AN INVESTIGATION OF Q-BALLS

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by

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Major: Physics and Mathematics

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Date: _____

AN ABSTRACT

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Abstract:

In this thesis we study the dynamics of a complex scalar field with a nonrenormalizable potential. In particular we study the existence of the non-topological soliton solutions known as Q-Balls. These objects have been proposed as candidates to solve the present baryon asymmetry, and arise in supersymetric theories, as well as abelian gauge theories, non-abelian gauge theores, non-commutative complex scalar field theories, etc. Formulating the existence problem using constrained minimization allows for this approach to translate easily into the numerical work that follows. With this framework the computation of soliton profiles, along with the associated angular momentum parameter are achieved. To best understand the work, we present the prerequisite classical field theory and an introduction to Q-Balls and Q-Vortex solitons.

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Dedication

I would like to dedicate this thesis to my dog, Jett.

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I would like to thank, first and foremost, Dr. Luciano M. Medina, the advisor for this work, for his continued support throughout this research and thesis writing process. Without his willingness to give so much time and effort to his students, I would not have learned as much as I did. His continued insight pushed my understanding and his generous supervision allowed me to take the research in directions I wished. For his generous nature I am extremely appreciative.

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Contents

Chapter 1

Introduction

> I do not keep up with the details of particle physics.
>
> Murray Gell-Mann
> *Particle Physicist*

In this chapter we will motivate the problem studied in this thesis by means of providing historical context, as well as recent developments that are related to the work done here.

1.1 Historical Context

The story of Q-Balls begins in 1834 with a much more basic study of waves and Scottish naval engineer John Scott Russell. Russell worked experimenting at the Union Canal where he attempted to to measure the relationship between the speed of a boat and its propelling force (which oftentimes were horses on land next to the canal). These measurements would allow one to then convert from horse power to steam power. One day while Russell was working he noted a rope had caught and caused the boat to [1]

> suddenly stop – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water,

> which continued its course along the channel without change of form or diminution of speed.

This caught Russell's attention and he followed it on horseback for nearly 2 miles before it became lost in the windings of the channel. While chasing the wave he noted an important fact that the solitary wave preserved its shape as it moved down the canal. So fascinated with what he had seen, Russell built a tank to further study this phenomenon. Over the next ten years he would study these objects and he was able to demonstrate they following qualities.

- These waves were stable and could travel long distances[1]
- The speed of the wave depends on the on the height of the wave and depth of the canal as $v = \sqrt{g(d+h)}$
- If a wave is too big for the depth of the water, it splits into two or more solitary waves
- Solitary waves cross each other "without change of any kind"

As with many other great discoveries, Russell's work was not taken seriously by the scientific community at the time because his results were not reproducible by means of Newtonian hydrodynamics.

Nearly 60 years passed before the idea of a solitary wave became popular when in 1895 Korteweg and de Vries [2] published a theory of shallow water waves that reproduced Russell's observations and essential qualities. In particular they found the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

which possessed solutions that fit Russell's criteria nearly exactly. This equation possesses a solution which maintains its shape because of the simple fact that the dispersive $\frac{\partial^3 u}{\partial x^3}$ term coordinates to cancel out the effects from the nonlinear $u\frac{\partial u}{\partial x}$ term. Along with this discovery, about 30 years later from the geometry of negatively curved surfaces [3], arose the equation

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u \tag{1.2}$$

[1] Waves you might find at a beach tend to flatten out over time, or peak and topple over.

and shortly after the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin u \tag{1.3}$$

coming from solid state physics [4] were both found to have solutions that possessed solitary wave solutions. Around this point the term "soliton" was coined because of their particle like nature, that is their ability to maintain their shape as they propagate, and their interaction dynamics.

As we have seen, solitons arise from many different areas of mathematics, physics and engineering. As we build richer and more complete models of the universe, and parts of it, we often run into nonlinearities which have the possibility of yielding soliton solutions should the right conditions be met. For this reason solitons are an important object to fully understand and study. As we will see they lead to rich areas of study for both mathematicians, physicists, and engineers as they play a role in mathematical frameworks, physical theories, and have many practical applications.

1.2 Q-Balls

Now that we have a basic understanding of a soliton, we can can ask if there is some sort of classification that we can do. Are all solitons essentially the same, or do some solitons arise from fundamentally different places than others?

The answer is of course yes, and at this point we have two main categories of solitons; *topological solitons* or *topological defects*, and *non-topological solitons*. As we noted in the previous section, solitons are stable and maintain their shape. Topological solitons inherit their stability from, not surprisingly, topological arguments. Technically, these soliton solutions are homotopically distinct from the vacuum state [5] and hence cannot be deformed into the trivial solution. This inability to deform them lends them a stability against decay. Non-topological solitons however are objects which are stabilised by other means. As is often the case, non-topological solitons are stabilised by their "Noether Charge" which we will learn about in Chapter 2. This charge stabilises them against decay and allows them to maintain their shape.

While topological defects are extremely important, and play a role in many quantum field theories, this thesis is concerned with non-topological solitons.

The history of Q-Balls begins with the great physicist Sidney Coleman and his paper "Q-Balls" [6], written in 1985. In this paper Coleman lays the groundwork for the existence of Q-Balls and their stability against decay and perturbations. Since this pioneering work there has been an enormous amount of work done pertaining to Q-Balls and their properties, and is of course not just limited to [7–15].

1.3 Applications

Since Coleman's first paper Q-Balls have celebrated much research and activity. Besides research into their properties, Q-Balls have been proposed as models of new physics in many areas.

In Alexander Kusenko's work [16] Q-Balls are shown to exist in Supersymmetric generalizations of the Standard Model of Particle Physics (one of the leading theories to go beyond the standard model). In this paper Kusenko shows how Q-Balls could have been created at a very early time in our universe ($\sim$ 1s). He then goes on to discuss the possible cosmological ramifications, and in [17] Kusenko and Shaposhnikov propose Q-Balls as a candidate for Dark Matter. In [18, 19], we even see Q-Balls aiding in the explanation of baryogenisis.[2]

[2]The unkown process that produced so many more baryons (matter), than antibaryons (antimatter).

Chapter 2

Field Theory

> Newton was the greatest genius that ever existed, and the most fortunate, for we cannot find more than once a system of the world to establish.
>
> Joseph-Louis Lagrange

In order to seriously discuss our problem at hand, we must first have a rudimentary understanding of some aspects of field theory, and more specifically Gauge theory and Noether's Theorem. We will develop the needed machinery starting from a basic understanding of classical mechanics, and the principle of least action.

2.1 The Principle of Stationary Action

As review, the equations of motion for a finite system of particles is given by minimizing the action of classical mechanics given by

$$S = \int L \, \mathrm{d}t \tag{2.1}$$

where the integral is taken over the path from the initial state of the system, to the final state of the system and L is the Lagrangian of the given system.

More precisely, the action S has domain $\mathrm{C}^1(\mathbb{R})$[1] and range $\mathbb{R}$ and hence $S : \mathrm{C}^1(\mathbb{R}) \to \mathbb{R}$ is a functional and should, more formally, be written

$$S\left[x(t)\right] = \int L(x(t), \dot{x}(t), t)\, \mathrm{d}t\,. \tag{2.2}$$

Suppose we are given the true path $x_{\text{true}}(t)$ the particle takes from time t_1 to t_2. The Principle of Stationary Action tells us that when we infinitesimally vary the path the action should not change to first order [20]. Symbolically this is written $\delta S = 0$. One may show, with use of the Calculus of Variations, that if $x(t)$ minimizes (or makes stationary) an action, then it must equivalently also satisfy the following Euler-Lagrange equation.

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \tag{2.3}$$

This is an extremely important result, as it gives us a differential equation, or equation of motion, from the abstract principle. This is helpful in many way, in particular when one wants to generalize the system in which one is discussing where writing down an equation of motion is oftentimes very difficult or impossible.

2.1.1 Noether's Theorem

Let our system be parametrized by coordinates q_i with corresponding velocities $\dot{q}_i$ with $i \in \{1, 2, \ldots, n\}$. We say $F(q_i, \dot{q}_i, t)$ is a *conserved quantity*, or *constant of motion* if it's total time derivative vanishes, i.e.,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial F}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t} = 0 \tag{2.4}$$

where $q_i(t)$ is taken along a path satisfying (2.3). This is to say the value of F does not change as we evolve our system in time.

Now suppose we have a one parameter family of maps

$$q_i(t) \to Q_i(t, s) \qquad s \in \mathbb{R} \tag{2.5}$$

[1] It is risky business to satisfy this domain as oftentimes is the case that in application it is disregarded and new spaces are taken as domains. However here we specify $\mathrm{C}^1(\mathbb{R})$ because of Theorem A.1.1 given in Appendix A.

such that $Q_i(t,0) = q_i(t)$. This transformation is said to be a *symmetry* of the Lagrangian L if

$$\frac{\partial}{\partial s} L(Q_i(t,s), \dot{Q}_i(t,s), t) = 0 \tag{2.6}$$

Theorem 2.1.1 (Noether's Theorem)**.** *For each such symmetry transformation as defined in in (2.6), there exists a corresponding conserved quantity.*

Proof. Expanding the symmetry definition (2.6) we have

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \frac{\partial L}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s} = 0 \tag{2.7}$$

This holds for all s, and in particular $s = 0$. Thus, using the fact that $Q_i(t,0) = q_i(t)$, we have

$$0 = \left.\frac{\partial L}{\partial s}\right|_{s=0} = \left.\frac{\partial L}{\partial q_i}\frac{\partial Q_i}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right|_{s=0} \tag{2.8}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\left.\frac{\partial Q_i}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right|_{s=0} \tag{2.9}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\right|_{s=0}\right) \tag{2.10}$$

Since this is true for all i, and by linearity of the time derivative we have conservation of $\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\partial \mathbf{Q}}{\partial s}$. □

We now see an example of this theorem in action and the true power behind Noether's ideas in terms of Classical Mechanics (later we will see a generalization of this theorem to deal with fields).

Example 1. Suppose we have a system of particles described by the following Lagrangian.

$$L = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^2 - \sum_{j \neq i} V(\mathbf{r}_i - \mathbf{r}_j) \tag{2.11}$$

Suppose we make the transformation of the coordinates $\mathbf{r}_i \to \mathbf{r}_i + s\mathbf{n}$ where $s \in \mathbb{R}$. We then immediately have $L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}_i + s\mathbf{n}, \dot{\mathbf{r}}_i, t)$ because

$\mathbf{n}$ does not change with time (nor space), and hence we have a conserved quantity. Noether's theorem then says

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \mathbf{n} = \sum_{i=1}^{n} \mathbf{p}_i \cdot \mathbf{n} = \mathbf{P} \cdot \mathbf{n} \tag{2.12}$$

where $\mathbf{P}$ is the total momentum of the system is conserved. Hence if we have homogeneity of space (that is, a potential term on dependent on the vector difference of objects), we have conservation of momentum in the direction of $\mathbf{n}$. However, the argument holds for all vectors $\mathbf{n}$ and hence we have conservation of momentum in all directions, and more simply, we have conservation of momentum.

This theorem has many important other consequences[2] and makes proving conservation laws, the thing physicists love most, much easier. In fact it turns out, every conservation law has a corresponding symmetry, however discrete symmetries do not depend on a continuous parameter and hence transformation such as parity transformations, i.e., $\mathbf{r}_i \to -\mathbf{r}_i$, do not have conservation laws in classical physics.

2.2 Adding a Few More Particles

Section 2.1 works great for most systems we wish to study, however there are many systems this approach does not work for. If one wishes to analyze the behavior of any continuous system, the approach outlined above fails, for there would be an infinite number of continuous variables. This, however, does not mean we cannot study these systems, but must find an alternative approach.

As an example we will study an infinitely long elastic rod which can undergo small longitudinal vibrations. In order to study the system we will first use a discretized version consisting of point particles, connected by springs, and then take the continuum limit.

[2]If space is isotropic, i.e. the Lagrangian is invariant under rotations around an axis, or differently $V(|\mathbf{r}_i - \mathbf{r}_j|)$ is only a function of the magnitude of the distance between objects, then you have conservation of total angular momentum. If the Lagrangian is invariant under time shifts $t \to t + a$ (homogeneity in time) then $H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ is conserved. This quantity is known as the Hamiltonian and is, in most systems, the total energy.

