2021 CMC 12B Solutions Document

Christmas Math Competitions

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1. **Answer (C):** We compute as follows:

$$(2+0+2+1)^2 - (2-0+2-1)(2+0-2+1) = 5^2 - 3 \cdot 1 = 22.$$

2. **Answer (B):** If $n \geq 4$, the list will consist of at least 4 consecutive integers, which means at least 1 of those integers is a multiple of 4. Clearly, a multiple of $4 = 2^2$ can never be semiprime, thus n is at most 3. We can show that $n = 3$ work: for example, consider $33 = 3 \cdot 11$, $34 = 2 \cdot 17$ and $35 = 5 \cdot 7$.

3. **Answer (A):** Since $2^{2021} > (2^{11})^2 = 2048^2 > 2021^2$, the given expression equals $2^{2021} - 2021^2$. We can see that the units digits of $2^1, 2^2, 2^3$, and 2^4 are $2, 4, 8$, and 6 respectively with the units digit repeating every four powers of 2. Thus, the units digit of 2^{2021} is 2. Since 2021^2 has units digit 1, the units digit of the expression is 1.

4. **Answer (C):** The term $\frac{\sqrt{2^n}}{2^n}$ can be rewritten as $\frac{2^{n/2}}{2^n} = \frac{1}{2^{n/2}}$. Clearly, the terms in the sum form an infinite geometric series, where the common ratio is $\frac{1}{2^{5/2}} \div \frac{1}{2^{1/2}} = \frac{1}{4}$. By the sum of an infinite geometric series formula, the desired sum is

$$\frac{\frac{1}{2^{1/2}}}{1 - \frac{1}{4}} = \frac{2\sqrt{2}}{3}.$$

5. **Answer (D):** Without loss of generality, assume $x \geq y \geq z$, so that we seek to minimize $\max(x, y, z) = x$. Note that

$$7! = 5040 = x \cdot y \cdot z \leq x \cdot x \cdot x \implies 5040 \leq x^3 \implies 18 \leq x.$$

If $x = 18$, then $yz = 280$. Since integers from 15 to 18 inclusive are not divisors of 280, $y, z \leq 14$. However, this implies that yz is at most $14^2 = 196 < 280$, contradiction.

The next smallest divisor of 5040 is 20, so we check $x = 20$. Indeed, we have $yz = 252$, in which we can take $y = 18$ and $z = 14$. Thus, the minimum value of x is 20.

6. **Answer (C):** By symmetry, Luke will have the same probability of reaching B in three minutes no matter which vertex he travels to in the first minute. In the second minute, Luke has a $\frac{2}{3}$ chance to travel to a vertex adjacent to B and a $\frac{1}{3}$ chance to return to A. If he returns to A, it will be impossible for him to travel to B in the third minute. Thus, Luke must be at a vertex adjacent to B after the second minute. In the third minute, Luke clearly must travel directly to B, which happens with $\frac{1}{3}$ probability. Thus, the desired probability is $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

7. **Answer (D):** Notice that M must be adjacent to one of the two Os and one of B or C. Assume the M is adjacent to O and B, and We will multiply by 2 at the end. Also, the O can either be to the left or to right of the M, so, assuming the O is to the left of the M, we will multiply by another factor of 2 at the end. Then, the sequence OMB can be treated as single block. Now, we must arrange a C, a O, and the OMB block, which can be done in 3! ways. Thus, the number of valid arrangements is $2 \cdot 2 \cdot 3! = 24$.

8. **Answer (A):** Let $\triangle ABC$ be the larger equilateral triangle and $\triangle DEF$ be the smaller equilateral triangle, where D, E, F lie on AB, BC, AC, respectively, and $\angle AFD = 45°$. Let G be the foot of the altitude from D to AC and s be the side length of $\triangle DEF$.

   Since $\triangle DFG$ is a 45-45-90 triangle, $DG = GF = \frac{s}{\sqrt{2}}$. In addition, $\triangle DAG$ is a 30-60-90 triangle, so $AG = \frac{s}{\sqrt{6}}$ and $AD = \frac{2s}{\sqrt{6}}$. Notice that by either rotational symmetry or the fact that $\triangle ADF$ and $\triangle CFE$ are congruent (since they are similar and $DF = EF$ as $\triangle DEF$ is equilateral), we have that $AD = FC$. Thus,

$$1 = AC = AG + GF + FC = \frac{s}{\sqrt{6}} + \frac{s}{\sqrt{2}} + \frac{2s}{\sqrt{6}}$$

   from which we get $s = \frac{\sqrt{6}-\sqrt{2}}{2}$. The requested sum is $6 + 2 + 2 = 10$.

