1. E
2. C
3. A
4. D
5. B
6. A
7. C
8. E
9. D
10. D
11. B
12. D
13. C
14. B
15. B
16. C
17. C
18. D
19. E
20. C
21. A
22. A
23. A
24. D
25. B

1. What is $\sqrt{6^4+2\cdot 6^2\cdot 8^2+8^4}$?

   (A) 10 **(B)** $10\sqrt{3}$ **(C)** $2\sqrt{1398}$ **(D)** $4\sqrt{481}$ **(E)** 100

   Factor the expression and then recognize the Pythagorean triple:

$$\sqrt{(6^2+8^2)^2}$$
$$6^2+8^2$$
$$10^2 = \boxed{\textbf{(E) } 100}$$

2. Joe wants to buy a shirt. He notices that the original price is \$25, but it is on sale for 20% off. The tax is 8% of the sale price. How much money does Joe need to pay for the shirt?

   (A) \$16.00 **(B)** \$18.40 **(C)** \$21.60 **(D)** \$22.00 **(E)** \$36.00

   Multiply the original price of \$25 by 0.8 for the sale price and then multiply by 1.08 to include tax:

$$\$25.00\cdot 0.8\cdot 1.08 = \boxed{\textbf{(C) } \$21.60}$$

3. Two fair six-sided dice numbered with integers from 1 through 6 are tossed. What is the probability that the sum and the product of the two values that show up on top of the dice are both odd?

   (A) 0 **(B)** $\frac{1}{16}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

   Recognize that if the sum of two integers is odd, one of the integers is even and the other is odd. If an odd integer is multiplied by an even integer, the product is always even. Therefore, the answer is $\boxed{\textbf{(A) } 0}$.

4. A picture frame is made by cutting out a $7\times 10\times 1$ rectangular prism from the center of a $9\times 12\times 1$ rectangular prism. What is the surface area of the picture frame?

   (A) 128 **(B)** 136 **(C)** 144 **(D)** 152 **(E)** 160

We can visualize that the top and bottom both have an area of 38. The outer sides of the picture frame have a total area of $9+9+12+12=42$. The inner sides have a total area of $7+7+10+10=34$. Add the areas for an answer of $38+38+42+34=\boxed{\textbf{(D) } 152}$.

5. Manuel chooses a random real number from the interval $[0, 2018]$. Nicole, Olivia, and Peter try to guess the number that Manuel chose. Their guesses are also random real numbers from the interval $[0, 2018]$. What is the probability that Nicole's guess is too low, Olivia's guess is too high, and Peter's guess is also too high?

   (A) $\frac{1}{16}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

   Call the numbers that Manuel, Nicole, Olivia, and Peter chose m, n, o, and p respectively. We can find that there are $4! = 24$ permutations of m, n, o, and p. There are two permutations that satisfy the conditions stated in the problem:

$$n < m < o < p$$
$$n < m < p < o$$

   The answer is $\frac{2}{24}$ which is simplified to $\boxed{\textbf{(B) } \frac{1}{12}}$.

6. n is a positive integer. The decimal representation of $n!$ ends in r zeros. The decimal representation of $n!+(n-1)!$ ends in s zeros. Which of the following values of n causes s to be greater than r?

   (A) 2014 **(B)** 2015 **(C)** 2016 **(D)** 2017 **(E)** 2018

   Rewrite the expression $n!+(n-1)!$ as $n(n-1)!+(n-1)!$. Then, factoring the expression results in $(n+1)(n-1)!$. We can see that 2015(2013!) is divisible by a larger power of 10 than $2014! = 2014(2013!)$ because the additional factor of 5 can multiply by a factor of 2. The answer is $\boxed{\textbf{(A) } 2014}$.

7. a and b are positive integers in the equation $a^2 + 4b = b^2 + a$, and $a > 1$. What is the only possible value of $a + b$?

   (A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 24

Move the terms with a to one side and the terms with b to the other side: $a^2-a=b^2-4b$. Factor both sides: $a(a-1)=b(b-4)$. We cans see that $a(a-1)=b(b-4)$ must have at least 2 pairs of factors. The factors in one pair must have a difference of 1 and the factors in the other pair must have a difference of 4. By inspection, we can see that $12=3\cdot 4=2\cdot 6$. Therefore, the answer is $4+6=\boxed{\textbf{(C) } 10}$.

8. Each element of the set $\{1,2,3,\ldots,9\}$ is placed into either set X or set Y but not both. All of the elements in set X make up a single arithmetic progression with at least three elements. All of the elements in set Y also make up a single arithmetic progression with at least three elements. For example, a possibility is $X=\{1,2,3\}$ and $Y=\{4,5,6,7,8,9\}$. How many distinct sets can X be equal to?