If we use η_i to denote particle i's displacement from it's equilibrium position, then the systems kinetic energy is the sum of each particles kinetic energy.

$$T = \frac{1}{2}\sum_{i\in\mathbb{Z}} m\dot{\eta}_i^2 \tag{2.13}$$

Here we let each particle have the same mass m. The potential energy is built up by summing over neighbors. In particular we take the distance in between neighbors, square and multiply by the spring constant as we know from Hooke's Law $V_{\text{spring}} = \frac{1}{2}kx^2$ for a mass on a spring.

$$V = \frac{1}{2}\sum_{i\in\mathbb{Z}} k\left(\eta_{i+1} - \eta_i\right)^2 \tag{2.14}$$

Combining (2.13) and (2.14) we obtain

$$L = T - V = \frac{1}{2}\sum_{i\in\mathbb{Z}} \left[m\dot{\eta}_i^2 - k\left(\eta_{i+1} - \eta_i\right)^2\right] \tag{2.15}$$

If, while the system at rest, the distance between the masses is a, then we can write (2.15) as

$$L = \frac{1}{2}\sum_{i\in\mathbb{Z}} a\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right]. \tag{2.16}$$

When we eventually take the continuum limit, we will need to understand what happens to this Lagrangian. The continuum limit in this system will mean taking $a \to 0$. As we do this $\frac{m}{a}$ approaches μ, mass per unit length, or linear mass density. The limiting value of the coefficient ka can be found by using Hooke's Law which states the extension of an elastic rod per unit length is directly proportional to the force exerted on that rod. This relation is written $F = Y\xi$ where Y is the Young's Modulus and ξ is the extension per unit length. The extension per unit length in the discretized system is $\xi = \frac{\eta_{i+1}-\eta_i}{a}$ and the force required to do the extension is

$$F = k(\eta_{i+1} - \eta_i) = ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right) = ka\xi \tag{2.17}$$

and hence ka corresponds to Young's Modulus. The displacements from equilibrium η_i will be promoted from being indexed by $\mathbb{Z}$, to being "indexed"

by $\mathbb{R}$. However, indexing by $\mathbb{R}$ corresponds to being a function of $\mathbb{R}$ and hence we write $\eta(x)$. Adding one to the index i then corresponds to moving over a distance a and hence

$$\frac{\eta_{i+1} - \eta_i}{a} = \frac{\eta(x+a) - \eta(x)}{a} \tag{2.18}$$

and in the limiting case $a \to 0$ this term approaches $\frac{\partial \eta}{\partial x}$[3] and a plays the role of $\mathrm{d}x$. Hence the sum over particles, becoming an integral over x and the Lagrangian reads

$$L = \frac{1}{2} \int_{\mathbb{R}} \left[\mu \left(\frac{\partial \eta}{\partial t} \right)^2 - Y \left(\frac{\partial \eta}{\partial x} \right)^2 \right] \mathrm{d}x \,. \tag{2.19}$$

We note that had we considered the system in three-dimensions, we would be integrating over the independent "indices" y and z as well. In general we will be able to write the Lagrangian as an integral over all space. That is

$$L = \iiint \mathcal{L} \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int \mathcal{L} \, \mathrm{d}^3 x \tag{2.20}$$

where $\mathcal{L}$ is defined as the *Lagrangian density* and for the elastic rod we read off the Lagrangian density to be

$$\mathcal{L} = \frac{1}{2} \left[\mu \left(\frac{\partial \eta}{\partial t} \right)^2 - Y \left(\frac{\partial \eta}{\partial x} \right)^2 \right] . \tag{2.21}$$

The action then reads

$$S = \iint \mathcal{L} \; \mathrm{d}^3 x \, \mathrm{d}t = \int \mathcal{L} \, \mathrm{d}^4 x \tag{2.22}$$

which treats space and time on equal footing.

We treat the Lagrangian density as a function of a field and it's derivatives. In the above example η is a field that describes, at each point, the displacement from equilibrium. The question is then, given a Lagrangian density can we formulate the equations of motion of the system purely in terms of it, and it's derivatives like we did with the Lagrangian.

[3] The only reason its not $\frac{\mathrm{d}\eta}{\mathrm{d}x}$ is because η depends on both time t and space x so partial derivatives are necessary.

2.3 Euler-Lagrange Equations Revisited

2.3.1 Some Notation

Before we continue to develop the Euler-Lagrange equations for the action written in terms of the Lagrangian density, it will be helpful to be familiar with some notation. A flat spacetime is modeled by a modified version of $\mathbb{R}^4$ because of the seeming difference between the three space dimensions and the single time. In the physics literature this space is normally denoted $\mathbb{R}^{1,3} = \mathbb{R} \times \mathbb{R}^3$. The underlying set is simply $\mathbb{R}^4$, however the inner product, and hence metric[4], is altered.

A point in this space (called an event) is denoted as (in units where $c = 1$) $(t, x, y, z) = (x^0, x^1, x^2, x^3) = x^\mu$ where the superscripts are indices, not powers. A simple abuse of notation will be used to allow x^μ to denote an event (vector) in our spacetime, and not just the μ^{th} coordinate. By convention, we will take all Greek indices $\mu, \nu, \lambda, \ldots$ to run over $\{0, 1, 2, 3\}$ (all spacetime dimensions) and Roman indices $i, j, k, \ldots$ to run over $\{1, 2, 3\}$ (the spatial dimensions). We will employ the Einstein Summation Convention which states that whenever you have an expression with a repeated index both upstairs x^μ and downstairs y_μ then that index is implied to be summed over. For example,

$$\alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3 = \sum_{i=1}^{3} \alpha_i x^i = \alpha_i x^i \tag{2.23}$$

This conventions is useful as it allows the manipulation of components without the extra baggage of a sum. A more practical example allows us to write Euclidean cross products as

$$\mathbf{u} \times \mathbf{v} = \varepsilon_{ijk} u^j v^k \mathbf{e}^i \tag{2.24}$$

where $\mathbf{e}^i$ is the standard basis for $\mathbb{R}^3$, and ε_{ijk} is the Levi-Civita Symbol which is defined as $+1$ if (i, j, k) is an even permutation of $(1, 2, 3)$ and -1 if it is an odd permutation. Otherwise the symbol is defined to be 0.

[4]The inner product induces a norm, which determines the metric by $d(x, y) = \|x - y\|$.

Lastly, we define the following symbols

$$\begin{aligned}\partial_\mu &\coloneqq \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \\ \partial^\mu &\coloneqq \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{\partial t}, \nabla\right)\end{aligned} \tag{2.25}$$

and note that we can very easily raise and lower all indices in the expressions aforementioned by "contracting" with metric of the space. In the work that follows we work in a flat Minkowski spacetime with metric $\eta_{\mu\nu}$ and signature $\mathrm{diag}(-,+,+,+)$[5]. Contracting with the metric means the following.

$$x_\mu = g_{\mu\nu}x^\nu \qquad A_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}A^{\alpha\beta} \tag{2.26}$$

This gives us another tool to manipulate indices, when in disguise this is really just matrix multiplication.

2.3.2 First Variation

We follow the same procedure as before in the derivation, by varying the action. Here, the field $\delta\varphi$ will be chosen so that it vanishes outside the region of spacetime we are considering.

$$\begin{aligned}\delta S &\coloneqq S[\varphi + \delta\varphi] - S[\varphi] \\ &= \int \mathcal{L}(\varphi + \delta\varphi, \partial_\mu\varphi + \partial_\mu\delta\varphi) - \mathcal{L}(\varphi, \partial_\mu\varphi)\,\mathrm{d}^4x \\ &= \int \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)\,\mathrm{d}^4x\end{aligned} \tag{2.27}$$

To continue we use the following fact

$$\delta(\partial_\mu\varphi) \coloneqq (\partial_\mu\varphi)(x^\mu + \delta a^\mu) - (\partial_\mu\varphi)(x^\mu) = \partial_\mu(\varphi(x^\mu + \delta a^\mu) - \varphi(x^\mu)) = \partial_\mu(\delta\varphi)$$

to write (2.27) as

$$\delta S = \int \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\delta\varphi)\,\mathrm{d}^4x\,. \tag{2.28}$$

[5]This is just shorthand for $\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

We then integrate by parts the second term, and because we chose $\delta\varphi$ to be 0 outside of our region of interest the boundary term goes to 0. Hence we have

$$\begin{aligned}\delta S &= \int \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi \,\mathrm{d}^4x \\ &= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right]\delta\varphi \,\mathrm{d}^4x \end{aligned} \tag{2.29}$$

In order for the action to be stationary on this field, δS must be 0 for *all* fields $\delta\varphi$. This meas the bracketed term must be 0 in order for the action to be stationary.

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0 \tag{2.30}$$

Hence we have arrived at the Euler-Lagrange equation for a Lagrangian density dependent on a field, and its derivatives. A slight generalization of this equation would be an Euler-Lagrange equation for a Lagrangian density dependent on multiple fields, i.e. $\mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$ for $i \in \mathbb{N}$. If this is the case, we can then vary the action with respect to each field, to obtain the above Euler-Lagrange equation for each independent field. That is we have

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = 0 \tag{2.31}$$

for each i. Deriving other Euler-Lagrange equations are possible, for example ones for Lagrangian densities dependent on higher derivatives, but we will not need them for our purpose.

For completeness we present an alternative derivation using the functional derivative and differential.

Definition 2.3.1. *Given a space X of functions which is closed under vector addition and scalar multiplication, we define the* ***functional derivative*** *of $F : X \to \mathbb{R}$, denoted as $\frac{\delta F}{\delta \psi}$, as*

$$\begin{aligned}\frac{\delta F}{\delta \psi} &:= \lim_{\varepsilon \to 0} \frac{F[\psi + \varepsilon\phi] - F[\psi]}{\varepsilon} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\psi + \varepsilon\phi]\bigg|_{\varepsilon=0} \end{aligned} \tag{2.32}$$

where we call ϕ the variation of ψ.

Using this new tool we can re-derive the Euler-Lagrange equations using S as our functional. First we calculate the functinal derivative.

$$\begin{aligned}\frac{\delta S}{\delta \varphi} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int \mathcal{L}(\varphi+\varepsilon\phi, \partial_\mu\varphi + \varepsilon\partial_\mu\phi)\,\mathrm{d}^4x\bigg|_{\varepsilon=0} \\ &= \int \frac{\partial\mathcal{L}}{\partial\varphi}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\phi\;\mathrm{d}^4x\end{aligned}$$

We now integrate by parts as before, and using the fact that our variation vanishes on the boundary of the region of spacetime we are considering we can write

$$\frac{\delta S}{\delta\varphi} = \int\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]\phi\,\mathrm{d}^4x\,. \tag{2.33}$$

Now the principal of stationary action tells us this variation of the action should be 0. If an integral is to be 0, then its integrand must be 0. The variation ϕ is arbitrary and in particular it is non-zero. Hence the only way for this integral to be 0 is for the bracketed term to be equal to 0 almost everywhere. We then obtain the same Euler-Lagrange equation as above for a single scalar field!

2.4 Aspects of Electromagnetism

Electromagnetism is fundamentally the study of the electric and magnetic fields. Mathematically these are vector fields $\mathbf{E}, \mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfying the following equations known as Maxwell's Equations (in Lorentz-Heaviside units).

$$\nabla\cdot\mathbf{E} = \rho \tag{2.34}$$
$$\nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0 \tag{2.35}$$
$$\nabla\cdot\mathbf{B} = 0 \tag{2.36}$$
$$\nabla\times\mathbf{B} - \partial_t\mathbf{E} = \mathbf{J} \tag{2.37}$$

These four equations completely describe Electromagnetic phenomena, where ρ is the charge density of the system, and $\mathbf{J}$ is the current density defined by the current per unit area. These equations describe classical Electrodynamics very well, but eqs. (2.34) to (2.37) tell a much larger story than at the surface.

To begin we use the familiar theorem from multi-variable calculus that states if a vector field's divergence is 0 everywhere, then it can be written as

the curl of some other vector field. The (current) in-existence of magnetic monopoles implies $\nabla \cdot \mathbf{B} = 0$ and hence we may write $\mathbf{B} = \nabla \times \mathbf{A}$.

One may then ask can we do the same for the electric field? Well neither the divergence, nor the curl is 0, so it makes our job slightly more difficult. That said, taking the curl of the vector field $\mathbf{F} = \mathbf{E} + \partial_t \mathbf{A}$ yields $\nabla \times \mathbf{F} = \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ where the last equality is by eq. (2.35). Because the curl is 0 everywhere it can be written (using another theorem from multi-variable calculus) as the gradient of a scalar field. Rearranging, we have arrived at $\mathbf{E} = -\nabla\varphi - \partial_t \mathbf{A}$ and hence a "potential" for $\mathbf{E}$.

Writing the electric and magnetic fields in terms of these potentials decreases the amount of information needed to solve problems. This can be seen readily as $\mathbf{E}(x, y, z, t) = (E_1(x, y, z, t), E_2(x, y, z, t), E_3(x, y, z, t))$ and also with the magnetic field. That is, 6 independent functions. However in our potential reformulation we have reduced the problem to finding the scalar field V as well as the three functions in the magnetic vector potential $\mathbf{A}$.

With these we can rewrite Maxwell's equations in terms of the the potentials. Below are the two inhomogeneous equations whereas the two homogeneous equations are satisfied by the constructions of the potentials.

$$-\nabla^2\varphi - \partial_t(\nabla \cdot \mathbf{A}) = \rho \tag{2.38}$$

$$-\nabla(\partial_t\varphi + \nabla \cdot \mathbf{A}) + (-\partial_t^2 + \nabla^2)\mathbf{A} = \mathbf{J} \tag{2.39}$$

These equations, along with the Lorentz force law

$$\begin{aligned} \mathbf{F} &= q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \\ &= q\left(-\nabla\varphi - \partial_t\mathbf{A} + \nabla(\mathbf{v} \cdot \mathbf{A})\right) \end{aligned} \tag{2.40}$$

allow us to construct the Lagrangian of Electrodynamics.

$$L_{\mathrm{EM}}(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\varphi(t, \mathbf{x}) + q\dot{\mathbf{x}} \cdot \mathbf{A}(t, \mathbf{x}) \tag{2.41}$$

2.5 Elementary Gauge Theory

With the electric and magnetic fields written in terms of "potential" functions

$$\mathbf{E} = -\nabla\varphi - \partial_t\mathbf{A} \tag{2.42}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.43}$$

we might wonder about the uniqueness of such fields φ and $\mathbf{A}$. Students of Electrodynamics know the electric potential φ is certainly not unique as we often chose it to be 0 at locations in spacetime that are convenient for solving the problem at hand. The picture is slightly more complicated in general, however. Suppose we are given electric and magnetic fields and have gone about finding potentials for those fields. If I then come along and make the transformation

$$\varphi \to \varphi' = \varphi - \partial_t \chi \tag{2.44}$$
$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \chi \tag{2.45}$$

we can see how the given electromagnetic fields change.

$$\begin{aligned}
\mathbf{B} \to \mathbf{B}' &= \nabla \times (\mathbf{A} + \nabla \chi) \\
&= \nabla \times \mathbf{A} + \cancelto{0}{\nabla \times (\nabla \chi)} \\
&= \mathbf{B} \\
\mathbf{E} \to \mathbf{E}' &= -\nabla (\varphi - \partial_t \chi) - \partial_t (\mathbf{A} + \nabla \chi) \\
&= -\nabla \varphi + \partial_t \nabla \chi - \partial_t \mathbf{A} - \partial_t \nabla \chi \\
&= \mathbf{E}
\end{aligned}$$

Surprisingly the electromagnetic fields are completely unchanged by these transformations for any given (differentiable) function $\chi(t, \mathbf{x})$. This ambiguity in the potentials describing the physical fields provide us with a way to simplify calculations. Suppose you are having trouble solving 2.38 because the second term $\nabla \cdot \mathbf{A}$ is getting in your way. Take χ to be any solution to $\nabla \cdot \mathbf{A} = -\nabla^2 \chi$ and make the transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \chi$ which makes $\nabla \cdot \mathbf{A}' = 0$. The equation $\nabla \cdot \mathbf{A} = -\nabla^2 \chi$ is simply Poisson's Equation and an existence proof will allow us to conclude such a χ exists as long as $\nabla \cdot \mathbf{A}$ dies off appropriately as we approach spatial infinity. This can be seen by the Green's Function solution to the problem which is simply

$$\chi(t, \mathbf{x}) = -\frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}^3 \mathbf{x}'$$

These transformations of the potentials are called **gauge transformations** and have no effect on the actual physics of what is happening (at least classically). When one imposes an extra condition which the potentials must satisfy, we call it choosing a gauge. Here are a few common examples.

