2020

PMC 10

DO NOT OPEN UNTIL AUGUST 15th, 2020

INSTRUCTIONS:

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question, 75-minute examination. All questions are multiple choice, with answers labeled A, B, C, D, and E; only one of these is the correct answer.
3. Every correct answer is worth 6 points, every blank answer is worth 1.5 points, and every incorrect answer is worth 0 points. Your score is the total number of points.
4. No aids other than scratch paper, graph paper, ruler, compass, eraser, pencil, and pen are permitted.
5. Figures are not necessarily drawn to scale.
6. For submitting your answers to the PMC 10, please PM kred9, CT17, Puddles_Penguin, and GrizzyProblemSolver79c on AoPS.
7. If there are any comments or questions regarding the test, please do not post them on the AoPS thread, but rather PM one of the organizers.

The PMC 10 is brought to you by:

CT17, **kred9**, **Puddles_Penguin**, **GrizzyProblemSolver79c**, **bissue**.

PMC 10

1. Do not discuss any part of this test anywhere, as it will lead to disqualification from all PMC contests.
2. For the sake of clarity, an example submission is shown; please note this is not indicative of the answers of the real test. Answers: AAAAAAA0AAAAAAABAAAAAAAA, where the 0 stands for a blank answer.
3. In addition to submitting your answers, in the PM, also add whether or not you would like to be anonymous on the leaderboard and any feedback you have regarding the test. After we receive all of this information, we will add you to the private discussion forum.

1. Compute $\frac{-1^{(-2^{-3})}+(-3^{(-1^{-2})})}{-1^{(-2^{-3})}-(-3^{(-1^{-2})})}$.

(A) -2 **(B)** $\frac{-1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 2

2. Let x, y, and z be positive integers such that $x+y=5$ and $y+z=8$. What is the maximum possible value of $\frac{z}{x}$?

(A) 1.75 **(B)** 2 **(C)** 2.5 **(D)** 3 **(E)** 4

3. Let $f(x) = \left(\sqrt{6}\right)^x$, and let $g(x) = x^{\sqrt{6}}$. Find k such that $f(g(f(\sqrt{6}))) = 6^k$.
(A) 36 **(B)** 54 **(C)** 72 **(D)** 108 **(E)** 216

4. Trapezoid $ABCD$ has $AB \parallel CD$ and $BC = CD = AD$. If $\angle ACB = 123°$, what is the measure of $\angle B$?

(A) 23° **(B)** 38° **(C)** 57° **(D)** 123° **(E)** 142°

5. Two concentric circles O_1 and O_2 have radii 7 and 15, respectively. A point P is chosen at random inside O_2. Let Q be the closest point to P on the circumference of O_1. What is the probability that $PQ > 5$?

(A) $\frac{9}{25}$ **(B)** $\frac{17}{45}$ **(C)** $\frac{4}{9}$ **(D)** $\frac{81}{176}$ **(E)** $\frac{16}{25}$

6. How many ordered triples (a, b, c) of nonnegative integers a, b, and c exist such that $a + b + c = 12$ and $a + ab + ac = 40$?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

7. What is the area of quadrilateral $ABCD$ if $AB = 8$, $BC = 20$, $CD = 5\sqrt{2}$, $\angle ABC = 90°$, and $\angle BCD = 45°$?

(A) 95 **(B)** 100 **(C)** 105 **(D)** 110 **(E)** 115

8. The rich owner of a igloo hires two house cleaners, each of who will come to the igloo on one day during a three day period. It is known that each cleaner has a 50% chance of coming on the first day, and among all three days they have a 30% chance of coming on the same day. The first day passes and neither cleaner arrives. After this, what is the probability that they will arrive on the same day given that each cleaner's arrival is independent of the other's?

(A) 5% **(B)** 10% **(C)** 20% **(D)** 30% **(E)** 70%

9. Six penguins stand in a circle. Suddenly, each of those penguins reach out to give high-fives to exactly 2 of the other five penguins, at random. On average, how many high-fives will occur? Note: A high-five occurs only when both penguins reach out to give high-fives to each other.

(A) 2 **(B)** $\frac{12}{5}$ **(C)** 3 **(D)** $\frac{24}{5}$ **(E)** 6

10. Square $ABCD$ has side length $\sqrt{13}$. Let P be a point on AB such that $PA = \sqrt{7}$, and let Q be a point on line DP such that $\angle DQB = 90°$. What is the distance from Q to the center of $ABCD$?

(A) $\sqrt{13} - \sqrt{7}$ **(B)** $\sqrt{\frac{13}{7}}$ **(C)** $\frac{\sqrt{13}}{2}$ **(D)** $\sqrt{6}$ **(E)** $\sqrt{\frac{13}{2}}$

11. Six penguins are sitting on an iceberg. Every five seconds, one penguin jumps off the iceberg into the water. The water is cold, so the penguin only stays in the water for six seconds, then returns back onto the iceberg. Over time, the average number of penguins on the iceberg approaches which of the following?

(A) 0 **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** $\frac{24}{5}$ **(E)** 5

12. In triangle ABC, $AB = 15$, $BC = 14$, and $AC = 13$. Let AD be the altitude from A to BC, and let CE bisect angle ACB. Suppose CE intersects AD at F. What is the area of triangle CFD?