   (A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

   We can list out the possibilities as follows:

$$\begin{gathered}
X=\{1,2,3\},\ Y=\{4,5,6,7,8,9\}\\
X=\{1,2,3,4\},\ Y=\{5,6,7,8,9\}\\
X=\{1,2,3,4,5\},\ Y=\{6,7,8,9\}\\
X=\{1,2,3,4,5,6\},\ Y=\{7,8,9\}\\
X=\{4,5,6,7,8,9\},\ Y=\{1,2,3\}\\
X=\{5,6,7,8,9\},\ Y=\{1,2,3,4\}\\
X=\{6,7,8,9\},\ Y=\{1,2,3,4,5\}\\
X=\{7,8,9\},\ Y=\{1,2,3,4,5,6\}\\
X=\{1,3,5,7,9\},\ Y=\{2,4,6,8\}\\
X=\{2,4,6,8\},\ Y=\{1,3,5,7,9\}
\end{gathered}$$

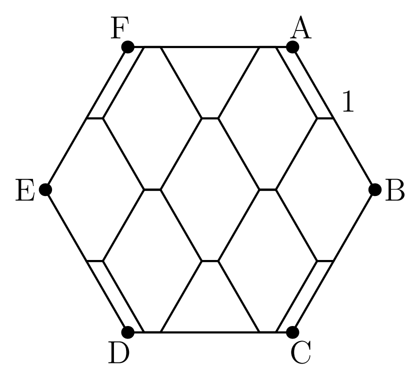
   There are $\boxed{\textbf{(E) } 10}$ possibilities.

9. Ashley and Ben are playing a game. At the beginning of the game, $n=1$. Starting with Ashley, the players then take turns multiplying n by an integer between 2 and 9, inclusive. The first player to get a product of at least 100 wins. If neither player makes any mistakes, how many different integers can Ashley multiply n by on her first turn in order to win?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

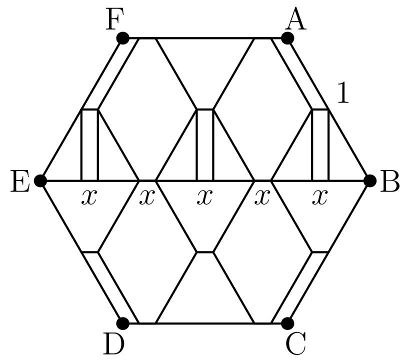
We can see that Ashley is able to win if it is her turn and $12 \leq n \leq 99$. When it is Ben's turn and $6 \leq n \leq 11$, Ben is forced to satisfy $12 \leq n \leq 99$, causing Ashley to win. If Ashley chooses 6, 7, 8, or 9, she will win. If Ashley chooses 1, 2, 3, 4, or 5, Ben is able to win by multiplying so that $6 \leq n \leq 11$, forcing Ashley to satisfy $12 \leq n \leq 99$. The answer is $\boxed{\textbf{(D)}\ 4}$.

10. Seven congruent equilangular hexagons are drawn inside regular hexagon $ABCDEF$ as shown. If $\overline{AB} = 1$, the perimeter of one of the smaller hexagons is equal to $\frac{a}{b}$ where a and b are relatively prime positive integers. What is $a + b$?



(A) 10 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 18

Notice that four sides of each of the smaller hexagons have a length of $\dfrac{1}{2}$. We can then draw in $\overline{EB}$. Draw a line from the midpoint of $\overline{EF}$ down to $\overline{EB}$ so that a 30-60-90 triangle with a hypotenuse along $\overline{EF}$ is formed. We can calculate that the shorter leg of the triangle has a length of $\dfrac{1}{4}$. Call the length of one of the shorter sides of the small hexagons x. By drawing five other 30-60-90 triangles (for a total of six, with two in each small hexagon that $\overline{EB}$ passes through), we can find that $EB = 6 \cdot \dfrac{1}{4} + 5x$.



Note that $EB = 2$. Solving for x in the equation $2 = 6 \cdot \frac{1}{4} + 5x$ results in $x = \frac{1}{10}$. Find the perimeter by calculating $4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{10} = \frac{11}{5}$. The answer is $11 + 5 = \boxed{\textbf{(D) } 16}$.

Alternate solution:

Notice that four sides of each of the smaller hexagons have a length of $\frac{1}{2}$. Set x to the length of one of the remaining sides. Set y to the distance between the two vertices that are surrounded by sides with lengths of $\frac{1}{2}$. Also, note that $EB = 2$. Solve for x in this system of equations:

$$2x + 3y = 2$$
$$4x + y = 1$$

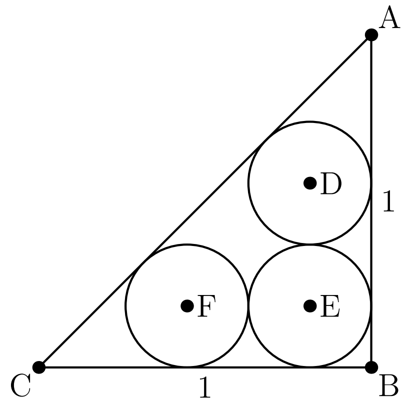
The result is $x = \frac{1}{10}$. Find the perimeter by calculating $4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{10} = \frac{11}{5}$. The answer is $11 + 5 = \boxed{\textbf{(D) } 16}$.