Coulomb Gauge	$\nabla \cdot \mathbf{A} = 0$
Lorenz Guage	$\nabla \cdot \mathbf{A} + \partial_t \varphi = 0$
Temporal (Weyl) Gauge	$\varphi = 0$

Table 2.1: Common Gauge Choices

For each choice of gauge there is a corresponding existence proof underlying the differential equations. Take, for example, the Lorenz Gauge which requires χ to satisfy the following inhomogeneous wave equation.

$$-\partial_t^2 \chi + \nabla^2 \chi = -\nabla \cdot \mathbf{A} - \partial_t \varphi$$

We can, again, use a Green's function solution to write down the most general solution to the problem that satisfies sensible boundary conditions, and is consistent with causality as

$$\chi(t, \mathbf{x}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}(t_\mathrm{r}, \mathbf{x}') + \partial_t \varphi(t_\mathrm{r}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}^3 \mathbf{x}'$$

where $t_\mathrm{r} \coloneqq t - |\mathbf{x} - \mathbf{x}'|$ is called the *retarded time*. Lastly, for the Temporal Gauge we simply choose χ to satisfy $\partial_t \chi = \varphi$ and when we make the gauge transformation, $\varphi' \equiv 0$ almost everywhere.

2.6 Symmetry of Lagrangian

With the idea of a gauge transformation, we can ask what happens to the Lagrangian if we perform such a transformation. If the transformations leave the physical electric and magnetic fields unchanged, then we would expect the Lagrangian to also remain unchanged considering it determines the equations of motion of the system. Remembering the EM Lagrangian 2.41 we have the following transformation.

$$\begin{aligned} L_\mathrm{EM} \to L'_\mathrm{EM} &= \frac{1}{2} m \dot{\mathbf{x}}^2 - q \left(\varphi - \partial_t \chi \right) + q \dot{\mathbf{x}} \cdot \left(\mathbf{A} + \nabla \chi \right) \\ &= \frac{1}{2} m \dot{\mathbf{x}}^2 - q\varphi + q \dot{\mathbf{x}} \cdot \mathbf{A} - q \partial_t \chi + q \dot{\mathbf{x}} \cdot \nabla \chi \\ &= L_\mathrm{EM} + q \left(\partial_t \chi + \dot{\mathbf{x}} \cdot \nabla \chi \right) \\ &= L_\mathrm{EM} + \frac{\mathrm{d}}{\mathrm{d}t} q \chi(t, \mathbf{x}) \end{aligned}$$

The characteristic of only changing by a total derivative is very important. Under the above gauge transformation the action 2.1 transforms to $S \to S + q\chi|_\partial$. If the region of space-time we wish to consider is finite, we can choose $\chi|_\partial$ to be some constant and hence the action only changes by a constant which does not affect the equations of motion. On the other hand if we consider all of space-time then χ should vanish at spatial infinity and hence the action is unaffected. Either way, the equations of motion are unaffected.

2.6.1 Symmetry of Lagrangian Density

Suppose we are studying the general Lagrangian density (herein referred to as the Lagrangian) dependent on a complex scalar field.

$$\mathcal{L} = -\partial_\mu \varphi \, \partial^\mu \overline{\varphi} - V(|\varphi|) \tag{2.46}$$

It is easy to see if we make the global substitution $\varphi \to \mathrm{e}^{\mathrm{i}\alpha}\varphi$ where $\alpha \in \mathbb{R}$, the Lagrangian remains invariant and the equations of motion are hence unaffected. It is natural to then ask what if α is not just a constant, but rather a function of space, i.e. $\alpha(x)$. Under this transformation the derivative of the field φ transforms as

$$\begin{aligned} \partial_\mu \varphi \to (\partial_\mu \varphi)' &= \partial_\mu \left(\mathrm{e}^{\mathrm{i}\alpha(x)} \varphi \right) \\ &= \mathrm{e}^{\mathrm{i}\alpha(x)} \left(\partial_\mu \varphi + \mathrm{i}\varphi \, \partial_\mu \alpha(x) \right) \end{aligned} \tag{2.47}$$

and hence it is clear this local transformation will not leave the Lagrangian invariant. The idea is then to build a new type of derivative what *would* leave the Lagrangian invariant. Such a derivative would transform according to

$$D_\mu \varphi \to (D_\mu \varphi)' = \mathrm{e}^{\mathrm{i}\alpha(x)} D_\mu \varphi$$

We can construct such a derivative by examining why we don't have symmetry in (2.47). The extra term can be cancelled by subtracting an associated term in our new derivative. In particular we can guess $D_\mu = \partial_\mu - \mathrm{i}A_\mu(x)$ where $A_\mu(x)$ is just some function of spacetime (for now). Under the local transformation this derivative transforms as follows.

$$\begin{aligned} D_\mu \varphi \to (D_\mu \varphi)' &= \left(\partial_\mu - \mathrm{i}A_\mu(x) \right) \mathrm{e}^{\mathrm{i}\alpha(x)} \varphi \\ &= e^{\mathrm{i}\alpha(x)} \left(\partial_\mu \varphi + \mathrm{i}\varphi \, \partial_\mu \alpha(x) - \mathrm{i}A_\mu(x)\varphi \right) \end{aligned}$$

But this still isn't right! What if this vector field A_μ that we added was in some way affected when we made the local gauge transformation? Even though this field seems physically superfluous, what if it had some effect on the dynamics of the system and hence would have it's own transformation law. If we force it's transformation law to respect the symmetry of the Lagrangian then we obtain

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha. \tag{2.48}$$

With this transformation the modified derivative above remains invariant, and hence so does the Lagrangian. Finally!

To recap, if we require a local gauge invariance we were forced to introduce a new field which is referred to as a *gauge field* and it's transformation law is specified by (2.48). This field played a starring role in the modified derivative, hereinafter referred to as the *gauge covariant derivative*. Hence the final Lagrangian we have arrived upon which respects the local gauge transformation is

$$\mathcal{L} = D_\mu \varphi D^\mu \varphi - V(|\varphi|). \tag{2.49}$$

When referring to this sort of problem physicists and mathematicians often say the Lagrangian has a $\mathsf{U}(1)$ symmetry. In general we have the following definition.

$$\mathsf{U}(n) \coloneqq \left\{ U \in \mathbb{C}^{n\times n} \,\middle|\, UU^\dagger = \mathbf{1} \right\} \tag{2.50}$$

where $\mathbb{C}^{n\times n}$ denotes n by n matrices with entries in $\mathbb{C}$, $U^\dagger$ denotes the Hermitian conjugate, and $\mathbf{1}$ denotes the identity matrix. This object is called the *unitary group*[6] and it forms an important set of matrices which is a group under matrix multiplication. In particular $\mathsf{U}(1)$ denotes 1 by 1 matrices, or complex numbers whose norm is 1, i.e. complex numbers on the unit circle. By saying the Lagrangian has a $\mathsf{U}(1)$ symmetry, means if we take an element from the group and multiply it by the field, the Lagrangian remains invariant.

Here we note an important homeomorphism (that is, a continuous bijection with continuous inverse) that allows us to better understand what a $\mathsf{U}(1)$

[6] An important subgroup of this group is unitary matrices whose determinant is $+1$, i.e. $\det U = 1$. These matrices indeed form a subgroup and is called the *special unitary group* denoted $\mathsf{SU}(n)$.

symmetry means. First we will define two new groups very much related to $\mathsf{U}(n)$ and $\mathsf{SU}(n)$. The *orthogonal group* is defined by the following

$$\mathsf{O}(n) \coloneqq \left\{ O \in \mathbb{R}^{n\times n} \,\middle|\, OO^{\intercal} = \mathbf{1} \right\}, \tag{2.51}$$

with an important subgroup $\mathsf{SO}(n)$ defined as matrices in $\mathsf{O}(n)$ with the additional property that $\det O = 1$. These matrices are then called the *special orthogonal group*. It is important to recall that rotation matrices satisfy $RR^{\intercal} = \mathbf{1}$ and additionally $\det R = 1$ (rotation here meaning proper rotation and not allowing inversions). Hence $\mathsf{SO}(2)$ corresponds to rotations of $\mathbb{R}^2$ and similarly in higher dimensions.

Getting back, the original Lagrangian (2.46) depends upon a complex scalar field φ which can always be written as $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ where φ_1, φ_2 are both real-valued scalar fields. The question is then "what does the action $\varphi \to \mathrm{e}^{\mathrm{i}\alpha}\varphi$ correspond to in terms of the real valued scalar fields?" Explicitly, using the correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, we have

$$\begin{aligned}
\varphi \to \mathrm{e}^{\mathrm{i}\alpha}\varphi &= \mathrm{e}^{\mathrm{i}\alpha}\left(\varphi_1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\varphi_2\right) \\
&= (\cos\alpha + \mathrm{i}\sin\alpha)\,(\varphi_1 + \mathrm{i}\varphi_2) \\
&= \varphi_1\cos\alpha - \varphi_2\sin\alpha + \mathrm{i}\,(\varphi_1\sin\alpha + \varphi_2\cos\alpha)
\end{aligned}$$

Now, using the fact that $\mathbb{C} \ni z = x + \mathrm{i}y \cong \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we can write the following.

$$\begin{aligned}
\varphi \to \mathrm{e}^{\mathrm{i}\alpha}\varphi &\cong \begin{pmatrix} \varphi_1\cos\alpha - \varphi_2\sin\alpha \\ \varphi_1\sin\alpha + \varphi_2\cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\
&= R_\alpha \boldsymbol{\varphi}
\end{aligned}$$

where R_α is the rotation matrix representing a rotation in $\mathbb{R}^2$ through an angle α and $\boldsymbol{\varphi} \coloneqq (\varphi_1, \varphi_2)^{\intercal}$. We note that this local transformation $\varphi \to \mathrm{e}^{\mathrm{i}\alpha}$ of a complex scalar field corresponds to a rotation of the corresponding real components. Hence an action of an element of $\mathsf{U}(1)$ on φ "is the same thing" as an action of an element of $\mathsf{SO}(2)$ on $\boldsymbol{\varphi}$. This correspondence is very important and can be made into a homeomorphism by defining $f : \mathsf{U}(1) \to$

$\mathsf{SO}(2)$ as

$$f\left(\mathrm{e}^{\mathrm{i}\alpha}\right) \coloneqq \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{2.52}$$

This homeomorphism, written $\mathsf{U}(1) \cong \mathsf{SO}(2)$, allows us to understand that our (complex) Lagrangian having a $\mathsf{U}(1)$ symmetry is the same thing as the Lagrangian, written in component form, having a rotational symmetry.

2.7 Noether's Theorem Revamped

As we saw in 2.1.1, Noether's Theorem provides a systematic way to study the effects of symmetries. Now that we have a more advanced study of classical fields, it is useful to translate theorem 2.1.1 into the new language of fields. We start with a definition about what symmetries really are in our new context of fields.

Definition 2.7.1. *A transformation of fields $\phi \to \phi + \delta\phi$ is said to be a symmetry of $\mathcal{L}$ if, upon transformation the Lagrangian changes by a total divergence.*

$$\delta\mathcal{L} = \partial_\mu F^\mu \tag{2.53}$$

Where $F^\mu(\phi)$ is an arbitrary collection of functions, of the transformation field, that decay to 0 at infinity.

It is important to note this is rather different than the symmetry definition for the particle Lagrangian where we said the Lagrangian musn't change under the transformation. This difference comes from the fact that the Lagrangian density if integrated over all space, and if there is an extra divergence, i.e. $\partial_\mu F^\mu$, then they get evaluated on the boundary and hence go to 0. With this definition we can now state the new version of Noether's Theorem.

Theorem 2.7.2. *Every continuous symmetry of the Lagrangian $\mathcal{L}$ yields a current $j^\mu(t, \mathbf{x})$ that satisfies*

$$\partial_\mu j^\mu = 0 \quad \iff \quad \frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{2.54}$$

Proof. Let $\phi \to \phi + \delta\phi$ be a symmetry transformation of $\mathcal{L}$. Now the Lagrangian, under this transformation, changes as follows.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{2.55}$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{2.56}$$

When the Euler-Lagrange equations are satisfied, the first term is 0 and hence the change in the Lagrangian is a total derivative. Since $\delta\mathcal{L} = \partial_\mu F^\mu$ we can define

$$j^\mu \coloneqq \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu \tag{2.57}$$

which satisfies $\partial_\mu j^\mu = 0$ by construction. □

In example 1 we saw that spatial translations led to conservation of momentum. We can then ask what happens when we shift spacetime.

