

# MATH3060: Mathematical Analysis III

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# 1 Fourier Series

## 1.1 Introduction to Fourier Series

### 1.1.1 Trigonometric Series

*Definition 1.1.* A **trigonometric series** on  $[-\pi, \pi]$  is a series of functions in the form

$$\sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where  $a_n, b_n \in \mathbb{R}$ . Furthermore, if  $a_n = 0$  for all  $n$ , the series is called a **sine series**. Similarly, if  $b_n = 0$  for all  $n$ , the series is called a **cosine series**.

Note that it is possible to pull out the zeroth term of the sum, making the series in the form

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

hence it can be assumed that  $b_0 = 0$ .

*Proposition 1.2.* Let  $(a_n), (b_n)$  be infinite series. If

$$|a_n|, |b_n| \leq \frac{C}{n^s}$$

for some  $C > 0$  and  $s > 1$ , then their corresponding series  $\sum_{n=0}^{\infty} |a_n|$  and  $\sum_{n=0}^{\infty} |b_n|$  are convergent.

*Proposition 1.3.* If  $\sum_{n=0}^{\infty} |a_n|$  and  $\sum_{n=0}^{\infty} |b_n|$  are convergent, then by Weierstrass M-test,

$$\sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is uniformly and absolutely convergent.

*Proposition 1.4.* Let  $\phi(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$  be a continuous function on  $[-\pi, \pi]$ . If  $\sum |a_n|, \sum |b_n| < \infty$ , then  $\phi(x)$  is  $2\pi$ -periodic.

### 1.1.2 Fourier Series

*Definition 1.5.* Let  $f$  be a  $2\pi$ -periodic function on  $\mathbb{R}$  which is Riemann integrable on  $[-\pi, \pi]$ , then the **Fourier series** (or **Fourier expansion**) of  $f$  is the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with Fourier coefficients of  $f$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ny) \, dy$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin(ny) \, dy$$

Note that  $a_0$  is actually the average of  $f$  over  $[-\pi, \pi]$ . Fourier series depends on the global information of  $f$  on  $[-\pi, \pi]$  instead of a point in  $f$ . Fourier series also depends only on  $f|_{(-\pi, \pi)}$ , which means the end points of the closed interval are independent.

*Proposition 1.6.* Let  $f_1, f_2$  are Fourier series where  $f_1 \equiv f_2$  almost everywhere on  $[-\pi, \pi]$ , then  $f_1$  and  $f_2$  are the same Fourier series.

### 1.1.3 Motivation of Fourier Series

Recall the form of a Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for all  $x \in \mathbb{R}$ . If  $f$  is uniformly convergent,

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx \\ &= a_0 \int_{-\pi}^{\pi} \cos(mx) \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx + b_n \int_{-\pi}^{\pi} \sin(nx) \cos(mx) \, dx \right) \end{aligned}$$

Note that

$$\int_{-\pi}^{\pi} \cos(mx) \, dx = \begin{cases} 2\pi & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx) \cos(mx) \, dx = 0 \quad \forall m, n \geq 1$$

which will deduce

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx$$

Using similar method but instead of  $\cos(mx)$ ,  $\sin(mx)$  will deduce

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx$$

## 1.2 Complex Fourier Series

### 1.2.1 Definition of Complex Fourier Series

*Definition 1.7.* Let  $f$  be a  $2\pi$ -periodic function on  $\mathbb{C}$  which is Riemann integrable on  $[-\pi, \pi]$ , then its **complex Fourier series** is a Fourier series of the form

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

where  $\{c_n\}_{-\infty}^{\infty}$  is a **bisequence** of complex numbers defined by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

for all integers  $n$ . Moreover,  $\sum_{-\infty}^{\infty} c_n e^{inx}$  is said to be convergent at  $x$  if

$$\lim_{N \rightarrow +\infty} \sum_{-N}^N c_n e^{inx}$$

exists.

Note that for a complex-valued function  $f = u + iv$ ,

$$\int_a^b f = \int_a^b u + i \int_a^b v$$

In other words,  $f$  is said to be integrable if both  $u$  and  $v$  are integrable.

### 1.2.2 Motivation of Complex Fourier Series

Recall the form of a complex Fourier series as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

for all  $x \in \mathbb{C}$ . If  $f$  converges nicely,

$$\int_{-\pi}^{\pi} e^{-imx} dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

Note that

$$\int_{-\pi}^{\pi} e^{i(n-m)x} dx = \begin{cases} 2\pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

which will deduce

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

### 1.2.3 Relations between Real and Complex Fourier Series

In this section the relationship between (real) Fourier series and complex Fourier series for a real-valued function  $f$  is discussed. Note that

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos(nx) - i \sin(nx)) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{aligned}$$

which will deduce

$$c_n = \begin{cases} (a_n - ib_n)/2 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \\ (a_{-n} + ib_{-n})/2 & \text{if } n \leq -1 \end{cases}$$

**Proposition 1.8.** Let  $f$  be a real-valued function, then its complex Fourier coefficient  $c_{-n} = \overline{c_n}$ .

**Proposition 1.9.** Let  $f$  be a  $2\pi$ -periodic real function which is differentiable on  $[-\pi, \pi]$  with  $f'$  integrable on  $[-\pi, \pi]$ . Denote the Fourier coefficients of  $f$  and  $f'$  by  $\{a_n(f), b_n(f); c_n(f)\}$  and  $\{a_n(f'), b_n(f'); c_n(f')\}$  respectively, then

$$\begin{cases} a_n(f') = nb_n(f) \\ b_n(f') = -na_n(f) \end{cases} \quad \text{and } c_n(f') = inc_n(f)$$

Note that one of the advantages of using complex Fourier series is to compute derivatives with more convenience.

## 1.3 Fourier Series and Extensions

### 1.3.1 Extensions of Periodic Functions

For any Riemann integrable function  $f$  on  $[-\pi, \pi]$ , one can define the Fourier coefficients to form a Fourier series. On the other hand, we can restrict  $f$  to  $(-\pi, \pi]$  and extend periodically to a  $2\pi$ -periodic function  $\tilde{f}$  on  $\mathbb{R}$ . The function  $f$  and its extension  $\tilde{f}$  have the same Fourier series.

Here the  $\sim$  symbol in

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

represents  $f(x)$  has the Fourier series on the right hand side. The equal sign  $=$  is not used since the series may not converge.

*Example 1.10.* Let  $f(x) = x$  be a function in  $[-\pi, \pi]$ . Find the Fourier series of  $f$ .

*Answer.* The Fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx = (-1)^{n+1} \frac{2}{n}$$

Therefore

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)$$

which is a sine series.

With the example above, the following proposition can be introduced by observation:



*Proposition 1.11.* Let  $f$  be a real function, then its Fourier series is a sine series if  $f$  is odd, and it is a cosine series if  $f$  is even.

Note that the Fourier series may not be the same as the original function, especially when there are discontinuous points like  $\pm\pi$ . Also, the convergence of Fourier series is not clear since some terms like  $\sum(1/n)$  does not converge.

### 1.3.2 Big-O and Little-O Notations

*Definition 1.12.* Let  $\{x_n\}$  be a sequence, then the **big-O notation**, denoted by  $O$ , paired with  $x_n$  is defined as

$$x_n = O(n^s) \Leftrightarrow |x_n| \leq Cn^s$$

for some constant  $C > 0$ , as  $n \rightarrow \infty$ . Similarly, the **little-O notation**, denoted by  $o$ , paired with  $x_n$  is defined as

$$x_n = o(n^s) \Leftrightarrow \frac{|x_n|}{n^s} \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Example 1.13.* Find the correlation of

$$x_n = \frac{2(-1)^{n+1}}{n} \sin(nx)$$

using big-O notation.

*Answer.* Since  $|x_n| \leq 2/n$ ,  $x_n = O(1/n)$ .

*Example 1.14.* Find the correlation of

$$x_n = \log(n)$$

using little-O notation.

*Answer.* Since  $|\log(n)|/n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $x_n = o(n)$ .

### 1.3.3 Fourier Series of Unusual Periodic Functions

Let  $f$  be a  $2T$ -periodic function. Note that  $g(x) = f(Tx/\pi)$  is a  $2\pi$ -periodic function, then

$$g(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx) \, dx$$

along with the substitution  $y = Tx/\pi$  implies

$$f(y) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi}{T}y\right) + b_n \sin\left(\frac{n\pi}{T}y\right) \right)$$

with

$$a_0 = \frac{1}{2T} \int_{-T}^T f(y) \, dy$$

$$a_n = \frac{1}{T} \int_{-T}^T f(y) \cos\left(\frac{n\pi}{T}y\right) \, dy$$

$$b_n = \frac{1}{T} \int_{-T}^T f(y) \sin\left(\frac{n\pi}{T}y\right) \, dy$$

Such Fourier series is called Fourier series of  $2T$ -periodic function  $f$ .

## 1.4 Convergence of Fourier Series

### 1.4.1 Riemann-Lebesgue Lemma

Recall the definition of a step function on  $[-\pi, \pi]$  as a function of the form

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

where  $-\pi = a_0 < a_1 < \cdots < a_N = \pi$ ,  $I_0 = [a_0, a_1]$  and  $I_j = (a_j, a_{j+1}]$  for  $1 \leq j \leq N-1$ . The characteristic function (or indicator function)

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

*Proposition 1.15.* For every step function  $s$  integrable on  $[-\pi, \pi]$ , there exists a constant  $C > 0$  depending on  $s$  such that

$$|a_n(s)|, |b_n(s)| \leq \frac{C}{n}$$

for all  $n \geq 1$ .  $a_n(s)$  and  $b_n(s)$  are Fourier coefficients of  $s$ .

*Proof.* Let

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

be the step function, then for  $n \geq 1$ ,

$$\begin{aligned} \pi a_n(s) &= \int_{-\pi}^{\pi} s(x) \cos(nx) \, dx \\ &= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos(nx) \, dx \\ &= \sum_{j=0}^{N-1} s_j \frac{\sin(na_{j+1}) - \sin(na_j)}{n} \end{aligned}$$

which implies  $|a_n(s)| \leq \frac{C}{n}$ . The proof for  $|b_n(s)|$  is similar.

*Proposition 1.16.* Let  $f$  be integrable on  $[-\pi, \pi]$ , then for all  $\epsilon > 0$ , there exists a step function  $s$  such that  $s \leq f$  on  $[-\pi, \pi]$  and

$$\int_{-\pi}^{\pi} (f - s) < \epsilon$$

*Proof.* Since  $f$  is Riemann integrable, the function can be approximated with Darboux lower sum. For all  $\epsilon > 0$ , there exists a partition  $-\pi = a_0 < a_1 < \dots < a_N = \pi$  such that

$$\int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) < \epsilon$$

where  $m_j = \inf \{f(x) \mid x \in [a_j, a_{j+1}]\}$ . Define the step function

$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j}$$

then  $s \leq f$  and

$$\int_{-\pi}^{\pi} s(x) \, dx = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j)$$

implies the result.

With the propositions above, the **Riemann-Lebesgue Lemma** can be introduced:

*Theorem 1.17.* The Fourier coefficients of any  $2\pi$ -periodic function  $f$  integrable on  $[-\pi, \pi]$  converge to 0 as  $n \rightarrow \infty$ .

*Proof.* By *Proposition 1.16*, for any  $\epsilon > 0$ , there exists a step function  $s$  such that  $s \leq f$  and

$$\int_{-\pi}^{\pi} (f - s) < \frac{\epsilon}{2}$$

On the other hand, by *Proposition 1.15*, there exists  $n_0 > 0$  such that

$$|a_n(s)| < \frac{\epsilon}{2}$$

for all  $n \geq n_0$ . For instance,  $n_0 = [2C/\epsilon] + 1$  with the constant  $C$  in *Proposition 1.15*. Note that

$$\begin{aligned} |a_n(f) - a_n(s)| &= \frac{1}{\pi} \left| \int_{-\pi}^{\pi} (f - s)(x) \cos(nx) \, dx \right| \\ &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} (f - s) \quad \text{as } f \geq s \\ &\leq \frac{\epsilon}{2\pi} \end{aligned}$$

Hence,

$$\begin{aligned} |a_n(f)| &\leq |a_n(s)| + |a_n(f) - a_n(s)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2\pi} < \epsilon \end{aligned}$$

for all  $n \geq n_0$ , which means  $a_n(f) \rightarrow 0$  as  $n \rightarrow \infty$ . The proof for  $b_n(f)$  is similar to that above.

### 1.4.2 Lipschitz Continuity at Points

*Definition 1.18.* Let  $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$  be a function which has a Fourier series, then the  **$n$ -th partial sum** of Fourier series of  $f$ , denoted by  $(S_n f)(x)$ , is given by

$$(S_n f)(x) = a_0 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$$

*Definition 1.19.* Let  $f$  be a function on  $[a, b]$ , then  $f$  is called **Lipschitz continuous** at a point  $x_0 \in [a, b]$  if there exists  $L > 0$  and  $\delta > 0$  such that

$$|f(x) - f(x_0)| \leq L |x - x_0|$$

for all  $|x - x_0| < \delta$ .

