2019 CMC 10A Solutions Document

Christmas Math Competitions

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1. **Answer (D):** We compute:

$$201 \times 9 + 20 \times 19 - 2 \times (0 + 1 + 9) = 1809 + 380 - 20 = 2169.$$

2. **Answer (E):** We compute: $1^6+2^5+3^4+4^3+5^2+6^1 = 1+32+81+64+25+6 = 209$. Since $209 = 11 \cdot 19$, 11 is a prime factor of the expression.

3. **Answer (A):** The smallest possible value of $a+b+c+d$ is $1+2+3+4 = 10$, while the largest possible value is $47+48+49+50 = 194$. Consider the expression $1+2+3+4$. We can keep increasing the 4 by 1 until we reach $1+2+3+50$. Next, we can keep increasing the 3 by 1 until we reach $1+2+49+50$. Then, we can keep increasing the 2 by 1 until we reach $1+48+49+50$. Finally, we can keep increasing the 1 by 1 until we reach $47+48+49+50$. Therefore, all integer values between 10 and 194 inclusive are obtainable, so there are $194 - 10 + 1 = 185$ possible values.

4. **Answer (D):** The number that is N% of 880 is $\frac{\text{N}}{100} \cdot 880 \implies \frac{44\text{N}}{5}$. Since this expression must be an integer, we let $\text{N} = 5k$ for some positive integer k. Then, our expression becomes $44k = 2^2 \cdot 11 \cdot k$. For this to be a perfect square, the exponent of each prime in the prime factorization must be even. Therefore, $k = 11$ and $\text{N} = 55$. We can check that 55% of 880 is $484 = 22^2$. The requested sum is $5 + 5 = 10$.

   To show that N is unique, we note that for the resulting number to be smaller than 880, we must have $\text{N} < 100$ or $k < 20$. Since the resulting number is a perfect square if $2^2 \cdot 11 \cdot k$ is a perfect square, k must be divisible by 11. But 11 is the only positive integer less than 20 that is divisible by 11, so N cannot equal another value less than 100. Thus, N is unique.

5. **Answer (B):** The ratio of the octagon's area to perimeter in inches is given by $\frac{5 \text{ in.}^2}{4 \text{ in.}} = \frac{5 \text{ in.}}{4}$.

   We note that 1 ft = 12 in.. Converting the unit of measurement to feet:

$$\frac{5 \text{ in.}}{4} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} = \frac{5 \text{ ft}}{48}$$

   Therefore, the requested ratio is $\frac{5}{48}$.

6. **Answer (B):** First, select one of the 2019 integers at random and then select one of the remaining 2018 integers at random. Note that for every positive integer k from 1 to 2019 except 1010, the integer $2020 - k$ can be selected after selecting k. Therefore,

if we select $k \neq 1010$ as the first number, there is a $\frac{1}{2018}$ chance that we will select $2020 - k$ as the second number. However, if we select $k = 1010$ as the first number, there is no chance that we will select 1010 again, since the two integers must be distinct. Therefore, $n = 1010$ and the requested sum is $1 + 0 + 1 + 0 = 2$.

7. **Answer (D):** Objectively, to maximize the number of gangsters alive, one gangster should be shot by as many gangster as possible to prevent other gangsters from dying. Clearly, all 10 of them cannot remain alive.

   If 9 gangsters remain alive, then one gangster must be shot by every gangster, including himself. However, a gangster clearly cannot shoot himself because he is not the gangster furthest away from himself.

   However, 8 gangsters can remain alive. This can be done by placing 9 of the gangsters close together and placing the 10th gangster arbitrarily far from the other gangsters. Then, the 10th gangster will shoot one of the first 9 gangsters, while the first 9 gangsters will all shoot the 10th. Therefore, at most 8 gangsters can remain alive.

