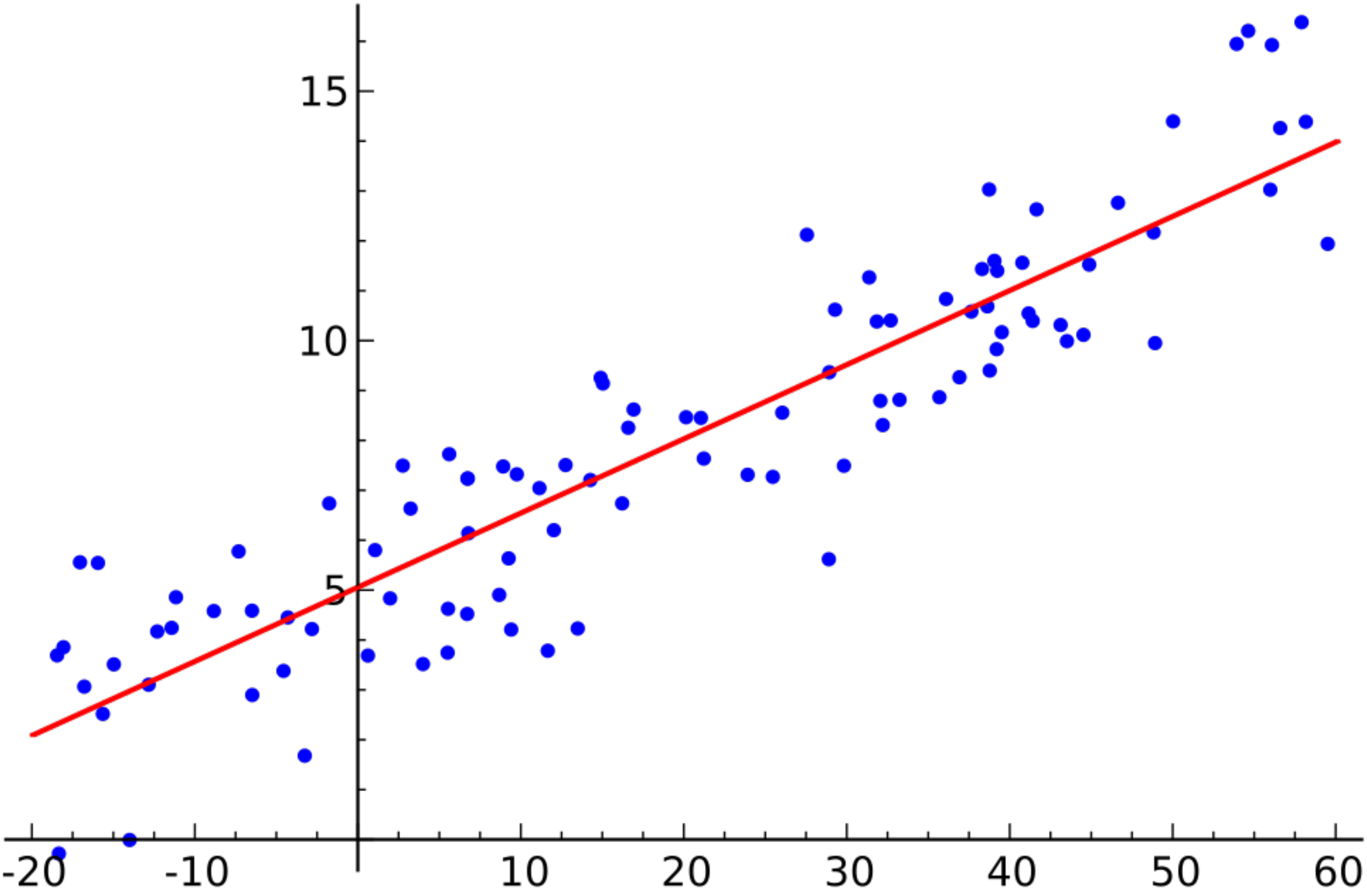
Statistical Regression With Python

Explain & Predict


15
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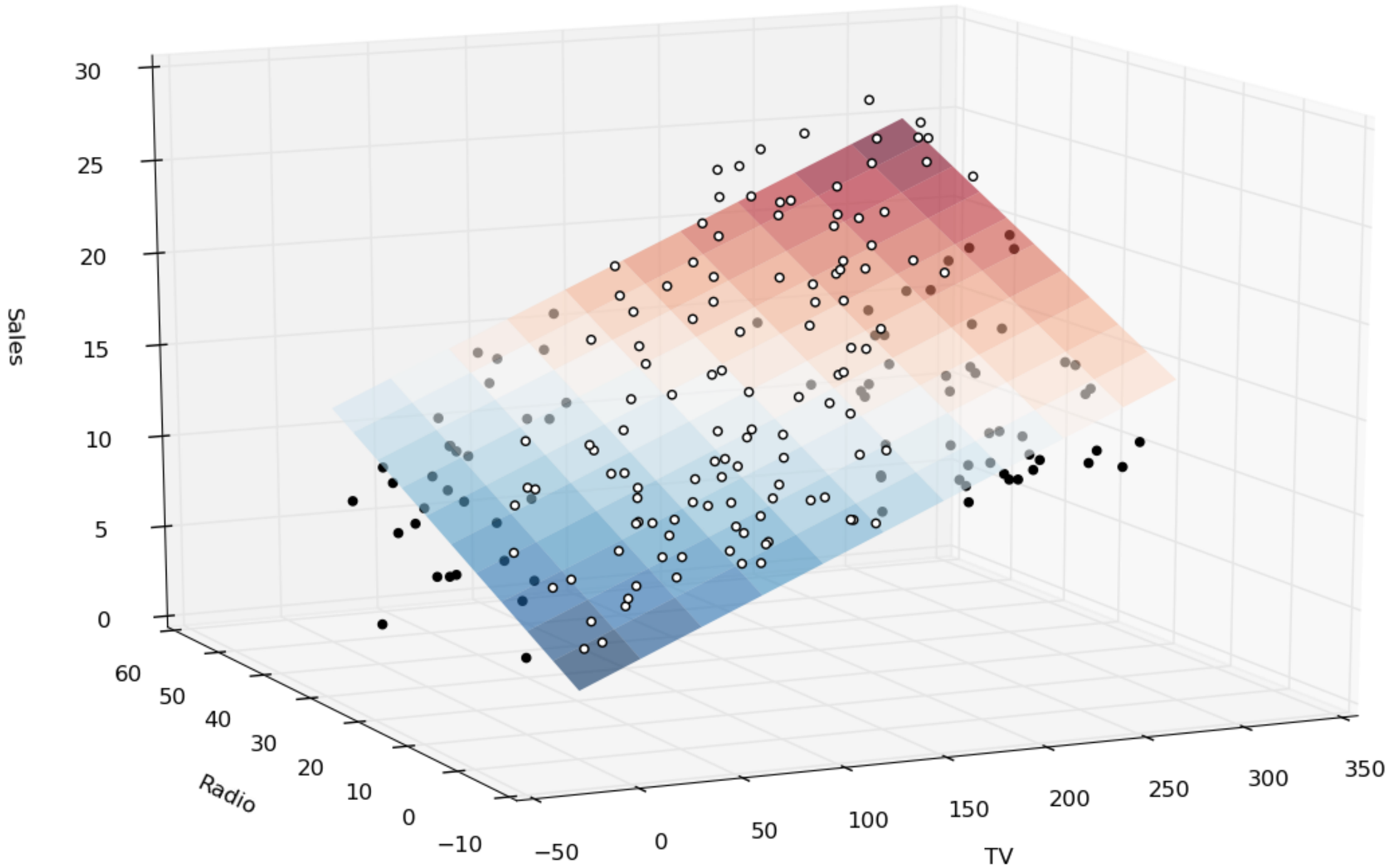
Sales
30
25
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15
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5
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Radio
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50
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20
10
0
−10
TV
−50
0
50
100
150
200
250
300
350

Explain & Predict

- $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
- A line.
- Explain by β, the slope.
- Predict by new x_i.
- "Simple linear regression model"

- $y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$
- A n-dim hyperplane.
- $\boldsymbol{\beta}$, a slope vector.
- New $\boldsymbol{x}_i$, a vector.
- "Multiple linear regression model"

How to find the “line”?

Mosky

- Python Charmer at Pinkoi.
- Has spoken at: PyCons in TW, MY, KR, JP, SG, HK, COSCUPs, and TEDx, etc.
- Countless hours on teaching Python.
- Own the Python packages: ZIPCodeTW, MoSQL, Clime, etc.
- http://mosky.tw/

Outline

- The Analysis Steps
  - Define Assumptions
  - Validate Assumptions
    - The Dataset: Fair
- Correlation Analysis
- Ordinary Least Squares
  - Models & Estimations
  - Understand Regression Result
- Model Specification Using the R Formula
- Covariance Types
- Outliers
- Correlation & Causation
- More Models & Estimations
  - Introduction
  - Logit Model

The PDF, Notebooks, and Packages

- The PDF and notebooks are available on https://github.com/moskytw/statistical-regression-with-python .
- The packages:
  - `$ pip3 install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn`

  Or:

  - `> conda install jupyter numpy scipy sympy matplotlib ipython pandas seaborn statsmodels scikit-learn`

Define Assumptions

- The regression analysis:
  - Suitable to measure the relationship between variables.
  - Can model most of the hypothesis testing. [ref]
  - Can predict.

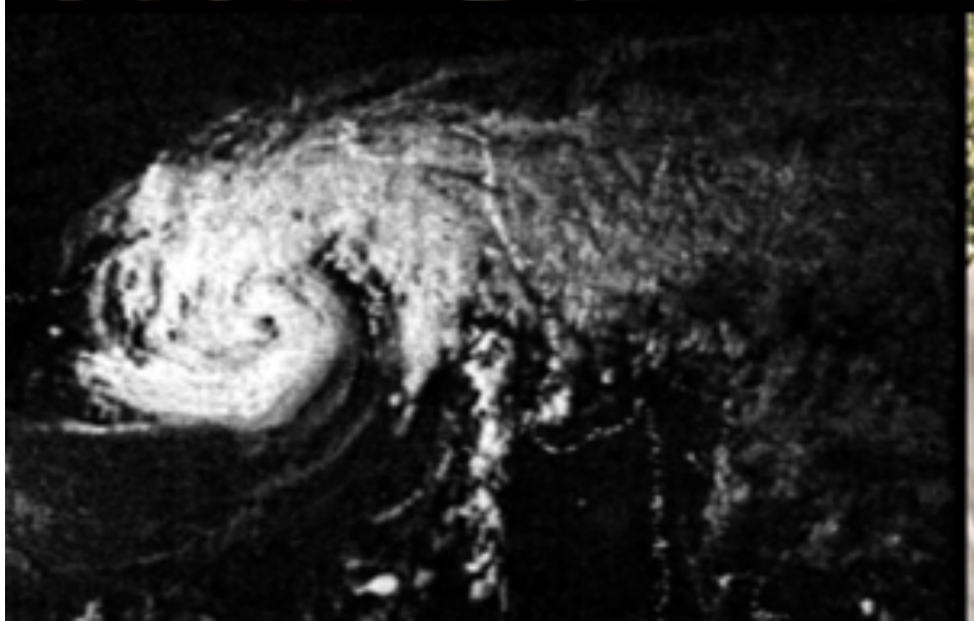
- "Years of marriage has association with children?"
- "Rates of marriage has association with affairs?"
- "Any background may have association with affairs?"

