2022 WMC 10

WMC Committee

This is the 2022 Winter Mathematics Competition for 10th graders and below, designed to imitate and provide practice for the MAA's AMC 10. The test was released on October 9, and submissions will be closed on November 13.

As with the MAA's AMC 10, you have 75 minutes to solve 25 multiple choice questions with answer choices A, B, C, D, and E. You will receive 6 points for every correct answer, 1.5 points for every answer left blank, and 0 points for every incorrect answer. The only aids permitted are writing utensils, erasers, blank scratch paper, straightedge, and compass; in particular, no graph paper, calculators, or any other computing devices are allowed. And of course, do not cheat or discuss problems outside the designated private discussion forum while the contest is ongoing; this may result in disqualification.

The 2022 WMC 10 was made possible by the contributions of problem-writers and test-solvers **A1001**, **bissue**, **capybara42**, **FalconMaster**, **ihatemath123**, **peace09**, **tenebrine**, and **v4913**.

1. What is the value of

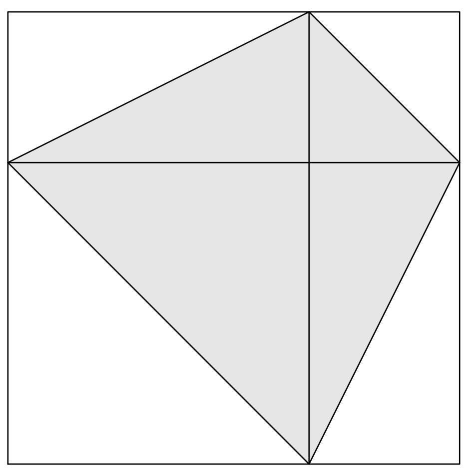
$$\frac{\left(\frac{\left(\frac{4}{3}\right)}{2}\right)}{1} \div \frac{4}{\left(\frac{3}{\left(\frac{2}{1}\right)}\right)}?$$

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

2. Two noncongruent rectangles with integer side lengths have the same area. What is the least possible difference between their perimeters?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

3. Two squares of side lengths 1 and 2 lie within a third square of side length 3, as shown below. What is the area of the shaded region?



(A) 4 **(B)** $4\frac{1}{2}$ **(C)** 5 **(D)** $5\frac{1}{2}$ **(E)** 6

4. How many positive integer multiples of 42 have 42 divisors?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** infinitely many

5. Consider the list of integers $1, 2, 2, 3, 4, 4, 5, \ldots, 999, 1000, 1000, 1001$, where every odd number between 1 and 1001 appears exactly once, and every even number between 1 and 1001 appears exactly twice. What is the mean of the list?

(A) 499 **(B)** $499\frac{1}{2}$ **(C)** 500 **(D)** $500\frac{1}{2}$ **(E)** 501

6. Justin flips a fair coin 4 times and records the outcomes. What is the probability that there is a sequence of 2 or more consecutive heads?

   (A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{5}{8}$

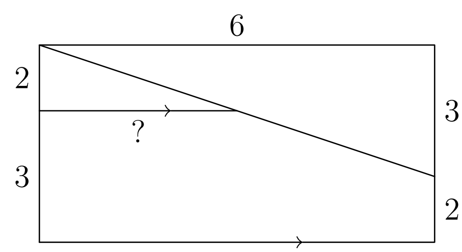
7. Abby is solving a system of equations in her homework, but the operations in the equations are missing, as shown below.

$$\begin{aligned} x \_ y &= 55 \\ y \_ z &= 89 \\ z \_ x &= 34 \end{aligned}$$

   Not bothering to clarify with her teacher, she fills in each blank with a $+$ or a $-$ at random. What is the probability that there is a solution to the resulting system of equations?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{8}$ **(E)** 1

8. What is the length of the segment marked with a question mark below?

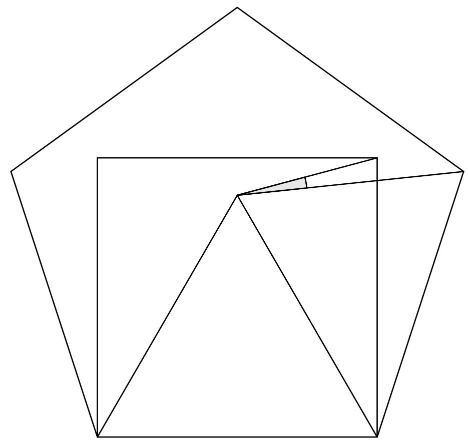


   (A) 2 **(B)** $2\frac{1}{2}$ **(C)** 3 **(D)** $3\frac{1}{2}$ **(E)** 4

9. The three-term sequences A and G are arithmetic and geometric, respectively, and they share the same middle term of k, where $k > 0$. The common difference of A and the common ratio of G are both 4, and the product of the terms of A is equal to the sum of the terms of G. What is k?

   (A) $\frac{\sqrt{70}}{2}$ **(B)** $\frac{5\sqrt{3}}{2}$ **(C)** $2\sqrt{5}$ **(D)** $\frac{\sqrt{85}}{2}$ **(E)** $\frac{3\sqrt{10}}{4}$

10. An equilateral triangle lies inside a square that lies inside a regular pentagon, and all three polygons share a side, as shown below. What is the degree measure of the marked angle?



(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

11. In how many ways can 18 indistinguishable adults and 6 indistinguishable children stand in a line such that any 2 children are separated by at least 3 adults?

