Quadrature for Parabolic Galerkin Boundary Element Method with Moving Surfaces

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I. Introduction

The successful implementation of the Galerkin Boundary Element Method hinges on the accurate and effective quadrature of the influence coefficients. Integrals for parabolic boundary integral operators must be performed in space and time where the integrals have singularities when source and evaluation times coincide. A set of transformations was derived that rendered the singular cases into smooth integrals that can be treated with standard tensor product Gauss quadrature rules. These transformations are particularly useful for problems with moving geometries where the time integration cannot be performed analytically, and are implemented numerically in the two dimensional case. Work was done previously for the fixed geometry case and for the elliptic problem [3], [6], but the time dependence of the moving geometry resulted in new types of singularities.

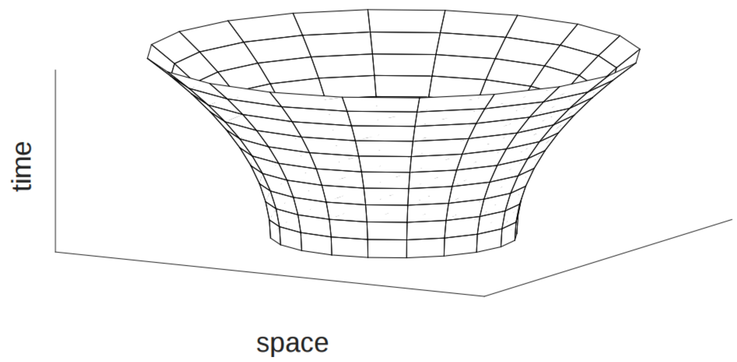


Figure: The two dimensional, moving geometry

Boundary element methods have been employed in thermal simulations with very particular industrial applications [4]. In order to properly cool or insulate objects such as electric car or cellphone batteries, a good model of heat flow in the battery is necessary. While solutions for modeling these instances exist, batteries can have very complicated geometries.In the case of the moving geometry, some interesting applications of our model would be in manufacturing, applied to the deformation of metal sheeting after the application of point heat sources such as laser cutting tools. The ability to model the numerical solutions of complicated heat flow problems is very useful, leading to new, safer and more efficient technology.

II. Problem

We assume a problem governed by the heat equation $\partial_t u - \nabla u = q$ defined on an interface $\Gamma(t)$ which grows in a prescribed fashion. With Ω_i defined as the interior of the interface Γ and T as the final time, the domain is defined in the following way

$$\Omega_T^e = \{(x,t) : x \in \mathbb{R}^2 \backslash \Omega(t), t \in (0,T)\},$$
$$\Omega_T^i = \{(x,t) : x \in \Omega(t), t \in (0,T)\},$$
$$\Gamma_T = \{(x,t) : x \in \Gamma(t), t \in (0,T)\},$$

on which we consider the heat equation:

$$\partial_t u = \Delta u \quad \text{in} \quad \Omega_T^i \quad \text{or} \quad \Omega_T^e,$$
$$u = f \quad \text{on} \quad \Gamma_T.$$

III. Discretization

The parabolic boundary value problem was reformulated as an integral equation on the boundary surface by way of Green's representation formula, where the thermal layer potential operators involved integrals over time in addition to the boundary surface. The following is the equation to be solved on the surface

$$\sigma \frac{1}{2} u = Ku(x,t) - V[\frac{\partial u}{\partial n} + uv](x,t),$$

with $\sigma = 1$ for the interior problem and $\sigma = -1$ for the exterior problem, and velocity of the boundary v. The single and double layer potentials are defined as the integrals

$$\mathcal{V}g(\mathbf{x},t) = \int_0^t \int_\Gamma G(\mathbf{x}-\mathbf{y}, t-\tau) g(\mathbf{y},\tau)\, ds(\mathbf{y}) d\tau,$$

$$\mathcal{K}g(\mathbf{x},t) = \int_0^t \int_\Gamma \frac{\partial}{\partial n_y} G(\mathbf{x}-\mathbf{y}, t-\tau) g(\mathbf{y},\tau)\, ds(\mathbf{y}) d\tau.$$

After performing singularity removing transformations on the integrals, the problem was solved using the Galerkin discretization method, which was selected for its unconditional stability. Previously, this problem was solved using the Nystrom method and stability became an issue for complicated geometries. In the time discretization, very small timesteps were needed to achieve convergence, which limited the complexity of the interface geometry [1]. The Galerkin matrices were populated using Gauss quadrature rules to compute the integrals, and were solved via block forward substitution.

IV. Results

The method was implemented on the Dirichlet problem, where the value of $\frac{\partial u}{\partial n}$ was assumed and the value of temperature u was solved for on the boundary. Based on previous work for the rate of convergence with the fixed geometry case, the best expected rate of convergence was $O(h_x + \sqrt{h_t})$, with $h_x = \frac{1}{N}$, N being the number of spatial discretization steps, and $h_t = \frac{1}{M}$ where M denotes the number of temporal discretization steps. [2], [5].

