2020 Mock

AMC 10A

DO NOT OPEN UNTIL WEDNESDAY, JULY 1, 2020

****Administration On An Earlier Date Will Disqualify Your Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the imaginary TEACHER'S MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE JULY 1, 2020. Nothing is needed from inside this package until July 1.
2. YOU must verify on the AMC 10 CERTIFICATION FORM that you followed all rules associated with the conduct of this exam.
3. The Answer Form may be mailed First Class to the AMC office no later than 24 hours following the exam, but it may not be graded as this is not an official AMC exam.
4. *The publication, reproduction, or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, AoPS, telepathy, or media of an type is a violation of the competition rules. Consequences include, but are not limited to, bad karma and increased likelihood of silly errors on the official AMC exam.*

The problems were written by scrabbler94 and test-solved by djmathman, P_Groudon, fidgetboss_4000, nikenissan, and IMadeYouReadThis.

Questions and comments about problems or solutions within this exam should be communicated via private message (PM) to scrabbler94.

PUBLICATIONS

A complete listing of previous publications may be found at:
`https://artofproblemsolving.com/wiki/index.php/User:Scrabbler94`.

scrabbler94's Mathematics Competitions

2nd Biannual

Mock AMC 10A

scrabbler94's Mathematics Competitions 10A

Wednesday, July 1, 2020 -
Wednesday, July 15, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. **No copies.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the scrabbler94 Mathematics Competitions (i.e., myself) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CSMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this AMC 10 will be invited to take the 1^{st} Annual Interplanetary Mathematics Examination (AIME) on January 1, 9001. More details about the AIME are on the back page of this packet.

The publication, reproduction, or communication of the problems or solutions during the period when students are allowed to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, AoPS, telepathy, or media of any type is a violation of the competition rules.

1. A grocery store sells candy bars for \$2.50 each. There is a 30% discount if five or more candy bars are bought. How many dollars will Nathan save if he buys five candy bars instead of four?

   (A) \$1.25 **(B)** \$1.50 **(C)** \$1.75 **(D)** \$2.50 **(E)** \$3.75

2. The ratio of boys to girls in a large sophomore class is $3:4$. When polled, it was found that exactly $\frac{1}{4}$ of the boys have a pet and $\frac{1}{3}$ of the girls have a pet. Out of all sophomores who have a pet, what percent of them are boys?

   (A) 36% **(B)** 40% **(C)** 50% **(D)** $56\frac{1}{4}$% **(E)** 64%

3. Boston and New York are 210 miles apart by train. A southbound train departs Boston for New York at an average speed of 70 miles per hour. Fifteen minutes later, a northbound train departs New York for Boston. The two trains each travel at a constant speed, and pass each other exactly halfway between Boston and New York. What is the average speed of the northbound train, in miles per hour?

   (A) 70 **(B)** 75 **(C)** 77 **(D)** 84 **(E)** $87\frac{1}{2}$

4. Ellie writes a list of five different prime numbers. Given that the average (arithmetic mean) of the numbers in her list is a prime number, what is the smallest possible value of the sum of the numbers in Ellie's list?

   (A) 25 **(B)** 35 **(C)** 55 **(D)** 65 **(E)** 85

5. What is the value of $\dfrac{2020!+2019!}{2018!+2017!}$?

   (A) 4076360 **(B)** 4076361 **(C)** 4078374 **(D)** 4078378
   (E) 4078380

24. Equiangular hexagon $ABCDEF$ has $AB = CD = EF = 28$ and $BC = DE = FA = 14$. Points M, N, O, P, Q, and R are the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$, respectively. There exists a unique circle ω which is tangent to line segments $\overline{MN}$, $\overline{OP}$, and $\overline{QR}$. What is the area of ω?

