July 2021 Mock AMC 12 Solutions

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1. **Answer (C):** We simplify each term first and then add/subtract:

$$2021 - 8 + 210 = 2223.$$

2. **Answer (B):** Let o denote an odd number in the line and e denote an even number in the line. For there to be two adjacent numbers that multiply to an odd number, we need two adjacent odd numbers. There are two available odd numbers, so there are 3 possible arrangements, namely $ooee, eooe$, and $eeoo$. In addition, there are 2 ways to assign 1 and 3 to the os and 2 ways to assign 2 and 4 to the es. Thus, there are $3 \cdot 2 \cdot 2 = 12$ arrangements.

3. **Answer (B):** Clearly, a must be divisible by 5 for the gcd to be 5. It then suffices to check all multiples of 5 up to 50. We can see that an even multiple of 5 causes the gcd to be at least 10, which is not allowed. Checking the odd multiples of 5, we have that $5, 15, 35$, and 45 all work, but 25 does not work because $\gcd(25, 50) = 25$. Thus, there are 4 positive integers a.

4. **Answer (A):** By the properties of a regular hexagon with a side length 1, the distance between lines AD and BC is $\frac{\sqrt{3}}{2}$. We can see MP is equidistant from AD and BC so that the height of $\triangle MNP$ is $\frac{\sqrt{3}}{4}$. In addition, MP is the average of BC and AD, which is $\frac{3}{2}$. Thus, $[MNP] = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{16}$.

5. **Answer (B):** Group up the numbers in pairs where the numbers add up to 7 : (1-6), (2-5), (3-4). The order in which Clark chooses the faces to write a number on it does not matter. Suppose he writes an arbitrary number on one face. There is a $\frac{1}{5}$ chance that Clark writes the correct number on the opposite face, since there are 5 integers available and only 1 of them is the correct number in the pair. Similarly, after Clark writes another arbitrary integer on a remaining face, there is a $\frac{1}{3}$ chance that he writes the correct number on the opposite face. Then, the remaining opposite face pair will be guaranteed to have the last remaining pair. Thus, the desired probability is $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$.

6. **Answer (C):** Note that $x = 0.6 + 0.\overline{1}$, where we will evaluate the two addends separately. Obviously, $0.6 = \frac{3}{5}$ and $0.\overline{1}$ is well known to equal $\frac{1}{9}$. Thus, $x = \frac{3}{5} + \frac{1}{9} = \frac{32}{45}$. The requested sum is 77.

7. **Answer (D):** Let $z = a + bi$ so that $z + 6 = (a + 6) + bi$. For the magnitudes to be equal, we require $\sqrt{a^2 + b^2} = \sqrt{(a + 6)^2 + b^2}$. Squaring both sides and simplifying, we have $a = -3$. This means the real part of z must be -3, while the imaginary part can be anything. This forms a vertical line.

8. **Answer (A):** We consider which statement is not satisfied in which the other two are. If the third statement is not satisfied, n must be divisible by 6 but not 5. There are 5 mulitples of 6 less than or equal to 30, but cannot count 30 as it is divisible by 5. Thus, there are 4 integers in this case. Similarly, if the second statement is not satisfied, there are $(3 - 1) = 2$ integers. Finally, if the first statement is not satisfied, there is 1 integer. Thus, there are $4 + 2 + 1 = 7$ integers n total.

9. **Answer (E):** Define E as the intersection of the diagonals of $ABCD$, O as the center of the circle, and T be the point where the circle touches BC. It is clear that $\triangle BEC$ is a $3-4-5$ right triangle. By the tangency, $\angle OTC = 90°$. Also, $\angle BCE = \angle OCT$. This implies $\triangle OTC \sim \triangle BEC$. Let r denote the radius of the circle. Clearly, $OT = r$ and $CO = CA - AO = 8 - r$. By the similarity, we have $\frac{CO}{BC} = \frac{OT}{BE} \implies r = 3$.

