Formulas for Bayesian quadratures using Wendland kernels

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**Abstract**

In this (incomplete) note, we provide formulas needed in the construction of Bayesian quadratures using Wendland kernels. Specifically, we focus on the setting where the dimensionality of the input space is 1, and the distribution is uniform on an interval.

# 1 Formulas

- $d = 1$.
- $\Omega := [a, b] \subset \mathbb{R}$ with $-\infty < a < b < +\infty$.
- $P$: the uniform distribution on $\Omega$.
- Design points $X_1, \ldots, X_n$: points in $\Omega$.
- Kernel: Wendland's function [44, Definition 9.11]. For $\tau \geq 0$, it is defined by

$$\phi_{d,k}(\tau) = \left(\mathcal{I}^k \phi_{\lfloor d/2 \rfloor + k + 1}\right)(\tau),$$

  where $\phi_\ell : [0, \infty) \to \mathbb{R}$ ($\ell := \lfloor d/2 \rfloor + k + 1$) is defined by

$$\phi_\ell(\tau) := (1 - \tau)_+^\ell,$$

  and the operator $\mathcal{I}$ is defined by [44, Definition 9.4]

$$(\mathcal{I}\phi)(\tau) := \int_r^\infty t\phi(t)dt, \quad \tau \geq 0$$

  for a function $\phi : [0, \infty) \to \infty$ such that $t \to \phi(t)t$ is in $L_1[0, \infty)$.
- Then the Wendland kernel is defined by

$$k_{d,k}(x, y) := \phi_{d,k}(\|x - y\|).$$

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- The corresponding RKHS is the Sobolev space $W_2^r(\mathbb{R}^d)$ of order $r = d/2 + k + 1/2$ [44, Theorem 10.35].
- Therefore, for instance, if $r = 4$ and $d = 1$, we have $k = 3$.
- Scaled version $k_{d,k,\delta}(x, y) := \phi_{d,k}(\|x - y\|/\delta)$ for $\delta > 0$. Assume $\delta \le (b - a)/2$.
- $\phi_{d,3}(r)$ is given as a polynomial [44, Corollary 9.14]

$$\phi_{d,3}(r) = (1 - r)_+^{\ell+3}(C_3 r^3 + C_2 r^2 + C_1 r + C_0),$$

where

$$\begin{aligned} \ell &:= \lfloor d/2 \rfloor + k + 1. \\ C_3 &:= \ell^3 + 9\ell^2 + 23\ell + 15 \\ C_2 &:= 6\ell^2 + 36\ell + 45 \\ C_1 &:= 15\ell + 45 \\ C_0 &:= 15. \end{aligned}$$

## 2 Explicit formula for the kernel mean with $k = 3$

- Let $x \in [a, b]$.
- Kernel:

$$\begin{aligned} k_{d,3,\delta}(x, y) &= \phi_{d,3}(|x - y|/\delta) \\ &= (1 - |x - y|/\delta)_+^{\ell+3} \left(C_3(|x - y|/\delta)^3 + C_2(|x - y|/\delta)^2 + C_1(|x - y|/\delta) + C_0\right) \\ &= (1 - |x - y|/\delta)_+^{\ell+3} \left(D_3|x - y|^3 + D_2|x - y|^2 + D_1|x - y| + D_0\right), \end{aligned}$$

where we defined the new constants as $D_3 := C_3\delta^{-3}$, $D_2 := C_2\delta^{-2}$, $D_1 := C_1\delta^{-1}$, and $D_0 := C_0$.

- We want to evaluate the value of the kernel mean

$$\begin{aligned} & \frac{1}{b - a} \int_a^b k_{d,3,\delta}(x, y) dy \\ = & \frac{1}{b - a} \int_a^b \phi_{d,3}(|x - y|/\delta) dy \\ = & \frac{1}{b - a} \int_a^b (1 - |x - y|/\delta)_+^{\ell+3} \left(D_3|x - y|^3 + D_2|x - y|^2 + D_1|x - y| + D_0\right) dy \\ = & \frac{1}{b - a} \int_a^x (1 - (x - y)/\delta)_+^{\ell+3} \left(D_3(x - y)^3 + D_2(x - y)^2 + D_1(x - y) + D_0\right) dy \qquad (1) \\ + & \frac{1}{b - a} \int_x^b (1 - (y - x)/\delta)_+^{\ell+3} \left(D_3(y - x)^3 + D_2(y - x)^2 + D_1(y - x) + D_0\right) dy. \qquad (2) \end{aligned}$$

- There are three cases.
  - Case 1: $[x-\delta, x+\delta] \subset [a,b]$.
  - Case 2: $x-\delta < a$.
  - Case 3: $x+\delta > b$.
- There are only these three cases, since $x-\delta < a$ and $x+\delta > b$ do not hold simultaneously from our assumption that $\delta \leq (b-a)/2$.
- Case 1: $[x-\delta, x+\delta] \subset [a,b]$. From symmetry, Eq.(1) is equal to Eq.(2). Thus we will focus on Eq.(1).