Note that both  $L$  and  $\delta$  may depend on the point  $x_0$ . Below is a proposition on extending Lipschitz continuity from a point to an interval:

*Proposition 1.20.* If  $f$  is Lipschitz continuous at  $x_0 \in [a, b]$  and  $f$  is bounded on  $[a, b]$ , then there exists  $L' > 0$  which may depend on  $x_0$  such that

$$|f(x) - f(x_0)| \leq L' |x - x_0|$$

for all  $x \in [a, b]$ .

*Proof.* By *Definition 1.19*, there exists  $L, \delta > 0$  such that

$$|f(x) - f(x_0)| \leq L |x - x_0|$$

for all  $|x - x_0| < \delta$ . If  $|x - x_0| \geq \delta$ , then

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x)| + |f(x_0)| \\ &\leq 2M \leq \frac{2M |x - x_0|}{\delta} \end{aligned}$$

where  $M = \sup_{[a, b]} |f| \geq 0$ . Pick  $L' = \max \{L, 2M/\delta\} > 0$ , then

$$|f(x) - f(x_0)| \leq L' |x - x_0|$$

for all  $x \in [a, b]$ .

### 1.4.3 Dirichlet Kernels

*Definition 1.21.* The **Dirichlet kernel**, denoted by  $D_n(z)$ , is defined by

$$D_n(z) = \begin{cases} (\sin((n + 1/2)z)) / (2\pi \sin(z/2)) & \text{if } z \neq 0 \\ (2n + 1) / (2\pi) & \text{if } z = 0 \end{cases}$$

*Proposition 1.22.* Below are the properties of Dirichlet kernels:

- (a) Integral of a Dirichlet kernel  $\int_{-\pi}^{\pi} D_n(z) \, dz = 1$ .
- (b)  $D_n(z)$  is even, continuous,  $2\pi$ -periodic on  $[-\pi, \pi]$  and

$$D_n\left(\frac{2k\pi}{2n+1}\right) = 0$$

for all  $k = -n, -n+1, \dots, n$ .

- (c) The maximum

$$\max_{[-\pi, \pi]} D_n(z) = D_n(0) = \frac{2n+1}{2\pi}$$

- (d) For all  $0 < \delta < \pi/2$ ,

$$\int_0^{\delta} |D_n(z)| \, dz \rightarrow +\infty$$

as  $n \rightarrow +\infty$

*Proof. Part (a)* This can be achieved by integrating

$$\int_{-\pi}^{\pi} \left( \frac{1}{2} + \sum_{k=1}^n \cos(kz) \right) \, dz$$

*Part (d)* Let  $0 < \delta < \pi/2$ , then for all  $n \in \mathbb{N}$ , there exists  $N \in \mathbb{N}$  such that

$$N < \frac{n+1/2}{\pi} \delta \leq N+1$$

where  $N \rightarrow \infty$  as  $n \rightarrow \infty$ . Note that

$$\begin{aligned}
\int_0^\delta |D_n(z)| \, dz &= \int_0^\delta \frac{|\sin(n + 1/2)z|}{2\pi |\sin(z/2)|} \, dz \\
&= \int_0^{(n+1/2)\delta} \frac{|\sin(t)|}{2\pi |\sin(t/(2n+1))|} \left( \frac{2 \, dt}{2n+1} \right) \quad \text{where } t = \left(n + \frac{1}{2}\right)z \\
&= \frac{1}{\pi} \int_0^{(n+1/2)\delta} \frac{\sin(t)}{t} \frac{t/(2n+1)}{|\sin(t/(2n+1))|} \, dt \\
&\geq \frac{1}{\pi} \int_0^{(n+1/2)\delta} \frac{\sin(t)}{t} \, dt \quad \text{since } \frac{\sin(x)}{x} < 1 \text{ for } 0 < x \\
&\geq \frac{1}{\pi} \int_0^{N\pi} \frac{\sin(t)}{t} \, dt \\
&= \frac{1}{\pi} \sum_{k=1}^N \int_{(k-1)\pi}^{k\pi} \frac{\sin(t)}{t} \, dt \\
&= \frac{1}{\pi} \sum_{k=1}^N \int_0^\pi \frac{|\sin(s)|}{s + (k-1)\pi} \, ds \quad \text{where } s = t - (k-1)\pi \\
&\geq \frac{1}{\pi} \sum_{k=1}^N \int_0^\pi \frac{|\sin(s)|}{k\pi} \, ds \quad \text{since } t \leq k\pi \\
&= \frac{1}{\pi^2} \left( \int_0^\pi |\sin(s)| \, ds \right) \sum_{k=1}^N \frac{1}{k} = \frac{2}{\pi^2} \sum_{k=1}^N \frac{1}{k}
\end{aligned}$$

But since the sum of harmonic series  $\sum_{k=1}^N (1/k)$  diverges when  $N \rightarrow \infty$  as  $n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \int_0^\delta |D_n(z)| \, dz = +\infty$$

With the definition and properties of Dirichlet kernels, the following proposition can be introduced.

*Proposition 1.23.* Let  $f$  be a  $2\pi$ -periodic function integrable on  $[-\pi, \pi]$ . Suppose that  $f$  is Lipschitz continuous at  $x$ , then the sequence  $\{S_n f(x)\}$  converges to  $f(x)$  as  $n \rightarrow +\infty$ .

*Proof.* Let  $f$  be a function that is Lipschitz continuous at a point  $x_0 \in [-\pi, \pi]$ . By splitting

$$(S_n(f))(x_0) - f(x_0) = I_1 + I_2$$

into integrals  $I_1$  and  $I_2$  concentrated in  $[-\delta, \delta]$  and essentially, outside the interval, respectively. Note that by *Proposition 1.22*,

*Example 1.24.* Let  $f(x) = x$  be a  $2\pi$ -periodic function integrable on  $[-\pi, \pi]$ . Its Fourier series

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

clearly implies that  $f(x)$  is Lipschitz continuous at any  $x \in (-\pi, \pi)$ .

*Proposition 1.25.* Let  $f$  be a  $2\pi$ -periodic function integrable on  $[-\pi, \pi]$ . Suppose that for  $x_0 \in [-\pi, \pi]$ , the following are satisfied:

(a) The left-hand limit and right-hand limit both exist, which is

$$f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x), f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x)$$

(b) There exists  $L > 0$  and  $\delta > 0$  such that

$$\begin{cases} |f(x) - f(x_0^+)| \leq L(x - x_0) & \text{where } 0 < x - x_0 < \delta \\ |f(x) - f(x_0^-)| \leq L(x_0 - x) & \text{where } 0 < x_0 - x < \delta \end{cases}$$

then

$$S_n f(x) \rightarrow \frac{f(x_0^+) + f(x_0^-)}{2}$$

as  $n \rightarrow +\infty$ .

*Example 1.26.* Let  $f(x) = x$  be a  $2\pi$ -periodic function integrable on  $[-\pi, \pi]$ , where  $f$  is discontinuous at  $x = \pi$ . Note that  $f(\pi^-) = \pi$  and  $f(\pi^+) = -\pi$ .

Now assume  $\delta = \frac{\pi}{2}$ . For  $0 < x - \pi < \delta$ ,

$$\begin{aligned} |f(x) - f(\pi^+)| &= |f(x - 2\pi) - (-\pi)| \\ &= |x - 2\pi + \pi| \\ &= x - \pi \leq L(x - \pi) \end{aligned}$$

where  $L = 1$ . The approach for  $0 < \pi - x < \delta$  is similar. Therefore, by *Proposition 1.25*,

$$S_n f(x) \rightarrow \frac{f(\pi^+) + f(\pi^-)}{2} = 0$$

as  $n \rightarrow +\infty$ .



### 1.4.4 Lipschitz Condition and Uniform Convergence

*Definition 1.27.* Let  $f$  be a function on  $[a, b]$ , then it is said to satisfy **Lipschitz condition** if there exists  $L > 0$  such that

$$|f(x) - f(y)| \leq L |x - y|$$

for all  $x, y \in [a, b]$ .

Note that Lipschitz condition is uniform since  $L$  is independent of any choice of  $x, y$ . Also, if  $f$  satisfies a Lipschitz condition,  $f$  is Lipschitz continuous at every point on  $[a, b]$ .

*Proposition 1.28.* Let  $f$  be a  $2\pi$ -periodic function satisfying a Lipschitz condition, then its Fourier series converge uniformly to  $f$  itself.

## 1.5 Weierstrass Approximation Theorem

### 1.5.1 Piecewise Linear Functions

Recall that a continuous function is piecewise linear if there exists a partition such that the function is linear within each subinterval.

*Proposition 1.29.* Let  $f$  be a continuous function on  $[a, b]$ , then for all  $\epsilon > 0$ , there exists a continuous and piecewise linear function  $g$  with  $g(a) = f(a)$ ,  $g(b) = f(b)$  such that

$$\|f - g\|_{\infty} < \epsilon$$

where

$$\|f - g\|_{\infty} = \sup_{[a,b]} |f(x) - g(x)|$$

### 1.5.2 Trigonometric Polynomials

*Definition 1.30.* A **trigonometric polynomial** is of the form  $P(\cos(x), \sin(x))$  where  $P(x, y)$  is a polynomial of 2 variables.

Note that a trigonometric polynomial is a finite Fourier series, and vice versa.

*Proposition 1.31.* Let  $f$  be a continuous function on  $[0, \pi]$ , then for all  $\epsilon > 0$ , there exists a trigonometric polynomial  $h$  such that  $\|f - h\|_{\infty} < \epsilon$ .

### 1.5.3 General Theorem

Below is the **Weierstrass Approximation Theorem**:

*Theorem 1.32.* Let  $f \in C[a, b]$ , then for all  $\epsilon > 0$ , there exists a polynomial  $q$  such that  $\|f - q\|_\infty < \epsilon$ .

## 1.6 Mean Convergence of Fourier Series

### 1.6.1 Bracket Products

*Definition 1.33.* Let  $f, g$  be Riemann integrable functions on  $[-\pi, \pi]$ , then the **bracket product** (or  **$L^2$ -product**,  **$L^2$  inner product**) of  $f$  and  $g$  is given by

$$\langle f, g \rangle_2 = \int_{-\pi}^{\pi} f(x)g(x) \, dx$$

Note that for complex functions, the bracket product is defined by

$$\langle f, g \rangle_2 = \int_{-\pi}^{\pi} f \bar{g}$$

*Definition 1.34.* Let  $f, g$  be Riemann integrable functions on  $[-\pi, \pi]$ , then the  **$L^2$ -norm** of  $f$  is given by

$$\|f\|_2 = \sqrt{\langle f, f \rangle_2}$$

Also, the  **$L^2$ -distance** between  $f$  and  $g$  is given by  $\|f - g\|_2$ .

### 1.6.2 Mean Convergence

*Definition 1.35.* Let  $f, f_n$  be Riemann integrable functions on  $[-\pi, \pi]$ , then  $f_n \rightarrow f$  in  **$L^2$ -sense** if  $\|f_n - f\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ .

This definition brings out why such idea is called mean convergence:

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} (f_n - f)^2 \rightarrow 0$$

is actually a variation of root mean square. Note that  $L^2$ -norm and  $L^2$ -distance are not norm and distance in a strict sense since

$$\begin{cases} \|f\|_2 = 0 & \not\Rightarrow f = 0 \\ \|f - g\|_2 = 0 & \not\Rightarrow f = g \end{cases}$$

in  $R[-\pi, \pi]$ . It is only true for almost everywhere. Also, although it is not hard to show that  $f_n \rightarrow f$  uniformly implies  $\|f_n - f\|_2 \rightarrow 0$ , its converse does not hold. Consider the following counterexample:

*Example 1.36.* Let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise} \end{cases}$$

be a function, then  $\|f_n\|_2^2 = \int_{-\pi}^{\pi} f_n^2 = 1/n$ , which tends to 0 as  $n \rightarrow \infty$ . In this way,  $f_n \rightarrow 0$  in  $L^2$ -sense. However,  $f_n \not\rightarrow 0$  uniformly or even pointwisely.

## 1.7 Applications to Fourier Series

### 1.7.1 Minimizers

Consider the functions on  $[-\pi, \pi]$

$$\begin{cases} \varphi_0 = \frac{1}{\sqrt{2\pi}} \\ \varphi_n = \frac{1}{\sqrt{\pi}} \cos(nx) \\ \psi_n = \frac{1}{\sqrt{\pi}} \sin(nx) \end{cases}$$

Note that

$$\begin{cases} \langle \varphi_m, \varphi_n \rangle_2 = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ \langle \psi_m, \psi_n \rangle_2 = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ \langle \varphi_m, \psi_n \rangle_2 = 0 \quad \text{for all } m, n \end{cases}$$

therefore

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}$$

can be regarded as an orthogonal basis in  $R[-\pi, \pi]$ .