    (A) $\dfrac{2023\pi}{12}$ **(B)** $\dfrac{833\pi}{4}$ **(C)** $\dfrac{700\pi}{3}$ **(D)** $\dfrac{1029\pi}{4}$ **(E)** 336π

25. How many ordered 11-tuples $(a_0, a_1, a_2, \ldots, a_{10})$ of integers satisfy the equation
    $$a_0 + 2a_1 + 2^2a_2 + \ldots + 2^{10}a_{10} = 2020$$
    where $0 \le a_i \le 2$ for all $0 \le i \le 10$?

    (A) 27 **(B)** 33 **(C)** 39 **(D)** 45 **(E)** 51

19. Let $f^1(x) = x^2 - 20$ for all real numbers x, and let $f^k(x) = f^1(f^{k-1}(x))$ for all integers $k \geq 2$. Let x_0 and x_1 be the smallest and largest real solutions to the equation $f^{2020}(x) = 0$, respectively. What is the largest integer less than or equal to $x_0^2 + x_1^2$?

(A) 32 **(B)** 40 **(C)** 41 **(D)** 49 **(E)** 50

20. Regular hexagon $ABCDEF$ has side length 1. It is rotated 30° about A to produce hexagon $AB'C'D'E'F'$. What is the area of the region common to both $ABCDEF$ and $AB'C'D'E'F'$?

(A) $12-6\sqrt{3}$ **(B)** $\frac{12-5\sqrt{3}}{2}$ **(C)** $\sqrt{3}$ **(D)** $\frac{2+\sqrt{3}}{2}$ **(E)** $\frac{16-7\sqrt{3}}{2}$

21. How many ways can six people of different heights stand in line such that for all $1 \leq k \leq 6$, the k^{th} tallest person must stand next to either the $(k+1)^{\text{th}}$ or $(k-1)^{\text{th}}$ tallest person (or both)? In particular, the tallest person must stand next to the second tallest person, and the shortest person must stand next to the second shortest person.

(A) 42 **(B)** 48 **(C)** 54 **(D)** 56 **(E)** 66

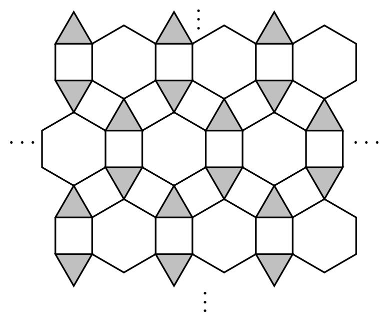
22. Parallelogram $ABCD$ has $AB = CD = 10$, $BC = AD = 6$, and $BD = 8$. Let O_1, O_2, O_3, and O_4 be the circumcenters of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$, respectively. What is the area of quadrilateral $O_1O_2O_3O_4$?

(A) 12 **(B)** $\frac{64}{3}$ **(C)** 24 **(D)** 27 **(E)** 36

23. Paige repeatedly rolls a fair six-sided die, and keeps a running total of the dice rolls she has obtained thus far. For example, if Paige's rolls are 2, 3, 1, 1, and 6 in that order, then her running total is 2, 5, 6, 7, then 13. If Paige rolls the die indefinitely, then the probability her running total equals 7 at some point in time equals $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the remainder when $m+n$ is divided by 1000?

(A) 7 **(B)** 241 **(C)** 585 **(D)** 733 **(E)** 929

6. An infinitely large floor is tessellated using the following pattern consisting of squares, regular hexagons, and equilateral triangles.



Which of the following is closest to the percentage of the total floor area covered by the triangles?

(A) 11% **(B)** 12% **(C)** 13% **(D)** 15% **(E)** 17%

7. The mean and median of Albert's seven quiz scores are both 84 points. After the teacher dropped his lowest two quiz scores, the mean increased by one point, while the median remained the same. Given that each of Albert's quiz scores is a whole number of points, what is the highest score that he could have earned on any of his seven quizzes?

(A) 88 **(B)** 89 **(C)** 90 **(D)** 91 **(E)** 92

8. If today is rainy, then Erin wears a raincoat. If today is rainy and Erin is wearing a raincoat, then Erin will not get wet outside. Which of the following statement(s) is logically implied from the above two statements?