11. How many numbers from the set $\{8^2, 18^2, 28^2, \dots, 2018^2\}$ have a remainder of 64 when divided by 100?

(A) 40 **(B)** 41 **(C)** 403 **(D)** 404 **(E)** 405

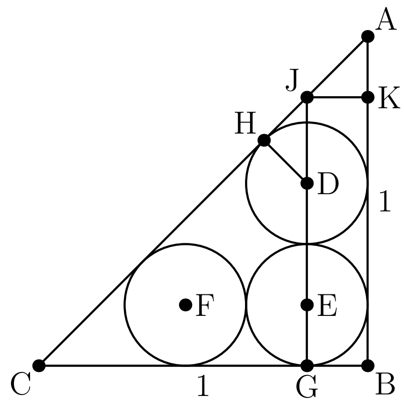
Rewrite the numbers as $(10n + 8)^2$. Multiply it out to $100n^2 + 160n + 64$. We can see that if $160n$ is a multiple of 100, $(10n + 8)^2$ has a remainder of 64 when divided by 100. $160n$ is a multiple of 100 if n is a multiple of 5. Therefore, the numbers $\{8^2, 58^2, 108^2, \ldots, 2008^2\}$ have a remainder of 64. There are $\boxed{\textbf{(B)}\ 41}$ numbers in that set.

12. In triangle ABC, $AB = BC = 1$ and $m\angle ABC = 90°$. Non-overlapping congruent circles D, E, and F are placed inside triangle ABC so that each circle is internally tangent to triangle ABC at two different points. If circle E is externally tangent to circles D and F, what is the radius of circle D?



(A) $\dfrac{2-\sqrt{2}}{4}$ **(B)** $\dfrac{5-\sqrt{2}}{23}$ **(C)** $3-2\sqrt{2}$ **(D)** $\dfrac{4-\sqrt{2}}{14}$ **(E)** $\dfrac{3-\sqrt{2}}{7}$

Add point G at the point tangency between circle E and $\overline{BC}$. Add point H at the point of tangency between circle D and $\overline{AC}$. Add point J on $\overline{AC}$ so triangle DHJ is a 45-45-90 triangle, and points J, D, E, and G are collinear. Add point K on $\overline{AB}$ so that $\angle AKJ = 90°$.



Let r represent the radius of circle D. $DJ = \sqrt{2}r$ so $GJ = (3+\sqrt{2})r$. Also, $BGJK$ is a rectangle so $BK = (3+\sqrt{2})r$. $JK = r$ and $AK = JK$ so $AK = r$. $AK+BK = AB = 1$ so $AB = (4+\sqrt{2})r = 1$. Solving for r results in $\boxed{\textbf{(D) } \frac{4-\sqrt{2}}{14}}$.

13. A three-digit positive integer n is called *triskaidekaphillic* if it satisfies at least one of the following conditions:

- n is evenly divisible by 13.
- n contains 13 in its digits. For example, 135 and 513 both satisfy this condition, but 153 and 531 don't.

How many three-digit positive integers are *triskaidekaphillic*?

(A) 78 **(B)** 86 **(C)** 87 **(D)** 94 **(E)** 95

$104 = 13 \cdot 8$ and $988 = 13 \cdot 76$ so there are 69 three-digit positive integers that are evenly divisible by 13. The numbers 131, 132, 133, $\ldots$, 139 are not divisible by 13 because 130 is divisible by 13. That means we can add them to 69: $69 + 9 = 78$. We can also find that the numbers 113, 213, 313, $\ldots$, 913 are not divisible by 13 because the numbers 100, 200, 300, $\ldots$, 900 are not divisible by 13. Add the numbers to 78: $78 + 9 = \boxed{\textbf{(C) } 87}$.

14. 6 students each took a 10-point science quiz. Each student received a score within the set $\{7, 8, 9, 10\}$, and each score in the set $\{7, 8, 9, 10\}$ was received by at least one

student. In how many ways could the students have scored?

(A) 1320 **(B)** 1560 **(C)** 2400 **(D)** 2640 **(E)** 5760

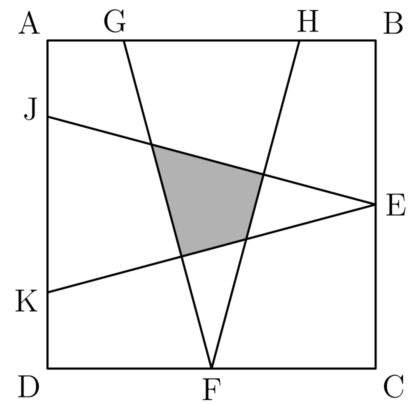
Notice that there are two possible cases when considering tied scores: either three students tie for the same score or two students tie for one score and two students tie for a different score.