9. **Answer (A):** Let n be the number of times the number being said increases between consecutive people. Since there are 8 instances where the number being said can increase or decrease, there are $8 - n$ instances where the number being said decreases.

   The first person says 1, so the number the 9^{th} person says is $1+1\cdot n+(-1)\cdot(8-n) = 2n-7$. Since this number must be greater than 0, we must have $4 \leq n \leq 8$. Notice that for any such value of n, there are $\binom{8}{n}$ ways that the game could play out, since the order in which the increases and decreases happen does not matter. As without regards to the condition there are $2^8 = 256$ ways the game could play out, the requested probability is

$$\frac{\binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}}{256} = \frac{163}{256}.$$

10. **Answer (D):** Let d and m be the number of days and months until Andre's 21^{st} birthday, respectively. We are given that $d = \lfloor m \rfloor^2$. Since any month has at least 28 days and at most 31, we have that

$$28\lfloor m \rfloor \leq d = \lfloor m \rfloor^2 \leq 31(\lfloor m \rfloor + 1) \implies 28 \leq \lfloor m \rfloor \leq 31.$$

   This is equivalent to at least 2 entire years (accounting for $2 \cdot 365 = 730$ days) and the first 4 months (accounting for $31+28+31+30 = 120$ days) as we do not have any leap year in this interval. We now proceed by casework on the value of $\lfloor m \rfloor$.

   - If $\lfloor m \rfloor = 28$, then $850 \leq d < 850 + 31$ as May has 31 days, but $\lfloor m \rfloor^2 = 784 \neq d$.
   - If $\lfloor m \rfloor = 29$, then $881 \leq d < 881 + 30$ as June has 30 days, but $\lfloor m \rfloor^2 = 841 \neq d$.
   - If $\lfloor m \rfloor = 30$, then $911 \leq d < 911 + 31$ as July has 31 days, but $\lfloor m \rfloor^2 = 900 \neq d$.
   - If $\lfloor m \rfloor = 31$, then $942 \leq d < 942 + 31$ as August has 31 days, and $\lfloor m \rfloor^2 = 961$ lies in this interval.

   Since $961 - 942 = 19$, we conclude that Andre's birthday is on the 19^{th} of August.

11. **Answer (E):** Recall the identity $\log(a^b) = b\log(a)$. Applying this ientity to Alice's expression, it equals:

$$\log\left(x^{\log\left(x^{\log x}\right)}\right) = \log\left(x^{\log x}\right) \cdot \log(x) = \log(x) \cdot \log(x) \cdot \log(x) = (\log(x))^3.$$

If $\log(x) = k$, then Alice's expression is k^3. Then, Billy's expression is $(k^k)^k = k^{k^2}$. We require $k^3 = k^{k^2}$. First, we check if $|k| = 1$. We find that both $k = 1$ and $k = -1$ satisfy the equation. Else, $|k| \neq 1$, implying that the exponents must be equal. We have $k^2 = 3 \implies k = \pm\sqrt{3}$. Therefore, $k \in \{1, -1, \sqrt{3}, -\sqrt{3}\}$. Then, the sum of all values of k^2 across all possible values of k is $1 + 1 + 3 + 3 = 8$.

12. **Answer (D):** Notice that the first and second smallest divisors of n must be 1 and 2. Indeed, if the second smallest divisor of n is not 2, then n is odd, thus it is impossible for n to have an even divisor and furthermore, impossible for $f(n)$ to be divisible by 4.

    Also, the third smallest divisor of n must be either 3 or 4. If it were higher than 4, then both n and thus also $f(n)$ would not be divisible by 4. We now proceed with casework:

    - If the third smallest divisor of n is 3, then $3 \mid n$ and $f(n) = \frac{n}{3}$. For this to be divisible by 4, we simply require $4 \mid n$ on top of $3 \mid n$. Thus, in this case $12 \mid n$.
    - If the third smallest divisor of n is 4, we must note that $3 \nmid n$. Then, $f(n) = \frac{n}{4}$, and for this to be divisible by 4 we need $16 \mid n$ on top of $3 \nmid n$.

