2020 MOCK AMC8 EB3 Solutions

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SOLUTION MANUAL - MOCK Test EB34GDFX3BMN

1. We have that without duplicates, there are 7 ways to pick the first letter, 6 ways to pick the second, 5 ways to pick the third, and etc until we reach 1. This is the multiplied quantity; 5040, and count the duplicates; there are two pairs, so $\frac{5040}{2!\cdot 2!} = \boxed{\textbf{A) 1260}}$.

2. We know that anything to the power of 0 is 1 (except 0), 1 to the power of anything is 1, and -1 to an even power is 1. As the problem calls for real values of p, we cannot consider i. We can write an equation for our first case: $49 - p^2 = 0$, so $p = 7, -7$. We can write a second equation: $p - 4 = 1$ so $p = 5$, and lastly, $p - 4 = -1$, so $p = 3$. But we have to check if $p^{(49-p^2)}$ is even! We have that $3^2 = 9$, and 40 is even, so we have the sevens cancel out, so $3 + 5 = \boxed{\textbf{E) 8}}$.

3. Let the variable defining the ratio be x. Then we have $2x + 3x = 200$, so $x = 40$. Therefore, we have $2x = \boxed{\textbf{(C) 80}}$.

4. After she adds x marbles, there will be $20 + x$ marbles in the bag. We are looking to minimize x, and that occurs when the color of marbles is closest to half of 20, or 8. Therefore, we have $\frac{8+x}{20+x} = \frac{1}{2}$, so $16 + 2x = 20 + x$ and $x = \boxed{\textbf{(B) 4}}$.

5. The minimum perimeter is 3, because we can't have side lengths of 0 or fractions. Let the sides of the triangle be (a, b, c). Then we have that $b \geq a$ and $c \geq a$. This is because we cannot have repeats, and the way to have repeats is to have a smaller number than a. By the triangle inequality, we have $a + b > c, a + c > b$, and $b + c > a$. Also, we have $3 \leq a + b + c < 15$. These inequalities are ugly, but we can say that because $b \geq a$ and $c \geq a$, $3 \leq a + a + a < 15$, so $1 \leq a < 5$. We have less cases now, so we can use casework.

CASE 1: $a = 1$

There are no special restrictions on b or c, as they can't be 0. b has to be less than 6 because the perimeter is less than 15, as b has to equal c or else it wouldn't make a triangle. Therefore the number of possibilities for $a = 1$ is $1 \cdot 6 \cdot 1 = 6$.

CASE 2: $a = 2$

We have: $b + c < 13$ and $b + c > 2$, so both b and c are greater than 1 (no repeats). We also have that $c \geq b$ as the easiest possible restriction that gets us no repeats. There are some special repetition restrictions, and we know that for $b \leq 6$, the only two possibilities for c are b and $b + 1$, by the triangle inequality.

We know that $(2, 6, 7)$ is not allowed, but $(2, 6, 6)$ is, so we subtract 1 from the total. There are 5 numbers to choose from inclusively between 2 and 5, so $2 \cdot 5 - 1 = 9$.

So there are a total of 9 possibilities for $a = 2$

CASE 3: $a = 3$

We have: $b + c < 12$, and $b + c > 3$, so both b and c are greater than 2, with no repetitions. As before, we have $c \geq b$. We know that if $b \leq 6$, the only two possibilities for the remaining side lengths are $b, b + 1$, and $b + 2$ again by the triangle inequality. Note that we cannot have a 6 because $3 + 6 + 6 = 15$.

Thus, there are 3 numbers inclusively between 3 and 5, so $3 \cdot 3 - 1$(excluding the $3 + 5 + 7$ possibility)$= 8$.

Case 4: $a = 4$

As case 2, we have $b + c < 11$, so our maximum case is $(4, 5, 5)$. We also have that both b and c are greater than 3, and we can have $c = b, b + 1, b + 2, b + 3$ by the triangle inequality. However, we cannot have $b + 3$, because $4 + 4 + 7 = 15$. We must leave out $5 + 2$ and $5 + 1$ for c, so:

There are 2 selections for b (4 and 5), so $2 \cdot 3 - 2 = 4$.

There are $4 + 8 + 9 + 6 = \boxed{\textbf{(D) 27}}$ triangles.