I: If today is not rainy, then Erin is not wet outside.

II: If Erin is not wet outside, then today is not rainy.

III: If Erin is wet outside, then today is not rainy.

IV: If Erin is wet outside, then Erin is not wearing a raincoat.

(A) I only **(B)** II only **(C)** III only **(D)** III and IV only
(E) I, III, and IV only

9. Square $ABCD$ has side length 1. Let $\mathcal{S}$ be the region consisting of all points P in the same plane as the square with the property that the sum of the distances from P to each of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$ is less than or equal to 3. What is the area of region $\mathcal{S}$?

(A) $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 7

10. Let $f(x) = |x-4|$ and $g(x) = x^2$ for all real numbers x. How many real numbers x satisfy $f(g(x)) = g(f(x))$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** Infinitely many

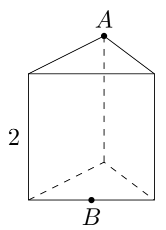
11. Positive integers $a, b, c \geq 2$ satisfy the equation $abc + ab + a = 64$. What is $a+b+c$?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** Cannot be determined

12. Joanna has \$372, consisting of two bills in each of the denominations \$1, \$5, \$10, \$20, \$50, and \$100. Using any combination of one or more of these bills, how many different monetary amounts can Joanna form? For example, she can form \$141 using one \$100 bill, two \$20 bills, and one \$1 bill.

(A) 221 **(B)** 222 **(C)** 223 **(D)** 224 **(E)** 225

13. A right triangular prism has all edges of length 2. An ant crawls on the exterior of the prism from point A to point B, where B is the midpoint of the edge opposite A as shown. What is the shortest possible distance the ant crawls?



(A) $\sqrt{7}$ **(B)** $\sqrt{7+2\sqrt{3}}$ **(C)** $\sqrt{8+2\sqrt{3}}$ **(D)** $\sqrt{13}$ **(E)** $2+\sqrt{3}$

14. Let M and m be the largest and smallest 4-digit positive integers, respectively, which are divisible by 45 and all of whose digits are odd. What is $\dfrac{M-m}{45}$?

(A) 168 **(B)** 176 **(C)** 184 **(D)** 196 **(E)** 198

15. The *Farey sequence* of order n is defined as the increasing sequence containing all common fractions between 0 and 1 which, when written in simplest form, have a denominator less than or equal to n. For example, the Farey sequence of order 5 is the sequence $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$. Let $\frac{a}{b}$ be the fraction immediately after $\frac{1}{3}$ in the Farey sequence of order 2020. What is $a+b$?

(A) 1346 **(B)** 1347 **(C)** 2690 **(D)** 2691 **(E)** 2693

16. A large number of children sit in a circle. They count the numbers 1, 2, 3, … in clockwise order, starting with the oldest child who counts the number "1." Once any child counts a number which is not divisible by 2, 3, or 5, that child leaves the circle and the next child continues with the next number. In particular, the oldest child leaves the circle after counting the number "1." They count until only one child remains, at which point they stop counting. Given that the number "121" was the last number counted, how many children were originally in the circle?

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

17. Let n be the smallest positive integer with the property that $\operatorname{lcm}(n, 2020!) = 2021!$, where $\operatorname{lcm}(a,b)$ denotes the least common multiple of a and b. How many positive factors does n have?

(A) 4 **(B)** 2021 **(C)** 2112 **(D)** 2156 **(E)** 2205

18. Robert writes all positive divisors of the number 216 on separate slips of paper, then places the slips into a hat. He randomly selects three slips from the hat, with replacement. What is the probability that the product of the numbers on the three slips Robert selects is a divisor of 216?

(A) $\frac{1}{64}$ **(B)** $\frac{81}{4096}$ **(C)** $\frac{25}{1024}$ **(D)** $\frac{225}{4096}$ **(E)** $\frac{25}{256}$