*Definition 1.37.* The  $(2n+1)$  dimensional vector subspace of  $R[-\pi, \pi]$  spanned by the first  $(2n+1)$  trigonometric functions, denoted by  $E_n$ , is defined by

$$E_n = \text{span} \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(kx), \frac{1}{\sqrt{\pi}} \sin(kx) \right\}_{k=1}^n$$

In general, if there is an orthogonal set (or orthogonal family)  $\{\phi_n\}_{n=1}^{\infty}$  in  $R[-\pi, \pi]$ , let

$$S_n = \text{span} \langle \phi_1, \phi_2, \dots, \phi_n \rangle$$

be an  $n$ -dimensional subspace spanned by the first  $n$  functions in the orthogonal set, then for any  $f \in R[-\pi, \pi]$ , the **minimization problem** is

$$\inf \{ \|f - g\|_2 \mid g \in S_n \}$$

*Proposition 1.38.* The unique minimizer of

$$\inf \{ \|f - g\|_2 \mid g \in S_n \}$$

is attained at the function

$$g = \sum_{k=1}^n \langle f, \phi_k \rangle_2 \phi_k \in S_n$$

*Proof.* Note that to minimize  $\|f - g\|_2$  is equivalent to minimize  $\|f - g\|_2^2$ . For all  $g \in S_n$ ,

$$g = \sum_{k=1}^n \beta_k \phi_k \Rightarrow \|f - g\|_2^2 = \int_{-\pi}^{\pi} \left| f - \sum_{k=1}^n \beta_k \phi_k \right|^2$$

Let  $\Phi(\beta) = \|f - g\|_2^2$ , then

$$\begin{aligned} \Phi(\beta) &= \int_{-\pi}^{\pi} \left| f - \sum_{k=1}^n \beta_k \phi_k \right|^2 \\ &= \left( \int_{-\pi}^{\pi} f^2 \right) - 2 \sum_{k=1}^{\infty} \left( \frac{\beta_k}{\sqrt{2}} \right) \left( \sqrt{2} \langle f, \phi_k \rangle_2^2 \right) + \sum_{k=1}^n \beta_k^2 \\ &\geq \left( \int_{-\pi}^{\pi} f^2 \right) - \sum_{k=1}^{\infty} \left( \frac{\beta_k^2}{2} + 2 \langle f, \phi_k \rangle_2^2 \right) + \sum_{k=1}^n \beta_k^2 \quad \text{since } 2ab \leq a^2 + b^2 \\ &= \left( \int_{-\pi}^{\pi} f^2 \right) - 2 \sum_{k=1}^n \langle f, \phi_k \rangle_2^2 + \frac{1}{2} \sum_{k=1}^n \beta_k^2 \rightarrow \infty \end{aligned}$$

as

$$\|\beta\| = \sqrt{\sum_{k=1}^n \beta_k^2} \rightarrow \infty$$

Hence  $\Phi(\beta)$  attains its minimum at some finite point  $\beta$ . By some calculus, the minimum required is given by  $\beta_k = \langle f, \phi_k \rangle_2$  for all  $1 \leq k \leq n$ .

Note that the minimizer  $g$  of  $\|f - g\|_2$  over  $S_n$  is called the **orthogonal projection** of  $f$  on  $S_n$ , denoted by  $P_n(f)$ . With the notation of orthogonal projection,

$$\text{dist}(f, S_n) = \|f, P_n(f)\|_2$$

*Corollary.* For a  $2\pi$ -periodic function  $f$  integrable on  $[-\pi, \pi]$  and  $n \geq 1$ ,  $\|f - S_n(f)\|_2 \leq \|f - g\|_2$  where  $S_n(f)$  represents the  $n$ -th partial sum of the Fourier series of  $f$ , for all

$$g = \alpha_0 + \sum_{k=1}^n (\alpha_k \cos(kx) + \beta_k \sin(kx))$$

with real coefficients.

*Proof.* By the definition of Fourier coefficients  $S_n(f) = P_n(f)$  of  $E_n$ ,

$$\begin{cases} a_0 = \frac{1}{\sqrt{2\pi}} \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle_2 \\ a_n \cos(nx) = \frac{1}{\sqrt{\pi}} \left\langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \right\rangle_2 \cos(nx) \\ b_n \sin(nx) = \frac{1}{\sqrt{\pi}} \left\langle f, \frac{1}{\sqrt{\pi}} \sin(nx) \right\rangle_2 \sin(nx) \end{cases}$$

### 1.7.2 Measure Zeroes of Fourier Series

*Theorem 1.39.* Let  $f$  be  $2\pi$ -periodic integrable function on  $[-\pi, \pi]$ , then the  $n$ -th partial sum of the Fourier series of  $f$  converges to  $f$  in  $L^2$ -sense. In other words,

$$\lim_{n \rightarrow \infty} \|S_n(f) - f\|_2 = 0$$

*Proof.* For any  $\epsilon > 0$ , there exists a  $2\pi$ -periodic Lipschitz continuous function  $g$  such that  $\|f - g\|_2 < \epsilon/2$ . This can be achieved by finding a step function approximating  $f$ . By *Proposition 1.28*, there exists  $N > 0$  such that

$$\|g - S_N(g)\|_\infty < \frac{\epsilon}{2\sqrt{2\pi}}$$

where  $\|\cdot\|_\infty$  represents the uniform convergence. This induces

$$\|g - S_N(g)\|_2 = \sqrt{\int_{-\pi}^{\pi} (g - S_N(g))^2} \leq \sqrt{2\pi \|g - S_N(g)\|_\infty^2} = \frac{\epsilon}{2}$$

By the corollary of *Proposition 1.38*,

$$\|f - S_N(f)\|_2 \leq \|f - S_N(g)\|_2 \leq \|f - g\|_2 + \|g - S_N(g)\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Finally, since  $E_N \subset E_n$  for all  $n \geq N$ ,

$$\|f - S_n(f)\|_2 \leq \|f - S_N(f)\|_2 < \epsilon$$

for any  $n \geq N$ , thus

$$\lim_{n \rightarrow \infty} \|S_n(f) - f\|_2 = 0$$

*Corollary.* Let  $f_1$  and  $f_2$  be  $2\pi$ -periodic integrable functions on  $[-\pi, \pi]$  with the same Fourier series, then  $f_1 = f_2$  almost everywhere, or  $f_1 = f_2$  except a set of measure zero. Furthermore, if  $f_1$  and  $f_2$  are both continuous on  $[-\pi, \pi]$ , then  $f_1 = f_2$ .

*Proof.* Let  $f = f_1 - f_2$ , then  $a_n(f) = b_n(f) = 0$  gives  $S_n(f) = 0$  for any  $n \geq 0$ . Therefore

$$\lim_{n \rightarrow \infty} \|S_n(f) - f\|_2 = 0 \Rightarrow \|f\|_2 = 0$$

and by theory of Riemann integrals,  $f = 0$  almost everywhere. If  $f_1, f_2$  are continuous,  $f^2_{\text{cts}} \geq 0 \Rightarrow f^2 \equiv 0$ .

Note that a set  $E$  is said to be measure zero if for any  $\epsilon > 0$ , there exists countably

many intervals  $\{I_k\}$  such that  $E \subset \bigcup_k I_k$  and  $\sum_k |I_k| < \epsilon$ .

### 1.7.3 Parseval's Identity

Below is the **Parseval's Identity**:

*Proposition 1.40.* Let  $f$  be a  $2\pi$ -periodic function  $f$  integrable on  $[-\pi, \pi]$ , then

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where  $a_0, a_n, b_n$  are Fourier coefficients of  $f$ .

*Proof.* Note that

$$\begin{cases} \sqrt{2\pi}a_0 = \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle_2 \\ \sqrt{\pi}a_n = \left\langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \right\rangle_2 & \text{for all } n \geq 1 \\ \sqrt{\pi}b_n = \left\langle f, \frac{1}{\sqrt{\pi}} \sin(nx) \right\rangle_2 & \text{for all } n \geq 1 \end{cases}$$

then by corollary of *Proposition 1.38*,

$$\begin{aligned} \langle f, S_N(f) \rangle_2 &= \langle (f - S_N(f)) + S_N(f), S_N(f) \rangle_2 \\ &= \langle S_N(f), S_N(f) \rangle_2 \quad \text{since } (f - S_N(f)) \text{ is orthogonal} \\ &= \int_{-\pi}^{\pi} \left( a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \right)^2 dx \end{aligned}$$

Finally by *Theorem 1.39*,

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} \|f - S_N(f)\|_2^2 \\ &= \lim_{N \rightarrow \infty} (\|f\|_2^2 - 2\langle f, S_N(f) \rangle_2 + \|S_N(f)\|_2^2) \\ &= \lim_{N \rightarrow \infty} (\|f\|_2^2 - 2\|S_N(f)\|_2^2 + \|S_N(f)\|_2^2) \\ &= \lim_{N \rightarrow \infty} (\|f\|_2^2 - \|S_N(f)\|_2^2) \end{aligned}$$

therefore

$$\|f\|_2^2 = \lim_{N \rightarrow \infty} \left( 2\pi a_0^2 + \pi \sum_{n=1}^N (a_n^2 + b_n^2) \right)$$

## 2 Metric Spaces

### 2.1 Introduction of Metric Spaces

#### 2.1.1 Definition of Metrics Spaces

*Definition 2.1.* Let  $X$  be a nonempty set, then a **metric** on  $X$  is a function

$$d : X \times X \rightarrow [0, +\infty)$$

such that the following properties is satisfied for all  $x, y, z \in X$ :

- (a) Metric is nonnegative, or  $d(x, y) \geq 0$ . Equality holds if and only if  $x = y$ .
- (b) Metric is symmetric, or  $d(x, y) = d(y, x)$ .
- (c) Metric is subadditive (satisfies triangle inequality), or  $d(x, y) \leq d(x, z) + d(z, y)$ .

The pair  $(X, d)$  is called a **metric space**.

*Definition 2.2.* Let  $(X, d)$  be a metric space, then the **metric ball** of radius  $r$  centered at  $x$ , denoted by  $B_r(x)$ , is defined as

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

#### 2.1.2 Examples of Metric Spaces

*Example 2.3.* Below are some examples of metric spaces:

- (a)  $(\mathbb{R}, |x - y|)$  is a metric space.
- (b) Let  $X = \mathbb{R}^n$ . Denote the metrics

$$\begin{cases} d_k(x, y) = \sqrt[k]{\sum |x_i - y_i|^k} \\ d_\infty(x, y) = \max |x_i - y_i| \end{cases}$$

where  $1 \leq i \leq n$ , then  $(\mathbb{R}^n, d_1)$ ,  $(\mathbb{R}^n, d_2)$ ,  $(\mathbb{R}^n, d_\infty)$  are metric spaces.

- (c) Let  $C[a, b]$  be the set of all (real) continuous functions on  $[a, b]$  and

$$\begin{cases} d_k(f, g) = \sqrt[k]{\int_a^b |f - g|^k} \\ d_\infty(f, g) = \max \{|f(x) - g(x)| \mid x \in [a, b]\} \end{cases}$$



for all  $f, g \in C[a, b]$ , then  $(C[a, b], d_1)$ ,  $(C[a, b], d_2)$ ,  $(C[a, b], d_\infty)$  are metric spaces.

*Example 2.4.* Let  $X = R[a, b]$  be the set of Riemann integrable functions on  $[a, b]$  and

$$d_1(f, g) = \int_a^b |f - g|$$

However, part (a) of *Definition 2.1* is not satisfied since  $d_1(f, g) = 0$  only implies  $f = g$  almost everywhere, but not exactly  $f = g$ .  $d_1$  is then not a suitable metric on  $R[a, b]$ .

In order to fix this problem, consider  $X = R[a, b] / \sim$  where  $\sim$  is an equivalent relation on  $R[a, b]$  defined by

$$f \sim g \Leftrightarrow f = g \text{ almost everywhere}$$

Denote

$$\bar{f} = \{g \in R[a, b] \mid f \sim g\}$$

and its corresponding metric

$$\tilde{d}_k(\bar{f}, \bar{g}) = d_k(f, g)$$

then  $(X, \tilde{d}_1)$  and  $(X, \tilde{d}_2)$  are metric spaces.

Note that  $\tilde{d}_2$  in the example above is in fact  $L^2$ -distance defined in the last section.

### 2.1.3 Normed Spaces

*Definition 2.5.* Let  $X$  be a nonempty set, then a **norm** on  $X$  is a function

$$\|\cdot\| : X \rightarrow [0, +\infty)$$

such that the following properties is satisfied for all  $x, y \in X$  and  $\alpha \in \mathbb{R}$ :

- (a) Norm is nonnegative, or  $\|x\| \geq 0$ . Equality holds if and only if  $x = 0$ .
- (b) Norm is absolutely scalable, or  $\|\alpha x\| = |\alpha| \|x\|$ .
- (c) Norm is subadditive, or  $\|x + y\| \leq \|x\| + \|y\|$ .

The pair  $(X, \|\cdot\|)$  is called a **normed space**. Furthermore, a metric  $d$  is said to be **induced** by the norm  $\|\cdot\|$  if  $d(x, y) = \|x - y\|$ .

*Example 2.6.* Below are some examples of norms:

- (a) Let  $\|x\|_k = \sqrt[k]{\sum |x_i|^k}$  and  $\|x\|_\infty = \max\{x_i\}$ , then  $\|\cdot\|_1$ ,  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are norms on  $\mathbb{R}^n$ .
- (b) Let  $\|f\|_k = \sqrt[k]{\int_a^b |f|^k}$  and  $\|f\|_\infty = \max\{f(x)\}$ , then  $\|\cdot\|_1$ ,  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are norms on  $C[a, b]$ .

Note that a norm can induce a metric, but not all metrics are induced from norm.