6. We start by thinking of what numbers don't result in real outputs when square rooted. We see that a square root of a negative number is not real, so we try to find numbers such that $x^2 < 3$. We see that $x < \sqrt{3}$ or $x > -\sqrt{3}$, and we must have x is an integer, so $-1 \leq x \leq 1$, so there are 3 eliminations for the square root part. The only time we have an impossible fraction is when we divide by 0, and there is only one possibility to make the denominator 0, or 2, so there are $\boxed{\textbf{(E) 4}}$ non-real inputs.

7. Let d be the distance, s be the speed, and t be the time for each of the two trains. We have that the distances must add up to 60, so we have $d = st$ and they are travelling for the same time, so $30t + 50t = 60$, and $t = \frac{3}{4}$ of an hour $= \boxed{\textbf{(D) 45}}$.

8. Let a_n be the nth term of this sequence. We notice that $a_n - a_{n-1} = n$.Thus, $a_{2021} + (-a_{2020}) + a_{2020} + (-a_{2019}) + ... + a_2 + (-a_1) = 2021 + 2020 + ... + 2$.Regrouping these terms, it is also equal to $a_{2021} - a_1$. However, by the triangular number formula, that means that $\frac{2021 \cdot 2022}{2} - 1 = a_{2021} - (-1)$, so using your simple operation calculator to calculate the expression on the left, $a_{2021} = 2043229$. So, $2 + 4 + 3 + 2 + 2 + 9 = \boxed{\textbf{(E) 22}}$

9. Our solution to this problem is too tedious to write in a solution, so here we use the answer choices. We consider the greatest option, 490. The

prime factorization of 490 is $2 \cdot 5 \cdot 7^2$. The number of powers of 7 in 2021! is $\lfloor \frac{2021}{7} \rfloor + \lfloor \frac{2021}{49} \rfloor + \lfloor \frac{2021}{343} \rfloor$, or 334. The number of powers of 7 in $490^4 90$ is $2 \cdot 490$, which is clearly bigger than 334. Thus, it is not choice E. We can repeat this, and we find that $\boxed{\textbf{(E) 360}}$ is the largest answer choice satisfying these criteria.

10. We could try using the answer choices, or we could try more conveniently using algebra. If $2n(n+1) = y^2$, we must also have $2x(x+1) = y^2$, where x is our desired answer. Thus, $2n(n+1) = 2x(x+1)$. Subtracting and dividing by two, $n(n+1) - x(x+1) = 0$. Factoring this, $(n-x)(n+x+1) = 0$. But we can easily see that one of these terms has to equal 0, so either $x = n$ or $x = -n-1$, and we have our answer, $\boxed{\textbf{E) f(-n-1)}}$.

11. We see that $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. We know $x + y = 1$, so $x^3 + y^3 = x^2 - xy + y^2$. Because $x^2 + y^2 = 2$, we have $x^3 + y^3 = 2 + xy$. To find xy, we square the first equation to get $x^2 + 2xy + y^2 = 1$ and $x^2 + y^2 = 2$, so $xy = -\frac{1}{2}$. If we multiply x+y and $(x+y)^2 - 3xy$, to get $x^3 + x^3$, we get $1 \cdot 5/2$, which is $\boxed{\textbf{(D) 5/2}}$

12. A diagonal forms a 45° angle with the side of the square, and when we fold the sides down we bisect the 45 degree angle, but there are two of the 'bisectings', so the top angle is 45°, and the bottom is 90°, as the bottom right side of the square doesn't get folded or altered in any way, so $360 - (90 + 45) = 225$, and by symmetry, the two remaining angles are equal, so $225/2 = \boxed{\textbf{(A) 112.5}}$.(One sentence xP)

13. A square number has a prime factorization of x^{2n}. Thus, it must have $2n+1$ factors, or an odd number. But squares of primes have 3 factors, so we count the largest prime less than $\sqrt{10,000}$. That happens when there are primes less than 100. We list the primes, and get that there are $\boxed{\textbf{(C) 25}}$ numbers with 3 factors less than 10,000.

14. We can write an equation with the Pythagorean theorem, where a is the shortest side and r is the ratio between the sides. We have: $a^2 + a^2r^2 = a^2r^4$, and dividing by a^2 we get $1 + r^2 = r^4$. Square rooting gives $1 + r = r^2$, so $r^2 - r = 1$, and completing the square gives $(r - \frac{1}{2})^2 = \frac{5}{4}$, and we get the square of our solution. In order to get our solution, we square root this (the positive solution), so we have $\sqrt{\sqrt{\frac{5}{4}} + \frac{1}{2}}$ which simplifies to $\sqrt{\frac{\sqrt{5}+1}{2}}$. Thus, $x = 5, z = 1$, and $y = 2$, so $5 + 2 = \boxed{\textbf{(E) 7}}$.