$$
\begin{aligned}
& (b-a) \times \text{Eq. } (1) \\
= & \int_a^x (1-(x-y)/\delta)_+^{\ell+3} \left(D_3(x-y)^3 + D_2(x-y)^2 + D_1(x-y) + D_0\right) dy \\
= & \int_{x-\delta}^x (1-(x-y)/\delta)_+^{\ell+3} \{D_3(x-y)^3 + D_2(x-y)^2 + D_1(x-y) + D_0\} dy \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+3} \{D_3(\delta-z)^3 + D_2(\delta-z)^2 + D_1(\delta-z) + D_0\} dz \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+3} \{D_3(\delta^3 - 3\delta^2 z + 3\delta z^2 - z^3) + D_2(\delta^2 - 2\delta z + z^2) + D_1(\delta - z) + D_0\} dz \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+3} \{z^3(-D_3) + z^2(3\delta D_3 + D_2) \\
& +z(-3\delta^2 D_3 - 2\delta D_2 - D_1) + (D_3\delta^3 + D_2\delta^2 + D_1\delta + D_0)\} dz \\
= & \int_0^\delta \{z^{\ell+6}\delta^{-\ell-6}(-C_3) + z^{\ell+5}\delta^{-\ell-5}(3C_3 + C_2) \\
& +z^{\ell+4}\delta^{-\ell-4}(-3C_3 - 2C_2 - C_1) + z^{\ell+3}\delta^{-\ell-3}(C_3 + C_2 + C_1 + C_0)\} dz \\
= & \delta \left( \frac{-C_3}{\ell+7} + \frac{3C_3 + C_2}{\ell+6} + \frac{-3C_3 - 2C_2 - C_1}{\ell+5} + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4} \right)
\end{aligned}
$$

  Therefore,

$$
\begin{aligned}
& \frac{1}{b-a} \int_a^b k_{d,3,\delta}(x,y) dy \\
= & \frac{2\delta}{b-a} \left( \frac{-C_3}{\ell+7} + \frac{3C_3 + C_2}{\ell+6} + \frac{-3C_3 - 2C_2 - C_1}{\ell+5} + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4} \right).
\end{aligned}
$$

- Case 2: $x-\delta < a$. We want to compute the sum of Eq.(1) and Eq.(2). Note the value of Eq.(2) is equal to that in Case 1. Therefore we have

$$
\text{Eq. } (2) = \frac{\delta}{b-a} \left( \frac{-C_3}{\ell+7} + \frac{3C_3 + C_2}{\ell+6} + \frac{-3C_3 - 2C_2 - C_1}{\ell+5} + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4} \right).
$$

Thus we focus on computing Eq.(1).

$$
\begin{aligned}
& (b-a) \times \text{Eq. (1)} \\
= \; & \int_a^x (1-(x-y)/\delta)_+^{\ell+3} \{D_3(x-y)^3 + D_2(x-y)^2 + D_1(x-y) + D_0\} dy \\
= \; & \int_{\delta-x+a}^{\delta} \left(\frac{z}{\delta}\right)^{\ell+3} \{D_3(\delta-z)^3 + D_2(\delta-z)^2 + D_1(\delta-z) + D_0\} dz \\
= \; & \int_{\delta-x+a}^{\delta} \left(\frac{z}{\delta}\right)^{\ell+3} \{D_3(\delta^3 - 3\delta^2 z + 3\delta z^2 - z^3) + D_2(\delta^2 - 2\delta z + z^2) + D_1(\delta - z) + D_0\} dz \\
= \; & \int_{\delta-x+a}^{\delta} \left(\frac{z}{\delta}\right)^{\ell+3} \{z^3(-D_3) + z^2(3\delta D_3 + D_2) \\
& \quad + z(-3\delta^2 D_3 - 2\delta D_2 - D_1) + (D_3\delta^3 + D_2\delta^2 + D_1\delta + D_0)\} dz \\
= \; & \int_{\delta-x+a}^{\delta} \{z^{\ell+6}\delta^{-\ell-6}(-C_3) + z^{\ell+5}\delta^{-\ell-5}(3C_3 + C_2) \\
& \quad + z^{\ell+4}\delta^{-\ell-4}(-3C_3 - 2C_2 - C_1) + z^{\ell+3}\delta^{-\ell-3}(C_3 + C_2 + C_1 + C_0)\} dz \\
= \; & \frac{-C_3}{\ell+7}\delta^{-\ell-6}[\delta^{\ell+7} - (\delta - x + a)^{\ell+7}] \\
& + \frac{3C_3 + C_2}{\ell+6}\delta^{-\ell-5}[\delta^{\ell+6} - (\delta - x + a)^{\ell+6}] \\
& + \frac{-3C_3 - 2C_2 - C_1}{\ell+5}\delta^{-\ell-4}[\delta^{\ell+5} - (\delta - x + a)^{\ell+5}] \\
& + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4}\delta^{-\ell-3}[\delta^{\ell+4} - (\delta - x + a)^{\ell+4}] \\
= \; & \frac{-C_3}{\ell+7}\delta[1 - (1-(x-a)/\delta)^{\ell+7}] + \frac{3C_3 + C_2}{\ell+6}\delta[1 - (1-(x-a)/\delta)^{\ell+6}] \\
& + \frac{-3C_3 - 2C_2 - C_1}{\ell+5}\delta[1 - (1-(x-a)/\delta)^{\ell+5}] + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4}\delta[1 - (1-(x-a)/\delta)^{\ell+4}]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,3,\delta}(x,y) dy \\
= \; & \frac{\delta}{b-a}\Bigg\{\left(\frac{-C_3}{\ell+7} + \frac{3C_3 + C_2}{\ell+6} + \frac{-3C_3 - 2C_2 - C_1}{\ell+5} + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4}\right) \\
& + \frac{-C_3}{\ell+7}[1 - (1-(x-a)/\delta)^{\ell+7}] + \frac{3C_3 + C_2}{\ell+6}[1 - (1-(x-a)/\delta)^{\ell+6}] \\
& + \frac{-3C_3 - 2C_2 - C_1}{\ell+5}[1 - (1-(x-a)/\delta)^{\ell+5}] + \frac{C_3 + C_2 + C_1 + C_0}{\ell+4}[1 - (1-(x-a)/\delta)^{\ell+4}]\Bigg\}.
\end{aligned}
$$

