2021

CMC 12A

DO NOT OPEN UNTIL SATURDAY, December 26, 2020

Christmas Math Competitions

Questions and comments about problems and solutions for this exam should be emailed to:

christmas.math.team@gmail.com

The 4th Annual CIME will be held on Saturday, January 2, 2021, with the alternate on Saturday, February 13, 2021. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be invited to take the 4th Annual Christmas American Math Olympiad (CAMO) or the Christmas Junior Math Olympiad (CJMO) on Saturday, January 9, 2021.

A complete listing of our previous publications may be found at our web site:

http://cmc.ericshen.net/

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE DECEMBER 26, 2020.
2. YOU must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on maa.org/amc under "AMC 12A") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, or digital media of any type during this period is a violation of the competition rules.

The **Christmas Math Competitions** *is made possible by the contributions of the following problem-writers and test-solvers:*

David Altizio, Allen Baranov, Ankan Bhattacharya, Luke Choi, Federico Clerici, Rishabh Das, Mason Fang, Raymond Feng, Preston Fu Valentio Iverson, Minjae Kwon, Benjamin Lee, Justin Lee, Kyle Lee, Kaiwen Li, Sean Li, Elliott Liu, Eric Shen, Albert Wang, Anthony Wang, Andrew Wen, Tovi Wen, Nathan Xiong, *and* Joseph Zhang

MAC CMC
Christmas Math Competitions

Christmas Math Competitions
4th Annual

CMC 12A

Christmas Math Contest 12A
Saturday, December 26, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from https://www.maa.org/math-competitions/amc-10-12/ and mark you answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the CMC, **you must submit your answers using the Submission Form found at http://cmc.ericshen.net/CMC-2021/. Only answers submitted to the Submission Form will be scored.**
4. Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, graph paper, protractors, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10 Answer Sheet from https://www.maa.org/math-competitions/amc-10-12/.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet should you choose to obtain one from https://www.maa.org/math-competitions/amc-10-12/.

The Committee on the Christmas Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 4th annual Christmas Invitational Math Examination (CIME) on Saturday, January 2, 2021 and Saturday, February 13, 2021. More details about the CIME are in the back of this test booklet.

1. What is the value of

$$\frac{21^2 + \frac{21}{20}}{20^2 + \frac{20}{21}}?$$

(A) $\dfrac{400}{441}$ **(B)** $\dfrac{20}{21}$ **(C)** 1 **(D)** $\dfrac{21}{20}$ **(E)** $\dfrac{441}{400}$

2. Two circles of equal radius r have an overlap of area 7π and the total area covered by the circles is 25π. What is the value of r?

(A) $2\sqrt{2}$ **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** 5 **(E)** $4\sqrt{2}$

3. A pyramid whose base is a regular n-gon has the same number of edges as a prism whose base is a regular m-gon. What is the smallest possible value of n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 12

4. There exists a positive integer N such that

$$\frac{\frac{1}{999} + \frac{1}{1001}}{2} = \frac{1}{1000} + \frac{1}{N}.$$

What is the sum of the digits of N?

(A) 36 **(B)** 45 **(C)** 54 **(D)** 63 **(E)** 72

5. For positive integers $m > 1$ and $n > 1$, the sum of the first m multiples of n is 2020. Compute $m + n$.

(A) 206 **(B)** 208 **(C)** 210 **(D)** 212 **(E)** 214

6. How many of the following statements are true for every parallelogram $\mathcal{P}$?

i. The perpendicular bisectors of the sides of $\mathcal{P}$ all share at least one common point.

ii. The perpendicular bisectors of the sides of $\mathcal{P}$ are all distinct.

iii. If the perpendicular bisectors of the sides of $\mathcal{P}$ all share at least one common point then $\mathcal{P}$ is a square.

iv. If the perpendicular bisectors of the sides of $\mathcal{P}$ are all distinct then these bisectors form a parallelogram.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

7. It is known that every positive integer can be represented as the sum of at most 4 squares. What is the sum of the 2 smallest positive integers which *cannot* be represented as a sum of fewer than 4 squares?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

24. In triangle $\triangle ABC$, $AB = 5$, $BC = 6$, and $CA = 7$. Let ω be the circle with diameter BC, and let $\overline{AE}$ and $\overline{AF}$ be the tangents from A to ω, such that segments BE and CF intersect at a point P inside $\triangle ABC$. What is AP?