*Example 2.7.* Let  $X$  be a nonempty set and

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

be a metric on  $X$ . Note that  $X$  is not necessary a vector space, so  $d$  is not induced by a norm. Moreover, even  $X$  is a vector space,

$$d(\alpha x, \alpha y) \neq |\alpha| d(x, y)$$

when  $|\alpha| \neq 1$  and  $x \neq y$ .

Such metric  $d$  in the above example is called a **discrete metric** on  $X$ .

### 2.1.4 Metric Subspaces

*Definition 2.8.* Let  $(X, d)$  be a metric space, then for any nonempty set  $Y \subset X$ ,  $(Y, d)$  is called a **metric subspace** of  $(X, d)$ .

Note that a metric subspace of a normed space may not be also a normed space, only if the subset is also a vector subspace.

## 2.2 Limits and Continuity

### 2.2.1 Limits and Convergence in Metric Spaces

With the understanding of metric spaces, one can extend the definition of limits and convergence to any metric space:

*Definition 2.9.* Let  $\{x_n\}$  be a sequence in a metric space  $(X, d)$ , then the sequence is said to be **converge** to  $x \in X$ , denoted by  $x_n \rightarrow x$ , if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

*Proposition 2.10.* Let  $\{x_n\}$  be a sequence in a metric space  $(X, d)$ . If  $x_n \rightarrow x$  and  $x_n \rightarrow y$ , then  $x = y$ .

*Example 2.11.* Below are some examples on convergence in metric spaces:

- (a) Convergence in  $(\mathbb{R}^n, d_2)$  is the usual convergence in advanced calculus.
- (b) Convergence in  $(C[a, b], d_\infty)$  is the uniform convergence of sequence of functions in  $C[a, b]$ .

### 2.2.2 Strength of Convergence

There are many metrics suitable for the same nonempty set  $X$ , so it is natural to think of comparing among those metrics.

*Definition 2.12.* Let  $d$  and  $\rho$  be different metrics defined on  $X$ , then  $\rho$  is said to be **stronger than**  $d$  (or  $d$  is **weaker than**  $\rho$ ) if there exists a constant  $C > 0$  such that

$$d(x, y) \leq C\rho(x, y)$$

for all  $x, y \in X$ .  $d$  and  $\rho$  are **equivalent** to each other if  $d$  is stronger and weaker than  $\rho$  at the same time. In other words, there exists  $C_1, C_2 > 0$  such that

$$d(x, y) \leq C_1\rho(x, y) \leq C_2d(x, y)$$

for all  $x, y \in X$ .

Note that the equivalence of metrics defined above is an equivalent relation.

*Proposition 2.13.* Let  $d$  and  $\rho$  be different metrics defined on  $X$ . If  $\rho$  is stronger than  $d$  and a sequence  $\{x_n\}$  converges in  $(X, \rho)$ , then the sequence also converges in  $(X, d)$  with the same limit. If  $\rho$  is equivalent to  $d$ , then  $\{x_n\}$  converges in  $(X, \rho)$  if and only if it converges in  $(X, d)$  also.

*Example 2.14.* Recall the metrics  $d_1$ ,  $d_2$  and  $d_\infty$  on  $\mathbb{R}^n$ , then

$$\begin{cases} d_1(x, y) \leq nd_\infty(x, y) \leq nd_1(x, y) \\ d_2(x, y) \leq \sqrt{n}d_\infty(x, y) \leq \sqrt{n}d_2(x, y) \end{cases}$$

shows that  $d_1$ ,  $d_2$  and  $d_\infty$  are equivalent metrics.

*Example 2.15.* Recall the metrics  $d_1$  and  $d_\infty$  on  $C[a, b]$ , then

$$d_1(f, g) \leq (b - a)d_\infty(f, g)$$

shows that  $d_\infty$  is stronger than  $d_1$ . However,  $d_1$  is not stronger than  $d_\infty$ , so  $d_1$  and  $d_\infty$  are not equivalent.

### 2.2.3 Continuity in Metric Spaces

*Definition 2.16.* Let  $f : (X, d) \rightarrow (Y, \rho)$  be a mapping between two metric spaces, then  $f$  is **continuous** at a point  $x \in X$  if  $f(x_n) \rightarrow f(x)$  in  $(Y, \rho)$  whenever  $x_n \rightarrow x$  in  $(X, d)$ .  $f$  is continuous on a set  $E \in X$  if it is continuous at every point in  $E$ .

*Proposition 2.17.* Let  $f : (X, d) \rightarrow (Y, \rho)$  be a mapping between two metric spaces and  $x_0 \in X$  be a point, then  $f$  is continuous at  $x_0$  if and only if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\rho(f(x), f(x_0)) < \epsilon \quad \text{for all } \{x \in X \mid d(x, x_0) < \delta\}$$

*Proposition 2.18.* Let  $f : (X, d) \rightarrow (Y, \rho)$  and  $g : (Y, \rho) \rightarrow (Z, m)$  be mappings between metric spaces, then if  $f$  is continuous at  $x$  and  $g$  is continuous at  $f(x)$ , then  $g \circ f$  is also continuous at  $x$ . Similarly, if  $f$  is continuous at  $X$  and  $g$  is continuous at  $Y$ , then  $g \circ f$  is also continuous at  $X$ .

*Example 2.19.* Let  $(X, d)$  be a metric space and  $A \subset X$  be a nonempty set. Further define  $\rho_A : X \rightarrow \mathbb{R}$  by

$$\rho_A(x) = \inf_{y \in A} d(y, x)$$

which is the shortest distance from  $x$  to the subset  $A$ . Show that

$$|\rho_A(x) - \rho_A(y)| \leq d(x, y)$$

for any  $x, y \in X$ .

*Answer.* For fixed  $x, y \in X$ , along with the definition of  $\rho_A$ , for all  $\epsilon > 0$ , there exists  $z \in A$  such that  $\rho_A(y) + \epsilon > d(z, y)$ . Hence

$$\rho_A(x) \leq d(z, x) \leq d(z, y) + d(y, x) < d(y, x) + \rho_A(y) + \epsilon$$

rearranging the equation gives

$$\rho_A(x) - \rho_A(y) < d(x, y) + \epsilon$$

Note that since  $x$  and  $y$  are interchangeable, and  $\epsilon$  is arbitrary,

$$|\rho_A(x) - \rho_A(y)| \leq d(x, y)$$

In fact the example above shows that  $\rho_A$  is continuous (and even Lipschitz continuous) since  $d(x_n, x) \rightarrow 0$  implies  $\rho_A(x_n) \rightarrow \rho_A(x)$ . This actually means there are many continuous functions on a metric space.

For simplicity, define

$$\begin{cases} d(x, F) = \inf \{d(x, y) \mid y \in F\} \\ d(E, F) = \inf \{d(x, y) \mid x \in E, y \in F\} \end{cases}$$

for subsets  $E$  and  $F$ .

## 2.3 Open and Closed Sets

### 2.3.1 Open Sets

*Definition 2.20.* Let  $(X, d)$  be a metric space, then a set  $G \subseteq X$  is called an **open set** if for any  $x \in G$ , there exists  $\epsilon > 0$  such that

$$B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\} \subset G$$

Note that  $\epsilon$  may vary depending on the choice of  $x$ , and the empty set  $\emptyset$  is considered an open set. Therefore, the proposition applies:

*Proposition 2.21.* Let  $(X, d)$  be a metric space and  $G_\alpha$  be a collection of open sets, then the following are true:

- (a)  $X$  and  $\emptyset$  are open sets.
- (b) Arbitrary union of open sets  $\bigcup_\alpha G_\alpha$  is an open set.

- (c) Finite intersection of open sets  $\bigcap_{i=1}^n G_i$  is an open set.

### 2.3.2 Closed Sets

*Definition 2.22.* Let  $(X, d)$  be a metric space, then a set  $F \subset X$  is called an **closed set** if  $X \setminus F$  is an open set.

*Proposition 2.23.* Let  $(X, d)$  be a metric space and  $F_\alpha$  be a collection of closed sets, then the following are true:

- (a)  $X$  and  $\emptyset$  are closed sets.
- (b) Finite union of closed sets  $\bigcup_{j=1}^n F_j$  is a closed set.
- (c) Arbitrary intersection of closed sets  $\bigcap_\alpha F_\alpha$  is a closed set.

*Corollary.* Let  $(X, d)$  be a metric space, then  $X$  and  $\emptyset$  are both open and closed.

### 2.3.3 Applications of Open and Closed Sets

*Proposition 2.24.* Let  $(X, d)$  be a metric space, then a sequence  $\{x_n\}$  converges to  $x$  if and only if for all open set  $G$  containing  $x$ , there exists  $n_0$  such that  $x_n \in G$  for all  $n \geq n_0$ .

*Proposition 2.25.* Let  $(X, d)$  be a metric space, then a set  $A \subset X$  is closed if and only if whenever  $\{x_n\} \subset A$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$  implies that  $x \in A$ .

*Proposition 2.26.* Let  $f : (X, d) \rightarrow (Y, \rho)$  be a mapping between metric spaces, then the following applies:

- (a)  $f$  is continuous at  $x$  if and only if for all open set  $G \subset Y$  containing  $f(x)$ ,  $f^{-1}(G)$  contains  $B_\epsilon(x)$  for some  $\epsilon > 0$ .
- (b)  $f$  is continuous at  $x$  if and only if for all open set  $G \subset Y$ ,  $f^{-1}(G)$  is open in  $X$ .

In this case,  $f$  is also continuous at  $x$  if and only if for all closed set  $F \subset Y$ ,  $f^{-1}(F)$  is closed in  $X$ .

## 2.4 Points in Metric Space

### 2.4.1 Boundary Points and Closures

*Definition 2.27.* Let  $E$  be a set in a metric space  $(X, d)$ , then a point  $x \in X$  (which is not necessary in  $E$ ) is called a **boundary point** of  $E$  if for all open set  $G \subset X$  containing  $x$ ,

$$G \cap E \neq \emptyset \text{ and } G \setminus E \neq \emptyset$$

In other words, this is satisfied when  $G \cap (X \setminus E) \neq \emptyset$ . The **boundary** of  $E$ , denoted by  $\partial E$ , is the set of boundary points of  $E$ . The **closure** of  $E$ , denoted by  $\overline{E}$ , is defined as  $\overline{E} = E \cup \partial E$ .

For the conditions of a boundary point in the definition above, it suffices to check  $G$  of the form  $B_\epsilon(x)$  for all small  $\epsilon > 0$ , or even  $B_{1/n}(x)$  for all  $n \geq 1$ . Also note that  $X$  and  $X \setminus E$  shares the same boundary no matter the choice of  $E$ , or

$$\partial E = \partial X \setminus E \quad \text{for all } E \subset X$$

### 2.4.2 Properties of Boundaries and Closures

*Proposition 2.28.* Below are some properties of boundaries and closures:

- (a) The boundary of an empty set is an empty set, or  $\partial \emptyset = \emptyset$ .
- (b) For all  $E \subset X$ ,  $\partial E$  is a closed set.
- (c) If  $E$  is a closed set,  $\overline{E} = E$ .

*Proposition 2.29.* Let  $E$  be a subset of a metric space  $(X, d)$ , then the following applies:

- (a)  $x \in \overline{E}$  if and only if  $B_r(x) \cap E \neq \emptyset$  for all  $r > 0$ .
- (b) If  $A \subset B$ ,  $\overline{A} \subset \overline{B}$  for all  $A, B \subset (X, d)$ .
- (c)  $\overline{E}$  is closed.
- (d)  $\overline{E}$  is the smallest closed set containing  $E$ , or  $\overline{E} = \bigcap \{G \subset X \mid G \text{ is closed and } E \subset G\}$ .

### 2.4.3 Interior Points

*Definition 2.30.* Let  $E$  be a subset of a metric space  $(X, d)$ , then a point  $x$  is called an **interior point** of  $E$  if there exists an open set  $G$  such that  $x \in G$  and  $G \subset E$ . The **interior** of  $E$ , denoted by  $E^0$ , is the set of interior points of  $E$ .

*Proposition 2.31.* Below are the properties of interiors:

- (a) Interior of  $E$ ,  $E^0$  is open.
- (b) Interior of  $E$  is the set without boundary, or  $E^0 = E \setminus \partial E$ .
- (c) Interior of  $E$ ,  $E^0 = X \setminus \overline{X \setminus E}$ .
- (d) Interior of  $E$ ,  $E^0 = \cup \{G \subset E \mid G \text{ is open}\}$ .

## 2.5 Elementary Inequalities for Functions

### 2.5.1 Young's Inequality

*Theorem 2.32.* By **Young's Inequality**, for  $a, b > 0$  and  $p > 1$ ,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1$$

Equality holds when  $a^p = b^q$ .

Note that  $q = \frac{p}{p-1} > 1$  is called the **conjugate** of  $p$ . Also specifically if  $p = 2$ , the inequality reduces to  $2ab \leq a^2 + b^2$ .