$N \backslash M$	20	40	80	160	320
20	0.53405	0.64236	0.50231	0.41774	0.41515
40	0.67848	0.47826	0.29248	0.19552	0.18003
80	0.74863	0.42057	0.22005	0.10834	0.10628
160	0.56828	0.54195	0.20950	0.08954	0.09409
320	0.49139	0.88415	0.33634	0.14624	0.08796

Figure: Errors from solving the Dirichlet problem on a moving, prescribed geometry. Numbers in the top row denote the temporal discretization, while the numbers in the column denote the time discretization.

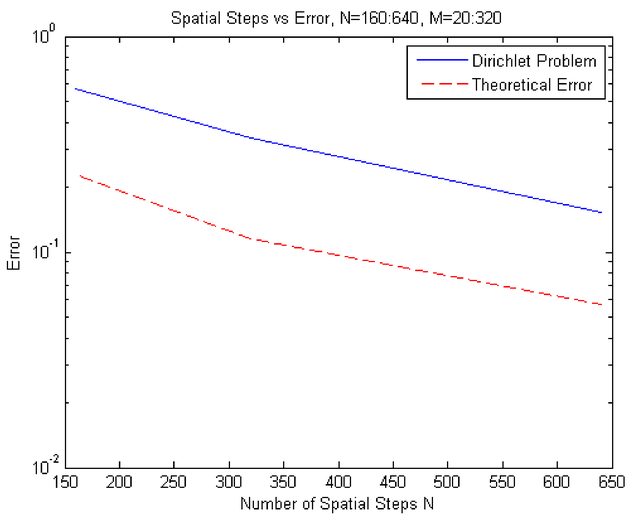


Figure: Comparison of the numerical convergence results with the best possible approximated convergence.

V. Future Work

The current implementation will be expanded to the three space dimensions, which will provide the opportunity to test the method on more interesting and practical test problems and geometries. Another topic of interest is the Stefan problem, which describes how the field moves the surface in time, incorporating an unknown moving geometry into the parabolic problem.

VI. References

[1] Case, Tausch (1988) *Applied Mathematics and Computation* **218** , pp. 11451–11460.

[2] Costabel (1988) *SIAM J. Sci Comput* **19** , pp. 613–626.

[3] Erichsen, Sauter (1998) *Computer Methods in Appl. Mech and Engrg.* **157** , pp. 215–224.

[4] Messner, Tausch, Schanz (2015) *SIAM J. Sci Comput* **37** , pp. A1554–A1576.

[5] Noon (1988) *PhD thesis, University of Maryland.*

[6] Sauter, Schwab (2004) *Randelementmethoden: Analyse, Numerik und Implementierung schneller Algorithmen.*

Error Tables

moving circular geometry r(t) = sqrt(1+t) x0 off center

max error P=5

N\M	10	20	40	80	160	320	640	720
10	0.2583674029	0.6612838429	0.9432552786	0.8349941214	0.7583659393	0.7615817385		
20	0.3118822985	0.5340518948	0.6423587491	0.5023123381	0.41773596	0.4151532944	0.4172002704	
40	0.353678366	0.6784804559	0.4782570703	0.2924796833	0.1955234662	0.1800267221		
80	0.3123400568	0.7486249044	0.4205703054	0.2200479673	0.108343403	0.1062829585	0.1072552261	0.1073912098
160	0.1717236696	0.568283871	0.5419533946	0.2095057065	0.0895370782	0.0940931941	0.0964274379	
320		0.491386856	0.8841518986	0.3363369724	0.1462395864	0.087956328	0.092390212	
640		0.4872905759	1.0908944394	0.4305365773	0.2875236958	0.152710552		
1280								
2560								

