2019 CMC 12B Solutions Document

Christmas Math Competitions

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1. **Answer (E):** We rationalize the denominators of both expressions:
$$\frac{\sqrt{2}-3}{2+\sqrt{3}}+\frac{\sqrt{2}+3}{2-\sqrt{3}}=\frac{(\sqrt{2}-3)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}+\frac{(\sqrt{2}+3)(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$
$$=(\sqrt{2}-3)(2-\sqrt{3})+(\sqrt{2}+3)(2+\sqrt{3}).$$
Then, we expand and combine like terms.
$$(2\sqrt{2}-\sqrt{6}-6+3\sqrt{3})+(2\sqrt{2}+\sqrt{6}+6+3\sqrt{3})=4\sqrt{2}+6\sqrt{3}$$

2. **Answer (D):** Clearly, if any of the absolute values of w, x, y, or z are greater than 2, the LHS will be too big. Therefore, $-1 \le w, x, y, z \le 1$, which means w^2, x^2, y^2, and z^2 are each either equal to 0 or 1. Clearly, among w^2, x^2, y^2, and z^2, two of them must be 1 and the other two must be 0. There are $\binom{4}{2} = 6$ ways to choose which ones are 1. For each of the two integers whose squares are equal to 1, we can choose the sign $\pm$ in 2 ways. Thus, there are $6 \cdot 2 \cdot 2 = 24$ quadruples.

3. **Answer (B):** The true value of the sum is 8.68. Fermat rounds each number as $3, 3, 1$, and 2, respectively, which gives him a sum of 9. Euler, on the other hand, rounds each number as $3.1, 2.7, 1.1$, and 1.7, respectively, which gives him a sum of 8.6. Since Euler is off by 0.08 and Fermat is off by 0.32, Euler is 4 times closer to the correct answer compared to Fermat.

4. **Answer (B):** We factor the numerator as follows:
$$x^4-3x^2+2=(x^2-1)(x^2-2)=(x+1)(x-1)(x+\sqrt{2})(x-\sqrt{2})$$
where the last step follows from difference of squares. We can also factor the denominator as $(x-1)(x-2)$. For the whole expression to be 0, the numerator must be 0, but to avoid the expression becoming undefined, the denominator cannot be 0.

   The valid x are $-1, \sqrt{2}$, and $-\sqrt{2}$, while $x = 1$ does not work because it causes the denominator to be 0. The requested sum is $-1+\sqrt{2}-\sqrt{2}=-1$.

5. **Answer (D):** There are 3 ways to choose which pair of people the boy will stand directly in between. WLOG, assume that the boy chooses to stand in between the two grandparents. There are 2 ways to choose which grandparent will be on the boy's left in which the other grandparent will be on the boy's right. Then, we treat the grandparents and boy as a single block, while the remaining four family members are four separate blocks. We can freely reorder these five blocks in 5! ways. Thus, there are $3 \cdot 2 \cdot 5! = 720$ possible arrangements.

6. **Answer (D):** At the beginning, cup A has $\frac{1}{2}$ cup of tea and cup B has $\frac{1}{2}$ cup of milk. During the first pouring, cup B transfers $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ cup of milk to cup A. Afterwards, cup A has $\frac{1}{6}$ cup of milk and $\frac{1}{2}$ cup of tea, while cup B has $\frac{1}{3}$ cup of milk.

   During the second pouring, cup A transfers $\frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$ cup of milk and $\frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$ cup of tea to cup B. Afterwards, cup A has $\frac{1}{10}$ cup of milk and $\frac{3}{10}$ cup of tea, which implies $m = 75$. In addition, cup B has $\frac{1}{15} + \frac{1}{3} = \frac{2}{5}$ cup of milk and $\frac{1}{5}$ cup of tea, which implies $n = 66.\overline{6}$.

   Therefore, $\frac{m}{n} = \frac{75}{200/3} = \frac{9}{8}$.