Example 2. Upon the shift of spacetime coordinates we have $\phi(x^\mu) \to \phi(x^\mu - \varepsilon^\mu)$. In order to make this transformation infinitesimal, or written in form $\phi + \delta\phi$ we Taylor expand the transformed field to read $\phi(x^\mu - \varepsilon^\mu) = \phi(x^\mu) - \varepsilon^\mu\partial_\mu\phi(x^\mu)$. Now the Lagrangian changes as

$$\mathcal{L} \to \mathcal{L}(x^\mu - \varepsilon^\mu) = \mathcal{L} - \varepsilon^\mu\partial_\mu\mathcal{L} \tag{2.58}$$

and hence we can use Noether's Theorem 2.7.2 to construct a conserved (Noether) current for this transformation

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left(-\varepsilon^\nu\partial_\nu\phi\right) + \varepsilon^\mu\mathcal{L} \tag{2.59}$$

$$= \varepsilon^\nu T^\mu{}_\nu \tag{2.60}$$

This new symbol $T^\mu{}_\nu$ is called the *energy-momentum tensor* and holds much of the important information about the system at hand. By construction (Noether's Theorem) it has divergence-less rows, i.e., $\partial_\mu T^\mu{}_\nu = 0$. Perhaps the most important component of this object is the 00 component.

$$T^{00} = -\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} = -\mathcal{H} \tag{2.61}$$

where $\mathcal{H}$ is the Hamiltonian density (akin the the Lagrangian density). This value is exactly the energy density, and integrating it yields the total energy i.e. $E = \int T^{00}\,\mathrm{d}^3x$. It is important to note all other components of this object are very familiar, such as momentum density and momentum currents.

Chapter 3

Mathematical Prerequisites

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
>
> John von Neumann

In this chapter we will present some mathematical necessities that will allow us to discuss our problem at a higher level of rigor and detail. The tools we will need are some partial differential equation theory, elements of functional analysiss, and ideas from the calculus of variations.

3.1 Function Spaces

Since this thesis is concerned with whether a certain model contains particular solutions, we will need to specify the space in which we are looking for solutions. Are we looking for continuous functions, Hölder continuous functions, or perhaps even functions that are discontinuous? In this section we will address these questions and motivate the space that is of particular interest.

In this chapter we will *mostly* work with functions of one variable because, in our Q-Ball work, that is all we need, but all definitions and theorems can be extended to functions of more variables. Most of the time the necessary background, and notation bogs down, and makes harder to see the meaning the definitions and theorems are trying to get at.

We begin with a few definitions that are of utmost importance in order to grasp further concepts. In the work to come, pertaining to Q-Balls we will need a notion of "size" that will measure the strength of the solutions. We will here on out assume the underlying structure of a vector space[1].

Definition 3.1.1. *A vector space V over the field $\mathbb{F}$ equipped with a function $\|\cdot\| : V \to \mathbb{F}$ is said to be a* ***normed vector space*** *if $\|\cdot\|$ satisfies the following properties.*

- $\|\alpha x\| = |\alpha|\|x\|$ (Absolutely Homogenuous)
- $\|x\| \geq 0$ and equals 0 if and only if $x = 0$ (Positive Definite)
- $\|x + y\| \leq \|x\| + \|y\|$ (Subadditive)[2]

Where $x, y \in V$ and $\alpha \in \mathbb{F}$.

With this definition, we see the fundamental ideas or notions of "length" embodied. We can now provide a definition which allows one to have a notion of "direction" in the space of question.

Definition 3.1.2. *A vector space V over the field $\mathbb{F}$ equipped with a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ is said to be an* ***inner product space*** *if $\langle\cdot,\cdot\rangle$ satisfies the following properties.*

- $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (Conjugate Symmetric)
- $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ (Linear in first entry)
- $\langle x, x\rangle \geq 0$ and equals 0 if and only if $x = 0$ (Positive Definite)

Where $x, y, z \in V$ and $\alpha, \beta \in \mathbb{F}$.

It is important to note here that once an inner product is defined on a vector space, we can immediately define a norm by $\|x\|_{\langle\cdot,\cdot\rangle} = \sqrt{\langle x, x\rangle}$.

The next definition provided allows us to "fill in the gaps". To give meaning to this statement look at $\mathbb{R}$. If we build $\mathbb{R}$ starting with the natural numbers, adding in the integers, and again adding in the quotients, then we are left with a set that resembles $\mathbb{R}$, but has lots of missing pieces. To fill those in, we take all Cauchy Sequences i.e., sequences such that for any $\varepsilon > 0$

[1] If it is not clear such a space is a vector space, we will provide an argument.

[2] Also known as, satisfying the Triangle Inequality.

there exists an $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $|a_n - a_m| < \varepsilon$, in $\mathbb{R}$ and add the limits to our existing set. This "fills in" the gaps and provides a much nicer space to work with. This motivates the following definition.

Definition 3.1.3. *A normed vector space $(V, \|\cdot\|)$ is **complete** if all Cauchy sequences converge with respect to the metric $d_{\|\cdot\|}(x, y) \coloneqq \|x - y\|$.*

Forcing the spaces we work with to be complete allows one to not worry about pedagogical counterexamples that make basic operations and constructions much more tedious and cumbersome. We can now add this condition to Definitions 3.1.1 and 3.1.2 to obtain further restricted spaces.

Definition 3.1.4. *A **Banach Space** is a complete, normed vector space.*

Definition 3.1.5. *A **Hilbert Space** is a complete, inner product space.*

It is important to remember that we can define completeness on an innerproduct space, because the inner product gives rise to the norm, which gives rise to the metric $d_{\|\cdot\|}$ defined in Definition 3.1.3.

3.1.1 L^p spaces

To begin our work with function spaces we introduce one of the most fundamental spaces, the space of Lebesgue integrable functions. In particular, given a measure space (S, Σ, μ)[3], for each $p \in [1, \infty)$ we define the Lebesgue norm of a function $f : S \to \mathbb{R}$ as

$$\|f\|_{L^p(S)}^p \coloneqq \int_S |f|^p \, \mathrm{d}\mu \,. \tag{3.1}$$

If this value is finite, we write $f \in L^p(S)$. Ideally these functions would form a space with some structure, so that we can add them, and multiply them by scalars, i.e., a vector space.

Lemma 3.1.6. *The set $L^p(\Omega)$ $(p \in [1, \infty))$ with addition and scalar multiplication defined by*

$$(f + g)(x) \coloneqq f(x) + g(x) \tag{3.2}$$

$$(\alpha f)(x) \coloneqq \alpha \cdot f(x). \tag{3.3}$$

[3]Here S is the base set, Σ is a collections of subsets of S that form a σ-algebra (a collections of subsets that contain the base set, is closed under complements, and closed under countable unions) and μ is a function $\mu : \Sigma \to [0, \infty]$ such that $\mu(E) \geq 0$, $\mu(\varnothing) = 0$, $\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) = \sum_{i \in \mathbb{N}} \mu(E_i)$, and finally we have the conditions that $\mu(S) = 1$.

forms a vector space.

Proof. We begin by showing $\alpha f \in L^p(\Omega)$.

$$\begin{aligned}\|\alpha f\|_{L^p(\Omega)} &= \left(\int_\Omega |\alpha f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \\ &\leq |\alpha| \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \\ &= |\alpha| \cdot \|f\|_{L^p(\Omega)} < \infty\end{aligned}$$

Since it's Lebesgue integral is finite, it is also in the space. We know show vector addition also yields a function within $L^p(\Omega)$.

$$\begin{aligned}\|f+g\|_{L^p(\Omega)} &= \left(\int_\Omega |f+g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Omega |2\max\left(|f|,|g|\right)|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \\ &= 2\left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \\ &= 2\|f\|_{L^p(\Omega)} < \infty\end{aligned}$$

Where we have taken $\max\left(|f|,|g|\right) = |f|$ arbitrarily. The rest of the vector space axioms are satisfied almost trivially, and hence for $p \in [1,\infty)$, we have that L^p is a vector space. □

Remark 1. For $p \in (0,1)$ the space L^p does *not* form a vector space because of the fact that the norm fails to be subadditive.

An extremely important, and I would argue the *best*[4] value of p is 2. This is because L^2 is a Hilbert space[5]. The functions that live here are often called square integrable functions and have applications across all areas of mathematics and are extremely prevalent in physics, in particular quantum mechanics.

[4]Not subjective whatsoever.

[5]Remember this is a complete inner product space, pretty much the best version of a vector space you can imagine.

These functions are extremely important, yet most of the time these functions are not even differentiable. As we will see in Section 4.4 the solutions we wish to work with need to be "differentiable" once, in some sense of the word. Thus, the space L^p is much too large, and we need to throw away some.

3.1.2 Sobolev Spaces

In this section we introduce the function spaces we will be working in for the following thesis. In order to introduce this space we will first introduce a weaker notion of differentiability that, again, expands our previous notions. Before this, we provide some more fundamental notions.

Definition 3.1.7. *Given a function a topological space X and a continuous function $f : X \to \mathbb{R}$ we define the* ***support*** *of f to be the closure of the set of points in X where the function does not vanish, i.e.,*

$$\operatorname{supp} f \coloneqq \operatorname{cl}\{x \in X | f(x) \neq 0\} = \operatorname{cl} f^{-1}(\mathbb{R} \setminus \{0\}). \tag{3.4}$$

A very important object is then a function which "lives" on a "small" or "confined" subset of the domain. The following definition makes this precise.

Definition 3.1.8. *A function $f : X \to \mathbb{R}$ is said to have* ***compact support*** *if* $\operatorname{supp} f$ *is a compact subset of X.*

To illustrate this concept we provide the following example.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} g(x) & x \in [a, b] \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

where $g \in \mathrm{C}^0([a, b])$. This is trivially a function of compact support as $\operatorname{supp} f = [a, b] \subset \mathbb{R}$ is a compact set.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $x \mapsto x^2$. It is easy to to see $\operatorname{supp} f = \mathbb{R} \setminus \{0\}$. This set is not only open, but also unbounded and hence not compact in $\mathbb{R}$. Hence this map does not have compact support.

An important characterization of functions with compact support is their behavior on or near the boundary of the compact subset on which the function lives. We provide the following lemma without proof.

Lemma 3.1.9. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function compactly supported on $U \subset \mathbb{R}$. Then f vanishes on the boundary of U i.e., $f|_{\partial U} = 0$.*

Before the definition we introduce two new spaces. Some of the "nicest" functions are ones that we can differentiate as many times as we'd like. To denote infinitely differentiable functions, we have $\mathrm{C}^\infty(U)$ for some open subset $U \subset \mathbb{R}$. A subset of these functions are the ones that vanish on the boundary of U, i.e., by Lemma 3.1.9 they have compact support and this is denoted $\mathrm{C}_c^\infty(U)$.

In Subsection 3.1.1 we saw L^p spaces. We now introduce a slightly larger set of functions called $L^p_{\mathrm{loc}}(U)$ to mean functions who are locally summable on U. This means functions which are Lebesgue integrable on all compact subsets of U. To fully understand this space we provide an example.

Example 5. The space $L^1(\mathbb{R})$ is defined by functions that satisfy

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty.$$

In order for a function ϕ to satisfy this property, it must decay to 0 as $|x| \to \infty$. The space $L^1_{\mathrm{loc}}(\mathbb{R})$ is much larger. Note compact subsets of $\mathbb{R}$ are all of the form $[a, b]$ for some $a, b \in \mathbb{R}$. Thus in order for ϕ to be in $L^1_{\mathrm{loc}}(\mathbb{R})$ we must have

$$\int_a^b |\phi(x)| \, \mathrm{d}x < \infty$$

for all $a, b \in \mathbb{R}$. This condition is much less restrictive as we now allow for objects such as constant functions, periodic functions, and any continuous function defined on $\mathbb{R}$.

With these new spaces we can now define our weakened notion of derivative.

Definition 3.1.10. *If $f, g \in L^1_{\mathrm{loc}}(U)$, we call g the **weak derivative** of f provided*

$$\int_U f\varphi' \, \mathrm{d}x = -\int_U g\varphi \, \mathrm{d}x \tag{3.6}$$

for all $\varphi \in \mathrm{C}_c^\infty(U)$.

This definition comes from integration by parts $\int u \, \mathrm{d}v = uv|_\partial - \int v \, \mathrm{d}u$ where the function we are integrating against, in our case φ' vanishes on

the boundary. Thus we can throw away the boundary term and we are left with another integral. This sense of derivative is much weaker than ordinary derivatives and allows us to weakly derive things like step functions or ramp functions to say the least. We can now easily prove the uniqueness of these objects.

Lemma 3.1.11. *The weak derivative of f, should it exist, is unique.*

Proof. Let us assume g_1, g_2 are both weak derivatives of f. Then we have

$$\int_U f\varphi' \,\mathrm{d}x = -\int_U g_1\varphi \,\mathrm{d}x = -\int_U g_2\varphi \,\mathrm{d}x \tag{3.7}$$

for all $\varphi \in \mathrm{C}_c^\infty(U)$ and hence, by linearity of the Lebesgue integral,

$$\int_U (g_1 - g_2)\,\varphi \,\mathrm{d}x = 0 \tag{3.8}$$

and thus $g_1 \overset{\text{a.e.}}{=} g_2$[6]. □

With the basic ideas of weak differentiability we can construct the Sobolev Space. Remember L^p spaces were far too big and contained functions we wished not to consider. Thus in defining Sobolev spaces we will first start in L^p and require some orders of weak derivatives also lie in various L^p spaces.