- Case 3: $x + \delta > b$. In this case, the value of the kernel mean can be computed similarly to

Case 2. The resulting value is

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,3,\delta}(x,y)dy \\
= \ & \frac{\delta}{b-a}\Bigg\{\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right) \\
& +\frac{-C_3}{\ell+7}[1-(1-(b-x)/\delta)^{\ell+7}]+\frac{3C_3+C_2}{\ell+6}[1-(1-(b-x)/\delta)^{\ell+6}] \\
& +\frac{-3C_3-2C_2-C_1}{\ell+5}[1-(1-(b-x)/\delta)^{\ell+5}]+\frac{C_3+C_2+C_1+C_0}{\ell+4}[1-(1-(b-x)/\delta)^{\ell+4}]\Bigg\}.
\end{aligned}
$$

## 2.1 Explicit formula for the worst case error for $k = 3$

- We need a formula for
$$\frac{1}{b-a}\int_a^b m_P(x)dx,$$
where
$$m_P(x) = \frac{1}{b-a}\int_a^b k_{d,3,\delta}(x,y)dy.$$

- Note that this value can be decomposed as
$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b m_P(x)dx \\
= \ & \frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx + \frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx + \frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx
\end{aligned}
$$

The above three terms correspond to the Cases 1, 2, and 3 discussed above. Therefore we can use the formulas we have derived:
$$
\begin{aligned}
& \frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx \\
= \ & \frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx \\
= \quad & \frac{\delta}{(b-a)^2}\int_a^{a+\delta}\Bigg\{\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right) \\
& +\frac{-C_3}{\ell+7}[1-(1-(x-a)/\delta)^{\ell+7}]+\frac{3C_3+C_2}{\ell+6}[1-(1-(x-a)/\delta)^{\ell+6}] \\
& +\frac{-3C_3-2C_2-C_1}{\ell+5}[1-(1-(x-a)/\delta)^{\ell+5}]+\frac{C_3+C_2+C_1+C_0}{\ell+4}[1-(1-(x-a)/\delta)^{\ell+4}]\Bigg\}dx. \\
= \quad & \frac{\delta}{(b-a)^2}\int_a^{a+\delta}\left\{2\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right)\right\}dx \\
& -\frac{\delta}{(b-a)^2}\int_a^{a+\delta}\Bigg\{\frac{-C_3}{\ell+7}(1-(x-a)/\delta)^{\ell+7}+\frac{3C_3+C_2}{\ell+6}(1-(x-a)/\delta)^{\ell+6} \\
& \quad +\frac{-3C_3-2C_2-C_1}{\ell+5}(1-(x-a)/\delta)^{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}(1-(x-a)/\delta)^{\ell+4}\Bigg\}dx \\
= \quad & \frac{2\delta^2}{(b-a)^2}\left\{\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right\} \\
& -\frac{\delta^2}{(b-a)^2}\left\{\frac{-C_3}{(\ell+7)(\ell+8)}+\frac{3C_3+C_2}{(\ell+6)(\ell+7)}+\frac{-3C_3-2C_2-C_1}{(\ell+5)(\ell+6)}+\frac{C_3+C_2+C_1+C_0}{(\ell+4)(\ell+5)}\right\} \\
= \quad & \frac{\delta^2}{(b-a)^2}\Bigg\{2\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right) \\
& -\left(\frac{-C_3}{(\ell+7)(\ell+8)}+\frac{3C_3+C_2}{(\ell+6)(\ell+7)}+\frac{-3C_3-2C_2-C_1}{(\ell+5)(\ell+6)}+\frac{C_3+C_2+C_1+C_0}{(\ell+4)(\ell+5)}\right)\Bigg\}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx \\
= \quad & \frac{\delta^2}{(b-a)^2}\Bigg\{2\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right) \\
& -\left(\frac{-C_3}{(\ell+7)(\ell+8)}+\frac{3C_3+C_2}{(\ell+6)(\ell+7)}+\frac{-3C_3-2C_2-C_1}{(\ell+5)(\ell+6)}+\frac{C_3+C_2+C_1+C_0}{(\ell+4)(\ell+5)}\right)\Bigg\}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b m_P(x)dx \\
= \;& \frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right) \\
& +\frac{2\delta^2}{(b-a)^2}\left\{2\left(\frac{-C_3}{\ell+7}+\frac{3C_3+C_2}{\ell+6}+\frac{-3C_3-2C_2-C_1}{\ell+5}+\frac{C_3+C_2+C_1+C_0}{\ell+4}\right)\right. \\
& \left.-\left(\frac{-C_3}{(\ell+7)(\ell+8)}+\frac{3C_3+C_2}{(\ell+6)(\ell+7)}+\frac{-3C_3-2C_2-C_1}{(\ell+5)(\ell+6)}+\frac{C_3+C_2+C_1+C_0}{(\ell+4)(\ell+5)}\right)\right\}
\end{aligned}$$