Validate Assumptions

- Collect data ...
  - The "Fair" dataset:
    - Fair, Ray. 1978. "A Theory of Extramarital Affairs," Journal of Political Economy, February, 45-61.
      - A dataset from 1970s.
      - Rows: 6,366
      - Columns: (next slide)
- The full version of the analysis steps: http://bit.ly/analysis-steps .

1. *rate_marriage*: 1~5; very poor, poor, fair, good, very good.
2. *age*
3. *yrs_married*
4. *children*: number of children.
5. *religious*: 1~4; not, mildly, fairly, strongly.
6. *educ*: 9, 12, 14, 16, 17, 20; grade school, some college, college graduate, some graduate school, advanced degree.
7. *occupation*: 1, 2, 3, 4, 5, 6; student, farming-like, white-colloar, teacher-like, business-like, professional with advanced degree.
8. *occupation_husb*
9. *affairs*: n times of extramarital affairs per year since marriage.

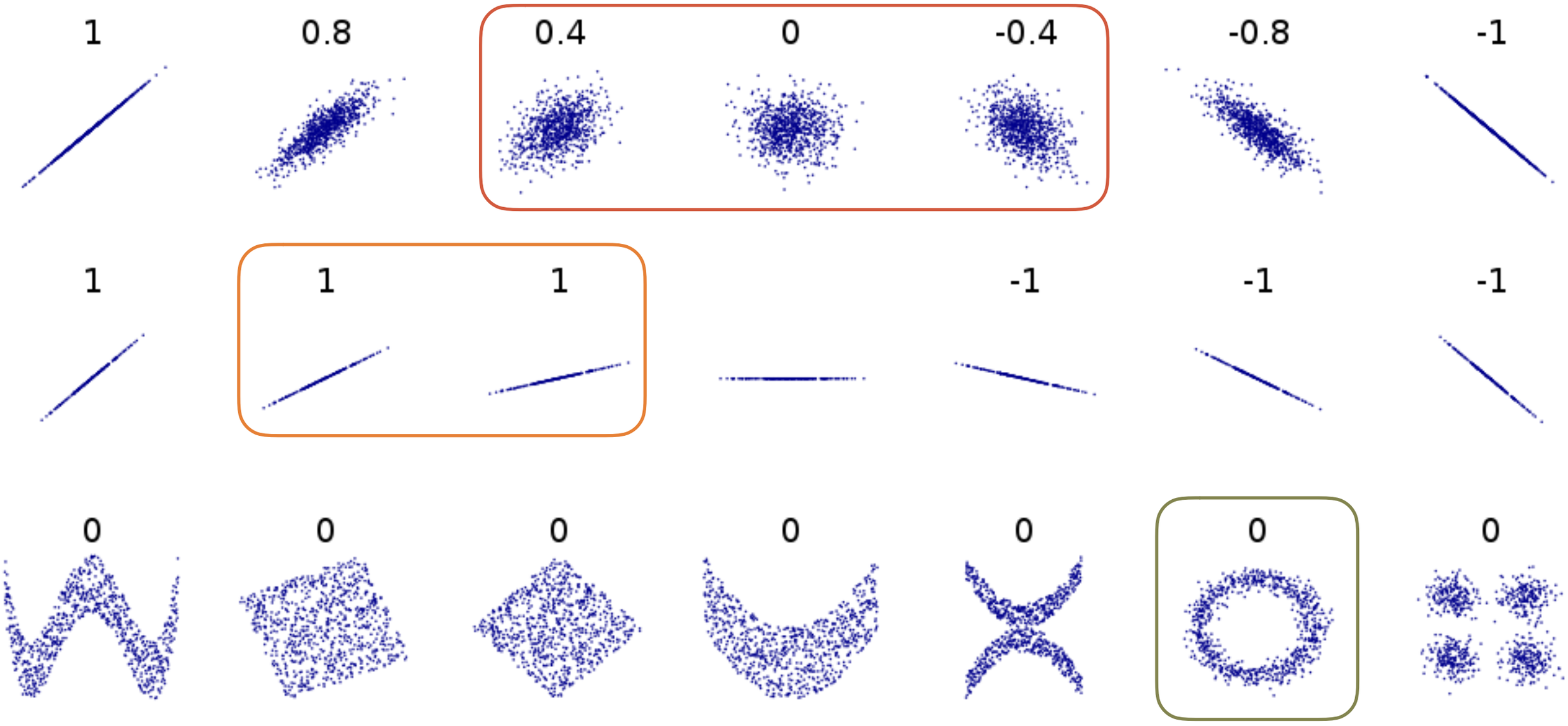

JESUS SAV
All up
with the
Holy
Ghost
& Fire

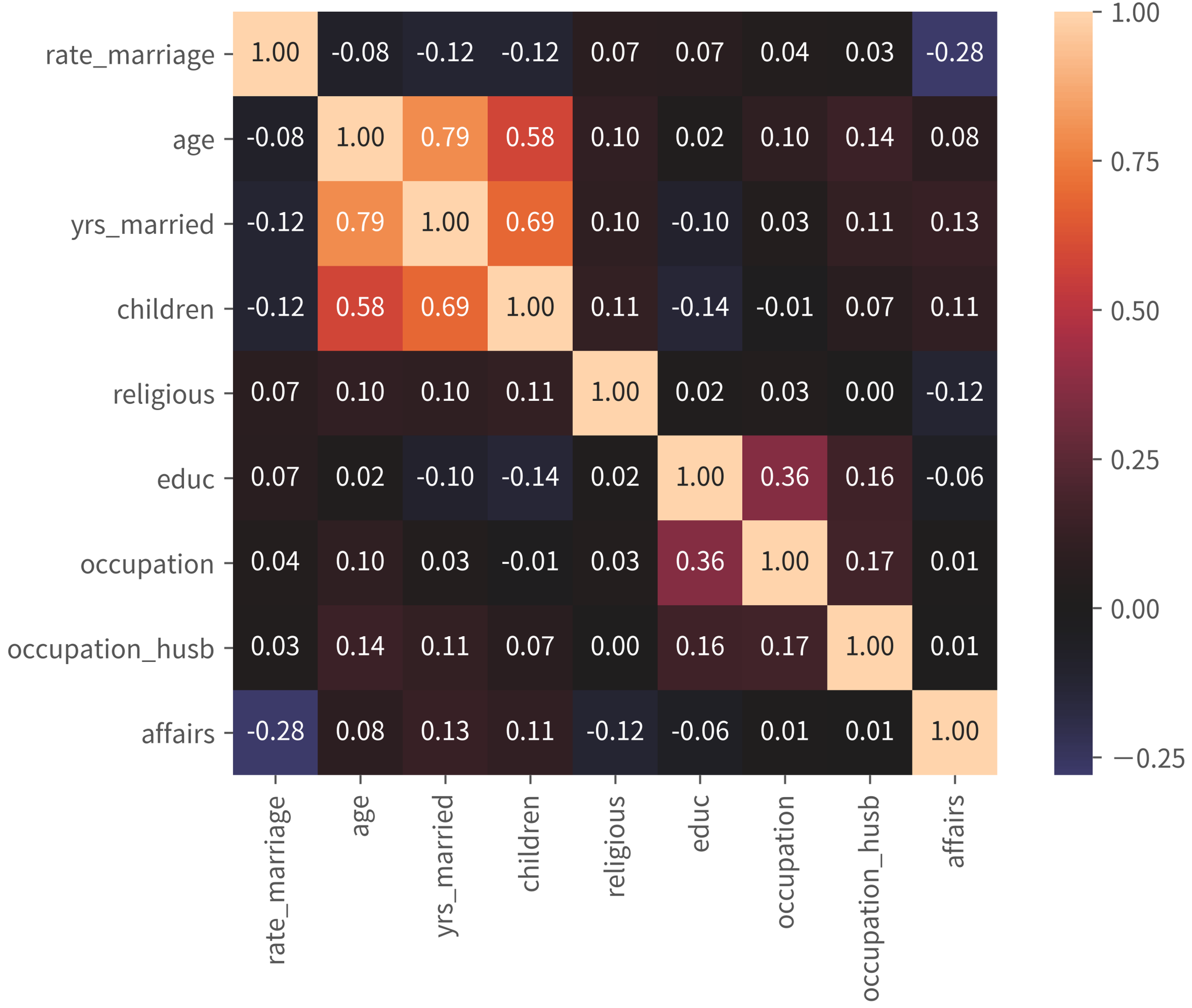


Correlation Analysis

- Measures "the linear tightness".
- **Pearson correlation coefficient**
  - For the variables whose distance is meaningful.
  - `df.corr()`
- **Kendall rank correlation coefficient**
  - For the variables whose order is meaningful.
  - `df.corr('kendall')`

Pearson Correlation Coefficient




rate_marriage 1.00 -0.08 -0.12 -0.12 0.07 0.07 0.04 0.03 -0.28
age -0.08 1.00 0.79 0.58 0.10 0.02 0.10 0.14 0.08
yrs_married -0.12 0.79 1.00 0.69 0.10 -0.10 0.03 0.11 0.13
children -0.12 0.58 0.69 1.00 0.11 -0.14 -0.01 0.07 0.11
religious 0.07 0.10 0.10 0.11 1.00 0.02 0.03 0.00 -0.12
educ 0.07 0.02 -0.10 -0.14 0.02 1.00 0.36 0.16 -0.06
occupation 0.04 0.10 0.03 -0.01 0.03 0.36 1.00 0.17 0.01
occupation_husb 0.03 0.14 0.11 0.07 0.00 0.16 0.17 1.00 0.01
affairs -0.28 0.08 0.13 0.11 -0.12 -0.06 0.01 0.01 1.00
1.00
0.75
0.50
0.25
0.00
−0.25
rate_marriage
age
yrs_married
children
religious
educ
occupation
occupation_husb
affairs

```
import statsmodels.api as sm
import statsmodels.formula.api as smf
import seaborn as sns


print(sm.datasets.fair.SOURCE,
      sm.datasets.fair.NOTE)

# -> Pandas's Dataframe
df_fair = sm.datasets.fair.load_pandas().data


df = df_fair
sns.heatmap(df.corr(method='kendall'),
            center=0, square=True,
            annot=True, fmt='.2f')
```