First, we can consider the case of two students tying for one score and two students tying for a different score. Choose four students out of six, one with each score in the set. $\binom{6}{4} = 15$. Now, arrange the four students in any order. There are $4! = 24$ possible orders. Now, one of the remaining students can have any score and the other student can have any remaining score. There are $4 \cdot 3 = 12$ ways for this to happen. Notice that it is possible to choose different students when initially choosing four students. Choosing either student works if two students are tied, so divide the result by 4. The total number of possibilities for this case is $\frac{15 \cdot 24 \cdot 12}{4} = 1080$.

Next, we can consider the case of three students tying for the same score. Choose four students out of six again. There are 15 possible combinations. Now, arrange the four students again. There are 24 possible orders. The two other students must be tied for the same score. There are 4 possible scores. Divide by 3 because initially choosing any student yields the same result if the students are tied. The total number of possibilities for this case is $\frac{15 \cdot 24 \cdot 4}{3} = 480$.

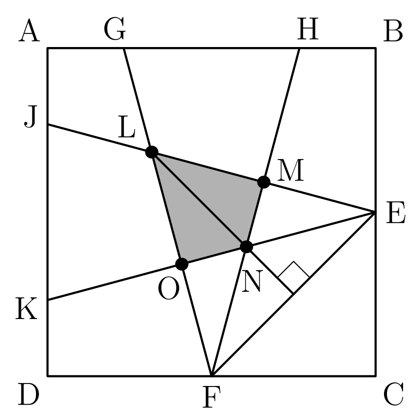
Add the possibilities for the two cases together: $1080 + 480 = \boxed{\textbf{(B) } 1560}$.

15. In the diagram shown below, square $ABCD$ has a side length of 10. Point E is the midpoint of $\overline{BC}$ and point F is the midpoint of $\overline{CD}$. If $m\angle EJD = m\angle EKA = m\angle FGB = m\angle FHA = 75°$, the area of the shaded quadrilateral is equal to $\frac{r\sqrt{t}}{s}$ where r, s, and t are positive integers, r and s are relatively prime, and t is not evenly divisible by the square of any integer greater than 1. What is $r + s + t$?



(A) 31 **(B)** 34 **(C)** 37 **(D)** 43 **(E)** 49

Call the intersection of $\overline{EJ}$ and $\overline{FG}$ point L, the intersection of $\overline{EJ}$ and $\overline{FH}$ point M, the intersection of $\overline{EK}$ and $\overline{FH}$ point N, and the intersection of $\overline{EK}$ and $\overline{FG}$ point O. The sum of the interior angles in a triangle is always $180°$, so $m\angle GFH = m\angle JEK = 30°$. $\overline{EJ}$ is perpendicular to $\overline{FH}$ because the slopes of the two line segments are negative reciprocals. Also, $\overline{EK}$ is perpendicular to $\overline{FG}$. Using this information, we can find that $m\angle FNO = m\angle ENM = 60°$. That means $m\angle MNO = 120°$ and $m\angle MLO = 60°$. Now, draw in $\overline{EF}$.



Triangle LFE is equilateral. We can verify this by seeing that $m\angle DFG + m\angle LFE + m\angle EFC = 180°$ if $m\angle LFE = 60°$. We can calculate that the area of triangle LFE is

$\dfrac{25\sqrt{3}}{2}$ because the base is $5\sqrt{2}$ and the height is $\dfrac{5\sqrt{6}}{2}$. Drawing an altitude from point L to $\overline{EF}$ splits triangle LFE into six congruent parts. Two out of the six parts make up quadrilateral $LMNO$ so the area of quadrilateral $LMNO$ is $\dfrac{1}{3}$ the area of triangle LFE. The area of quadrilateral $LMNO$ is $\dfrac{25\sqrt{3}}{6}$, so the answer is $25+3+6 = \boxed{\textbf{(B) } 34}$.

16. Anna and Beth are 10 kilometers apart from each other on a road. They both start walking toward each other. Anna walks at 4 kilometers per hour, and Beth walks at 5 kilometers per hour. After a while, Anna runs at 10 kilometers per hour, while Beth still walks at 5 kilometers per hour. Anna and Beth both stopped when they met each other. If the distance that Anna walked is equal to the distance that Anna ran, the total distance that Anna traveled in kilometers is equal to $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m+n$?