    There are $\frac{720}{12} = 60$ integers in the interval $[1, 720]$ satisfying the first case and similarly $\frac{720}{16} - \frac{720}{48} = 30$ integers satisfying the second case. Notice that there is no overlap between the two cases since the first case requires $3 \mid n$, but the second one forces $3 \nmid n$. Thus, the answer is $60 + 30 = 90$.

13. **Answer (C):** Let $P(x)$ be the polynomial and r, s, and t be its roots. We assume WLOG that the sum of the reciprocals of s and t add up to r, that is to say, $r = \frac{1}{s} + \frac{1}{t} \implies rst = s + t$. Using Vieta's formulas on $P(x)$, we have $r + s + t = \frac{11}{2}$ and $rst = \frac{a}{2}$. Thus, $\frac{a}{2} = \frac{11}{2} - r \implies a = 11 - 2r$. Then,

    $$P(x) = 2x^3 - 11x^2 + 18x - (11 - 2r).$$

    However, r is a root of $P(x)$, so $P(r) = 0$. We have $P(r) = 2r^3 - 11r^2 + 20r - 11 = 0$.

    By Rational Root Theorem, we can find that $r = 1$ is a solution. It turns out that the other two values of r are nonreal, which we must rule out, since each root of $P(x)$ is real. Since $a = 11 - 2r$, the value of a is 9.

    The other two roots of $P(x)$ are $\frac{3}{2}$ and 3.

14. **Answer (D):** Let $A_n(x) = (x-1)(x-2)\cdots(x-n)$. Jensen's collection of polynomials consists of $A_1(x), A_2(x), ..., A_{2021}(x)$.

    Consider the factor $(x-m)$. The factor appears in the polynomials $A_m, A_{m+1}, ..., A_{2021}$. If m is odd, the factor $(x-m)$ shows up in an odd number of polynomials. This means that no matter how we split up the two groups, we cannot cancel every factor of $(x-m)$ with $\frac{Q(x)}{P(x)}$. For example, consider $m = 2019$. The factor $(x - 2019)$ shows up in exactly 3 different polynomials. For all factors of $(x - 2019)$ to be cancelled in $\frac{Q(x)}{P(x)}$, we need $(x-2019)$ to appear in $Q(x)$ with the same multiplicity as $P(x)$. But this is impossible, since 3 is odd. Thus, every factor in the form $(x - m)$ for each odd integer m is forced to appear at least once in $\frac{Q(x)}{P(x)}$. This causes the lower bound of the degree of $\frac{Q(x)}{P(x)}$ to be 1011 to account for all odd integers from 1 to 2021.

    We can show that 1011 is achievable by letting $A_1, A_3, A_5, ..., A_{2021}$ be in the second group and letting $A_2, A_4, A_6, ..., A_{2020}$ be in the first group. Since $\frac{A_n(x)}{A_{n-1}(x)} = x - n$, the resulting polynomial $\frac{Q(x)}{P(x)}$ will equal $(x-1)(x-3)(x-5)\cdots(x-2021)$, which has degree 1011.

15. **Answer (B):** Note that by the definition of the floor and fractional part, $\lfloor x \rfloor + \{x\} = x$ for all real x. Thus,

$$\lfloor x \rfloor^2 - \{x\}^2 = (\lfloor x \rfloor + \{x\})(\lfloor x \rfloor - \{x\}) = x(\lfloor x \rfloor - \{x\}) = \frac{2020}{2021}x^2.$$

Therefore, $x = 0$ or $\lfloor x \rfloor - \{x\} = \frac{2020}{2021}x$. We find that $x = 0$ is also a solution to the latter equation, so we only consider solutions from the latter equation to avoid overcounting.

The equation is equal to $2021(\lfloor x \rfloor - \{x\}) = 2020(\lfloor x \rfloor + \{x\}) \implies \lfloor x \rfloor = 4041\{x\}$.

Because $0 \le \{x\} < 1$, we have $4041 \cdot 0 \le \lfloor x \rfloor < 4041 \cdot 1 \implies 0 \le \lfloor x \rfloor \le 4040$. For each value of $\lfloor x \rfloor$ in the interval $[0, 4040]$, we uniquely determine an appropriate value of $\{x\}$ through the equation $\lfloor x \rfloor = 4041\{x\}$. Thus, $n = 4041$. The requested remainder is 1.