10. **Answer (B):** As the clock functions from $3:00$ pm to $4:00$ pm, the minute hand rotates a full $360°$ from the 0 minutes position back to the 0 minutes position. However, the hour hand rotates from the 15 minutes position to the 20 minutes position. Let t be the number of minutes after $3:00$ for which T occurs. The minute hand is obviously at the t minutes position. However, the hour hand is at the $(15+\frac{t}{12})$ minutes position. This is because the hour hand starts at the 15 minute position and travels a $\frac{1}{12}$th of the way around the clock within the time from $3:00$ to $4:00$.

Setting the two expressions equal to each other, we have $t = 15 + \frac{t}{12} \implies t = \frac{180}{11} \approx 16$. Thus, T is closest to $3:16$ pm.

11. **Answer (B):** By Pythagorean Theorem and difference of squares, $AB = 28$. Let I be the incenter of $\triangle ABC$. By definition I is equidistant from each of the three sides of the triangle. Note that each point along the line segment AI is equidistant from AB and AC. Similarly, each point along the line segment BI is equidistant from AB and BC. It becomes clear that the desired set of points is precisely $\triangle ABI$.

Let r denote the inradius of $\triangle ABC$. The desired probability is given by $\frac{[ABI]}{[ABC]} = \frac{0.5 \cdot r \cdot AB}{rs}$, where s is the semiperimeter of $\triangle ABC$. The value of r cancels out and then we can work out the fraction equals $\frac{2}{9}$. The requested sum is 11.

12. **Answer (C):** Note that m and n are positive integers both at least 3. It is well-known that the interior angle measure of a regular k-gon is given by $180 - \frac{360}{k}$. To that end, we have

$$\left(180 - \frac{360}{m}\right) - \left(180 - \frac{360}{n}\right) = 24 \implies 15\left(\frac{1}{n} - \frac{1}{m}\right) = 1 \implies mn - 15m + 15n = 0.$$

By Simon's Favorite Factoring Trick, the equation becomes $(m+15)(15-n) = 225$. As $m+15$ is clearly a positive integer, we must have $15-n$ be a positive integer too. We set $15-n$ to positive factors of 225 under 15, namely $1, 3, 5, 9$. This returns $n = 6, 10, 12, 14$ for a sum of 42.

13. **Answer (E):** Note that 4 does not necessarily have to be in the subset for it to be the median.

Case 1: 4 is in the subset (in which it is the median)

We can choose some number (possibly none) of elements to add into the subset below 4. However many elements we choose to add to the subset that are below 4, there must be that same number of elements above 4 so that 4 ends up being the median. There are $\binom{3}{k}$ ways to choose the elements below 4 and $\binom{3}{k}$ ways to choose the elements above 4. Summing from $0 \le k \le 3$, there are $\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20$ subsets in this case.

Case 2: 4 is not in the subset

Then, for 4 to be the median, the subset must have an even number of elements, the middle two of which have an average of 4. We will break into subcases on the middle two numbers.

Case 2.1: 3 and 5 are the middle numbers

Similar to case 1, we must choose some number of elements below 3 to include in the subset. However, we must choose that same number of elements above 5. Thus, there are $\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 6$ subsets in this case.

Case 2.2: 2 and 6 are the middle numbers

Clearly, there are $\binom{1}{0}^2 + \binom{1}{1}^2 = 2$ subsets in this case.

Case 2.3: 1 and 7 are the middle numbers

The only subset is $\{1, 7\}$, so there is 1 subset in this case.

In total, there are $20 + 6 + 2 + 1 = 29$ subsets.

14. **Answer (D):** Let a be the probability that coin A lands on heads when flipped by itself.