Models & Estimations

- Models
  - $y = X\beta + \varepsilon$
  - Like simple, multiple, logit, etc.
- Estimations
  - How to estimate the $\hat{\beta}$? For example, OLS:

$$y = X\hat{\beta}$$

$$S(b) = \sum_{i=1}^{n} (y_i - x_i^{\mathrm{T}} b)^2 = (y - Xb)^{\mathrm{T}}(y - Xb)$$

$$\hat{\beta} = \operatorname{argmin}_{b \in \mathbb{R}^p} S(b) = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

affairs ~ rate_marriage

OLS Regression Results

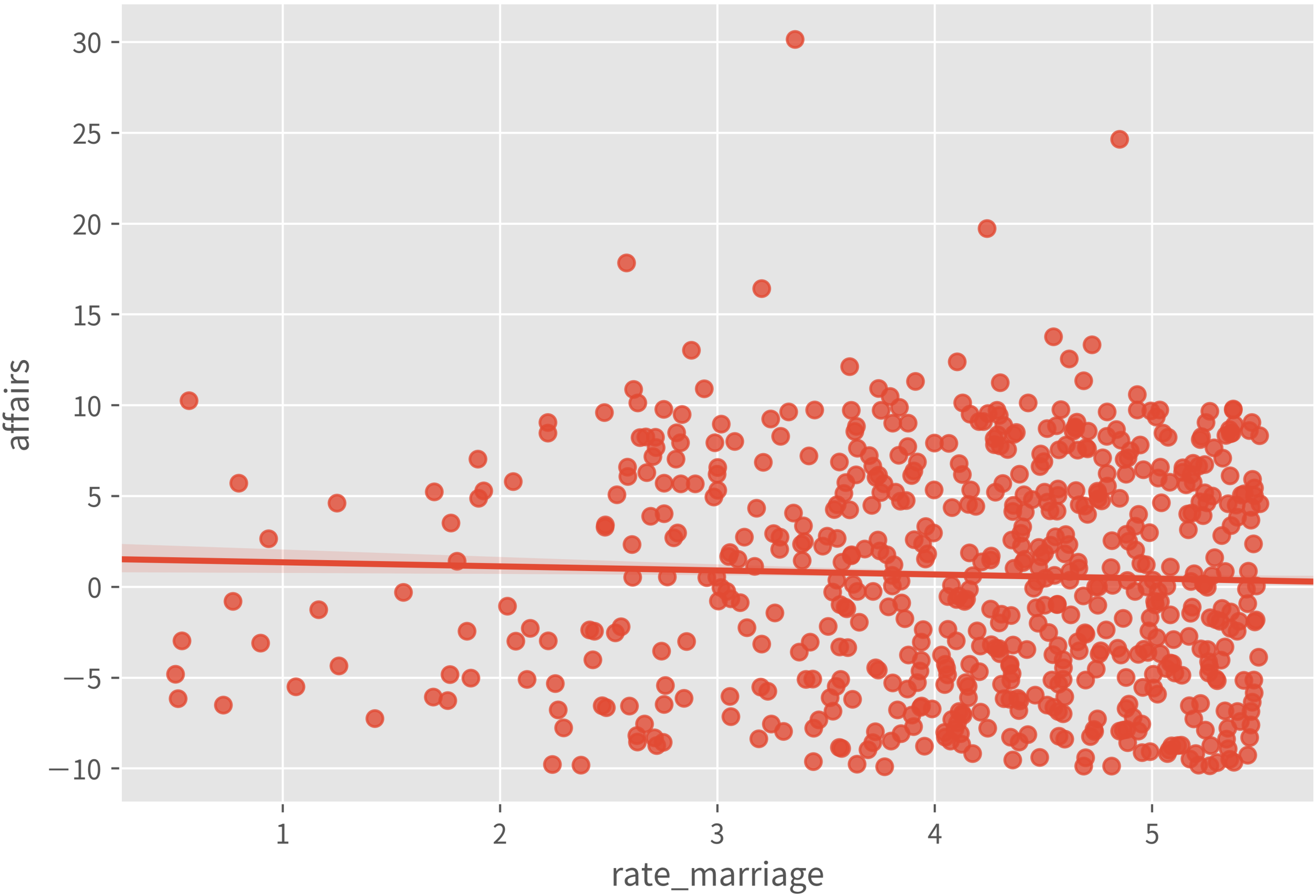
Dep. Variable:	affairs	**R-squared:**	0.032
Model:	OLS	**Adj. R-squared:**	0.032
Method:	Least Squares	**F-statistic:**	208.4
Date:	Fri, 26 Apr 2019	**Prob (F-statistic):**	1.66e-46
Time:	23:25:02	**Log-Likelihood:**	-13959.
No. Observations:	6366	**AIC:**	2.792e+04
Df Residuals:	6364	**BIC:**	2.794e+04
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353

Omnibus:	9443.528	**Durbin-Watson:**	1.606
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

- Using the R formula implementation in Python, Patsy:

$$
\begin{aligned}
& y \sim x \\
\equiv\ & y \sim 1 + x \\
\equiv\ & y = \beta_0 1 + \beta_1 x + \varepsilon
\end{aligned}
$$


affairs
30
25
20
15
10
5
0
−5
−10
1
2
3
4
5
rate_marriage

```
df_fair_sample = df_fair.sample(
    frac=0.1, random_state=20190425
)


df = df_fair
(smf
 .ols('affairs ~ rate_marriage', df)
 .fit()
 .summary())


df = df_fair_sample
sns.regplot(data=df, x='rate_marriage', y='affairs',
            x_jitter=1/2, y_jitter=20/2)
```

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Adj. R-squared

- ≡ *explained var. by X / var. of y and adjusted by no. of X*
- ∈ [0, 1], usually.
- Can compare among models.
- 0.032 is super bad.

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Prob(F-statistics)

- $= \mathrm{P}(data \mid all\ coefs\ are\ zero)$
- Accept the model if low enough.
- "Low enough" is "< 0.05" in convention.

OLS Regression Results

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Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

Log-Likelihood

- Higher is better.
- Negative, usually.
- Can compare among models when the datasets are the same.
- Also check likelihood-ratio test.

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Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

Large Sample or Normality

- No. Observations, or
  - $\geq$ 110~200 [ref]
- Normality of Residuals
  - Prob(*Omnibus*) ≥ 0.05
  - $\wedge$ Prob(JB) ≥ 0.05
- To construct interval estimates correctly, e.g., hypothesis tests on coefs, confidence intervals.

OLS Regression Results

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Df Model:	1		

Covariance Type:	nonrobust

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Omnibus:	9443.528	**Durbin-Watson:**	1.606
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

Cond. No.

- Measures the degree of multicollinearity.
- Multicollinearity increases the std err,
  i.e., decreases efficiency.
- If ≥ 30, check:
  - Any variable has redundant information? Like fat % and weight. Drop one.
  - If no, the model is good.
  - Or other suggestions.

OLS Regression Results

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Time:	23:25:02	**Log-Likelihood:**	-13959.
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Df Residuals:	6364	**BIC:**	2.794e+04
Df Model:	1		
Covariance Type:	nonrobust		

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Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

P>|t|

- $= \mathrm{P}(data \mid the\ coef\ is\ zero)$
- Accept the coef if low enough.
- Drop the term, otherwise.

Coef & Confidence Intervals

OLS Regression Results

Dep. Variable:	affairs	**R-squared:**	0.032
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Covariance Type:	nonrobust		

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Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

- "The rate_marriage and affairs have a negative relationship, the strength is -0.41, and 95% confidence interval is [-0.46, -0.35]."

OLS Regression Results

Dep. Variable:	affairs	**R-squared:**	0.036
Model:	OLS	**Adj. R-squared:**	0.035
Method:	Least Squares	**F-statistic:**	58.57
Date:	Sat, 27 Apr 2019	**Prob (F-statistic):**	1.25e-48
Time:	15:27:20	**Log-Likelihood:**	-13946.
No. Observations:	6366	**AIC:**	2.790e+04
Df Residuals:	6361	**BIC:**	2.794e+04
Df Model:	4		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	1.2017	0.218	5.524	0.000	0.775	1.628
C(rate_marriage)[T.2.0]	0.4141	0.247	1.679	0.093	-0.069	0.897
C(rate_marriage)[T.3.0]	0.1696	0.228	0.743	0.457	-0.278	0.617
C(rate_marriage)[T.4.0]	-0.5268	0.222	-2.370	0.018	-0.963	-0.091
C(rate_marriage)[T.5.0]	-0.8535	0.222	-3.853	0.000	-1.288	-0.419

Omnibus:	9436.269	**Durbin-Watson:**	1.612
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5218576.884
Skew:	8.915	**Prob(JB):**	0.00
Kurtosis:	142.127	**Cond. No.**	21.0

affairs ~ C(rate_marriage)

- `y ~ C(x)`
- If 1, affairs is 1.20.
- If 5, affairs is 1.20 - 0.85.
- The 2, 3, 4 are not significant to 1.

Code Categorical Variables

- The str or bool is treated as categorical by default.
- The C function:

$$
\begin{aligned}
& y \sim \mathrm{C}(x) \\
\equiv\ & y \sim 1 + (x_1 + x_2 + \ldots + x_i) - x_1 \\
\equiv\ & y \sim 1 + x_2 + \ldots + x_i \\
\equiv\ & y = \beta_0 1 + \beta_2 x_2 + \ldots + \beta_i x_i + \varepsilon
\end{aligned}
$$

  - The x_1 is chosen as the reference level automatically.

- For example:

$$
\begin{aligned}
& \mathrm{C}(rate\_marriage \in \{1, 2, 3, 4, 5\}) \\
\equiv\ & 1 + rate\_marriage\_2 \in \{0, 1\} + \ldots + rate\_marriage\_5 \in \{0, 1\}
\end{aligned}
$$

OLS Regression Results

Dep. Variable:	affairs	**R-squared:**	0.036
Model:	OLS	**Adj. R-squared:**	0.035
Method:	Least Squares	**F-statistic:**	58.57
Date:	Sat, 27 Apr 2019	**Prob (F-statistic):**	1.25e-48
Time:	17:20:18	**Log-Likelihood:**	-13946.
No. Observations:	6366	**AIC:**	2.790e+04
Df Residuals:	6361	**BIC:**	2.794e+04
Df Model:	4		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
C(rate_marriage)[1.0]	1.2017	0.218	5.524	0.000	0.775	1.628
C(rate_marriage)[2.0]	1.6157	0.116	13.925	0.000	1.388	1.843
C(rate_marriage)[3.0]	1.3713	0.069	19.963	0.000	1.237	1.506
C(rate_marriage)[4.0]	0.6748	0.046	14.762	0.000	0.585	0.764
C(rate_marriage)[5.0]	0.3482	0.042	8.333	0.000	0.266	0.430

Omnibus:	9436.269	**Durbin-Watson:**	1.612
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5218576.884
Skew:	8.915	**Prob(JB):**	0.00
Kurtosis:	142.127	**Cond. No.**	5.21

affairs ~ 0 + C(rate_marriage)

- `y ~ 0 + C(x)`
  - Code without a reference.
- To calculate the mean of each group.