7. **Answer (B):** Let $A = (1,1), B = (13,-5), C = (1,4), D = (13,4)$, and $E = (x,4)$. Note that $\angle ACE = \angle BDE = 90°$. In addition, due to the reflection of the laser when it hits the mirror, $\angle AEC = \angle BED$. Therefore, $\triangle AEC \sim \triangle BED$. Since $AC = 3$ and $BD = 9$, it immediately follows that $CE : ED = 3 : 9$. Since, $CD = 12$, $CE = 3$ and $x = 4$.

8. **Answer (D):** Note that w and z are interchangeable in the equations and the expression we want to find. We have $w^2 + z^2 = 0 \implies w = \pm zi$. Let $w = zi$. Plugging this into $w + z = 24 + 8i$ and solving, we have:

   $$zi + z = 24 + 8i \implies z(1+i) = 24 + 8i$$

   $$z(1+i)(1-i) = (24+8i)(1-i) \implies 2z = 32 - 16i \implies z = 16 - 8i$$

   Then, $w = 8 + 16i$ and $(w,z) = (8+16i, 16-8i)$. If we were to solve the case for $w = -zi$, the values of w and z are switched. It then follows that $\text{Re}(w) \cdot \text{Re}(z) = 128$.

9. **Answer (D):** Clearly, the graph G of the equation $x^{1000} + y^{1000} = 1$ is symmetric across the x-axis, y-axis, and the line $y = x$ (because x and y are interchangeable).

   Let S be the square centered at $(0,0)$ in the coordinate plane with corners at $(\pm 1, \pm 1)$. Note that if $|x| > 1$, then $x^{1000} > 1$, but if $x \leq 1$, $x^{1000} \leq 1$. Therefore, $|x| \leq 1$ and similarly, $|y| \leq 1$. It follows that G is contained completely within S.

   We need x and y to be slightly under 1 in the graph G. If the values of $|x|$ and $|y|$ are not close enough to 1, then $x^{1000} + y^{1000}$ will be much less than 1 due to how quickly x^{1000} decreases. Therefore, intuitively G should be very close to the shape of S, which is has an area of 4.

10. **Answer (D):** Distribute the pennies first, then the dimes, then the quarters. By stars and bars, there are $\binom{8}{2} = 28$ ways to distribute the pennies. Now, note that once we have distributed the pennies, we must distribute 6 of the dimes in the exact same way that we distributed the pennies. For example, if Andrew, Bob, and Charlie receive 2, 1, and 3 pennies, respectively, we will initially give them 2, 1, and 3 dimes, respectively. This way, each person will be forced to receive at least as many dimes as pennies, no matter how we distribute the remaining dimes. This will use up 6 of the dimes, leaving 2 dimes to distribute in $\binom{4}{2} = 6$ ways. Then, by the same process, we must use up 8 of the quarters to match each quarter with a dime. The remaining quarter can be distributed in 3 ways.

    Our total is $28 \cdot 6 \cdot 3 = 504$ ways.

11. **Answer (C):** Clearly, $P(a) = P(b) = P(c) = 0$. To find a polynomial with roots at a^2, b^2, and c^2, we must look at the function $P(\sqrt{x}) = 0$. We have $P(\sqrt{x}) = x\sqrt{x} - 3\sqrt{x} + 1 = 0$. However, this is not a polynomial because some of the terms are of

a non-integer degree. We can group all the terms containing a square root and then square the expression so that each terms is of an integer degree:

$$(x-3)\sqrt{x} = -1$$

$$(x^2-6x+9)(x) = 1$$

$$x^3-6x^2+9x-1 = 0$$

$$(x-2)^3-3(x-2)+1 = 0$$

Therefore, the polynomial with roots at a^2, b^2, and c^2 is given by $P(x-2)$.

12. **Answer (B):** The coordinates of P and Q obviously satisfy both equations. Therefore, if we were to linearly combine the two equations, the graph of the resulting equation will pass through both P and Q. Objectively, we should combine the two equations so that it creates the equation of a line. This can be done by subtracting the two equations to cancel out the x^2 and y^2.

    After subtracting, the equation ultimately becomes $y = -x + \frac{1}{2}$. The y-intercept is clearly $\frac{1}{2}$.