(A) 11 **(B)** 17 **(C)** 19 **(D)** 29 **(E)** 41

We can make a system of equations based on speed $= \dfrac{\text{distance}}{\text{time}}$. Let d be the distance that Anna walked or the distance that Anna ran (Both are equal.). Let t_w be the time that Anna walked, and let t_r be the time that Anna ran. These three equations are needed:

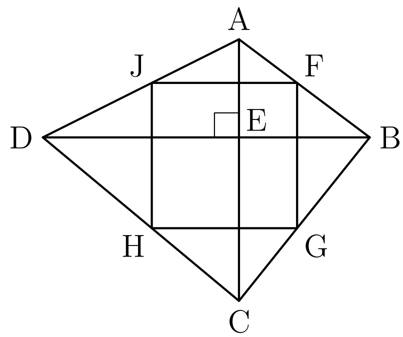
$$4 = \frac{d}{t_w}$$
$$10 = \frac{d}{t_r}$$
$$5 = \frac{10-2d}{t_w+t_r}$$

Solving for $2d$ results in $2d = \dfrac{16}{3}$. The answer is $16+3 = \boxed{\textbf{(C) } 19}$.

17. The diagonals of quadrilateral $ABCD$ are perpendicular to each other. Let point E be the intersection of the diagonals. If $AE = 3$, $BE = 4$, $CE = 5$, and $DE = 6$, the side length of a square inscribed in quadrilateral $ABCD$ with two sides parallel to $\overline{AC}$ and the other two sides parallel to $\overline{BD}$ is equal to $\frac{a}{b}$ where a and b are relatively prime positive integers. What is $a+b$?

(A) 37 **(B)** 39 **(C)** 49 **(D)** 61 **(E)** 155

Call the vertex on $\overline{AB}$ point F, the vertex on $\overline{BC}$ point G, the vertex on $\overline{CD}$ point H, and the vertex on $\overline{DA}$ point J.



Notice that triangle JAF is similar to triangle DAB, triangle FBG is similar to triangle ABC, triangle GCH is similar to triangle BCD, and triangle HDJ is similar to triangle CDA. We can calculate that $\frac{4}{10}$ of $\overline{JF}$ is within triangle ABE. Similarly, $\frac{3}{8}$ of $\overline{FG}$ is within triangle ABE. Make point E $(0,0)$. The slope of a line passing through point E and point F is $\frac{\frac{3}{8}}{\frac{4}{10}} = \frac{15}{16}$. Make this system of equations:

$$y = \frac{15}{16}x$$
$$y = -\frac{3}{4}x + 3$$

Solving for x results in $x = \frac{16}{9}$. Multiply $\frac{16}{9} \cdot \frac{10}{4} = \frac{40}{9}$ for an answer of $40+9 = \boxed{\textbf{(C) } 49}$.

18. How many integer values of n between 1 and 100, inclusive, cause the value of the following expression to be an integer?

$$\frac{\sqrt[2]{n^{3n}}}{\sqrt[3]{n^{2n}}}$$

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

First, simplify the expression to $\sqrt[6]{n^{5n}}$. We can see that $\sqrt[6]{n^{5n}}$ is an integer if n is the 6th power of an integer or a multiple of 6. In addition, n can be a multiple of 2 and the cube of an integer or a multiple of 3 and the square of an integer.

Calculate that there are 16 multiples of 6 between 1 and 100, inclusive. There are 2 numbers equal to the 6th powers of integers: $1^6 = 1$ and $2^6 = 64$. There are 2 numbers that are multiples of 3, the squares of integers, and are not previously included: 9 and 81. There is 1 number that is a multiple of 2, the cube of an integer, and is not previously included: 8. Adding together the calculations results in an answer of $16 + 2 + 2 + 1 = \boxed{\textbf{(D) } 21}$.

19. In the xy-coordinate plane, three distinct points with integer x- and y- coordinates are randomly chosen within the area bounded by $-1 \le x \le 1$ and $-1 \le y \le 1$. What is the probability that the three points, when connected, form a triangle with an area of 1?

(A) $\frac{4}{21}$ **(B)** $\frac{5}{21}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{8}{21}$

We can see that there are 3 ways to obtain an area of 1: a right triangle with leg lengths of 1 and 2, an obtuse triangle with side lengths of 1, $\sqrt{5}$, and $2\sqrt{2}$, or an isosceles right triangle with leg lengths of $\sqrt{2}$.

To calculate the number of possibilities for a right triangle with leg lengths of 1 and 2, we can see that we need a 1×2 "rectangle" of 4 points and we need to then choose 3 of them. There are 4 ways to do this, and there are 4 total "rectangles" to choose from, for a total of 16 possibilities.

To calculate the number of possibilities for the obtuse triangle, we can see that the side with a length of 1 must be on the edge. There are 8 possibilities.

To calculate the number of possibilities for the isosceles right triangle, we can see that for each 1×2 "rectangle", there are 2 possibilities. This creates 8 total possibilities.

There are 32 possible ways to create a triangle with an area of 1. $\binom{9}{3} = 84$ so the answer is $\boxed{\textbf{(E) } \frac{8}{21}}$.