affairs

2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25

1.0
2.0
3.0
4.0
5.0

rate_marriage

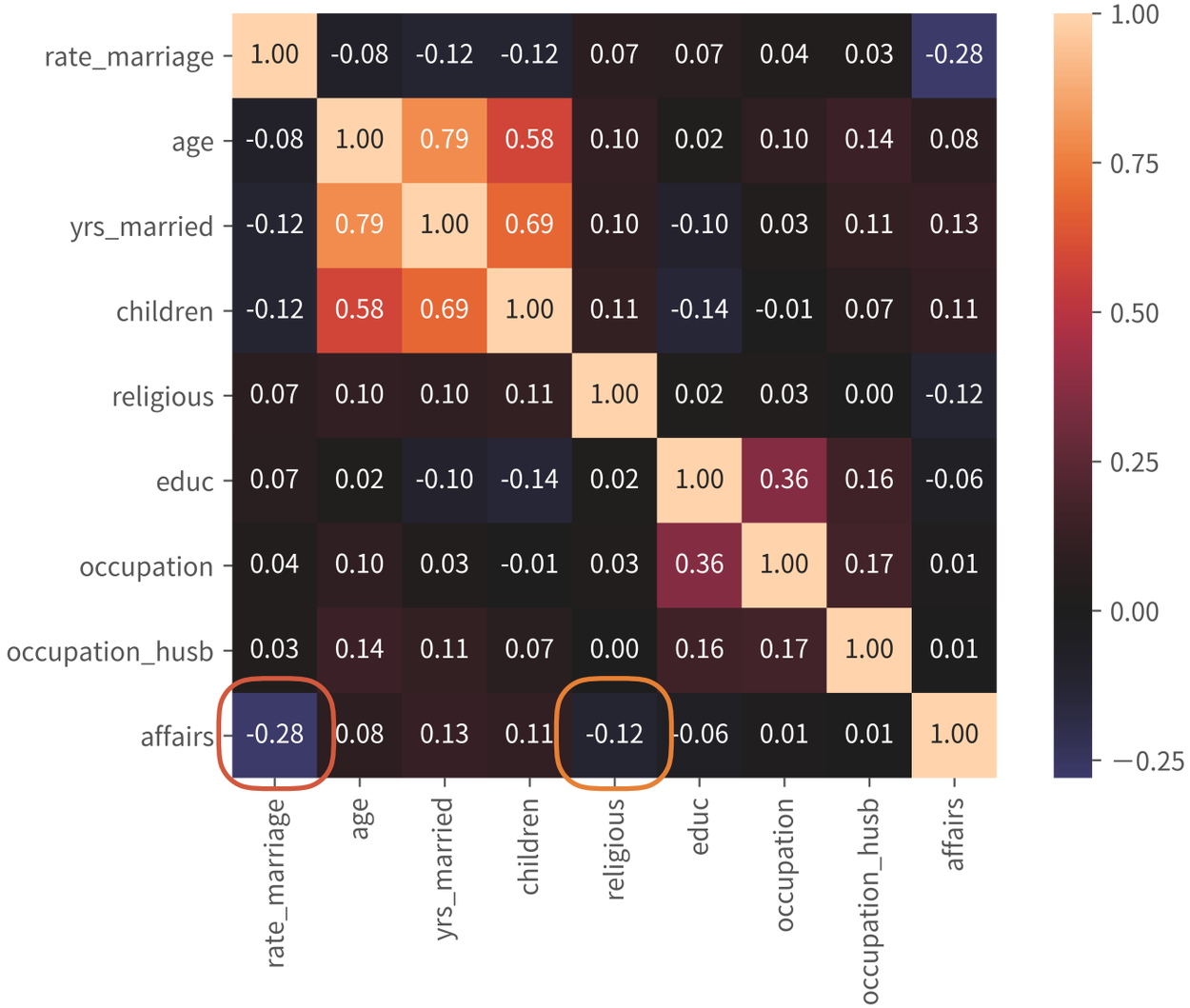
```
df = df_fair
(smf
 .ols('affairs ~ C(rate_marriage)', df)
 .fit()
 .summary())
```

```
df = df_fair
(smf
 .ols('affairs ~ 0 + C(rate_marriage)', df)
 .fit()
 .summary())
```

```
df = df_fair
sns.pointplot(data=df, x='rate_marriage', y='affairs')
```

More Ways to Code Categorical Variables

- `y ~ C(x, Treatment(reference='A'))`
  - Specify the reference level.
- `affairs ~ 0 + C(rate_marriage, Diff)`
  - Compare each level with the preceding level.
- `affairs ~ 0 + C(rate_marriage, Sum)`
  - Compare each level with the mean-of-means.
- Check the full reference.



Interaction

- "The low rate_marriage with high religious has a lower affairs?"

$$y \sim x * z$$
$$\equiv y = \beta_0 1 + \beta_1 x + \beta_2 z + \beta_3 xz + \varepsilon$$

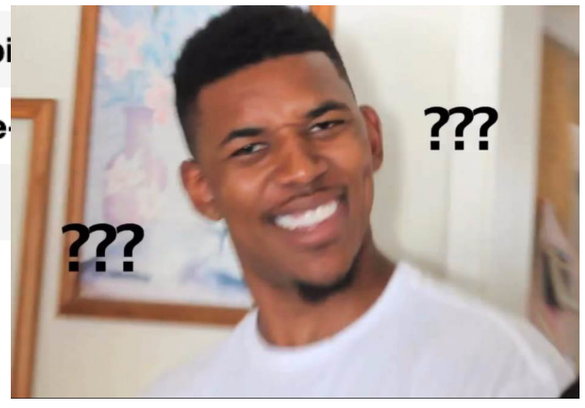
affairs ~ rate_marriage*religious

OLS Regression Results

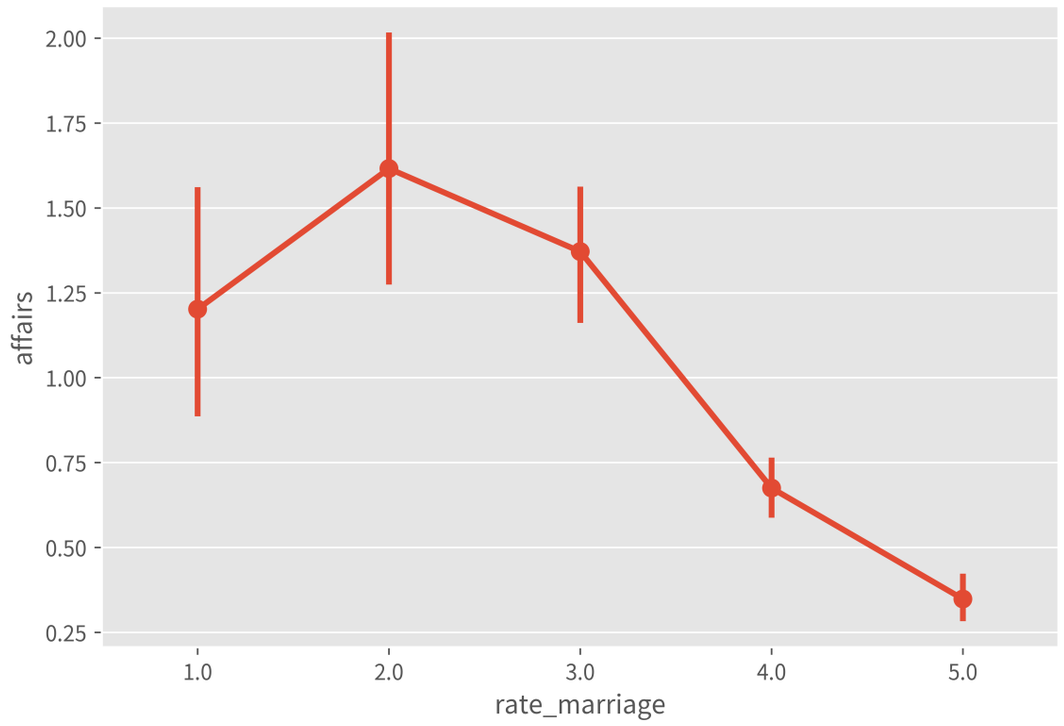
Dep. Variable:	affairs	**R-squared:**	0.048
Model:	OLS	**Adj. R-squared:**	0.048
Method:	Least Squares	**F-statistic:**	107.8
Date:	Fri, 26 Apr 2019	**Prob (F-statistic):**	4.55e-68
Time:	23:25:03	**Log-Likelihood:**	-13904.
No. Observations:	6366	**AIC:**	2.782e+04
Df Residuals:	6362	**BIC:**	2.784e+04
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	4.6495	0.347	13.395	0.000	3.969	5.330
rate_marriage	-0.7891	0.082	-9.622	0.000	-0.950	-0.628
religious	-0.9846	0.138	-7.122	0.000	-1.256	-0.714
rate_marriage:religious	0.1681	0.032	5.209	0.000	0.105	0.231

Omnibus:	9399.882	**Durbi**	
Prob(Omnibus):	0.000	**Jarque**	
Skew:	8.843		
Kurtosis:	141.848		



- The model may be wrong, since the relationship is not linear.



	coef	std err	z	P>|z|	[0.025	0.975]
Intercept	1.3341	0.498	2.678	0.007	0.358	2.311
C(rate_marriage)[T.2.0]	1.9654	0.921	2.134	0.033	0.160	3.771
C(rate_marriage)[T.3.0]	1.0479	0.634	1.654	0.098	-0.194	2.290
C(rate_marriage)[T.4.0]	-0.3282	0.517	-0.635	0.525	-1.341	0.685
C(rate_marriage)[T.5.0]	-0.6430	0.522	-1.232	0.218	-1.666	0.380
C(religious)[T.2.0]	0.1143	0.627	0.182	0.855	-1.115	1.343
C(religious)[T.3.0]	-0.3413	0.529	-0.646	0.519	-1.377	0.695
C(religious)[T.4.0]	-0.6082	0.588	-1.035	0.301	-1.760	0.544
C(rate_marriage)[T.2.0]:C(religious)[T.2.0]	-1.8103	1.028	-1.761	0.078	-3.825	0.204
C(rate_marriage)[T.3.0]:C(religious)[T.2.0]	-1.1905	0.754	-1.580	0.114	-2.668	0.286
C(rate_marriage)[T.4.0]:C(religious)[T.2.0]	-0.3499	0.646	-0.542	0.588	-1.615	0.916
C(rate_marriage)[T.5.0]:C(religious)[T.2.0]	-0.4682	0.647	-0.723	0.469	-1.737	0.800
C(rate_marriage)[T.2.0]:C(religious)[T.3.0]	-1.8707	0.950	-1.968	0.049	-3.734	-0.008
C(rate_marriage)[T.3.0]:C(religious)[T.3.0]	-0.9741	0.665	-1.464	0.143	-2.278	0.330
C(rate_marriage)[T.4.0]:C(religious)[T.3.0]	-0.1333	0.549	-0.243	0.808	-1.209	0.943
C(rate_marriage)[T.5.0]:C(religious)[T.3.0]	-0.0448	0.553	-0.081	0.935	-1.128	1.039
C(rate_marriage)[T.2.0]:C(religious)[T.4.0]	-2.2192	0.997	-2.225	0.026	-4.174	-0.265
C(rate_marriage)[T.3.0]:C(religious)[T.4.0]	-1.1008	0.726	-1.517	0.129	-2.523	0.321
C(rate_marriage)[T.4.0]:C(religious)[T.4.0]	-0.0946	0.608	-0.156	0.876	-1.286	1.097
C(rate_marriage)[T.5.0]:C(religious)[T.4.0]	0.0196	0.608	0.032	0.974	-1.173	1.212

affairs ~ C(rate_marriage)*C(religious)

- Hmmm ...
- TL;DR by ANOVA.

	sum_sq	df	F	PR(>F)
Intercept	32.952681	1.0	7.171451	0.007426
C(rate_marriage)	117.831212	4.0	6.410865	0.000038
C(religious)	11.838124	3.0	0.858772	0.461713
C(rate_marriage):C(religious)	111.850861	12.0	2.028497	0.018427
Residual	29159.748241	6346.0	NaN	NaN

- Looks like C(*religious*) isn't helping to explain. Drop it.

```
res = res_1
df = pd.DataFrame(dict(params=res.params,
                       pvalues=res.pvalues))
df[df.pvalues < 0.05].sort_values('params')
```

	params	pvalues
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-2.827311	0.000450
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-2.211959	0.005104
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-1.708937	0.000059
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-1.695969	0.037348
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-1.315414	0.001136
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-1.076195	0.010052
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.702780	0.000006
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-0.588575	0.000194
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.474564	0.001321
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.386073	0.016820
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.353891	0.027721
Intercept	1.334104	0.007407
C(rate_marriage)[T.2.0]	1.965381	0.032863

