqrt{73}}{16}\}$. So the sum of all possible values of AC is

$$\frac{2}{\frac{2}{3}} + \frac{2}{\frac{\sqrt{73}+3}{16}} = 3 + \frac{\sqrt{73}-3}{2} = \frac{\sqrt{73}+3}{2}.$$

The requested sum is $73 + 3 + 2 = 78$.

24. **Answer (E):** Define the sequence $\{p_n\}_{0 \le n \le 10}$ such that $p_0 = 0$ and for all integers $1 \le i \le 10$, $p_i = \sum_{k=1}^{i} (a_k)$. Note that there is a bijection between $\{a_i\}$ and $\{p_i\}$.

Then, $|a_i + a_{i+1} + \cdots + a_j| \le 2 \implies |p_j - p_{i-1}| \le 2$ for every ordered pair of integers (i, j) such that $1 \le i \le j \le 10$.

If we let $i' = i - 1$, we can revise our condition to $|p_j - p_{i'}| \le 2$ for $0 \le i' < j \le 10$. This condition means that no two terms in the sequence $\{p_n\}_{0 \le n \le 10}$ are more than 2 apart, and we already know that $p_0 = 0$. Let A be the set of sequences $\{p_n\}_{0\leq n\leq 10}$ such that for $1 \leq n \leq 10$, $p_n \in \{0,1,2\}$. Define B and C similarly for $p_n \in \{-1,0,1\}$ and $p_n \in \{-2,-1,0\}$, respectively.

By PIE, we wish to compute:

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|.$$

Clearly, $|A| = |B| = |C| = 3^{10}$, while $|A\cap B| = |B\cap C| = 2^{10}$, $|A\cap C| = 1$, and $|A\cap B\cap C| = 1$. Therefore, $|A\cup B\cup C| = 3^{11} - 2^{11} \equiv 99 \pmod{100}$.

25. **Answer (D):** Define S_k as the set $\{0,1,2,3,4,...,12\}$ with the element k removed.

Work in $\mathbb{F}_{13}$, meaning the coefficients of the polynomials will be taken mod 13. Apply Lagrange interpolation on $P(0), P(1), ..., P(12)$:

$$P(x) = \sum_{i=0}^{12}\left(P(i)\cdot\prod_{j\in S_i}\frac{x-j}{i-j}\right)$$

Note that $\prod_{j\in S_i}(i-j) = 12\cdot 11\cdot 10\cdot ...\cdot 1 = 12!$ as if we randomly fix i, $(i-j)$ where $j\in S_i$ achieves each value integer value from 1 to 12 inclusive. However, by Wilson's Theorem, $12! = -1$. Thus, for all $0\leq i\leq 12$, $\prod_{j\in S_i}(i-j) = -1$. Therefore,

$$P(x) = -\sum_{i=0}^{12}\left(P(i)\cdot\prod_{j\in S_i}(x-j)\right).$$

It is sufficient for the leading coefficient of P to be 1 so that polynomial is monic, since by Lagangre interpolation, we guarantee that the expanded RHS will equal the appropriate polynomial if we arbitrarily select the values of each $P(i)$. Therefore, $\sum_{i=0}^{12} P(i) = -1$.

By the condition that $P(x)+1$ or $P(x)-1$ is divisible by 13 for every integer x, we must have $P(i)\in\{1,-1\}$ for each positive integer $0\leq i\leq 12$. Thus, among $P(0), P(1), ..., P(12)$ there are six 1's and seven -1's, so the answer is $\binom{13}{6} = 1716$.