## 3 Explicit formula for the kernel mean with $k = 2$

- Let $x \in [a, b]$.
- Constants $C_2 := \ell^2 + 4\ell + 3$, $C_1 := 3\ell + 6$, and $C_0 := 3$.
- Kernel:

$$\begin{aligned}
k_{d,2,\delta}(x, y) &= \phi_{d,2}(|x-y|/\delta) \\
&= (1-|x-y|/\delta)_+^{\ell+2}\left(C_2(|x-y|/\delta)^2 + C_1(|x-y|/\delta) + C_0\right) \\
&= (1-|x-y|/\delta)_+^{\ell+2}\left(D_2|x-y|^2 + D_1|x-y| + D_0\right),
\end{aligned}$$

where we defined the new constants as $D_2 := C_2\delta^{-2}$, $D_1 := C_1\delta^{-1}$, and $D_0 := C_0$.

- We want to evaluate the value of the kernel mean

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b k_{d,2,\delta}(x,y)dy \\
=\;& \frac{1}{b-a}\int_a^b \phi_{d,2}(|x-y|/\delta)dy \\
=\;& \frac{1}{b-a}\int_a^b (1-|x-y|/\delta)_+^{\ell+2}\left(D_2|x-y|^2 + D_1|x-y| + D_0\right)dy \\
=\;& \frac{1}{b-a}\int_a^x (1-(x-y)/\delta)_+^{\ell+2}\left(D_2(x-y)^2 + D_1(x-y) + D_0\right)dy \qquad (3) \\
+\;& \frac{1}{b-a}\int_x^b (1-(y-x)/\delta)_+^{\ell+2}\left(D_2(y-x)^2 + D_1(y-x) + D_0\right)dy. \qquad (4)
\end{aligned}$$

- There are three cases.
  - Case 1: $a \le x-\delta < x+\delta \le b$.
  - Case 2: $x - \delta < a$.

– Case 3: $x+\delta > b$.

- Case 1: $a \leq x-\delta < x+\delta \leq b$. From symmetry, Eq.(3) is equal to Eq.(4). Thus we will focus on Eq.(3).

$$
\begin{aligned}
& (b-a) \times \text{Eq. (3)} \\
= & \int_a^x (1-(x-y)/\delta)_+^{\ell+2} \left(D_2(x-y)^2 + D_1(x-y) + D_0\right) dy \\
= & \int_{x-\delta}^x (1-(x-y)/\delta)_+^{\ell+2} \{D_2(x-y)^2 + D_1(x-y) + D_0\} dy \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+2} \{D_2(\delta-z)^2 + D_1(\delta-z) + D_0\} dz \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+2} \{D_2(\delta^2 - 2\delta z + z^2) + D_1(\delta-z) + D_0\} dz \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+2} \{z^2 D_2 + z(-2\delta D_2 - D_1) + (D_2\delta^2 + D_1\delta + D_0)\} dz \\
= & \int_0^\delta \{z^{\ell+4}\delta^{-\ell-4} C_2 + z^{\ell+3}\delta^{-\ell-3}(-2C_2 - C_1) + z^{\ell+2}\delta^{-\ell-2}(C_2 + C_1 + C_0)\} dz \\
= & \delta \left( \frac{C_2}{\ell+5} + \frac{-2C_2 - C_1}{\ell+4} + \frac{C_2 + C_1 + C_0}{\ell+3} \right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{b-a} \int_a^b k_{d,2,\delta}(x,y) dy \\
= & \frac{2\delta}{b-a} \left( \frac{C_2}{\ell+5} + \frac{-2C_2 - C_1}{\ell+4} + \frac{C_2 + C_1 + C_0}{\ell+3} \right).
\end{aligned}
$$