16. **Answer (E):** Let E be the foot of the altitude from A to DC and F be the foot of the altitude from B to DC so that $EF = AB = 4$. By the symmetry of an isosceles trapezoid, $DE = FC$. Let $x = DE = FC$. In addition, let the altitude from B to AC intersect DC at G. Because BG, bisects the area of trapezoid $ABCD$, we require $2[BCG] = [ABCD]$. Let h be the height of the trapezoid and $k = CG$. Then, $[BCG] = \frac{1}{2} \cdot h \cdot k$ and $[ABCD] = h \cdot \frac{4+(2x+4)}{2}$. Plugging these two equations into $2[BCG] = [ABCD]$ and simplifying, we have $k = x + 4$. Thus, $CG = CE$, which implies $E = G$.

Since we are given BG is perpendicular to AC, we have BE is perpendicular to AC. Now, consider trapezoid $ABCE$ and let AC and BE intersect at X. If $\angle ABX = \theta$, then $\angle BEA = \angle BAX = 90^\circ - \theta$ and $\angle CAE = \theta$ as well as $\angle ACE = 90^\circ - \theta$. This implies $\triangle CEA \sim \triangle EAB$. Then, $\frac{AE}{EC} = \frac{AB}{AE} \implies \frac{h}{x+4} = \frac{4}{h}$. By right triangle $\triangle BFC$ with $BF = h$ and $FC = x$, we have $x^2 + h^2 = 25$. Solving the system of equations gives $x^2 + 4x - 9 = 0$ or $x = -2 + \sqrt{13}$. Then, $CD = 2x + 4 = 2\sqrt{13}$.

17. **Answer (B):** We must have the 5 values $|a_2 - a_1|, |a_3 - a_2|, ..., |a_6 - a_5|$ all be distinct. Due to the numbers available in the permutation, $|a_{k+1} - a_k|$ can only equal $1, 2, ..., 5$. Thus, as k ranges from 1 to 5, $|a_{k+1} - a_k|$ must equal each of $1, 2, ..., 5$ exactly once.

The absolute difference of 5 must be achieved, which only occurs if the 1 and 6 are adjacent to each other. WLOG, always assume the 1 comes directly before the 6. We will multiply by 2 for symmetry at the end. Now, the absolute difference of 4 must be achieved, which is only possible if either the 1 and 5 are adjacent or the 2 and 6 are adjacent (but not both).

Note that if $\pi = (a_1, a_2, ..., a_6)$ is a valid permutation, then $\pi' = (7-a_1, 7-a_2, ..., 7-a_6)$ is also a valid permutation. Thus, if the subsequence, $1, 6, 2$ appears in π, the reverse order of π' will contain $5, 1, 6$. This process works the other way around too (if π contains $5, 1, 6$, the reverse order of π' will contain $1, 6, 2$). Thus, we will assume π contains the subsequence $1, 6, 2$ and multiply by a factor of 2 at the end.

Now, the difference of 3 must be achieved by either 2 and 5 being adjacent or 1 and 4 being adjacent (it is clearly impossible for the 3 and 6 to be adjacent).

Case 1: 1 and 4 are adjacent.

We can list out that the only permutations that work are $(4, 1, 6, 2, 3, 5)$, $(5, 3, 4, 1, 6, 2)$, and $(3, 5, 4, 1, 6, 2)$. There are 3 permutations in this case.

Case 2: 2 and 5 are adjacent.

We can list out that the only permutations that work are $(4, 3, 1, 6, 2, 5)$, $(3, 1, 6, 2, 5, 4)$, and $(1, 6, 2, 5, 3, 4)$. There are 3 permutations in this case.

Thus, there are $2 \cdot 2 \cdot (3 + 3) = 24$ valid permutations.

18. **Answer (B):** Let $d(P, \triangle XYZ)$ denote the distance from a point P to the plane containing $\triangle XYZ$. We seek the value of $d(L, \triangle IJK)$. To find this distance, we will compute the volume of tetrahedron $IJKL$ in two ways, that is to say $[IJK] \cdot d(L, \triangle IJK) = [IKL] \cdot d(J, \triangle IKL)$.

    Clearly, $\triangle IJK$ is an equilateral triangle with side length $\sqrt{5^2 + 5^2} = 5\sqrt{2}$. Thus, $[IJK] = \frac{(5\sqrt{2})^2 \cdot \sqrt{3}}{4} = \frac{25\sqrt{3}}{2}$. Clearly, the plane containing $\triangle IKL$ is parallel to the plane containing $ABCD$ and the planes are separated by a distance of 5. Thus, $d(J, \triangle IKL) = 5$. Clearly, $KL = 5$ and the distance from I to the line containing KL is also 5. Thus, $[IKL] = \frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

    Then, $d(L, \triangle IJK) = \frac{5}{\sqrt{3}}$. The requested sum is $5 + 3 = 8$.

