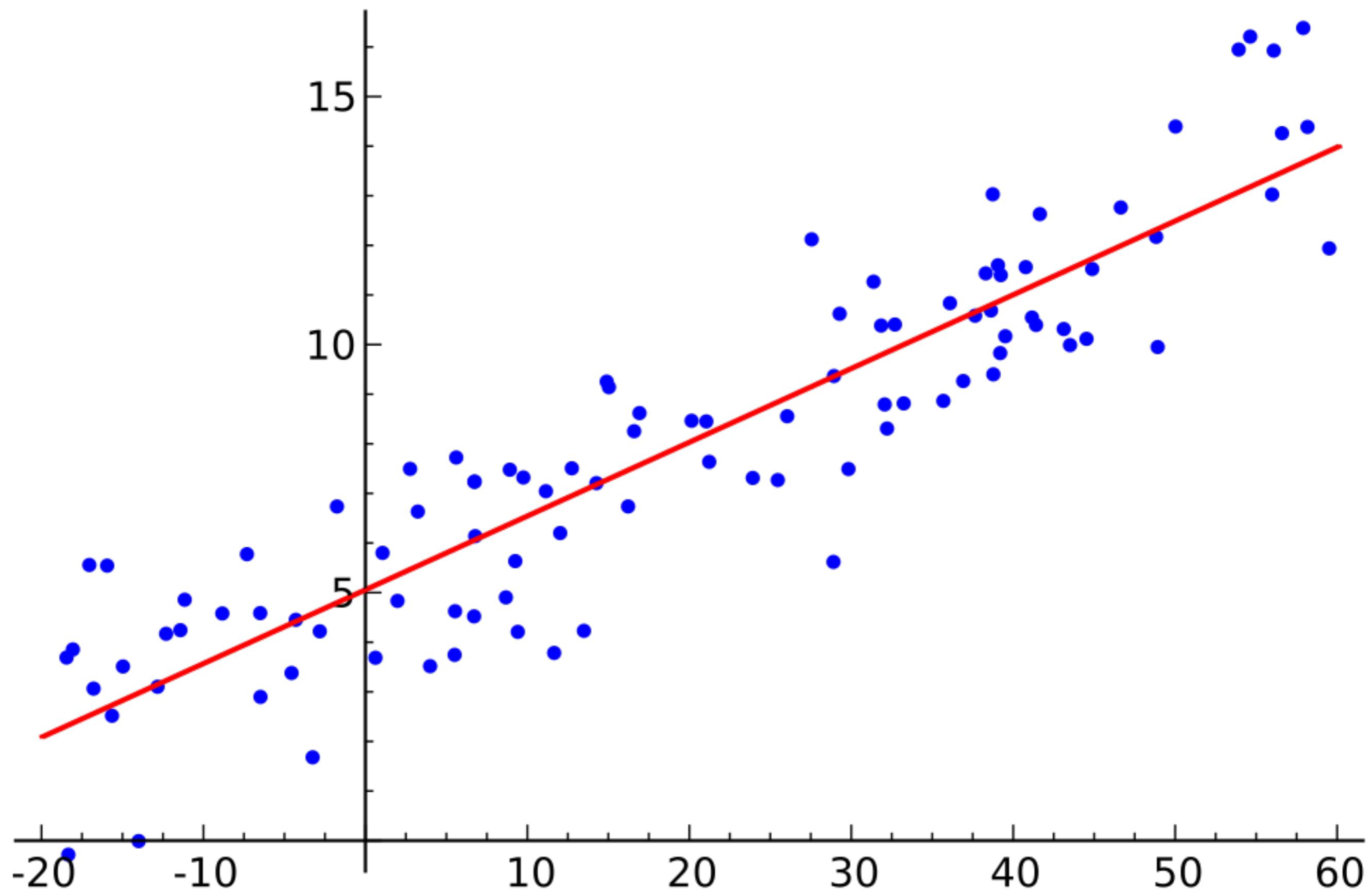


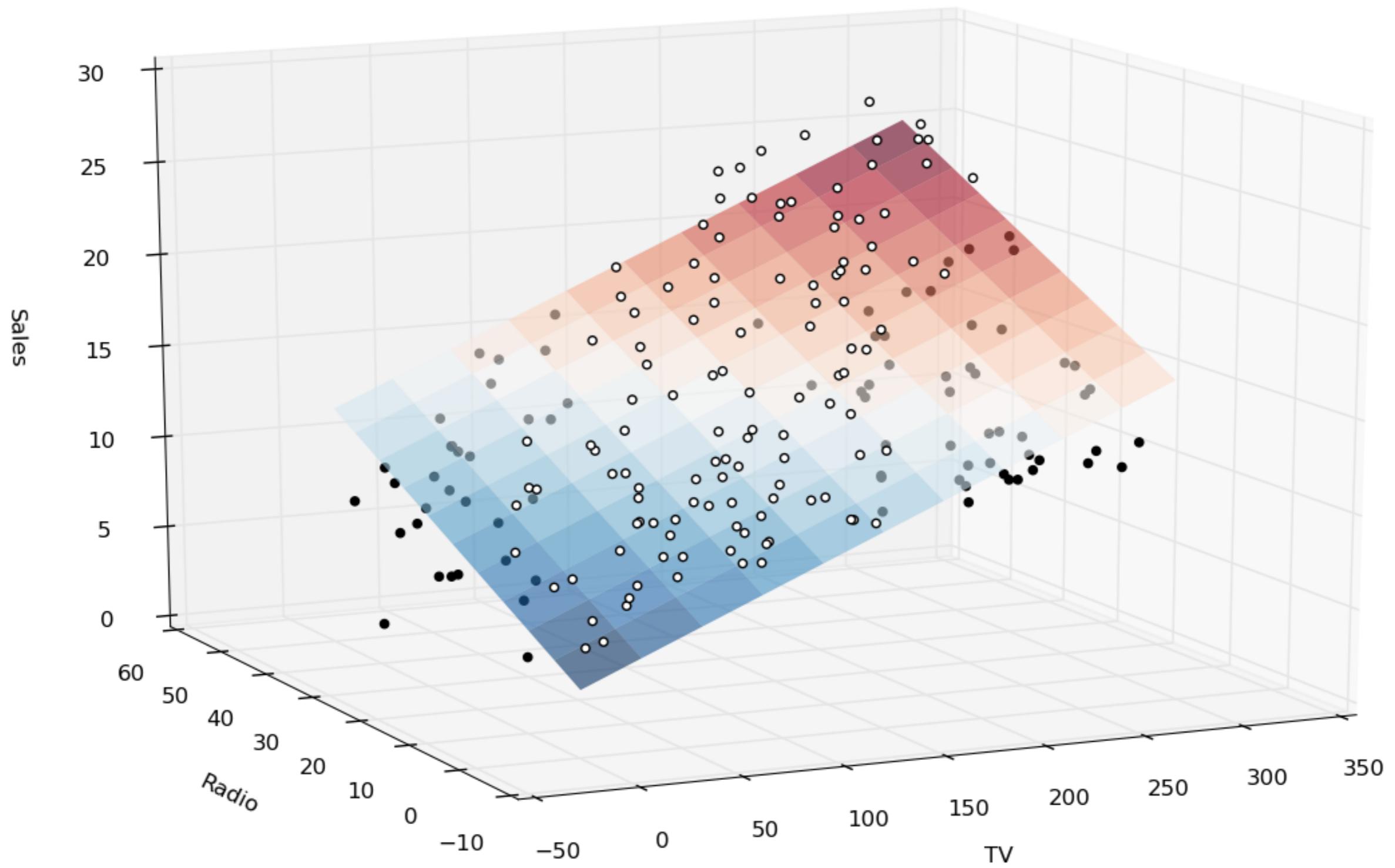


# Statistical Regression With Python

---

*Explain & Predict*





# Explain & Predict

---

- $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
- Explain by  $\beta$ , the slope.
- Predict by new  $x_i$ .
- “Simple linear regression model”
- $y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$
- $\beta$ , a slope vector.
- New  $x_i$ , a vector.
- “Multiple linear regression model”

# Mosky

---



- Python Charmer at Pinkoi.
- Has spoken at: PyCons in TW, MY, KR, JP, SG, HK, COSCUPs, and TEDx, etc.
- Countless hours on teaching Python.
- Own the Python packages: ZIPCodeTW, MoSQL, Clime, etc.
- <http://mosky.tw/>

# Outline

---

- The Analysis Steps
  - Define Assumptions
  - Validate Assumptions
    - The Dataset: Fair
- Correlation Analysis
- Ordinary Least Squares
  - Models & Estimations
  - Understand Regression Result
- Model Specification Using the R Formula
- Covariance Types
- Outliers
- Correlation & Causation
- More Models & Estimations
  - Introduction
  - Logit Model

# The PDF, Notebooks, and Packages

---

- The PDF and notebooks are available on <https://github.com/moskytw/statistical-regression-with-python> .
- The packages:
  - \$ pip3 install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn
- Or:
  - > conda install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn

# Define Assumptions

---

- The regression analysis:
  - Suitable to validate the “**relationship**” between variables.
  - Can model most of the hypothesis testing. [ref]
  - Can predict.

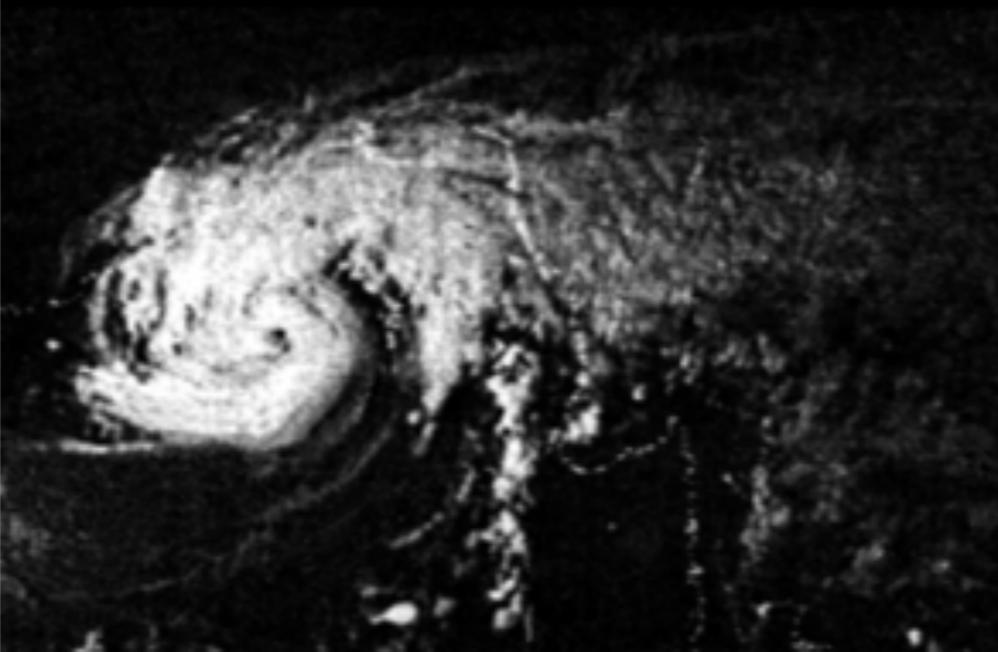
- “Years of marriage has association with children?”
- “Rate of marriage has association with affairs?”
- “Any background may have association with affairs?”

# Validate Assumptions

---

- Collect data ...
- The “Fair” dataset:
  - Fair, Ray. 1978. “A Theory of Extramarital Affairs,” Journal of Political Economy, February, 45-61.
  - A dataset from 1970s.
  - Rows: 6,366
  - Columns: (next slide)
- The full version of the analysis steps:  
<http://bit.ly/analysis-steps> .

1. *rate\_marriage*: 1~5; very poor, poor, fair, good, very good.
2. *age*
3. *yrs\_married*
4. *children*: number of children.
5. *religious*: 1~4; not, mildly, fairly, strongly.
6. *educ*: 9, 12, 14, 16, 17, 20; grade school, some college, college graduate, some graduate school, advanced degree.
7. *occupation*: 1, 2, 3, 4, 5, 6; student, farming-like, white-collar, teacher-like, business-like, professional with advanced degree.
8. *occupation\_husb*
9. *affairs*: n times of extramarital affairs per year since marriage.