20. q and r are positive integers in the following equation. There is only one possible value of $q+r$. Let $S(n)$ represent the sum of the digits of positive integer n. What is $S(q+r)$?

$$\frac{q}{r} + \frac{r}{q} = \frac{qr}{144}$$

(A) 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 11

Multiply both sides by qr and then solve for q in this way:

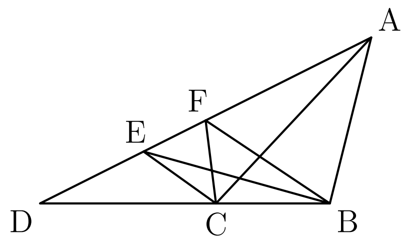
$$\begin{aligned}
144q^2 + 144r^2 &= q^2r^2 \\
144q^2 - q^2r^2 + 144r^2 &= 0 \\
q^2(144 - r^2) + 144r^2 &= 0 \\
q^2(144 - r^2) &= -144r^2 \\
q^2 &= \frac{-144r^2}{144 - r^2} \\
q^2 &= \frac{144r^2}{r^2 - 144} \\
q &= \frac{12r}{\sqrt{r^2 - 12^2}}
\end{aligned}$$

Notice that the denominator of the right side of the equation is the formula for calculating the leg of a right triangle using the Pythagorean Theorem. A right triangle with a hypotenuse of r and a leg with a length of 12 is needed. There aren't many such triangles to choose from. 5, 12, 13; 9, 12, 15; 12, 16, 20; and 12, 35, 37 are the only options. We can calculate q by plugging in different values for r to find that 9, 12, 15 results in $q = 20$ and $r = 15$. Also, 12, 16, 20 results in $q = 15$ and $r = 20$, which gives the same answer.

Add q and r together: $15 + 20 = 35$. Take the sum of the digits of 35: $3 + 5 = \boxed{\textbf{(C)}\ 8}$

Note: Calculating the Pythagorean triples and proving that those are the only ones with a leg length of 12 is more difficult, but the most basic Pythagorean triples are easily recognizable.

21. In the following diagram, $m\angle BAC = m\angle BFC = 40^\circ$, $m\angle ABF = 80^\circ$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?



(A) 12° **(B)** 15° **(C)** 24° **(D)** 30° **(E)** 48°

Recognize that quadrilateral $ABCF$ is cyclic because $m\angle BAC = m\angle BFC$. We can figure out that $m(\overset{\frown}{BC}) = 80^\circ$, and $m(\overset{\frown}{AF}) = 160^\circ$. Let point G be the intersection of $\overset{\frown}{FC}$ and $\overline{BE}$. Set x to $m\angle DBE$. We can figure out that $m(\overset{\frown}{FG}) = m(\overset{\frown}{CG}) = 2x$ because $m\angle FBE = m\angle DBE = x$. $m\angle FEB = \dfrac{m(\overset{\frown}{AB}) - m(\overset{\frown}{FG})}{2}$ so we can set up the equation $2x = \dfrac{m(\overset{\frown}{AB}) - 2x}{2}$. Solving for $m(\overset{\frown}{AB})$ in terms of x results in $m(\overset{\frown}{AB}) = 6x$. $m(\overset{\frown}{AB}) + m(\overset{\frown}{FC}) = 120^\circ$ because the arcs around a circle must add up to 360°. That means we can set up the equation $2x + 2x + 6x = 120^\circ$. Solving for x results in $x = 12^\circ$. Lastly, $m\angle ADB = \dfrac{m(\overset{\frown}{AB}) - m(\overset{\frown}{FC})}{2}$. Plugging in the values results $\boxed{\textbf{(A) } 12^\circ}$.

22. The quadratic equations $y = ax^2 + 20x + 32$ and $y = x^2 + bx + 16$ share the same vertex in the xy-coordinate plane. If a and b are integers, what is $a + b$?

(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 21

The vertex form of a quadratic equation is $a(x - h)^2 + k$, and the vertex is (h, k). We can calculate that the coefficient of the quadratic term is a, the coefficient of the linear term is $-2ah$, and the constant term is $h^2 + k$. From $y = ax^2 + 20x + 32$, we can find that $20 = -2ah$ and $32 = ah^2 + k$. From $y = x^2 + bx + 16$, we can find that $16 = h^2 + k$. Solve the following system of equations for a:

$$20 = -2ah$$
$$32 = ah^2 + k$$
$$16 = h^2 + k$$

The only possible integer value of a is 5. When $a = 5$, $h = -2$. h is the x-coordinate of the vertex. The x-coordinate of the vertex is also equal to $-\frac{b}{2a}$ in any quadratic, where a is the coefficient of the quadratic term, and b is the coefficient of the linear term. Therefore, we can solve the equation $-2 = -\frac{b}{2 \cdot 1}$ for b. The result is $b = 4$. $a = 5$ and $b = 4$, so the answer is $5 + 4 = \boxed{\textbf{(A) } 9}$.

23. In triangle ABC, $AB = AC = 5$ and $BC = 6$. Point D is placed on $\overline{AB}$ and point E is placed on $\overline{AC}$ so that $BD = DE$ and $m\angle BDE = 90°$. BD is equal to $\frac{u}{v}$ where u and v are relatively prime positive integers. What is $u + v$?

