2021

CMC 12B

DO NOT OPEN UNTIL SATURDAY, January 30, 2021

Christmas Math Competitions

Questions and comments about problems and solutions for this exam should be emailed to:

`christmas.math.team@gmail.com`

The 4th Annual CIME will be held on Saturday, January 2, 2021, with the alternate on Saturday, February 13, 2021. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate regardless of your score on this competition. All students will be invited to take the 4th Annual Christmas American Math Olympiad (CAMO) or the Christmas Junior Math Olympiad (CJMO) on Saturday, January 9, 2021.

A complete listing of our previous publications may be found at our web site:

`http://cmc.ericshen.net/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent CMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE JANUARY 30, 2021.
2. YOU must not verify on the CMC 10/12 COMPETITION CERTIFICATION FORM (found on `maa.org/amc` under "AMC 12B") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedeX333X or UPS is strongly recommended.
4. The publication, asexual reproduction, sexual reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the definite (but not indefinite) integrity of the results. Dissemination via phone, email, raven, or digital media of any type during this period is a violation of the competition rules.

The **Christmas Math Competitions** *is made possible by the contributions of the following problem-writers and test-solvers:*

David Altizio, Allen Baranov, Ankan Bhattacharya, Luke Choi, Federico Clerici, Rishabh Das, Mason Fang, Raymond Feng, Preston Fu Valentio Iverson, Minjae Kwon, Benjamin Lee, Justin Lee, Kyle Lee, Kaiwen Li, Sean Li, Elliott Liu, Eric Shen, Albert Wang, Anthony Wang, Andrew Wen, Tovi Wen, Nathan Xiong, *and* Joseph Zhang

MAC CMC
Christmas Math Competitions

Christmas Math Competitions
4th Annual

CMC 12B

Christmas Math Contest 12B
Saturday, January 30, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/` and mark you answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the CMC, **you must submit your answers using the Submission Form found at `http://cmc.ericshen.net/CMC-2021/`. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, computing devices, graph paper, protractors, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily not drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-10-12/`.
8. When you give yourself the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-10-12/`.

The Committee on the Christmas Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

All students will be invited to take the 4th annual Christmas Invitational Math Examination (CIME) on Saturday, January 2, 2021 and Saturday, February 13, 2021. More details about the CIME are in the back of this test booklet.

1. What is the value of

$$(2+0+2+1)^2-(2-0+2-1)(2+0-2+1)?$$

(A) 0 **(B)** 13 **(C)** 22 **(D)** 23 **(E)** 25

2. A positive integer is a *semiprime* if it is the product of two distinct primes. Suppose m, n are positive integers such that $m+1$, $m+2$, …, $m+n$ are all semiprimes. What is the largest possible value of n?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

3. What is the units digit of $|2021^2-2^{2021}|$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

4. What is the value of

$$\frac{\sqrt{2^1}}{2^1}+\frac{\sqrt{2^5}}{2^5}+\frac{\sqrt{2^9}}{2^9}+\frac{\sqrt{2^{13}}}{2^{13}}+\frac{\sqrt{2^{17}}}{2^{17}}+\cdots?$$

(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{2\sqrt{2}}{3}$ **(D)** 1 **(E)** $\sqrt{2}$

5. If x, y, z are positive integers such that

$$x\cdot y\cdot z=7!,$$

what is the smallest possible value of $\max(x,y,z)$?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 21

6. Luke the Lizard is crawling along the edges of a cube. He starts at vertex A, and each minute, he moves to one of the three vertices adjacent to the vertex he is currently on with equal probability. What is the probability that after three minutes, he is at vertex B, which is opposite A?

(A) 0 **(B)** $\frac{1}{9}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{1}{3}$ **(E)** 1

7. How many ways can the letters of the word $COMBO$ be arranged such that the M is adjacent to both a vowel and a consonant?

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 32

23. Let $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n. For example, $\varphi(6) = 2$ and $\varphi(15) = 8$. For how many positive integers $1 < n < 100$ does $\varphi(n) \mid (5^n + 1)$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

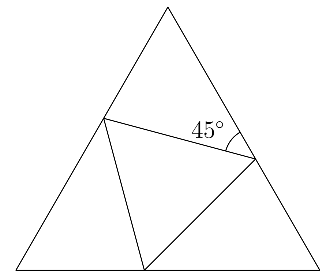
24. Let $p_1 < p_2 < \cdots < p_{24}$ be the distinct permutations of 1234 in increasing order. For positive integers $1 \le i < j \le 24$, we define $d_{(i,j)} = p_j - p_i$. For example, $d_{(1,24)} = 4321 - 1234 = 3087$. Over all possible choices of (i, j), how many distinct values of $d_{(i,j)}$ are there?

(A) 92 **(B)** 94 **(C)** 96 **(D)** 98 **(E)** 100

25. Let $ABCD$ be a quadrilateral with $AB = 8$ and $CD = 11$. If ABC has area 80 and DBC has area 72, then what is the least possible area of $ABCD$?

(A) 96 **(B)** 108 **(C)** 112 **(D)** 128 **(E)** 152

8. An equilateral triangle is inscribed in another equilateral triangle as in the diagram below. If the side length of the outer equilateral triangle is 1, the side length of the inner triangle can be represented as $\frac{\sqrt{a}-\sqrt{b}}{c}$ for positive integers a, b, and c such that a and b are square free. What is $a + b + c$?



(A) 10 **(B)** 12 **(C)** 13 **(D)** 15 **(E)** 16

9. Nine friends play a game. The first person says 1. For $1 \le n \le 8$, if the nth person says k then the $(n + 1)$th person will say either $k - 1$ or $k + 1$ with equal probability. What is the probability that the number the 9th person says is positive?