19. **Answer (A):** First, we will show that $\gcd(m, k) = 1$. Assume that for the sake of contradiction that $\gcd(m, k) = d > 1$. Then, since d divides k, we must have $m^n + n^m \equiv 0 \pmod{d}$. But d divides m as well, so $n^m \equiv 0 \pmod{d}$. Clearly, if we let $n = 1$ (which is relatively prime to k no matter what k is) the congruence will not be satisfied regardless of what m is unless $d = 1$. However, this contradicts $d > 1$. Thus, $\gcd(m, k) = 1$.

    Again, since 1 is relatively prime to k no matter what k is, take $n = 1$ to get $m + 1 \equiv 0 \pmod{k} \implies m \equiv -1 \pmod{k}$.

    Then, we can take $n = m$ (which is allowed no matter what k is since $\gcd(m, k) = 1$) to get $2m^m \equiv 2(-1)^m \equiv 0 \pmod{k} \implies k \mid 2 \implies k = 2$. We can easily check that $k = 2$ works, so the requested sum is 2.

20. **Answer (A):** We will count the complement; we will find the probability that $f(n)$ is odd.

    Case 1: n has a digit equal to 0

    Since 0 is the smallest digit possible, we need some odd digit to appear more times than the 0 in n. This is only possible if one digit is 0 and the other two are the same odd digit. There are 5 ways to choose which odd digit we use and 2 ways to choose which digit the 0 occupies in the 3-digit number (we can't have leading 0s). Thus, there are 10 integers n in this case.

    Case 2: all of digits of n are nonzero

    We will perform subcases based on how often the digit equal to $f(n)$ appears among the digits of n.

    Case 2.1: the digit equal to $f(n)$ appears once among the digits of n

    In this case, no digit may appear more than once among the digits of n. If o is the odd digit such that $f(n) = o$, then we must choose two distinct digits greater than o. If $o = 1$, there are $\binom{8}{2}$ ways to do so. If $o = 3$, there are $\binom{6}{2}$ ways to do so. Continuing this pattern, there are

    $$3! \cdot \left(\binom{8}{2} + \binom{6}{2} + \binom{4}{2} + \binom{2}{2}\right) = 300$$

    integers n in this subcase, where we multiply by 3! to account for permuting the digits.

    Case 2.2: the digit equal to $f(n)$ appears twice among the digits of n

    Then, as long as the digit appearing twice is odd, $f(n)$ will be odd no matter what the remaining digit is (as long as it is not 0). There are 5 ways to select the odd digit appearing twice, 8 ways to select another nonzero digit, and 3 ways to permute the digits. Thus, there are $5 \cdot 8 \cdot 3 = 120$ integers n in this subcase.

Case 2.3: the digit equal to $f(n)$ appears thrice among the digits of n

Clearly, $n \in \{111, 333, 555, 777, 999\}$ for a total of 5 integers n in this subcase.

Across all cases, there are $10 + 300 + 120 + 5 = 435$ three-digit integers such that $f(n)$ is odd. Since this is the complement, there are $900 - 435 = 465$ integers such that $f(n)$ is even. Thus, the desired probability is $\frac{465}{900} = \frac{31}{60}$.

21. **Answer (C):** Obviously, $a_1 \geq 0$ and $d \geq 0$ (or else the sequence will inevitably become negative). Then, $a_2 = a_1 + d$ and $a_5 = a_1 + 4d$. Plugging these two equations into $a_5^2 - a_2^2 = 432$ and simplifying, we have

$$2a_1d + 5d^2 = 144 \implies a_1 = \frac{72}{d} - \frac{5d}{2}.$$

Since $a_6 = a_1 + 5d$, we have $a_6 = \frac{5d}{2} + \frac{72}{d}$. Recall that $d \geq 0$, which implies both $\frac{5d}{2}$ and $\frac{72}{d}$ are nonnegative, so we may apply the AM-GM inequality. Thus,

$$\frac{a_6}{2} = \frac{\frac{5d}{2} + \frac{72}{d}}{2} \geq \sqrt{\frac{5d}{2} \cdot \frac{72}{d}} = \sqrt{180} \implies a_6 \geq 2\sqrt{180} \implies a_6^2 \geq 720.$$

We will quickly check that there exists nonnegative values of d and a_1 so that $a_6^2 = 720$ is achievable. By the AM-GM equality case, we have $\frac{5d}{2} = \frac{72}{d} \implies d = \frac{12}{\sqrt{5}}$. Then, with $a_6 = a_1 + 5d$, we have $a_1 = 0$, which is valid. Thus, the minimum value of a_6^2 is 720.