(A) 151 **(B)** 157 **(C)** 163 **(D)** 229 **(E)** 239

First, draw point F so that the points B, D, E, and F form a square. Make point G the midpoint of $\overline{BC}$. We can see that $BG = CG = 3$ and $AG = 4$. $m\angle ABG + m\angle BAG = 90°$, and $m\angle ABG + m\angle FBG = 90°$, so $m\angle FBG = m\angle BAG$. Call the intersection of $\overline{BC}$ and $\overline{EF}$ point H. Triangle FHB is similar to triangle GBA because all angles are congruent. We can find that $BH = \frac{5}{4}BD$ and $FH = \frac{3}{4}BD$. That means $EH = \frac{1}{4}BD$. We can find that triangle EHC is similar to triangle ABC. Using the proportion $\frac{\frac{1}{4}BD}{HC} = \frac{5}{6}$, we can find that $HC = \frac{3}{10}BD$. Solve for BD in the equation $\frac{5}{4}BD + \frac{3}{10}BD = 6$. $BD = \frac{120}{31}$ so the answer is $120 + 31 = \boxed{\textbf{(A) } 151}$.

24. n is a nonnegative integer with up to four digits. If n has less than four digits, leading zeros are added to n until n has four digits. Let $R(n)$ represent n with its digits reversed after adding leading zeros to n if necessary. For example, $R(476) = R(0476) = 6740$ and $R(1453) = 3541$. The function f is defined by $f(x) = |x - R(x)|$. Which integer is closest to the average value of $f(x)$ for all nonnegative integer values of x less than 10000?

(A) 2997 **(B)** 3297 **(C)** 3300 **(D)** 3326 **(E)** 3594

Let $R_2(n)$ represent n with its digits reversed where n is a nonnegative integer with up to 2 digits. If n only has one digit, a leading zero is added. The function g is defined by $g(x) = |x - R_2(x)|$. We can calculate the average value of $g(x)$ for all nonnegative integer values of x less than 100 by recognizing that $0, 11, 22, \ldots, 99$ results in a $g(x)$ value of 0; $10, 21, 32, \ldots, 98$ and $1, 12, 23, \ldots, 89$ results in a $g(x)$ value of 9; $20, 31, 42, \ldots, 97$ and $2, 13, 24, \ldots, 79$ results in a $g(x)$ value of 18; etc. We can see that there are 10 values of x that result in $g(x) = 0$ and $20 - 2a$ values of x that result in $g(x) = 9a$, where a is an positive integer less than 10. That means the average value of $g(x)$ for all nonnegative integer values of x less than 100 is the following:

$$\frac{((9 \cdot 1) \cdot (2 \cdot 9)) + ((9 \cdot 2) \cdot (2 \cdot 8)) + ((9 \cdot 3) \cdot (2 \cdot 7)) + \cdots + ((9 \cdot 9) \cdot (2 \cdot 1))}{100}$$

That is equal to the following:

$$\frac{18((1 \cdot 9) + (2 \cdot 8) + (3 \cdot 7) + \cdots + (9 \cdot 1))}{100}$$

That is also equal to the following:

$$\frac{18(2(1 \cdot 9 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 6) + 5 \cdot 5)}{100}$$

We can find that $2(1 \cdot 9 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 6) + 5 \cdot 5 = 165$ so simplifying the expression results in 29.7.

Now, notice that for the number $\underline{abcd}$, where $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ are nonnegative digits, the average value of $f(x)$ is $29.7 \cdot 10 = 297$ if $\underline{a} = \underline{d}$. For each number where $\underline{b} = \underline{c}$, we can calculate using similar methods that the average values of $f(x)$ is the following:

$$\frac{1998(2(1 \cdot 9 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 6) + 5 \cdot 5)}{100}$$

Rewrite 1998 as $2000-2$ and use the value of 165 again for $2(1 \cdot 9 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 6) + 5 \cdot 5$:

$$\frac{(2000 - 2)(165)}{100}$$

Simplify the expression to 3296.7. We can find that $f(x) = |999(a-d) + 90(b-c)|$. That means $\dfrac{f(\underline{abcd}) + f(\underline{acbd})}{2}$ will always equal $999|\underline{a} - \underline{d}|$ if $\underline{a} \neq \underline{d}$. In turn, the average $f(x)$ value where $\underline{a} \neq \underline{d}$ and $\underline{b} \neq \underline{c}$ is equal to the average $f(x)$ value where $\underline{a} \neq \underline{d}$ and $\underline{b} = \underline{c}$. To find the average value for any number where $\underline{a} \neq \underline{d}$, use the previous expression with a denominator of 90:

$$\frac{(2000-2)(165)}{90}$$

Note that the average value of $f(x)$ if $\underline{a} = \underline{d}$ is simply 297. 90% of the numbers satisfy $\underline{a} \neq \underline{d}$ and 10% of the numbers satisfy $\underline{a} = \underline{d}$ so the answer is the following:

$$\frac{0.9(2000-2)(165)}{90} + 0.1(297)$$

Use the previous value of 3296.7 to help find the value of this expression. This simplifies to 3326.4. The closest integer is $\boxed{\textbf{(D) } 3326}$.