Definition 3.1.12. *The **Sobolev Space** $W^{k,p}(U)$ ($k, p \in \mathbb{N}$) consists of functions $f \in L^p_{\mathrm{loc}}(U)$ such that $f^{(i)}$ exists in the weak sense and also belongs to $L^p(U)$.*

This space admits a natural norm that makes it a Banach space (3.1.4).

$$\|f\|^p_{W^{k,p}(U)} \coloneqq \sum_{i=0}^{k} \int_U |f^{(i)}|^p \,\mathrm{d}x \tag{3.9}$$

An immediate important subset of the Sobolev space is $W_0^{k,p}(U)$ which denotes the closure of $\mathrm{C}_c^\infty(U)$ in $W^{k,p}(U)$. Another particularly interesting case is when $p = 2$, and in this case we write $H^k(U) = W^{k,2}(U)$ where we use the letter H because this space inherits a completeness, and an inner product from L^2 and hence it is a Hilbert space.

[6]The "a.e." over the "=" means these functions are equal **a**lmost **e**verywhere, or put differently, they are the same in the integral sense, but not necessarily pointwise.

3.2 The Calculus of Variations

In this section we present the underlying ideas that allow us to solve (4.48). This method provides an alternative way to view partial differential equations that is often insightful and lends itself to Lagrangian systems.

The basic idea of partial differential equations is to solve

$$D[u] = 0 \tag{3.10}$$

where D is some differential operator. The calculus of variations treats this differential operator as the "derivative" of some functional $I[\cdot]$, i.e., $I'[u] = A[u]$. The fundamental problem of PDE's (3.10) is then written as $I'[u] = 0$. The advantage this method has over attempting to directly solve PDE's is that often times finding critical points may be easier than a direct proof of existence of solutions.

To better understand the basic idea of this we provide an illustrative example.

Example 6. Suppose we wish to solve $u'' = \sin u$ for some function $u \in \mathrm{C}^2(\mathbb{R})$. Directly this problem can be very difficult, so let us try and find a functional, whose derivative represents the operator $A = \partial_x^2 - \sin(\cdot)$. If we define

$$I[u] = \int \frac{1}{2}(u')^2 - \cos u \, \mathrm{d}x \tag{3.11}$$

then take its functional derivative as we learned in Chapter 2 we see the Euler-Lagrange equations of this functional yields the original partial differential equation. Hence if we are able to find minima of $I[\cdot]$, then we have found solutions to the original problem.

In this example we found I to be the integral of another function. As this is normally the case, we denote by L, the "*Lagrangian*" of such a functional. Thus for the above example we have $L = \frac{1}{2}(u')^2 - \cos u$. This function is called the Lagrangian simply because resembles the action equation we saw in Equation (2.1).

3.2.1 When does inf = min?

In this section we study when a functional attains its minima. If it does, then inf = min, so really you can stop reading this section now since I answered

the main question. In this section we will study further properties which ensure a functional does attain its minima.

It is easy to see that given, even an infinitely differentiabl function, it need not attain its minima as seen in $f(x) = \mathrm{e}^x$. Thus, in order to force it to attain its minima we must control the function at "large" input. A way to do this is to say, as $|x| \to \infty$ we also have $f(x) \to +\infty$. The following definition generalizes this to functionals.

Definition 3.2.1. *A functional I with Lagrangian L is said to be coercive if, for a given q, there exist constants α and β such that*

$$L(f', f, x) \geq \alpha |f'|^q - \beta. \tag{3.12}$$

This definition immediately leads to an alternative, but equivalent description in terms of the functional.

Lemma 3.2.2. *A functional is coercive if and only if it satisfies a coercivity condition i.e.,*

$$I[f] \geq \delta \|f'\|^q_{L^q(U)} - \gamma \tag{3.13}$$

Proof. To show these are equivalent, we simply integrate the Lagrangian and use the inequality.

$$\begin{aligned} I[f] &= \int_U L(f', f, x)\,\mathrm{d}x \\ &\geq \int_U \alpha |f'|^q - \beta\,\mathrm{d}x \\ &= \delta \|f'\|^q_{L^q(U)} - \gamma \end{aligned}$$

□

Chapter 4

An introduction to Q-Balls

> The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.
>
> Sidney Coleman
> *Inventor of Q-Balls*

With the tools from Chapter 2 we now have the tools to break down our problem at hand. In this chapter we will present a model with soliton solutions and then define and study $\mathcal{Q}$-balls and $\mathcal{Q}$-vortices.

4.1 *The* Lagrangian

In this section we first and foremost introduce the Lagrangian of study. We then proceed to apply techniques learned through Chapter 2 to better understand the model.

Through the rest of this thesis we study the following Lagrangian of a complex scalar field in $3+1$ dimensions.

$$\mathcal{L} \coloneqq -\partial_\mu \Phi \, \partial^\mu \overline{\Phi} - V(|\Phi|) \tag{4.1}$$

This Lagrangian is special as even without specifying the potential we have a $\mathsf{U}(1)$ symmetry $\Phi \to \mathrm{e}^{\mathrm{i}\alpha}\Phi$ as we saw in Section 2.6.1. While here we

work with the abelian group $\mathsf{U}(1)$, Q-Balls have been developed with more general gauge groups as in [21]. With more ideas about symmetry from Noether's Theorem and section 2.7 we can better understand this symmetry. In infinitesimal form ($\alpha \ll 1$) this symmetry is (using the Taylor expansion of e^x) $\Phi \to \Phi + i\alpha\Phi$ and $\overline{\Phi} \to \overline{\Phi} - i\alpha\overline{\Phi}$. Under this infinitesimal form, the Lagrangian changes as follows.

$$\begin{aligned}
\mathcal{L} &\to -\partial_\mu(\Phi + i\alpha\Phi)\,\partial^\mu(\overline{\Phi} - i\alpha\overline{\Phi}) - V(|\Phi + i\alpha\Phi|) && (4.2)\\
&= -\left(\partial_\mu\Phi + i\alpha\partial_\mu\Phi\right)\left(\partial^\mu\overline{\Phi} - i\alpha\overline{\Phi}\right) - V(|\Phi|) && (4.3)\\
&= -\partial_\mu\Phi\,\partial^\mu\overline{\Phi} + \cancelto{0}{\alpha^2\partial_\mu\Phi\,\partial^\mu\overline{\Phi}} - V(|\Phi|) && (4.4)\\
&= \mathcal{L} && (4.5)
\end{aligned}$$

Where we have used the fact that $|\Phi + i\alpha\Phi| = \left[(\Phi + i\alpha\Phi)\left(\overline{\Phi} - i\alpha\overline{\Phi}\right)\right]^{1/2} = \left[\Phi\overline{\Phi} + \alpha^2\Phi\overline{\Phi}\right]^{1/2}$ and with the fact that $\alpha \ll 1$ we can throw away higher powers of α to conclude $|\Phi + i\alpha\Phi| = |\Phi|$. Thus, as expected this transformation has no effect on the Lagrangian and hence we can trivially write $\delta\mathcal{L} = 0$. For simplicity the Noether's Theorem in Chapter 2 was for a Lagrangian dependent on one field. For our Lagrangian, we treat Φ and $\overline{\Phi}$ as separate fields[1] and in a more generalized Noether's Theorem we simply add terms to the current as follows.

$$\begin{aligned}
\alpha j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Phi})}\delta\overline{\Phi} && (4.6)\\
&= -\partial^\mu\overline{\Phi}\,(i\alpha\Phi) - \partial^\mu\Phi\left(-i\alpha\overline{\Phi}\right) && (4.7)\\
&= \alpha i\left(\overline{\Phi}\partial^\mu\Phi - \Phi\partial^\mu\overline{\Phi}\right) && (4.8)
\end{aligned}$$

It is important to note here how we went from (4.6) to (4.7) as $\partial_\mu\overline{\Phi}$ does not appear in the original Lagrangian (4.1). As stated in section 2.3.1 we can raise and lower indices by contracting with the underlying metric. Thus we can rewrite (4.1) as

$$\mathcal{L} = -\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\overline{\Phi} - V(|\Phi|) \tag{4.9}$$

where $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$ and the inverse of $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \eta^{\mu\nu}$.

[1] Written in terms of real components we have $\Phi = \phi_1 + i\phi_2$ and $\overline{\Phi} = \phi_1 - i\phi_2$ which are clearly linearly independent functions as long as ϕ_2 is non-trivial.

Now we can take $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Phi})}$ as we have a $\partial_\mu \overline{\Phi}$ term in (4.9). Explicitly this is

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Phi})} = \eta^{\nu\mu}\partial_\nu \Phi \tag{4.10}$$

$$= \partial^\mu \Phi \tag{4.11}$$

Now that we have constructed the Noether current for the $\mathsf{U}(1)$ internal symmetry, we can construct what is called the *Noether Charge*. We know the current satisfies

$$\frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{4.12}$$

by Theorem 2.7.2. Thus, we can define the Noether Charge

$$Q = \int_{\mathbb{R}^3} j^0 \, \mathrm{d}^3 x \tag{4.13}$$

which we will show does not depend on time, and hence is a constant of motion. To show this we will simply take the time derivative of (4.13) and show it is equal to 0.

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^3} j^0 \, \mathrm{d}^3 x \tag{4.14}$$

$$= \int_{\mathbb{R}^3} \frac{\partial j^0}{\partial t} \, \mathrm{d}^3 x \tag{4.15}$$

$$= -\int_{\mathbb{R}^3} \nabla \cdot \mathbf{j} \, \mathrm{d}^3 x \tag{4.16}$$

We can now use the Divergence Theorem to write this integral as a surface term over the boundary of integration. However since $\mathbf{j}$ is supposed to be well behaved it decays to 0 at infinity, and thuse the surface term is 0. Thus we conclude $\frac{\partial Q}{\partial t} = 0$. This is very important as constants of motion are extremely helpful not only to solve problems, but also as deeper insights into the problem. Note that this argument was provided without any reference to a specific Lagrangian, as long as we have a current, we have an associated charge. Thus we can work out what the Noether Charge is for (4.1). We first put all indices downstairs as these correspond to our regular derivatives (derivatives with respect to vectors, rather than derivatives with respect to co-vectors, or linear functionals, or dual-vectors or whatever you'd like to call

them).

$$j^\mu = \mathrm{i}\left(\overline{\Phi}\partial^\mu\Phi - \Phi\partial^\mu\overline{\Phi}\right) \tag{4.17}$$
$$= \mathrm{i}\eta^{\mu\nu}\left(\overline{\Phi}\partial_\nu\Phi - \Phi\partial_\nu\overline{\Phi}\right) \tag{4.18}$$
$$j^0 = \mathrm{i}\eta^{0\nu}\left(\overline{\Phi}\partial_\nu\Phi - \Phi\partial_\nu\overline{\Phi}\right) \tag{4.19}$$

In order to proceed we recognize the only component of the "vector" $\eta^{0\nu}$ that is non-vanishing is the first component $\eta^{00} = -1$ so our sum over ν is really just one term. Thus our Noether Charge is given as

$$Q = \int j^0 \,\mathrm{d}^3x \tag{4.20}$$
$$= \mathrm{i}\int\left(\Phi\dot{\overline{\Phi}} - \overline{\Phi}\dot{\Phi}\right)\mathrm{d}^3x \tag{4.21}$$

In terms of a quantized theory Q often plays the role of either some sort of electric charge, or particle number. Thus we only have these fundamental facts of nature because of these symmetries of the Lagrangian and Noether Theorem!

The last construction before moving on is the Energy-Momentum Tensor as we saw in Example 2. This is an important object for many reasons, but it will give us the energy density which we be a vital piece of our problem. From (2.59) we have

$$T^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Phi})}\partial^\nu\overline{\Phi} + \eta^{\mu\nu}\mathcal{L} \tag{4.22}$$
$$= \partial^\mu\overline{\Phi}\partial^\nu\Phi + \partial^\mu\Phi\partial^\nu\overline{\Phi} + \eta^{\mu\nu}\mathcal{L} \tag{4.23}$$

With this object we can find all sorts of physics quantities that might be of interest such as $T_{00} \iff$ Energy, $T_{0\varphi} \iff$ Angular Momentum, etc,.

4.2 Soliton Ansatz

In order to proceed we make some fundamental assumption about our solution Φ and it's structure. As nice as it would be to have Φ be independent of time, we would have an accompanying vanishing Noether Charged which we worked so hard to get. Thus we require our solution to indeed have a time dependence, but only a harmonic one $\mathrm{e}^{\mathrm{i}\omega t}$. We also want spherically

symmetric solutions as the objects we are studying are Q-Balls. To recap, we force our solutions into the form

$$\Phi(t, \mathbf{x}) = \phi(r)\mathrm{e}^{\mathrm{i}\omega t} \tag{4.24}$$

where $\phi(r)$ is a real field. Under this spherically symmetric soliton ansatz the implicit time dependence of many of our objects disappears. For example our Lagrangian (4.1) now reads

$$\begin{aligned}
\mathcal{L} &= -\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\overline{\Phi} - V(|\Phi|) && (4.25)\\
&= |\dot{\Phi}|^2 - |\nabla\Phi|^2 - V(|\Phi|) && (4.26)\\
&= \omega^2\phi^2 - \phi_r^2 - V(\phi) && (4.27)
\end{aligned}$$

which is becoming much more "differential equation"-esque. Most importantly though, $\mathcal{L}$ does not have any implicit or explicit t dependence and hence we have conservation of energy.[2]

The energy density (T^{00}) is also very important and we can now calculate that with (4.24) in mind. Remembering (4.23) we have

$$\begin{aligned}
T^{00} &= 2|\dot{\Phi}|^2 - \mathcal{L} && (4.28)\\
&= 2|\dot{\Phi}|^2 - |\dot{\Phi}|^2 + |\nabla\Phi|^2 + V(|\Phi|) && (4.29)\\
&= \omega^2\phi^2 + \phi_r^2 + V(\phi) && (4.30)
\end{aligned}$$

where ϕ_r denotes $\frac{\mathrm{d}\phi}{\mathrm{d}r}$.[3] With this we can easily calculate the total energy of the system.

$$\begin{aligned}
E &= \int T^{00}\,\mathrm{d}^3x && (4.31)\\
&= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}r\left(\omega^2\phi^2 + \phi_r^2 + V(\phi)\right)r^2\sin\theta && (4.32)\\
&= 4\pi\int_0^{\infty} r^2\left(\omega^2\phi^2 + \phi_r^2 + V(\phi)\right)\mathrm{d}r && (4.33)
\end{aligned}$$

[2]This fact is proved twice. First we can attack it via Noether's Theorem because of the symmetry $t \to t + s$ which yields a conserved current which ends up having a Noether Charge of the energy. Or, we can show that the Lagrangian being independent of time is equivalent to the Hamiltonian being time-independent. If the Hamiltonian is independent of time, then surely the energy is constant by construction of the Hamiltonian.