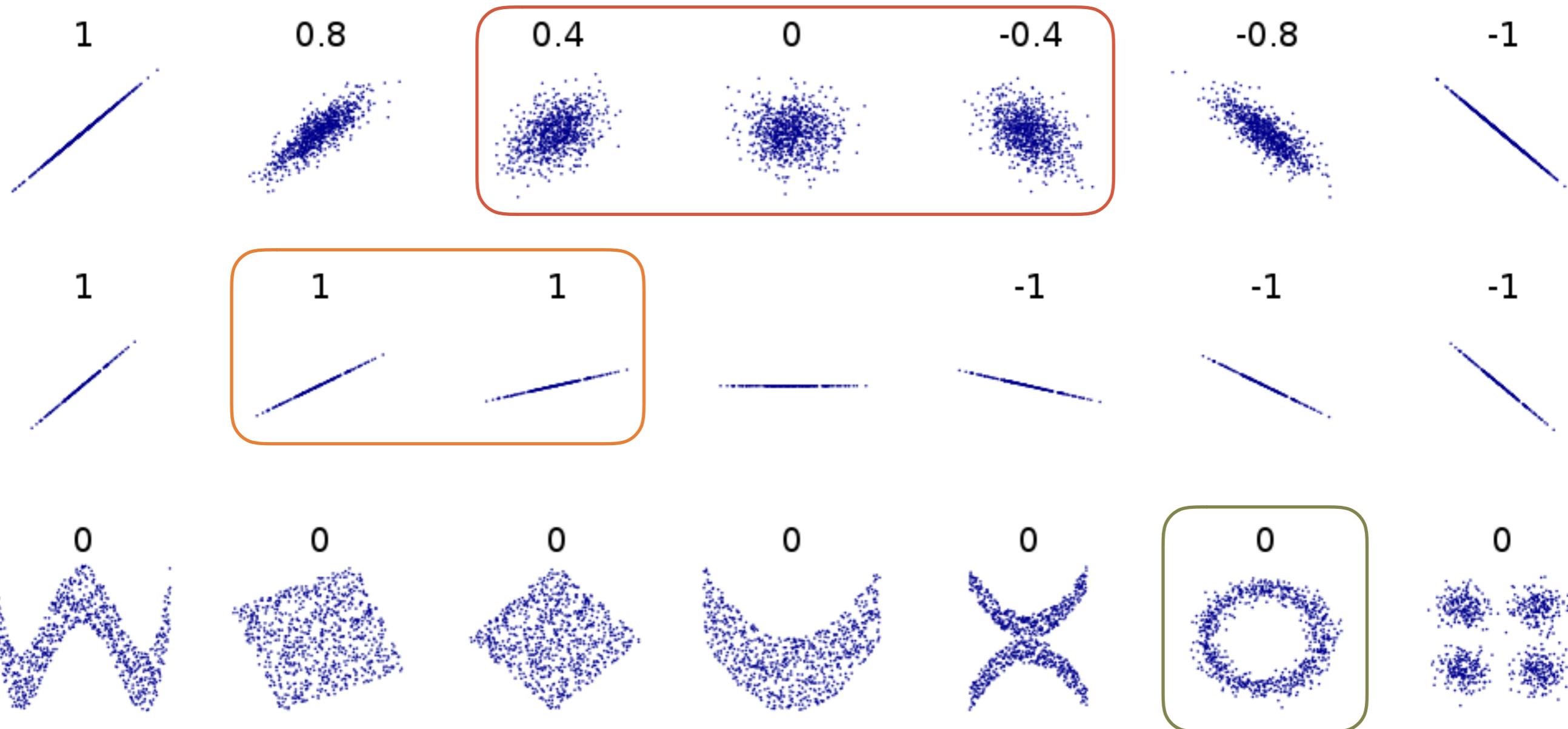
# Correlation Analysis

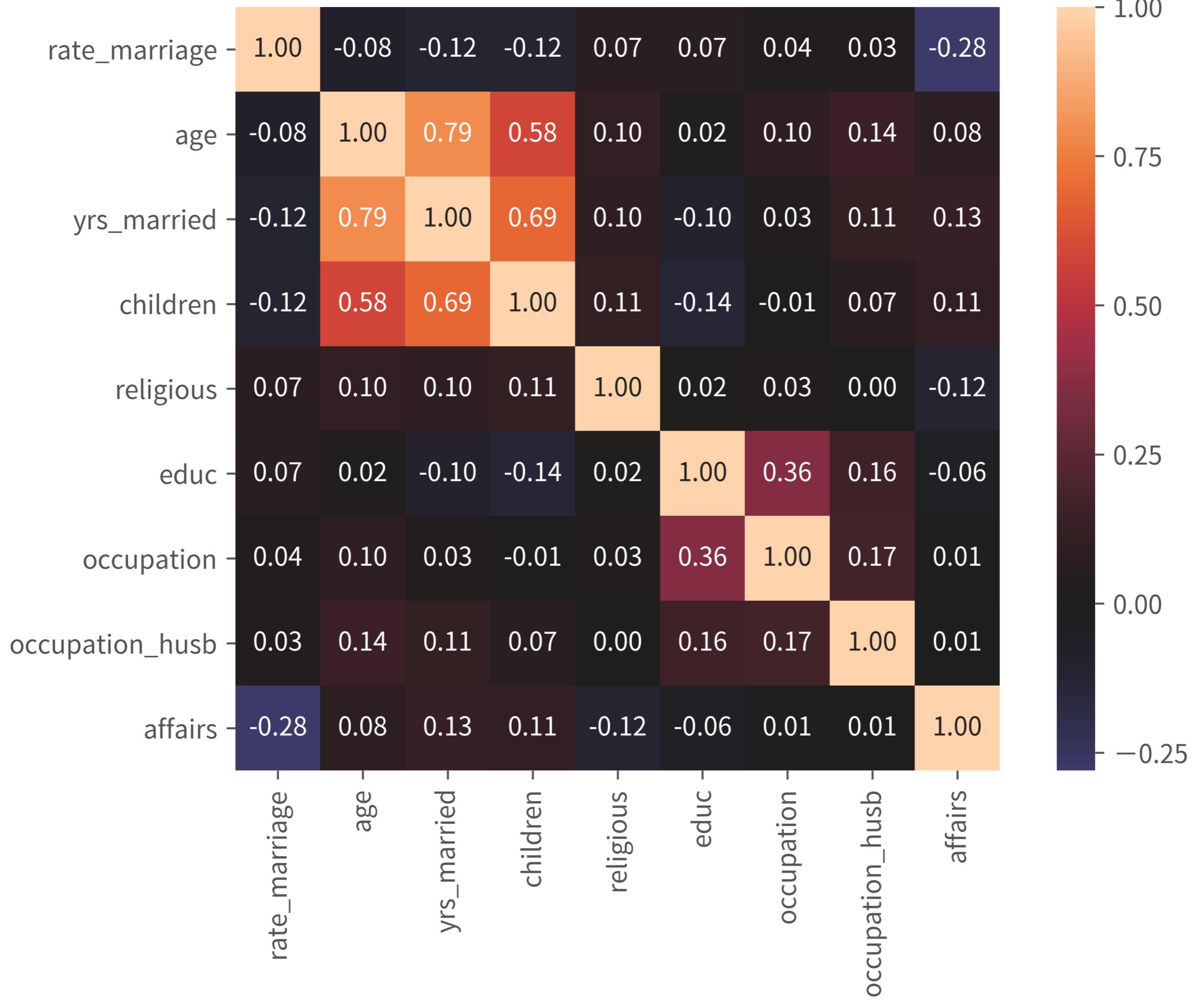
---

- Pearson correlation coefficient
  - For the variables whose **distance is meaningful**.
  - `df.corr()`
- Kendall rank correlation coefficient
  - For the variables whose **order is meaningful**.
  - `df.corr('kendall')`
- Measures “tightness”.

# Pearson Correlation Coefficient

---





```
import statsmodels.api as sm
import seaborn as sns

print(sm.datasets.fair.SOURCE,
      sm.datasets.fair.NOTE)

# -> Pandas's Dataframe
df_fair = sm.datasets.fair.load_pandas().data

df = df_fair
sns.heatmap(df.corr(method='kendall'),
            center=0, square=True,
            annot=True, fmt='.2f')
```

# Models & Estimations

---

## ► Models

- $y = X\beta + \varepsilon$
- Like simple, multiple, logit, etc.

## ► Estimations

- How to estimate the  $\hat{\beta}$ ? For example, **OLS**:

$$y = X\hat{\beta}$$

$$S(b) = \sum_{i=1}^n (y_i - x_i^T b)^2 = (y - Xb)^T (y - Xb)$$

$$\hat{\beta} = \operatorname{argmin}_{b \in \mathbb{R}^p} S(b) = (X^T X)^{-1} X^T y$$

# Model Specification Using the R Formula

---

- Using the R formula implementation in Python, Patsy:

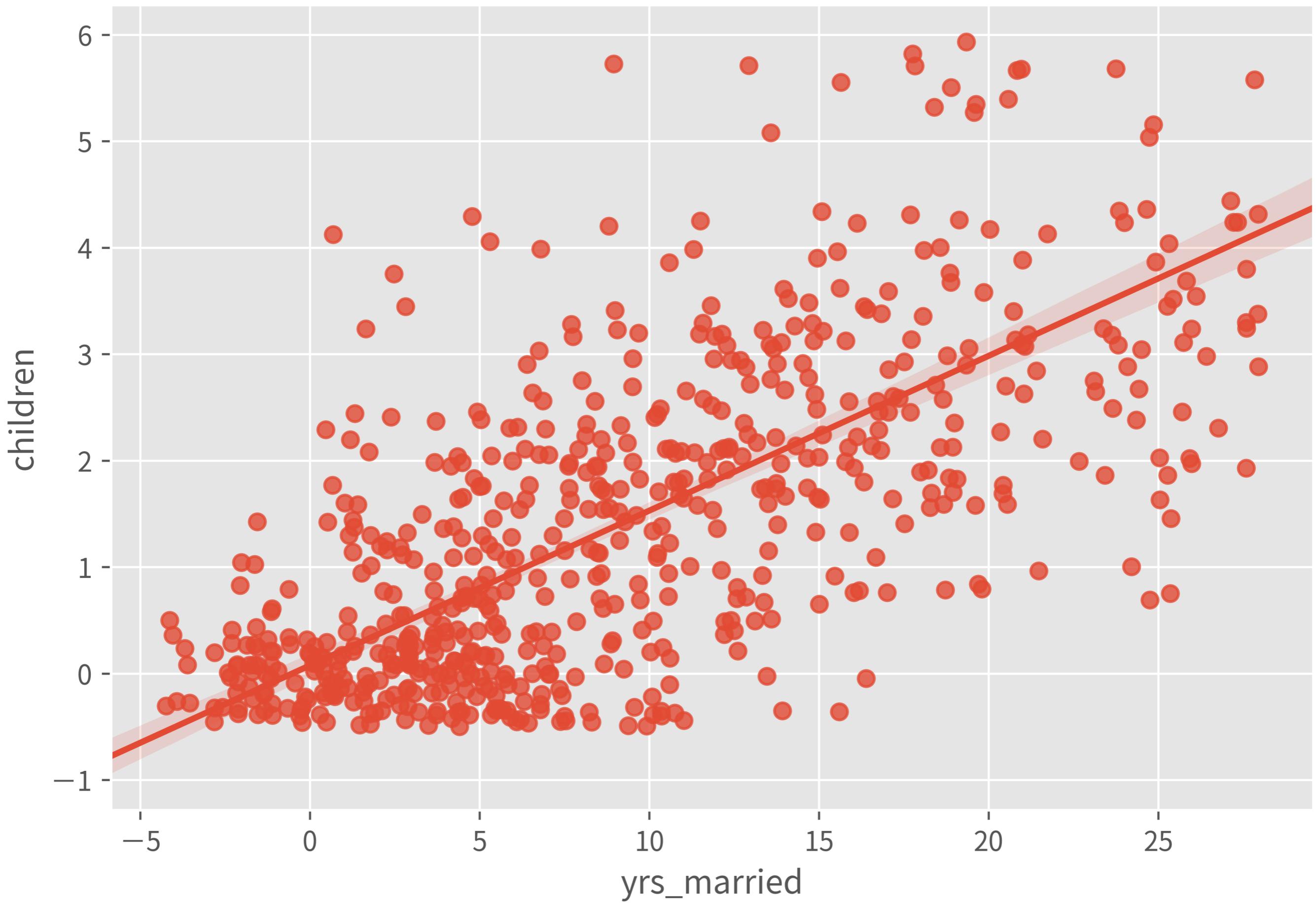
$$y \sim x$$

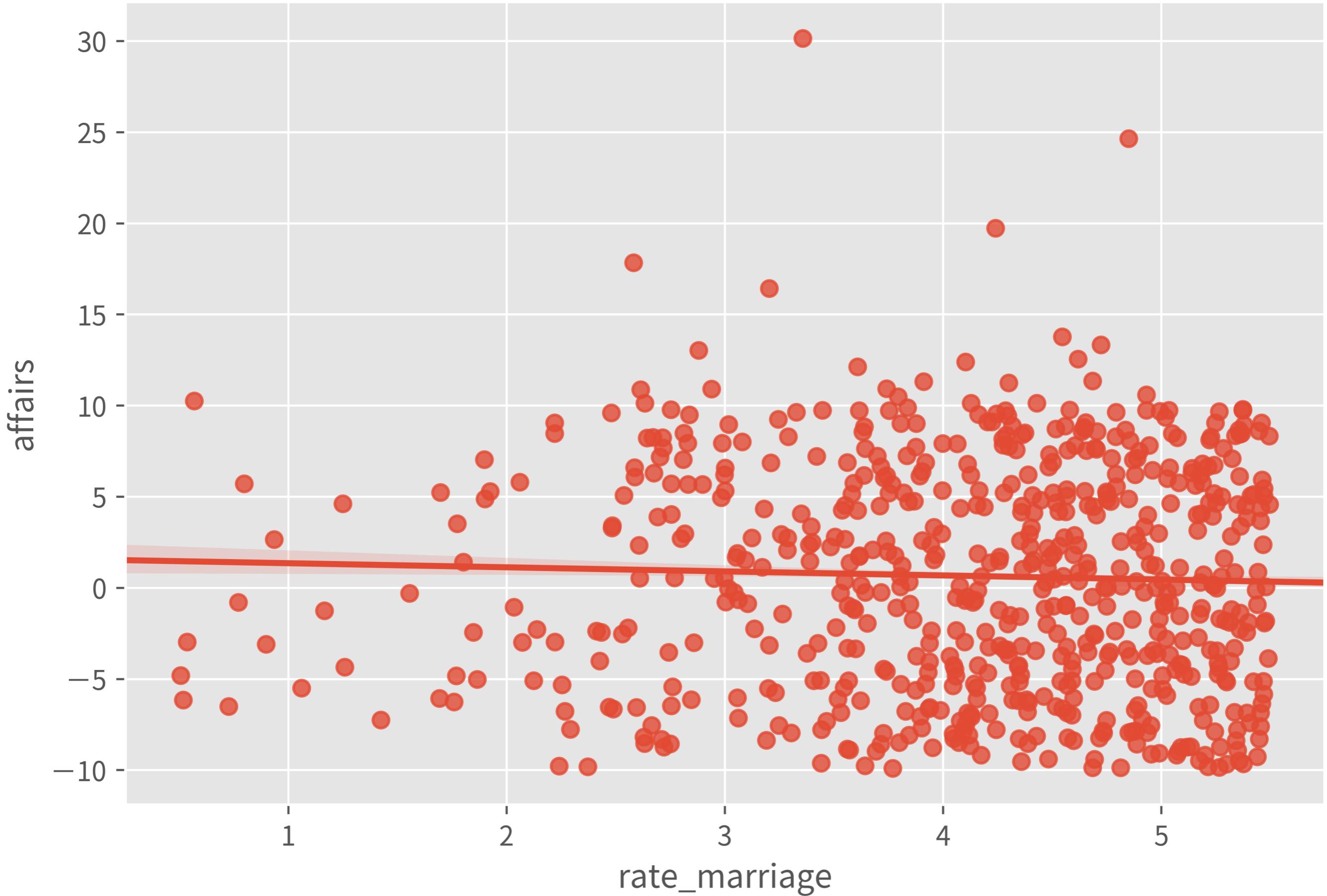
$$\equiv y \sim 1 + x$$

$$\equiv y = \beta_0 1 + \beta_1 x + \varepsilon$$

- For example:

- `affairs ~ rate_marriage`





## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
<b>Df Residuals:</b>	6364	<b>BIC:</b>	2.794e+04			
<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>		1.606		
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>		5215639.758		
<b>Skew:</b>	8.930	<b>Prob(JB):</b>		0.00		
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>		19.5		

```
df_fair_sample = df_fair.sample(  
    frac=0.1, random_state=20190425  
)  
  
df = df_fair_sample  
sns.regplot(data=df, x='yrs_married', y='children',  
            x_jitter=10/2, y_jitter=1/2)
```

```
df = df_fair_sample  
sns.regplot(data=df, x='rate_marriage', y='affairs',  
            x_jitter=1/2, y_jitter=20/2)
```

```
df = df_fair  
(smf  
    .ols('affairs ~ rate_marriage', df)  
    .fit()  
    .summary())
```

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
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<b>Df Model:</b>	1					
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	coef	std err	t	P> t	[0.025	0.975]
<b>Intercept</b>	2.3825	0.119	19.969	0.000	2.149	2.616
<b>rate_marriage</b>	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606			
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<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

## Adj. R-squared

- .....
- $\equiv$  explained var. by  $X$  / var. of  $y$  and adjusted by no. of  $X$
- $\in [0, 1]$ , usually.
- Can compare among models.
- 0.032 is super bad.

# Prob(F-statistics)

OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
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<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	<b>coef</b>	<b>std err</b>	<b>t</b>	<b>P&gt; t </b>	[ <b>0.025</b>	<b>0.975]</b>
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606			
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5215639.758			
<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

- $\equiv P(\text{data} \mid \text{all coeffs are zero})$
- Trust the coeffs if low enough.
- “Low enough” is “ $< 0.05$ ” in convention.

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
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<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

## Log-Likelihood

- .....
- Higher is better.
- Negative, usually.
- Can compare among models when the datasets are the same.
- Also check likelihood-ratio test.

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
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<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606			
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5215639.758			
<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

# Large Sample or Normality

- To construct interval estimates correctly, e.g., hypothesis tests on coefs, confidence intervals.
- **No. Observations**
  - $\geq 110 \sim 200$  [ref]
- **Normality of Residuals**
  - $\text{Prob}(\text{Omnibus}) \geq 0.05$
  - $\wedge \text{Prob}(JB) \geq 0.05$

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
<b>Df Residuals:</b>	6364	<b>BIC:</b>	2.794e+04			
<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606			
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5215639.758			
<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

## Cond. No.

- .....
- Measures the degree of multicollinearity.
- Multicollinearity increases the std err, i.e., **decreases efficiency**.
- If  $\geq 30$ , check:
  - Any variable has **redundant information**? Like fat % and weight. Drop one.
  - Dummy variable trap?
  - If no, the model is good.
  - Other suggestions.

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04
<b>Df Residuals:</b>	6364	<b>BIC:</b>	2.794e+04
<b>Df Model:</b>	1		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353

<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5215639.758
<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5

P>|t|

- .....
- $\equiv P(\text{data} \mid \text{the coef is zero})$
- Exclude the  $x$  whose p-value is not low enough.

# Coef & Confidence Intervals

OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
<b>Df Residuals:</b>	6364	<b>BIC:</b>	2.794e+04			
<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
<b>Omnibus:</b>	9443.528	<b>Durbin-Watson:</b>	1.606			
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5215639.758			
<b>Skew:</b>	8.930	<b>Prob(JB):</b>	0.00			
<b>Kurtosis:</b>	142.083	<b>Cond. No.</b>	19.5			

- “The rate\_marriage and affairs has negative relationship, the strength is -0.4081, and 95% confidence interval is [-0.464, -0.353].”

# Code Categorical Variables

---

- The **str or bool** is treated as **categorical** by default.
- Or use **the C function**:

$$\begin{aligned}y &\sim \text{C}(c) \\ \equiv y &\sim 1 + (\cancel{\beta_1} + c_2 + \dots + c_i) - \cancel{\beta_1} \\ \equiv y &= \beta_0 1 + (\cancel{\beta_1} c_1 + \beta_2 c_2 + \dots + \beta_i c_i) - \cancel{\beta_1} c_1 + \varepsilon\end{aligned}$$

- The  $x_1$  is chosen as reference level automatically.
- For example:

$$\begin{aligned}\text{C}(\text{rate\_marriage} \in \{1, 2, 3, 4, 5\}) \\ \equiv 1 + \text{rate\_marriage\_2} \in \{0, 1\} + \dots + \text{rate\_marriage\_5} \in \{0, 1\}\end{aligned}$$

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.036
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.035
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	58.57
<b>Date:</b>	Sat, 27 Apr 2019	<b>Prob (F-statistic):</b>	1.25e-48
<b>Time:</b>	15:27:20	<b>Log-Likelihood:</b>	-13946.
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.790e+04
<b>Df Residuals:</b>	6361	<b>BIC:</b>	2.794e+04
<b>Df Model:</b>	4		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
<b>Intercept</b>	1.2017	0.218	5.524	0.000	0.775	1.628
<b>C(rate_marriage)[T.2.0]</b>	0.4141	0.247	1.679	0.093	-0.069	0.897
<b>C(rate_marriage)[T.3.0]</b>	0.1696	0.228	0.743	0.457	-0.278	0.617
<b>C(rate_marriage)[T.4.0]</b>	-0.5268	0.222	-2.370	0.018	-0.963	-0.091
<b>C(rate_marriage)[T.5.0]</b>	-0.8535	0.222	-3.853	0.000	-1.288	-0.419

<b>Omnibus:</b>	9436.269	<b>Durbin-Watson:</b>	1.612
<b>Prob(Omnibus):</b>	0.000	<b>Jarque-Bera (JB):</b>	5218576.884
<b>Skew:</b>	8.915	<b>Prob(JB):</b>	0.00
<b>Kurtosis:</b>	142.127	<b>Cond. No.</b>	21.0

## affairs ~ C(rate\_marriage)

- .....
- If 1, affairs is 1.2017.
- If 5, affairs is 1.2017-0.8535.
- The 2, 3, 4 are not significant to the reference level.

## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.036
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.035
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	58.57
<b>Date:</b>	Sat, 27 Apr 2019	<b>Prob (F-statistic):</b>	1.25e-48
<b>Time:</b>	17:20:18	<b>Log-Likelihood:</b>	-13946.
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.790e+04
<b>Df Residuals:</b>	6361	<b>BIC:</b>	2.794e+04
<b>Df Model:</b>	4		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
C(rate_marriage)[1.0]	1.2017	0.218	5.524	0.000	0.775	1.628
C(rate_marriage)[2.0]	1.6157	0.116	13.925	0.000	1.388	1.843
C(rate_marriage)[3.0]	1.3713	0.069	19.963	0.000	1.237	1.506
C(rate_marriage)[4.0]	0.6748	0.046	14.762	0.000	0.585	0.764
C(rate_marriage)[5.0]	0.3482	0.042	8.333	0.000	0.266	0.430