8. **Answer (D):** Clearly, $AC = 10$ by Pythagorean Theorem on $\triangle ACD$. Consider the incircle of $\triangle ACD$. Its radius is equal to the area of $\triangle ACD$ divided by its semiperimeter. The area of $\triangle ACD$ is $\frac{1}{2} \cdot 6 \cdot 8 = 24$. Since the semiperimeter is 12, the radius of the incircle is 2. Therefore, the distance from I to DC is 2 and the distance from I to AB is $6 - 2 = 4$. The distance from I to AD is 2, so the distance from I to BC is $8 - 2 = 6$. Therefore, by Pythagorean Theorem, $BI^2 = 4^2 + 6^2 = 52$.

9. **Answer (D):** For $\sqrt{2} \cdot x$ to be an integer, x must be in the form $\frac{n}{\sqrt{2}}$ for some integer n. Then,
$$0 \leq x \leq 100 \implies 0 \leq \frac{n}{\sqrt{2}} \leq 100 \implies 0 \leq n \leq 100\sqrt{2}.$$
   The value of $\sqrt{2}$ is often approximated as 1.414, so $141 < 100\sqrt{2} < 142 \implies 0 \leq n \leq 141$. There are 142 possible values of n, which gives us 142 possible values of x.

10. **Answer (D):** Note that $3^{20} = 9^{10}$ and $8^{20} = 16^{15}$. Our inequality becomes $9^{10} < n^n < 16^{15}$. Clearly, the values of n that satisfy this inequality are $10, 11, 12, 13, 14$, and 15, which have a sum of 75.

11. **Answer (C):** Note that for all real numbers k, $1 \circ k = 1 - \frac{1}{k} + \frac{1}{k} = 1$. Therefore, $1 \circ (2 \circ (\cdots (2018 \circ 2019))) = 1$.

12. **Answer (B):** The shaded area is given by the area of kite $NCMF$ minus the area of kite $NPMQ$. Since the area of a kite is half of the product of its diagonals, our desired area is $\frac{1}{2}(NM \cdot CF - NM \cdot PQ)$.

    Clearly, $NM = \sqrt{3}$ and $CF = 2$. In addition, $PQ = 1$ because $ABPQ$ is a square. Therefore, the desired area is $\frac{1}{2}(\sqrt{3} \cdot 2 - \sqrt{3} \cdot 1) = \frac{\sqrt{3}}{2}$.

13. **Answer (C):** Note that $r = p^q + q^p \geq 2^2 + 2^2 = 8$. Therefore, $r \neq 2$, so r must be an odd prime. Since the exponentiation of an integer preserves its parity, one of (p, q) must be odd and the other must be even. WLOG, assume that $p = 2$ and that q is odd. We will multiply by 2 at the end to take symmetry into account.

    We have $2^q + q^2 = r$. Consider this equation in mod 3. The expression 2^q is $1 \pmod 3$ when q is even and $2 \pmod 3$ when q is odd. However, since q is odd, $2^q \equiv 2 \pmod 3$. By Fermat's Little Theorem, $q^2 \equiv 1 \pmod 3$ if q is not a multiple of 3. However, if q is a multiple of 3, $q^2 \equiv 0 \pmod 3$.

    If q is not a multiple of 3, $2 + 1 \equiv 0 \equiv r \pmod 3$, which implies 3 divides r. Since r is prime, $r = 3$, but $r \geq 8$, which is a contradiction.

However, if q is a multiple of 3, we must have $q = 3$. This gives $r = 17$, which is prime. We have one ordered triple for $p = 2$ and multiply by 2 for symmetry, there are 2 ordered triples.

14. **Answer (C):** We wish to determine which of the intervals in the answer choices contains the smallest positive integer N such that $10^N > 4^{1000} = 2^{2000}$. Since N is minimized, we can approximate $10^N \approx 2^{2000}$.

    We know that $2^3 < 10 < 2^4$. Raising the inequality to the power of N, we get:

    $$2^{3N} < 10^N < 2^{4N} \implies 2^{3N} < 2^{2000} < 2^{4N} \implies 3N < 2000 < 4N$$

    $$3N < 2000 \implies N < 667$$
    $$2000 < 4N \implies 500 < N$$

    Therefore, $500 < N < 667$. The interval that best matches our bounds on N is $[550, 650]$. It turns out that the exact value of $\min(N)$ is 603.