13. **Answer (C):** Take the equation in mod a. This tells us that $b \equiv 0 \pmod{a}$ or that a is a divisor of b. Let $b = am$ for some positive integer m. We have $a^2 + am = a(2019 - am) \implies a + m = 2019 - am$. We get $m = \frac{2019-a}{a+1}$ which is only an integer if $\frac{2019-a}{a+1} = \frac{2020}{a+1} - 1$ is an integer.

    Clearly, $a+1$ must be a positive integer divisor of 2020. However, we must make sure that $a > 0$ and $m > 0$. The condition $a > 0$ is fulfilled if $a + 1 > 1$. The condition $m > 0$ tells us that $\frac{2020}{a+1} - 1 > 0$ or $2020 > a + 1$. Among the divisors of 2020, $a + 1$ may equal any of them except 1 or 2020. Since $2020 = 2^2 \cdot 5 \cdot 101$ has 12 divisors, our answer is $12 - 2 = 10$.

14. **Answer (B):** Let $f(n)$ be the sum of the numbers on the whiteboard after n seconds.

    On the board, consider an arbitrary number x written on the board after n seconds that is not one of the two 1s on either end of the string of numbers. That number x will be added with the two numbers on either side of x and x will be present in the list of numbers after $n+1$ seconds. This contributes $3x$ to the sum of numbers after $n+1$ seconds. However, the two 1s on either end of the string of numbers are adjacent to one other number each. Therefore, both 1s will contribute 2 to the total sum for $n+1$ seconds instead of 3. Overall, the sum for $n+1$ is triple that of n, but we must subtract 2 because the 1s on either end of the string are doubled instead of tripled. So $f(n+1) = 3f(n) - 2$ for all $n \geq 0$ with our base case of $f(0) = 2$.

    We can recognize that $f(n) = 3^n + 1$ for all $n \geq 0$, so $f(10) = 59{,}050$. Among the answer choices, this is closest to $60{,}000$.

15. **Answer (D):** Let N be the midpoint of CD, and let E be the point in 3D space where A and B coincide after the folding. Now, we have tetrahedron $EMCD$, and we wish to find its height with respect to base MCD. To find this desired height, we simply need to find the altitude from E to MN in $\triangle EMN$.

    By inspection, $MN = 13$ and $ME = MA = 5$. After the folding, we have that $\triangle ECD$ is a triangle with lengths $DC = 10$ and $ED = EC = 13$. When we drop the altitude from E to DC in $\triangle ECD$, the foot is N. Using Pythagorean Theorem, we find that $EN = 12$, so $\triangle EMN$ is a $5-12-13$ right triangle. It then follows that the length of the altitude from E to MN in $\triangle EMN$ is $\frac{5 \cdot 12}{13} = \frac{60}{13}$.

16. **Answer (C):** In each valid way to shade the squares in the grid, there will always be the same number of shaded squares in each row and column. Therefore, if we were to choose the two squares in the first column to shade, we will always have the same number of ways to shade squares in the rest of the grid. WLOG, suppose we shade the two topmost squares in the first column. We will multiply by $\binom{4}{2} = 6$ at the end.

    Now, we will determine how to shade the second column. The cases will be dependent on what types of squares are immediately to the left of the shaded squares in the second column.

    Suppose both shaded squares in the second column have a shaded square immediately to their left. Then, we must shade all the squares in the 2×2 area in the bottom right corner of the grid. This gives us 1 valid way.

    Suppose one of the shaded squares in the second column has a shaded square immediately to its left, while the other shaded square has an unshaded square immediately to its left. WLOG, in the second column, shade the first an third squares. We will multiply by 4 later. Clearly, in the third and fourth columns, the two squares in the first row must be unshaded, while the two squares in the fourth row must be shaded. In the third column, we can choose either the second or third square to shade, which determines which square to shade in the fourth column. There are $4 \cdot 2 = 8$ valid ways in this case.

    Lastly, suppose both shaded squares in the second column have an unshaded square immediately to their left. Then, we can choose any two squares in the third column to shade and then the two shaded squares in the fourth column are uniquely determined. There are $\binom{4}{2} = 6$ valid ways in this case.

