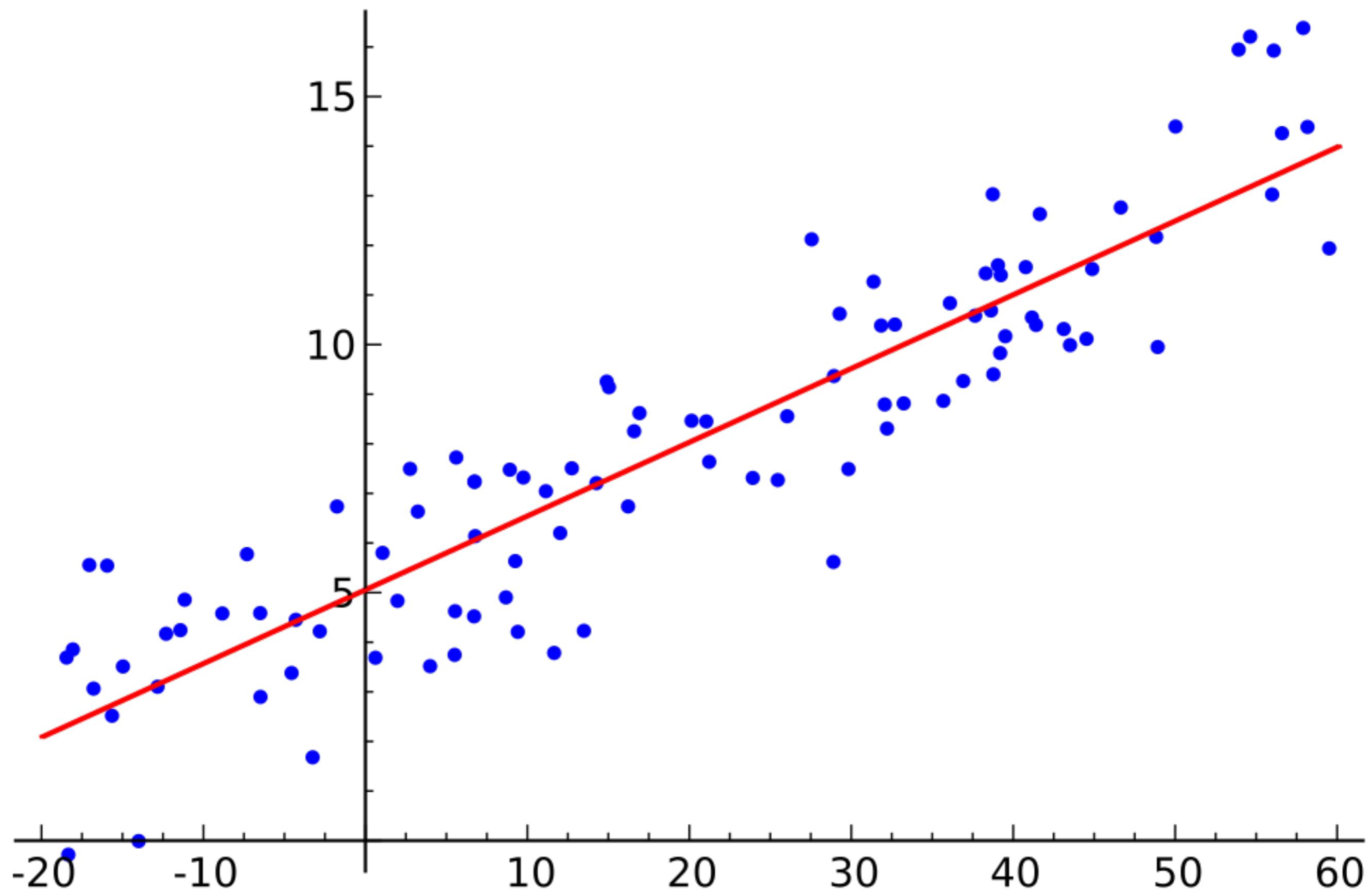
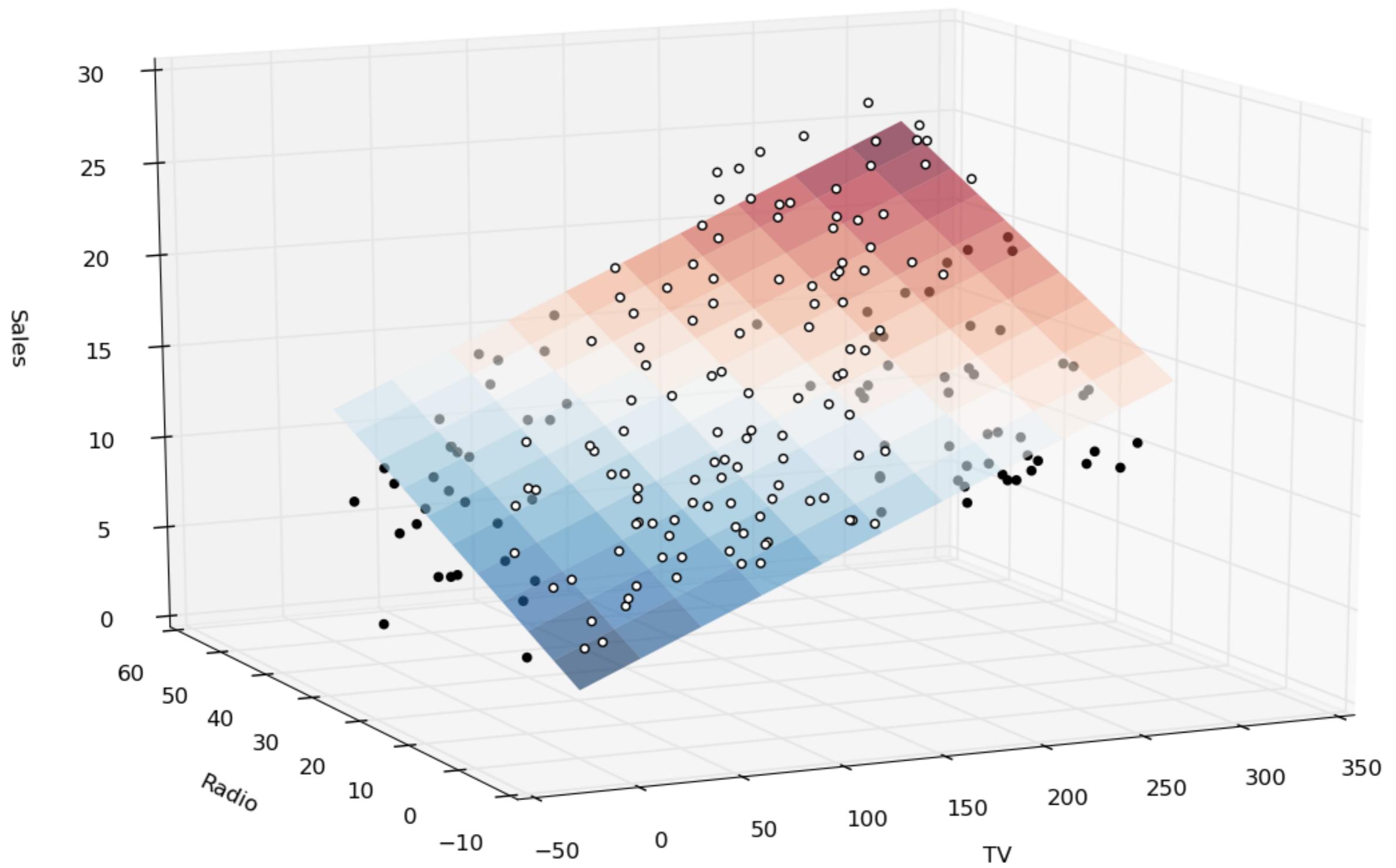




Statistical Regression With Python

Explain & Predict





Explain & Predict

- $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
- A line.
- Explain by β , the slope.
- Predict by new x_i .
- “Simple linear regression model”
- $y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$
- A n-dim hyperplane.
- $\boldsymbol{\beta}$, a slope vector.
- New x_i , a vector.
- “Multiple linear regression model”

How to find the “line”?



Mosky

- Python Charmer at Pinkoi.
- Has spoken at: PyCons in TW, MY, KR, JP, SG, HK, COSCUPs, and TEDx, etc.
- Countless hours on teaching Python.
- Own the Python packages: ZIPCodeTW, MoSQL, Clime, etc.
- <http://mosky.tw/>

Outline

- The Analysis Steps
 - Define Assumptions
 - Validate Assumptions
 - The Dataset: Fair
- Correlation Analysis
- Ordinary Least Squares
 - Models & Estimations
 - Understand Regression Result
- Model Specification Using the R Formula
- Covariance Types
- Outliers
- Correlation & Causation
- More Models & Estimations
 - Introduction
 - Logit Model

The PDF, Notebooks, and Packages

- The PDF and notebooks are available on <https://github.com/moskytw/statistical-regression-with-python> .
- The packages:
 - \$ pip3 install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn
- Or:
 - > conda install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn

Define Assumptions

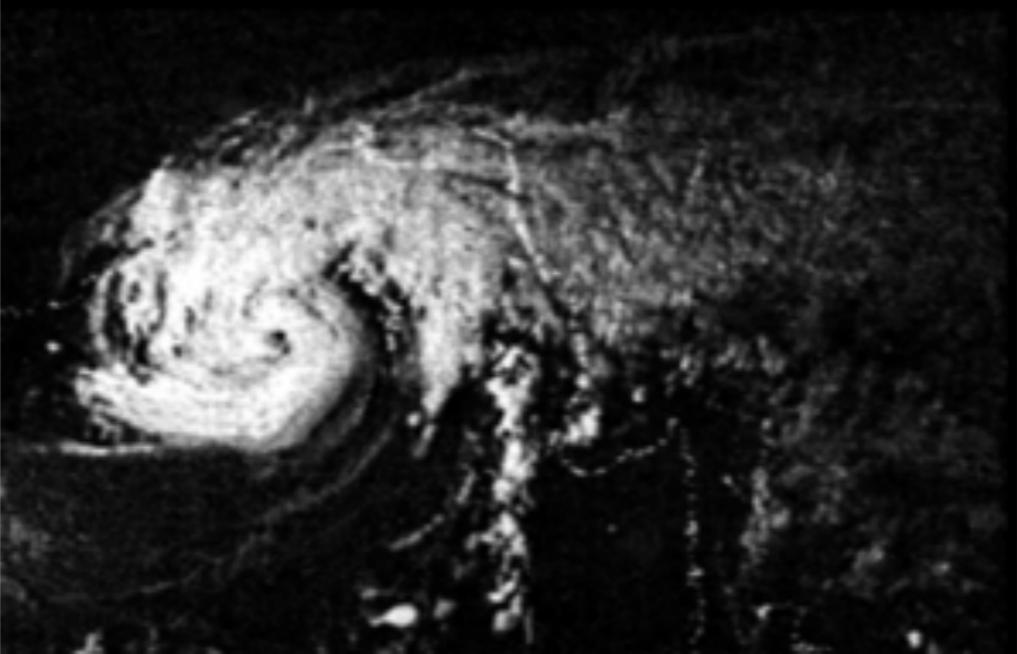
- The regression analysis:
 - Suitable to measure the **relationship** between variables.
 - Can model most of the hypothesis testing. [ref]
 - Can predict.

- “Years of marriage has association with children?”
- “Rates of marriage has association with affairs?”
- “Any background may have association with affairs?”

Validate Assumptions

- Collect data ...
- The “Fair” dataset:
 - Fair, Ray. 1978. “A Theory of Extramarital Affairs,” Journal of Political Economy, February, 45-61.
 - A dataset from 1970s.
 - Rows: 6,366
 - Columns: (next slide)
- The full version of the analysis steps:
<http://bit.ly/analysis-steps> .