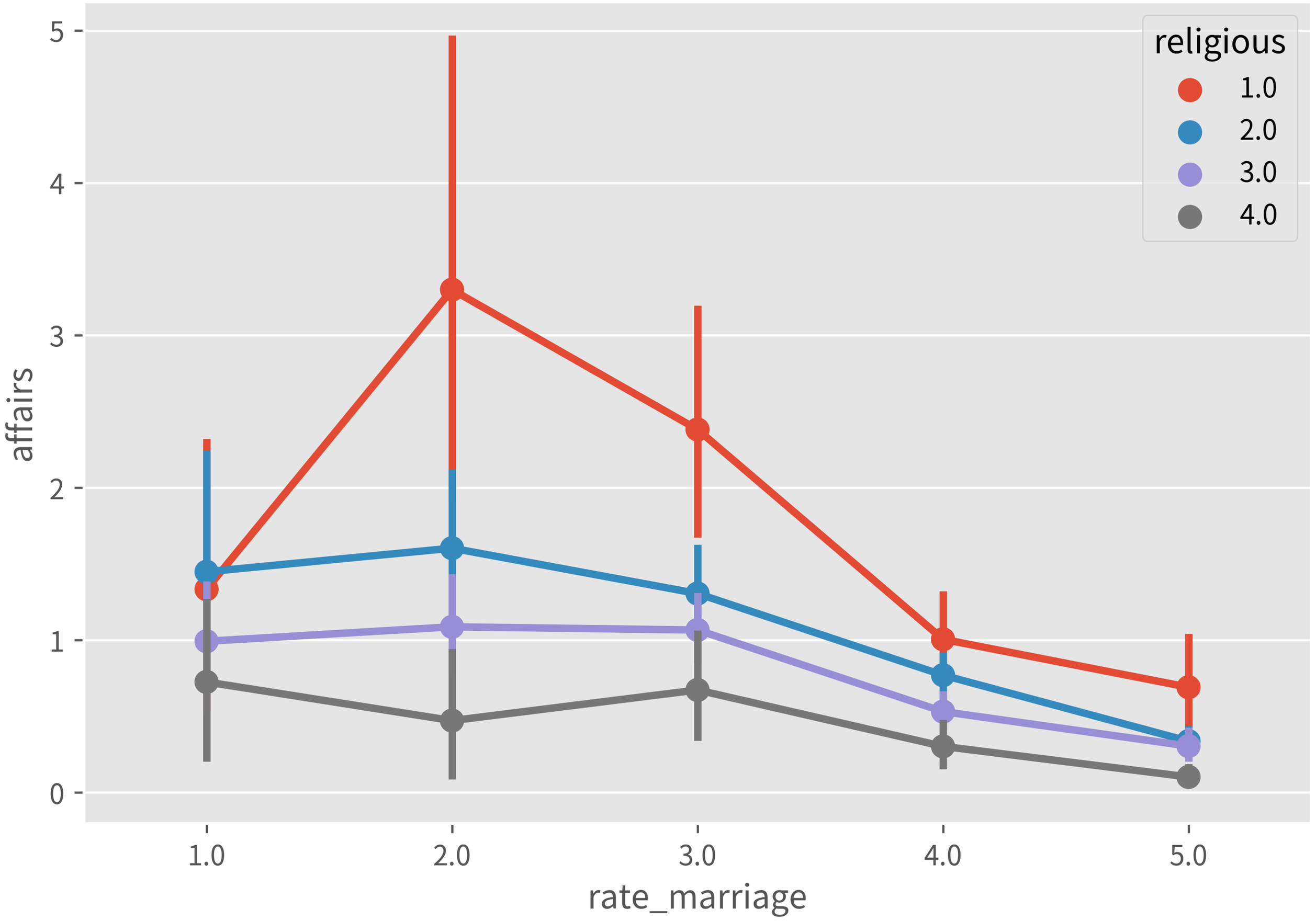
- `affairs ~ C(rate_marriage)*C(religious) - C(religious)`

Or:

- `affairs ~ C(rate_marriage) + C(religious):C(rate_marriage)`

$$y \sim x : z$$
$$\equiv y = \beta_0 + \beta_1 xz + \varepsilon$$

- "The low rate_marriage with high religious has a lower affairs?" Yes!


religious
1.0
2.0
3.0
4.0
affairs
0
1
2
3
4
5
1.0
2.0
3.0
4.0
5.0
rate_marriage

```
df = df_fair
(smf
 .ols('affairs ~ rate_marriage*religious', df)
 .fit()
 .summary())
```

```
df = df_fair
res = (smf
       .ols('affairs'
            '~ C(rate_marriage)*C(religious)', df)
       .fit())
display(res.summary(),
        # type III is suitable to unbalanced dataset
        # ref: http://bit.ly/3typess
        sm.stats.anova_lm(res, typ=3))
```

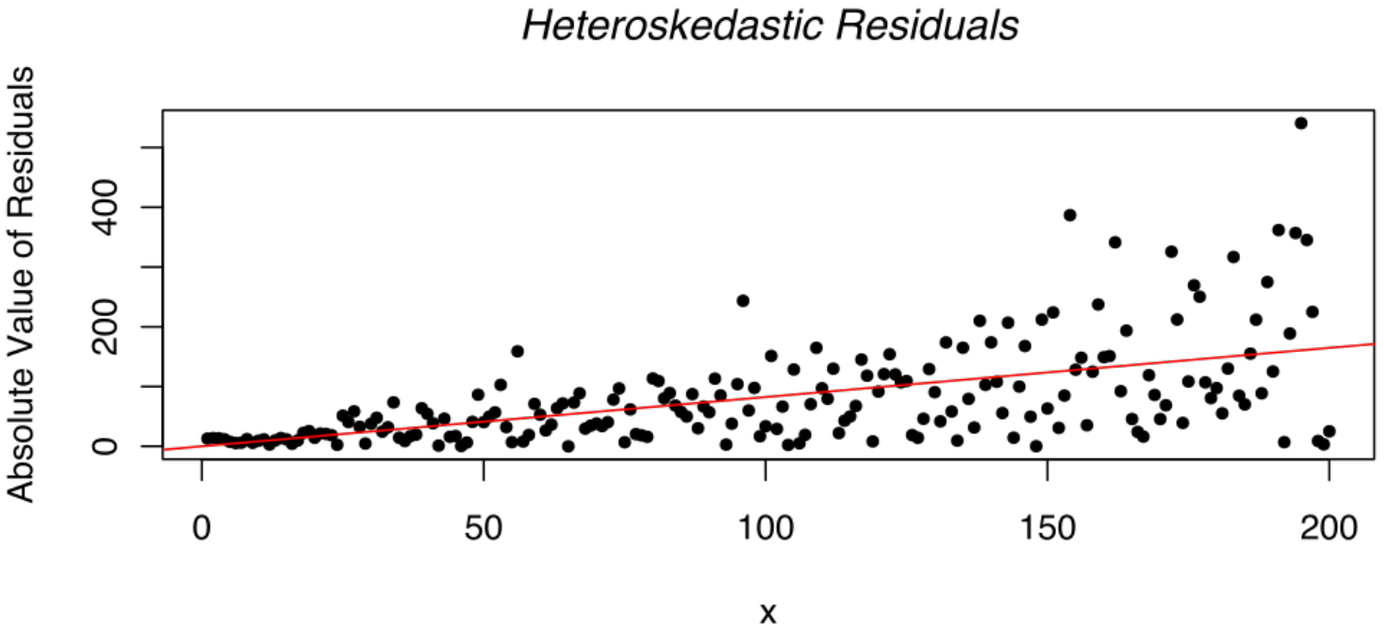
```
df = df_fair
res = (smf
       .ols('affairs'
            '~ C(rate_marriage)'
            '+ C(rate_marriage):C(religious)', df)
       .fit())
display(res.summary(),
        sm.stats.anova_lm(res, typ=3))
```

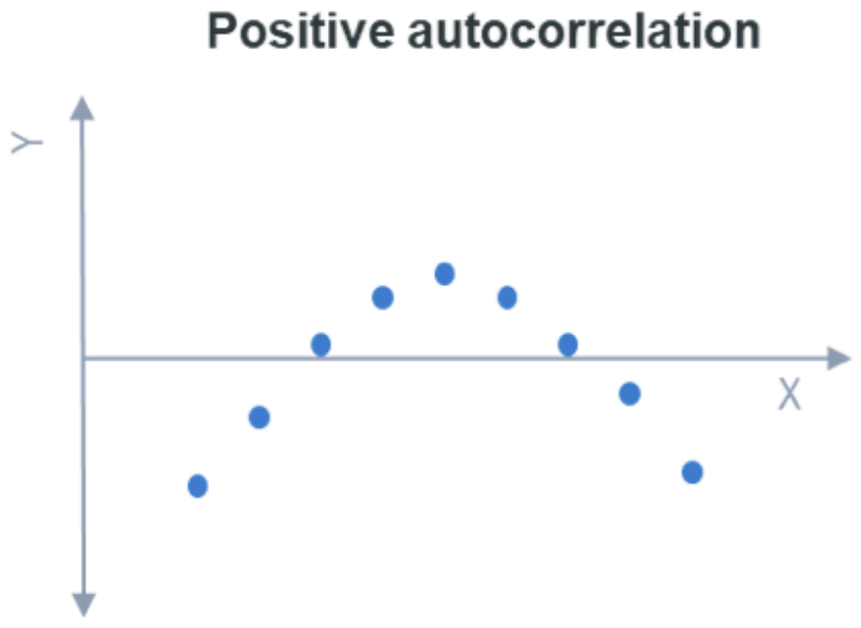
```
df = df_fair
sns.pointplot(data=df,
              x='rate_marriage',
              y='affairs',
              hue='religious')
```

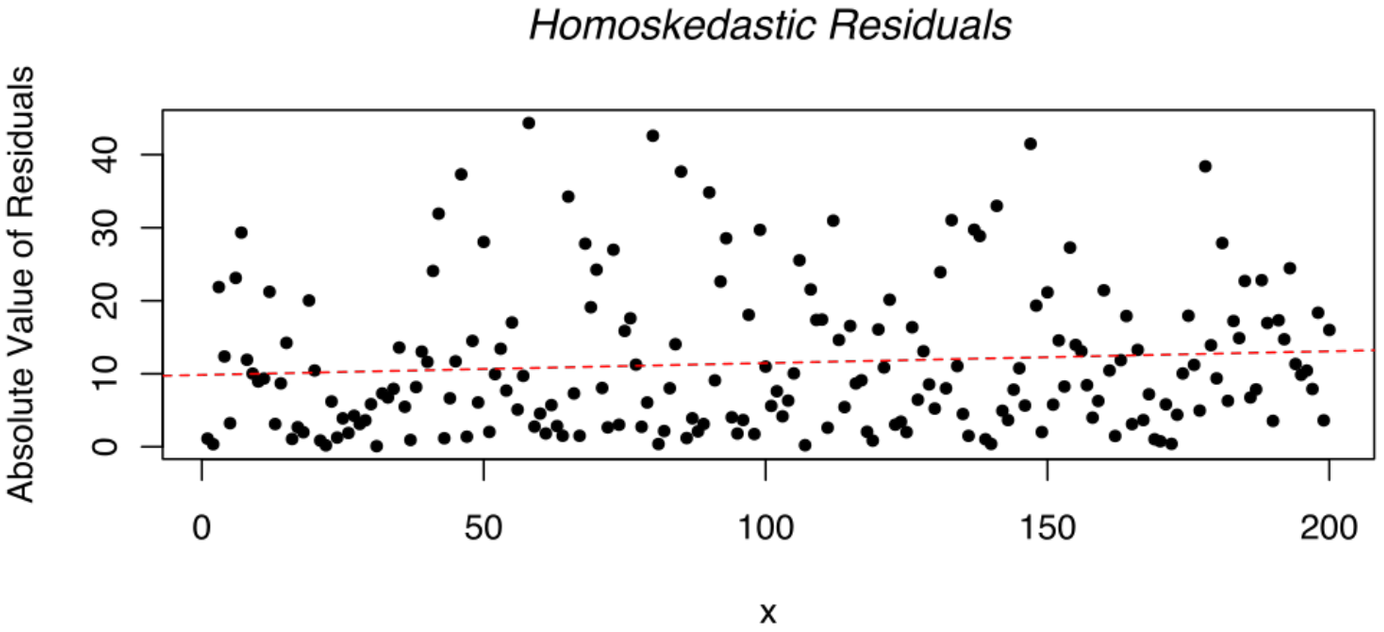
More Operators: Transformation & Control Variables