Omnibus: 9436.269 Durbin-Watson: 1.612

Prob(Omnibus): 0.000 Jarque-Bera (JB): 5218576.884

Skew: 8.915 Prob(JB): 0.00

Kurtosis: 142.127 Cond. No. 5.21

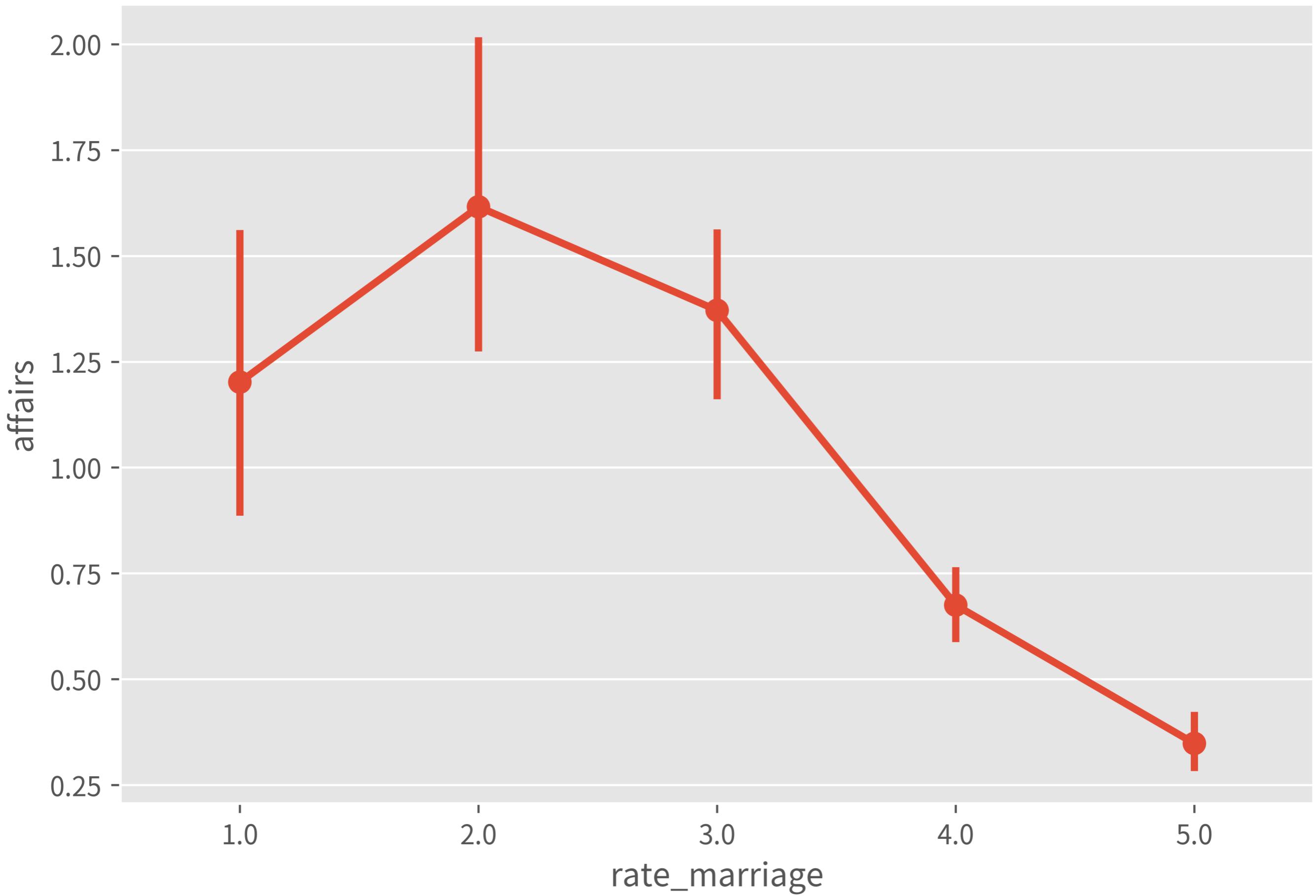
$$\text{affairs} \sim 0 + C(\text{rate\_marriage})$$

.....

➤  $y \sim 0 + C$

➤ Code without a reference.

➤ To calculate the mean of each group.



```
df = df_fair  
sns.pointplot(data=df, x='rate_marriage', y='affairs')
```

```
df = df_fair  
(smf  
.ols('affairs ~ C(rate_marriage)', df)  
.fit()  
.summary())
```

```
df = df_fair  
(smf  
.ols('affairs ~ 0 + C(rate_marriage)', df)  
.fit()  
.summary())
```

# Other Ways to Code Categorical Variables

---

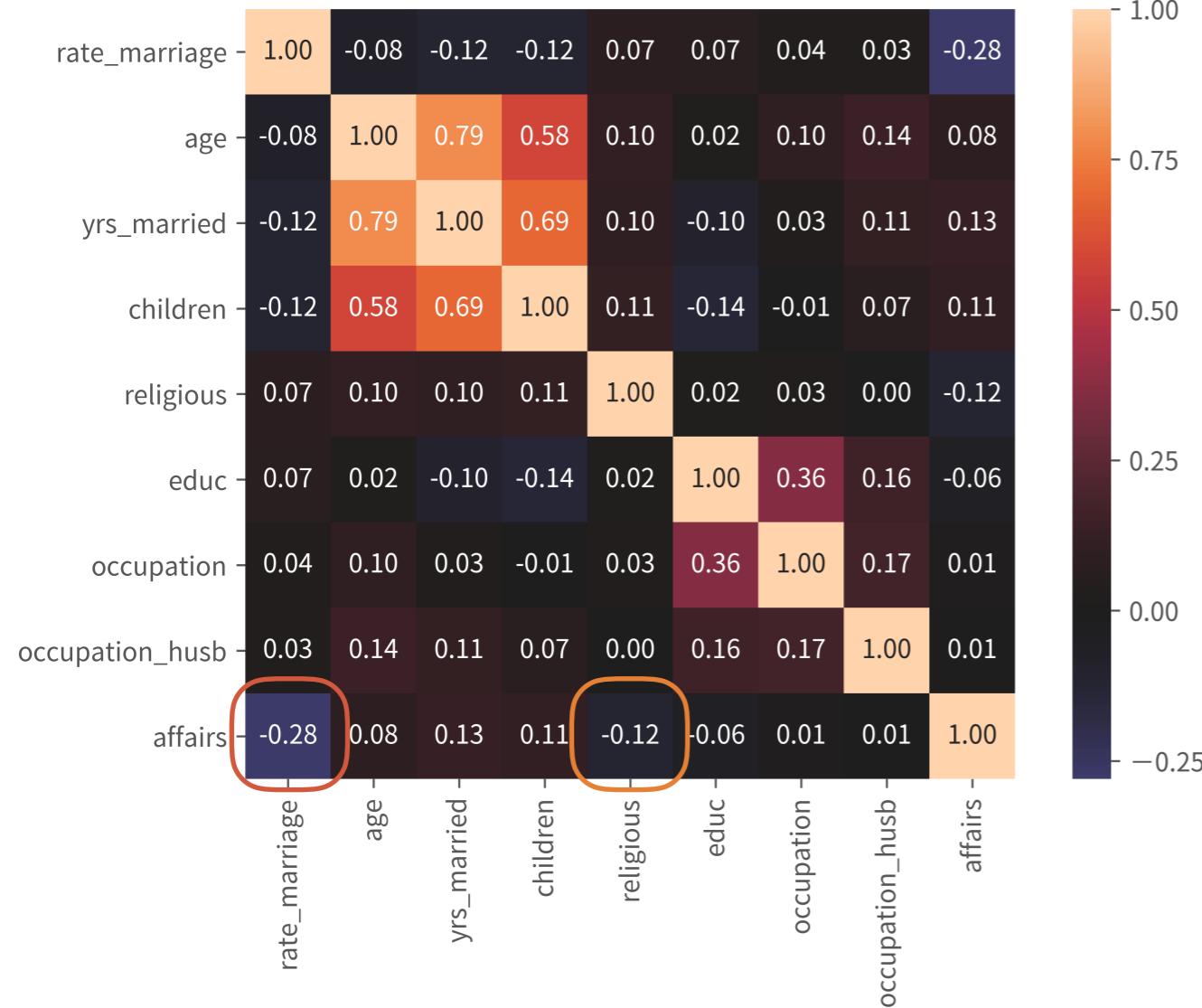
- `y ~ C(x, Treatment(reference='A'))`
  - Specify the reference level.
- `affairs ~ 0 + C(rate_marriage, Diff)`
  - Compare each level **with the preceding level**.
- `affairs ~ 0 + C(rate_marriage, Sum)`
  - Compare each level **with the mean-of-means**.
- Check the full reference.

# Interaction

- .....
- “The **low rate\_marriage** with **high religious** has stronger negative relationship with affairs?”

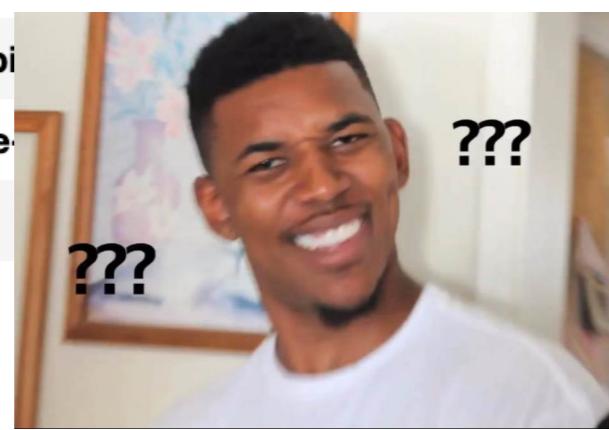
$$y \sim x^* z$$

$$\equiv y = \beta_0 1 + \beta_1 x + \beta_2 z + \beta_3 xz + \epsilon$$



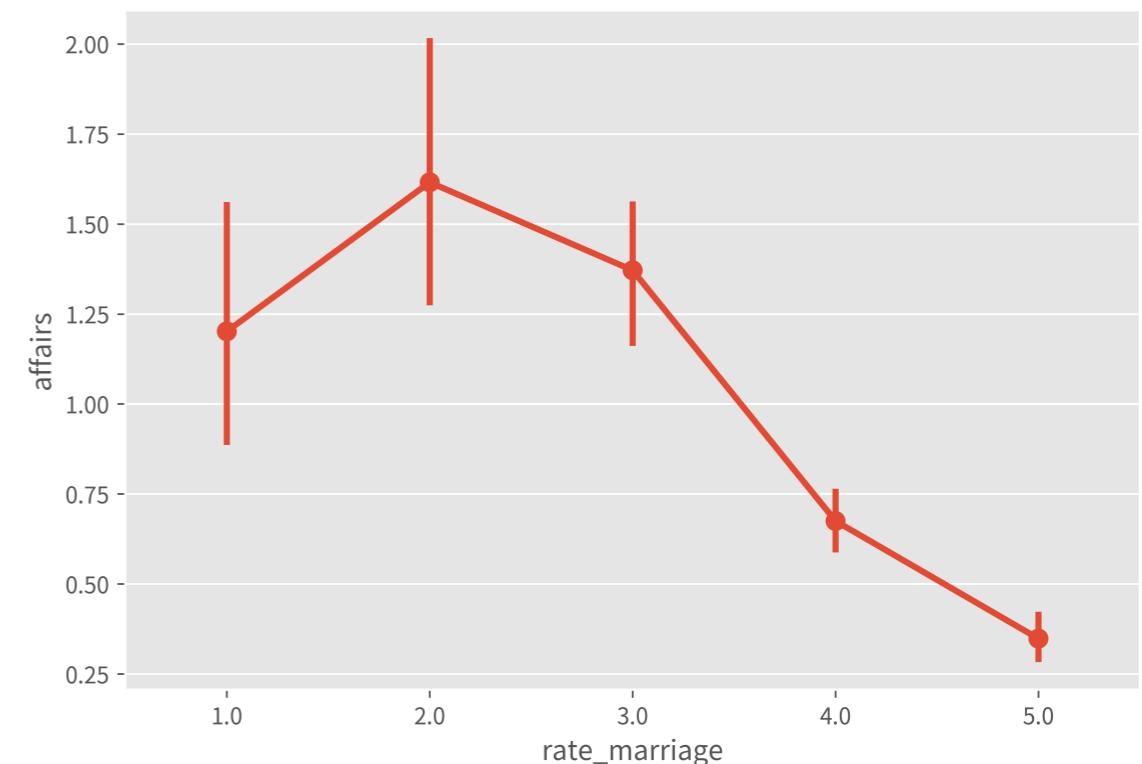
## OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.048			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.048			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	107.8			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	4.55e-68			
<b>Time:</b>	23:25:03	<b>Log-Likelihood:</b>	-13904.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.782e+04			
<b>Df Residuals:</b>	6362	<b>BIC:</b>	2.784e+04			
<b>Df Model:</b>	3					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	4.6495	0.347	13.395	0.000	3.969	5.330
rate_marriage	-0.7891	0.082	-9.622	0.000	-0.950	-0.628
religious	-0.9846	0.138	-7.122	0.000	-1.256	-0.714
rate_marriage:religious	0.1681	0.032	5.209	0.000	0.105	0.231
Omnibus:	9399.882	Durbin-Watson:	1.87			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	100.00			
Skew:	8.843	Kurtosis:	141.848			