(A) 3 **(B)** $\sqrt{19}$ **(C)** $\sqrt{21}$ **(D)** $2\sqrt{7}$ **(E)** 6

25. In a set of 15 points, no 4 of which are cyclic and no 3 of which are collinear, there are exactly 2021 pairs $(\mathcal{C}, P)$ where $\mathcal{C}$ is a circle that passes through 3 of the points and P is a point lying strictly inside $\mathcal{C}$. What is the maximum number of convex quadrilaterals with vertices among the 15 points?

(A) 637 **(B)** 656 **(C)** 678 **(D)** 715 **(E)** 734

8. There are 8 scoops of ice cream, two of each flavor: vanilla, strawberry, chocolate, and mint. If scoops of the same flavor are not distinguishable, how many ways are there to distribute one scoop of ice cream to each of 5 different people?

(A) 420 **(B)** 450 **(C)** 600 **(D)** 1620 **(E)** 2520

9. Suppose points A, B, C, D, E, and F lie on a line such that $AB = 1$, $BC = 4$, $CD = 9$, $DE = 16$, $EF = 25$, and $FA = 23$. What is CF?

(A) 14 **(B)** 18 **(C)** 21 **(D)** 25 **(E)** 26

10. A soccer league consists of 20 teams and every pair of teams play each other once. If the game is a draw then each team receives one point, otherwise the winner receives 3 points while the loser receives 0 points. If the total number of points scored by all the teams was 500, how many games ended in a draw?

(A) 70 **(B)** 75 **(C)** 100 **(D)** 115 **(E)** 120

11. Suppose that $ABCD$ is a parallelogram whose diagonals intersect at P. Let X and Y be the feet from P to AD and AB, respectively. If $AX = 1$, $AY = 2$, and $BY = 5$, then compute DX.

(A) $\sqrt{22}$ **(B)** $2 + 2\sqrt{2}$ **(C)** $2\sqrt{6}$ **(D)** $\frac{11}{2}$ **(E)** 6

12. Let a, b be not necessarily distinct positive integers chosen uniformly at random from the set $\{1, 2, ..., 100\}$. Find the probability that

$$\left\lfloor \sqrt{a} \right\rfloor + \left\lceil \sqrt{b} \right\rceil = \left\lceil \sqrt{a} \right\rceil + \left\lfloor \sqrt{b} \right\rfloor,$$

where for real x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the least integer greater than or equal to x.

(A) $\frac{17}{25}$ **(B)** $\frac{39}{50}$ **(C)** $\frac{81}{100}$ **(D)** $\frac{41}{50}$ **(E)** $\frac{21}{25}$

13. Jerry writes an equation on a board. However, Mario comes along and erases some of the digits, and replaces them with the variables a, b, c, d. On the board remains

$$\log_a \overline{b2c} = d$$

for $a \neq 1$ and $b \neq 0$. Find the sum of all possible values of d.

(A) 13 **(B)** 17 **(C)** 18 **(D)** 20 **(E)** 24

14. Let a and b be real numbers such that $z_1 = 1010 + 2021i$, $z_2 = 2020 + 1011i$ and $z_3 = az_1 + bz_2$ are the vertices of an equilateral triangle in the complex plane. What is the sum of all possible values of $a + b$?