25. A particle starts at the origin in the xy-coordinate plane. A *move* is defined as going from (x, y) to any one of $(x, y+1)$, $(x+1, y+1)$, $(x+1, y)$, $(x+1, y-1)$, $(x, y-1)$, $(x-1, y-1)$, $(x-1, y)$, or $(x-1, y+1)$ with one additional rule: the x- and y-coordinates of the particle must always stay between 0 and 5, inclusive. The particle stops forever when it gets to $(5, 5)$. In how many ways can the particle can get to $(5, 5)$ using no more than 7 moves?

(A) 471 **(B)** 473 **(C)** 475 **(D)** 477 **(E)** 479

Note that there are only 8 possible directions to move at most: $\uparrow \nearrow \rightarrow \searrow \downarrow \swarrow \leftarrow \nwarrow$.

Having the same combination of arrows always ends at the same location. The order of the arrows does not matter as long as the particle never goes outside the area with x- and y- coordinates between 0 and 5. We can see that the fastest route is $\nearrow \nearrow \nearrow \nearrow \nearrow$. We can change the arrows in the following ways:

- replace $\nearrow$ with $\rightarrow \uparrow$
- replace $\nearrow \nearrow$ with $\rightarrow \rightarrow \uparrow \uparrow$

- add $\swarrow\nearrow$
- add $\rightarrow\leftarrow$
- add $\uparrow\downarrow$
- add $\nwarrow\searrow$
- replace $\nearrow$ with $\rightarrow\rightarrow\nwarrow$
- replace $\nearrow$ with $\uparrow\uparrow\searrow$

First, start with the "replace $\nearrow$ with $\rightarrow\uparrow$" case. There are 4 $\nearrow$s, 1 $\rightarrow$, and 1 $\uparrow$. We can see that there is no case where the particle can go outside the range of coordinates. Think of this as placing one arrow in any of 5 locations and the other arrow in any of 6 locations. The total number of combinations is $5 \cdot 6 = 30$.

The second case is "replace $\nearrow\nearrow$ with $\rightarrow\rightarrow\uparrow\uparrow$". There are 3 $\nearrow$s, 2 $\rightarrow$s, and 2 $\uparrow$s. Think of this as choosing 2 locations out of 5 for the $\rightarrow$s, and then 2 locations out of 7 for the $\uparrow$s. The total number of combinations is $\binom{5}{2} \cdot \binom{7}{2} = 210$.

The third case is "add $\swarrow\nearrow$". Note that this case has the particle staying on the $x = y$ line. There are only 4 possible places where the particle could go in the $\swarrow$ direction: $(1, 1)$; $(2, 2)$; $(3, 3)$; and $(4, 4)$. Going to $(5, 5)$ would cause the particle to stop.

The fourth case is "add $\rightarrow\leftarrow$". Note that $\leftarrow$ cannot be at the beginning or the end because it would immediately go outside the range of coordinates if $\leftarrow$ is at the beginning, and it must go to $(6, 5)$ before going in the $\leftarrow$ direction to go to $(5, 5)$. There are 5 $\nearrow$s, 1 $\rightarrow$, and 1 $\leftarrow$. First, add in the right arrows. Then, add the left arrows in any location that is not at the beginning or the end. There seems to be $6 \cdot 5 = 30$ combinations, but the $\nearrow\nearrow\nearrow\nearrow\nearrow\leftarrow\rightarrow$ combination does not work because the particle stops at $(5, 5)$. There are actually $30 - 1 = 29$ combinations.

The fifth case is symmetrical to the fourth case so there are also 29 combinations.

The sixth case is "add $\nwarrow\searrow$". There are 5 $\nearrow$s, 1 $\nwarrow$, and 1 $\searrow$. Neither the $\nwarrow$ nor the $\searrow$ can be at the beginning or the end, so there are $4 \cdot 5 = 20$ combinations.

The seventh case is the "replace $\nearrow$ with $\rightarrow\rightarrow\nwarrow$" case. There are 4 $\nearrow$s, 2$\rightarrow$s, and 1 $\nwarrow$. The $\nwarrow$ cannot be at the beginning or the end so there are $\binom{6}{2} \cdot 5 = 75$ combinations.

The eighth case is symmetrical to the seventh case so there are also 75 combinations.

There is also the possibility of $\nearrow\nearrow\nearrow\nearrow\nearrow$.

Add the possibilities together: $30 + 210 + 4 + 29 + 29 + 20 + 75 + 75 + 1 = \boxed{\textbf{(B)}\ 473}$.