1 Question 6

1.1 Part a

$$\begin{aligned}
\mathbf{y}^t P\mathbf{y} &= \mathbf{y}^T A^T A\mathbf{y} \\
&= (A\mathbf{y})^T A\mathbf{y} \\
&= ||A\mathbf{y}||^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{z}^t Q\mathbf{z} &= \mathbf{z}^T AA^T\mathbf{z} \\
&= (A^T\mathbf{z})^T A^T\mathbf{z} \\
&= ||A^T\mathbf{z}||^2
\end{aligned}$$

Squared norm is always non-negative, so we conclude that $\mathbf{y}^t P\mathbf{y} \geq 0$ and $\mathbf{z}^t Q\mathbf{z} \geq 0$

P and Q are symmetric matrices, so their eigenvalues must be real. Suppose P has a real eigenvalue λ with corresponding eigenvector x. Then

$$\begin{aligned}
P\mathbf{x} &= \lambda\mathbf{x} \\
\mathbf{x}^T P\mathbf{x} &= \lambda\mathbf{x}^T\mathbf{x} \\
\mathbf{x}^T P\mathbf{x} &= \lambda||\mathbf{x}||^2
\end{aligned}$$

Having earlier shown that the above LHS is always non-negative, the RHS also must be non-negative. Therefore, all eigenvalues of P (and similarly of Q) are non-negative.

1.2 Part b

If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue λ,

$$\begin{aligned}
P\mathbf{u} &= \lambda\mathbf{u} \\
A^T A\mathbf{u} &= \lambda\mathbf{u} \\
AA^T A\mathbf{u} &= \lambda A\mathbf{u} \\
QA\mathbf{u} &= \lambda A\mathbf{u}
\end{aligned}$$

Therefore, $A\mathbf{u}$ is an eigenvector of Q.

If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ,

$$\begin{aligned}
Q\mathbf{v} &= \mu\mathbf{v} \\
AA^T\mathbf{v} &= \mu\mathbf{v} \\
A^T AA^T\mathbf{v} &= \mu A^T\mathbf{v} \\
PA^T\mathbf{u} &= \mu A^T\mathbf{v}
\end{aligned}$$

Therefore, $A^T\mathbf{v}$ is an eigenvector of P
$\mathbf{v}$ is an m-dimensional vector and $\mathbf{u}$ is an n-dimensional vector.

1.3 Part c

As shown in the previous part, if $\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{Q}$, then $A^T\mathbf{v_i}$ is an eigenvector of P. So for some non-negative μ we have:

$$Q\mathbf{v_i} = \mu_i\mathbf{v_i}$$
$$PA^T\mathbf{v_i} = \mu_i A^T\mathbf{v_i}$$

Assuming that for any i, $||\mathbf{v_i}|| = 1$. First we calculate the norm of $A^T\mathbf{v_i}$:

$$||A^T\mathbf{v_i}||^2 = \mathbf{v_i}^T AA^T\mathbf{v_i} = \mu\mathbf{v_i}^T\mathbf{v_i} = \mu_i||\mathbf{v_i}||^2 = \mu_i$$

Some algebraic manipulation:

$$\begin{aligned}
P(A^T\mathbf{v_i}) &= \mu_i(A^T\mathbf{v_i}) \\
A^TA(A^T\mathbf{v_i}) &= \mu_i(A^T\mathbf{v_i}) \\
(\mathbf{v_i}^TA)A^TA(A^T\mathbf{v_i}) &= \mu_i(\mathbf{v_i}^TA)(A^T\mathbf{v_i}) \\
(\mathbf{v_i}^TA)A^TA(A^T\mathbf{v_i}) &= \mu_i||A^T\mathbf{v_i}||^2 \\
\frac{(\mathbf{v_i}^TA)A^TA(A^T\mathbf{v_i})}{||A^T\mathbf{v_i}||} &= \mu_i||A^T\mathbf{v_i}|| \\
\mathbf{v_i}^TAA^TA\mathbf{u_i} &= \mu_i^{\frac{3}{2}} \\
\mathbf{v_i}^TQA\mathbf{u_i} &= \mu_i^{\frac{3}{2}} \\
\mathbf{v_i}(\mathbf{v_i}^TQ)A\mathbf{u_i} &= \mu_i^{\frac{3}{2}}\mathbf{v_i} \\
\mathbf{v_i}(Q\mathbf{v_i})^TA\mathbf{u_i} &= \mu_i^{\frac{3}{2}}\mathbf{v_i} \\
\mathbf{v_i}(\mu_i\mathbf{v_i})^TA\mathbf{u_i} &= \mu_i^{\frac{3}{2}}\mathbf{v_i} \\
\mu_i\mathbf{v_i}\mathbf{v_i}^TA\mathbf{u_i} &= \mu_i^{\frac{3}{2}}\mathbf{v_i} \\
||\mathbf{v_i}||A\mathbf{u_i} &= \mu_i^{\frac{1}{2}}\mathbf{v_i} \\
\implies A\mathbf{u_i} &= \gamma_i\mathbf{v_i}
\end{aligned}$$

Here $\gamma_i = \sqrt{\mu_i}$. Since μ_i is non-negative, γ_i must also be non-negative. Hence proved.

1.4 Part d

For any i, j,

$$\mathbf{v_i}^TA\mathbf{u_j} = \mathbf{v_i}^T(\gamma_j\mathbf{v_j}) = \gamma_j\mathbf{v_i}^T\mathbf{v_j}$$

So for $i \neq j$, $\mathbf{v_i}^TA\mathbf{u_j} = 0$
and for $i = j$, $\mathbf{v_i}^TA\mathbf{u_j} = \gamma_i$

So for $i, j = 1 \dots m$, writing above result in matrix form:

$$[\boldsymbol{v}_1|\boldsymbol{v}_2|\boldsymbol{v}_3|...|\boldsymbol{v}_m]^T A[\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3|...|\boldsymbol{u}_m] = diag(\gamma_m, \dots, \gamma_m)$$

Let $\mathcal{T} = diag(\gamma_m, \dots, \gamma_m)$, then:

$$U^T AV = \mathcal{T}$$
$$UU^T AVV^T = U\mathcal{T}V$$

Since the eigenvectors are orthogonal, the matrices U and V are orthogonal matrices, ie, $U^T U = I$ and $V^T V = I$. Hence, we have $A = U\mathcal{T}V$.