22. **Answer (A):** Let X be the foot of the altitude from Q to line PA and Y be the foot of the altitude from Q to CD. Let s be the side length of the hexagon. We are given that $AQ = \frac{s}{2}$. Clearly, $\triangle QAX$ is a $30-60-90$ triangle because $\angle PAQ = 120° \implies \angle QAX = 60°$, so $QX = \frac{s\sqrt{3}}{4}$. By the properties of a regular hexagon, the distance between parallel lines AF and CD is given by $s\sqrt{3}$, which implies $QY = \frac{3s\sqrt{3}}{4}$.

Now, let $\angle PQX = \theta$. Since $\angle PQR = 90°$, we have $\angle RQY = 90° - \theta$. Then, $\triangle RYQ \sim \triangle QXP$ by AAA similarity. Thus, $\frac{QR^2}{QY^2} = \frac{PQ^2}{PX^2}$. We have $QR^2 = 16$, $QY^2 = \frac{27s^2}{16}$, $PQ^2 = 9$, and $PX^2 = PQ^2 - XQ^2 = 9 - \frac{3s^2}{16}$. Therefore, $s^2 = \frac{768}{97}$. The requested sum is $768 + 97 = 865$.

23. **Answer (E):** Recall that if the prime factorization of n is $\prod_{m=1}^{r}(p_m^{e_m})$, then $\varphi(n) = \prod_{m=1}^{r}(p_m^{e_m-1}(p_m - 1))$. First, note that $5^n + 1 \equiv 2 \pmod 4$, implying that $4 \nmid \varphi(n)$. If n is divisible by more than two odd primes p_1 and p_2, then the product for $\varphi(n)$ contains the factors $(p_1 - 1)$ and $(p_2 - 1)$, which are both even, implying $4 \mid \varphi(n)$. In addition, we can clearly see that if n is divisible by both 4 and an odd prime, $4 \mid \varphi(n)$. Therefore, it is necessary (but not sufficient) for n to be in the form $n = p^k$ or $n = 2p^k$ for some prime p and nonnegative integer k (we ignore $n = 1$, as the given range does not include 1).

Case 1: If $n = p^k$, then $p^{k-1}(p-1) \mid 5^{p^k} + 1$. If $k > 1$, we have

$$5^{p^k} + 1 \equiv 5 + 1 \equiv 6 \equiv 0 \pmod p \implies p = 2, 3,$$

which gives the solutions $n = 2, 4$ and $n = 3^k$ for all k. We can easily check these work by LTE.

Otherwise, $k = 1$ and $p > 3$; let q be any prime dividing $p - 1$ (and in turn, $5^p + 1$) so that

$$5^p \equiv -1 \pmod q \implies 5^{2p} \equiv 1 \pmod q \implies \text{ord}_q(5) \mid 2p.$$

But by Fermat's Little Theorem and properties of order, $\text{ord}_q(5) \mid q - 1$. Because $p > 3$, $p - 1$ is composite and q being prime implies $q < p - 1 \implies q - 1 < p - 2$. Thus,

$\text{ord}_q(5) \le q-1 < p-2 < p$. This means $\text{ord}_q(5)$ cannot contain a factor of p, as p is prime. We have

$$\text{ord}_q(5) \mid 2 \implies 5^2 \equiv 1 \pmod{q} \implies q \mid 24 \implies q = 2, 3.$$

This means $p = 2^a 3^b + 1$ for some nonnegative integers $a \in \{0, 1\}$ and b. We can't have $a \ge 2$ or $p-1$ is divisible by 4, implying $4 \mid \varphi(n)$. Since $p > 3 \implies a = 1$, we have $p = 2 \cdot 3^b + 1$. Recall that $\varphi(p) = p-1$ for any prime p. Plugging this back in, we have $2 \cdot 3^b \mid 5^{2 \cdot 3^b + 1} + 1$. Obviously, 2 divides the expression, but $\nu_3(5^{2\cdot 3^b+1}+1) = 1$ by LTE, so $b \le 1$ and $n = p \in \{3, 7\}$. We can easily check that 7 is a new solution which works.