- Case 2: $x - \delta < a$. We want to compute the sum of Eq.(3) and Eq.(4). Note the value of Eq.(4) is equal to that in Case 1. Therefore we have

$$
\text{Eq. (4)} = \frac{\delta}{b-a} \left( \frac{C_2}{\ell+5} + \frac{-2C_2 - C_1}{\ell+4} + \frac{C_2 + C_1 + C_0}{\ell+3} \right).
$$

Thus we focus on computing Eq.(3). Similarly to Case 1, we have

$$
\begin{aligned}
& (b-a) \times \text{Eq. (3)} \\
= & \int_a^x (1-(x-y)/\delta)_+^{\ell+2} \{D_2(x-y)^2 + D_1(x-y) + D_0\} dy \\
= & \int_{\delta-x+a}^{\delta} \{z^{\ell+4}\delta^{-\ell-4}C_2 + z^{\ell+3}\delta^{-\ell-3}(-2C_2 - C_1) + z^{\ell+2}\delta^{-\ell-2}(C_2+C_1+C_0)\} dz \\
= & \frac{C_2}{\ell+5}\left(1-(1-(x-a)/\delta)^{\ell+5}\right)\delta + \frac{-2C_2-C_1}{\ell+4}\left(1-(1-(x-a)/\delta)^{\ell+4}\right)\delta \\
& + \frac{C_2+C_1+C_0}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+4}\right)\delta
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,2,\delta}(x,y)dy \\
= & \frac{\delta}{b-a}\Bigg\{\left(\frac{C_2}{\ell+5} + \frac{-2C_2-C_1}{\ell+4} + \frac{C_2+C_1+C_0}{\ell+3}\right) \\
& + \frac{C_2}{\ell+5}\left(1-(1-(x-a)/\delta)^{\ell+5}\right) + \frac{-2C_2-C_1}{\ell+4}\left(1-(1-(x-a)/\delta)^{\ell+4}\right) \\
& \quad + \frac{C_2+C_1+C_0}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+4}\right)\Bigg\}.
\end{aligned}
$$

- Case 3: $x + \delta > b$. In this case, the value of the kernel mean can be computed similarly to Case 2. The resulting value is

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,2,\delta}(x,y)dy \\
= & \frac{\delta}{b-a}\Bigg\{\left(\frac{C_2}{\ell+5} + \frac{-2C_2-C_1}{\ell+4} + \frac{C_2+C_1+C_0}{\ell+3}\right) \\
& + \frac{C_2}{\ell+5}\left(1-(1-(b-x)/\delta)^{\ell+5}\right) + \frac{-2C_2-C_1}{\ell+4}\left(1-(1-(b-x)/\delta)^{\ell+4}\right) \\
& \quad + \frac{C_2+C_1+C_0}{\ell+3}\left(1-(1-(b-x)/\delta)^{\ell+4}\right)\Bigg\}.
\end{aligned}
$$

### 3.1 Explicit formula for the worst case error for $k = 2$

- We need a formula for

$$\frac{1}{b-a}\int_a^b m_P(x)dx,$$

where

$$m_P(x) = \frac{1}{b-a}\int_a^b k_{d,2,\delta}(x,y)dy.$$

- Note that this value can be decomposed as

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b m_P(x)dx \\
=\ & \frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx + \frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx + \frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx
\end{aligned}$$

The above three terms correspond to the Cases 1, 2, and 3 discussed above. Therefore we can use the formulas we have derived:

$$\begin{aligned}
&\frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx \\
=\ & \frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx \\
=\ & \frac{\delta}{(b-a)^2}\int_a^{a+\delta}\Bigg\{\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right) \\
&+\frac{C_2}{\ell+5}\left(1-(1-(x-a)/\delta)^{\ell+5}\right)+\frac{-2C_2-C_1}{\ell+4}\left(1-(1-(x-a)/\delta)^{\ell+4}\right) \\
&\quad+\frac{C_2+C_1+C_0}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+4}\right)\Bigg\}dx \\
=\ & \frac{2\delta^2}{(b-a)^2}\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right) \\
&-\frac{\delta^2}{(b-a)^2}\left(\frac{C_2}{(\ell+5)(\ell+6)}+\frac{-2C_2-C_1}{(\ell+4)(\ell+5)}+\frac{C_2+C_1+C_0}{(\ell+3)(\ell+4)}\right) \\
=\ & \frac{\delta^2}{(b-a)^2}\Bigg\{2\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right) \\
&-\left(\frac{C_2}{(\ell+5)(\ell+6)}+\frac{-2C_2-C_1}{(\ell+4)(\ell+5)}+\frac{C_2+C_1+C_0}{(\ell+3)(\ell+4)}\right)\Bigg\}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx \\
=\ & \frac{\delta^2}{(b-a)^2}\Bigg\{2\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right) \\
&-\left(\frac{C_2}{(\ell+5)(\ell+6)}+\frac{-2C_2-C_1}{(\ell+4)(\ell+5)}+\frac{C_2+C_1+C_0}{(\ell+3)(\ell+4)}\right)\Bigg\}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b m_P(x)dx \\
= \quad & \frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right) \\
& +\frac{2\delta^2}{(b-a)^2}\left\{2\left(\frac{C_2}{\ell+5}+\frac{-2C_2-C_1}{\ell+4}+\frac{C_2+C_1+C_0}{\ell+3}\right)\right. \\
& \left.-\left(\frac{C_2}{(\ell+5)(\ell+6)}+\frac{-2C_2-C_1}{(\ell+4)(\ell+5)}+\frac{C_2+C_1+C_0}{(\ell+3)(\ell+4)}\right)\right\}
\end{aligned}
$$

# 4 Explicit formula for the kernel mean with $k = 1$

- Let $x \in [a, b]$.
- Constants $C_1 := \ell + 1$, and $C_0 := 1$.
- Kernel:

$$
\begin{aligned}
k_{d,1,\delta}(x,y) &= \phi_{d,1}(|x-y|/\delta) \\
&= (1-|x-y|/\delta)_+^{\ell+1}\left(C_1(|x-y|/\delta)+C_0\right) \\
&= (1-|x-y|/\delta)_+^{\ell+1}\left(D_1|x-y|+D_0\right),
\end{aligned}
$$

  where we defined the new constants as $D_1 := C_1\delta^{-1}$, and $D_0 := C_0$.

- We want to evaluate the value of the kernel mean

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,1,\delta}(x,y)dy \\
= \quad & \frac{1}{b-a}\int_a^b \phi_{d,1}(|x-y|/\delta)dy \\
= \quad & \frac{1}{b-a}\int_a^b (1-|x-y|/\delta)_+^{\ell+1}\left(D_1|x-y|+D_0\right)dy \\
= \quad & \frac{1}{b-a}\int_a^x (1-(x-y)/\delta)_+^{\ell+1}\left(D_1(x-y)+D_0\right)dy \qquad (5) \\
+ \quad & \frac{1}{b-a}\int_x^b (1-(y-x)/\delta)_+^{\ell+1}\left(D_1(y-x)+D_0\right)dy. \qquad (6)
\end{aligned}
$$

- There are three cases.
  - Case 1: $a \le x-\delta < x+\delta \le b$.
  - Case 2: $x-\delta < a$.

– Case 3: $x+\delta > b$.

- Case 1: $a \le x-\delta < x+\delta \le b$. From symmetry, Eq.(5) is equal to Eq.(6). Thus we will focus on Eq.(5).

$$\begin{aligned}
& (b-a) \times \text{Eq. (5)} \\
= \;& \int_a^x (1-(x-y)/\delta)_+^{\ell+1} (D_1(x-y)+D_0)\, dy \\
= \;& \int_{x-\delta}^x (1-(x-y)/\delta)_+^{\ell+1} \{D_1(x-y)+D_0\} dy \\
= \;& \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+1} \{D_1(\delta-z)+D_0\} dz \\
= \;& \int_0^\delta \left(\frac{z}{\delta}\right)^{\ell+1} \{z(-D_1)+(D_1\delta+D_0)\} dz \\
= \;& \int_0^\delta \{z^{\ell+2}\delta^{-\ell-2}(-C_1)+z^{\ell+1}\delta^{-\ell-1}(C_1+C_0)\} dz \\
= \;& \delta\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)
\end{aligned}$$

Therefore,

$$\frac{1}{b-a}\int_a^b k_{d,1,\delta}(x,y)dy = \frac{2\delta}{b-a}\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right).$$

- Case 2: $x-\delta < a$. We want to compute the sum of Eq.(5) and Eq.(6). Note the value of Eq.(6) is equal to that in Case 1. Therefore we have

$$\text{Eq. (6)} = \frac{\delta}{b-a}\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right).$$