## affairs ~ rate\_marriage\*religious

- .....
- The model may be wrong, since **the relationship is not linear.**



# affairs ~ C(rate\_marriage)\*C(religious)

	coef	std err	z	P> z	[0.025	0.975]
Intercept	1.3341	0.498	2.678	0.007	0.358	2.311
C(rate_marriage)[T.2.0]	1.9654	0.921	2.134	0.033	0.160	3.771
C(rate_marriage)[T.3.0]	1.0479	0.634	1.654	0.098	-0.194	2.290
C(rate_marriage)[T.4.0]	-0.3282	0.517	-0.635	0.525	-1.341	0.685
C(rate_marriage)[T.5.0]	-0.6430	0.522	-1.232	0.218	-1.666	0.380
C(religious)[T.2.0]	0.1143	0.627	0.182	0.855	-1.115	1.343
C(religious)[T.3.0]	-0.3413	0.529	-0.646	0.519	-1.377	0.695
C(religious)[T.4.0]	-0.6082	0.588	-1.035	0.301	-1.760	0.544
C(rate_marriage)[T.2.0]:C(religious)[T.2.0]	-1.8103	1.028	-1.761	0.078	-3.825	0.204
C(rate_marriage)[T.3.0]:C(religious)[T.2.0]	-1.1905	0.754	-1.580	0.114	-2.668	0.286
C(rate_marriage)[T.4.0]:C(religious)[T.2.0]	-0.3499	0.646	-0.542	0.588	-1.615	0.916
C(rate_marriage)[T.5.0]:C(religious)[T.2.0]	-0.4682	0.647	-0.723	0.469	-1.737	0.800
C(rate_marriage)[T.2.0]:C(religious)[T.3.0]	-1.8707	0.950	-1.968	0.049	-3.734	-0.008
C(rate_marriage)[T.3.0]:C(religious)[T.3.0]	-0.9741	0.665	-1.464	0.143	-2.278	0.330
C(rate_marriage)[T.4.0]:C(religious)[T.3.0]	-0.1333	0.549	-0.243	0.808	-1.209	0.943
C(rate_marriage)[T.5.0]:C(religious)[T.3.0]	-0.0448	0.553	-0.081	0.935	-1.128	1.039
C(rate_marriage)[T.2.0]:C(religious)[T.4.0]	-2.2192	0.997	-2.225	0.026	-4.174	-0.265
C(rate_marriage)[T.3.0]:C(religious)[T.4.0]	-1.1008	0.726	-1.517	0.129	-2.523	0.321
C(rate_marriage)[T.4.0]:C(religious)[T.4.0]	-0.0946	0.608	-0.156	0.876	-1.286	1.097
C(rate_marriage)[T.5.0]:C(religious)[T.4.0]	0.0196	0.608	0.032	0.974	-1.173	1.212

	sum_sq	df	F	PR(>F)
Intercept	32.952681	1.0	7.171451	0.007426
C(rate_marriage)	117.831212	4.0	6.410865	0.000038
C(religious)	11.838124	3.0	0.858772	0.461713
C(rate_marriage):C(religious)	111.850861	12.0	2.028497	0.018427
Residual	29159.748241	6346.0	NaN	NaN

- Looks like C(religious) isn't helping to explain in this model. Drop it.

```

res = res_1
df = pd.DataFrame(dict(params=res.params,
                       pvalues=res.pvalues))
df[df.pvalues < 0.05].sort_values('params')

```

	params	pvalues
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-2.827311	0.000450
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-2.211959	0.005104
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-1.708937	0.000059
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-1.695969	0.037348
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-1.315414	0.001136
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-1.076195	0.010052
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.702780	0.000006
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-0.588575	0.000194
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.474564	0.001321
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.386073	0.016820
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.353891	0.027721
Intercept	1.334104	0.007407
C(rate_marriage)[T.2.0]	1.965381	0.032863

$$\text{affairs} \sim C(\text{rate\_marriage}) + C(R):C(L)$$

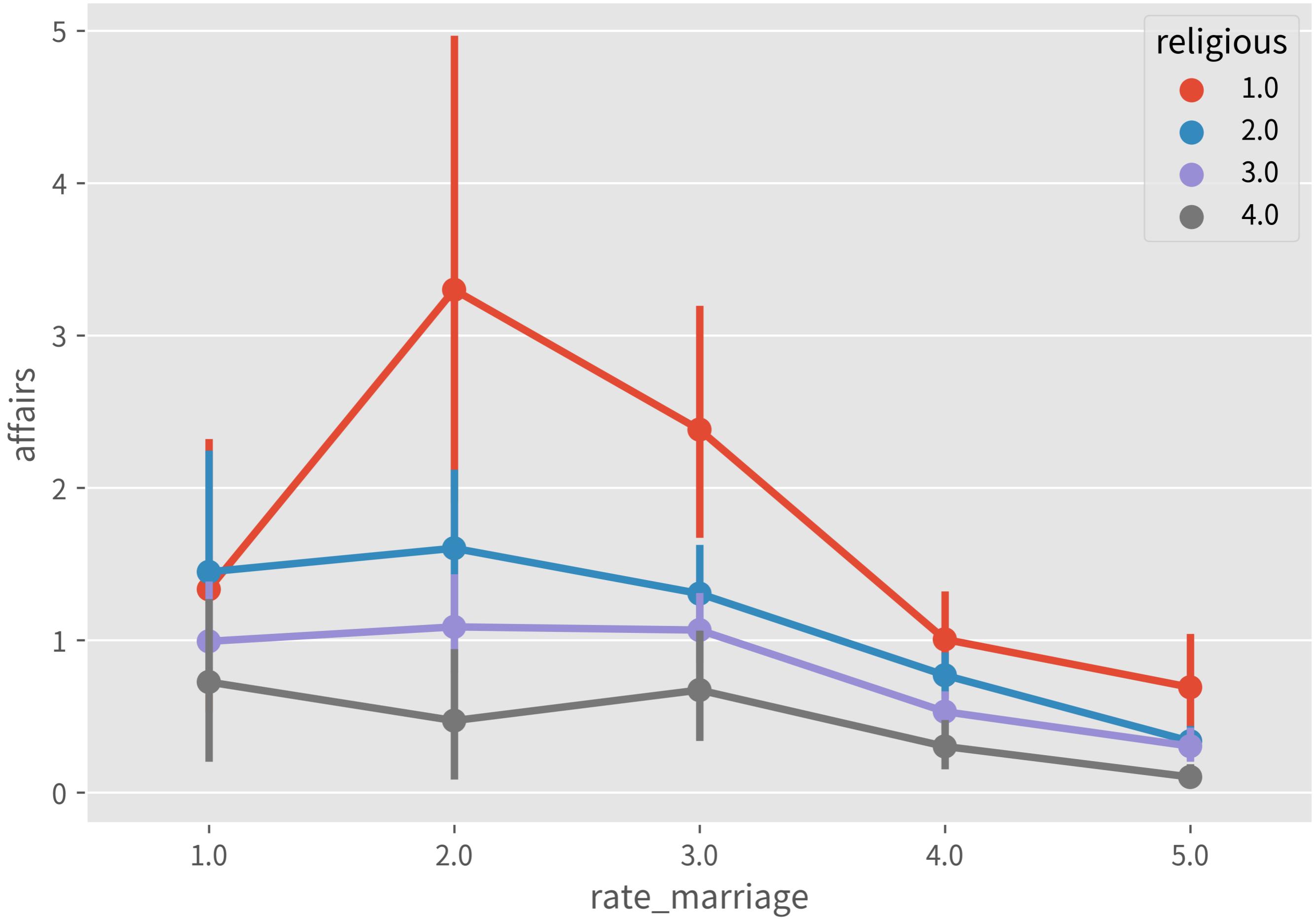
.....

$$y \sim x : z$$

$$\equiv y = \beta_0 + \beta_1 xz + \varepsilon$$

► Stronger!

►  $\equiv A \sim C(R)*C(L) - C(L)$



```
df = df_fair
(smf
.ols('affairs ~ rate_marriage*religious', df)
.fit()
.summary())
```

```
df = df_fair
res = (smf
.ols('affairs'
      '~ C(rate_marriage)*C(religious)', df)
.fit())
display(res.summary(),
        # type III is suitable to unbalanced dataset
        # ref: http://bit.ly/3typess
        sm.stats.anova_lm(res, typ=3))
```

```
df = df_fair
res = (smf
       .ols('affairs'
             '~ C(rate_marriage)'
             '+ C(rate_marriage):C(religious)', df)
       .fit())
display(res.summary(),
        sm.stats.anova_lm(res, typ=3))
```

```
df = df_fair
sns.pointplot(data=df,
               x='rate_marriage',
               y='affairs',
               hue='religious')
```

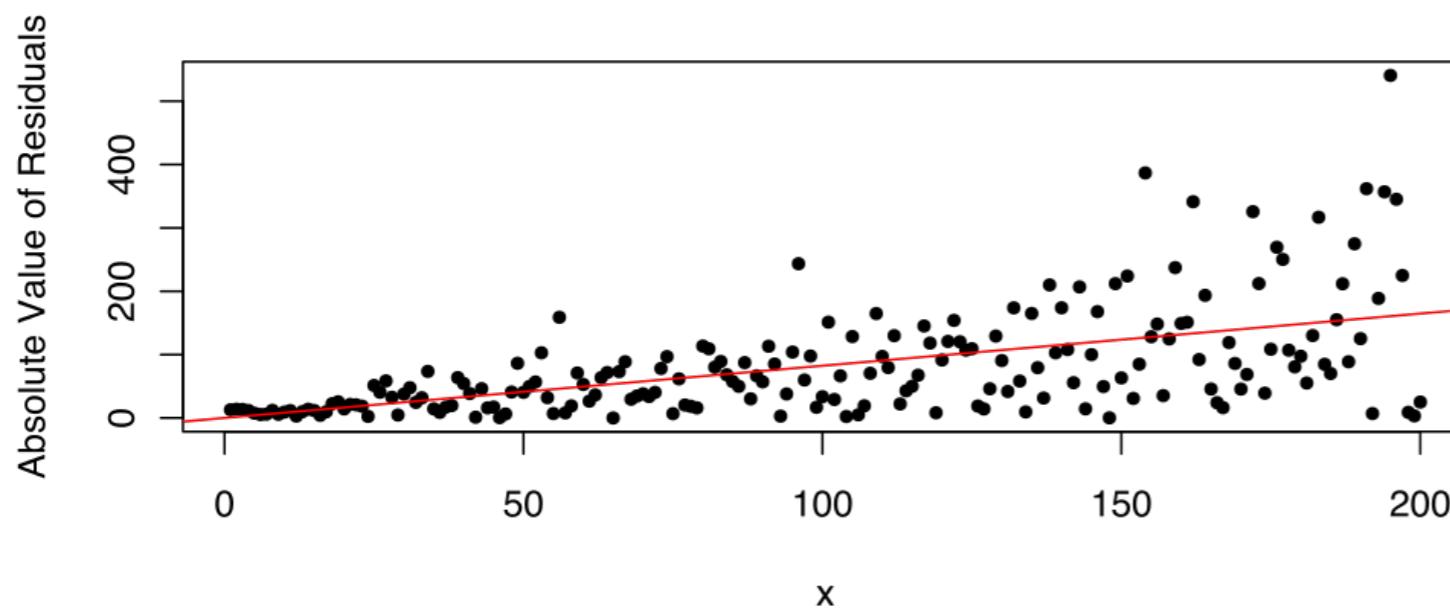
# More Operators: Transformation & Control Variables

---

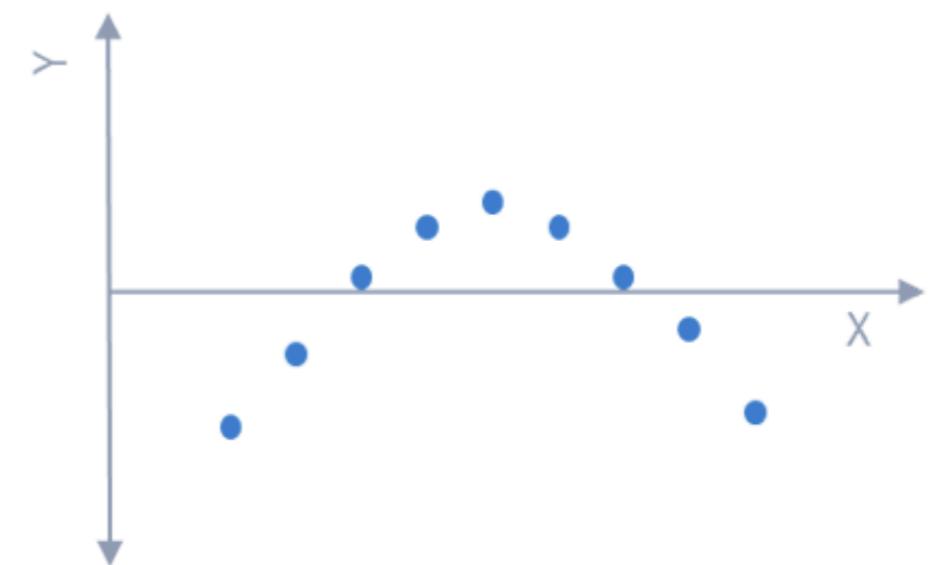
- `np.log(y) ~ x`
  - If  $y$  and  $x$  have a better linear relationship after transform.
  - Note that:
    - $\log(y) = \hat{\beta}_1x_1 + \hat{\beta}_2x_2$
    - $y = \exp(\hat{\beta}_1x_1 + \hat{\beta}_2x_2)$
    - $y = \exp(\hat{\beta}_1x_1) \times \exp(\hat{\beta}_2x_2)$
- `np.sqrt(y) ~ x`
- $y \sim I(x*z)$ 
  - True multiplication.