(A) $\frac{13}{3}$ **(B)** $\frac{13}{2}$ **(C)** $\frac{25}{3}$ **(D)** $\frac{25}{2}$ **(E)** 13

13. Let z be a factor of 20^{20}. For how many values of z is z^5 neither a factor nor a multiple of 20^{20}?

(A) 32 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 512

14. Square $ABCD$ has side length 1. Suppose another square with side length 1 is situated in the same plane as $ABCD$ such that one of its vertices is exactly r units from A, and one of its vertices is the midpoint of BC. What is the difference between the maximum and minimum possible values of r?

(A) 2 **(B)** $\frac{2+\sqrt{5}}{2}$ **(C)** $\sqrt{5}$ **(D)** $\frac{7-\sqrt{5}}{2}$ **(E)** $\sqrt{2}+1$

15. For all positive integers n, let $P(n)$ denote the smallest positive integer with exactly 2^n positive integer factors. What is the smallest positive integer a for which there exists no positive integer b such that $P(b)$ has a digits?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

16. Let S be the set of all positive real numbers n such that $\frac{1}{n}$ can be expressed in decimal form as $a.b$ where a is a nonnegative integer and b is a nonzero digit. Given that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+... = \frac{\pi^2}{6}$, the sum of the squares of the elements in S can be expressed as $k\pi^2$. Compute k.

(A) $\frac{3}{2}$ **(B)** $\frac{5}{3}$ **(C)** 15 **(D)** $\frac{33}{2}$ **(E)** $\frac{50}{3}$

17. Ipegunn has two copies of each of the numbers 1, 2, 3, and 4. He wishes to arrange them in a row such that any two adjacent numbers differ by at most one. How many ways can he do this, given that two copies of the same number are indistinguishable? Note: Two arrangements are the same if one is the other but in reverse order, for example 11223344 and 44332211 are the same arrangement.

(A) 8 **(B)** 11 **(C)** 13 **(D)** 14 **(E)** 16

18. Parallelogram $ABCD$ has $AB = CD = 10$, $BC = AD = 12$, and $\angle ABC = 60°$. Point P is chosen at random inside $ABCD$, and PE, PF, PG, and PH are the altitudes from P to lines AB, BC, CD, and AD, respectively. What is the probability that E, F, G, and H all lie on the perimeter of $ABCD$?

(A) $\frac{4}{15}$ **(B)** $\frac{14}{45}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

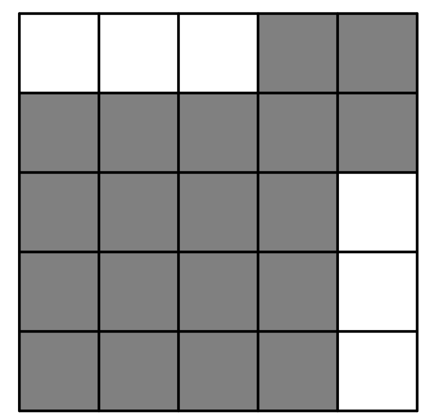
19. A circle with radius 2 and an equilateral triangle with side length s are concentric, with the triangle entirely inside the circle. The three circles whose diameters are the sides of the triangle intersect the larger circle at 6 points equally spaced around its circumference. Find s.

(A) $\sqrt{13}-1$ **(B)** $\sqrt{7}$ **(C)** $\sqrt{33}-3$ **(D)** $2\sqrt{2}$ **(E)** 3

20. An infinite geometric series with initial term a and common ratio b sums to $b - a$. What is the maximum possible value of a?

(A) $\frac{1}{6}$ **(B)** $3-2\sqrt{2}$ **(C)** $\frac{\sqrt{3}}{9}$ **(D)** $\frac{1}{5}$ **(E)** $\sqrt{5}-2$

21. In the diagram below, some of the cells of a 5×5 grid are shaded. How many ways are there to place 10 identical coins in different shaded cells such that every row and every column of the 5×5 grid contain exactly two coins?



(A) 51 **(B)** 54 **(C)** 57 **(D)** 60 **(E)** 72

22. Points A and B lie on unit circle ω with center O such that $AB = 1$. Let A' be the reflection of A over B, and let T be a point on ω such that $A'T$ is tangent to ω. Let α be the sum of all possible values of $\angle BA'T$. Compute the ratio of the lengths of the longer leg to the shorter leg in a right triangle with one acute angle measure α.

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{2}$ **(E)** $2\sqrt{3}$

23. For all positive integers n, let $f(n)$ denote the remainder when $(n+1)^{(n+2)^{(n+3)^{\cdot^{\cdot^{\cdot^{(2n-1)^{2n}}}}}}}$ is divided by n^2. Compute $f(101)$.

(A) 1 **(B)** 102 **(C)** 405 **(D)** 506 **(E)** 10101

24. Let $f(x) = \frac{1}{2}x^2 + \frac{1}{7}x + \frac{1}{4}$. Compute the number of ordered pairs (a, b) of positive integers, both less than 140, such that $a^2b^2 f\left(\frac{1}{a} + \frac{1}{b}\right)$ is also an integer.

(A) 1324 **(B)** 1500 **(C)** 1761 **(D)** 1900 **(E)** 2145

25. Let a positive integer be a rainbow number if it has exactly nine distinct nonzero digits. For all rainbow numbers x, let $f(x) \neq x$ be another rainbow number such that $|f(x) - x|$ is as small as possible. If a rainbow number k is chosen at random, compute the probability that $|f(k) - k| < 40$.

(A) $\frac{13}{18}$ **(B)** $\frac{61}{84}$ **(C)** $\frac{185}{252}$ **(D)** $\frac{31}{42}$ **(E)** $\frac{47}{63}$