### 2.5.2 Holder's Inequality

For the following inequality, denote the norm

$$\|f\|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}$$

*Theorem 2.33.* Let  $f, g \in R[a, b]$  be Riemann integrable functions and  $p > 1$ , then by **Holder's Inequality**,

$$\int_a^b |f(x)g(x)| \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \left( \int_a^b |f(x)|^q \, dx \right)^{1/q}$$

where  $q$  is the conjugate of  $p$ . Equality holds when one of the following conditions is satisfied:

- (a)  $f$  or  $g$  equals to 0 almost everywhere.
- (b) There exists a constant  $\lambda > 0$  such that  $|g(x)|^q = \lambda |f(x)|^p$  almost everywhere.

Note that Holder's Inequality can be written in norm form  $\|fg\|_1 = \|f\|_p \|g\|_q$ . Holder's Inequality also holds for limiting cases  $(p, q) \rightarrow (1, \infty)$  and  $(p, q) \rightarrow (\infty, 1)$ .



### 2.5.3 Minkowski's Inequality

*Theorem 2.34.* By **Minkowski's Inequality**, for any  $f, g \in R[a, b]$  and  $p > 1$ ,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Equality holds when one of the following conditions is satisfied:

- (a)  $f$  or  $g$  equals to 0 almost everywhere.
- (b)  $\|f\|_p, \|g\|_p > 0$  and there exists a constant  $\lambda > 0$  such that  $g(x) = \lambda f(x)$  almost everywhere.

### 3 Contraction Mapping Principle

#### 3.1 Complete Metric Space

##### 3.1.1 Definition of Complete Metric Space

*Definition 3.1.* Let  $(X, d)$  be a metric space, then a sequence  $\{x_n\}$  in  $(X, d)$  is a **Cauchy sequence** if for any  $\epsilon > 0$ , there exists  $n_0$  such that  $d(x_n, x_m) < \epsilon$  for all  $n, m > n_0$ .

*Definition 3.2.* Let  $(X, d)$  be a metric space, then the metric space is **complete** if every Cauchy sequence in the metric space converges. A subset  $E$  is complete if the induced metric subspace  $(E, d)$  with  $d = d|_{E \times E}$  is complete. In other words, every Cauchy sequence in  $E$  converges with limit in  $E$ .

Note that convergent sequence is a Cauchy sequence.

*Proposition 3.3.* Let  $(X, d)$  be a metric space, then the following applies:

- (a) Every complete set in  $X$  is closed.
- (b) If  $X$  is complete, then every closed set in  $X$  is complete.

*Example 3.4.* Below are the examples of complete metric space:

- (a)  $(\mathbb{R}, \text{standard})$  is complete.
- (b)  $[a, b]$ ,  $(-\infty, b]$  and  $[a, \infty)$  are complete.

Below are the counterexamples of complete metric space:

- (a)  $[a, b)$  where  $b$  is finite, is not complete since  $x_n = b - 1/n \rightarrow b \notin [a, b)$ .
- (b)  $\mathbb{Q}$  is not complete.

##### 3.1.2 Completion of Metric Spaces

*Definition 3.5.* A metric space  $(X, d)$  is said to be **isometrically embedded** in metric space  $(Y, \rho)$  if there exists a mapping  $\Phi : X \rightarrow Y$  such that  $d(x, y) = \rho(\Phi(x), \Phi(y))$ . If such mapping exists,  $\Phi$  is called an **isometric embedding** (or a **metric preserving map**) from  $(X, d)$  to  $(Y, \rho)$ .

Note that  $\Phi$  must be injective and continuous.

*Definition 3.6.* Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, then  $(Y, \rho)$  is called a completion of  $(X, d)$  if the following statements are satisfied:

- (a)  $(Y, \rho)$  is complete.

(b) There exists an isometric embedding  $\Phi$  such that the closure  $\overline{\Phi(X)} = Y$ .

*Example 3.7.* Let  $(X, d) = (\mathbb{Q}, \text{induced metric})$  and  $(Y, \rho) = (\mathbb{R}, \text{standard})$ . Since  $\mathbb{Q} \subset \mathbb{R}$ ,  $(Y, \rho)$  is complete. Further let  $\Phi : (X, d) \rightarrow (Y, \rho)$  where  $\Phi(q) = q$ , since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ,  $\overline{\Phi(\mathbb{Q})} = \overline{\mathbb{Q}} = \mathbb{R}$ . Therefore,  $(Y, \rho)$  is a completion of  $(X, d)$ .

*Theorem 3.8.* Every metric space has a completion.

Note that the definition of isometric embedding can be extended to bijection as below:

*Definition 3.9.* Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, then they are called **isometric** if there exists a bijective isometric embedding between  $(X, d)$  and  $(Y, \rho)$ .

Note that the inverse of a bijective isometric embedding is also an isometric embedding. Also, the two metric spaces are regarded as the same if they are isometric.

*Theorem 3.10.* If metric spaces  $(Y, \rho)$  and  $(Y', \rho')$  are both completions of a metric space  $(X, d)$ , then  $(Y, \rho)$  and  $(Y', \rho')$  are isometric. In other words, completion is unique up to isometry.

## 3.2 Introduction to Contraction Mapping Principle

### 3.2.1 General Theorem

*Definition 3.11.* Let  $(X, d)$  be a metric space, then a map  $T : (X, d) \rightarrow (X, d)$  is called a **contraction** if there exists a constant  $\gamma \in (0, 1)$  such that

$$d(Tx, Ty) \leq \gamma d(x, y)$$

for all  $x, y \in X$ . A point  $x \in X$  is called a **fixed point** of  $T$  if  $Tx = x$ .

Note that  $Tx$  is the notation for  $T(x)$  but not the multiplication. With the definition of a contraction, below is the **Contraction Mapping Principle** (or the **Banach Fixed Point Theorem**):

*Theorem 3.12.* Every contraction in a complete metric space admit a fixed point.

### 3.2.2 Perturbation of Identity

*Definition 3.13.* A normed space  $(X, \|\cdot\|)$  is a **Banach space** if it is complete as a metric space with respect to the induced metric  $d(x, y) = \|x - y\|$  for all  $x, y \in X$ .

*Example 3.14.* Below are some examples of Banach space:

- (a)  $(\mathbb{R}^n, \|\cdot\|_p)$  is a Banach space if  $p > 1$ .
- (b)  $(C[a, b], \|\cdot\|_\infty)$  is a Banach space.

*Theorem 3.15.* Let  $(X, \|\cdot\|)$  be a Banach space, and  $\Phi : \overline{B_r(x_0)} \rightarrow X$  satisfies  $\Phi(x_0) = y_0$ . Suppose that  $\Phi = \text{Id}_X + \Psi$  such that there exists a constant  $\gamma \in (0, 1)$  such that

$$\|\Psi(x_2) - \Psi(x_1)\| \leq \gamma \|x_2 - x_1\|$$

for all  $x_1, x_2 \in \overline{B_r(x_0)}$ , then by **Perturbation of Identity**, for all  $y \in \overline{B_R(y_0)}$  where  $R = (1 - \gamma)r$ , there exists unique  $x \in \overline{B_r(x_0)}$  such that  $\Phi(x) = y$ .

*Example 3.16.* Show that  $3x^4 - x^2 + x = -0.05$  has a real root.

*Proof.* Notice that  $3x^4 - x^2 + x = 0$  has a root  $x = 0$ . Let  $\Phi(x) = x + \Psi(x)$  where  $\Psi(x) = 3x^4 - x^2$ , then  $\Phi(0) = 0$ . For  $x_1, x_2 \in \overline{B_r(0)}$ ,

$$\begin{aligned} |\Psi(x_1) - \Psi(x_2)| &= |3x_1^4 - x_1^2 - 3x_2^4 + x_2^2| \\ &= |3(x_1^4 - x_2^4) - (x_1^2 - x_2^2)| \\ &= |3(x_1^3 + x_1^2x_2 + x_2^2x_1 + x_2^3) - (x_1 + x_2)| |x_1 - x_2| \\ &= |12r^3 + 2r| |x_1 - x_2| \end{aligned}$$

Choose  $r > 0$  such that  $\gamma = 12r^3 + 2r < 1$  and  $R = (1 - \gamma)r \geq 0.05$  so that  $-0.05 \in \overline{B_R(0)}$ . Pick  $r = 1/4$ , then  $\gamma = 11/16$  and  $R = 5/64$ . By Perturbation of Identity, for all  $y \in \overline{B_R(0)}$ , there exists  $x \in \overline{B_r(0)}$  such that  $\Phi(x) = y$ . Therefore, there exists a real root for  $3x^4 - x^2 + x = -0.05$  since  $-0.05 \in \overline{B_R(0)}$ .

The example above can be summarized into the following proposition:

*Proposition 3.17.* Let  $\Phi(x) = x + \Psi(x)$  where  $\Psi(x) : U \rightarrow \mathbb{R}^n$  be a  $C^1$ -function on some open set  $U \subset \mathbb{R}^n$  containing 0, such that

$$\Psi(0) = 0 \text{ and } \lim_{x \rightarrow 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0$$

for all  $i, j$ , then there exists  $r > 0$  and  $R > 0$  such that for all  $y \in B_R(0)$ ,  $\Phi(x) = y$  has a unique solution  $x \in B_r(0)$ .

## 3.3 Inverse Function Theorem

### 3.3.1 Introduction to Inverse Function Theorem

Recall the chain rule: let  $G : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $F : V \subset \mathbb{R}^m \rightarrow \mathbb{R}^l$  be differentiable functions where  $U, V$  open in  $\mathbb{R}^n, \mathbb{R}^m$  respectively, and  $G(U) \subset V$ . Then  $H = F \circ G :$

$U \rightarrow \mathbb{R}^l$  differentiable and  $DH(x) = DF(G(x))DG(x)$  where

$$DG(x) = \left( \frac{\partial G_i}{\partial x_j}(x) \right)_{i,j}$$

and similarly for  $DF$  and  $DH$ . Besides the proposition is required:

**Proposition 3.18.** Let  $F : B \rightarrow \mathbb{R}^n$  be  $C^1$  function, where  $B$  is a ball in  $\mathbb{R}^n$ , then for any  $x_1, x_2 \in B$ ,

$$F(x_1) - F(x_2) = \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) dt \right) \cdot (x_1 - x_2)$$

in component form  $F = (F_1, \dots, F_n)$ . In other words,

$$F_i(x_1) - F_i(x_2) = \sum_{j=1}^n \left( \int_0^1 \frac{\partial F_i}{\partial x_j}(x_2 + t(x_1 - x_2)) dt \right) (x_1 - x_2)_j$$

Finally, recall that if  $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at a point  $p$  in an open set  $U$  of  $\mathbb{R}^n$ , then

$$F(p+x) - F(p) = DF(p)x + o(|x|)$$

for all  $x = (x_1, \dots, x_n)$  sufficiently small (or  $|x|$  small) where  $o(|x|)$  is a remaining term such that

$$\frac{o(|x|)}{|x|} \rightarrow 0 \text{ as } |x| \rightarrow 0$$

**Definition 3.19.** The condition in Inverse Function Theorem that  $DF(x_0)$  is invertible is called the **nondegeneracy condition**.

Note that nondegeneracy condition is necessary for the differentiability of local inverse.

**Proposition 3.20.** Let  $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  function where  $U$  is an open set and  $x_0 \in U$ . Suppose there exists open  $V$  such that  $x_0 \in V \subset U$  and  $F|_V$  has a differentiable inverse, then  $DF(x_0)$  is nonsingular (or invertible).

Below is the **Inverse Function Theorem**:

**Theorem 3.21.** Let  $F : U \rightarrow \mathbb{R}^n$  be a  $C^1$  map from an open set  $U \rightarrow \mathbb{R}^n$ . Suppose  $x_0 \in U$  and  $DF(x_0)$  is invertible (as a matrix or linear transformation), then there exists open sets  $V, W$  containing  $x_0, F(x_0)$  respectively such that the restriction of  $F$  on  $V$  is a bijection onto  $W$  with a  $C^1$  inverse.

Moreover, the inverse is  $C^k$  when  $F$  is  $C^k$  where  $1 \leq k \leq \infty$ , in  $V$ .

*Proof. Part (a)* Consider the special case where  $x_0 = 0, y_0 = F(x_0) = F(0) = 0$ , then  $DF(0) = I$ , which is the identity. Let  $\Psi(x) = -x + F(x)$ . As  $0 \in U$  and  $U$  is open, there exists  $r_0 > 0$  such that  $\overline{B_{r_0}(0)} \subset U$ . Then

$$\Psi(x_1) - \Psi(x_2) = -x_1 + F(x_1) + x_2 - F(x_2)$$

By *Proposition 3.18*,

$$\begin{aligned} \Psi(x_1) - \Psi(x_2) &= \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) dt \right) \cdot (x_1 - x_2) - (x_1 - x_2) \\ &= \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) dt - I \right) \cdot (x_1 - x_2) \\ &= \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) - DF(0) dt \right) \cdot (x_1 - x_2) \end{aligned}$$

Since  $F$  is  $C^1$ , for all  $\epsilon > 0$ , there exists  $0 < r \leq r_0$  such that

$$\|DF(x) - DF(0)\| < \epsilon$$

for all  $x \in \overline{B_r(0)}$ , where

$$\|(b_{ij})\| = \sqrt{\sum_{i,j} b_{ij}^2}$$

for any  $n \times n$  matrix  $(b_{ij})$ .