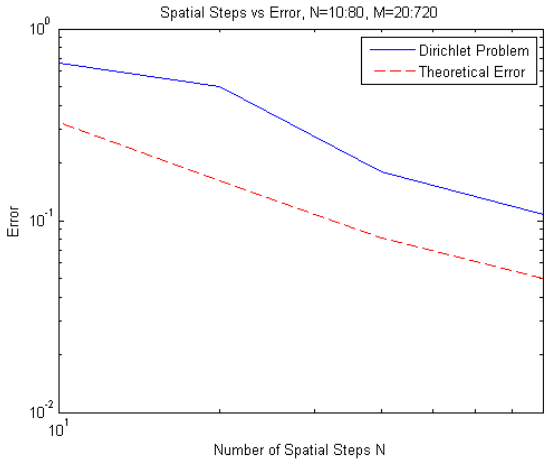
L2 error P=5

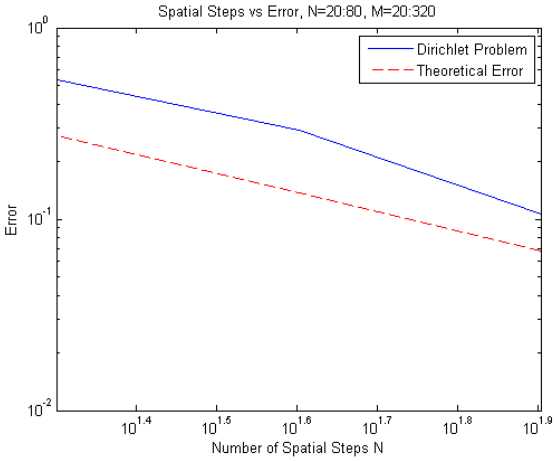
N\M	10	20	40	80	160	320	640	720
10	0.3084555817	0.3181785822	0.2598442438	0.2171640611	0.2109192773	0.2094801195		
20	0.2476529717	0.256125157	0.1722257241	0.1185365476	0.1066432278	0.1033724735	0.1027787894	
40	0.2326273591	0.2410385568	0.1483962127	0.0873257159	0.0706510318	0.0656172022		
80	0.232911186	0.2366060905	0.1415968846	0.0776166312	0.0588173277	0.052831782	0.0513916545	0.0513091422
160	0.2671525074	0.2451577185	0.1462665608	0.0742309827	0.0548372141	0.048881964	0.0475765316	
320		0.2653142746	0.1758642418	0.0792534109	0.0537144235	0.0471556346	0.046189798	
640		0.2864878601	0.2023712779	0.090658048	0.0599625939	0.0480638252		
1280								
2560								

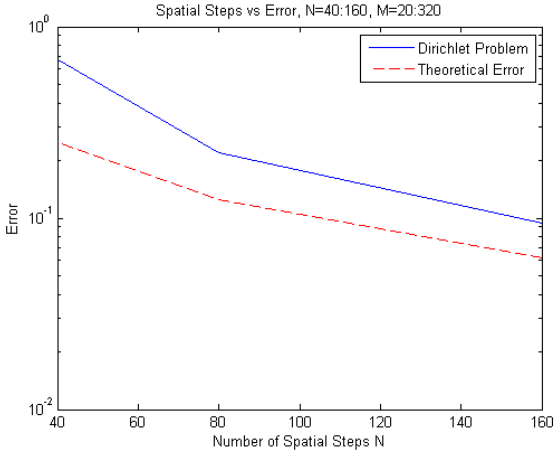
N = 320 M = 40 p = 7
maxerr = 0.5881004548
l2 error = 0.1493801802

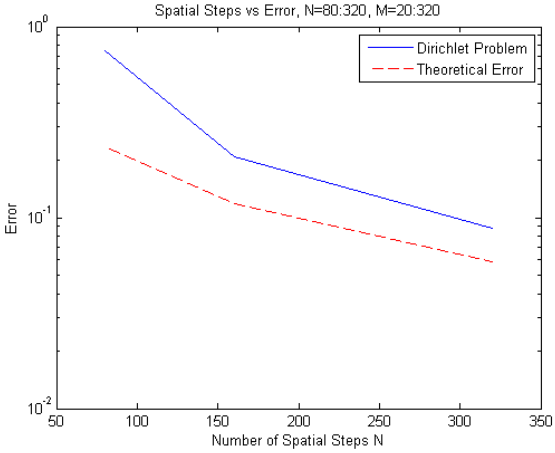
N = 320 M = 160 p = 7
maxerr = 0.1120451202
l2 error = 0.0534028486

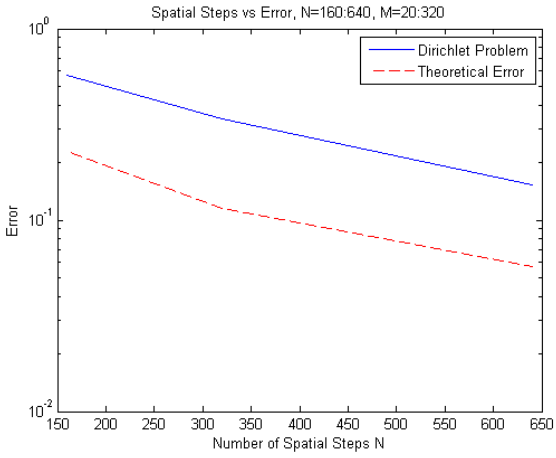
Error Plots


Spatial Steps vs Error, N=10:80, M=20:720
Dirichlet Problem
Theoretical Error
Error
Number of Spatial Steps N


Spatial Steps vs Error, N=20:80, M=20:320
Dirichlet Problem
Theoretical Error
Error
Number of Spatial Steps N


Spatial Steps vs Error, N=40:160, M=20:320
Dirichlet Problem
Theoretical Error
Error
Number of Spatial Steps N


Spatial Steps vs Error, N=80:320, M=20:320
Dirichlet Problem
Theoretical Error
Error
Number of Spatial Steps N


Spatial Steps vs Error, N=160:640, M=20:320
Dirichlet Problem
Theoretical Error
Error
Number of Spatial Steps N