15. We can let the denominators be x and y, so we have $\frac{1}{x} + \frac{1}{y} = \frac{1}{8}$. Next, we simplify, finding the LCM of x and y, which is, most simply, xy. So, $\frac{x+y}{xy} = \frac{1}{8}$. Getting rid of the fractions, we have $8x + 8y = xy$, and $xy - 8x - 8y = 0$. Completing the SFFT, we have $xy - 8x - 8y + 64 = 64$, so $(x-8)(y-8) = 64$. Next, we use the rational root theorem to figure out that x and y are 8 more

than factors of 64. The factors of 64 are $1, 2, 4, 8, 16, 32,$ and 64, and as all of these are real, there are 7 different pairs (x, y). However, $\frac{1}{y} + \frac{1}{x}$ is implied the same as $\frac{1}{x} + \frac{1}{y}$, so there are really $7 - 3 = \boxed{\textbf{(C) 4}}$ distinct pairs $\frac{1}{x}, \frac{1}{y}$.

16. As the negative signs are not uniform, we rewrite the series such that we only have -3s in the numerator:

$\left(\frac{-3}{12} + \frac{-3}{20}\right) + \left(\frac{-3}{30} + \frac{-3}{42}\right) + ... + \left(\frac{-3}{(n+2)(n+3)} + \frac{-3}{(n+3)(n+4)}\right)$

We would definitely like to turn this into a telescoping series. As the multiples of the denominators of the fractions are one apart and one multiple stays the same as n gets bigger, we want to create a scenario where we have additive reciprocals in our fractions and they cancel out, leaving us with only the first and last terms. In our case, the last term will approach 0, as the denominator getting bigger as it approaches ∞ yields a smaller fraction quantity. We look for fractions such that $\frac{A}{(n+2)} + \frac{B}{(n+3)} = \frac{-3}{(n+2)(n+3)}$. First, we add the two fractions. Finding the common denominator, $(n + 2)(n + 3)$, we have $\frac{B(n+2)+A(n+3)}{(n+2)(n+3)} = \frac{-3}{(n+2)(n+3)}$, so $B(n + 2) + A(n + 3) = -3$. Next, we expand, yielding $Bn + 2B + An + 3A = -3$, so $(B + A)n + 2B + 3A = -3$. We can see -3 as $0n - 3$, so $B + A = 0$ and $2B + 3A = -3$, and solving we get $A = -3$ and $A = 3$, and checking, B does equal $-A$, so we are on the path to success! Rewriting these fractions, we do get a scenario similar to the one hoped for:

$\left(\frac{-3}{3} + \frac{3}{4} + \frac{-3}{4} + \frac{3}{5}\right) + \left(\frac{-3}{5} + \frac{3}{6} + \frac{-3}{6} + \frac{3}{7}\right) + ... + \left(\frac{-3}{(n+2)} + \frac{3}{(n+3)} + \frac{-3}{(n+3)} + \frac{3}{(n+4)}\right)$

As shown, the $\frac{3}{(n+3)} + \frac{-3}{(n+3)}$ cancel out to 0, so we are left with the first term, -1, and the last term, the limit of $\frac{3}{(n+4)}$, when the denominator approaches ∞, which is 0, as the $\frac{3}{(n+?)}$ is losing value every time the denominator goes up. So the answer is $0 + -1 = \boxed{\textbf{(B) -1}}$

17. Let's express the numbers using modular arithmetic:

$1 \equiv 1 \pmod 4$
$2 \equiv 2 \pmod 4$
$3 \equiv 3 \pmod 4$
$4 \equiv 0 \pmod 4$
$5 \equiv 1 \pmod 4$
$6 \equiv 2 \pmod 4$
$7 \equiv 3 \pmod 4$

Count:
1 number with R0
2 numbers with R1
2 numbers with R2
2 numbers with R3

Let's look at the possible combinations.