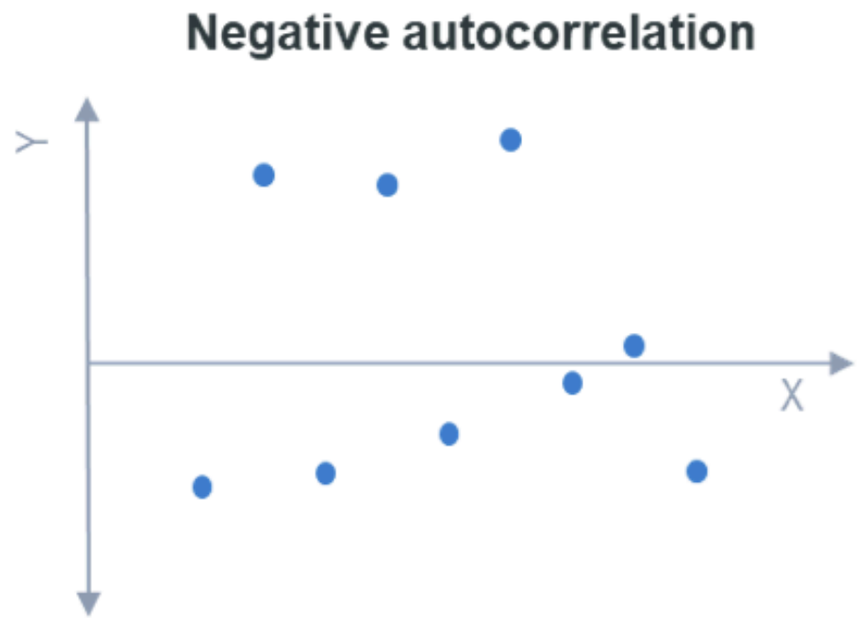
- `np.log(y) ~ x`
  - If y and x have a better linear relationship after transform.
  - Note that:
    - $\log(y) = \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$
    - $y = \exp(\hat{\beta}_1 x_1 + \hat{\beta}_2 x_2)$
    - $y = \exp(\hat{\beta}_1 x_1) \times \exp(\hat{\beta}_2 x_2)$
- `np.sqrt(y) ~ x`
- `y ~ I(x*z)`
  - True multiplication.

- `y ~ z_1 + ... + x_1 + ...`
  - The z_i and x_i are both independent variables.
  - If we don't interest in z_i, but they can carry some effects and clarify the effects of x_i, we call z_i "control variables".
    - For example:
      - `GMV ~ month + group`
- Check the full reference.









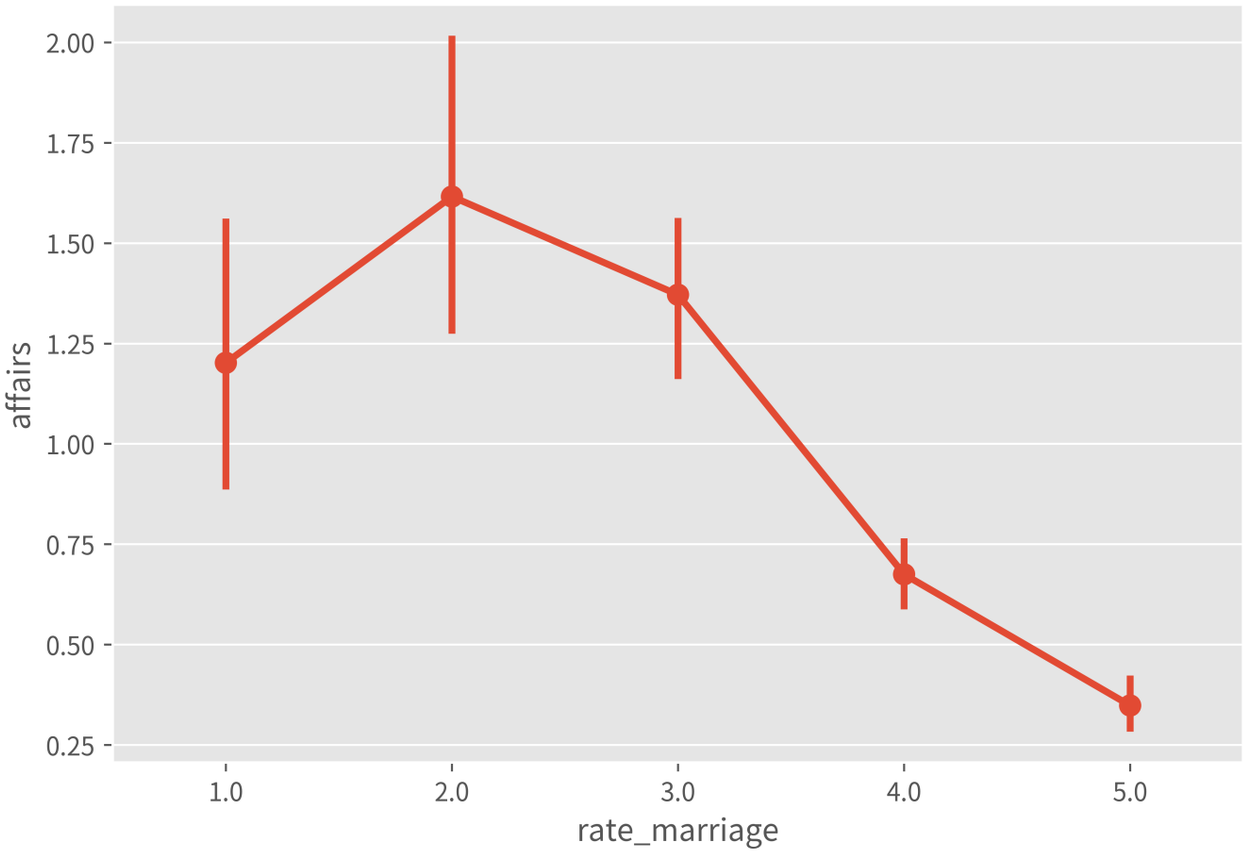
Covariance Types of Errors

Covariance Types of Errors

- **Spherical Errors**
  - ≡ Homoscedasticity & no autocorrelation
- **Heteroscedasticity** ≡ no homoscedasticity
- **Autocorrelation** ≡ serial correlation

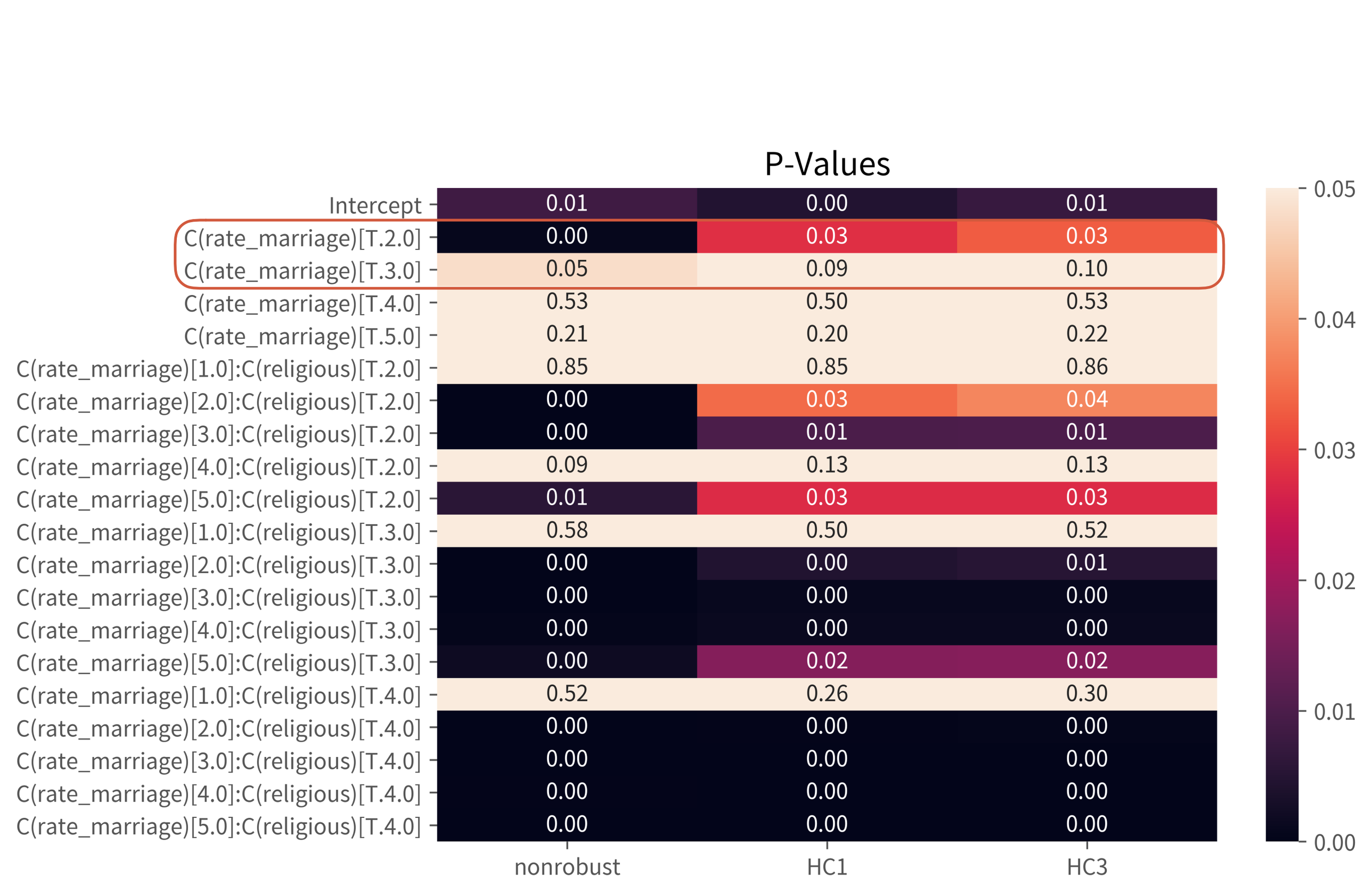
- If spherical errors, the model is good.
- If **not** spherical errors, the std errs are wrong.
  - So the interval estimates are wrong, including hypothesis tests on coefs, confidence intervals.

Heteroscedasticity



- Use HC std errs (heteroscedasticity-consistent standard errors) to correct.
- If $N \leq 250$, use HC3. [ref]
- If $N > 250$, consider HC1 for the speed.
- Also suggest to use by default.
- `.fit(cov_type='HC3')`

  ← The confidence intervals vary among groups. The heteroscedasticity exists.