15. **Answer (D):** Since $p(a) = 2$, $p(p(a)) = 17 \implies p(2) = 17$. Since $p(p(a)) = 17$, $p(p(p(a))) = 167 \implies p(17) = 167$.

    Let $p(x) = mx + n$ for real constant m and n. Plugging in $x = 2$ gives us $17 = 2m + n$. Plugging in $x = 17$ gives us $167 = 17m + n$. Solving these equations, $(m, n) = (10, -3)$. Since $p(a) = 2$, we have $2 = 10a - 3 \implies a = \frac{1}{2}$.

16. **Answer (D):** Let R denote a rightwards move and U denote an upwards move. First, arrange all 9 of the Rs in a line: $RRRRRRRRR$. We have 10 spaces to insert a U, namely at the beginning of the string, at the end of the string, or in between two Rs. We can only insert up to 1 U in each of these 10 spaces, since each U must be followed by an R. In addition, a U as the last character is allowed. There are $\binom{10}{5} = 252$ ways to choose the spaces for the U and hence, the number of possible sequences.

17. **Answer (C):** Let E' be the rotation of E $180°$ around D. Then, $BECE'$ is a parallelogram because BC and EE' bisect each other. It then follows that $\angle BEC = \angle BE'C$.

    Then, $\angle BEC + \angle BAC = \angle BE'C + \angle BAC = 180°$. It then follows that $BE'CA$ is cyclic. By Power of a Point with respect to D, $DE' \cdot DA = BD \cdot CD$. Because $DE' = DE$, we have $DE \cdot DA = BD \cdot CD$. D is the midpoint of BC, so we have $BD = CD = \frac{\sqrt{7}}{2}$. Finally, $DE \cdot DA = \frac{7}{4}$. The requested sum is $7 + 4 = 11$.

18. **Answer (D):** Let k be the number of seconds before the expected sum of the outputted numbers is at least 280. Clearly, one output has no effect on the others, so the expected sum of the outputted numbers is the sum of the expected numbers from each second. At the nth second, the expected output is the average of the numbers $\{1, 2, 3, ..., n\}$ or $\frac{n+1}{2}$. Summing this expression from 1 to k, we have:

    $$\frac{1+1}{2} + \frac{2+1}{2} + \frac{3+1}{2} + \cdots + \frac{k+1}{2} = \frac{k(k+3)}{4} \geq 280 \implies k^2 + 3k - 1120 \geq 0$$

    By testing values of k or by factoring $k^2 + 3k - 1120 = (k - 32)(k + 35) \geq 0$, we find that $k = 32$ is the smallest value of k that satisfies the inequality.

19. **Answer (C):** Expanding out the expression, we get $2(ab+ac+ad+bc+bd+cd)$. For this expression to be divisible by 4, $ab + ac + ad + bc + bd + cd$ must be even. Clearly, a, b, c, and d each have a $\frac{1}{2}$ chance of being odd and a $\frac{1}{2}$ chance of being even. Since,

a, b, c, and d are all interchangeable in the expression, we do casework on the number of elements in the set $\{a, b, c, d\}$ that are even. There are $2^4 = 16$ possible outcomes.

If none of the elements are even, each of the addends are odd, which results in an even expression. There is 1 way to select 0 elements.

If only 1 of the elements is even, then 3 of the addends will be even, while the other 3 will be odd. This results in an odd expression.

If 2 of the elements are even, then every addend except one of them will be even. The odd addend is the one that is the product of the two odd elements. This results in an odd expression.

If 3 or 4 of the elements are even, then each addend will be even, which results in an even expression. There are 4 ways to select 3 elements and 1 way to select 4 elements. Therefore, our probability is $\frac{1+4+1}{16} = \frac{3}{8}$.