Define b similarly for coin B. We are given that $a \cdot b = \frac{8}{35}$ and $(1-a)(1-b) = \frac{1}{7}$. We simplify the second equation:

$$1 - a - b + ab = \frac{1}{7} \implies a + b = \frac{38}{35}.$$

We have that $a + b = \frac{38}{35}$ and $a \cdot b = \frac{8}{35}$. By Vieta's formulas, a and b are the two roots of the quadratic $x^2 - \frac{38x}{35} + \frac{8}{35}$. However, flipping coin A and obtaining heads is more likely than flipping coin B and obtaining heads, so a will be the larger of the two roots of the quadratic.

Using the quadratic formula, we find that $a = \frac{4}{5}$ and $b = \frac{2}{7}$. The quantity we seek is $a(1-b)$, which is equal to $\frac{4}{7}$.

15. **Answer (D):** We claim that the only x that satisfy the condition are primes and prime powers.

Obviously, if we take $x = p$ for some prime p, none of $1, 2, 3, ..., p-1$ are divisible by p. Thus, $\text{lcm}(1, 2, 3, ..., p)$ will be divisible by p and $\text{lcm}(1, 2, 3, ..., p-1)$ will not, making the two expressions unequal. Similarly, for $x = p^k$, where p is prime and $k \geq 2$, none of $1, 2, 3, ..., p^k - 1$ are divisible by p^k and thus, the two expressions will be unequal.

Now, it suffices to show no other x satisfy the condition. Prime factorize x as $p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k}$. Note that $p_1^{e_1}, p_2^{e_2}, ..., p_k^{e_k}$ are all distinct integers less than x, that is to say each appears among $1, 2, 3, ..., x-1$. Thus, the two LCMs remain equal even when x is added.

The valid x in the given range are $31, 32, 37, 41, 43, 47, 49$ for which the requested sum is 280.

16. **Answer (C):** Let AB and CD intersect at X, F be the foot of the altitude from C to AD, M be the midpoint of BC, and N be the midpoint of CD. In addition, let $\angle BXC = \theta$ so that $\angle MXC = \angle FCD = \frac{\theta}{2}$. Since $CF = 5$ and $CD = \sqrt{26}$, we have $FD = 1$ in which $\tan(\frac{\theta}{2}) = \frac{1}{5}$ by $\triangle CFD$. Then, by similar triangles $\triangle CFD$ and $\triangle XMC$, there exists a positive real number x such that $MC = x$, $BC = 2x$ and $XC = x\sqrt{26}$. Also, by equilateral triangle $\triangle CDE$, we have $CN = \frac{\sqrt{26}}{2}$ and $EN = \frac{\sqrt{78}}{2}$.

By the double angle formula, we have $\tan(\theta) = \frac{5}{12}$. Then, by right triangle $\triangle XNE$ and breaking XN into $XC + CN$, we have:

$$\tan(\theta) = \frac{EN}{XC + CN} \implies \frac{5}{12} = \frac{\frac{\sqrt{78}}{2}}{x\sqrt{26} + \frac{\sqrt{26}}{2}} \implies BC = 2x = \frac{12\sqrt{3}}{5} - 1.$$

The requested sum is $12 + 3 + 5 + 1 = 21$.

17. **Answer (B):** Since n is odd, all of its divisors are odd, implying $\tau(n)$ is odd. Since $\tau(n)$ is only odd if and only if n is a perfect square, we have that n is an odd perfect square. Note that $21^2 < 500 < 23^2$, so there are not many perfect squares to check.