If the first number is 0: (0, 1, 3) or (0, 2, 2).
There are $1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot \frac{1}{2} = 5$ possibilities
If the first number is 1: (1, 1, 2)
There are $2 \cdot 1 \cdot \frac{1}{2} \cdot 2 = 2$ possibilities
If the first number is 2: (2, 3, 3)
There are $2 \cdot 2 \cdot 1 \cdot \frac{1}{2} = 2$ possibilities
If the first number is 3: No non-repetitive combinations.

We divide by two because there are overcountings of certain combinations when we use the multiplication principle.

So, there are 9 possibilities for favorable. The possible is $\binom{7}{3} = 35$, so $\frac{9}{35} \approx \boxed{\textbf{(C) 26 \%}}$

18. We start by noticing that ratios imply similar triangles, so we look for similar triangles. Another thing that yields similar triangles are lines forming the same angles, so we also look for those. We notice that because the radius is always perpendicular to its tangent line, we have that $\triangle ABX \sim \triangle AOP$ by AA similarity because $\angle A = \angle A$ and $\angle APO = \angle AXB = 90°$. Similarity statements give us ratios, so we write ratios based on the desired lengths. First, we let the radius of the circle be r, and AP be x. From our similarity, $\frac{AX}{AP} = \frac{BX}{OP}$, so $\frac{AX}{x} = \frac{BX}{r}$. Cross-dividing, $\frac{AX}{BX} = \frac{x}{r}$. As we are also given that $\frac{x}{r} = \frac{4}{3}$, we substitute, getting $\frac{AX}{BX} = \frac{4}{3}$. By the Pythagorean theorem, $\frac{AB}{BX} = \frac{5}{3}$. We know that $BX = BP$, so $\frac{AB}{BP} = \frac{AB}{BX} = \frac{5}{3}$. Thus, $\frac{AP}{BX} = \frac{AB-BP}{BX} = \frac{AB-BX}{BX} = \frac{AB}{BX} - 1 = \frac{2}{3}$. However, $\frac{AP}{OP} = \frac{4}{3}$, so $\frac{OP}{BP} = \boxed{\textbf{(B) 1/2}}$

19. We notice that the toothpicks are merely arrangements of vertical and horizontal columns and rows. If there are x columns and rows on each side of the square arrangement, we have that there are $x+1$ rows and columns, as there are 2 spaces on either end of the toothpick borders, and $x-1$ in between. Each row and column has length x. Fortunately, these rows and columns do not have any intersection, so we can say that $x(x+1) + x(x+1)$ is the number of toothpicks in the figure, or $2x(x+1) = 144$. Therefore, $x(x+1) = 72$, and it's clear to see that $x = \boxed{\textbf{(A) 8}}$.

20. As the solution to 7, we let d be distance, t be time, and s be speed (mph). We have: $d = st$. While it's not clear how to approach this with one equation, we can approach this with two: $d = 600(t+7)$ and $d = 350(t+120)$. But, we don't have the same units, so we must convert miles per hour to minutes! As the per in miles per hour really means miles / hours, we have $\frac{miles}{hours \cdot 60}$ or $\frac{miles}{hours} \cdot \frac{1}{60}$, or $\frac{speed}{60}$. So, $d = 10(t+7)$ and $d = \frac{35}{6}(t+120)$. Expanding, we subtract the equations and get $0 = \frac{25}{6}t - 630$, so $t = 630 \cdot 6 \cdot \frac{1}{25} = 151.2$. Next, we multiply this by 10 and get 1512, and add 70 to get our answer, $\boxed{\textbf{(D) 1582}}$.

21. The numbers that have four digits in base 7 are 343_10 to 2400_10, and in base 13 they are 2197_10 to $13^5 - 1_10$. Clearly, 2400 is less than $13^5 - 1_10$, so

the average of all such numbers is the average of the upper and lower bounds of $2197+2198+2199+...2400$, or $\boxed{\textbf{(A) 4597/2}}$.

22. We start by listing the first few rounds. $2015, 2014, 2013$ by the first round becomes $2012, 2015, 2014$. By the second round, it is $2013, 2012$, and 2015. By the third round, it is $2014, 2013$, and 2012. We see now that everything has decreased by 1 in a matter of 3 rounds. We can see that all of the -3's and +1's are just differently arranged, so all of the terms decrease by 1 every 3 rounds, and this pattern holds. Now, we must find how many rounds it takes for one of the players to give away their last token. Notice that the number of rounds to go from $2015, 2014, 2013$ and $3, 2, 1$ is $2012 \cdot 3$. However, we see that $2, 1, 0$ is never an obtainable combination, because the player with 3 tokens will give away 2 two the other players and 1 to the discard pile, ending the game with 0 tokens. Thus, the answer is $2012 \cdot 3 + 1 = \boxed{\textbf{(E) 6037}}$.