20. **Answer (C):** Let l_1 be the line $y = 2x$ for $x \geq 0$ and l_2 be the line $y = -2x$ for $x \leq 0$. Clearly, A and C lie on l_1, while B lies on l_2. In addition, let l_3 be the line passing through M that is perpendicular to l_1. It follows that B is the intersection of l_2 and l_3.

    Since lines l_1 and l_3 are perpendicular, the slope of l_3 is the negative reciprocal of the slope of l_1. Therefore, the slope of l_3 is $-\frac{1}{2}$. Since, l_3 passes through $(3, 6)$, the equation of l_3 is given by $y = -\frac{1}{2}x + \frac{15}{2}$. Since l_2 has the equation $y = -2x$, $B = (-5, 10)$. Because $\triangle ABC$ is equilateral, $AC = BM \cdot \frac{2}{\sqrt{3}}$. Therefore, $[ABC] = \frac{BM^2}{\sqrt{3}}$. By the distance formula, $BM^2 = (3 - (-5))^2 + (6 - 10)^2 = 80$. Therefore, $[ABC] = \frac{80}{\sqrt{3}}$ or $[ABC] = \frac{80\sqrt{3}}{3}$.

21. **Answer (C):** Let the path have a southeast arrows, b horizontal arrows, and c southwest arrows. Then, $a + b + c = 8$. Since James must travel down 5 rows, we must have $a + c = 5$, as only the southeast and southwest arrows cause James to travel down a row. Lastly, to ensure that James ends up on AB among all the lines parallel to AB, we must have $b = c$. Solving these equations gives $(a, b, c) = (2, 3, 3)$.

    Let the point at the bottom left corner of the grid be C. We must make sure that our path does not go out of the bounds of $\triangle ABC$. Clearly, the directions James can travel in prevent us from ever crossing AC. For James to cross BC, he must travel down more than 5 rows. Our composition of directional arrows ensures that he will travel down exactly 5 rows. Therefore, James can't cross BC. However, if at any point in the path, James has traveled more times horizontally than he has in the southwest direction, he will cross AB.

    Consider a string of characters consisting of 3 bs and 3 cs. We will count how many strings there are such that at every point in the string, there are always at least as many cs as bs. There are 5 such strings, namely *cbcbcb*, *cbccbb*, *ccbcbb*, *ccbbcb*, and *cccbbb*. The 2 as can be inserted anywhere in these 5 strings to construct a possible path.

    Suppose we have 8 blank characters in a row. We have $\binom{8}{2} = 28$ ways to choose the locations of the 2 as. Then, we choose any of the 5 strings of bs and cs and insert the characters in order in the remaining 6 spaces. This gives $28 \cdot 5 = 140$ paths total.

22. **Answer (D):** Clearly, $\triangle ABC$ is obtuse in $\angle ABC$ since $AB^2 + BC^2 = 121 + 169 < 400 = AC^2$.

    Since $\angle ADC = 90°$, by the Pythagorean Theorem $(AB + BD)^2 + CD^2 = AC^2 \Rightarrow (11 + BD)^2 + CD^2 = 400$ and $BD^2 + CD^2 = BC^2 = 169$.

    Expanding the first equation, $121 + 22BD + BD^2 + CD^2 = 400$. Plugging in the second equation, $121 + 22BD + 169 = 400 \Rightarrow 22BD = 110 \Rightarrow BD = 5$.

Then, $\triangle ABC \sim \triangle EBD$ because $\angle CBA = \angle DBE$ and $\angle EDB = \angle BCA$ by cyclic quadrilateral $ACDE$. Therefore, $DE = AC \cdot \frac{BD}{BC} = 20 \cdot \frac{5}{13} = \frac{100}{13}$.

23. **Answer (B):** We claim that for positive integers a, b, x, and y, $a \star b = x \star y$ if and only if $a + b = x + y$. Our base case for $a + b = x + y = 2$ holds, as $(a, b) = (1, 1)$ is the only ordered pair of positive integers such that $a + b = 2$.