Case 2: If $n = 2p^k$, then $p^{k-1}(p-1) \mid 25^{p^k} + 1$. If $k > 1$, we have

$$25^{p^k} + 1 \equiv 25 + 1 \equiv 26 \equiv 0 \pmod{p} \implies p = 2, 13,$$

but if $13 \mid n$ then $4 \mid 12 \mid \varphi(n)$, contradiction. We already covered when n is a power of 2, so there are no new solutions.

Otherwise, $k = 1$; let q be any prime dividing $p - 1$. We have

$$q \mid p-1 \mid 25^p + 1 \implies \text{ord}_q(25) \mid 2p.$$

Again, since $\text{ord}_q(25) \mid q-1 \implies \text{ord}_q(25) \le q-1 < p$ (implying $\text{ord}_q(25)$ can't contain a factor of p), we get

$$\text{ord}_q(25) \mid 2 \implies q \mid 624 \implies q = 2, 3, 13.$$

As before, $13 \nmid n \implies q = 2, 3$, so using similar reasoning to the first case we get $p = 2 \cdot 3^b + 1$. Then, $2 \cdot 3^b \mid 25^{2\cdot 3^b+1} + 1$. However, 3 clearly will never divide $25^{2\cdot 3^b+1}+1$, implying $b = 0$ and thus, $p = 3$, giving the new solution $n = 6$, which works.

Thus, all possible n are $n = 2, 4, 6, 7$ and $n = 3^k$ for positive k. Using $1 < n < 100$ gives $n = 2, 3, 4, 6, 7, 9, 27, 81$, so the answer is 8.

24. **Answer (E):** Let $p_j = \overline{a_1a_2a_3a_4}$ and $p_i = \overline{b_1b_2b_3b_4}$, where the a_k and b_k represent digits. Define $p_k = b_k - a_k$ for $1 \le k \le 4$. Then, $d_{(i,j)} = 1000p_1 + 100p_2 + 10p_3 + p_4$ with $-3 \le p_k \le 3$ for $1 \le k \le 4$. Note that $p_1 + p_2 + p_3 + p_4 = (a_1 + a_2 + a_3 + a_4) - (b_1 + b_2 + b_3 + b_4) = 0$. Then, $p_1 + 3, p_2 + 3, p_3 + 3, p_4 + 3$ sum to 12 and each range from 0 to 6.

There are $\binom{15}{3}$ ways to choose four nonnegative integers (n_1, n_2, n_3, n_4) that sum to 12 by Stars and Bars. However, since none of the integers can be 7 or greater, we assume $n_1 \ge 7$. Thus, can write $(n_1' + 7) + n_2 + n_3 + n_4 = 12$ for some nonnegative integer n_1' and so there are $\binom{8}{3}$ ways to choose the four numbers. There are four ways to choose which is at least 7 (no overlap), so we have $\binom{15}{3} - 4\binom{8}{3} = 455 - 4 \cdot 56 = 231$ possible values for $d_{(i,j)}$. However, for some of these values, there does not exist an (i, j) that equals that value. In addition, some of these values may be nonpositive. Call the tuple (p_1, p_2, p_3, p_4) achievable if there exists an (i, j) such that $d_{(i,j)}$ produces that set and unachievable otherwise. Clearly, all permutations of an achievable tuple are achievable by permuting the digits in a linked fashion. For example, $1243 - 1234$ produces $(0, 0, 1, -1)$, but we can also produce $(0, 1, -1, 0)$ by moving the digits around as such: $1432 - 1342$.

Case 1: The p_k set contains 0

Case 1a: It contains $(0, 0)$. Then, $\{p_1, p_2, p_3, p_4\}$ is a permutation of $\{0, 0, p, -p\}$, where $0 \le p \le 3$. Obviously, $p = 0$ is forbidden because that implies p_i and p_j are the same, which is not allowed. However, for the other values, take $(a_3, a_4) = (p+1, 1), (b_3, b_4) = (1, p+1), a_1 = b_1$, and $a_2 = b_2$.

Case 1b: It contains $(0, 1)$ but not $(0, 0)$ ($(0, -1)$ is identical by negating). The p_k set must be a permutation of $\{0, 1, 1, -2\}$ or $\{0, 1, 2, -3\}$. However, both are produced by $1342 - 1234$ and $3241 - 3124$.