23. Let us make abbreviations for all the possible events for purposes of communication.

B = Black card drawn, R = Red card drawn, T = Coin tells to Toss, D = Coin tells to return the card to the Deck.

We want all of the cards to be red. so we have R_{RR}. The last toss does not matter, so we eliminate that. Next, for the blank slots, we could have any combination of T's and D's, so there are 2*2 = 4 combinations, specifically RDRDR, RDRTR, RTRDR, and RTRTR.

CASE 1: RDRDR

We return all of the red cards to the deck, so the probability of drawing a red card is always $\frac{6}{9} = \frac{2}{3}$ in this scenario. Thus, we have $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{6}$.

CASE 2: RDRTR

For the first two red cards drawn, the deck is unchanged, as we return the red card drawn to the deck, but for the third red card, the second card was tossed, so we have one less card from total and favorable. Thus, the total probability is $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{6-1}{9-1} = \frac{5}{96}$.

CASE 3: RTRDR

This probability is different than case 2, although they are rearrangements of each other. This is because the first toss changes the deck, which returning it to the deck The probability of getting a red on the first draw: $\frac{2}{3}$ Toss, which has a probability of $\frac{1}{4}$ Probability of getting a red after a toss: $\frac{6-1}{9-1} = \frac{5}{8}$ Deck, which has a probability of $\frac{3}{4}$ Probability of getting a red now: $\frac{5}{8}$. This gives $\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{5}{8} \cdot \frac{3}{4} \cdot \frac{5}{8} = \frac{25}{512}$.

CASE 4: RTRTR

As before, we do separate probabilities, then multiply them out. The probability of getting the first R is $\frac{6}{9}$, or $\frac{2}{3}$. Next, the probability of getting a toss is $\frac{1}{4}$. Third, there are 2 reds and 8 cards left after the toss, so the probability of getting a red after one toss is $\frac{5}{8}$. Fourth, the probability of getting a toss is

$\frac{1}{4}$, and finally, fifth, the probability of getting a red after 2 tosses is $\frac{4}{7}$, as we decrease by one card after the toss. Thus, $\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{5}{8} \cdot \frac{1}{4} \cdot \frac{4}{7} = \frac{5}{336}$.

Adding all these results together, we get $\frac{3037}{10752} \rightarrow \boxed{\textbf{(D) 13789}}$

24. We have several cases for the repetition length. As 7 is prime, and we cannot have 1 or 7 repetitions, we can always truncate the number.

CASE 1: 2 per repetition

9 ways to pick the first digit, 10 ways to pick the second, 1 way as all of the others are auto-filled in, so $9 \cdot 10 = 90$. Then, we subtract 9 for all of the 1 digit cases, so there are 81.

CASE 2: 3 per repetition

As above, we have 9 for the first digit, 10 for the second, 10 for the third, so 900 ways in all. Then we subtract 9 for all of the 1 digit cases, so there are 891.

We cannot have greater than $\frac{7}{2} = 3.5$ numbers per untruncated repetition, as there must be at least two repetitions, so in all, there are $900 + 90 - 18 = \boxed{\textbf{(A) 972}}$ seven digit groovy numbers.

25. There are several cases, as the candy bars must remain in the box. As the candy bars are indistinguishable, we don't have to write each separate case. We know that the number of candy bars distributed cannot be greater than 5, but no multiple of 5 added to a multiple of 4 is 24 with these restrictions, so we can narrow it down to 3 cases each with 2^6 possibilities: Every child receives 1 or 2 candy bars, every child receives 2 or 3 candy bars, and every child receives 3 or 4 candy bars. The total number of cases is 192, however we have double counted the cases where all children receive two candy bars and all children receive three candy bars, so the real solution is $192 - 2 = \boxed{\textbf{(B) 190}}$.

Credits

Solution 15 by SigmaPiE

Answer Key

1. A
2. E
3. C
4. B
5. D
6. E
7. D
8. E
9. B
10. E
11. D
12. A
13. C
14. E
15. C
16. B
17. C
18. B
19. A
20. D
21. A
22. E
23. D
24. C
25. B