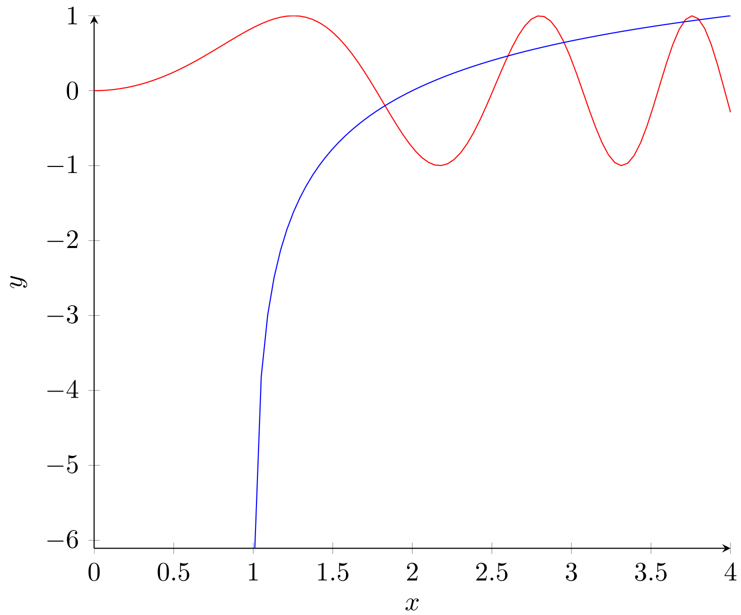
Obviously, $\tau(1) = 1$, so 1 is refactorable. We can see that $\tau(3^2) = 3$, so 9 is also refactorable. Note that $\tau(p^2) = 3$ for some prime p and that $3 \nmid p^2$ if $p \neq 3$. Thus, we can immediately rule out all squares of a prime besides 3^2. This narrows our search down to $9^2, 15^2$, and 21^2.

Since $\tau(9^2) = 5$, 81 is not refactorable. However, $\tau(15^2) = \tau(21^2) = 9$, which divides both 15^2 and 21^2. Thus, we have 4 odd refactorable integers less than 500.

18. **Answer (A):** Let $f(x) = \sin(x^2)$ and $g(x) = \log_2(\log_2(x))$ so that we seek the number of solutions to $f(x) = g(x)$. We can that the graph of $f(x)$ consists of a series of humps that alternate in sign. Specifically, the humps alternate in sign at the zeroes of $f(x)$. The precise location of the humps' apexes does not matter; only where the humps start and stop. We can see that the zeroes of $f(x)$ occur at $x = \sqrt{m\pi}$ for some integer m.

Since $f(x) \leq 1$ due to the sin function, we require $g(x) \leq 1 \implies x \leq 4$. Checking the zeroes of $f(x)$ in that range, we find $x = \sqrt{\pi}, \sqrt{2\pi}, ..., \sqrt{5\pi}$, since $\sqrt{5\pi} \approx \sqrt{15.70} < 4$.

Obviously, $g(x)$ is a strictly increasing function with a zero at $x = 2$. A rough graph of the two functions then becomes apparent. See the figure below. There are 5 intersection points, which corresponds to 5 solutions.



19. **Answer (E):** Let $T(i, j)$ denote the set of all permutations with $a_i = j$ and $a_j = i$, where $1 \le i < j \le 6$. By Principle of Inclusion-Exclusion, N is given by

$$(|T(1,2)| + |T(1,3)| + ...) - (|T(1,2) \cap T(3,4)| + |T(1,2) \cap T(3,5)| + ...)$$
$$+(|T(1,2) \cap T(3,4) \cap T(5,6)| + |T(1,2) \cap (3,5) \cap T(4,6)| + ...).$$

Each term will be calculated separately.

For the terms in the form $|T(A, B)|$, there are $\binom{6}{2}$ ways to choose (A, B). Since $a_A = B$ and $a_B = A$, there are 4! ways to permute the remaining elements.

For the terms in the form $|T(A, B) \cap T(C, D)|$, we have $\frac{\binom{6}{2} \cdot \binom{4}{2}}{2!}$ ways to choose (A, B) and (C, D), where we divide by 2! because the order in which we choose (A, B) and (C, D) does not matter. For example, choosing $(1, 2)$ for one pair and then choosing $(3, 4)$ is the same as choosing it the other way around. We know that a_A, a_B, a_C, and a_D are all uniquely determined. Thus, there are 2! ways to permute the remaining elements.

Finally, for the terms in the form $|T(A, B) \cap T(C, D) \cap T(E, F)|$, there are $\frac{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{3!}$ ways to choose the three pairs (again, without regards to the order we choose them in). After the pairs are chosen, every element is uniquely determined in the permutation.