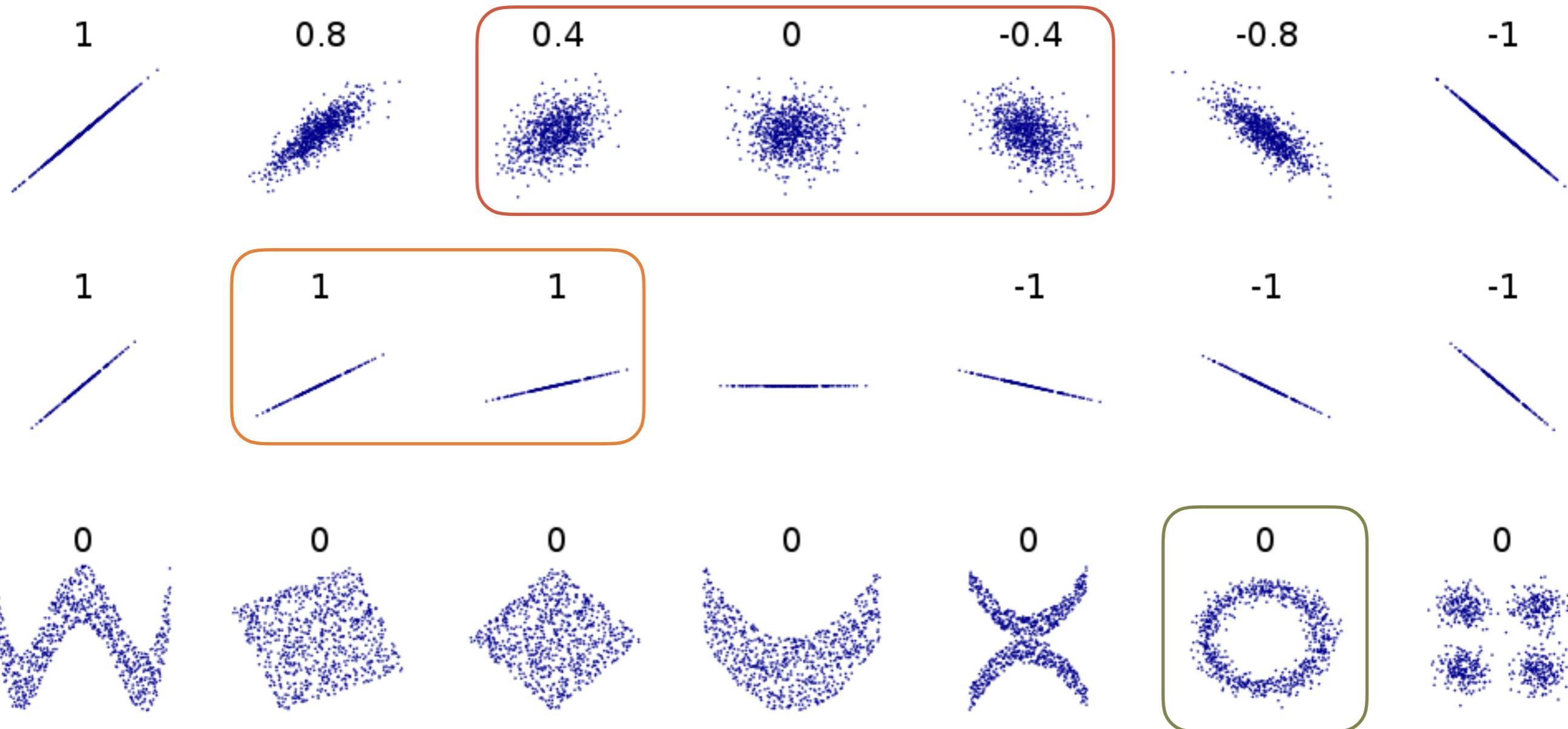
1. *rate_marriage*: 1~5; very poor, poor, fair, good, very good.
2. *age*
3. *yrs_married*
4. *children*: number of children.
5. *religious*: 1~4; not, mildly, fairly, strongly.
6. *educ*: 9, 12, 14, 16, 17, 20; grade school, some college, college graduate, some graduate school, advanced degree.
7. *occupation*: 1, 2, 3, 4, 5, 6; student, farming-like, white-collar, teacher-like, business-like, professional with advanced degree.
8. *occupation_husb*
9. *affairs*: n times of extramarital affairs per year since marriage.

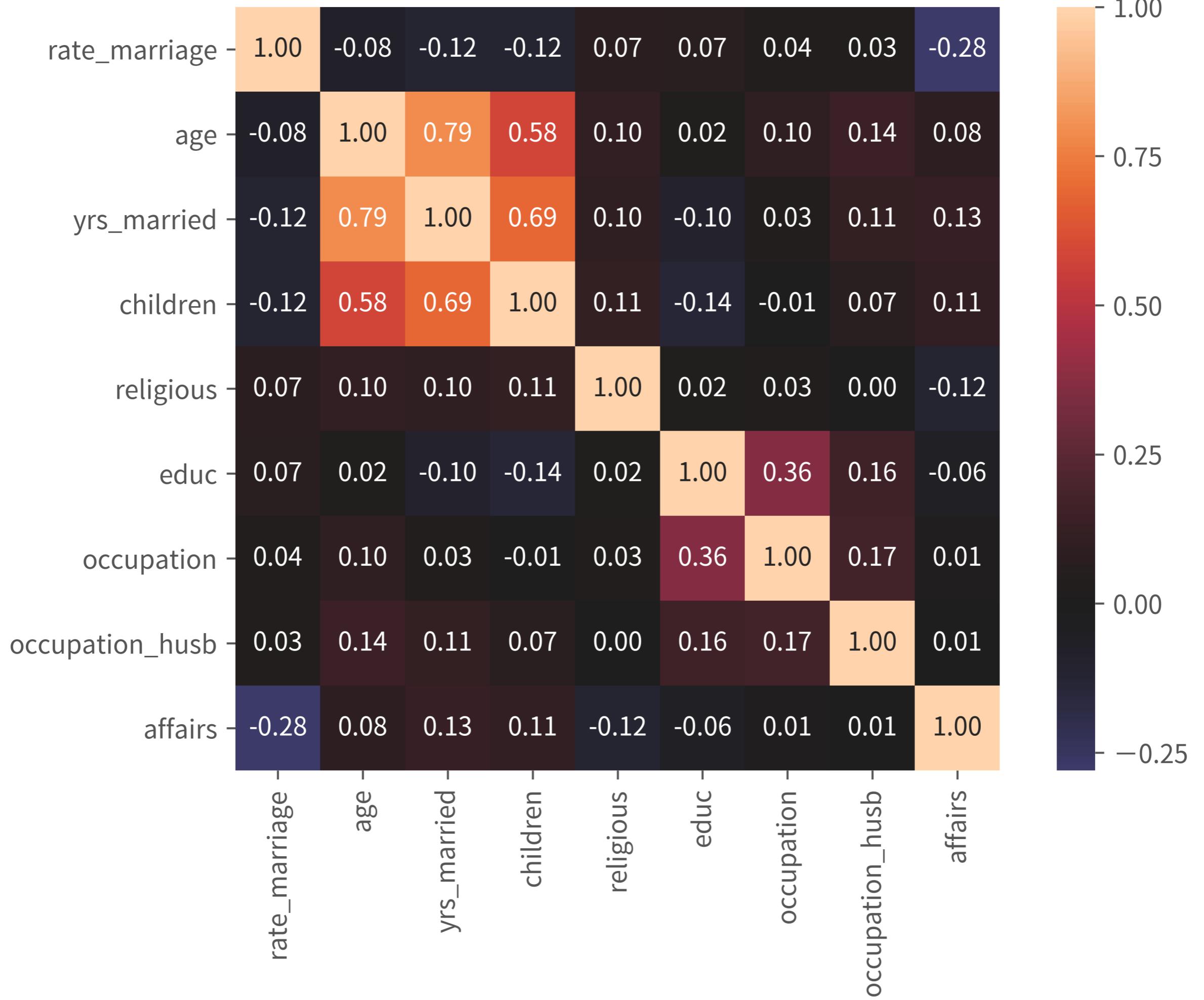


Correlation Analysis

- Measures “the linear tightness”.
- Pearson correlation coefficient
 - For the variables whose **distance is meaningful**.
 - `df.corr()`
- Kendall rank correlation coefficient
 - For the variables whose **order is meaningful**.
 - `df.corr('kendall')`

Pearson Correlation Coefficient





```
import statsmodels.api as sm
import seaborn as sns

print(sm.datasets.fair.SOURCE,
      sm.datasets.fair.NOTE)

# -> Pandas's Dataframe
df_fair = sm.datasets.fair.load_pandas().data

df = df_fair
sns.heatmap(df.corr(method='kendall'),
            center=0, square=True,
            annot=True, fmt='.2f')
```

Models & Estimations

► Models

- $y = X\beta + \varepsilon$
- Like simple, multiple, logit, etc.

► Estimations

- How to estimate the $\hat{\beta}$? For example, **OLS**:

$$y = X\hat{\beta}$$

$$S(b) = \sum_{i=1}^n (y_i - x_i^T b)^2 = (y - Xb)^T (y - Xb)$$

$$\hat{\beta} = \operatorname{argmin}_{b \in \mathbb{R}^p} S(b) = (X^T X)^{-1} X^T y$$

Model Specification Using the R Formula

- Using the R formula implementation in Python, Patsy:

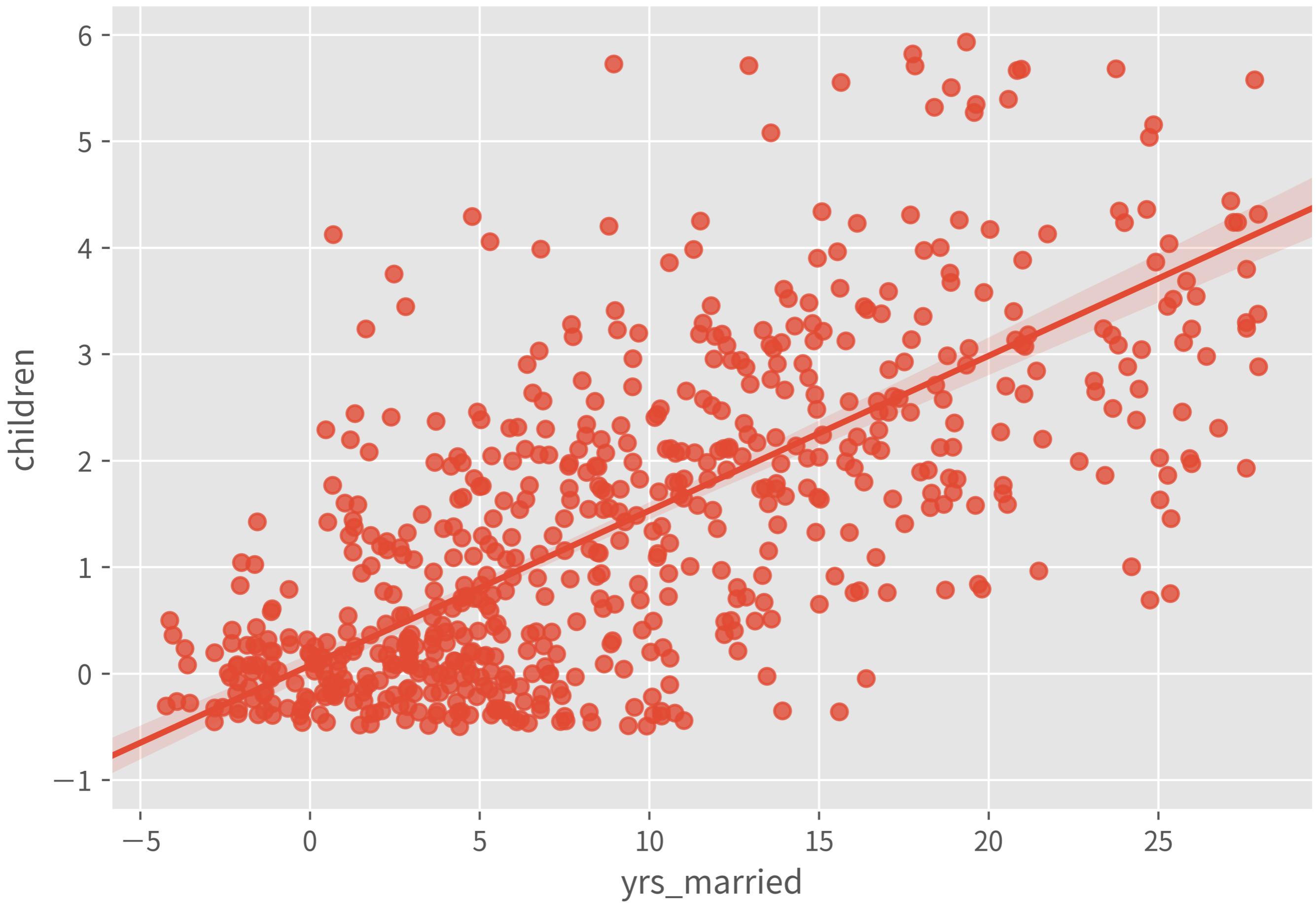
$$y \sim x$$

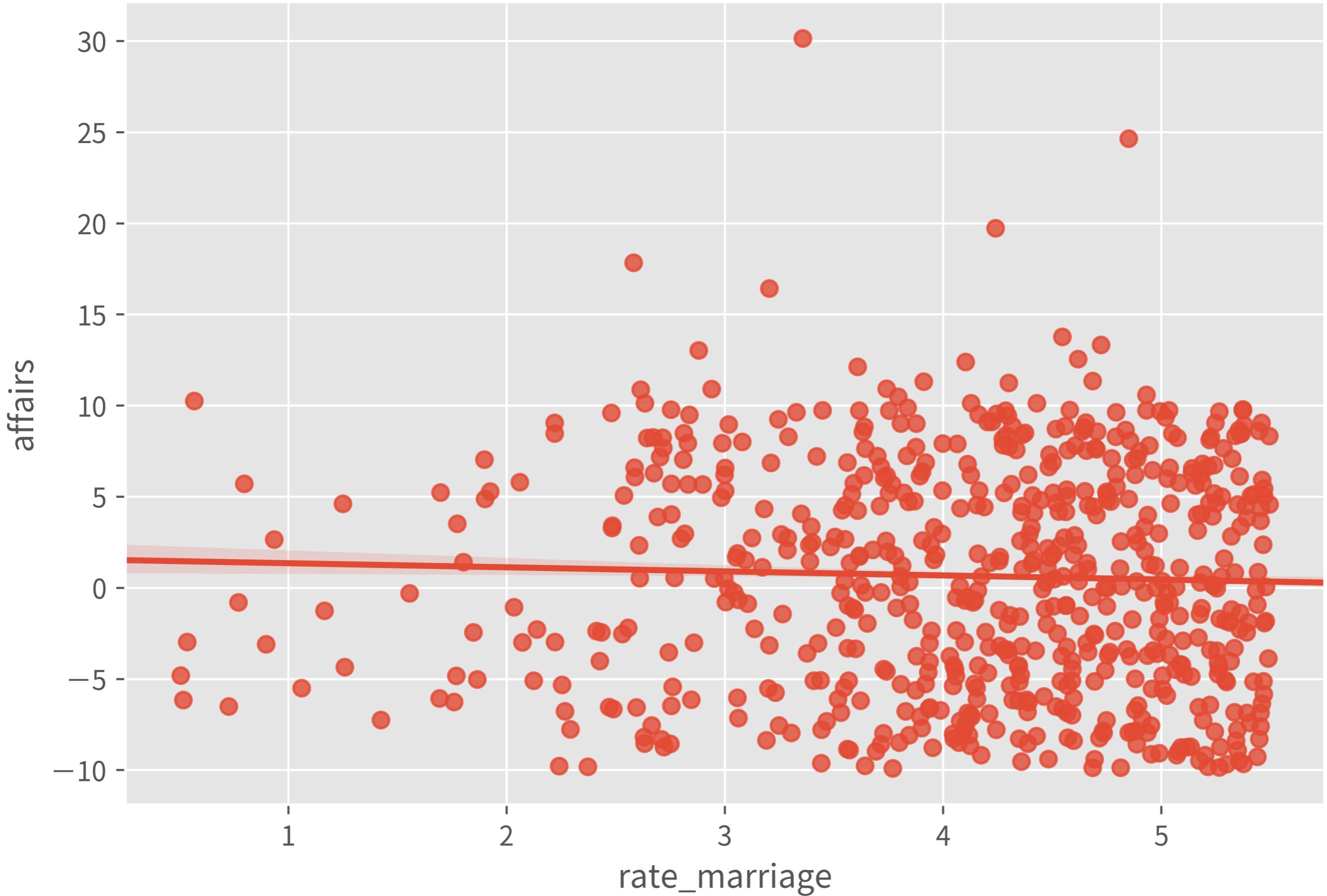
$$\equiv y \sim 1 + x$$

$$\equiv y = \beta_0 1 + \beta_1 x + \varepsilon$$

- For example:

- `affairs ~ rate_marriage`





OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:		1.606		
Prob(Omnibus):	0.000	Jarque-Bera (JB):		5215639.758		
Skew:	8.930	Prob(JB):		0.00		
Kurtosis:	142.083	Cond. No.		19.5		

```
df_fair_sample = df_fair.sample(  
    frac=0.1, random_state=20190425  
)  
  
df = df_fair_sample  
sns.regplot(data=df, x='yrs_married', y='children',  
            x_jitter=10/2, y_jitter=1/2)
```

```
df = df_fair_sample  
sns.regplot(data=df, x='rate_marriage', y='affairs',  
            x_jitter=1/2, y_jitter=20/2)
```

```
df = df_fair  
(smf  
    .ols('affairs ~ rate_marriage', df)  
    .fit()  
    .summary())
```

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

Adj. R-squared

-
- \equiv explained var. by X / var. of y and adjusted by no. of X
- $\in [0, 1]$, usually.
- Can compare among models.
- 0.032 is super bad.

Prob(F-statistics)

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t 	[0.025	0.975
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

- $\equiv P(\text{data} \mid \text{all coeffs are zero})$
- Trust the coeffs if low enough.
- “Low enough” is “ < 0.05 ” in convention.

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

Log-Likelihood

-
- Higher is better.
- Negative, usually.
- Can compare among models when the datasets are the same.
- Also check likelihood-ratio test.

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

Large Sample or Normality

- No. Observations, or
➤ $\geq 110 \sim 200$ [ref]
- Normality of Residuals
 - $\text{Prob}(\text{Omnibus}) \geq 0.05$
 - $\wedge \text{Prob}(JB) \geq 0.05$
- To construct interval estimates correctly, e.g., hypothesis tests on coeffs, confidence intervals.

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

Cond. No.

-
- Measures the degree of multicollinearity.
 - Multicollinearity increases the std err, i.e., decreases efficiency.
 - If ≥ 30 , check:
 - Any variable has redundant information? Like fat % and weight. Drop one.
 - If no, the model is good.
 - Or other suggestions.

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032
Model:	OLS	Adj. R-squared:	0.032
Method:	Least Squares	F-statistic:	208.4
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46
Time:	23:25:02	Log-Likelihood:	-13959.
No. Observations:	6366	AIC:	2.792e+04
Df Residuals:	6364	BIC:	2.794e+04
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353

Omnibus:	9443.528	Durbin-Watson:	1.606
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758
Skew:	8.930	Prob(JB):	0.00
Kurtosis:	142.083	Cond. No.	19.5

P>|t|

.....
➤ $\equiv P(\text{data} \mid \text{the coef is zero})$

➤ Drop the x whose p-value is not low enough.

Coef & Confidence Intervals

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

- “The rate_marriage and affairs has negative relationship, the strength is -0.41, and 95% confidence interval is [-0.46, -0.35].”

Code Categorical Variables

- The str or bool is treated as categorical by default.
- Or use the C function:

$$y \sim C(x)$$

$$\equiv y \sim 1 + (\cancel{x_1} + x_2 + \dots + x_i) - \cancel{x_1}$$

$$\equiv y = \beta_0 1 + (\cancel{\beta_1 x_1} + \beta_2 x_2 + \dots + \beta_i x_i) - \cancel{\beta_1 x_1} + \varepsilon$$

- The x_1 is chosen as reference level automatically.
- For example:

$$C(rate_marriage \in \{1, 2, 3, 4, 5\})$$

$$\equiv 1 + rate_marriage_2 \in \{0, 1\} + \dots + rate_marriage_5 \in \{0, 1\}$$

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.036			
Model:	OLS	Adj. R-squared:	0.035			
Method:	Least Squares	F-statistic:	58.57			
Date:	Sat, 27 Apr 2019	Prob (F-statistic):	1.25e-48			
Time:	15:27:20	Log-Likelihood:	-13946.			
No. Observations:	6366	AIC:	2.790e+04			
Df Residuals:	6361	BIC:	2.794e+04			
Df Model:	4					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	1.2017	0.218	5.524	0.000	0.775	1.628
C(rate_marriage)[T.2.0]	0.4141	0.247	1.679	0.093	-0.069	0.897
C(rate_marriage)[T.3.0]	0.1696	0.228	0.743	0.457	-0.278	0.617
C(rate_marriage)[T.4.0]	-0.5268	0.222	-2.370	0.018	-0.963	-0.091
C(rate_marriage)[T.5.0]	-0.8535	0.222	-3.853	0.000	-1.288	-0.419
Omnibus:	9436.269	Durbin-Watson:	1.612			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5218576.884			
Skew:	8.915	Prob(JB):	0.00			
Kurtosis:	142.127	Cond. No.	21.0			

affairs ~ C(rate_marriage)

-
- $y \sim C(x)$
- If 1, affairs is 1.20.
- If 5, affairs is 1.20-0.85.
- The 2, 3, 4 are not significant to the reference level, which is the 1.

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.036
Model:	OLS	Adj. R-squared:	0.035
Method:	Least Squares	F-statistic:	58.57
Date:	Sat, 27 Apr 2019	Prob (F-statistic):	1.25e-48
Time:	17:20:18	Log-Likelihood:	-13946.
No. Observations:	6366	AIC:	2.790e+04
Df Residuals:	6361	BIC:	2.794e+04
Df Model:	4		
Covariance Type:	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
C(rate_marriage)[1.0]	1.2017	0.218	5.524	0.000	0.775	1.628
C(rate_marriage)[2.0]	1.6157	0.116	13.925	0.000	1.388	1.843
C(rate_marriage)[3.0]	1.3713	0.069	19.963	0.000	1.237	1.506
C(rate_marriage)[4.0]	0.6748	0.046	14.762	0.000	0.585	0.764
C(rate_marriage)[5.0]	0.3482	0.042	8.333	0.000	0.266	0.430

Omnibus: 9436.269 Durbin-Watson: 1.612

Prob(Omnibus): 0.000 Jarque-Bera (JB): 5218576.884

Skew: 8.915 Prob(JB): 0.00

Kurtosis: 142.127 Cond. No. 5.21

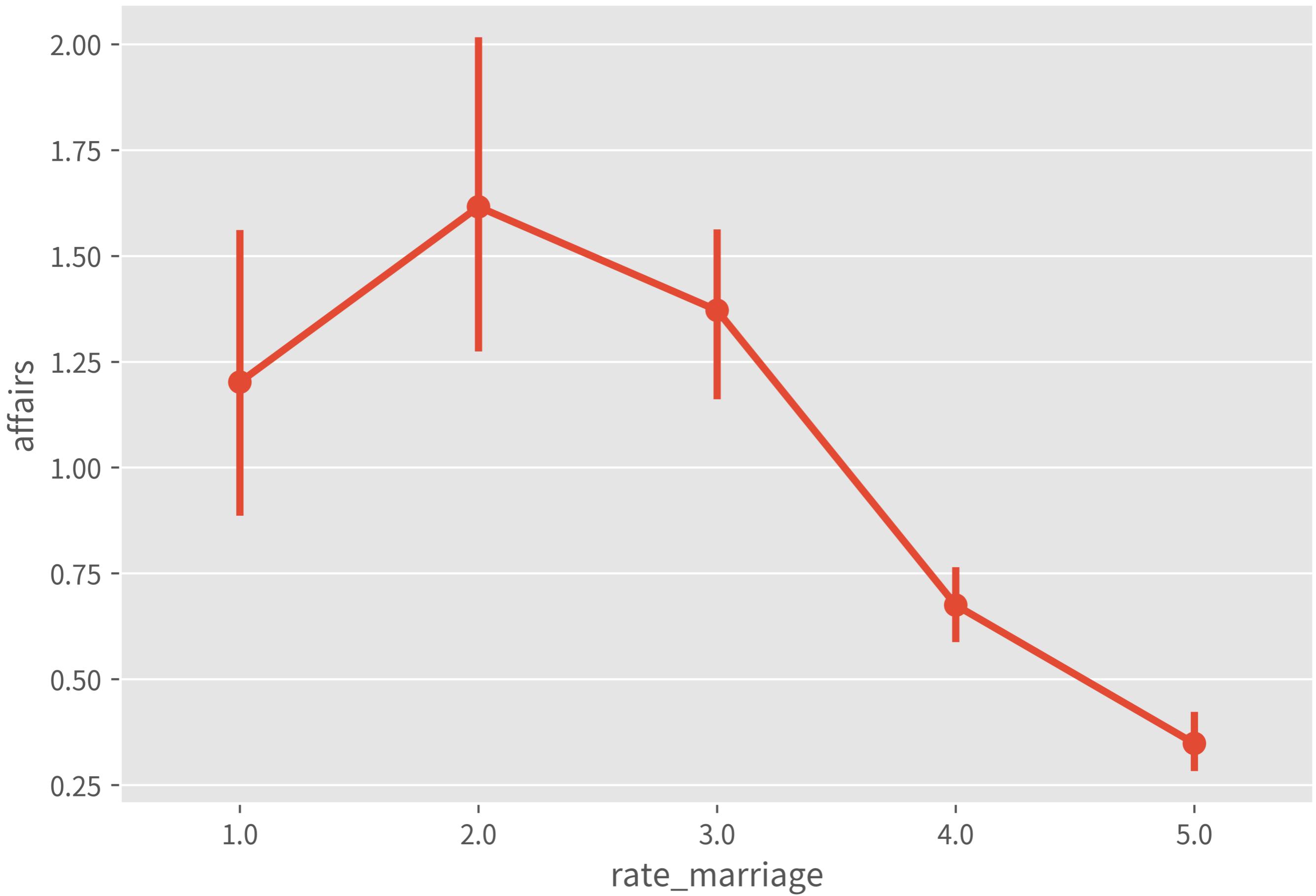
$$\text{affairs} \sim 0 + C(\text{rate_marriage})$$

.....