    Assume that our inductive hypothesis holds true for a, b, x, and y. Then, $a \star b = x \star y$ implies that $(a + 1) \star b = (x + 1) \star y$. By the second property of the operation, $(a + 1) \star b = (x + 1) \star y = a \star b + a + b = x \star y + x + y$. By our inductive hypothesis, $a \star b + a + b = x \star y + x + y$ implies that $a + b = x + y$. This completes the induction for our proposition. In addition, because our steps are reversible, the converse of our proposition holds true as well. Therefore, $a \star b = x \star y$ if and only if $a + b = x + y$.

    For a fixed value of c, $a \star b = c$ must be satisfied by exactly 25 ordered pairs of positive integers (a, b). Therefore, we must find a positive integer m such that $a + b = m$ has exactly 25 ordered pairs of positive integer solution. Clearly, $m = 26$ and our ordered pairs are $(25, 1), (24, 2), (23, 3), ..., (2, 24), (1, 25)$. Therefore, $c = 25 \star 1$.

    Let $f(n) = n \star 1$ for all $n \geq 1$. We are given that $f(1) = 2$, and we seek $f(25)$. By the second property of the function, $a \star b$, $(n+1) \star 1 = n \star 1 + n + 1$ or $f(n+1) = f(n) + (n+1)$.

    Therefore, $f(25) = 2 + (2 + 3 + 4 + ... + 24 + 25) = 326$.

24. **Answer (D):** Let O_1, O_2, O_3 be the centers of $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. By the tangency conditions, $O_1O_2 = 2 + 3 = 5$, $O_1O_3 = 6 - 2 = 4$ and $O_2O_3 = 6 - 3 = 3$, that is to say, O_3 lies on Γ_2 and since $3^2 + 4^2 = 5^2$, $\triangle O_1O_2O_3$ is right in O_3. Hence, since $O_3B = O_3C = 6$, $\triangle BO_3C$ is a right isosceles triangle.

    Let O be the circumcenter of $\triangle ABC$; by the definition of circumcenter, $OA = OB$, so $\triangle AOB$ is isosceles too. Since $\triangle ABO_1$ and $\triangle ACO_2$ are isosceles, $\angle BAO_1 + \angle CAO_2 = \angle ABO_1 + \angle ACO_2 = \frac{1}{2}\angle AO_1O_3 + \frac{1}{2}\angle AO_2O_3 = 45°$. Hence, $\angle BAC = 135°$.

    Finally, by the fact that the angle at the center is twice the angle at the circumference, considering arc BC not containing A, the concave $\angle BOC = 2\angle BAC = 270°$, hence $\angle BOC = 90°$ and BO_3CO is a square, that is to say, the circumradius of $\triangle ABC$ is $R = 6$.

25. **Answer (A):** Since $x = \lfloor x \rfloor + \{x\}$ for all real x, rewrite the definition of $f(x)$ as $\lfloor x \rfloor(\lfloor x \rfloor + 2\{x\})$. Since $0 \leq \{x\} < 1$, we have:

    $$\lfloor x \rfloor^2 \leq f(x) = \lfloor x \rfloor(\lfloor x \rfloor + 2\{x\}) < \lfloor x \rfloor^2 + 2\lfloor x \rfloor < (\lfloor x \rfloor + 1)^2$$

    $$\lfloor x \rfloor^2 \leq f(x) < (\lfloor x \rfloor + 1)^2$$

    Since $18^2 < 334 < 343 < 19^2$, we have $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 18$. In addition, because $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$, $\alpha - \beta = \{\alpha\} - \{\beta\}$.

    We know that $9 = f(\alpha) - f(\beta) = 18(18 + 2\{\alpha\}) - 18(18 + 2\{\beta\})$. Then,

    $$9 = 18(18 + 2\{\alpha\}) - 18(18 + 2\{\beta\}) \implies \frac{1}{4} = \{\alpha\} - \{\beta\}$$

    Hence, our answer is $\frac{1}{4}$.