(A) 60 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

12. The six-digit base-two integer $ABCDEF_{\text{two}}$ and the six-digit base-ten integer $ABCDEF_{\text{ten}}$ are both multiples of 6. What is the value of the six-digit base-six integer $ABCDEF_{\text{six}}$, expressed in base-ten?

(A) 7818 **(B)** 7998 **(C)** 8028 **(D)** 9078 **(E)** 9128

13. In triangle ABC with area 60, side $\overline{BC}$ is extended past C to point D such that $BC = CD$. Let M be the midpoint of segment $\overline{AD}$, and let N be the intersection of segments $\overline{AC}$ and $\overline{BM}$. What is the area of triangle AMN?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 20 **(E)** 30

14. Let n be a randomly chosen divisor of 30^6. What is the probability that $2n$ has more divisors than $3n$ and $5n$?

(A) $\frac{12}{49}$ **(B)** $\frac{13}{49}$ **(C)** $\frac{14}{49}$ **(D)** $\frac{15}{49}$ **(E)** $\frac{16}{49}$

15. Three diagonals are drawn in a regular hexagon of side length 1, splitting the hexagon into several regions. What is the least possible area of such a region?

(A) $\frac{\sqrt{3}}{24}$ **(B)** $\frac{\sqrt{3}}{18}$ **(C)** $\frac{\sqrt{3}}{16}$ **(D)** $\frac{\sqrt{3}}{12}$ **(E)** $\frac{\sqrt{3}}{8}$

16. Three rational fractions have denominators of 20, 21, and 30 when expressed in lowest terms. What is the least possible denominator that the sum of the three fractions, expressed in lowest terms, could have?

(A) 1 **(B)** 7 **(C)** 28 **(D)** 84 **(E)** 420

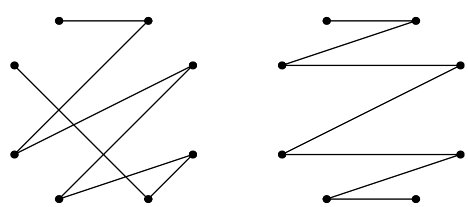
17. In rectangle $ABCD$ with $AB = 12$ and $BC = 9$, points W, X, Y, and Z lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that AW and CX are integers. If $WXYZ$ is a parallelogram with area 50, what is its perimeter?

(A) $11\sqrt{5}$ **(B)** $12\sqrt{5}$ **(C)** $13\sqrt{5}$ **(D)** $14\sqrt{5}$ **(E)** $15\sqrt{5}$

18. A field trip group has 3 adults, 3 boys, and 4 girls. How many (possibly empty) subsets of the 10 people can pose for a photo if at least one adult is left out to take the photo, one adult can't pose alone with one child, and each child refuses to pose with only children of the opposite gender? For example, there can be 1 adult, 1 boy, and 1 girl, but there can't be only 1 boy and 1 girl.

(A) 807 **(B)** 814 **(C)** 821 **(D)** 828 **(E)** 835

19. Ari the ant visits each vertex of a convex octagon exactly once in some order, crawling in straight lines from vertex to vertex. What is the probability that his path self-intersects? For example, the left path shown below is valid, but the right path is invalid.



(A) $\frac{103}{105}$ **(B)** $\frac{62}{63}$ **(C)** $\frac{311}{315}$ **(D)** $\frac{104}{105}$ **(E)** $\frac{313}{315}$

20. A positive integer N is said to be *nearly-square* if $\sqrt{N}$ can be expressed in simplest radical form as $a\sqrt{b}$, where a and b are positive integers greater than 1 with $a > b$. How many of the first 1000 positive integers are nearly-square?

(A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

21. The *Centinacci sequence* $\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$ is defined by $C_n = 0$ for $n < 0$, $C_0 = 1$, and $C_n = C_{n-1} + C_{n-2} + \cdots + C_{n-100}$ for $n > 0$. What is the largest positive integer k such that 2^k divides C_{123}?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

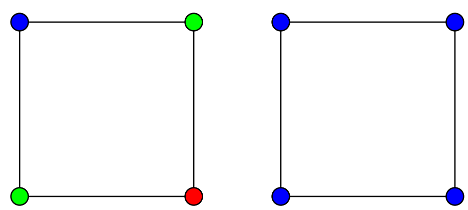
22. Three right circular cones each with base radius 15 and height 20 share the same vertex, and their bases are pairwise tangent. The height of the smallest possible fourth cone that contains the other three cones in its interior can be expressed in the form $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$?

    (A) 316 **(B)** 317 **(C)** 318 **(D)** 319 **(E)** 320

23. For all nonnegative integers n, let $f(n)$ be the least number of pennies, dimes, and quarters needed to amount to n cents in total. What is the largest positive integer k such that $f(n)=k$ for some n between 0 and 99?

    (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

24. Each vertex of a cube is colored red, green, or blue. Then, for each face, define its *color-count* to be the unordered list of the frequencies of all colors present amongst its vertices. For example, the left face shown below has color-count $1,1,2$ (equivalently $1,2,1$ or $2,1,1$), while the right face has color-count 4.

    For how many such colorings is the color-count of each face the same? (Two colorings are considered identical if one can be rotated to match the other.)

    (A) 30 **(B)** 33 **(C)** 36 **(D)** 39 **(E)** 42

25. A negligibly small grasshopper hops onto a random point on a horizontal sidewalk with meter-long tiles separated by negligibly thin grooves. It proceeds to hop in the horizontal direction with each hop of a constant length less than a meter, which it chooses uniformly at random. Exactly 4 of its hops (including the initial hop onto the sidewalk) land in the first tile it hops across. The probability that exactly 4 hops land in the second tile it hops across can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

    (A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44