Thus we focus on computing Eq.(5). Similarly to Case 1, we have

$$\begin{aligned}
& (b-a) \times \text{Eq. (5)} \\
= \;& \int_a^x (1-(x-y)/\delta)_+^{\ell+1} \{D_1(x-y)+D_0\} dy \\
= \;& \int_{\delta-x+a}^\delta \{z^{\ell+2}\delta^{-\ell-2}(-C_1)+z^{\ell+1}\delta^{-\ell-1}(C_1+C_0)\} dz \\
= \;& \frac{-C_1}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+3}\right)\delta+\frac{C_1+C_0}{\ell+2}\left(1-(1-(x-a)/\delta)^{\ell+2}\right)\delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b k_{d,1,\delta}(x,y)dy \\
= \quad &\frac{\delta}{b-a}\Bigg\{\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right) \\
&+\frac{-C_1}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+3}\right)+\frac{C_1+C_0}{\ell+2}\left(1-(1-(x-a)/\delta)^{\ell+2}\right)\Bigg\}.
\end{aligned}$$

- Case 3: $x+\delta>b$. In this case, the value of the kernel mean can be computed similarly to Case 2. The resulting value is

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b k_{d,1,\delta}(x,y)dy \\
= \quad &\frac{\delta}{b-a}\Bigg\{\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right) \\
&+\frac{-C_1}{\ell+3}\left(1-(1-(b-x)/\delta)^{\ell+3}\right)+\frac{C_1+C_0}{\ell+2}\left(1-(1-(b-x)/\delta)^{\ell+2}\right)\Bigg\}.
\end{aligned}$$

## 4.1 Explicit formula for the worst case error for $k=1$

- We need a formula for
$$\frac{1}{b-a}\int_a^b m_P(x)dx,$$
where
$$m_P(x)=\frac{1}{b-a}\int_a^b k_{d,1,\delta}(x,y)dy.$$

- Note that this value can be decomposed as

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b m_P(x)dx \\
= \quad &\frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx+\frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx+\frac{1}{b-a}\int_{b-\delta}^b m_P(x)dx
\end{aligned}$$

The above three terms correspond to the Cases 1, 2, and 3 discussed above. Therefore we can use the formulas we have derived:

$$\begin{aligned}
&\frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx \\
= \quad &\frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right).
\end{aligned}$$

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx \\
= \quad & \frac{\delta}{(b-a)^2}\int_a^{a+\delta}\left\{\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)\right. \\
& \left.+\frac{-C_1}{\ell+3}\left(1-(1-(x-a)/\delta)^{\ell+3}\right)+\frac{C_1+C_0}{\ell+2}\left(1-(1-(x-a)/\delta)^{\ell+2}\right)\right\}dx \\
= \quad & \frac{2\delta^2}{(b-a)^2}\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)-\frac{\delta^2}{(b-a)^2}\left(\frac{-C_1}{(\ell+3)(\ell+4)}+\frac{C_1+C_0}{(\ell+2)(\ell+3)}\right) \\
= \quad & \frac{\delta^2}{(b-a)^2}\left\{2\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)-\left(\frac{-C_1}{(\ell+3)(\ell+4)}+\frac{C_1+C_0}{(\ell+2)(\ell+3)}\right)\right\}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx \\
= \quad & \frac{\delta^2}{(b-a)^2}\left\{2\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)-\left(\frac{-C_1}{(\ell+3)(\ell+4)}+\frac{C_1+C_0}{(\ell+2)(\ell+3)}\right)\right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{b-a}\int_{a}^{b} m_P(x)dx \\
= \quad & \frac{2\delta(b-a-2\delta)}{(b-a)^2}\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right) \\
& +\frac{2\delta^2}{(b-a)^2}\left\{2\left(\frac{-C_1}{\ell+3}+\frac{C_1+C_0}{\ell+2}\right)-\left(\frac{-C_1}{(\ell+3)(\ell+4)}+\frac{C_1+C_0}{(\ell+2)(\ell+3)}\right)\right\}
\end{aligned}
$$

# 5 Explicit formula for the kernel mean with $k = 0$

- Let $x \in [a, b]$.
- Kernel:

$$
\begin{aligned}
k_{d,0,\delta}(x,y) &= \phi_{d,0}(|x-y|/\delta) \\
&= (1-|x-y|/\delta)_+^{\lfloor d/2\rfloor+1}.
\end{aligned}
$$

- We want to evaluate the value of the kernel mean

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,0,\delta}(x,y)dy \\
= & \frac{1}{b-a}\int_a^b \phi_{d,0}(|x-y|/\delta)dy \\
= & \frac{1}{b-a}\int_a^b (1-|x-y|/\delta)_+^{\lfloor d/2\rfloor+1}\,dy \\
= & \frac{1}{b-a}\int_a^x (1-(x-y)/\delta)_+^{\lfloor d/2\rfloor+1}\,dy \qquad (7)\\
+ & \frac{1}{b-a}\int_x^b (1-(y-x)/\delta)_+^{\lfloor d/2\rfloor+1}\,dy. \qquad (8)
\end{aligned}
$$

- There are three cases.
  - Case 1: $a \le x-\delta < x+\delta \le b$.
  - Case 2: $x-\delta < a$.
  - Case 3: $x+\delta > b$.