    There are $6 \cdot (1 + 8 + 6) = 90$ ways total.

17. **Answer (A):** Let $n = \overline{abcde}$. We will first consider n being a multiple of 8. Since 1000 is divisible by 8, only $\overline{cde}$ has an effect on n being a multiple of 8. Since $100 \equiv 4 \pmod 8$, if c is odd, then we need $\overline{de} \equiv 4 \pmod 8$. However, if c is even, then we need $\overline{de} \equiv 0 \pmod 8$.

    We need $\overline{de}$ to be divisible by 4, since 100 is divisible by 4. Overall, $\overline{de}$ has $\frac{100}{4} = 25$ possible values, 13 of which are $0 \pmod 8$ and 12 of which are $4 \pmod 8$. However, we need to subtract all the values where d or e is 0, namely $00, 04, 08, 20, 40, 60$, and 80. From this list, 4 of these numbers are $0 \pmod 8$ and 3 of these are $4 \pmod 8$. So we have 18 acceptable values of $\overline{de}$, 9 of which are $0 \pmod 8$ and 9 of which are $4 \pmod 8$.

    We have 9 choices for c. No matter what the parity of c is, there will always be 9 possible values of $\overline{de}$ that will let $\overline{cde}$ be divisible by 8. Now, to let n be divisible by 3, we can randomly select b, and we choose a such that $a + b + c + d + e \equiv 0 \pmod 3$. There are always 3 such choices for a regardless of what the other digits are.

    Hence, we have a total of $3 \cdot 9 \cdot 9 \cdot 9 = 3^7$ values of n, which has 8 divisors.

18. **Answer (C):** By the symmetry of the four quadrants in the coordinate plane, we only need to consider the first quadrant. Let A be the bottom right corner of the oasis in the first quadrant, and let B be the top left corner of the oasis in the first quadrant. In addition, let $O = (0, 0), C(1, 0)$, and $D = (0, 1)$. By symmetry across the line $y = x$, $\angle BOD = \angle AOC$. Clearly, the man will eventually encounter the oasis if and only if he chooses a direction bounded by rays OB and OA. Since the probability he will encounter the oasis is $\frac{1}{3}$ and $\angle DOC = 90°$, it follows that $\angle BOA = 30°$ and then $\angle BOD = \angle AOC = 30°$.

For $\angle AOC = 30°$, point A must lie on the line $y = \tan(30°)x$ or $y = \frac{1}{\sqrt{3}}x$. Clearly, in terms of s, the coordinates of A are $\left(1+\frac{s}{2}, 1-\frac{s}{2}\right)$. Solving the equation $1-\frac{s}{2} = \frac{1}{\sqrt{3}} \cdot \left(1+\frac{s}{2}\right)$ gives us $s = 4-2\sqrt{3}$.

19. **Answer (B):** For $\gcd(a, c)$ and $\gcd(b, c)$ to both greater than 1, c must share a prime factor with a and a prime factor with b. The shared prime factor between a and c is not necessarily the same as the shared prime factor between b and c.

Let $g(a, b)$ denote the smallest prime that divides both a and b if $\gcd(a, b) > 1$ and undefined if $\gcd(a, b) = 1$. In addition, let $h(a)$ denote the smallest prime that divides a for $a > 1$.

It follows that:

$$f(a,b) = \begin{cases} \min(g(a,b), h(a)\cdot h(b)), & \text{if } \gcd(a,b) > 1 \\ h(a)\cdot h(b), & \text{if } \gcd(a,b) = 1 \end{cases}$$

Since neither $55 = 5 \cdot 11$ nor $77 = 7 \cdot 11$ are prime, $f(x, y) = h(x) \cdot h(y) = 55$ and $f(y, z) = h(y) \cdot h(z) = 77$. Clearly, $(h(x), h(y), h(z)) = (5, 11, 7)$. However, we need to avoid changing the value of $f(x, y)$ and $f(y, z)$ through the min function. Therefore, we must make sure $g(x, y) > h(x) \cdot h(y) = 55$ when $\gcd(x, y) > 1$ and $g(y, z) > h(y) \cdot h(z) = 77$ when $\gcd(y, z) > 1$. The only prime that necessarily divides y with our current information is 11. Therefore, if x is divisible by 11, it will cause $g(x, y) = 11 > 55$, which is a contradiction. Similarly, if z is divisible by 11, it will cause $g(y, z) = 11 > 77$, which is a contradiction. Therefore, neither x nor z may be divisible by 11.