Thus,
$$N = \binom{6}{2} \cdot 4! - \frac{\binom{6}{2} \cdot \binom{4}{2}}{2!} \cdot 2! + \frac{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{3!} = 360 - 90 + 15 = 285.$$

The requested sum is 15.

20. **Answer (C):** Let $n = 3^a \cdot b$, where a is a nonnegative integer and b is a positive integer not divisible by 3. Then, because 3^a and b are relatively prime, the multiplicity of the σ function implies $\sigma(3^a \cdot b) = \sigma(3^a) \cdot \sigma(b)$. Also, note that since the divisors of 3^a are $1, 3, ..., 3^a$, we have $\sigma(3^a) = \frac{3^{a+1}-1}{2}$ by the sum of a geometric series. To that end, the equation becomes

$$\sigma(n) + \sigma(3n) = \sigma(3^a) \cdot \sigma(b) + \sigma(3^{a+1}) \cdot \sigma(b) = \sigma(b) \cdot (\sigma(3^a) + \sigma(3^{a+1})) = \sigma(b) \cdot (6 \cdot 3^a - 1) = 954.$$

Now, we seek divisors of $954 = 2 \cdot 3^2 \cdot 53$ in the form $(6 \cdot 3^a - 1)$ for some nonnegative integer a. We can see that $(6 \cdot 3^a - 1)$ is relatively prime to 2 and 3 regardless of what a is, so it must divide 53. We can see that setting it equal to 1 does not give a nonnegative integer a, but setting it equal to 53 implies $a = 2$. This in turn implies $\sigma(b) = 18$, so it suffices to find the number of positive integers b that satisfy that. Since the sum of divisors includes b and 1, we know that $\sigma(b) \geq b + 1$, so we only need to check $b \leq 17$. Also, we are told b is not divisible by 3.

We find that $b = 10$ and $b = 17$ are the only solutions; thus, there are 2 solutions.

21. **Answer (C):** Define M as the midpoint of AB, and define N as the midpoint of AC. Because $BM = 3$ and $BC = 4$, it follows that $MC = 5$. In addition, we can use Pythagorean Theorem on $\triangle ABC$ to find that $AC = 2\sqrt{13}$. Because $AN = NC = BN$, we have that $BN = \sqrt{13}$.

Using the property that the centroid splits the medians in a $2:1$ ratio, we find that $MG = \frac{5}{3}$ and $GN = \frac{\sqrt{13}}{3}$. We can use Power of a Point to find that $AM \cdot MB = DM \cdot MG$ or $3 \cdot 3 = DM \cdot \frac{5}{3}$, which leads to $DM = \frac{27}{5}$. It then follows that $DG = \frac{106}{15}$. Power of a Point also tells us that $AN \cdot NC = GN \cdot NE$ or $\sqrt{13} \cdot \sqrt{13} = \frac{\sqrt{13}}{3} \cdot NE$, which leads to $NE = 3\sqrt{13}$. It then follows that $GE = \frac{10\sqrt{13}}{3}$.

We seek $[DGE]$. Consider $\triangle DGE$ and $\triangle MGE$. Both triangles have the same height but have different bases. We can then write $\frac{[DGE]}{[MGE]} = \frac{DG}{MG}$. Now, consider $\triangle MGE$ and $\triangle MGN$. Again, both triangles have the same height but different bases. Therefore, $\frac{[MGE]}{[MGN]} = \frac{GE}{NG}$. We multiply the two equations: $\frac{[DGE]}{[MGE]} \cdot \frac{[MGE]}{[MGN]} = \frac{DG}{MG} \cdot \frac{GE}{NG}$.