- Case 1: $a \le x-\delta < x+\delta \le b$. From symmetry, Eq.(7) is equal to Eq.(8). Thus we will focus on Eq.(7).

$$
\begin{aligned}
& (b-a)\times \text{Eq. (7)} \\
= & \int_a^x (1-(x-y)/\delta)_+^{\lfloor d/2\rfloor+1}\,dy \\
= & \int_{x-\delta}^x (1-(x-y)/\delta)_+^{\lfloor d/2\rfloor+1}\,dy \\
= & \int_0^\delta \left(\frac{z}{\delta}\right)^{\lfloor d/2\rfloor+1}\,dz \\
= & \frac{\delta}{\lfloor d/2\rfloor+2}.
\end{aligned}
$$

  Therefore,

$$
\frac{1}{b-a}\int_a^b k_{d,0,\delta}(x,y)dy = \frac{2\delta}{(b-a)(\lfloor d/2\rfloor+2)}.
$$

- Case 2: $x-\delta < a$. We want to compute the sum of Eq.(7) and Eq.(8). Note the value of Eq.(8) is equal to that in Case 1. Therefore we have

$$
\text{Eq. (8)} = \frac{\delta}{(b-a)(\lfloor d/2\rfloor+2)}.
$$

Thus we focus on computing Eq.(7). Similarly to Case 1, we have

$$\begin{aligned}
& (b-a) \times \text{Eq. (7)} \\
= & \int_a^x (1-(x-y)/\delta)_+^{\lfloor d/2 \rfloor+1} dy \\
= & \int_{\delta-x+a}^{\delta} \left(\frac{z}{\delta}\right)^{\lfloor d/2 \rfloor+1} dz \\
= & \frac{\delta}{\lfloor d/2 \rfloor+2} \left(1-(1-(x-a)/\delta)^{\lfloor d/2 \rfloor+2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,0,\delta}(x,y)dy \\
= & \frac{\delta}{b-a}\left\{\frac{1}{\lfloor d/2 \rfloor+2} + \frac{1}{\lfloor d/2 \rfloor+2}\left(1-(1-(x-a)/\delta)^{\lfloor d/2 \rfloor+2}\right)\right\} \\
= & \frac{\delta}{(b-a)(\lfloor d/2 \rfloor+2)}\left\{2-(1-(x-a)/\delta)^{\lfloor d/2 \rfloor+2}\right\}.
\end{aligned}$$

- Case 3: $x+\delta > b$. In this case, the value of the kernel mean can be computed similarly to Case 2. The resulting value is

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b k_{d,0,\delta}(x,y)dy \\
= & \frac{\delta}{(b-a)(\lfloor d/2 \rfloor+2)}\left\{2-(1-(b-x)/\delta)^{\lfloor d/2 \rfloor+2}\right\}.
\end{aligned}$$

## 5.1 Explicit formula for the worst case error for $k=0$

- We need a formula for

$$\frac{1}{b-a}\int_a^b m_P(x)dx,$$

where

$$m_P(x) = \frac{1}{b-a}\int_a^b k_{d,0,\delta}(x,y)dy.$$

- Note that this value can be decomposed as

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b m_P(x)dx \\
= & \frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx + \frac{1}{b-a}\int_a^{a+\delta} m_P(x)dx + \frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx
\end{aligned}$$

The above three terms correspond to the Cases 1, 2, and 3 discussed above. Therefore we can use the formulas we have derived:

$$\frac{1}{b-a}\int_{a+\delta}^{b-\delta} m_P(x)dx = \frac{2\delta(b-a-2\delta)}{(b-a)^2(\lfloor d/2\rfloor+2)}$$

$$\begin{aligned}
& \frac{1}{b-a}\int_{a}^{a+\delta} m_P(x)dx \\
= \; & \frac{\delta}{(b-a)^2(\lfloor d/2\rfloor+2)}\int_{a}^{a+\delta}\left(2-(1-(x-a)/\delta)^{\lfloor d/2\rfloor+2}\right)dx \\
= \; & \frac{\delta^2}{(b-a)^2(\lfloor d/2\rfloor+2)}\left(2-\frac{1}{\lfloor d/2\rfloor+3}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \frac{1}{b-a}\int_{b-\delta}^{b} m_P(x)dx \\
= \; & \frac{\delta^2}{(b-a)^2(\lfloor d/2\rfloor+2)}\left(2-\frac{1}{\lfloor d/2\rfloor+3}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{b-a}\int_{a}^{b} m_P(x)dx \\
= \; & \frac{2\delta(b-a-2\delta)}{(b-a)^2(\lfloor d/2\rfloor+2)}+\frac{2\delta^2}{(b-a)^2(\lfloor d/2\rfloor+2)}\left(2-\frac{1}{\lfloor d/2\rfloor+3}\right)
\end{aligned}$$

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