➤ $y \sim 0 + C(x)$

➤ Code without a reference.

➤ To calculate **the mean of each group**.



```
df = df_fair  
sns.pointplot(data=df, x='rate_marriage', y='affairs')
```

```
df = df_fair  
(smf  
.ols('affairs ~ C(rate_marriage)', df)  
.fit()  
.summary())
```

```
df = df_fair  
(smf  
.ols('affairs ~ 0 + C(rate_marriage)', df)  
.fit()  
.summary())
```

Other Ways to Code Categorical Variables

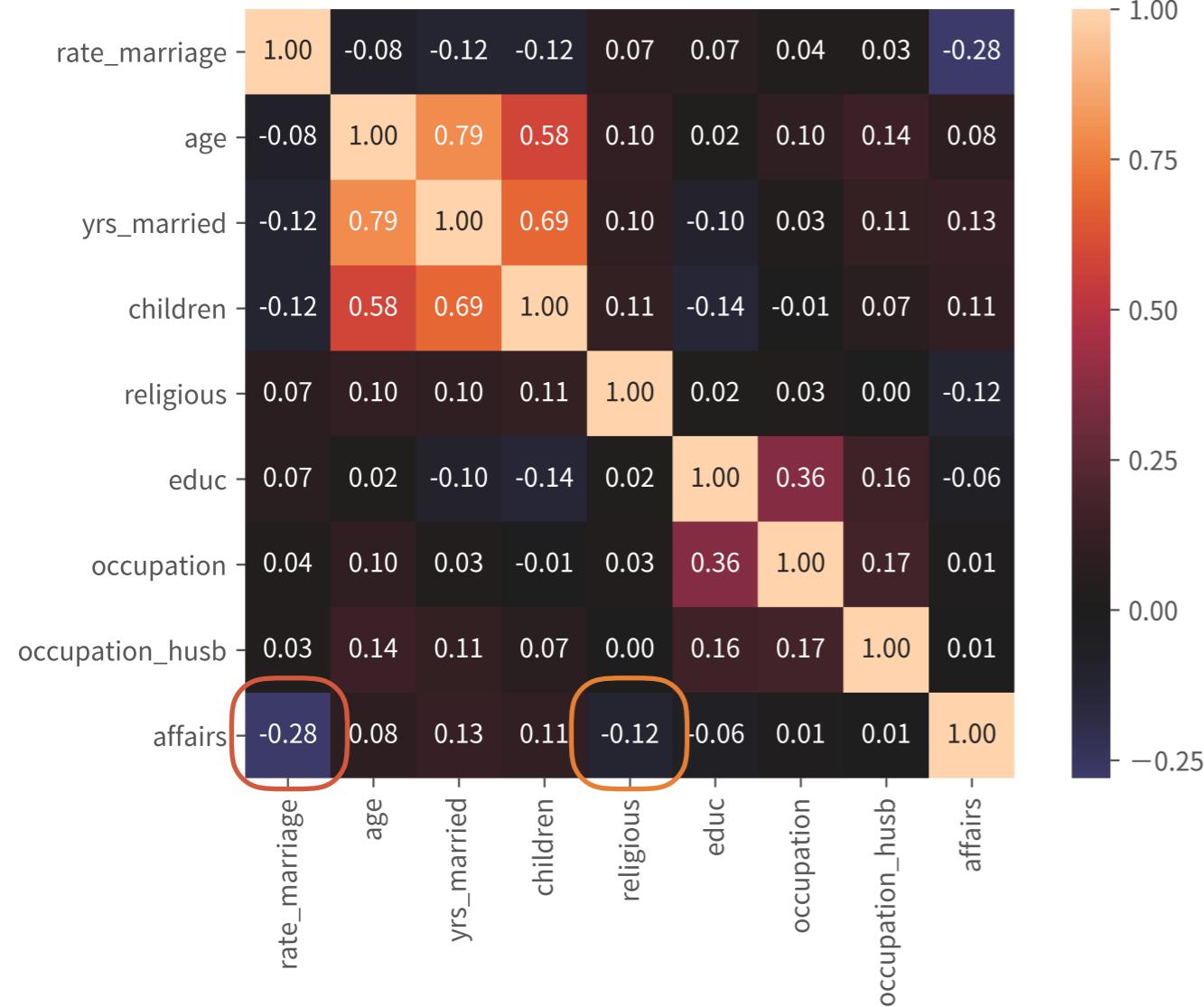
- `y ~ C(x, Treatment(reference='A'))`
 - Specify the reference level.
- `affairs ~ 0 + C(rate_marriage, Diff)`
 - Compare each level **with the preceding level**.
- `affairs ~ 0 + C(rate_marriage, Sum)`
 - Compare each level **with the mean-of-means**.
- Check the full reference.

Interaction

-
- “The **low rate_marriage** with **high religious** has stronger negative relationship with affairs?”

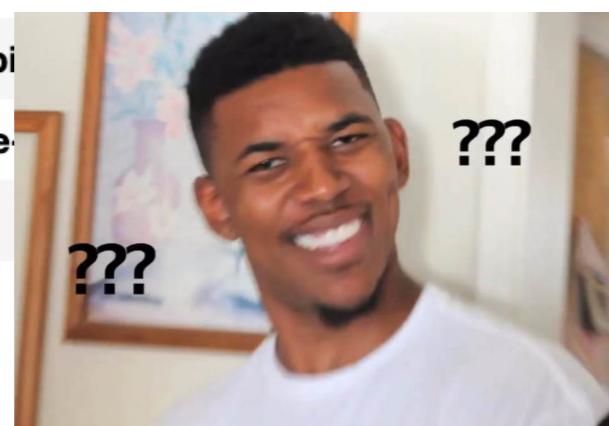
$$y \sim x^* z$$

$$\equiv y = \beta_0 1 + \beta_1 x + \beta_2 z + \beta_3 xz + \epsilon$$



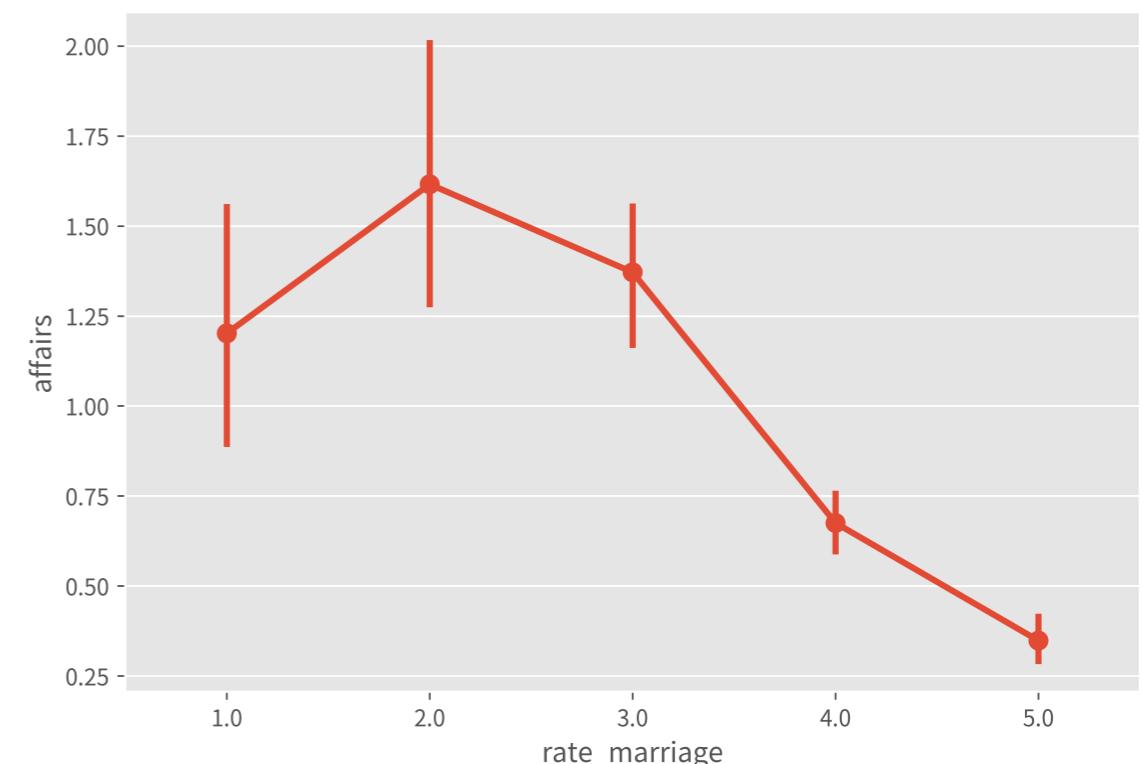
OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.048			
Model:	OLS	Adj. R-squared:	0.048			
Method:	Least Squares	F-statistic:	107.8			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	4.55e-68			
Time:	23:25:03	Log-Likelihood:	-13904.			
No. Observations:	6366	AIC:	2.782e+04			
Df Residuals:	6362	BIC:	2.784e+04			
Df Model:	3					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	4.6495	0.347	13.395	0.000	3.969	5.330
rate_marriage	-0.7891	0.082	-9.622	0.000	-0.950	-0.628
religious	-0.9846	0.138	-7.122	0.000	-1.256	-0.714
rate_marriage:religious	0.1681	0.032	5.209	0.000	0.105	0.231
Omnibus:	9399.882	Durbin-Watson:	1.96			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	100.00			
Skew:	8.843	Kurtosis:	141.848			



affairs ~ rate_marriage*religious

-
- The model may be wrong, since **the relationship is not linear.**



affairs ~ C(rate_marriage)*C(religious)

	coef	std err	z	P> z	[0.025	0.975]
Intercept	1.3341	0.498	2.678	0.007	0.358	2.311
C(rate_marriage)[T.2.0]	1.9654	0.921	2.134	0.033	0.160	3.771
C(rate_marriage)[T.3.0]	1.0479	0.634	1.654	0.098	-0.194	2.290
C(rate_marriage)[T.4.0]	-0.3282	0.517	-0.635	0.525	-1.341	0.685
C(rate_marriage)[T.5.0]	-0.6430	0.522	-1.232	0.218	-1.666	0.380
C(religious)[T.2.0]	0.1143	0.627	0.182	0.855	-1.115	1.343
C(religious)[T.3.0]	-0.3413	0.529	-0.646	0.519	-1.377	0.695
C(religious)[T.4.0]	-0.6082	0.588	-1.035	0.301	-1.760	0.544
C(rate_marriage)[T.2.0]:C(religious)[T.2.0]	-1.8103	1.028	-1.761	0.078	-3.825	0.204
C(rate_marriage)[T.3.0]:C(religious)[T.2.0]	-1.1905	0.754	-1.580	0.114	-2.668	0.286
C(rate_marriage)[T.4.0]:C(religious)[T.2.0]	-0.3499	0.646	-0.542	0.588	-1.615	0.916
C(rate_marriage)[T.5.0]:C(religious)[T.2.0]	-0.4682	0.647	-0.723	0.469	-1.737	0.800
C(rate_marriage)[T.2.0]:C(religious)[T.3.0]	-1.8707	0.950	-1.968	0.049	-3.734	-0.008
C(rate_marriage)[T.3.0]:C(religious)[T.3.0]	-0.9741	0.665	-1.464	0.143	-2.278	0.330
C(rate_marriage)[T.4.0]:C(religious)[T.3.0]	-0.1333	0.549	-0.243	0.808	-1.209	0.943
C(rate_marriage)[T.5.0]:C(religious)[T.3.0]	-0.0448	0.553	-0.081	0.935	-1.128	1.039
C(rate_marriage)[T.2.0]:C(religious)[T.4.0]	-2.2192	0.997	-2.225	0.026	-4.174	-0.265
C(rate_marriage)[T.3.0]:C(religious)[T.4.0]	-1.1008	0.726	-1.517	0.129	-2.523	0.321
C(rate_marriage)[T.4.0]:C(religious)[T.4.0]	-0.0946	0.608	-0.156	0.876	-1.286	1.097
C(rate_marriage)[T.5.0]:C(religious)[T.4.0]	0.0196	0.608	0.032	0.974	-1.173	1.212