We will now find all possible values of $f(x, z)$. If we let $f(x, z) = h(x) \cdot h(z)$, then $f(x, z) = 35$. This is easily achievable with $(x, y, z) = (5, 11, 7)$.

However, if we let $g(x, z) < h(x) \cdot h(z) = 35$, then we consider $f(x, z) = p$ for some prime $p < 35$. Since $h(z) = 7$, we must have $p \geq 7$. However, recall earlier that x and z are not divisible by 11, so $p \neq 11$.

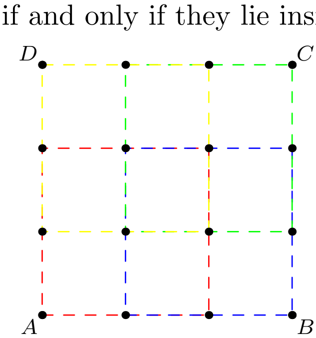
Checking the primes in the interval $7 \leq p < 35$ aside from 11, we have $p \in \{7, 13, 17, 19, 23, 29, 31$ Each of these values $f(x, z)$ can be achieved with $(x, y, z) = (5p, 11, 7p)$.

Therefore, $f(x, z) \in \{7, 13, 17, 19, 23, 29, 31, 35\}$ for a total of 8 values.

20. **Answer (E):** Theoretically, the smallest possible value of AP_1 occurs when AP_1 is perpendicular to BC. Similarly, the smallest possible value of BP_2 occurs when BP_2 is perpendicular to AC. Fortunately, $AP_1B = AP_2B = 90°$ implies that ABP_1P_2 is cyclic, so these positions of P_1 and P_2 are valid. Therefore, $AP_1 \perp BC$ and $BP_2 \perp AC$.

By Heron's formula, $[ABC] = \frac{15\sqrt{7}}{4}$. It then follows that $AP_1 = \frac{2\cdot[ABC]}{4} = \frac{15\sqrt{7}}{8}$ and $BP_2 = \frac{2\cdot[ABC]}{6} = \frac{5\sqrt{7}}{4}$. By Pythagorean Theorem, $AP_2 = \frac{15}{4}$ and $BP_1 = \frac{5}{8}$. It then follows that $AP_2 + BP_1 = \frac{35}{8}$. The requested sum is $35 + 8 = 43$.

21. **Answer (B):** To start with, divide the 3×3 array of points in four 2×2 squares. We will now prove that it is possible to select 4 of these points such that the distance between them is at most $2\sqrt{2}$ if and only if they lie inside the same 2×2 square.



Clearly if the 4 points lie within the same square, the maximum distance between two of them is at most $2\sqrt{2}$, and this happens when they lie on opposite vertices of one of the main diagonals of the square. Suppose then for the sake of contradiction that it is possible to select 4 points, not all of them lying inside of the same square, such that the maximum distance between two of them is at most $2\sqrt{2}$. We distinguish two cases: If one of these points is either A, B, C or D, consider the only 2×2 square passing trough it; if WLOG such point is A, notice that for any other point P lying outside of the red square, we have $AP \geq 3$, contradiction. If none of these points is A, B, C or D, at least one of them lies on a side of $ABCD$ (otherwise they are 4 inmost points of the square, but we supposed they were not lying inside the same 2×2 square). Then, consider the two 2×2 squares passing trough it; if the four points are not all contained inside the same 2×2 square, then there exists a pair of points lying on opposites sides of $ABCD$. If so, however, their distance is ≥ 3, contradiction. Hence, we can use the Principle of Inclusion / Exclusion: there are

$$4 \cdot \binom{9}{4} - 4 \cdot \binom{6}{4} + 1 \cdot \binom{4}{4} = 504 - 60 + 1 = 445$$

such possibilities, where $4 \cdot \binom{9}{4}$ is the number of ways to choose 4 points lying inside a 2×2 square, $4 \cdot \binom{6}{4}$ is the number of ways to choose 4 points lying in the intersection of two 2×2 squares (so, lying in one of the four 1×2 rectangles where at least two squares intersects) and $1 \cdot \binom{4}{4}$ is the number of ways of choosing 4 points lying in the intersection of three (and it's the same as four) 2×2 squares.