Since  $\overline{B_r(0)}$  is convex,  $x_1, x_2 \in \overline{B_r(0)}$  implies  $x_2 + t(x_1 - x_2) \in \overline{B_r(0)}$ . Hence for all  $\epsilon > 0$ , there exists  $0 < r \leq r_0$  such that

$$\|DF(x_2 + t(x_1 - x_2)) - DF(0)\| < \epsilon$$

for all  $x_1, x_2 \in \overline{B_r(0)}$  and  $t \in (0, 1)$ . Therefore choosing  $\epsilon = 1/2$  gives

$$|\Psi(x_1) - \Psi(x_2)| \leq \frac{1}{2} |x_1 - x_2|$$

for all  $x_1, x_2 \in \overline{B_r(0)}$ .

*Part (b)* Choose  $r > 0$  as in Part (a), then for all  $y \in B_{r/2}(0)$ , there exists  $x \in B_r(0)$  such that  $F(x) = y$ . This is true because of Perturbation of Identity (*Theorem 3.15*) with  $\epsilon = 1/2$ . The local inverse  $G$  of  $F$ ,

$$G : B_{r/2}(0) \rightarrow G(B_{r/2}(0)) \subset B_r(0)$$

satisfies

$$|G(y_1) - G(y_2)| \leq \frac{1}{1-\epsilon} |y_1 - y_2| = 2 |y_1 - y_2|$$

for all  $y_1, y_2 \in B_{r/2}(0)$  with  $G(B_{r/2}(0))$  open in  $B_r(0)$ .

*Part (c)* Since  $DF(0) = I$ , assume that  $DF(x)$  is invertible for all  $x \in B_r(0)$  for  $r > 0$  given in Part (a). Further let  $W = B_{r/2}(0) = B_R(0)$ , and  $V = G(W) \ni 0$ , then  $G : W \rightarrow V$  (and similarly  $F : V \rightarrow W$ ). If  $G$  is differentiable, by chain rule  $DF(G(y))DG(y) = I$  for all  $y \in W$ . Rewriting the equation gives  $DG(y) = (DF)^{-1}(G(y))$ .

For any  $y_1 \in W$  such that  $y_1 + y \in W$ ,

$$y = (y_1 + y) - y_1 = F(G(y_1 + y)) - F(G(y_1))$$

let  $x_1 = G(y_1 + y)$  and  $x_2 = G(y_1)$ , then by *Proposition 3.18*,

$$\begin{aligned} y &= F(x_1) - F(x_2) = \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) dt \right) \cdot (x_1 - x_2) \\ &= \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) - DF(x_2) dt \right) \cdot (x_1 - x_2) + DF(x_2)(x_1 - x_2) \end{aligned}$$

Hence

$$\begin{aligned} (DF)^{-1}(x_2)y &= (DF)^{-1}(x_2) \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) - DF(x_2) dt \right) \cdot (x_1 - x_2) \\ &\quad + (x_1 - x_2) \end{aligned}$$

In other words,  $G(y_1 + y) - G(y_1) = (DF)^{-1}(G(y_1))y + R$  where

$$R = (DF)^{-1}(x_2) \left( \int_0^1 DF(x_2 + t(x_1 - x_2)) - DF(x_2) dt \right) \cdot (x_1 - x_2)$$

By Part (b),  $|x_1 - x_2| \leq 2|y|$ , so  $|x_1 - x_2| \rightarrow 0$  as  $|y| \rightarrow 0$  and

$$\frac{|R|}{|y|} \leq 2 \|(DF)^{-1}(x_2)\| \int_0^1 \|DF(x_2) - DF(x_2 + t(x_1 - x_2))\| dt$$

With the assumption that  $F$  is  $C^1$ ,

$$\lim_{|y| \rightarrow 0} \frac{|R|}{|y|} = 0$$

Therefore  $G(y_1+y)-G(y) = (DF)^{-1}(G(y_1))y + o(|y|)$  which implies  $G$  is differentiable at  $y_1 \in W$  and  $DG(y_1) = (DF)^{-1}(G(y_1))$ .

Finally, for the special case, it is assumed that  $DF$  is continuous and invertible on  $B_r(0)$ , then by linear algebra  $(DF)^{-1}$  is also continuous. Then  $DG(y) = (DF)^{-1}(G(y))$  is also continuous, and implies  $G$  is  $C^1$ . Using induction and differentiating the identity  $DG(y) = (DF)^{-1}(G(y))$  will finish the fact that  $F$  is  $C^k$  implies  $G$  is  $C^k$ .

### 3.3.2 Diffeomorphisms

*Definition 3.22.* Let  $F : V \rightarrow W$  be a  $C^k$  map where  $V$  and  $W$  are open sets in  $\mathbb{R}^n$ , then  $F$  is called a  **$C^k$ -diffeomorphism** if  $F^{-1}$  exists and is also  $C^k$ .

With the definition of diffeomorphisms, the Inverse Function Theorem can be rephrased as follows:

*Theorem 3.23.* Let  $F : U \rightarrow \mathbb{R}^n$  be a  $C^k$  map from an open set  $U \rightarrow \mathbb{R}^n$ . Suppose  $x_0 \in U$  and  $DF(x_0)$  is invertible (as a matrix or linear transformation), then  $F$  is a  $C^k$ -diffeomorphism between some open sets  $V, W$  of  $x_0, F(x_0)$  respectively.

Also, if  $F : V \rightarrow W$  is a  $C^k$ -diffeomorphism, then for all function  $\varphi : W \rightarrow \mathbb{R}$ , there corresponds a function  $\psi = \varphi \circ F : V \rightarrow \mathbb{R}$ . Conversely, for all function  $\psi : V \rightarrow \mathbb{R}$ , there corresponds a function  $\varphi = \psi \circ F^{-1} : W \rightarrow \mathbb{R}$ . Moreover,  $\varphi$  is  $C^k$  if and only if  $\psi$  is  $C^k$ . Thus every  $C^k$ -diffeomorphism gives rise to a **local  $C^k$ -change of coordinates**.

### 3.3.3 Examples of Inverse Function Theorem

Below are some examples about the Inverse Function Theorem:

*Example 3.24.* Let  $F : (0, \infty), (-\infty, \infty) \rightarrow \mathbb{R}^2$  such that  $F(r, \theta) = (r \cos(\theta), r \sin(\theta))$ , then

$$DF = \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix}$$

is invertible for all  $(r, \theta)$ . By the Inverse Function Theorem,  $F$  is locally invertible at every point  $(r, \theta) \in (0, \infty) \times (-\infty, \infty)$ . However,  $F$  is not globally invertible as  $F(r, \theta + 2\pi) = F(r, \theta)$  implies it is not injective.

*Example 3.25.* Let  $U$  be an open interval  $(a, b) \in \mathbb{R}$ , then a  $C^1$  function  $f : (a, b) \rightarrow \mathbb{R}$  with  $f' \neq 0$  implies  $f$  is strictly increasing or decreasing, so global inverse exists. Therefore 1-dimensional case has stronger result than higher dimensions.



### 3.3.4 Implicit Function Theorem

A theorem similar to Inverse Function Theorem is the **Implicit Function Theorem**:

*Theorem 3.26.* Let  $U$  be an open set in  $\mathbb{R}^n \times \mathbb{R}^m$ , and  $F : U \rightarrow \mathbb{R}^m$  is a  $C^1$  map. Suppose that  $(x_0, y_0) \in U$  satisfies  $F(x_0, y_0) = 0$  and  $D_y F(x_0, y_0)$  is invertible in  $\mathbb{R}^m$ , then the following applies:

- (a) There exists an open set of the form  $V_1 \times V_2 \in U$  containing  $(x_0, y_0)$  and a  $C^1$  map

$$\varphi : V_1 \subset \mathbb{R}^n \times V_2 \subset \mathbb{R}^m$$

with  $\varphi(x_0) = y_0$  such that  $F(x, \varphi(x)) = 0$  for all  $x \in V_1$ .

- (b)  $\varphi : V_1 \rightarrow V_2$  is  $C^k$  when  $F$  is  $C^k$  where  $1 \leq k \leq \infty$ .

- (c) Assume  $DF_y$  is invertible in  $V_1 \times V_2$ , then if  $\psi : V_1 \rightarrow V_2$  is another  $C^1$  map satisfying  $F(x, \psi(x)) = 0$ , then  $\psi \equiv \varphi$ .

Note that if

$$F = \begin{pmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) \end{pmatrix}$$

then

$$D_y F = \begin{pmatrix} \partial F_1 / \partial y_1 & \cdots & \partial F_1 / \partial y_m \\ \vdots & \ddots & \vdots \\ \partial F_m / \partial y_1 & \cdots & \partial F_m / \partial y_m \end{pmatrix}$$

is an  $m \times m$  matrix and can be regarded as a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^m$ . In general, for a map  $F$  such that  $DF(x_0, y_0)$  has rank  $m$ , then one can rearrange the independent variables to make the  $m \times m$  submatrix corresponding to the last  $m$  columns of the Jacobian matrix invertible, which is the situation in the theorem. Hence the condition  $DF_y(x_0, y_0)$  is invertible in the Implicit Function Theorem can be generalized to  $\text{rank} DF(x_0, y_0) = m$ .

## 3.4 Picard-Lindelof Theorem

### 3.4.1 Initial Value Problems

*Definition 3.27.* Let  $f$  be a function defined on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$$

where  $(t_0, x_0) \in \mathbb{R}^2$  and  $a, b > 0$ . An **initial value problem** (or **Cauchy problem**) is of the form

$$\begin{cases} dx/dt = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

This means one has to find  $x(t)$  defined in an interval

$$x : [t_0 - a', t_0 + a'] \rightarrow [x_0 - b, x_0 + b]$$

for some  $0 < a' \leq a$  such that  $x(t)$  is differentiable,  $x(t_0) = x_0$  and

$$\frac{dx}{dt}(t) = f(t, x(t))$$

for all  $t \in [t_0 - a', t_0 + a']$ .

*Example 3.28.* Consider the initial value problem

$$\begin{cases} dx/dt = 1 + x^2 \\ x(0) = 0 \end{cases}$$

Note that  $f(t, x) = 1 + x^2$  is smooth on  $[-a, a] \times [-b, b]$  for any  $a, b > 0$ , but the solution  $x(t) = \tan(t)$  is defined only on  $(-\pi/2, \pi/2)$ . Therefore, even for a nice function  $f$ , it is still possible that  $a' < a$ .

### 3.4.2 Picard-Lindelof Theorem for Differential Equations

Recall the definition of Lipschitz condition:

*Definition 3.29.* Let  $f$  be a function defined in  $R : [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ , then  $f$  satisfies the Lipschitz condition (uniform in  $t$ ) if there exists a Lipschitz constant  $L > 0$  such that for all  $(t, x_1), (t, x_2) \in R$ ,

$$|f(t, x_1) - f(t, x_2)| \leq L |x_1 - x_2|$$

Also recall the properties related to Lipschitz condition:

*Proposition 3.30.* The following statements are true:

- (a)  $f(t, \cdot)$  is Lipschitz continuous in  $x$  for all  $t \in [t_0 - a, t_0 + a]$ .
- (b) If  $L$  is a Lipschitz constant for  $f$ , then any  $L' > L$  is also a Lipschitz constant.
- (c) Continuity does not imply Lipschitz continuity. For example,  $f(t, x) = tx^{1/2}$  is

continuous but not Lipschitz continuous near 0.

- (d) If  $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$  and  $f(t, x) : R \rightarrow \mathbb{R}$  is  $C^1$ , then  $f(t, x)$  satisfies the Lipschitz condition. In fact, for some  $y \in [x_0 - b, x_0 + b]$ ,

$$|f(t, x_1) - f(t, x_2)| = \left| \frac{\partial f}{\partial x}(t, y)(x_2 - x_1) \right|$$

Hence  $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$  for

$$L = \max \left\{ \left| \frac{\partial f}{\partial x}(t, x) \right| \mid (t, x) \in R \right\}$$

*Proposition 3.31.* Under assumption of *Theorem 3.30*, every solution  $x$  of the initial value problem from  $[t_0 - a', t_0 + a']$  to  $[x_0 - b, x_0 + b]$  satisfies the equation

$$x(t) = x_0 + \int_{t_0}^t f(t, x(t)) dt$$

Conversely, every  $x(t) \in C[t_0 - a', t_0 + a']$  satisfying the equation above is  $C^1$  and solves the initial value problem.

*Proof.* This is a result of Fundamental Theorem of Calculus.

Below is the **Picard-Lindelof Theorem**:

*Theorem 3.32.* Let  $f$  be a continuous function on  $R : [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$  where  $(t_0, x_0) \in \mathbb{R}^2$  and  $a, b > 0$ . If  $f$  satisfies Lipschitz condition on  $R$  (uniform in  $t$ ), then there exists  $a' \in (0, a]$  and  $x \in C^1[t_0 - a', t_0 + a']$  such that

$$x_0 - b \leq x(t) \leq x_0 + b$$

for all  $t \in [t_0 - a', t_0 + a']$  and solving the initial value problem. Furthermore,  $x$  is the unique solution in  $[t_0 - a', t_0 + a']$ .