.....

	sum_sq	df	F	PR(>F)
Intercept	32.952681	1.0	7.171451	0.007426
C(rate_marriage)	117.831212	4.0	6.410865	0.000038
C(religious)	11.838124	3.0	0.858772	0.461713
C(rate_marriage):C(religious)	111.850861	12.0	2.028497	0.018427
Residual	29159.748241	6346.0	NaN	NaN

➤ Looks like C(religious) isn't helping to explain. **Drop it.**

```

res = res_1
df = pd.DataFrame(dict(params=res.params,
                       pvalues=res.pvalues))
df[df.pvalues < 0.05].sort_values('params')

```

	params	pvalues
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-2.827311	0.000450
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-2.211959	0.005104
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-1.708937	0.000059
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-1.695969	0.037348
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-1.315414	0.001136
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-1.076195	0.010052
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.702780	0.000006
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-0.588575	0.000194
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.474564	0.001321
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.386073	0.016820
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.353891	0.027721
Intercept	1.334104	0.007407
C(rate_marriage)[T.2.0]	1.965381	0.032863

.....

- affairs ~ C(rate_marriage)*C(religious)
- C(religious)

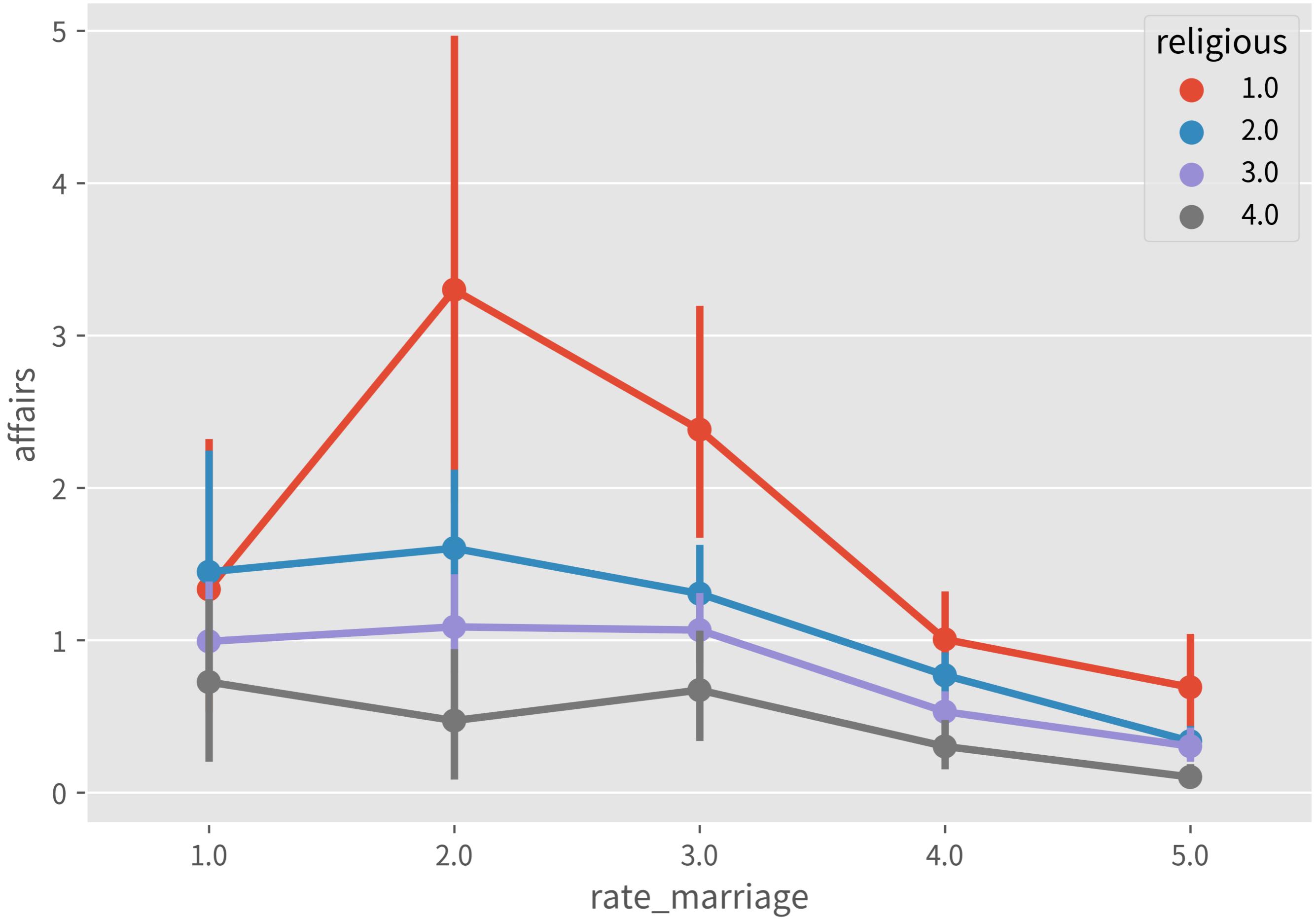
Or:

- affairs ~ C(rate_marriage)
+ C(religious):C(rate_marriage)

$$y \sim x : z$$

$$\equiv y = \beta_0 + \beta_1 xz + \varepsilon$$

- “The low rate_marriage with high religious has stronger negative relationship with affairs?” Yes!



```
df = df_fair
(smf
.ols('affairs ~ rate_marriage*religious', df)
.fit()
.summary())
```

```
df = df_fair
res = (smf
.ols('affairs'
      '~ C(rate_marriage)*C(religious)', df)
.fit())
display(res.summary(),
        # type III is suitable to unbalanced dataset
        # ref: http://bit.ly/3typess
        sm.stats.anova_lm(res, typ=3))
```

```
df = df_fair
res = (smf
       .ols('affairs'
             '~ C(rate_marriage)'
             '+ C(rate_marriage):C(religious)', df)
       .fit())
display(res.summary(),
        sm.stats.anova_lm(res, typ=3))
```

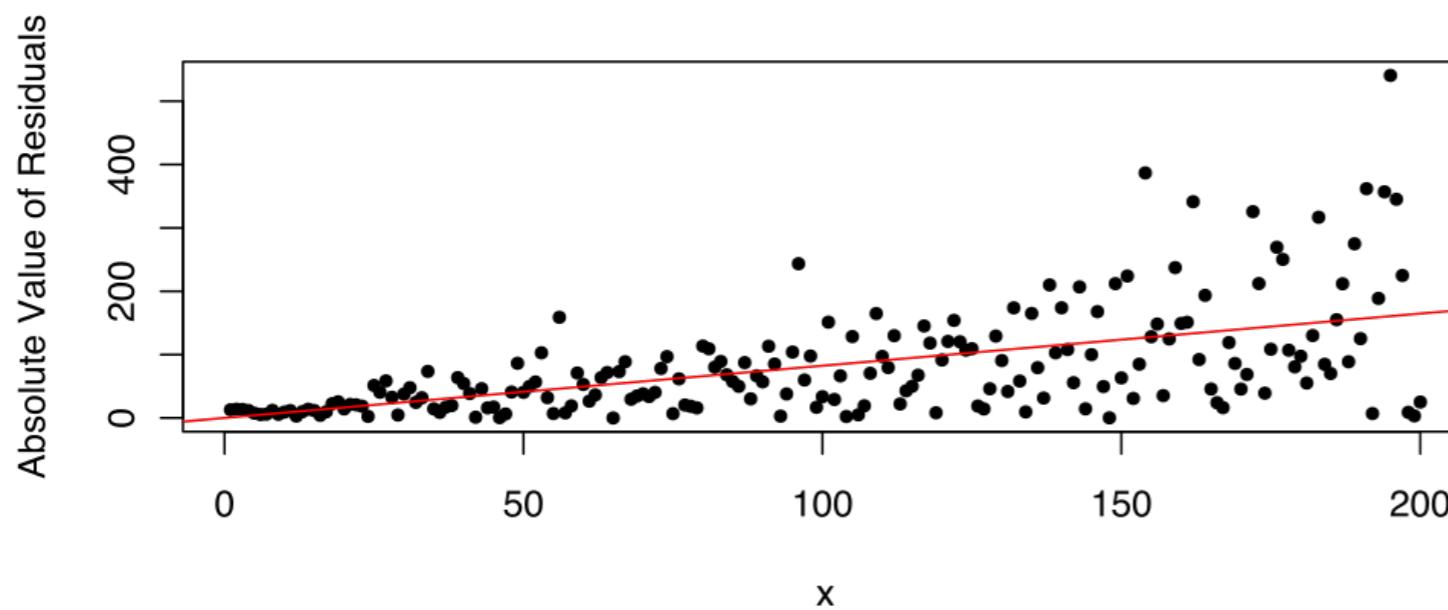
```
df = df_fair
sns.pointplot(data=df,
               x='rate_marriage',
               y='affairs',
               hue='religious')
```

More Operators: Transformation & Control Variables

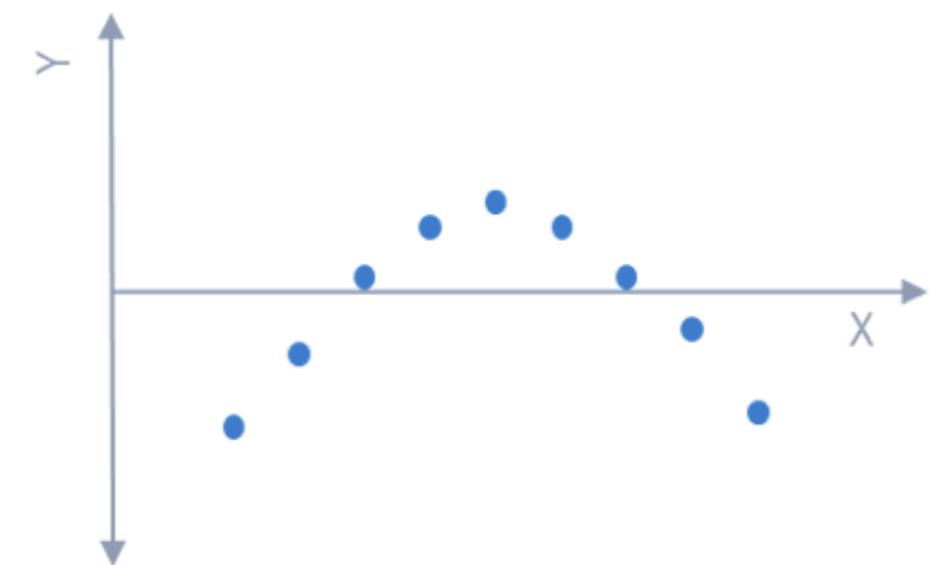
- `np.log(y) ~ x`
 - If y and x have a better linear relationship after transform.
 - Note that:
 - $\log(y) = \hat{\beta}_1x_1 + \hat{\beta}_2x_2$
 - $y = \exp(\hat{\beta}_1x_1 + \hat{\beta}_2x_2)$
 - $y = \exp(\hat{\beta}_1x_1) \times \exp(\hat{\beta}_2x_2)$
- `np.sqrt(y) ~ x`
- $y \sim I(x*z)$
 - True multiplication.