- $y \sim z_1 + \dots + x_1 + \dots$
- $z_i$  and  $x_i$  are both independent variables.
- If we don't interest in  $z_i$ , but add them to carry some effects and clarify the effects of  $x_i$ , we call  $z_i$  “**control variables**”.
- For example:
  - `conversion_rate ~ month + group`
- Check the full reference.

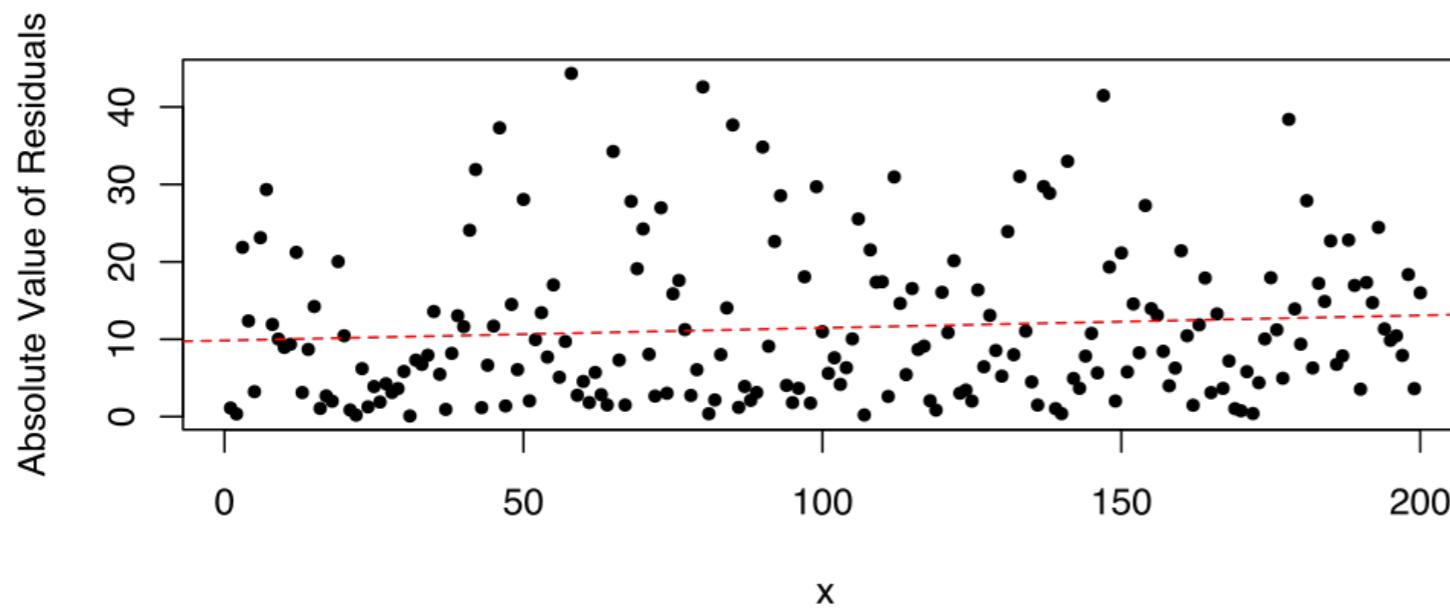
*Heteroskedastic Residuals*



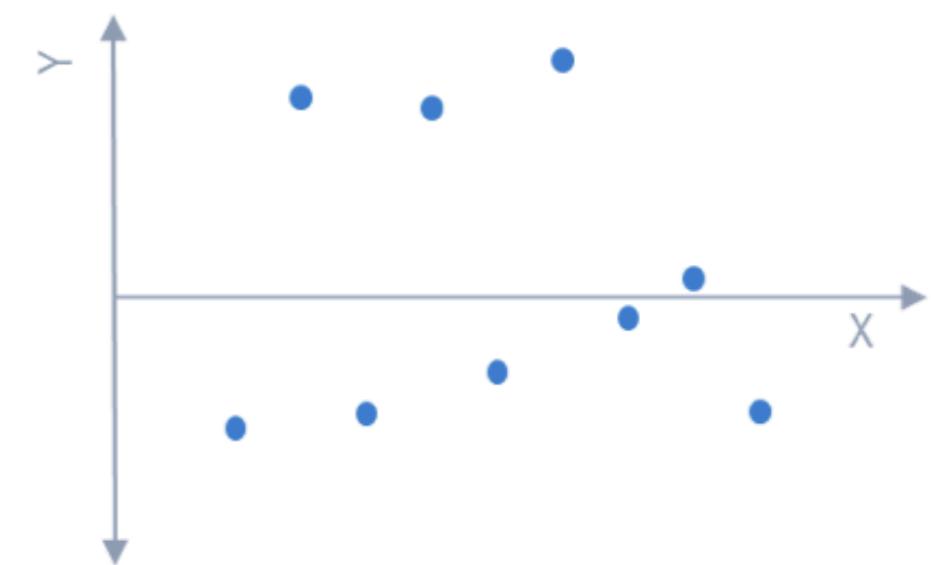
**Positive autocorrelation**



*Homoskedastic Residuals*



**Negative autocorrelation**



*Covariance Types of Errors*

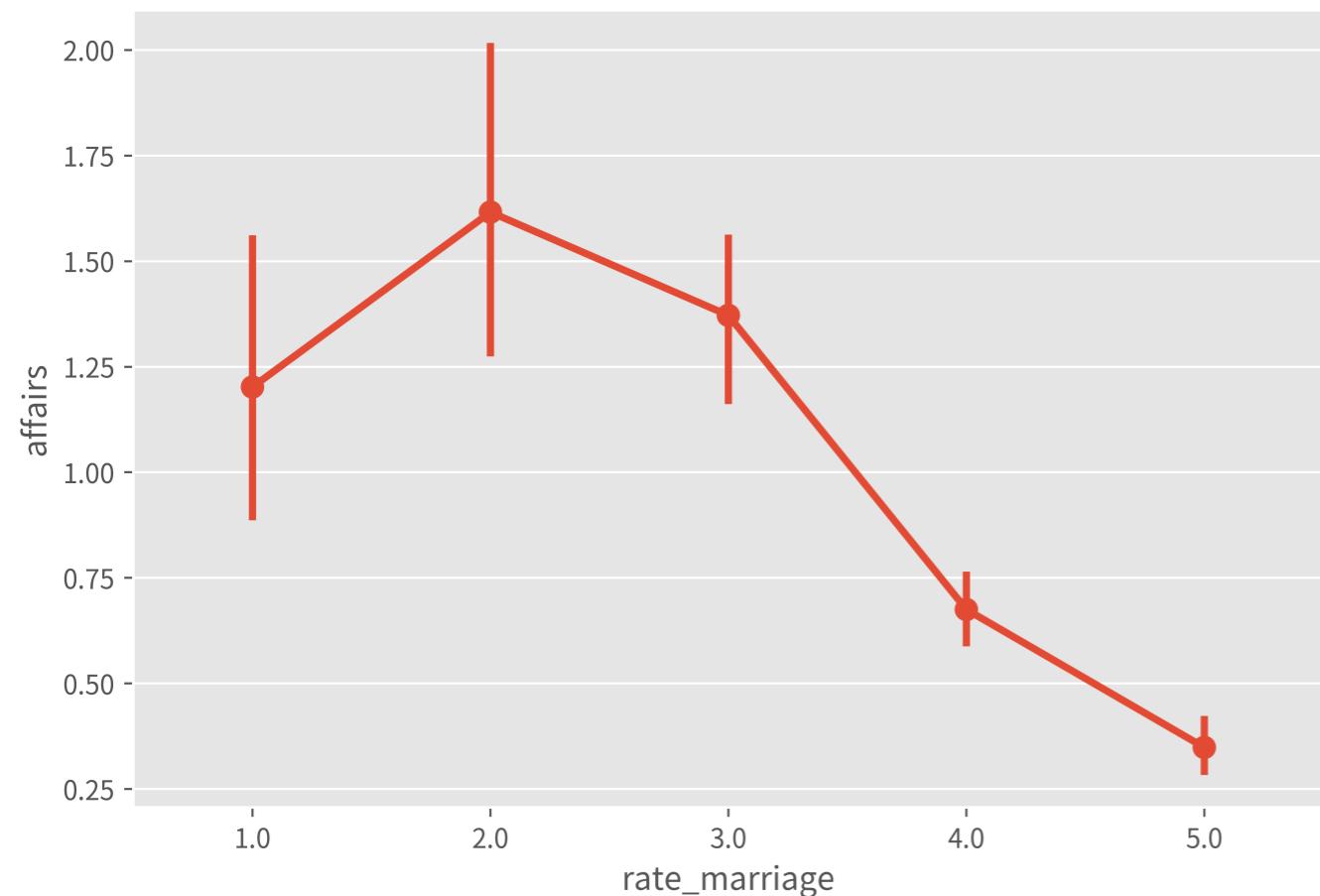
# Covariance Types of Errors

---

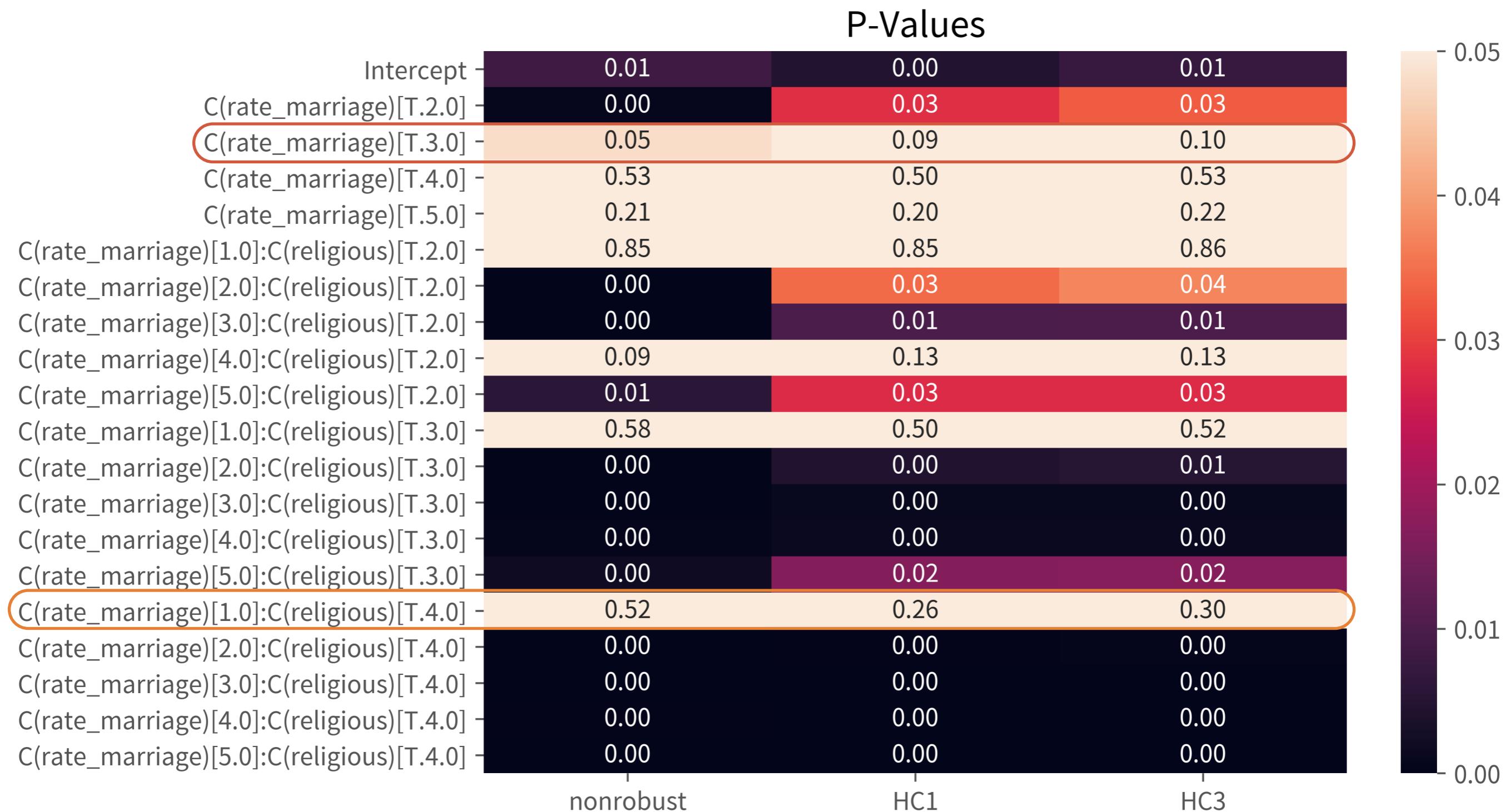
- **Spherical Errors**
  - $\equiv$  Homoscedasticity & no autocorrelation
- **Heteroscedasticity**
- **Autocorrelation**  $\equiv$  serial correlation
- If spherical errors, the model is good.
- If not spherical errors, the std errs are wrong.
  - So the interval estimates are **wrong**, including hypothesis tests on coeffs, confidence intervals.