P-Values
Intercept
C(rate_marriage)[T.2.0]
C(rate_marriage)[T.3.0]
C(rate_marriage)[T.4.0]
C(rate_marriage)[T.5.0]
C(rate_marriage)[1.0]:C(religious)[T.2.0]
C(rate_marriage)[2.0]:C(religious)[T.2.0]
C(rate_marriage)[3.0]:C(religious)[T.2.0]
C(rate_marriage)[4.0]:C(religious)[T.2.0]
C(rate_marriage)[5.0]:C(religious)[T.2.0]
C(rate_marriage)[1.0]:C(religious)[T.3.0]
C(rate_marriage)[2.0]:C(religious)[T.3.0]
C(rate_marriage)[3.0]:C(religious)[T.3.0]
C(rate_marriage)[4.0]:C(religious)[T.3.0]
C(rate_marriage)[5.0]:C(religious)[T.3.0]
C(rate_marriage)[1.0]:C(religious)[T.4.0]
C(rate_marriage)[2.0]:C(religious)[T.4.0]
C(rate_marriage)[3.0]:C(religious)[T.4.0]
C(rate_marriage)[4.0]:C(religious)[T.4.0]
C(rate_marriage)[5.0]:C(religious)[T.4.0]
0.01 0.00 0.01
0.00 0.03 0.03
0.05 0.09 0.10
0.53 0.50 0.53
0.21 0.20 0.22
0.85 0.85 0.86
0.00 0.03 0.04
0.00 0.01 0.01
0.09 0.13 0.13
0.01 0.03 0.03
0.58 0.50 0.52
0.00 0.00 0.01
0.00 0.00 0.00
0.00 0.00 0.00
0.00 0.02 0.02
0.52 0.26 0.30
0.00 0.00 0.00
0.00 0.00 0.00
0.00 0.00 0.00
0.00 0.00 0.00
nonrobust
HC1
HC3
0.05
0.04
0.03
0.02
0.01
0.00

OLS Regression Results

Dep. Variable:	affairs	**R-squared:**	0.032
Model:	OLS	**Adj. R-squared:**	0.032
Method:	Least Squares	**F-statistic:**	208.4
Date:	Fri, 26 Apr 2019	**Prob (F-statistic):**	1.66e-46
Time:	23:25:02	**Log-Likelihood:**	-13959.
No. Observations:	6366	**AIC:**	2.792e+04
Df Residuals:	6364	**BIC:**	2.794e+04
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	2.3825	0.119	19.969	0.000	2.149	2.616
rate_marriage	-0.4081	0.028	-14.436	0.000	-0.464	-0.353

Omnibus:	9443.528	**Durbin-Watson:**	1.606
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	5215639.758
Skew:	8.930	**Prob(JB):**	0.00
Kurtosis:	142.083	**Cond. No.**	19.5

Autocorrelation

- Durbin-Watson
  - 2 is no autocorrelation.
  - [0, 2)
    is positive autocorrelation.
  - (2, 4]
    is negative autocorrelation.
  - [1.5, 2.5]
    are relatively normal. [ref]
- Use HAC std err.
- `.fit(cov_type='HAC', cov_kwds=dict(maxlag=tau))`

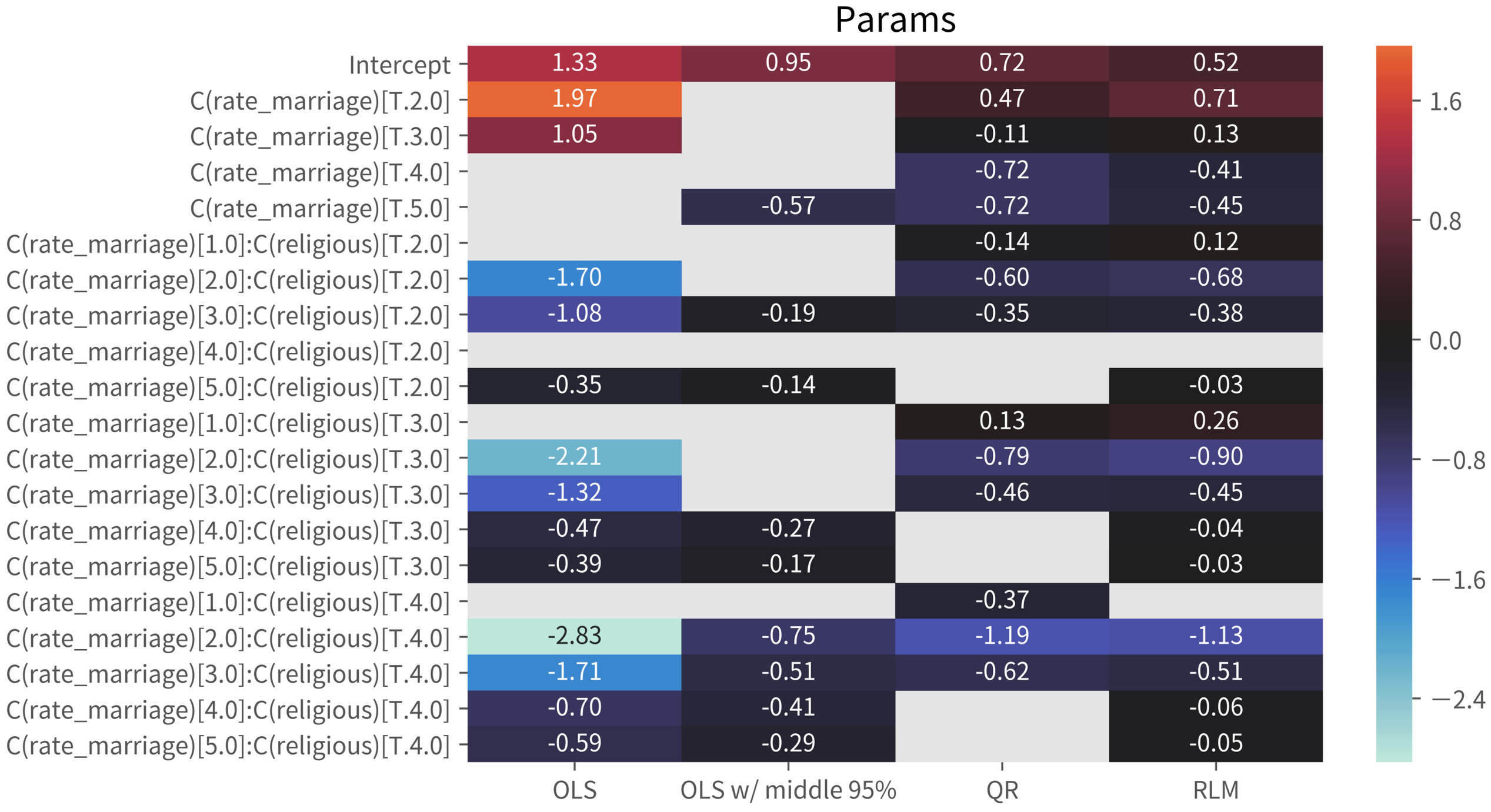
Other Covariance Types

- `cluster`
  - Assume each group has spherical errors.
- `hac-groupsum`
  - Sum by the time label and then process.
- `hac-panel`
  - Process by the groups and then aggregate.
- Check the full references.

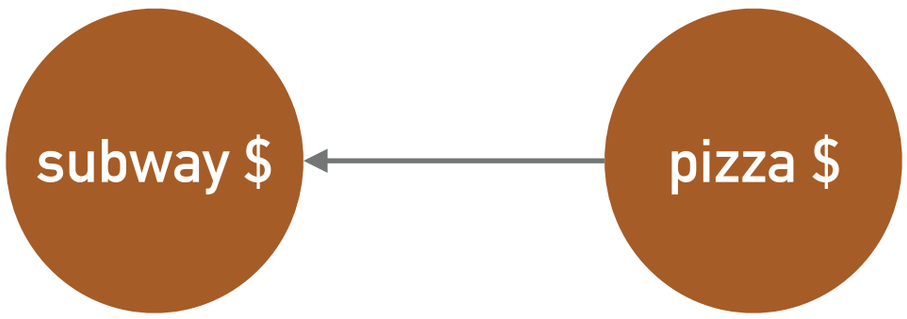
Outliers

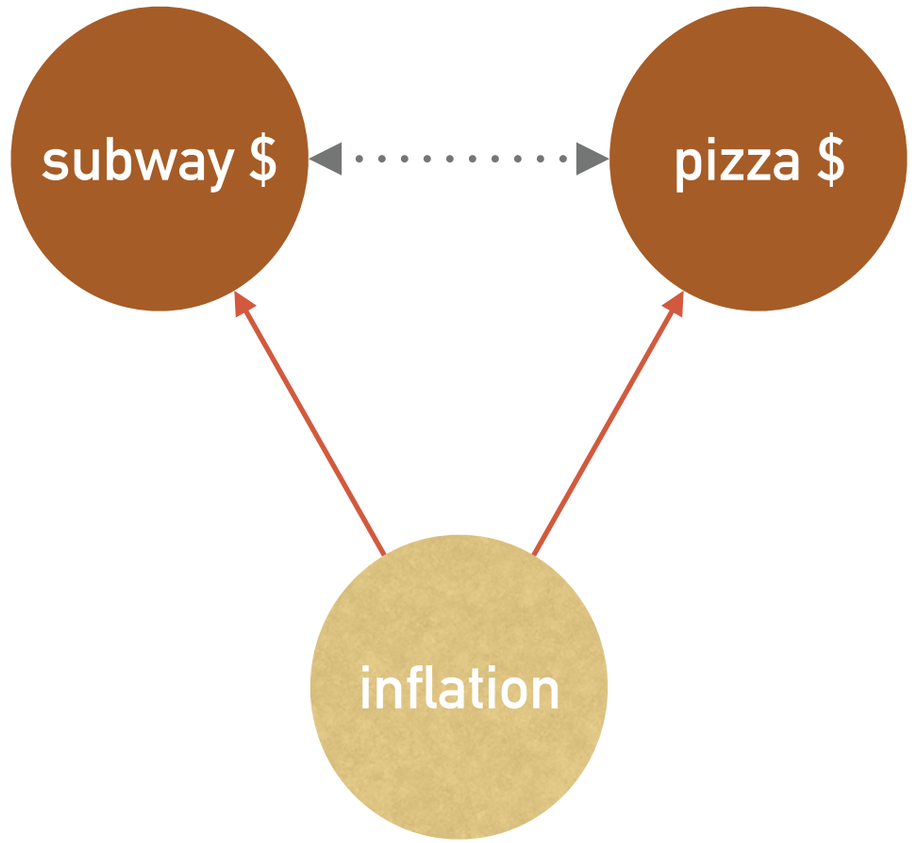
- An outlier may be the most interesting observation.
  - Consider to include more variables to explain.
  - Consider the possible non-linear relationship.
  - Consider the outlier-insensitive models.
- Drop observations only when you have a good reason, like:
  - Typo.
  - Not representative of the intended study population.
- Report fully, including:
  - The preprocess steps.
  - The models with and without outliers.