- $y \sim z_1 + \dots + x_1 + \dots$
- The z_i and x_i are both independent variables.
- If we don't interest in z_i , but they can carry some effects and clarify the effects of x_i , we call z_i “**control variables**”.
- For example:
 - $\text{GMV} \sim \text{month} + \text{group}$
- Check the full reference.

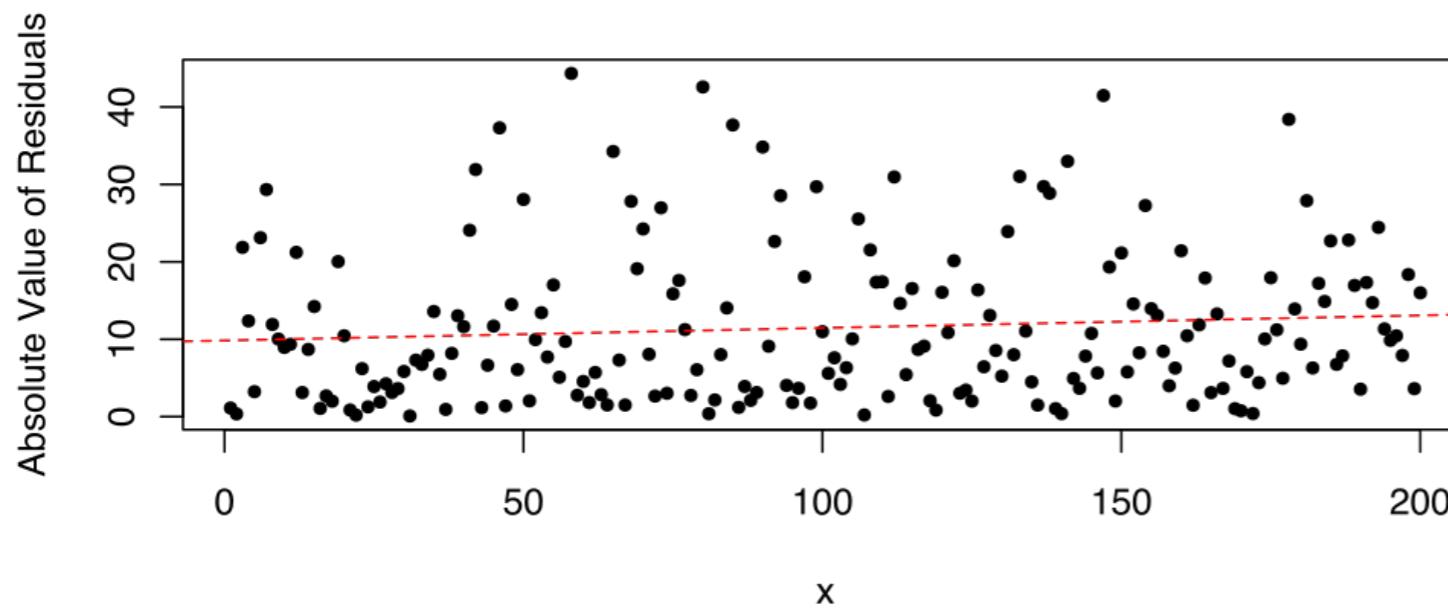
Heteroskedastic Residuals



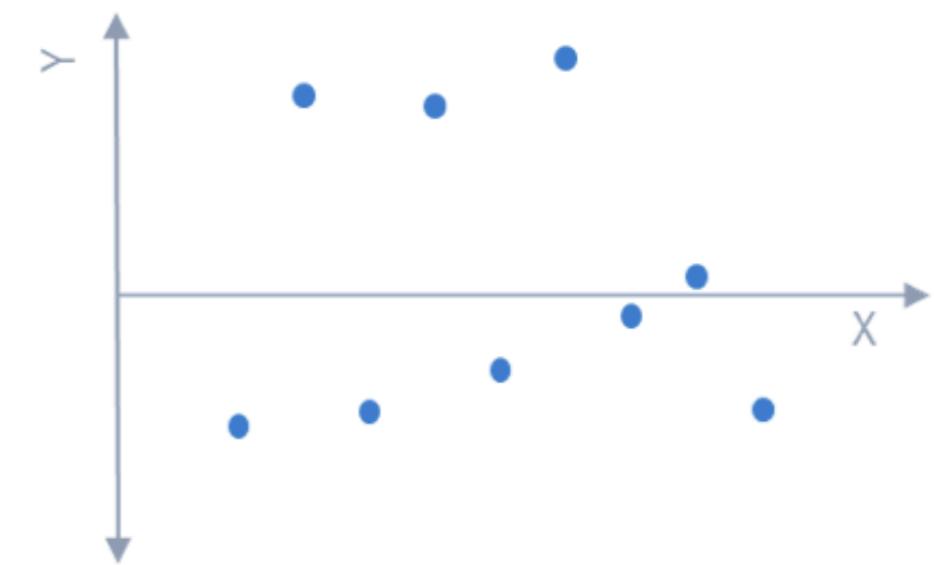
Positive autocorrelation



Homoskedastic Residuals



Negative autocorrelation

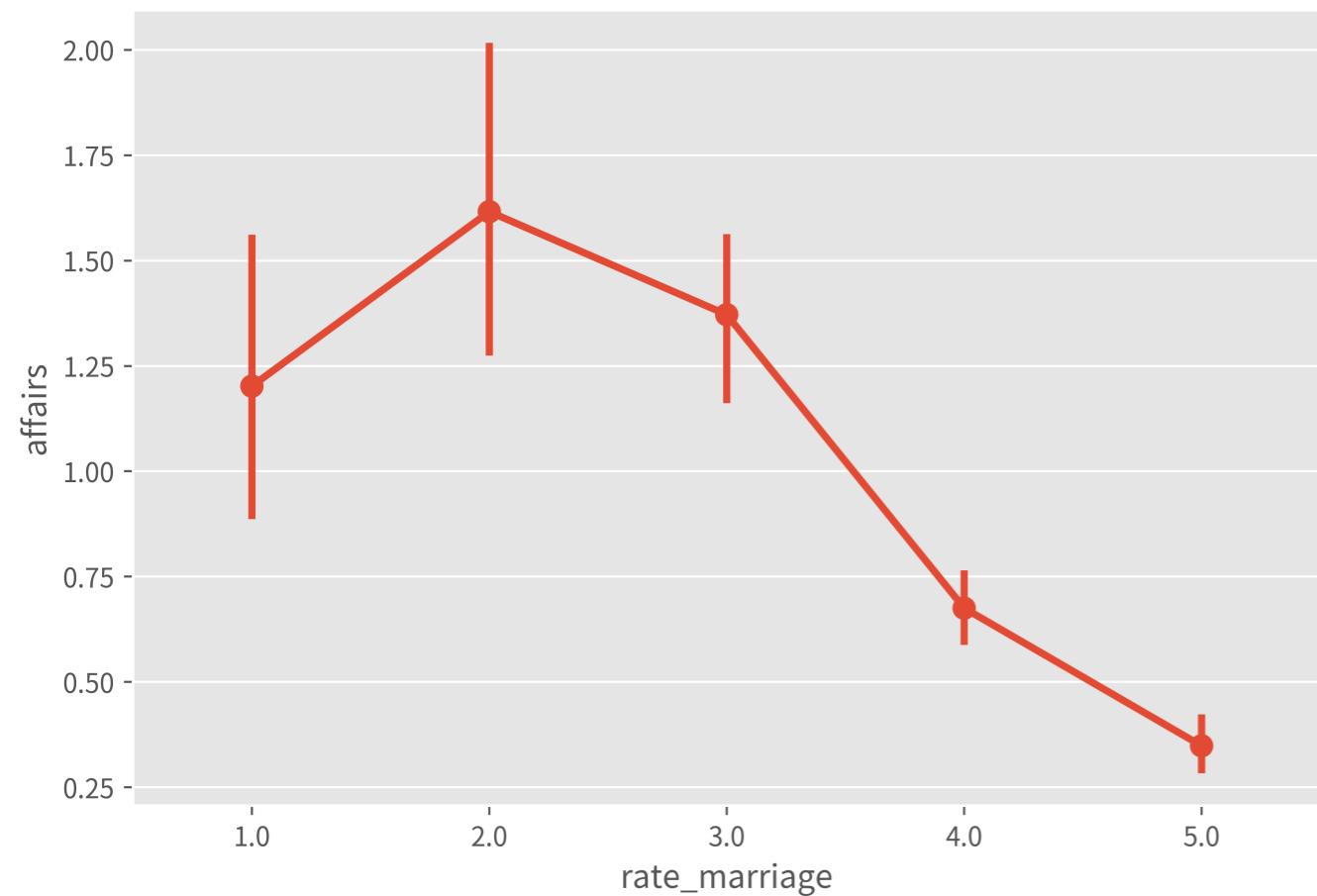


Covariance Types of Errors

Covariance Types of Errors

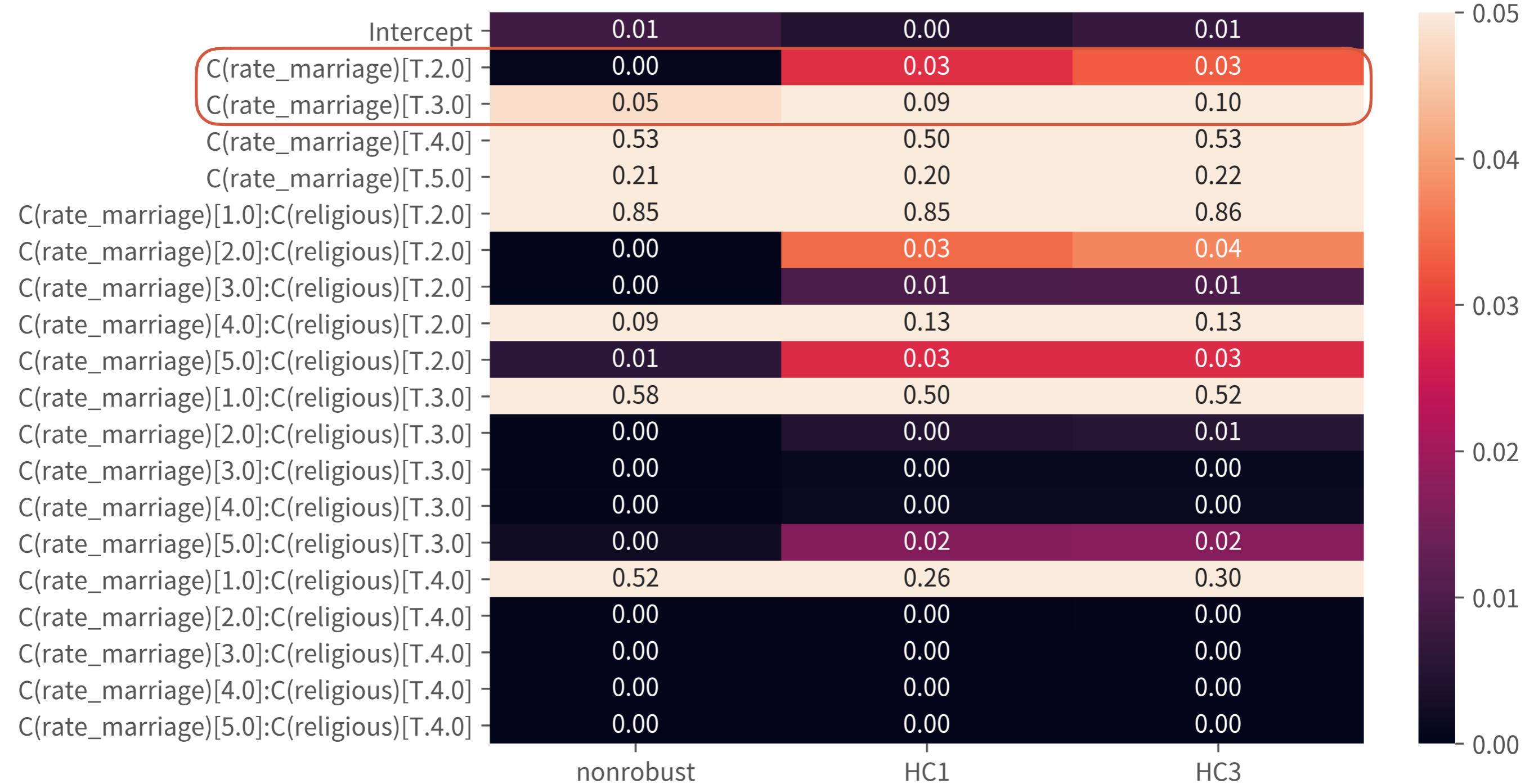
- Spherical Errors
 - \equiv Homoscedasticity & no autocorrelation
- Heteroscedasticity
- Autocorrelation \equiv serial correlation
- If spherical errors, the model is good.
- If not spherical errors, the std errs are wrong.
 - So the interval estimates are wrong, including hypothesis tests on coeffs, confidence intervals.