# Heteroscedasticity

---



- Use HC std errs  
(heteroscedasticity-consistent standard errors) to correct.
- If  $N \leq 250$ , use HC3. [ref]
- If  $N > 250$ , consider HC1 for the speed.
- Also suggest to use by default.
- `.fit(cov_type='HC3')`  
← The confidence intervals vary among groups. The heteroscedasticity exists.



# Autocorrelation

OLS Regression Results

<b>Dep. Variable:</b>	affairs	<b>R-squared:</b>	0.032			
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.032			
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	208.4			
<b>Date:</b>	Fri, 26 Apr 2019	<b>Prob (F-statistic):</b>	1.66e-46			
<b>Time:</b>	23:25:02	<b>Log-Likelihood:</b>	-13959.			
<b>No. Observations:</b>	6366	<b>AIC:</b>	2.792e+04			
<b>Df Residuals:</b>	6364	<b>BIC:</b>	2.794e+04			
<b>Df Model:</b>	1					
<b>Covariance Type:</b>	nonrobust					
	coef	std err	t	P> t	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353
Omnibus:	9443.528	Durbin-Watson:	1.606			
Prob(Omnibus):	0.000	Jarque-Bera (JB):	5215639.758			
Skew:	8.930	Prob(JB):	0.00			
Kurtosis:	142.083	Cond. No.	19.5			

- Durbin-Watson
  - 2 is no autocorrelation.
  - [0, 2) is positive autocorrelation.
  - (2, 4] is negative autocorrelation.
  - [1.5, 2.5] are relatively normal. [\[ref\]](#)
- Use HAC std err.
- `.fit(cov_type='HAC', cov_kwds=dict(maxlag=tau))`

# Other Covariance Types

---

- cluster
  - Assume each group has spherical errors.
- hac-groupsum
  - Sum by the time label and then process.
- hac-panel
  - Process by the groups and then aggregate.
- Check the full references.

# Outliers

---

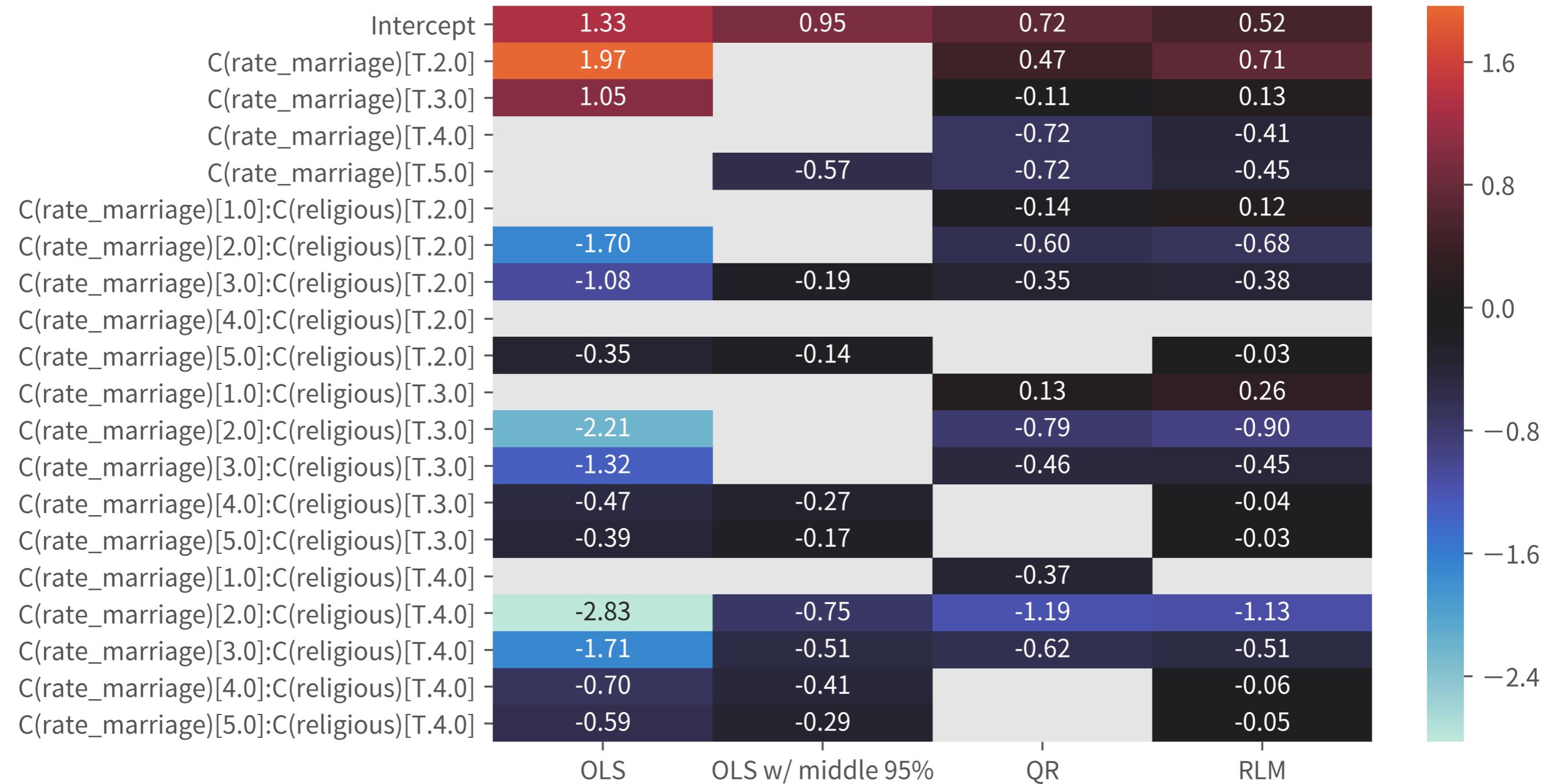
- An outlier may be the most interesting observation.
- Consider to include more variables to explain.
- Consider the non-linear relationship.
- Consider outlier-insensitive models.
- Do not drop observation without a good reason.
- Some good reasons:
  - Typo.
  - Not representative of the intended study population.
- Report the models with and without outliers, or;
- Include the preprocess steps in the report.

# Preprocess Methods for Outliers

---

- Quantile regression: estimates the **median** rather than mean.
- Robust regression: **robust to outliers**, but slower.
- Keep middle n%: changes the intended study population.
- OLS
  - Outliner test
  - Influence report

Params



```
df = df_fair
```

```
alpha = 0.05
```

```
a = df.affairs.quantile(alpha/2)
```

```
b = df.affairs.quantile(1-alpha/2)
```

```
df = df[(df.affairs >= a) & (df.affairs <= b)]
```

```
df_fair_middle95 = df
```

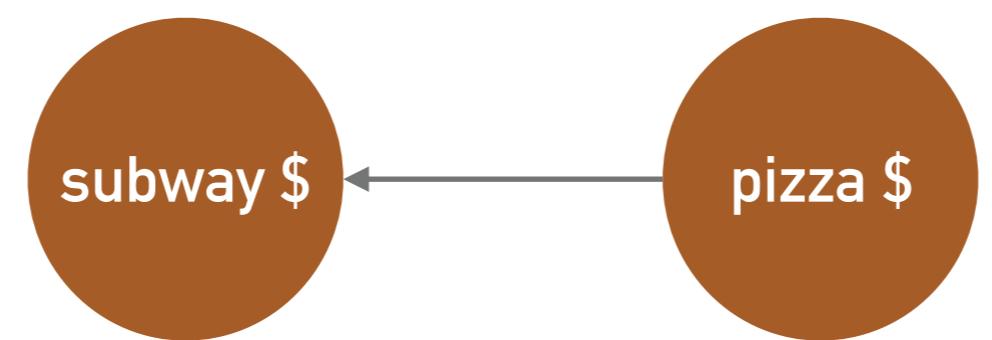
```
df = df_fair
```

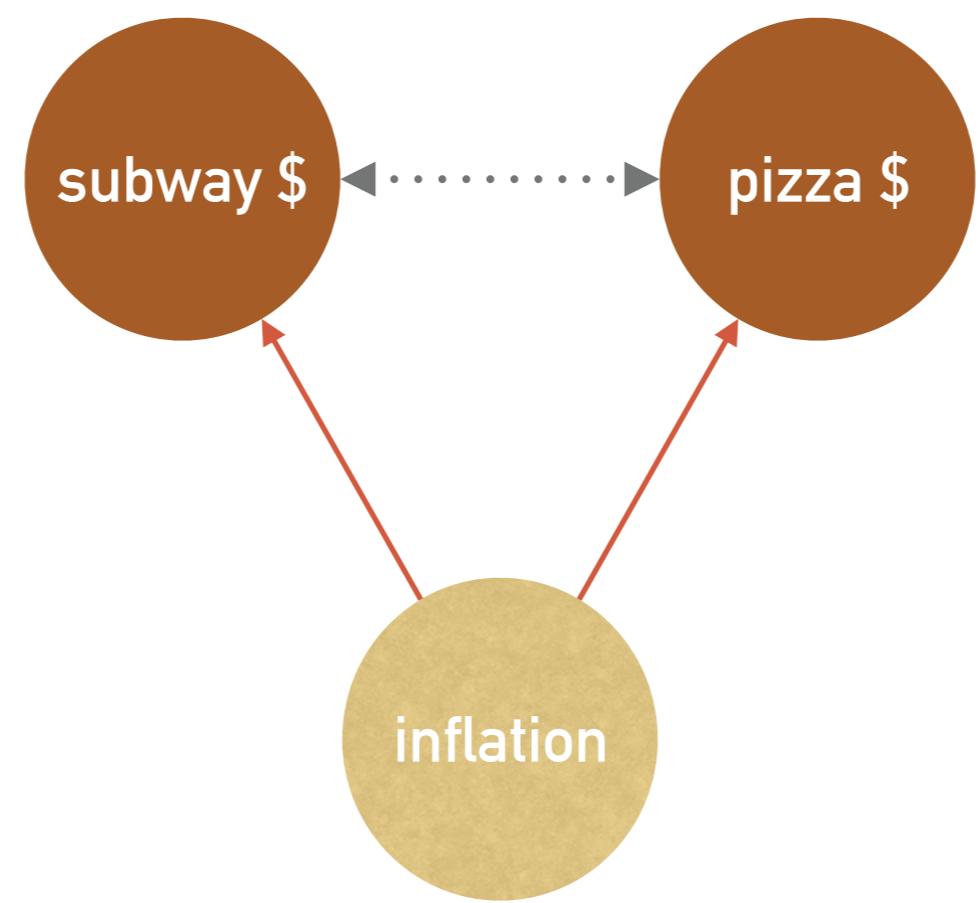
```
smf.ols(formula, df).fit().summary()
```

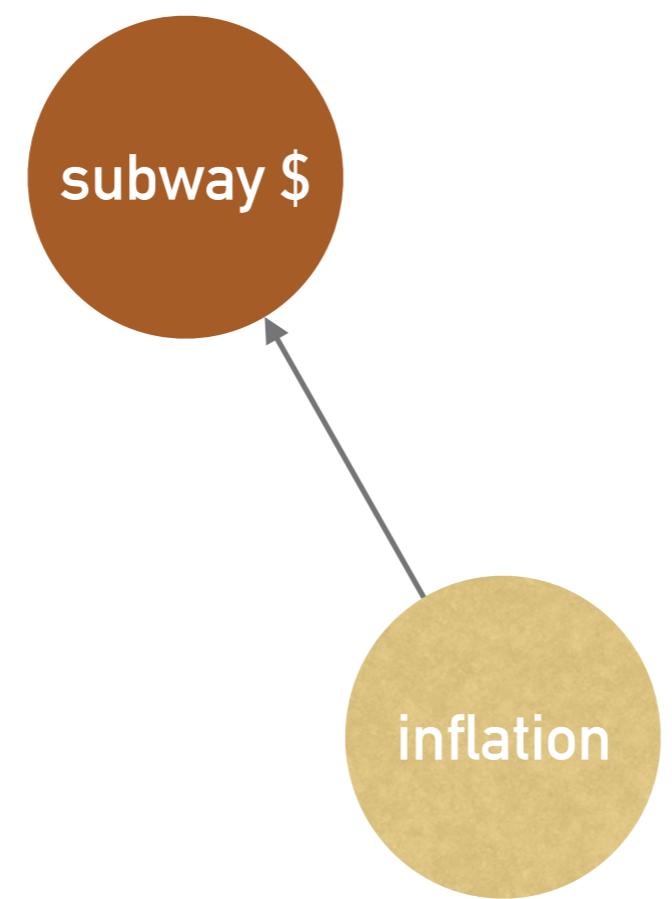
```
smf.ols(formula, df_fair_middle95).fit().summary()
```

```
smf.quantreg(formula, df).fit().summary()
```

```
smf.rlm(formula, df).fit().summary()
```







# Correlation Does Not Imply Causation

---

- $y \sim x :=$  “ $y$  has association with  $x$ ”
- $y \leftarrow x :=$  “ $y$  because  $x$ ”
- $y \sim x$  may be:
  - $y \leftarrow x$
  - $y \rightarrow x$
  - $z \rightarrow y \wedge z \rightarrow x$
- So  $y \sim x$  doesn't implies  $y \leftarrow x$ .
- *A good study design*  $\wedge$   $y \sim x$  can implies  $y \leftarrow x$ .