Henceforth, we can assume the p_k set does not contain a 0.

Case 2: The p_k set contains ± 1

By negation, we will assume that the set must contain 1 and may or may not contain -1. Clearly, the p_k set must have a negative number or the sum of all the p_k cannot be 0.

Case 2a: it contains $(1, -1)$. Then, the set is a permutation of $\{1, -1, p, -p\}$. $p = 1$ is produced by $2143 - 1234$, $p = 2$ from $2341 - 1423$, and $p = 3$ from $3241 - 2314$.

Case 2b: it contains $(1, -2)$ but not $(1, -1)$. Then, it must be $(1, -2, 3, -2)$ in some order produced by $3241 - 2413$.

Case 2c: it contains $(1, -3)$ but neither of the previous cases. Then, it must be $(1, -3, 1, 1)$ produced by $2143 - 1432$.

Henceforth, we can assume the p_k set does not contain 0 or ± 1.

Case 3: The p_k set contains ± 2

By negation, we will assume that the set must contain 2 and may or may not contain -2. The set is a permutation of $\{2, -2, 2, -2\}, \{2, -2, 3, -3\}$, or $\{3, -3, 3, -3\}$. The first one is possible by $3142 - 1324$. However, the others are unachievable. A subset of $\{3, -3\}$ forces $(a_x, b_x) = (4, 1)$ and $(a_y, b_y) = (1, 4)$ for some x and y. However, it becomes impossible to fulfill the other differences in the set, as the remaining numbers, 2 and 3, have no way of making a difference of ± 2 or ± 3.

We found that all the unachievable tuples are permutations of $\{0, 0, 0, 0\}, \{2, -2, 3, -3\}$, and $\{3, -3, 3, -3\}$. There is 1 way to permute the first set, 24 ways to permute the second set, and 6 ways to permute the third set. Thus, there are now $231 - 31 = 200$ achievable tuples. However, half of these yield negative values; we can easily switch $\overline{a_1a_2a_3a_4}$ and $\overline{b_1b_2b_3b_4}$ to control the sign of the difference if necessary. Thus, $d_{(i,j)}$ assumes $200 \cdot \frac{1}{2} = 100$ distinct values.

25. **Answer (B):** We take 3 cases.

Case 1: If $AB \parallel CD$, note that $[ABCD] = [ABD] + [DBC] = [ABC] + [DBC] = 152$.

Otherwise, let $P = AB \cap CD$. Set $PB = x$ and $PC = y$ and $\angle BPC = \theta$.

Case 2: Suppose B is between P and A. By area ratios, $\frac{[ABC]}{[APC]} = \frac{AB}{AP}$. We have $[APC] = \frac{1}{2} \sin\theta \cdot (x+8) \cdot y$ by Law of Sines, which implies $[ABC] = 80 = \frac{1}{2} \cdot 8 \cdot (y \sin\theta)$. Similarly, via the area ratios $\frac{[DBC]}{[DBP]} = \frac{CD}{PD}$, we can derive $\frac{1}{2} \cdot 11 \cdot (x \sin\theta) = 72$. Then, we have

$$\begin{aligned}
[ABCD] = [PAD] - [PBC] &= \frac{1}{2} \cdot \sin\theta \cdot ((x+8)(y+11) - xy) \\
&= \frac{1}{2} \cdot \sin\theta \cdot (8y + 11x + 88) \\
&= 80 + 72 + 44 \sin\theta \\
&\geq 152.
\end{aligned}$$

Case 3: Suppose A is between P and B. Again, we use area ratios $\frac{[ABC]}{[PBC]} = \frac{BA}{BP}$ and $\frac{[DBC]}{[PBC]} = \frac{CD}{CP}$ to derive $\frac{1}{2} \cdot 8 \cdot (y \sin\theta) = 80$ and $\frac{1}{2} \cdot 11 \cdot (x \sin\theta) = 72$. Then, we have

$$\begin{aligned}[ABCD] = [PBC] - [PAD] &= \frac{1}{2} \cdot \sin\theta \cdot (xy - (x-8)(y-11)) \\ &= \frac{1}{2} \cdot \sin\theta \cdot (8y + 11x - 88) \\ &= 80 + 72 - 44\sin\theta \\ &\geq 152 - 44 \\ &= 108.\end{aligned}$$

Equality occurs when $\theta = 90^\circ$, which is clearly possible.

Thus, the minimum is 108.