Heteroscedasticity



- Use **HC std errs** (heteroscedasticity-consistent standard errors) to correct.
- If $N \leq 250$, use **HC3**. [ref]
- If $N > 250$, consider HC1 for the speed.
- Also suggest to **use by default**.
- **.fit(cov_type='HC3')**
 - ← The confidence intervals vary among groups. The heteroscedasticity exists.

P-Values



Autocorrelation

OLS Regression Results

Dep. Variable:	affairs	R-squared:	0.032			
Model:	OLS	Adj. R-squared:	0.032			
Method:	Least Squares	F-statistic:	208.4			
Date:	Fri, 26 Apr 2019	Prob (F-statistic):	1.66e-46			
Time:	23:25:02	Log-Likelihood:	-13959.			
No. Observations:	6366	AIC:	2.792e+04			
Df Residuals:	6364	BIC:	2.794e+04			
Df Model:	1					
Covariance Type:	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

- Durbin-Watson
 - 2 is no autocorrelation.
 - [0, 2) is positive autocorrelation.
 - (2, 4] is negative autocorrelation.
 - [1.5, 2.5] are relatively normal. [\[ref\]](#)
- Use HAC std err.
- `.fit(cov_type='HAC', cov_kwds=dict(maxlag=tau))`

Other Covariance Types

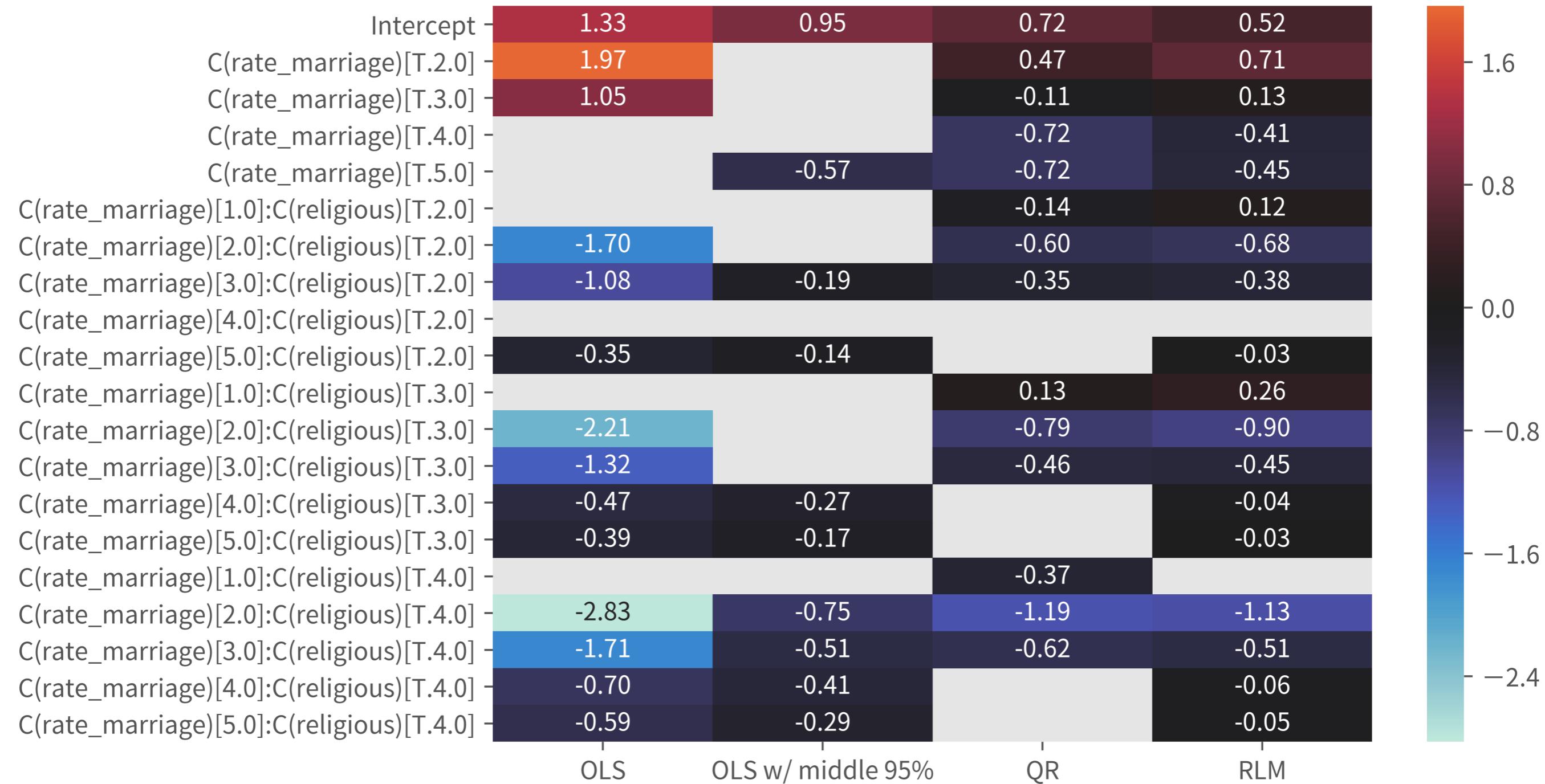
- cluster
 - Assume each group has spherical errors.
- hac-groupsum
 - Sum by the time label and then process.
- hac-panel
 - Process by the groups and then aggregate.
- Check the full references.

Outliers

- An outlier may be the most interesting observation.
 - Consider to include more variables to explain.
 - Consider the possible non-linear relationship.
 - Consider outlier-insensitive models.
- Do not drop observation without a good reason, like:
 - Typo.
 - Not representative of the intended study population.
- Report fully, like:
 - The preprocess steps.
 - The models with and without outliers.

- Quantile regression: estimates the **median** rather than mean.
- Robust regression: **robust to outliers**, but slower.
- Keep middle n%: changes the intended study population.
- OLS
 - Outliner test
 - Influence report

Params



```
df = df_fair
```

```
alpha = 0.05
```

```
a = df.affairs.quantile(alpha/2)
```

```
b = df.affairs.quantile(1-alpha/2)
```

```
df = df[(df.affairs >= a) & (df.affairs <= b)]
```

```
df_fair_middle95 = df
```

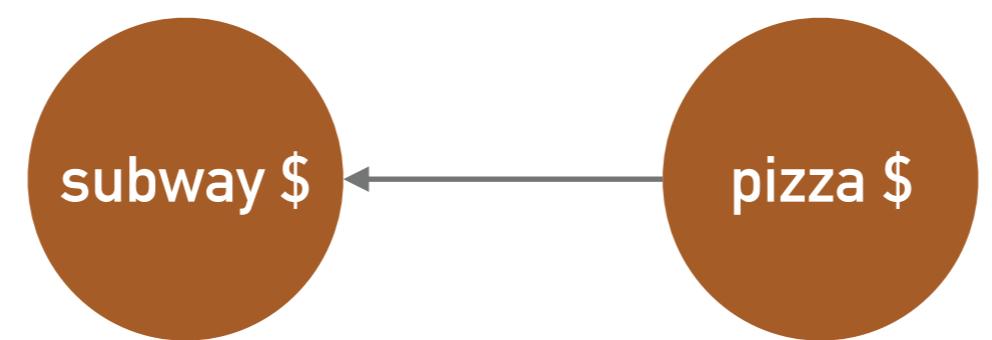
```
df = df_fair
```

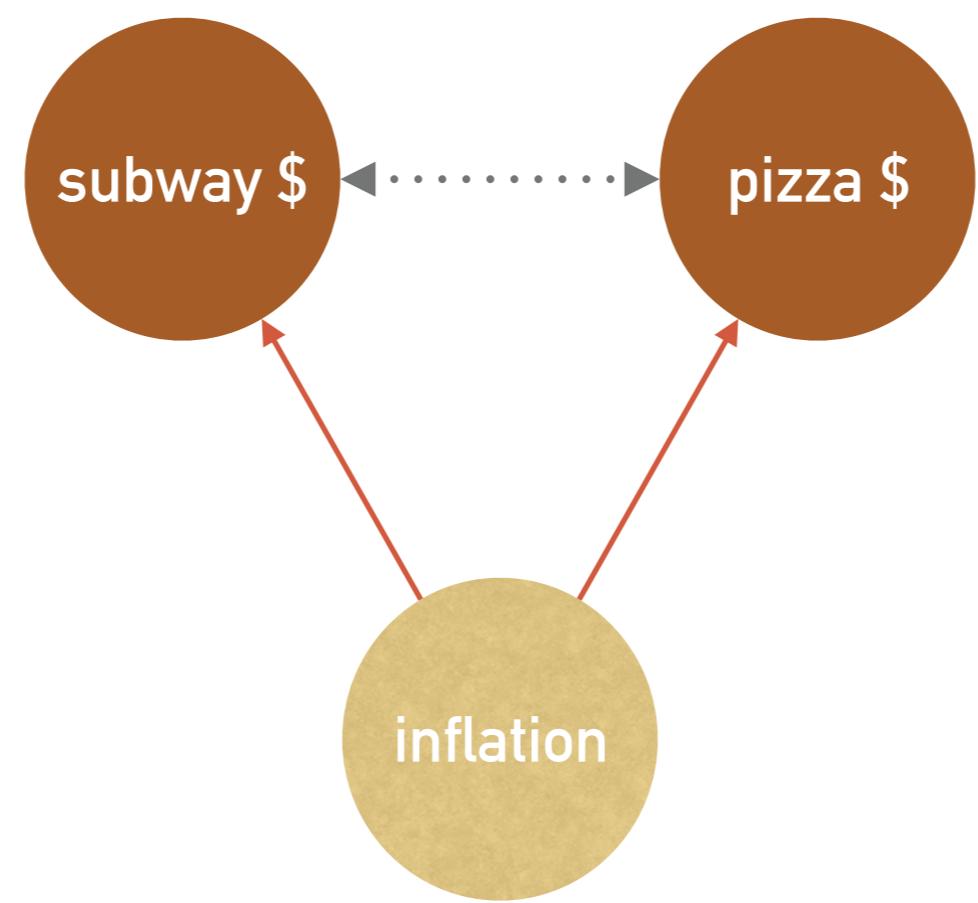
```
smf.ols(formula, df).fit().summary()
```

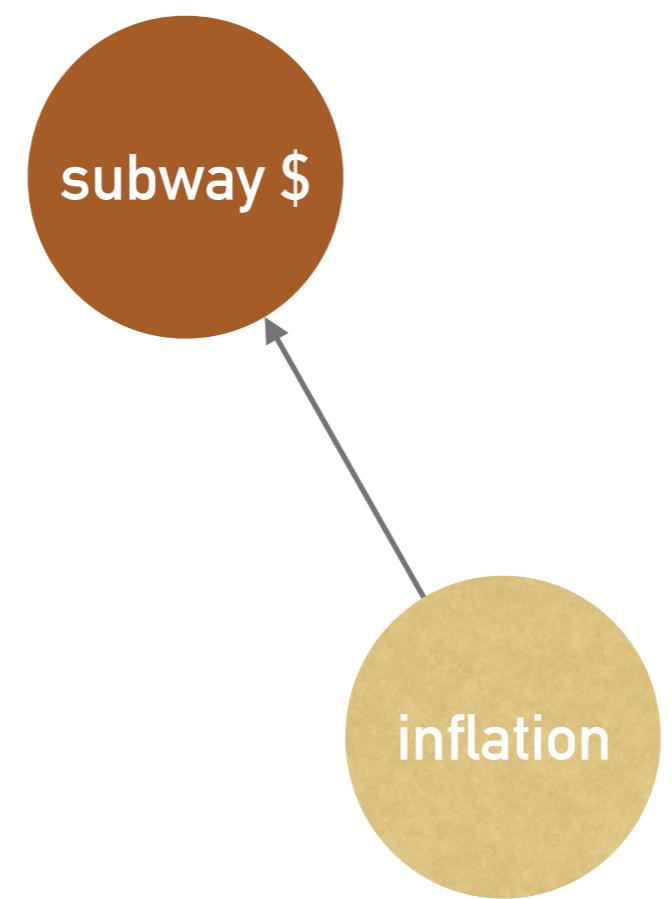
```
smf.ols(formula, df_fair_middle95).fit().summary()
```

```
smf.quantreg(formula, df).fit().summary()
```

```
smf.rlm(formula, df).fit().summary()
```







Correlation Does Not Imply Causation

- $y \sim x :=$ “ y has association with x ”
- $y \leftarrow x :=$ “ y because x ”
- $y \sim x$ may be:
 - $y \leftarrow x$
 - $y \rightarrow x$
 - $z \rightarrow y \wedge z \rightarrow x$
- So $y \sim x$ doesn't implies $y \leftarrow x$.
- *A good research design* \wedge $y \sim x$ can implies $y \leftarrow x$.