[3]I just don't like to commit atrocities such as ϕ'^2.

The conserved charge (4.21) is then

$$Q = \mathrm{i}\int \left(-\mathrm{i}\omega\phi^2 - \mathrm{i}\omega\phi^2\right) \mathrm{d}^3x \tag{4.34}$$

$$= 2\omega \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}r\, \phi^2 r^2 \sin\theta \tag{4.35}$$

$$= 8\pi\omega \int_0^{\infty} r^2\phi^2\, \mathrm{d}r\,. \tag{4.36}$$

Now of course one is very interested in the equations of motion of a system once a model is proposed. Thus we investigate the Euler-Lagrange equations (2.3) for our Lagrangian. Note there will be two because our Lagrangian is a function of both Φ and $\overline{\Phi}$. Here we present the ELE for the conjugate field so the field equation comes out in terms of the field Φ rather than $\overline{\Phi}$.

$$\frac{\partial\mathcal{L}}{\partial\overline{\Phi}} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Phi})} \tag{4.37}$$

Computing these derivative we have

$$\frac{\partial\mathcal{L}}{\partial\overline{\Phi}} = -\frac{\mathrm{d}V(|\Phi|)}{\mathrm{d}|\Phi|}\frac{\mathrm{d}|\Phi|}{\mathrm{d}\overline{\Phi}} \tag{4.38}$$

$$= -V'\frac{\Phi}{2|\Phi|} \tag{4.39}$$

In order to compute $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Phi})}$ we use the fact that we can exchange the upper and lower indices in the Lagrangian (4.1) to have the first term read $-\partial^\mu\Phi\partial_\mu\overline{\Phi}$. Then it is easy to compute the right hand side as follows.

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Phi})} = -\partial^\mu\Phi \tag{4.40}$$

Hence we can write our full Euler-Lagrange equation as

$$V'\frac{\Phi}{2|\Phi|} = \partial_\mu\partial^\mu\Phi \tag{4.41}$$

On the right hand side we have the operator $\partial_\mu\partial^\mu$ which we can easily compute using (2.25) to see $\partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$ where ∇^2 is the Laplacian for spatial

coordinates. Hence, in spherical coordinates[4] (with a radially symmetric Φ, that is if $\Phi(t, r)$) we have

$$V' \frac{\Phi}{2|\Phi|} = -\ddot{\Phi} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \Phi' \right) \tag{4.42}$$

$$= -\ddot{\Phi} + \frac{2\Phi'}{r} + \Phi'' \tag{4.43}$$

where $\dot{f}$ denotes a time derivative and f' denotes a spatial, or in our case r derivative. Hence upon substituting our spherical symmetric ansatz (4.24) we have the following differential equation.

$$\omega^2 \phi + \frac{2\phi'}{r} + \phi'' = \frac{V'}{2} \tag{4.44}$$

Since we often only care about the global behavior of the potential, we can arbitrarily scale it, and hence absorb the factor 2 into the potential so the right hand side of (4.44) reads V'.

4.3 Adding a Twist

With the basic idea of Q-vortices behind us it is then natural to ask, as done in [22], if there exist spinning generalizations of the objects studied in Section 4.2. To study this possibility we posit a solution Φ in $3 + 1$[5] dimensions to have a particular dependence on an angular variable as

$$\Phi(t, \mathbf{x}) = \phi(r) \, \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}N\theta} \tag{4.45}$$

where we are now working in cylindrical coordinates (t, r, θ, z). As before we calculate some derivatives in order to write the field equations. The potential term is exactly the same as before so we leave out the repetition and continue to the Laplacian which in cylindrical coordinates reads $\nabla^2 = r^{-1} \partial_r \left(r \partial_r \right) + r^{-2} \partial_\theta^2 + \partial_z^2$.

$$\partial_\mu \partial^\mu \Phi = -\ddot{\Phi} + \nabla^2 \Phi \tag{4.46}$$

$$= -\ddot{\Phi} + \frac{\Phi_r}{r} + \Phi_{rr} + \frac{1}{r^2} \Phi_{\theta\theta} \tag{4.47}$$

[4] In spherical coordinates the radial part of the Laplacian is $\frac{1}{r^2} \partial_r \left(r^2 \partial_r \right)$.

[5] We are indeed working in 4-spacetime dimensions, but we are not allowing our solution to have a z dependence. This is because we are attempting to model a spinning Q-vortex solution travelling in the z-direction.

And hence with our spinning soliton ansatz (4.45) our differential equation reads

$$\phi'' + \frac{\phi'}{r} + \omega^2\phi - \frac{N^2}{r^2}\phi = V' \tag{4.48}$$

where we have used the same scaling argument as used above on Equation (4.44).

We can also go ahead and calculate the energy of this system. However because this system travels along the z-axis we calculate the energy per unit length. This simply means integrating the energy density T^{00} over a 2-dimensional plane rather than all of $\mathbb{R}^3$.

$$T^{00} = |\dot{\Phi}|^2 + |\nabla\Phi|^2 + V(|\Phi|) \tag{4.49}$$

$$= \omega^2\phi^2 + \left\| \begin{pmatrix} \phi_r \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}N\theta} \\ \frac{\mathrm{i}N}{r}\phi \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}N\theta} \end{pmatrix} \right\|^2 + V(\phi) \tag{4.50}$$

$$= \omega^2\phi^2 + \phi_r^2 + \frac{N^2}{r^2}\phi^2 + V(\phi) \tag{4.51}$$

Hence the total energy per unit length is give by

$$E_\ell = \int T^{00}\,\mathrm{d}^2x \tag{4.52}$$

$$= \int_0^{2\pi}\mathrm{d}\theta \int_0^\infty \mathrm{d}r\; r\left(\omega^2\phi^2 + \phi_r^2 + \frac{N^2}{r^2}\phi^2 + V(\phi)\right) \tag{4.53}$$

$$= 2\pi\int_0^\infty r\left(\omega^2\phi^2 + \phi_r^2 + \frac{N^2}{r^2}\phi^2 + V(\phi)\right)\mathrm{d}r \tag{4.54}$$

Before we go on to discuss the angular momentum of this system we note that this formula for the total energy is extremely important as it provides us with the boundary conditions $\phi(0) = 0$ and $\phi(r) \to 0$ as $r \to \infty$. This is implied by (4.54) as if this was not the case then the third term would blow up and we would not have a system with finite energy which is physically unrealizable.

What is of real interest to us is the angular momentum of this system. In order to calculate that we will first calcuate the Noether Charge (4.13) which is almost identical to the previous calculation in Section 4.2 although

here we calculate the charge per unit length.

$$Q_\ell = \int j^0 \,\mathrm{d}^2x \tag{4.55}$$

$$= 2\omega \int_0^{2\pi} \int_0^\infty r \,\mathrm{d}r \,\phi^2 \tag{4.56}$$

$$= 4\pi\omega \int_0^\infty r\phi^2 \,\mathrm{d}r \tag{4.57}$$

We can now extract the angular momentum per unit length from the energy momentum tensor if we integrate $T^{0\theta}$. We fist calculate magnitude of the value to see how much angular momentum these objects have.

$$|T^{0\theta}| = \dot{\overline{\Phi}}\Phi_\theta + \dot{\Phi}\overline{\Phi}_\theta \tag{4.58}$$

$$= (-\mathrm{i}\omega\phi)(\mathrm{i}N\phi) + (\mathrm{i}\omega\phi)(-\mathrm{i}N\phi) \tag{4.59}$$

$$= 2N\omega\phi^2 \tag{4.60}$$

Hence, the angular momentum per unit length is give by

$$J_\ell = \int T^{0\theta} \,\mathrm{d}^2x = 2\omega N \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty r\phi^2 \,\mathrm{d}r \tag{4.61}$$

$$= 4\pi\omega N \int_0^\infty r\phi^2 \,\mathrm{d}r = NQ_\ell \tag{4.62}$$

What is important about this formula is that for a given Q_ℓ, the angular momentum is purely determined by N.

Now that we all of the basic quantities calculate we can revisit (4.48). In order to continue the work at hand we must specify a potential in which we wish to study. Before we do this, we look to the conditions set forth on V as shown in [6]. Here Coleman shows the parameter ω must satisfy

$$\min_\phi \frac{2V(\phi)}{\phi^2} \leq \omega^2 < \left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\right|_{\phi=0} \tag{4.63}$$

due to considerations energy considerations. These arguments can be found in Appendix A. In order for ω to have a non-empty set to vary in (4.63) must be satisfied. The only renormalizable potential in the theory given by $V = \frac{1}{2}\mu^2\phi^2 + \lambda\phi^4$ does not satisfy this condition, and hence nonrenormalizable potentials must be considered [6]. In this work we take the potential to read

$$V(\phi) \coloneqq \lambda\left(\phi^6 - a\phi^4 + b\phi^2\right). \tag{4.64}$$

Before we continue some remarks are in order. In field theory, suppose we are given a potential $U(\varphi) = \alpha_0 + \alpha_1\varphi + \alpha_2\varphi^2 + \cdots + \alpha_n\varphi^n$. Then the "quadratic self coupling" is lingo for α_2 and in a theory with massive particles it is the square of the mass m^2. This value can also be found as $m^2 \coloneqq U''(0)$. Hence for the potential given in (4.64) we have $m^2 = V''(0) = 2\lambda b$ which is also equal to $\frac{2V(\phi)}{\phi^2}\Big|_{\phi=0}$. Hence we can interpret the condition (4.63) as saying the the function $\frac{2V(\phi)}{\phi^2}$ must dip below m^2 in order for ω to range in a non-empty set. The reasons for this are given in Appendix A.

As convention we take $V(0) = 0$, which is always possible as adding a constant to the potential is immaterial. We also wish m^2 to be positive. Since $m^2 = 2\lambda b$, either $\lambda, b \in \mathbb{R}_+$ or $\lambda, b \in \mathbb{R}_-$. However, we wish $V(0)$ to be a global minimum, and hence λ must be positive, and by the above, so must b. In order for (4.63) to be satisfied, the potential must have a local minima at $\tilde{\phi} \neq 0$, and hence a must also be positive and non-zero.

The last condition we impose on the potential is that it must not dip below 0 as to have a global minima at a point $\tilde{\phi} \neq 0$. The following argument ensures this is the case for $\phi \neq 0$.

$$\begin{aligned} V(\phi) = \lambda\left(\phi^6 - a\phi^4 + b\phi^2\right) &> 0 \\ \phi^4 - a\phi^2 + b &> 0 \\ x^2 - ax + b &> 0 \end{aligned}$$

where $x = \phi^2$, and hence the last polynomial written down has no real roots provided it is always greater than 0. Hence the discriminant is negative, which imposes $a^2 - 4b < 0$, or $b > \frac{a^2}{4}$. In recap we have $\lambda, a, b \in \mathbb{R}_+$, and $b > \frac{a^2}{4}$.

4.4 Constrained Minimization

In order to show the existence of solutions to (4.48) we will make use of the Calculus of Variations. In particular we will define a functional I and constraint functional J so that when I is minimized with respect to J we obtain the nonlinear eigenvalue problem $\delta I = \chi\delta J$.

To begin we define the action functional $I : W_0^{1,2}(0, R) \to \mathbb{R}$ as

$$I[\phi] \coloneqq \int_0^R \left[\frac{1}{2}\left(r\phi_r^2 + \frac{N^2}{r}\phi^2\right) + r\lambda\left(\phi^6 - a\phi^4 + b\phi^2\right)\right] \mathrm{d}r \tag{4.65}$$

and constraint functional $J : W_0^{1,2}(0, R) \to \mathbb{R}$,

$$J[\phi] \coloneqq 2\pi \int_0^R r\phi^2 \, \mathrm{d}r \, . \tag{4.66}$$

Using

$$I_0 \coloneqq \inf_{\phi} \{ I[\phi] \ \mid \ J[\phi] = J_0 \in (0, \infty) \} \tag{4.67}$$

we can now prove this problem is well defined.

Before we prove the well defined-ness we prove a few minor results which will aid in the proof.