# Suggested Wording

---

- Causation
  - “Affect” “Influence”
  - “Predict” “Prediction”, in regression analysis especially.
- Correlation
  - “Associate” “Association”
  - “Correlate” “Correlation”, in correlation analysis especially.

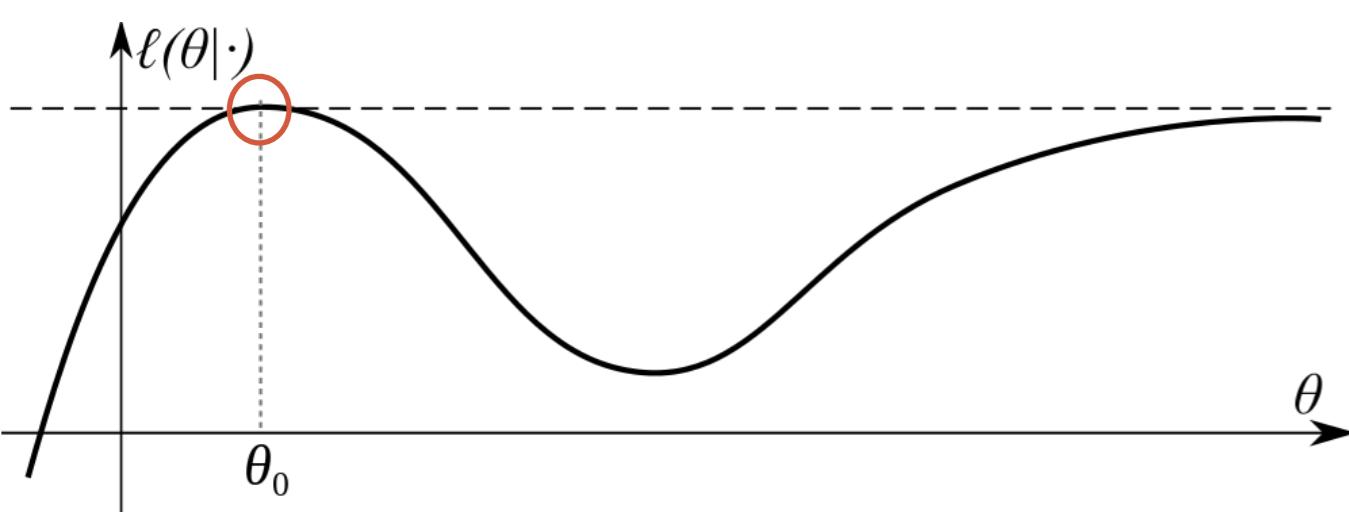
# More Models

---

- Discrete Models, e.g., **Logit Model**:
  - $y \in \{0, 1\}$
- Mixed Model for **group** and **subject** effect.
  - $y = X\beta + Zu + \varepsilon$
- Time Series Models, e.g., **Autoregressive Model**:
  - $x_t = c + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \dots + \varphi_p x_{t-p} + \varepsilon_t$
- Check all the models that StatsModels supports.

# More Estimations

---



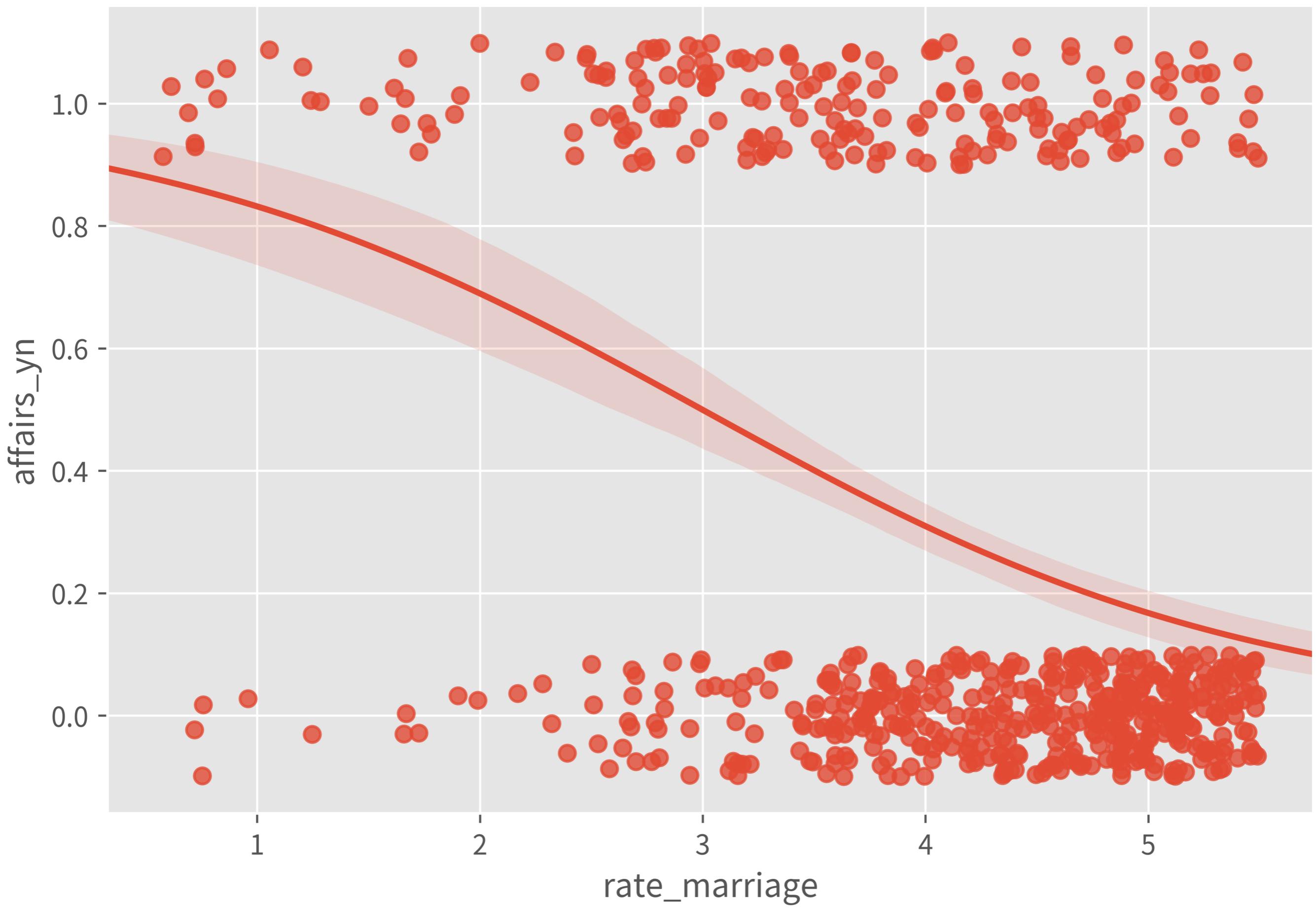
- MLE, Maximum Likelihood Estimation.

← Usually find by numerical methods.

- TSLS, Two-Stage Least Squares.

- $y \leftarrow (x \leftarrow z)$

- Handle the endogeneity:  
 $E[\varepsilon | X] \neq 0.$



Dep. Variable:	affairs_yn	No. Observations:	6366			
Model:	Logit	Df Residuals:	6346			
Method:	MLE	Df Model:	19			
Date:	Mon, 29 Apr 2019	Pseudo R-squ.:	0.1003			
Time:	22:48:50	Log-Likelihood:	-3601.0			
converged:	True	LL-Null:	-4002.5			
		LLR p-value:	4.598e-158			
	coef	std err	z	P> z	[0.025	0.975]
Intercept	0.6931	0.500	1.386	0.166	-0.287	1.673
C(rate_marriage)[T.2.0]	0.2231	0.581	0.384	0.701	-0.915	1.362
C(rate_marriage)[T.3.0]	-0.2122	0.523	-0.405	0.685	-1.238	0.813
C(rate_marriage)[T.4.0]	-1.2009	0.512	-2.345	0.019	-2.205	-0.197
C(rate_marriage)[T.5.0]	-1.6664	0.512	-3.256	0.001	-2.669	-0.663
C(rate_marriage)[1.0]:C(religious)[T.2.0]	0.5596	0.641	0.873	0.383	-0.696	1.816
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-0.1398	0.345	-0.405	0.686	-0.817	0.537
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-0.2657	0.184	-1.443	0.149	-0.626	0.095
C(rate_marriage)[4.0]:C(religious)[T.2.0]	-0.0294	0.132	-0.222	0.824	-0.288	0.230
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.4791	0.140	-3.427	0.001	-0.753	-0.205
C(rate_marriage)[1.0]:C(religious)[T.3.0]	0.4769	0.629	0.758	0.448	-0.756	1.710
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-0.5656	0.349	-1.622	0.105	-1.249	0.118
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-0.3412	0.188	-1.811	0.070	-0.710	0.028
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.4342	0.134	-3.239	0.001	-0.697	-0.171
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.6133	0.137	-4.487	0.000	-0.881	-0.345
C(rate_marriage)[1.0]:C(religious)[T.4.0]	0.2231	0.975	0.229	0.819	-1.687	2.133
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-1.3218	0.504	-2.622	0.009	-2.310	-0.334
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-0.7105	0.286	-2.486	0.013	-1.271	-0.150
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.7732	0.210	-3.675	0.000	-1.186	-0.361
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-1.3503	0.212	-6.355	0.000	-1.767	-0.934

# Logit Model

- The coef is log-odds.
- Use  $\exp(x)/(\exp(x) + 1)$  to transform back to probability:
  - $0.6931 \rightarrow 67\%$
  - " $-1.6664 \rightarrow 27\%$ "
  - " $-1.3503 \rightarrow 9\%$ "
- Or:
- `.predict(dict(rate_marriage=[1, 5, 5], religious=[1, 1, 4]))`

```
df = df_fair
df = df.assign(affairs_yn=(df.affairs > 0).astype(float))
df_fair_2 = df
```

```
df = df_fair_2.sample(frac=0.1, random_state=20190429)
sns.regplot(data=df, x='rate_marriage', y='affairs_yn',
             logistic=True,
             x_jitter=1/2, y_jitter=0.2/2)
```

```
df = df_fair_2
(smf
.logit('affairs_yn'
       ' ~ C(rate_marriage)'
       '+ C(rate_marriage):C(religious)', df)
.fit()
.summary())
```

# Recap

---

- Choose the method by the assumption.
- Correlation Analysis: Gets an overview.
- Understand Regression Result:
  - Plotting, Adj. R-squared, Cond. No., Durbin-Watson, etc.
- Model Specification Using the R Formula:
  - $y \sim \theta + x$
  - $y \sim x * z$
  - $y \sim x : z$
- Covariance Types: Use HC3 by default.
- Let's explain and predict efficiently! 