(A) $\frac{163}{256}$ **(B)** $\frac{41}{64}$ **(C)** $\frac{165}{256}$ **(D)** $\frac{83}{128}$ **(E)** $\frac{167}{256}$

10. On the 31^{st} December 2020, Andre realizes that the number of days until his 21^{st} birthday is the square of the number of entire months leading to it. When does his birthday fall?

(A) May 16^{th} **(B)** Jun 17^{th} **(C)** Jul 18^{th} **(D)** Aug 19^{th} **(E)** Sep 20^{th}

11. Alice and Billy are told by their teacher to compute "$\log x$ to the power of $\log x$ to the power of $\log x$" for some constant x, and where log is the base-10 logarithm. Alice computes

$$\log\left(x^{\log\left(x^{\log x}\right)}\right)$$

while Billy computes

$$\left((\log x)^{\log x}\right)^{\log x},$$

but surprisingly, they get the same answer. Let S be the sum of $(\log x)^2$ over all possible values of x. Find S.

(A) 1 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 8

12. Suppose n is a composite positive integer, and let $f(n)$ be the third-largest divisor of n. For how many positive integers $n \le 720$ is $f(n)$ divisible by 4?

(A) 45 **(B)** 60 **(C)** 72 **(D)** 90 **(E)** 120

13. Let $2x^3 - 11x^2 + 18x - a$ be a polynomial with real roots such that the sum of the reciprocals of two of the roots equals the third root. What is a?

(A) $\frac{9}{4}$ **(B)** $\frac{9}{2}$ **(C)** 9 **(D)** $\frac{81}{4}$ **(E)** $\frac{81}{2}$

14. Jensen has polynomials $(x-1)$, $(x-1)(x-2)$, $\ldots$, $(x-1)(x-2)\cdots(x-2020)(x-2021)$ and he decides to split them in two groups. Let $P(x)$ and $Q(x)$ denote the product of all the polynomials in the first and in the second group, respectively. Given that $Q(x)$ is divisible by $P(x)$, what is the smallest possible degree of $\frac{Q(x)}{P(x)}$?

(A) 1 **(B)** 3 **(C)** 1009 **(D)** 1011 **(E)** 2021

15. There are n values of x which satisfy

$$\lfloor x \rfloor^2 = \{x\}^2 + \frac{2020}{2021}x^2.$$

What is the remainder when n is divided by 5? (Here, $\lfloor \bullet \rfloor$ is the greatest integer function and $\{\bullet\}$ is the fractional part function.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

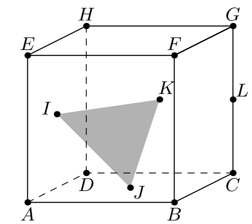
16. Let $ABCD$ be an isosceles trapezoid with bases AB and CD, such that $AB < CD$. Let ℓ be the altitude from B to AC. Suppose that $AD = BC = 5$, $AB = 4$, and ℓ bisects the area of $ABCD$. Compute CD.

(A) $3\sqrt{5}$ **(B)** $4\sqrt{3}$ **(C)** 7 **(D)** $5\sqrt{2}$ **(E)** $2\sqrt{13}$

17. For a permutation of 1, 2, 3, 4, 5, 6, let a_k denote the kth element of the permutation. Suppose that for all integers $1 \leq i < j \leq 5$, $|a_{i+1} - a_i| \neq |a_{j+1} - a_j|$. How many permutations exist in this manner?

(A) 18 **(B)** 24 **(C)** 28 **(D)** 30 **(E)** 36

18. In the figure below, $ABCDEFGH$ is a cube with side length 10. If I is the center of $ADHE$, J is the center of $ABCD$, K is the center of $CDHG$, and L is the midpoint of $\overline{CG}$, then the distance between L and the plane formed by $\triangle IJK$ can be written as $\frac{m}{\sqrt{n}}$ for positive integers m, n such that n is square-free. Find $m + n$.



(A) 7 **(B)** 8 **(C)** 11 **(D)** 12 **(E)** 13

19. Define a positive integer $k \geq 2$ to be *artistic* if there exists a natural number m such that for any positive integer n that is relatively prime to k, we have

$$m^n + n^m \equiv 0 \pmod{k}.$$

Determine the sum of all artistic numbers.

(A) 2 **(B)** 5 **(C)** 8 **(D)** 13 **(E)** 18

20. Let $f(n)$ be a function that takes in a positive integer n and outputs the digit that occurs most frequently in n, with ties going to the smaller digit. For example, $f(123) = 1$ and $f(2020) = 0$. A three-digit positive integer n is selected at random. What is the probability that $f(n)$ is even?

(A) $\frac{31}{60}$ **(B)** $\frac{469}{900}$ **(C)** $\frac{24}{45}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{559}{900}$

21. An infinite arithmetic sequence of nonnegative real numbers a_1, a_2, a_3, $\ldots$ satisfies $a_5^2 - a_2^2 = 432$. Find the minimum value of a_6^2.

(A) 540 **(B)** 648 **(C)** 720 **(D)** 768 **(E)** 840

22. Triangle $\triangle PQR$ with $PQ = 3$, $QR = 4$, $RP = 5$ is drawn inside a regular hexagon $ABCDEF$ with P on segment FA, Q the midpoint of segment AB, and R on segment CD. Given that $AB^2 = \frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

(A) 865 **(B)** 866 **(C)** 867 **(D)** 868 **(E)** 869