Lemma 4.4.1. *The differential equation (4.48) may be treated as a nonlinear eigenvalue problem such that ω^2 appears in the Lagrange multiplier of (4.67).*

Proof. To prove this we will first find the first variation of each functional.

$$\begin{aligned}
\delta I &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I[\phi + \varepsilon\psi]\bigg|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^R \frac{1}{2} \left(r \left(\phi_r + \varepsilon\psi_r\right)^2 + \frac{N^2}{r} \left(\phi + \varepsilon\psi\right)^2 \right) \\
&\quad + r\lambda \left(\left(\phi + \varepsilon\psi\right)^6 - a \left(\phi + \varepsilon\psi\right)^4 + b \left(\phi + \varepsilon\psi\right)^2 \right) \mathrm{d}r \Big|_{\varepsilon=0} \\
&= \int_0^R r\phi_r\psi_r + \frac{N^2}{r}\phi\psi + r\lambda \left(6\phi^5 - 4a\phi^3 + 2b\phi\right) \psi \, \mathrm{d}r \\
&= \int_0^R \left[-\left(r\phi_r\right)_r + \frac{N^2}{r}\phi + r\lambda \left(6\phi^5 - 4a\phi^3 + 2b\phi\right) \right] \psi \, \mathrm{d}r
\end{aligned} \tag{4.68}$$

Where, in moving from the second to last, to last line we have integrated by parts on the first term and used the fact that ϕ and ψ vanish on the boundary.

$$\begin{aligned}
\delta J &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[\phi + \varepsilon\psi]\bigg|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} 2\pi \int_0^R r \left(\phi + \varepsilon\psi\right)^2 \mathrm{d}r \\
&= 4\pi \int_0^R r\phi\psi \, \mathrm{d}r
\end{aligned} \tag{4.69}$$

Hence when we examine the subsequent Lagrange multiplier equation $\delta I = \chi \delta J$, or equivalently $\delta I - \chi \delta J = 0$ we obtain

$$\int_0^R \left[-\left(r\phi_r\right)_r + \frac{N^2}{r}\phi + r\lambda\left(6\phi^5 - 4a\phi^3 + 2b\phi\right) - 4\pi r\chi\phi \right] \psi \, \mathrm{d}r = 0. \quad (4.70)$$

This must be true for all ψ, so the only way this can happen is if the bracketed term vanishes. Thus, expanding the first term, and dividing through by r we obtain

$$\frac{\phi_r}{r} + \phi_{rr} - \frac{N^2}{r^2}\phi - \lambda\left(6\phi^5 - 4a\phi^3 + 2b\phi\right) + 4\pi\chi\phi \quad (4.71)$$

which gives us the equation $4\pi\chi = \omega^2$ for the ω^2 value in terms of the Lagrange multiplier. □

Lemma 4.4.2. *For all real functions ϕ the following inequality holds.*

$$\phi^6 - a\phi^4 + b\phi^2 \geq \phi^2\left(b - \frac{a^2}{4}\right) \quad (4.72)$$

Proof. Because of the *amazing* result that all real numbers squared are non-negative we have $\left(\phi^2 - \frac{a}{2}\right)^2 \geq 0$. And now, some manipulations.

$$\left(\phi^2 - \frac{a}{2}\right)^2 \geq 0$$

$$\phi^4 - a\phi^2 + \frac{a^2}{4} + b \geq b$$

$$\phi^6 - a\phi^4 + b\phi^2 \geq \phi^2\left(b - \frac{a^2}{4}\right)$$

□

Lemma 4.4.3. *Given a function $\phi : [0, R] \to \mathbb{R}$ satisfying $\phi(0) = 0 = \phi(R)$, we have the following integral inequality.*

$$\int_0^R r\phi^2 \, \mathrm{d}r \leq R^2 \int_0^R \frac{\phi^2}{r} \, \mathrm{d}r \quad (4.73)$$

Proof. We begin with the simple fact that for $r \in (0, R)$, we have $r^2 \leq R^2$. Multiplying by ϕ^2 changes nothing, and we then divide both sides by r to give $r\phi^2 \leq R^2\frac{\phi^2}{r}$. By monotonicity[6] of the integral from 0 to R we have

$$\int_0^R r\phi^2 \, \mathrm{d}r \leq R^2 \int_0^R \frac{\phi^2}{r} \, \mathrm{d}r \quad (4.74)$$

[6]Monotonicity meaning if $f \leq g$, then $\int f \leq \int g$.

which proves the desired fact. □

We now have the tools to prove the functional (4.65) is coercive.

Theorem 4.4.4. *The functional defined in (4.65) is coercive.*

Proof. Using Lemma 4.4.2 and 4.4.3 we can immediately write the following inequality.

$$I[\phi] \geq \int_0^R \frac{1}{2}\left(r\phi_r^2 + r\frac{N^2}{R^2}\phi^2\right) + \lambda r\left(b - \frac{a^2}{4}\right)\phi^2 \,\mathrm{d}r \tag{4.75}$$

Grouping the ϕ^2 terms we can use the constraint (4.66) (which is prescribed ahead of time) to write

$$I[\phi] \geq \frac{1}{2}\int_0^R r\phi_r^2 \,\mathrm{d}r + \underbrace{\left[\frac{N^2}{R^2} + \lambda\left(b - \frac{a^2}{4}\right)\right]}_{\geq 0} J_0 \tag{4.76}$$

$$\geq \frac{1}{2}\int_0^R r\phi_r^2 \,\mathrm{d}r \tag{4.77}$$

We now claim the proof is effectively done. As in the definition of coercivity (3.2.1), we have written $I[\phi] \geq \delta\|\phi_r\|^2_{L^2(0,R)} - \gamma$ with $\gamma = 0$, and $\delta = \frac{1}{2}$. The only concern a careful reader might have is that we are integrating $r\,\mathrm{d}r$ as opposed to something like $\mathrm{d}x\,\mathrm{d}y$, but this is simply the measure we must work with in handling the problem at hand given in polar coordinates. □

With the coercivity of $I[\,\cdot\,]$ proved, we can use Theorem 2 in Section 8.2 of [23] to say that if $I[\,\cdot\,]$ is also *lower semi-continuous*, then the existence problem is well defined and has a solutions in $W_0^{1,2}(0,R)$. By proof by intimidation, it is ***clearly*** true, and every good student should know that $I[\,\cdot\,]$ is lower semi-continuous!

Chapter 5

Numerical Investigations of Q-vortex solutions

> So, ultimately, in order to understand nature it may be necessary to have a deeper understanding of mathematical relationships. But the real reason is that the subject is enjoyable...
>
> Richard Feynman

In this Chapter we use the constrained minimization problem put forth in Section 4.4 to numerically solve the equation of motion. This numerical solution provides further evidence for the existence of Q-Vortex solitons and the profiles allow us to better understand the nature of the objects.

5.1 Finite Element Formalism

As we saw in Chapter 4, in particular Section 4.4 the constrained minimization approach to the problem is well defined because of the coercivity of the functional $I[\,\cdot\,]$ along with its lower semi-continuity. This means we can utilize it as a tool to numerically solve the equation of motion (4.48) if we can reduce the infinite dimensional domain $W_0^{1,2}(0,R)$ to finite dimensional.

To attack this problem we will begin by taking a subspace $A \subset W_0^{1,2}(0,R)$ spanned by n vectors $\{\psi_i\}_{i=1}^n$. We take the ψ's to be orthonormal with respect to the following inner product

$$\langle \alpha, \beta \rangle \coloneqq 2\pi \int_0^P r\alpha\beta \, \mathrm{d}r \tag{5.1}$$

where $\alpha, \beta \in A$. The form of (5.1) is suggested by the Noether Charge we saw in Chapter 4 and $\langle \varphi, \varphi \rangle = \|\varphi\|^2$ corresponds to almost exactly the Noether Charge (4.57). Using this this basis we approximate functions $f \in W_0^{1,2}(0,P)$ simply as a linear combinations

$$f = \sum_{i=1}^n a_i \psi_i \tag{5.2}$$

with all $a_i \in \mathbb{R}$. Inserting this approximation into our functional (4.65) we can define a new multivariable function

$$F(\mathbf{a}) \coloneqq I[f] = I\left[\sum_{i=1}^n a_i \psi_i\right] \tag{5.3}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)^\intercal \in \mathbb{R}^n$ is called the variational vector. Now our problem of minimizing the action over an infinite dimensional function space becomes one of finite dimension. That is, now we want to minimize a multivariable function $F : \mathbb{R}^n \to \mathbb{R}$ which is easy (most of the time) numerically. To further restrict our study we look for solutions with finite charge i.e.

$$\langle f, f \rangle = \|f\|^2 = J_0 < \infty. \tag{5.4}$$

Upon inserting (5.2) into (5.4) we obtain the following condition[1] on the variational vector $\mathbf{a}$.

$$\sum_{i=1}^n a_i^2 = J_0 \iff \|\mathbf{a}\|_{\mathbb{R}^n}^2 = J_0 \tag{5.5}$$

This condition realizes our problem now as a constrained minimization problem where, given solitonic charge J_0, we can find the solution if we minimize

[1] using the fact that the inner product is linear in both entries, and the fact that the basis is orthonormal i.e., $\langle \psi_i, \psi_j \rangle = \delta_{ij}$.

over all functions satisfying (5.5). Written out, our problem (4.67) has been simplified to

$$\mathbf{a}_{\min,J_0} \coloneqq \min_{\mathbf{a}\in\mathbb{R}^n} \left\{ F(\mathbf{a}) \;\middle|\; \|\mathbf{a}\|^2_{\mathbb{R}^n} = J_0 \right\}. \tag{5.6}$$

We have used the notation $\mathbf{a}_{\min,J_0}$ to mean the vector $\mathbf{a}$ that minimizes F for a prescribed J_0.

Now we note two very important facts. First, F is a continuous function. This can be seen by expanding (5.3) and writing it as a sum over the basis. Since the basis is known ahead of time this expansion yields a polynomial in the a_i's and since it is a polynomial, it is clearly continuous. Second, the constraint (5.5) defines a compact set, in particular a sphere of radius $\sqrt{J_0}$ centered at the origin. With a continuous function defined over a compact set, we know the minima must be attained and hence this problem is well defined and has a solution.

5.2 Implementation

The code written to implement the constrained minimization problem (5.6) was written in Python with use of NumPy and SciPy[2]. We now explain, and ellucidate the process we used to obtain the numerical solutions.

We begin with NumPy arrays initialized to $\psi_i = \sin \frac{i\pi x}{r_{\max}}$ with i running in the set $\{1, 2, \ldots, n\}$ before we orthonormalize them using the the Gram-Schmidt Procedure, and the inner product given in (5.1). All integration is handled using Simpson's Rule, and the `scipy.integrate.simps` built in function. In the plot below we see the result of Gram-Schmidt orthonormalization procedure under this inner product for a basis of size 10.

[2]Both of which are pronounced with a "pi" sound, not "pee". Num"pi", not numpee please.

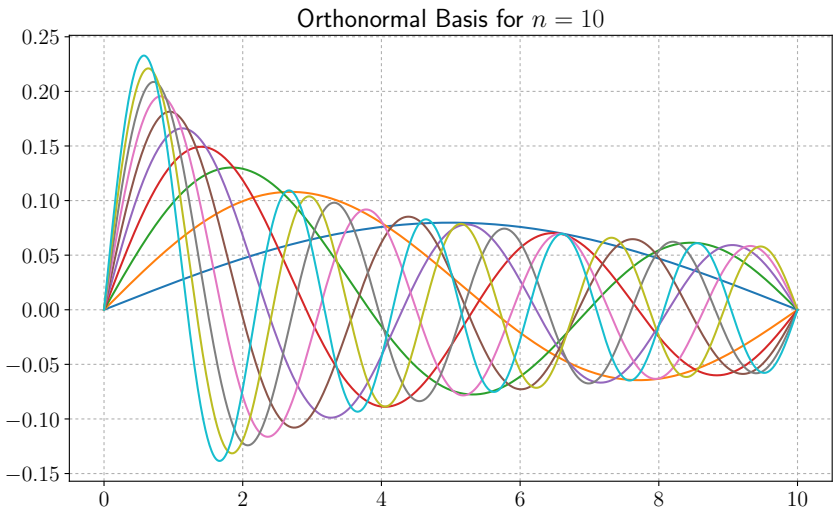

Figure 5.1: 10 Orthonormal Basis Functions with $r_{\max} = 10$

With the basis at hand we can build the objective function (5.3), that is the function we wish to minimize. To satisfy (5.5) we begin with the initial guess given by $\left(\sqrt{\frac{J_0}{n}}, \sqrt{\frac{J_0}{n}}, \ldots, \sqrt{\frac{J_0}{n}}\right)$. With an objective function, an initial guess, and a constraint, we can use SciPy's `scipy.optimize.minimize` to carry out the minimization[3]. To reproduce the results in [22], in particular, Figure 6, we chose parameters $a = 2, b = 1.1, \lambda = 1$ and with $J_0 = 30$ we obtain the following solution.

[3]In order to have this function succeed for large values of n, that is a large basis set, the maximum number of iterations must be increased to reach the minimum.