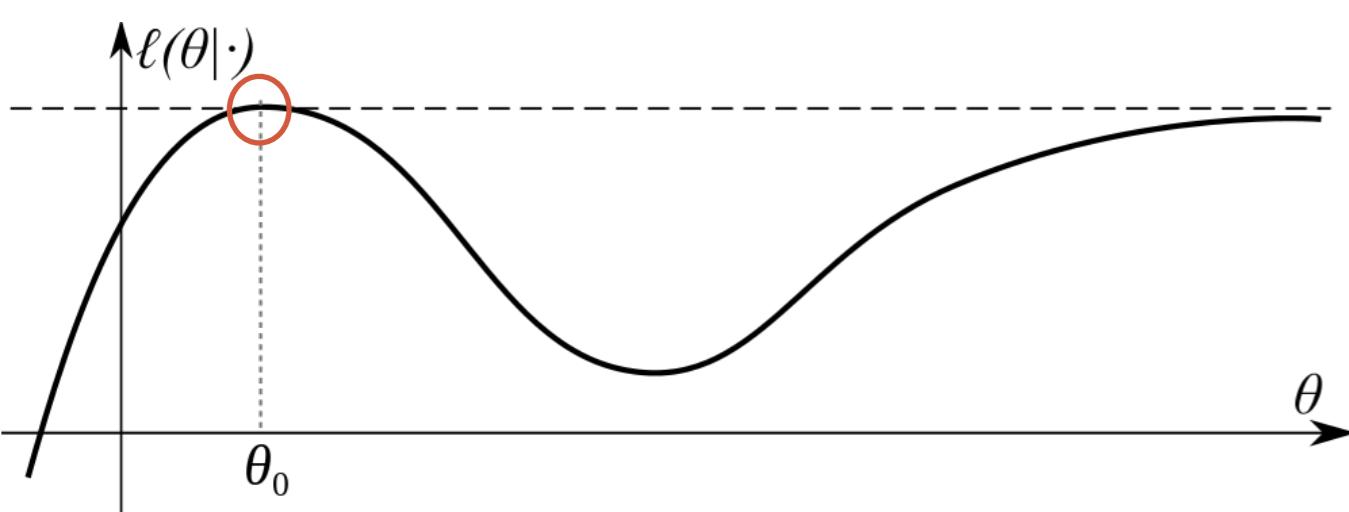
Suggested Wording

- “Relationship”
 - Any relationship, so it can't be wrong.
- Correlation
 - “Associate” “Association”, relatively conservative.
 - “Correlate” “Correlation”, usually in correlation analysis.
- Causation
 - “Predict” “Prediction”.
 - “Affect” “Influence”, the most strong wording.

More Models

- Discrete Models, like **logit model**:
 - $y \in \{0, 1\}$
- Mixed Model for estimate both **subject** and **group** effect:
 - $y = X\beta + Zu + \varepsilon$
- Time Series Models, like **autoregressive model**:
 - $x_t = c + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \dots + \varphi_p x_{t-p} + \varepsilon_t$
- Check all the models that StatsModels supports.

More Estimations



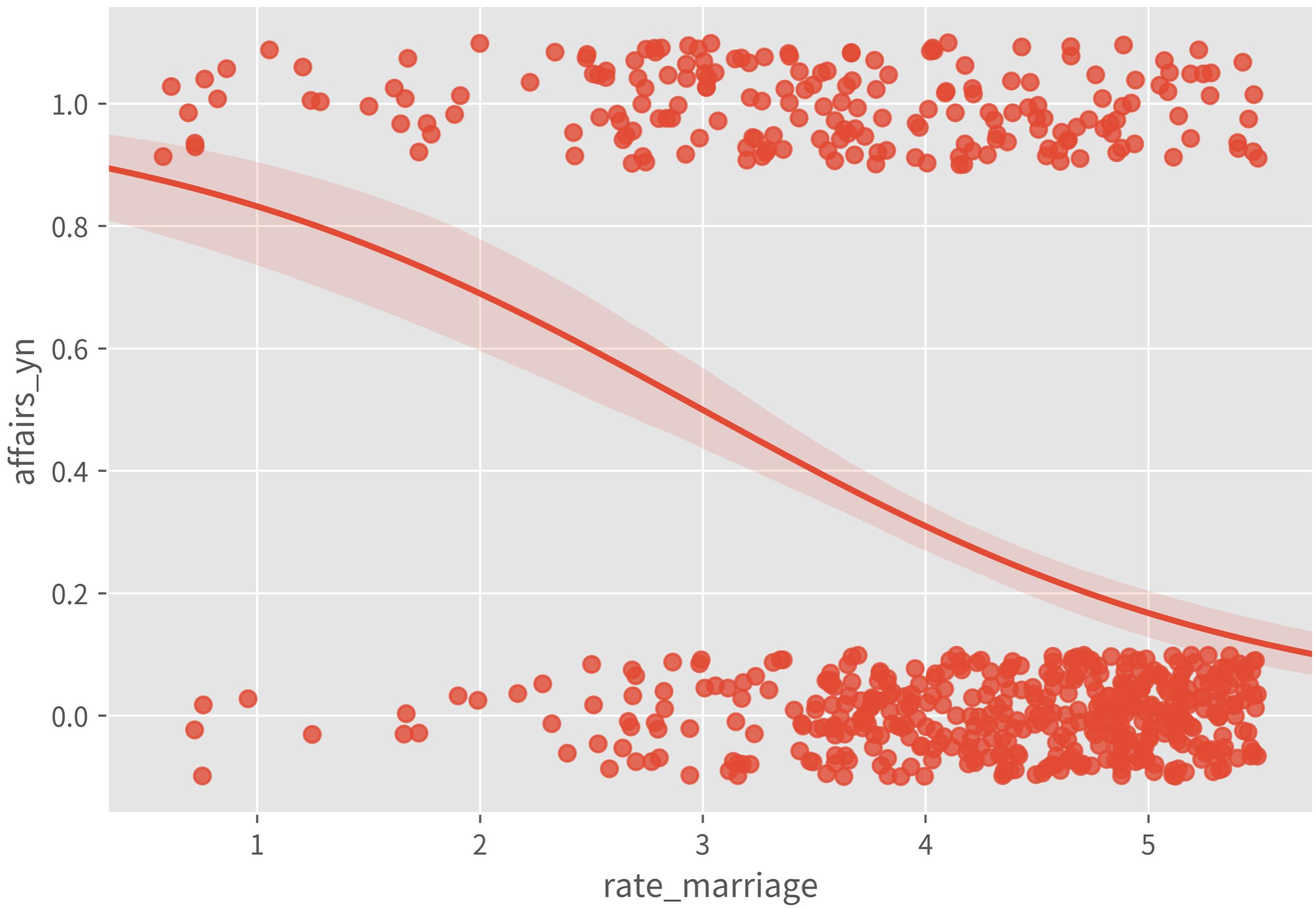
- **MLE**, Maximum Likelihood Estimation.

← Usually find by numerical methods.

- **TSLS**, Two-Stage Least Squares.

- $y \leftarrow (x \leftarrow z)$

- Handle the endogeneity:
 $E[\varepsilon | X] \neq 0$.



Dep. Variable:	affairs_yn	No. Observations:	6366			
Model:	Logit	Df Residuals:	6346			
Method:	MLE	Df Model:	19			
Date:	Mon, 29 Apr 2019	Pseudo R-squ.:	0.1003			
Time:	22:48:50	Log-Likelihood:	-3601.0			
converged:	True	LL-Null:	-4002.5			
		LLR p-value:	4.598e-158			
	coef	std err	z	P> z	[0.025	0.975]
Intercept	0.6931	0.500	1.386	0.166	-0.287	1.673
C(rate_marriage)[T.2.0]	0.2231	0.581	0.384	0.701	-0.915	1.362
C(rate_marriage)[T.3.0]	-0.2122	0.523	-0.405	0.685	-1.238	0.813
C(rate_marriage)[T.4.0]	-1.2009	0.512	-2.345	0.019	-2.205	-0.197
C(rate_marriage)[T.5.0]	-1.6664	0.512	-3.256	0.001	-2.669	-0.663
C(rate_marriage)[1.0]:C(religious)[T.2.0]	0.5596	0.641	0.873	0.383	-0.696	1.816
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-0.1398	0.345	-0.405	0.686	-0.817	0.537
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-0.2657	0.184	-1.443	0.149	-0.626	0.095
C(rate_marriage)[4.0]:C(religious)[T.2.0]	-0.0294	0.132	-0.222	0.824	-0.288	0.230
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.4791	0.140	-3.427	0.001	-0.753	-0.205
C(rate_marriage)[1.0]:C(religious)[T.3.0]	0.4769	0.629	0.758	0.448	-0.756	1.710
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-0.5656	0.349	-1.622	0.105	-1.249	0.118
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-0.3412	0.188	-1.811	0.070	-0.710	0.028
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.4342	0.134	-3.239	0.001	-0.697	-0.171
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.6133	0.137	-4.487	0.000	-0.881	-0.345
C(rate_marriage)[1.0]:C(religious)[T.4.0]	0.2231	0.975	0.229	0.819	-1.687	2.133
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-1.3218	0.504	-2.622	0.009	-2.310	-0.334
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-0.7105	0.286	-2.486	0.013	-1.271	-0.150
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.7732	0.210	-3.675	0.000	-1.186	-0.361
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-1.3503	0.212	-6.355	0.000	-1.767	-0.934

Logit Model

- The coef is log-odds.
- Use $\exp(x)/(\exp(x) + 1)$ to transform back to probability:

➤ 0.6931 → 67%

➤ " - 1.6664 → 27%

➤ " - 1.3503 → 9%

Or:

- `.predict(dict(rate_marriage=[1, 5, 5], religious=[1, 1, 4]))`

```
df = df_fair
df = df.assign(affairs_yn=(df.affairs > 0).astype(float))
df_fair_2 = df
```

```
df = df_fair_2.sample(frac=0.1, random_state=20190429)
sns.regplot(data=df, x='rate_marriage', y='affairs_yn',
             logistic=True,
             x_jitter=1/2, y_jitter=0.2/2)
```

```
df = df_fair_2
(smf
.logit('affairs_yn'
       ' ~ C(rate_marriage)'
       '+ C(rate_marriage):C(religious)', df)
.fit()
.summary())
```

Recap

- Choose the method by the assumption.
- Get an overview by correlation analysis.
- Understand Regression Result:
 - Plotting, Adj. R-squared, Cond. No., Durbin-Watson, etc.
- Model Specification Using the R Formula:
 - $y \sim C(x)$
 - $y \sim x^*z$
- Covariance Types: Use HC3 by default.
- Let's explain and predict efficiently! 