*Proof.* For  $a' > 0$  to be chosen later, let

$$X = \{\varphi \in C[t_0 - a', t_0 + a'] \mid \varphi(t_0) = x_0, \varphi(t) \in [x_0 - b, x_0 + b]\}$$

with uniform metric  $d_\infty$  on  $X$ . Note that  $X$  is a closed subset in the complete metric space  $(C[t_0 - a', t_0 + a'], d_\infty)$ , so  $(X, d_\infty)$  is complete.

Define  $T$  on  $X$  by

$$(T\varphi)(t) = x_0 + \int_{t_0}^t f(s, \varphi(s)) ds$$

Note that it is well-defined since  $\varphi(s) \in [x_0 - b, x_0 + b]$ . To show  $T\varphi \in X$ , one requires  $(T\varphi)(t) \in [x_0 - b, x_0 + b]$ . Let  $M = \sup \{|f(t, x)| \mid (t, x) \in R\}$ , then for all  $t \in [t_0 - a', t_0 + a']$ ,

$$\begin{aligned} |(T\varphi)(t) - x_0| &= \left| \int_{t_0}^t f(s, \varphi(s)) \, ds \right| \\ &\leq M |t - t_0| \leq Ma' \end{aligned}$$

Choose  $0 < a' \leq b/M$  gives  $|(T\varphi)(t) - x_0| \leq b$  and so  $T\varphi \in X$ . Notice that  $T : X \rightarrow X$  is a mapping from  $(X, d_\infty)$  to itself. For contraction,

$$\begin{aligned} |(T\varphi_1 - T\varphi_2)(t)| &= \left| (x_0 + \int_{t_0}^t f(s, \varphi_1(s)) \, ds) - (x_0 + \int_{t_0}^t f(s, \varphi_2(s)) \, ds) \right| \\ &\leq \int_{t_0}^t |f(s, \varphi_1(s)) - f(s, \varphi_2(s))| \, ds \\ &\leq L \int_{t_0}^t |\varphi_1(s) - \varphi_2(s)| \, ds \\ &\leq L |t - t_0| \sup_{[t_0 - a', t_0 + a']} \{|\varphi_1(s) - \varphi_2(s)|\} \\ &\leq La' d_\infty(\varphi_1, \varphi_2) \end{aligned}$$

Therefore if  $La' = \gamma < 1$ ,  $T$  is a contraction since  $d_\infty(T\varphi_1, T\varphi_2) \leq \gamma d_\infty(\varphi_1, \varphi_2)$ .

In conclusion, if  $0 < a' < \min \{a, b/M, 1/L\}$ , then  $T$  is a contraction on a complete metric space. By Contraction Mapping Principle,  $T$  admits a unique fixed point  $x(t) \in X$ .

Note that the existence part of Picard-Lindelof Theorem still holds with  $f(t, x)$  being continuous only. However, the solution may not be unique. Consider the following example:

*Example 3.33.* Let  $f(t, x) = |x|^{1/2}$  on  $\mathbb{R} \times \mathbb{R}$ . Note that  $f$  is continuous but not Lipschitz continuous. Then the initial value problem

$$\begin{cases} dx/dt = |x|^{1/2} \\ x(0) = 0 \end{cases}$$

has solutions

$$x_1 = 0, x_2 = \frac{1}{4} |t| t$$

for all  $t \in \mathbb{R}$ .

On the other hand, uniqueness of Picard-Lindelof Theorem holds regardless of the size of the interval of existence.

### 3.4.3 Picard-Lindelof Theorem for Systems

*Theorem 3.34.* Consider the initial value problem

$$\begin{cases} d\mathbf{x}/dt = \mathbf{f}(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in [x_1 - b, x_1 + b] \times \cdots \times [x_n - b, x_n + b]$$

$$\mathbf{x}_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{f}(t, x) = \begin{pmatrix} f_1(t, x) \\ \vdots \\ f_n(t, x) \end{pmatrix} \in C^1(R)$$

with

$$R = [t_0 - a, t_0 + a] \times [x_1 - b, x_1 + b] \times \cdots \times [x_n - b, x_n + b]$$

satisfying the Lipschitz condition (uniform in  $t$ ),

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq L |\mathbf{x} - \mathbf{y}|$$

for all  $(t, \mathbf{x}), (t, \mathbf{y}) \in R$  and some constant  $L > 0$ . There exists a unique solution  $\mathbf{x} \in C^1[t_0 - a', t_0 + a']$  with

$$\mathbf{x}(t) \in [x_1 - b, x_1 + b] \times \cdots \times [x_n - b, x_n + b]$$

for all  $t \in [t_0 - a', t_0 + a']$  to the initial value problem, where  $a'$  satisfies

$$0 < a' < \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$$

with

$$M = \max_{j=1, \dots, n} \sup_R |f_j(t, \mathbf{x})|$$

Note that the Picard-Lindelof Theorem for systems can be applied to initial value problems for higher order ordinary differential equations:

$$\begin{cases} d^m x / dt^m = f(t, x, dx/dt, \dots, d^{m-1}x/dt^{m-1}) \\ x(t_0) = x_0 \\ dx/dt(t_0) = x_1 \\ \vdots \\ d^{m-1}x/dt^{m-1} = x_{m-1} \end{cases}$$

by letting

$$\mathbf{x} = \begin{pmatrix} x \\ dx/dt \\ \vdots \\ d^{m-1}x/dt^{m-1} \end{pmatrix}$$

then

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} dx/dt \\ d^2x/dt^2 \\ \vdots \\ d^m x/dt^m \end{pmatrix} = \mathbf{f}(t, \mathbf{x})$$

with

$$\mathbf{x}(t_0) = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix}$$

## 4 Space of Continuous Functions

### 4.1 Arzela-Ascoli Theorem

#### 4.1.1 Compact Sets

*Definition 4.1.* Let  $(X, d)$  be a metric space, then the vector space of all bounded continuous functions is denoted by

$$C_b(X) = \{f \in C(X) \mid |f(x)| \leq M, \forall x \in X, \exists M\}$$

It is simple to see that  $C_b(X) \subset C(X)$ , where  $C(X)$  is the set of continuous functions on  $X$ .

*Example 4.2.* If  $G$  is a nonempty bounded open set in  $\mathbb{R}^n$ , then  $C_b(\overline{G}) = C(\overline{G})$  as  $\overline{G}$  is closed and bounded, then  $f \in C(\overline{G})$  has to be bounded.

Recall that a norm  $\|\cdot\|$  on a real vector space  $X$  is defined by the following properties:

- (a)  $\|x\| \geq 0$  is nonnegative, and  $\|x\| = 0$  if and only if  $x = 0$ .
- (b)  $\|\alpha x\| = |\alpha| \|x\|$  for all  $\alpha \in \mathbb{R}$ .
- (c) Triangle inequality holds, or  $\|x + y\| \leq \|x\| + \|y\|$

A vector space with norm  $(X, \|\cdot\|)$  is called a norm space. Note that a norm space has a natural metric  $d(x, y) = \|x - y\|$ .

*Definition 4.3.* Let  $C_b(X)$  be the vector space of all bounded continuous functions, then the **supnorm** is a norm on  $C_b(X)$  defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

It is always assumed  $C_b(X)$  with metric  $d_\infty(f, g) = \|f - g\|_\infty$  given by the supnorm.

*Proposition 4.4.*  $(C_b(X), d_\infty)$  is a complete metric space, for any metric space  $(X, d)$ .

Note that  $(C_b(X), d_\infty)$  is a Banach space since it is a complete normed vector space.  $C_b(X)$  is usually of infinite dimensional (for example  $X = \mathbb{R}^n$  or a subset with nonempty interior in  $\mathbb{R}^n$  like  $X = [0, 1]$ ), but it also could be of finite dimensional (for example  $X = \{p_1, \dots, p_n\}$  as a finite set of discrete metrics, which gives  $X \rightarrow \mathbb{R}^n$  a linear bijection).

A reason for studying  $C_b(X)$  instead of  $C(X)$  is the fact that  $C(X)$  may contain unbounded function and the supnorm is not defined (for example  $X = \mathbb{R}$ ). However, in some cases, it is still possible to define a metric on  $C(X)$ :

*Example 4.5.* Let  $X = \mathbb{R}^n$  and  $\overline{B_n(0)} = \{|x| \leq n\}$  for all positive integers  $n$ . For all  $f \in C(\mathbb{R}^n)$ , define

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{\infty, \overline{B_n(0)}}}{1 + \|f - g\|_{\infty, \overline{B_n(0)}}}$$

where  $\|\cdot\|_{\infty, \overline{B_n(0)}}$  is the supnorm on the closed ball  $\overline{B_n(0)}$ , then  $d$  is a complete metric on  $C(\mathbb{R}^n)$ .

Finally, recall the Bolzano-Weierstrass Theorem in  $\mathbb{R}^n$ :

*Theorem 4.6.* Every bounded sequence has a convergent subsequence. Similarly, every bounded set contains a convergent sequence.

$C_b(X)$  may not have Bolzano-Weierstrass property. Consider the following example:

*Example 4.7.* Observe that  $C_b([0, 1]) = C[0, 1]$ . Let  $f_n(x) = x^n$  where  $x \in [0, 1]$  for all  $n$ , then  $\|f_n\|_{\infty} = 1$ . The pointwise limit

$$f_n(x) \rightarrow \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

implies that no subsequence converges in  $C_b[0, 1]$ .

Because of this, further condition are required to find convergent sequences in subsets of  $C_b(X)$ .

*Definition 4.8.* Let  $(X, d)$  be a metric space. A set  $E \subset X$  is called a **precompact** set if every sequence in  $E$  contains a convergent subsequence with limit in  $X$  (which is not necessary in  $E$ ). If the limit is further restricted within  $E$ , then  $E$  is called a **compact** set.

*Proposition 4.9.* A compact set is a closed precompact set.

*Proof.* Let  $(X, d)$  be a metric space and  $\{x_n\} \subset E$  be a sequence in  $E \subset X$ . If  $E$  is precompact, there exists a subsequence  $\{x_{n_j}\}$  with limit  $z \in X$ . If  $E$  is closed, the limit  $z \in E$ , which implies compactness.

Also recall that by Bolzano-Weierstrass Theorem,  $E \subset \mathbb{R}^n$  is precompact implies  $E$  is bounded. Therefore  $E$  is compact implies  $E$  is closed and bounded.



### 4.1.2 Equicontinuity

*Definition 4.10.* Let  $(X, d)$  be a metric space. A subset  $C$  of  $C(X)$  is said to be **equicontinuous** if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - f(y)| < \epsilon$  for all  $f \in C$  and  $x, y \in X$  where  $d(x, y) < \delta$ .

In fact, equicontinuity is based on uniform continuity, but at the same time extends  $\delta$  to fulfill every function  $f \in C$ . Therefore, equicontinuity implies every function in  $C$  is uniformly continuous. Then, it is simple to see that if  $C$  is equicontinuous, any  $C' \subset C$  is also equicontinuous.

There are other ways to show that a set is equicontinuous. Recall that a function  $f$  is Holder continuous if there exists a Holder exponent  $\alpha \in (0, 1)$  such that

$$|f(x) - f(y)| \leq L |x - y|^\alpha$$

for some constant  $L$ .  $f$  is Lipschitz continuous if the equation holds for  $\alpha = 1$ . A set  $C$  is equicontinuous too if every function  $f \in C$  is Holder continuous or Lipschitz continuous.

Another method for equicontinuity requires the following definition:

*Definition 4.11.* A set  $C$  is said to be **convex** (in  $\mathbb{R}^n$ ) if  $x + t(y - x) \in C$  for all  $x, y \in C$  and  $t \in [0, 1]$ .

*Proposition 4.12.* Let  $C$  be a subset of  $C(\overline{G})$  where  $\overline{G}$  is a convex in  $\mathbb{R}^n$ . Suppose that each function in  $C$  is differentiable and there is a uniform bound on their partial derivatives:

$$\left\| \frac{\partial f}{\partial x_i} \right\|_\infty \leq M$$

for all  $f \in C(\overline{G})$  and  $i$ , then  $C$  is equicontinuous.

*Proposition 4.13.* Let  $A = \{z_j\}$  be a countable set and  $f_n : A \rightarrow \mathbb{R}$  where  $n = 1, 2, \dots$  be a sequence of functions defined on  $A$ . Suppose for each  $z_j \in A$ ,  $\{f_n(z_j)\}$  is a bounded sequence in  $\mathbb{R}$ , then there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that for all  $z_j \in A$ ,  $\{f_{n_k}(z_j)\}$  is convergent.

### 4.1.3 Ascoli's Theorem

Below is the **Ascoli's Theorem**:

**Theorem 4.14.** Suppose  $G$  is a bounded nonempty open set in  $\mathbb{R}^m$ , then a set  $\mathcal{E} \subset C(\overline{G}) = C_b(\overline{G})$  is precompact if  $\mathcal{E}$  is bounded (in supnorm) and equicontinuous.