(A) -6062 **(B)** -3031 **(C)** 2 **(D)** 3031 **(E)** 6062

15. For some reals a, b, c, d, a polynomial $P(x) = x^3 + ax^2 + bx + 10$ has three distinct real roots in an arithmetic progression, with each root precisely 1 less than a root of $Q(x) = x^3 - 7x^2 + cx + d$. If b can be written as $-\frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

(A) 88 **(B)** 89 **(C)** 90 **(D)** 91 **(E)** 92

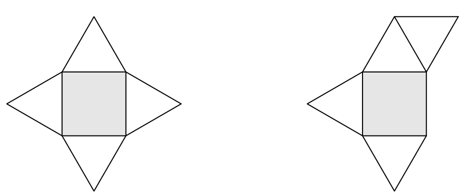
16. Let $ABCD$ be a quadrilateral inscribed in a circle ω. Diagonals AC and BD meet at P such that $BP = 4$ and $DP = 7$. Point Q is chosen such that $\triangle CPQ$ is equilateral. What is the area of $\triangle APQ$?

(A) 9 **(B)**12 **(C)** $7\sqrt{3}$ **(D)** $14 - \sqrt{3}$ **(E)** $12 + \sqrt{3}$

17. We say that two quadratic polynomials $P(x)$ and $Q(x)$ with leading coefficient 1 are *friends* if for every root r of $P(x)$, $Q(r)$ is also a root of $P(x)$. If $P(x) = x^2 - 4x + 1$, what is the maximum value of $Q(0)$ over all friends of $P(x)$?

(A) 1 **(B)** $3 - \sqrt{3}$ **(C)** $3 + \sqrt{3}$ **(D)** 5 **(E)** 6

18. Pyramid $ABCDE$ with square base $BCDE$ and apex A has edge lengths all equal to 1 with the square base sitting atop a table. Some of the edges of the pyramid are cut so that it is opened out flat to its net (in one piece) on the table without any folds. Two possibilities for the net are shown below, and nets are considered the same if they differ by a rotation but different if they differ by a reflection. How many distinct nets are possible?



(A) 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 16

19. Consider the function

$$f(x) = \underbrace{2\sin(2\sin(2\sin(\ldots 2\sin(2\sin(x))\ldots)))}_{200 \text{ sin's}}.$$

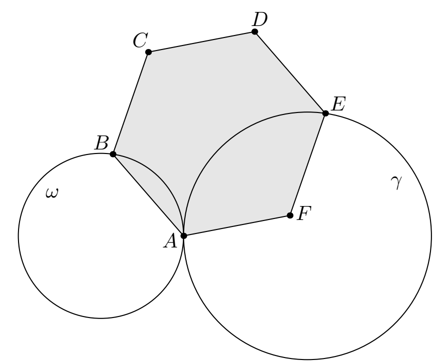
Over the interval $0 \le x \le \pi$, $f(x)$ attains its maximum value M a total of N times. Determine the largest integer no greater than $M + N$.

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

20. For a certain positive integer n, there are exactly 2021 ordered pairs of positive divisors (d_1, d_2) of n for which d_1 and d_2 are relatively prime. What is the sum of all possible values of the number of divisors of n?

(A) 1535 **(B)** 1536 **(C)** 1537 **(D)** 1538 **(E)** 1539

21. Circles ω and γ have radii 2 and 3 respectively, and are externally tangent at A. Let $ABCDEF$ be a non-degenerate regular hexagon such that B lies on ω and E lies on γ. The area of this hexagon can be written as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers with a, c coprime and b square free. Find $a + b + c$.



(A) 80 **(B)** 81 **(C)** 82 **(D)** 83 **(E)** 84

22. Find the number of integers n between 2 and 50, inclusive, such that the product of any two distinct divisors of n is either a multiple or divisor of n.

(A) 21 **(B)** 25 **(C)** 28 **(D)** 33 **(E)** 36

23. How many ordered tuples of positive integers $(a_1, a_2, ..., a_n)$ are there such that

$$\sum_{k=1}^{n} a_k = 12 \quad \text{and} \quad \sum_{k=1}^{\ell} (-1)^k a_k \le \sum_{k=1}^{n} (-1)^k a_k = 0$$

for all $\ell \in \{1, 2, ..., n-1\}$?

(A) 64 **(B)** 128 **(C)** 132 **(D)** 144 **(E)** 256