Cancelling the $[MGE]$ and then plugging in the side lengths we found earlier, our equation now becomes:

$$\frac{[DGE]}{[MGN]} = \frac{DG}{MG} \cdot \frac{GE}{NG} \implies \frac{[DGE]}{[MGN]} = \frac{\frac{106}{15}}{\frac{5}{3}} \cdot \frac{\frac{10\sqrt{13}}{3}}{\frac{\sqrt{13}}{3}} \implies [DGE] = [MGN] \cdot \frac{212}{5}$$

We now seek $[MGN]$. Because MN is parallel to BC, $\triangle MNG$ is similar to $\triangle CBG$ with a $1:2$ ratio due to the properties of the centroid. It then follows that the height of $\triangle MNG$ is $\frac{1}{3} \cdot 3$ with base $MN = 2$. Therefore, $[MGN] = 1$, implying $[DGE] = \frac{212}{5}$. The requested sum is 217.

22. **Answer (D):** Noticing the coefficients of $1, -6$, and 15, we are reminded that these are $\binom{6}{0}$, $-\binom{6}{1}$, and $\binom{6}{2}$. Seeing this, we should try to force in the perfect sixth.

After doing so, we discover that: $(x-1)^6 = -x^3 - 3x^2 - 3x - 1$. We recognize the RHS as $-(x+1)^3$, so we have $(x-1)^6 = -(x+1)^3$.

Taking the cube root of both sides and rearranging terms, we get $x^2 - x + 2 = 0$. Using the quadratic formula, we find that $x = \frac{1 \pm i\sqrt{7}}{2}$, so the requested sum is 10.

23. **Answer (E):** Let O_n be the center of sphere S_n and let T be the point where S_1 and S_2 are tangent to each other. Define point A to be the foot of the perpendicular from T onto plane P_1 and define B to be the tangency point of P_1 and S_3. Consider the plane P_3 that is tangent to both S_1 and S_2 and contains point T. We note that P_3 is perpendicular to both P_1 and P_2. Let C be the intersection of these three planes. If we take the 2-dimensional cross section of plane P_3, S_1 and S_2 both map to a single circle of radius 6 inscribed in a 60° angle (because the dihedral angle between P_1 and P_2 is 60° and P_3 is perpendicular to both P_1 and P_2). S_3 maps to a circle of radius r inscribed in the same 60° and is intersecting with the circle of radius 6.

Now, consider triangle $\triangle CAT$ on the cross section. O_3 lies somewhere along CT and B is the foot of the altitude from O_3 to line AC. Drop the altitude from O_3 onto AT and call the foot D. Because of the tangency of the spheres, we know that $\angle ACT$ is half the dihedral angle and that $\angle CAT = 90^\circ$ because of the tangency. This means that $\angle O_3TA = 60^\circ$. Therefore, $\triangle DTO_3$ is a 30-60-90 triangle. Because $AT = 6$ and $O_3B = r$, $DT = 6 - r$.

Consider the triangle $O_1O_2O_3$. This triangle is isosceles with T being the foot of the altitude from O_3 to O_1O_2. With Pythagorean Theorem, we find that TO_3 is $\sqrt{r^2+12r}$. But we found earlier that $\triangle DTO_3$ is a 30-60-90 triangle and $DT = 6-r$. Therefore, we can write $\sqrt{r^2+12r} = 2(6-r)$.

We square both sides and solve for r with the quadratic formula. We take the smaller root (because $r < 6$) to get $r = 10 - 2\sqrt{13}$. The requested sum is $10 + 2 + 13 = 25$.

24. **Answer (B):** First, we claim that after k is chosen, Matthew's sequence will be invariant mod $(k-1)$. Note that for $k = 10$, we can recognize the process Matthew is doing as taking the digital root of m. It is well-known that in base-10, the digital root of m is congruent to m modulo 9. However, we will analyze why that is the case.

Take an arbitrary base-10 number $(a_n a_{n-1} ... a_1 a_0)_{10}$ where $0 \leq a_i \leq 9$ for $0 \leq i \leq n$.