Note that Ascoli's Theorem remains valid for bounded and equicontinuous subsets of  $C(G)$  where  $G$  is not necessary to take closure. This is because equicontinuity implies uniform continuity of  $G$ , which can be further extended to uniform continuity of  $\overline{G}$ . However, boundedness of the domain  $\overline{G}$  cannot be removed:

**Example 4.15.** Let  $\overline{G} = [0, \infty) \subset \mathbb{R}$ , then take  $\varphi \in C^1[0, 1]$  such that  $\varphi \not\equiv 0$  and  $\varphi(x) = 0$  when  $x \in [0, 1] \setminus [1/2, 3/4]$ . Further define

$$f_n(x) = \begin{cases} \varphi(x - n) & \text{if } x \in [n, n + 1] \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that  $f_n \in C(\overline{G})$  and

$$\|f_n\|_{\infty, \overline{G}} = \|\varphi\|_{\infty, [0, 1]} > 0$$

Thus  $\mathcal{E} = \{f_n\}$  is a bounded subset of  $C(\overline{G})$ . By chain rule,

$$\left\| \frac{df_n}{dx} \right\|_{\infty, \overline{G}} = \left\| \frac{d\varphi}{dx} \right\|_{\infty, [0, 1]} > 0$$

Then **Proposition 4.12** states that  $\mathcal{E}$  is also equicontinuous.

Suppose there exists a subsequence  $\{f_{n_j}\}$  of  $\{f_n\}$  converges to the same  $f \in C(\overline{G})$  in  $d_\infty$ . In other words,  $f_{n_j} \rightarrow f$  uniformly on  $\overline{G}$  implies pointwise convergence  $f_{n_j}(x) \rightarrow f(x)$  for all  $x \in \overline{G}$ . However, for fixed  $x$ ,  $f_n(x) = 0$  for all  $n \geq x$ , so it is expected to have

$$\lim_{j \rightarrow +\infty} f_{n_j}(x) \rightarrow 0$$

which shows that  $f(x) = 0$  for all  $x \in \overline{G}$ . This is a contraction since

$$0 < \|\varphi\|_{\infty, [0, 1]} = \|f_{n_j}\|_{\infty, \overline{G}} = \|f_{n_j} - f\|_{\infty, \overline{G}} \rightarrow 0$$

Therefore  $\mathcal{E}$  is bounded and equicontinuous, but Ascoli's Theorem doesn't hold.

#### 4.1.4 Arzela's Theorem

Below is the **Arzela's Theorem**, which is the converse of Ascoli's Theorem:

**Theorem 4.16.** Suppose  $G$  is a bounded nonempty open set in  $\mathbb{R}^m$ , then every precompact set in  $C(\overline{G})$  must be bounded and equicontinuous.

## 4.2 Applications to Ordinary Differential Equations

### 4.2.1 Improvement to Picard-Lindelof Theorem

Consider the initial value problem

$$\begin{cases} dx/dt = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

with  $f$  being continuous (but not necessary Lipschitz) on  $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ . Of course this is not expected to give a unique result, but existence can be proved. The idea of proof is as follows:

- (1) By Weierstrass Approximation Theorem (on  $\mathbb{R}^2$ ), there exists a sequence  $\{p_n\}$  of polynomials such that  $d_\infty(p_n, f) \rightarrow 0$  (in  $C(R)$ ).
- (2) By Picard-Lindelof Theorem, since every  $p_n$  satisfies Lipschitz condition (uniform in  $t$ ), there exists  $a'_n > 0$  with

$$a'_n = \min \left\{ a, \frac{b}{M_n}, \frac{1}{L_n} \right\}$$

where  $M_n = \|p_n\|_{\infty, R}$  and  $L_n$  be Lipschitz constant of  $p_n$  on  $R$ , such that there exists a unique solution  $x_n \in C^1[t_0 - a'_n, t_0 + a'_n]$  to the approximated initial value problem

$$\begin{cases} dx_n/dt = p_n(t, x_n) \\ x_n(t_0) = x_0 \end{cases}$$

for all  $t \in [t_0 - a'_n, t_0 + a'_n]$ .

- (3) By Ascoli's Theorem, there exists a convergent subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow x$  for some function  $x(t)$ . It is hoped that such  $x$  is the required solution.

However, since  $f$  is not assumed to satisfy the Lipschitz condition, one cannot expect  $\{L_n\}$  is bounded. In fact,  $\{L_n\}$  is unbounded, otherwise  $f$  satisfies Lipschitz condition. Here

$$a'_n = \min \left\{ a, \frac{b}{M_n}, \frac{1}{L_n} \right\} \rightarrow 0$$

then there is no proper interval for existence of the solution. On the other hand, as  $p_n \rightarrow f$  in  $(C(R), d_\infty)$ ,  $M_n \leq M$  for some  $M > 0$ . Therefore, in order to implement the plan above, it is required to improve Picard-Lindelof Theorem:

**Proposition 4.17.** Under the setting of Picard-Lindelof Theorem, there exists a unique solution  $x(t)$  on the interval  $[t_0 - a', t_0 + a']$  with  $x(t) \in [x_0 - b, x_0 + b]$ , where  $a'$  is any number satisfying

$$0 < a' < a^* = \min \left\{ a, \frac{b}{M} \right\}$$

### 4.2.2 Cauchy-Peano Theorem

Below is the **Cauchy-Peano Theorem**:

*Theorem 4.18.* Consider the initial value problem

$$\begin{cases} dx/dt = f(t, x) \\ x_{t_0} = x_0 \end{cases}$$

where  $f$  is continuous on  $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ , then there exists  $a' \in (0, a)$  and a  $C^1$  function

$$x : [t_0 - a, t_0 + a] \rightarrow [x_0 - b, x_0 + b]$$

solving the initial value problem.

## 4.3 Baire Category Theorem

### 4.3.1 Denseness

*Definition 4.19.* Let  $(X, d)$  be a metric space, then a set  $E \subset X$  is said to be **dense** if for all  $x \in X$  and  $\epsilon > 0$ ,  $B_\epsilon(x) \cap E \neq \emptyset$ .

Note that  $X$  is naturally dense in  $(X, d)$ , and if  $E$  is dense in  $X$ , its closure  $\overline{E} = X$ .

*Definition 4.20.* Let  $(X, d)$  be a metric space, then a set  $E \subset X$  is said to be **nowhere dense** if its closure does not contain any ball. In other words,  $\overline{E}$  has empty interior.

*Example 4.21.* Set of integers  $\mathbb{Z}$  is nowhere dense in  $\mathbb{R}$ . However, although set of rationals  $\mathbb{Q}$  has empty interior, its closure  $\overline{\mathbb{Q}} = \mathbb{R}$  has nonempty interior, so  $\mathbb{Q}$  is not nowhere dense.

*Proposition 4.22.* Let  $(X, d)$  be a metric space and  $E \subset X$  be a set, then  $E$  is nowhere dense if and only if  $X \setminus \overline{E}$  is dense in  $X$ .

*Proof.* If  $E$  is nowhere dense, for all  $x \in X$  and any  $r > 0$ ,  $B_r(x) \not\subset \overline{E}$ , which implies  $B_r(x) \cap (X \setminus \overline{E}) \neq \emptyset$ , so  $X \setminus \overline{E}$  is dense. The converse follows the reverse order.

*Definition 4.23.* Let  $(X, d)$  be a metric space, then a point  $x \in X$  is called an **isolated point** if  $\{x\}$  is open in  $X$ .

Note that  $\{x\}$  is always closed in a metric space. Therefore,  $\{x\}$  is both open and closed if and only if  $x$  is an isolated point.

*Proposition 4.24.* Let  $(X, d)$  be a metric space, then the following applies:

- (a) If  $E$  is nowhere dense in  $X$ ,  $\overline{E}$  is nowhere dense in  $X$ . Also,  $E'$  is nowhere dense in  $X$  if  $E' \subset E$ .
- (b) The union of finitely many nowhere dense sets in  $X$  is nowhere dense in  $X$ .
- (c) If  $(X, d)$  has no isolated point, then every finite set is nowhere dense.

Now consider the following example in infinite dimensional normed spaces:

*Example 4.25.* Let  $M[a, b]$  be a space of bounded functions on  $[a, b]$ , then

$$\|f\|_\infty = \sup_{[a, b]} |f(x)|$$

is well-defined and is a norm on  $M[a, b]$ . It is clear that  $(C[a, b], d_\infty)$  is a metric (and vector) subspace of  $(M[a, b], d_\infty)$ . Show that  $C[a, b]$  is nowhere dense in  $M[a, b]$ .

*Answer.* Note that  $C[a, b]$  is closed because uniform limit of continuous functions is continuous. It is left to show that for all  $B_\epsilon^\infty(f) \subset M[a, b]$ ,

$$B_\epsilon^\infty(f) \cap (M[a, b] \setminus C[a, b]) \neq \emptyset$$

If  $f \in M[a, b] \setminus C[a, b]$ , the result is already achieved. For any  $f \in C[a, b]$ , let

$$g(x) = \begin{cases} f(x) + \epsilon/2 & \text{if } x \in [a, b] \cap \mathbb{Q} \\ f(x) - \epsilon/2 & \text{if } x \in [a, b] \setminus \mathbb{Q} \end{cases}$$

such that  $\|g - f\|_\infty = \epsilon/2$  implies  $g \in B_\epsilon^\infty(f)$ . Since both  $[a, b] \cap \mathbb{Q}$  and  $[a, b] \setminus \mathbb{Q}$  are dense in  $[a, b]$ ,

$$\limsup_{x \rightarrow a} g(x) = f(a) + \frac{\epsilon}{2}$$

and

$$\liminf_{x \rightarrow a} g(x) = f(a) - \frac{\epsilon}{2}$$

shows that  $g \in M[a, b] \setminus C[a, b]$ . Therefore  $B_\epsilon^\infty(f) \cap (M[a, b] \setminus C[a, b]) \neq \emptyset$ , and  $C[a, b]$  is nowhere dense in  $M[a, b]$ .

### 4.3.2 First Category and Second Category

*Definition 4.26.* Let  $(X, d)$  be a metric space, then a set  $E \subset X$  is called **first category** (or **meager**) if it can be expressed as a countable union of nowhere dense sets. If  $E$  is not of first category, then  $E$  is called **second category**.

$E$  is said to be **residual** if its complement is of first category.

*Proposition 4.27.* Let  $(X, d)$  be a metric space, then the following applies:

- (a) Every subset of a set of first category is of first category.
- (b) The union of countable many sets of first category is of first category.
- (c) If  $(X, d)$  has no isolated point, every countable subset of  $X$  is of first category.

With the proposition above, a similar proposition for residual sets can be made by taking complements:

*Proposition 4.28.* Let  $(X, d)$  be a metric space, then the following applies:

- (a) Every subset containing a residual set is residual.
- (b) The intersection of countable many residual sets is residual.
- (c) If  $(X, d)$  has no isolated point, complement of any countable set is residual.

*Example 4.29.* Let  $(\mathbb{R}, d_1)$  be a metric space. Since  $\mathbb{R}$  has no isolated points,  $\{q\}$  is nowhere dense for any  $q \in \mathbb{Q}$ , so  $\mathbb{Q}$  is of first category since it is a countable union of  $\{q\}$ . On the other hand, the set of irrational numbers  $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$  is residual in  $\mathbb{R}$ .

### 4.3.3 General Theorem

Below is the **Baire Category Theorem**:

*Theorem 4.30.* Any set of first category in a complete metric space has empty interior. In other words, any countable intersection of open dense sets in a complete metric space is dense.

With Baire Category Theorem, there are some corollaries to follow:

*Corollary.* Let  $(X, d)$  be a complete metric space. Suppose that  $X = \bigcup_{n=1}^{\infty} E_n$  with  $E_n$  are closed subsets. Then at least one of there  $E_n$  has nonempty interior.

*Corollary.* A set of first category in a complete metric space cannot be a residual set, and vice versa.

#### 4.3.4 Applications of Baire Category Theorem

*Proposition 4.31.* Let  $f \in C[a, b]$  be differentiable at  $x$ , then it is Lipschitz continuous at  $x$ .

*Proof.* By assumption, for any  $\epsilon = 1 > 0$ , there exists  $\delta_0 > 0$  such that for all  $y \in (x - \delta_0, x + \delta_0) \setminus \{x\}$  and  $y \in [a, b]$ ,

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 1$$

implies

$$|f(y) - f(x)| \leq (1 + |f'(x)|) |y - x|$$

for all  $y \in (x - \delta_0, x + \delta_0) \cap [a, b]$ . If  $[a, b] \setminus (x - \delta_0, x + \delta_0) = \emptyset$ , it is already done. Consider for  $y \in [a, b] \setminus (x - \delta_0, x + \delta_0)$  if such set is nonempty,  $|y - x| \geq \delta_0$ , hence

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y)| + |f(x)| \\ &\leq 2 \|f\|_{\infty} \\ &\leq \frac{2 \|f\|_{\infty}}{\delta_0} |y - x| = L' |y - x| \end{aligned}$$

Finally, let  $L = \max \{1 + |f'(x)|, L'\}$ , then  $|f(y) - f(x)| \leq L |y - x|$  for all  $y \in [a, b]$ .

With the proposition above, the following theorem can be introduced:

*Theorem 4.32.* The set of all continuous, nowhere differentiable functions forms a residual set in  $C[a, b]$  and hence dense in  $C[a, b]$ .

## References

The following are the references of the context of this document:

- (a) Professor(s) associated to *MATH3060: Mathematical Analysis III*
- (b) Elias M. Stein, Rami Shakarchi, *Fourier Analysis: An Introduction (Princeton Lectures in Analysis)*, Princeton, 2003
- (c) Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill (3rd Edition), 1976