22. **Answer (C):** Intuitively, the roots of the quadratic $x^2 + F_{100}x - F_{100}^2 - 1 = 0$ should be Fibonacci numbers. In particular, we claim that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all integers $n \geq 1$. We will prove this identity by induction. Checking the base case with $n = 2$, we have $F_3F_1 - F_2^2 = 1 \implies 2 \cdot 1 - 1^2 = 1$, which holds.

Now, we will show that if we assume the induction hypothesis is true for $n = k$, it will also hold for $n = k + 1$:

$$\begin{aligned}
F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - (F_k + F_{k-1})^2 \\
&= F_{k+1}F_k - 2F_kF_{k-1} - F_{k-1}^2 \\
&= (F_k + F_{k-1})F_k - 2F_kF_{k-1} - F_{k-1}^2 \\
&= F_k^2 - F_kF_{k-1} - F_{k-1}^2 \\
&= F_k^2 - F_{k-1}(F_k + F_{k-1}) \\
&= F_k^2 - F_{k+1}F_{k-1} \\
&= (-1) \cdot (F_{k+1}F_{k-1} - F_k^2) \\
&= (-1)^{k+1}
\end{aligned}$$

Since the induction hypothesis is true for $n = k + 1$ given that it is true for $n = k$, the induction is complete. (This identity is more commonly known as Cassini's identity)

By the identity, $F_{100}^2 + 1 = F_{99}F_{101}$. In addition, we can rewrite F_{100} as $F_{101} - F_{99}$.

The quadratic becomes $x^2 + (F_{101} - F_{99})x - F_{99}F_{101} = 0$, which can be factored as $(x + F_{101})(x - F_{99}) = 0$. Now, we wish to find $N \equiv F_{99} + F_{101} \pmod{11}$. Quickly checking, the Fibonacci numbers repeat with a period of 10 in modulo 11 and $F_1 \equiv F_9 \equiv 1 \pmod{11}$. Therefore, $N \equiv 2 \pmod{11}$.

23. **Answer (E):** For $p(x)-2017 \geq 0$ for all real numbers x, $p(x)-2017$ must have an even multiplicity of its roots. Therefore, $p(x)-2017$ is in the form $a(x-r)^2(x-s)^2(x-t)^2$ for real numbers a, r, s, and t and $a > 0$. We are given that $p(x) - 2017 = 0$ when $x \in \{2014, 2015, 2016\}$, so $\{r, s, t\} = \{2014, 2015, 2016\}$.

    Since $p(x) - 2017 = a((x-2014)(x-2015)(x-2016))^2$, we can plug in $x = 2017$ to find a:
$$p(2017) - 2017 = a((2017-2014)(2017-2015)(2017-2016))^2$$
$$2018 - 2017 = a \cdot 6^2$$
$$a = \frac{1}{36}$$
    Finally, $p(2019) = 2017 + \frac{1}{36} \cdot ((2019-2014)(2019-2015)(2019-2016))^2 = 2117$.

24. **Answer (E):** Let $f(n)$ be the number of distinct ordered pairs (a, b) achieved over all colorings with n dots. For $n = 1$, we always have $(a, b) = (0, 0)$, so $f(1) = 1$.

    Any pair of adjacent dots must be both white, both black, or different colors. For a given coloring, let c be the number of adjacent dots that are different colors. If two adjacent dots are different colors, draw a divider between the two dots. Then, the region of dots bounded between any two dividers are all the same color and the next dot past the divider is the opposite color from the dots in the region. Since c can also be interpreted as the number of dividers, we need c to be even. If c is odd, then we would have two adjacent regions that are both the same color, which is a contradiction. Since $a + b + c = n$, it follows that $a + b$ must have the same parity as n.