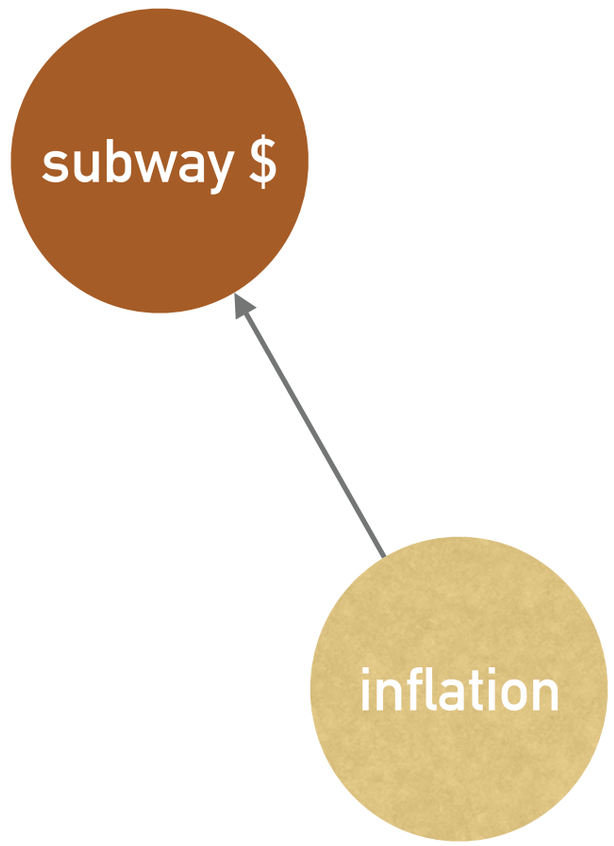
- Quantile regression: estimates the median rather than mean.
- Robust regression: robust to outliers, but slower.
- Keep middle n%: changes the intended study population.
- OLS
- Outliner test
- Influence report


Params
Intercept 1.33 0.95 0.72 0.52
C(rate_marriage)[T.2.0] 1.97 0.47 0.71
C(rate_marriage)[T.3.0] 1.05 -0.11 0.13
C(rate_marriage)[T.4.0] -0.72 -0.41
C(rate_marriage)[T.5.0] -0.57 -0.72 -0.45
C(rate_marriage)[1.0]:C(religious)[T.2.0] -0.14 0.12
C(rate_marriage)[2.0]:C(religious)[T.2.0] -1.70 -0.60 -0.68
C(rate_marriage)[3.0]:C(religious)[T.2.0] -1.08 -0.19 -0.35 -0.38
C(rate_marriage)[4.0]:C(religious)[T.2.0]
C(rate_marriage)[5.0]:C(religious)[T.2.0] -0.35 -0.14 -0.03
C(rate_marriage)[1.0]:C(religious)[T.3.0] 0.13 0.26
C(rate_marriage)[2.0]:C(religious)[T.3.0] -2.21 -0.79 -0.90
C(rate_marriage)[3.0]:C(religious)[T.3.0] -1.32 -0.46 -0.45
C(rate_marriage)[4.0]:C(religious)[T.3.0] -0.47 -0.27 -0.04
C(rate_marriage)[5.0]:C(religious)[T.3.0] -0.39 -0.17 -0.03
C(rate_marriage)[1.0]:C(religious)[T.4.0] -0.37
C(rate_marriage)[2.0]:C(religious)[T.4.0] -2.83 -0.75 -1.19 -1.13
C(rate_marriage)[3.0]:C(religious)[T.4.0] -1.71 -0.51 -0.62 -0.51
C(rate_marriage)[4.0]:C(religious)[T.4.0] -0.70 -0.41 -0.06
C(rate_marriage)[5.0]:C(religious)[T.4.0] -0.59 -0.29 -0.05
OLS
OLS w/ middle 95%
QR
RLM
1.6
0.8
0.0
0.8
1.6
2.4

```
df = df_fair

alpha = 0.05
a = df.affairs.quantile(alpha/2)
b = df.affairs.quantile(1-alpha/2)

df = df[(df.affairs >= a) & (df.affairs <= b)]
df_fair_middle95 = df


df = df_fair
smf.ols(formula, df).fit().summary()
smf.ols(formula, df_fair_middle95).fit().summary()
smf.quantreg(formula, df).fit().summary()
smf.rlm(formula, df).fit().summary()
```


subway $
pizza $


subway $
pizza $
inflation


subway $
inflation

Correlation Does Not Imply Causation

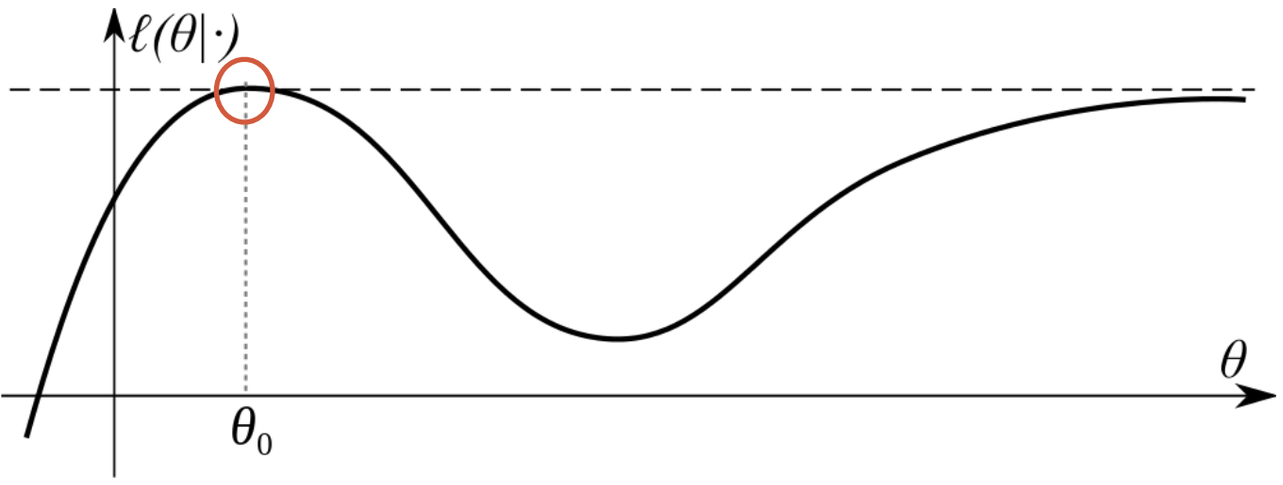
- $y \sim x$:= "y has association with x"
- $y \leftarrow x$:= "y because x"
- $y \sim x$ may be:
  - $y \leftarrow x$
  - $y \rightarrow x$
  - $z \rightarrow y \wedge z \rightarrow x$
- So $y \sim x$ doesn't implies $y \leftarrow x$.
- But usually we want the causation, not only correlation.
- Use a good study design to construct a causation inference.

Suggested Wording

- “Relationship”
  - Any relationship, so it can't be wrong.
- Correlation
  - “Associate” “Association”, relatively conservative.
  - “Correlate” “Correlation”, usually in correlation analysis.
- Causation
  - “Predict” “Prediction”.
  - “Affect” “Influence”, the most strong wording.

More Models

- Discrete Models, like logit model:
  - $y \in \{0, 1\}$
- Mixed Model for estimate both subject and group effect:
  - $y = X\beta + Zu + \varepsilon$
- Time Series Models, like autoregressive model:
  - $x_t = c + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \ldots + \varphi_p x_{t-p} + \varepsilon_t$
- Check all the models that StatsModels supports.



More Estimations

- **MLE**, Maximum Likelihood Estimation.

  ← Usually find by numerical methods.

- **TSLS**, Two-Stage Least Squares.

- $y \leftarrow (x \leftarrow z)$

- Handle the endogeneity: $\mathrm{E}[\varepsilon|X] \neq 0$.

affairs_yn

1.0
0.8
0.6
0.4
0.2
0.0

1
2
3
4
5

rate_marriage

Logit Regression Results

Dep. Variable:	affairs_yn	**No. Observations:**	6366
Model:	Logit	**Df Residuals:**	6346
Method:	MLE	**Df Model:**	19
Date:	Mon, 29 Apr 2019	**Pseudo R-squ.:**	0.1003
Time:	22:48:50	**Log-Likelihood:**	-3601.0
converged:	True	**LL-Null:**	-4002.5
		LLR p-value:	4.598e-158

	coef	std err	z	P>|z|	[0.025	0.975]
Intercept	0.6931	0.500	1.386	0.166	-0.287	1.673
C(rate_marriage)[T.2.0]	0.2231	0.581	0.384	0.701	-0.915	1.362
C(rate_marriage)[T.3.0]	-0.2122	0.523	-0.405	0.685	-1.238	0.813
C(rate_marriage)[T.4.0]	-1.2009	0.512	-2.345	0.019	-2.205	-0.197
C(rate_marriage)[T.5.0]	-1.6664	0.512	-3.256	0.001	-2.669	-0.663
C(rate_marriage)[1.0]:C(religious)[T.2.0]	0.5596	0.641	0.873	0.383	-0.696	1.816
C(rate_marriage)[2.0]:C(religious)[T.2.0]	-0.1398	0.345	-0.405	0.686	-0.817	0.537
C(rate_marriage)[3.0]:C(religious)[T.2.0]	-0.2657	0.184	-1.443	0.149	-0.626	0.095
C(rate_marriage)[4.0]:C(religious)[T.2.0]	-0.0294	0.132	-0.222	0.824	-0.288	0.230
C(rate_marriage)[5.0]:C(religious)[T.2.0]	-0.4791	0.140	-3.427	0.001	-0.753	-0.205
C(rate_marriage)[1.0]:C(religious)[T.3.0]	0.4769	0.629	0.758	0.448	-0.756	1.710
C(rate_marriage)[2.0]:C(religious)[T.3.0]	-0.5656	0.349	-1.622	0.105	-1.249	0.118
C(rate_marriage)[3.0]:C(religious)[T.3.0]	-0.3412	0.188	-1.811	0.070	-0.710	0.028
C(rate_marriage)[4.0]:C(religious)[T.3.0]	-0.4342	0.134	-3.239	0.001	-0.697	-0.171
C(rate_marriage)[5.0]:C(religious)[T.3.0]	-0.6133	0.137	-4.487	0.000	-0.881	-0.345
C(rate_marriage)[1.0]:C(religious)[T.4.0]	0.2231	0.975	0.229	0.819	-1.687	2.133
C(rate_marriage)[2.0]:C(religious)[T.4.0]	-1.3218	0.504	-2.622	0.009	-2.310	-0.334
C(rate_marriage)[3.0]:C(religious)[T.4.0]	-0.7105	0.286	-2.486	0.013	-1.271	-0.150
C(rate_marriage)[4.0]:C(religious)[T.4.0]	-0.7732	0.210	-3.675	0.000	-1.186	-0.361
C(rate_marriage)[5.0]:C(religious)[T.4.0]	-1.3503	0.212	-6.355	0.000	-1.767	-0.934

Logit Model

- The coef is log-odds.
- Use $\exp(x)/(\exp(x)+1)$ to transform back to probability:
  - 0.6931 → 67%
  - " - 1.6664 → 27%
  - " - 1.3503 → 9%

Or:

- `.predict(dict(rate_marriage=[1, 5, 5], religious=[1, 1, 4]))`

```
df = df_fair
df = df.assign(affairs_yn=(df.affairs > 0).astype(float))
df_fair_2 = df
```

```
df = df_fair_2.sample(frac=0.1, random_state=20190429)
sns.regplot(data=df, x='rate_marriage', y='affairs_yn',
            logistic=True,
            x_jitter=1/2, y_jitter=0.2/2)
```

```
df = df_fair_2
(smf
 .logit('affairs_yn'
        '~ C(rate_marriage)'
        '+ C(rate_marriage):C(religious)', df)
 .fit()
 .summary())
```

Recap

- Use the correlation analysis to get an overview.
- Plotting, Adj. R-squared, Cond. No., Durbin-Watson, etc.
- `y ~ C(x)`
  - For categorical variables.
- `y ~ x*z`
  - For investigating the interaction.
- Use HC3 by default.
- Correlation does not imply causation.
- Let's explain and predict efficiently! 📈