When we expand the base-10 representation of this number, we get:

$$10^n \cdot a_n + 10^{n-1} \cdot a_{n-1} + ... + 10 \cdot a_1 + a_0.$$

Because $10^n \equiv 1 \pmod 9$ for all nonnegative integers n, taking mod 9 of the base-10 representation of the number gives us:

$$a_n + a_{n-1} + ... + a_1 + a_0.$$

This is the sum of the digits of the number.

Generalizing this result, we can take a generic base-k integer $(a_n a_{n-1} ... a_1 a_0)_k$ where $0 \leq a_i \leq (k-1)$ for $0 \leq i \leq n$. We take the expansion modulo $(k-1)$:

$$k^n \cdot a_n + k^{n-1} \cdot a_{n-1} + ... + k \cdot a_1 + a_0 \equiv a_n + a_{n-1} + ... + a_1 + a_0 \pmod{(k-1)},$$

since all powers of k are equivalent to 1 in modulo $(k-1)$. Thus, we find in modulo $(k-1)$, a number is congruent to the sum of its digits when written base-k.

Next, we claim that Matthew's sequence is non-increasing and stops decreasing when it reaches a number within the interval $[1, k-1]$.

Let's set an inequality comparing the base-k expansion of a number and the sum of its digits in base-k.

$$k^n \cdot a_n + k^{n-1} \cdot a_{n-1} + ... + k \cdot a_1 + a_0 \stackrel{?}{=} a_n + a_{n-1} + ... + a_1 + a_0$$

When we subtract both sides by $a_n + a_{n-1} + ... + a_1 + a_0$, we find that

$$(k^n - 1) \cdot a_n + (k^{n-1} - 1) \cdot a_{n-1} + ... + (k-1) \cdot a_1 \stackrel{?}{=} 0$$

Since $k \geq 2$, we have that $(k^i - 1)$ is a positive integer when i is a positive integer. Also, we know that the a_i terms are nonnegative. Therefore, the LHS must be nonnegative. This implies a number is always greater than or equal to the sum of its digits when written in base-k for $k \geq 2$, meaning the sequence never increases. Notice that the a_0 cancelled upon subtraction, so a_0 can be anything.

$$(k^n - 1) \cdot a_n + (k^{n-1} - 1) \cdot a_{n-1} + ... + (k-1) \cdot a_1 \geq 0$$

Because all the a_i terms are nonnegative and all the $(k^i - 1)$ terms must be positive integers, equality only occurs when $a_1, a_2, ... a_{n-1}, a_n$ are all 0. More specifically, equality occurs when the number is a single digit in base-k, that is to say, it is in interval $[1, k-1]$. Thus, the sequence will always decrease until it reaches the interval $[1, k-1]$ in which the sequence becomes constant. Then, Matthew will have the same number on the board for two consecutive minutes in which that is when he stops.

With our findings, we have that $f(k)$ is the unique integer j such that $1 \leq j \leq k-1$ and $2021^{2021} \equiv j \pmod{k-1}$.

Noticing that 2016 is divisible by 6, 7, and 8, we need to evaluate 5^{2021} in mod 6, 7, and 8.

In mod 6, clearly, $5^{2021} \equiv 5 \pmod 6$, since $5^2 \equiv 1 \pmod 6$. Thus, $f(7) = 5$. In mod 7, we can use Fermat's Little Theorem to find that $5^6 \equiv 1 \pmod 7$, so we analyze $5^5 \pmod 7$, which is 3. Thus, $f(8) = 3$. Finally, in mod 8, $5^2 \equiv 1 \pmod 8$, so $5^{2021} \equiv 5 \pmod 8$, implying $f(9) = 5$.

Thus, $A = 75$, lying in the interval $[51, 100]$.

25. **Answer (D):** First, we should try to understand what the rule. Call a set of three adjacent chairs in the circle *apt* if more than one of the chairs is red. To create an apt set, there must be two or three red chairs among three adjacent chairs.

Take a 13 character string with R for red and B for blue and assume that character 13 and character 1 are connected.