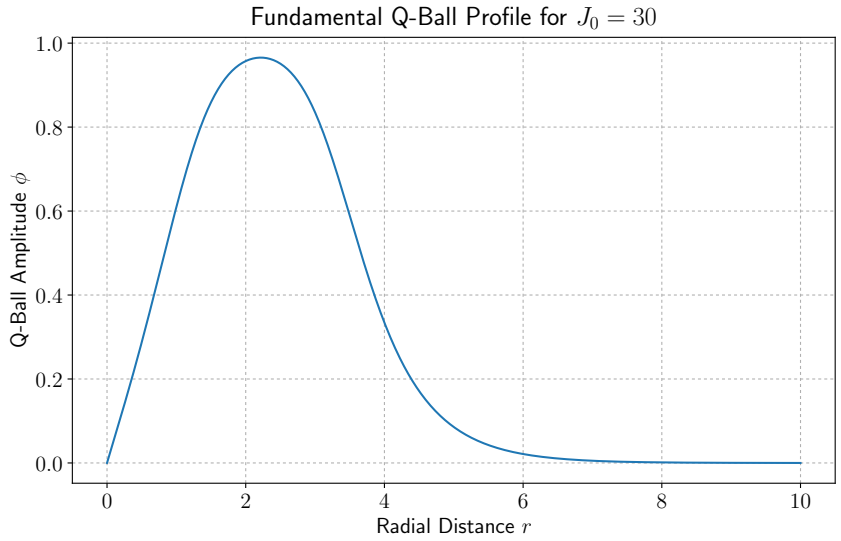

Figure 5.2: $N = 1$ Excitation of Q-Vortex Solution

This plot was generated with 50 basis functions and an x-axis sampled 1000 times, i.e. $\mathrm{d}x = 0.01$. With this solution we can then go on to calculate the Lagrange multiplier ω^2 that we saw in Lemma 4.4.1. To calculate ω^2, we take the δI and δJ as defined in Lemma 4.4.1, and take the test function ψ to be exactly ϕ. This may seem like cheating, however $\delta I = \chi \delta J$ should hold for all ψ, and in particular $\psi = \phi$. We can then numerically calculate δI and δJ, calculate the Lagrange multiplier by taking the quotient $\chi = \frac{\delta I}{\delta J}$, and then using the relation $4\pi\chi = \omega^2$ we have $\omega = \sqrt{4\pi\chi}$. For the plot above this yields $\omega = 0.7216$.

We can then run this procedure for multiple values of N to further understand how the Q-Vortex behaves with larger N values. We note how, as we increase N, the peak of the soliton moves outward similar to how objects with more angular momentum tend to increase their radii.

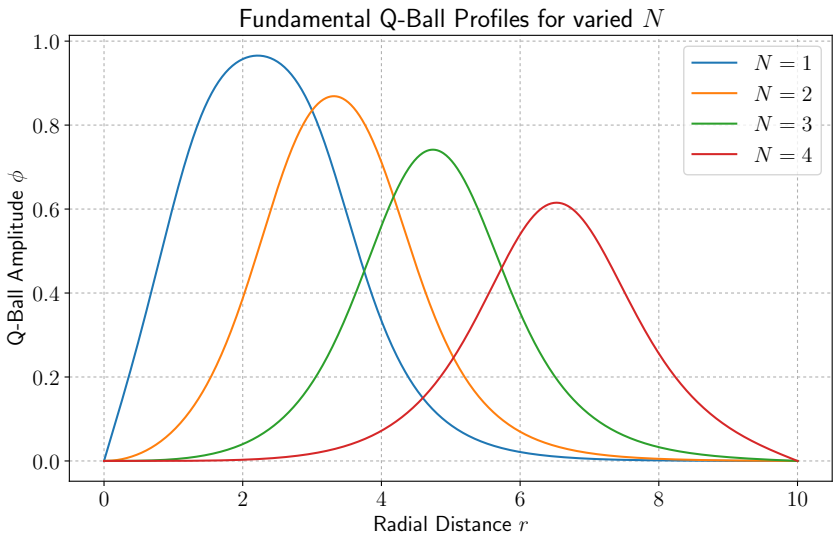

Figure 5.3: First Four Fundamental Q-Vortex Solutions

Now that we have solutions at hand, it is important to understand the error associated with them. To measure the error we use the following idea and metric. If the solutions computed above are accurate, they should of course satisfy the equation of motion (4.48). Thus we can add everything to one side, multiply by r^2, square the differential equation, and integrate[4].

$$\text{error} = \int_0^R \left(r\left(r\phi_r\right)_r + \omega^2 r^2 \phi - N^2\phi - \lambda r^2 \left(6\phi^5 - 4a\phi^3 + 2b\phi\right)\right)^2 \mathrm{d}r \quad (5.7)$$

We can then tabulate the associated values for each solution shown in Figure 5.3.

N	ω	error
1	0.72156	0.00626
2	0.85271	0.00084
3	1.01243	0.00021
4	1.16983	0.01622

Table 5.1: Parameters for $J_0 = 30$

[4]We multiply by r^2 to make our lives easier so we don't have to deal with the inverse powers of r numerically.

It is then natural to ask, for a given N, how does omega change as we vary the power J_0. Below we see that as the power is increased the soliton peak flattens and moves outward.

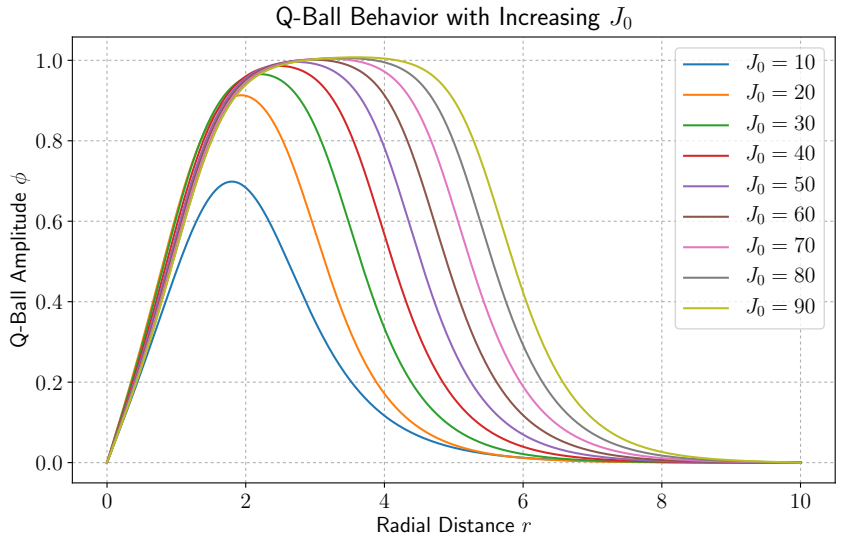

Figure 5.4: Flat Top Emergence of Q-Vortex Soliton

The parameters for this plot is shown in the table below.

J_0	ω	error
10	1.06898	0.00002
20	0.80058	0.00498
30	0.72156	0.00626
40	0.68074	0.00137
50	0.65481	0.00447
60	0.63783	0.03409
70	0.62205	0.02264
80	0.61019	0.00427
90	0.60444	0.06596

Table 5.2: Parameters for $N = 1$

It is easy to see in this table that as the power increases, omega not only

decreases, but it looks as though it tends to some number. To investigate this further we performed a finer scan to understand how the power J_0 effects ω.

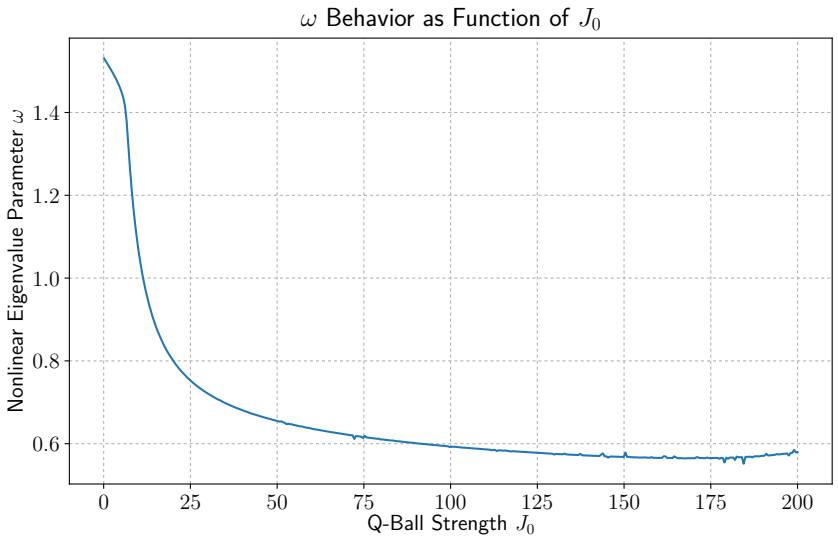

Figure 5.5: ω vs. J_0 Parameter Space Scan

As this plot shows, as we increase the J_0, ω falls very quickly at first, and then more slowly as it dips slightly below 0.6. In particular the minimum this curve achieves (not including small deviations) is roughly 0.57.

Chapter 6

Conclusion

> Le but de cette thèse est de munir son auteur du titre de Docteur. - The goal of this thesis is to furnish its author with the title of Doctor
>
> Adrien Douady
> *Beginning of A. Douady's thesis*

Throughout this thesis we have accomplished many things. We began with a thorough introduction to field theory, assuming only a rudimentary understanding of classical mechanics. Here we developed ideas that utilized in nearly all areas of theoretical, and mathematical physics and are paramount for understanding current research. The highlights were Gauge Theory, the field theoretic Lagrangian, and of course Noether's Theorem, all of which play a very important role in the work that followed.

To best understand the work that followed, a chapter of the necessary mathematical prerequisites followed where we introduced fundamental ideas from functional analysis and the calculus of variations in order to discuss our problem at hand with a level of rigor.

With the fundamentals from both mathematics, and physics we were able to introduce Q-Balls. These extremely imporant objects were developped systematically using the tool from the previous chapters that allowed us to calculate many important physics quantities such as total energy, angular momentum, and the stabilizing Noether charge. Once the basic quantities

were calculated, the constrained minimization problem was introduced, and we proved it was well defined.

The fact that the problem was well defined allowed us to move forward and to look for solutions numerically. Using a finite element formalism we reduced the problem to a multivariable calculus minimization problem which is easily solved numerically. We were then able to compute Q-Ball profiles and their associated parameters which allow us to understand the objects much better. With the constrained minimization problem set up we were able to explore parameter space not previously studied in the literature. In particular we studied how the Q-Ball changes as one varies the "strength" or "power" of the object.

While the end of this thesis has come, the work on the problem is certainly not. There is much more to explore in this field, and in particular this study of Q-Balls. Further questions are, which potentials have solutions, and are they qualitatively different from the ones found here? Are the solutions found unique?[1] Can we decrease the numerical error? There are of course many more interesting questions to be had, but unfortunately we only had a finite time to answer some.

[1]One way to explore this is to test a variety of points satisfying the constraint and seeing if they all lead to the same solution.

Appendix A

Some (possibly) Useful Facts

> The noblest ambition is that of leaving behind something of permanent value.
>
> G.H. Hardy
> *A Mathematicians Apology*

In this chapter we provide a few "random" facts, theorems and arguments that are not paramount to this thesis, but help it along.

A.1 Some Theorems

Theorem A.1.1. *Let $X \subseteq \mathbb{R}^n$ be open, and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$. If $L : [\alpha, \beta] \times X \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, then the integral*

$$S[x(t)] \coloneqq \int_\alpha^\beta L(t, x(t), \dot{x}(t))\,\mathrm{d}t \tag{A.1}$$

exists for all $x \in \mathrm{C}^1([\alpha, \beta], X)$.

This theorem justifies calling $\mathrm{C}^1(\mathbb{R})$ the domain of the action as we did in Section 2.1. The next theorem is a slightly more rigorous justification as to when the Euler-Lagrange equations 2.3 hold.

Theorem A.1.2. *If $x \in \mathrm{C}^1([\alpha, \beta], X)$ is a local extremum of S as defined above, and moreover the function*

$$[\alpha, \beta] \ni t \mapsto \frac{\partial}{\partial \dot{x}} L(t, x(t), \dot{x}(t)) \tag{A.2}$$

is continuously differentiable, then the following equation holds.

$$\frac{\partial}{\partial x}L(t,x(t),\dot{x}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}L(t,x(t),\dot{x}(t)) \tag{A.3}$$

Theorem A.1.3 (Poincaré Inequality)**.** *Let $p \in [1,\infty)$ and Ω a bounded subset of $\mathbb{R}$. Then there exists a constant C, dependent on Ω and p, so that for every function $u \in W^{1,2}(\Omega)$ we have the following inequality.*

$$\|u\|_{L^p(\Omega)} \leq C\|u'\|_{L^p(\Omega)} \tag{A.4}$$

A.2 Some Arguments

In this section we recreate, with more detail, the original arguments for the existence of Q-Balls as proposed in [6]. We

As Q-Balls were proposed in [6] these objects are described by a field ϕ that is some constant value inside a volume B, and 0 outside. There exact energy is given by (4.33), and in this approximation reduces to

$$E = \frac{1}{2}\omega^2\phi^2 B + VB \tag{A.5}$$

and similarly for the Noether Charge (4.36) we have the following.

$$Q = \omega\phi^2 B \tag{A.6}$$

The fundamental idea of a Q-Ball is that it's energy configuration is lower than that of the separate particles, and hence we should minimize the energy for a given value of the charge Q. In order to do this we eliminate ω by inserting (A.6) into (A.5) to obtain

$$E = \frac{1}{2}\frac{Q^2}{\phi^2 B} + VB. \tag{A.7}$$

Now we minimize this with respect to the volume B to find it is minimized when

$$B = \frac{Q}{\sqrt{2\phi^2 V}}. \tag{A.8}$$

With this volume the energy then reads

$$E = \frac{1}{2}\frac{Q^2}{\phi^2}\frac{\sqrt{2\phi^2 V}}{Q} + \frac{VQ}{\sqrt{2\phi^2 V}} \tag{A.9}$$

$$= Q\sqrt{2\phi^2 V}\left(\frac{1}{2\phi^2} + \frac{V}{2\phi^2 V}\right) \tag{A.10}$$

$$= Q\sqrt{\frac{2V}{\phi^2}}. \tag{A.11}$$

The last step is then to minimize this value with respect to ϕ in order to obtain

$$E = \min_{\phi} Q\sqrt{\frac{2V}{\phi^2}} \tag{A.12}$$

which is where the restriction on ω comes from as seen in (4.63). The upper bound for ω arises from considering the qualitative shape of the potential, and a more detailed description of its derivation can be found in [6, 12].

This construction of Q-Balls allows the energy configuration of the "Q-matter" as Coleman calls it to be lower than that of the separate particles floating around in space.

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