    We claim that the only achievable ordered pairs (a, b) satisfy each of the following conditions:

    - $a + b \leq n$
    - If $a + b < n$, then (a, b) is achievable as long as $a + b$ has the same parity as n
    - If $a + b = n$, then $(a, b) \in \{(n, 0), (0, n)\}$

    The first condition trivially must hold true.

    We can construct a coloring for any ordered pair (a, b) with $a + b < n$ that satisfies the second condition as follows: first, color $a + 1$ adjacent white dots, and call this region of dots Region 1. Immediately after the white dots, color $b + 1$ adjacent black dots, and call this region of dots Region 2. If $a + b < n$, then $a + b$ is at most $n - 2$ and $(a + 1) + (b + 1)$ is at most n, which means that we have enough dots available for $a + 1$ adjacent white dots and $b + 1$ adjacent black dots. After the black dots, if there are dots still left to color, color the first dot white and then alternate the colors of the dots between black and white until we reach Region 1. Since $(a + 1) + (b + 1)$ has the same parity as n, the last dot before we reach Region 1 will be black. We have a pairs of adjacent white dots from Region 1 and b pairs of adjacent black dots from Region 2, while every other pair of adjacent dots have different colors. Therefore, this construction achieves every ordered pair (a, b) that satisfies the second condition.

    For the ordered pairs (a, b) that satisfies the third condition, we know that $c = 0$, so the coloring contains only one color. Therefore, only $(n, 0)$ and $(0, n)$ are achievable when $a + b = n$.

    Now, we will determine a formula for n.

    If n is odd and $n > 1$, we do casework on the value of $a+b$. In general, for a nonnegative integer m, the equation $a+b = m$ has $m+1$ solutions in nonnegative integers. If $a+b = 1$, we have $(1 + 1)$ ordered pairs (a, b), and if $a + b = 3$, we have $(3 + 1)$ ordered pairs. We continue in this fashion until $a+b=n-2$, while $a+b=n$ gives us 2 more ordered pairs. So for odd n with $n>1$, $f(n)=((1+1)+(3+1)+(5+1)+...+((n-2)+1))+2=\frac{n^2}{4}+\frac{7}{4}$.

Similarly, for even n, $f(n)=((0+1)+(2+1)+(4+1)+...+((n-2)+1))+2=\frac{n^2}{4}+2$.

So $f(n)\leq\frac{n^2}{4}+2$ regardless of the parity of n and we wish to find the smallest positive integer n such that $f(n)\geq 2019$. Solving $2019\leq\frac{n^2}{4}+2$ tells us that $n\geq 2\sqrt{2017}$ and $2\sqrt{2017}>2\cdot 44 \implies n>88$. We can check that $f(89)=1982$, but $f(90)=2027$, so $n=90$ is the smallest n that works.

25. **Answer (E):** Let $AF=5x$ and $BE=8x$ for some positive real number x. We have that $\frac{BC}{BE}=\frac{AB}{AF}=\frac{7}{x}$. In addition, $\angle CBE=\angle BAF=90°$. By SAS similarity, $\triangle CBE\sim\triangle BAF$.

If we let $\angle ABF=\theta$, then $\angle BCE=\theta$ and $\angle BEC=90°-\theta$. As a result, $\angle BPE=90°$ and $\angle BPC=90°$. Since $\angle BPC$ being a right angle is invariant, the locus of P is the semicircle ω with diameter BC that lies completely within rectangle $ABCD$. Let O be the midpoint of BC or the center of ω. The length AP is minimized when P is the intersection of AO and ω.

Because BC and AD are parallel, we have $\triangle POB\sim\triangle PAF$. Since $BO=OP$, then $AP=AF$ by the similar triangles. In addition, $AP=AO-OP=AO-28$. By Pythagorean Theorem on $\triangle ABO$, $AO=7\sqrt{41}$. So $AP=AF=7\sqrt{41}-28$. The requested sum is $7+41+28=76$.