If we let characters 1, 2, and 3 be R, we get: RRR. However, the sets of characters $\{13, 1, 2\}$, $\{1, 2, 3\}$, and $\{2, 3, 4\}$ are all apt sets, regardless of what color chairs 13 and 4 get painted. However, we only want 2 apt sets. Therefore, we can never have 3 red chairs in a row.

Now, let's consider two red chairs in a group of three adjacent chairs. We can either have: RR or RBR.

Case 1: Characters 1 and 2 are RR
Sets $\{13, 1, 2\}$ and $\{1, 2, 3\}$ are both apt sets, regardless of what characters 13 and 3 are. Now, we have to make sure that no other set of three adjacent chairs has more than one red chair in it. By inspection, characters 12, 13, 3, and 4 must be blue. So we have: $B - B - R - R - B - B$. (The hyphens are there just to make the string easier to read)

We cannot have any more apt sets, this 6 character sequence of the string cannot appear anywhere else in the string, so we do not have to worry about the string being the same after some rotation that is not the identity rotation.

Note that there is no restriction on the blue chairs, so the blue chairs on either end of that string have no effect on the remaining 7 chairs. Therefore, we can safely put the remaining 7 chairs in a line and count how many ways there are to paint those 7 chairs in a line such that each red chair in the line must have at least two blue chairs in between that red chair and the next red chair.

Now, we will do subcases based on the number of red chairs. If there are 0 red chairs, the 7 chairs must all be blue, so we have 1 case here. If there is 1 red chair, it can be any of the 7 chairs without restriction, so we have 7 cases here.

If there are 2 red chairs (and 5 blue chairs), there must be at least 2 blue chairs in between the 2 red chairs. Place down chairs in the following order: $R - B - B - R$. Now, we have to insert the remaining 3 blue chairs into the partitions formed by the red chairs. There are 3 sections where we can insert blue chairs, so by Stars and Bars, we have $\binom{5}{2} = 10$ arrangements.

For 3 red chairs, there is only 1 arrangement, namely $R - B - B - R - B - B - R$.

Thus, we have $1 + 7 + 10 + 1 = 19$ arrangements here.

Case 2: Characters 1, 2, and 3 are $R - B - R$

We have one apt set being $\{1, 2, 3\}$, but we need one more. We can't have $R - B - R - R$, but we can have $R - B - R - B - R$.

Alternatively, we can have two separate substrings that are both $R - B - R$ and have at least two blue chairs in between the two substrings in both directions. The $R - B - R$ substrings will be our apt sets. If the two $R - B - R$ substrings are not separated by at least two blue chairs on both sides, we will produce excessive apt sets.

Case 2.1: We have $R - B - R - B - R$

Like with case 1, there must be two consecutive blue chairs on both ends of that string, so our string is now $B - B - R - B - R - B - R - B - B$.

Also, like with case 1, we already have our 2 apt sets. We can't have any more, so this subsequence in the string cannot appear more than once and thus, rotations of the string are irrelevant.

Now we must paint 4 chairs in a line such that each red chair in the line must have at least two blue chairs in between that red chair and the next red chair.

Again, casework on the number of red chairs gives us $1 + 4 + 1 = 6$ arrangements for this case.

Case 2.2: We have two separate substrings that are both $R - B - R$ and have at least two blue chairs in between the two substrings in both directions.

To take rotations into account, assume the blue chair in one of the $R - B - R$ substrings occupies character 2. Thus, the blue chair of the other $R - B - R$ substring can occupy up to character 8 at the furthest.

Case 2.2.1: The second $R - B - R$ substring occupies $\{6, 7, 8\}$

Only chair 11 does not have its color pre-determined. It can be either color, so there are 2 configurations in this case.

Case 2.2.2: The second $R - B - R$ substring occupies $\{7, 8, 9\}$

We can see that every chair has its color predetermined, so there is only 1 configuration in this case.

In total, there are, $19 + 6 + 2 + 1 = 28$ configurations.