# *Research in Undergraduate Mathematics Education I: Qualifying Exam Notes*

## Consistent Themes in Literature

The following ideas were prominent in the literature that we read this semester.

1) Functions (Sfard, Sierpinska, Bridenbach, Thompson, Oehrtman et al.)
    a. Covariation vs. Correspondence
    b. Action, Process, Object
2) Pain (Sierpinska, Tall & Vinner, Stigler & Hiebert, Sfard)
    a. Development of Understanding (Thompson & Thompson, Carlson)
3) Cultural Script (Stigler & Hiebert, Thompson & Thompson, Thompson)
    a. Norms and Expectations (Stigler & Hiebert, Thompson)
4) Covariational Reasoning (Thompson & Thompson, Thompson, Carlson, Oehrtman et al., Saldanha & Thompson)
5) Communication
    a. Perspective of Student (Thompson & Thompson, Oehrtman, Carlson, Tall & Vinner)
6) Historical Development of Mathematics (Sfard, Sierpinska, Thompson)
7) Doing vs. Understanding (Carlson, Thompson & Thompson)
8) Role of Theory on Constructs (Lobato & Siebert, Ellis)
9) Conceptually Oriented Instruction (Thompson, Ellis, Thompson & Thompson)
10) Additive vs. Multiplicative Reasoning (Saldanha & Thompson, Thompson & Thompson, Thompson).

## Key Points in Literature

The literature summarized below is presented in the same order in which it was read during RUME I. You should be able to expound of the key points given below.

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap.* New York: Free Press.

- Stigler and Hiebert advocate for greater attention to teaching as opposed to teachers. They contend that teaching is a *system* and a *cultural activity*.
- Framing teaching as a cultural activity is not only significant to understand how teachers develop their instructional strategies, but also creates a lens from which to assess the countless reform efforts that have not produced results.
- Lesson study emerged as a recommendation for gradual professional development for teachers in the United States. Stigler and Hiebert make it clear that there is currently no mechanism for improving education in the United States.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279-303.

- This paper was predominantly about the significance of communication between a teacher and a student.
- "Bill's quantitative conceptualizations appeared to be encapsulated in the language of numbers, operations, and procedures. He thus had no other means outside the language of mathematical symbolism and operations to express his conceptualizations." "…that language served him poorly when trying to communicate with children who knew the tokens of his language but had not constructed the meanings and images that Bill had constructed to go along with them" (p. 301).
- In this paper, Thompson & Thompson argue, "For curricular reform to happen in classrooms, teachers must teach from the basis of a conceptual curriculum. That means they must be sensitive to children's thinking during instruction, and they must shape their instructional actions so that children actually hear what is intended" (p. 280).
- Bill was unable to effectively communicate with Ann because his explanations relied heavily on symbolic representation. For Ann, this symbolic representation did not have the same meaning that Bill associated with it.

Thompson, P. W., & Thompson, A. G. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

- This paper "presents Pat's session with Ann, analyzing his instructional actions in light of his agenda for having Ann build a scheme of operations by which to understand distance, time, and speed" (p. 2).
- Pat took an approach that relied heavily on developing the idea of covariation between distance traveled and time elapsed. Proportionality between covarying quantities was heavily emphasized also. We note that Pat's presentation of these ideas was not encapsulated in the language of numbers and operations. Rather, Pat used diagrammatic representations to convey the ideas of covariation and proportionality.
- "How one teaches a subject is influenced greatly by the many ways one understands it" (p. 19).
- "Teachers' images—the loose ensemble of actions, operations, and ways of thinking that come to mind unawarely—of what they wish students to learn, and the language in which they have captured those images, play important roles in what teachers do, what they teach, and how they influence students' understandings" (p. 19).
- "Pat's actions were highly image-oriented and his language was deliberately chosen to help Ann in two ways: to become oriented likewise and to form, in fact, those images" (p. 19).

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

- Teaching Experiment with student JJ using the OVER & BACK program.
- Aim of the study was to determine the re-constitutions that occur in students' understanding of rate and ratio with respect to several constructs in existing literature (e.g. quantitative reasoning, quantity, quantification, quantitative operation, numerical operation, ratio, rate, motion, internalization, interiorization, mental operation, scheme, and reflective abstraction).
- One learns and comprehends based on existing schemes and understandings.
- The goal is not to get students to solve problems but to help them construct understandings that enable them to solve problems.
- Thompson advocates for an emphasis on an individual's reasoning and their construction of mathematical objects.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

- Four day teaching experiment with with six mid-range performing fifth-grade students. Students were interviewed after the teaching experiment.
- The primary focus of the study was to investigate students' conceptions of difference as a quantitative structure and their ability to cope with relationally complex situations.
- The focus of the interviews (conducted after the four day teaching experiment) was to determine students' ability to conceive of a quantitative difference independently of an evaluation process.
- Tasks during the teaching experiment were meant to promote students' analysis of complexly described situations and emphasize the issue of relational structure of a situation. Accordingly, these tasks generally involved a comparison of two comparisons (i.e. the difference of differences).
- Students were unabel to conceive a quantitative difference independently of any particular calculation employed to evaluate it.

Lobato, J. & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematics Behavior, 21*(1), 87-116.

- Teaching experiment involving nine students from 8$^{th}$ to 10$^{th}$ grade mathematics classes. Only one student (Terry) served as the focus of the paper.
- A distinction between traditional transfer and actor-oriented transfer is made.
- The surface/structure distinction that characterizes the traditional transfer paradigm is problematic as it assumes a level of quantitative reasoning proficiency among students.

- Particular attention is paid to explicating the facility of the actor-oriented transfer paradigm that emphasizes an individual's capacity to recognize similarities between different situations.
- There are methodological implications for operating under the traditional and actor-oriented transfer paradigms.
- The actor-oriented paradigm "seeks to understand the process by which individuals generate their own similarities between problems" (p. 89).
- Particular attention is paid, however, to the limitation of the traditional transfer paradigm that emphasizes the application of mathematical content to contextualized tasks.
- Transfer may not be identified using the traditional transfer paradigm when actor-oriented transfer has occurred.
- Investigating the process of transfer by having students reason with contextual tasks is problematic because appropriate quantitative reasoning skills are assumed and various contextual tasks require different types of quantitative reasoning.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition & Instruction, 25*(4), 439-478.

- This study was conducted in two parts. The first part consisted of classroom observations and semi-structured interviews while the second consisted of a teaching experiment.
- The objective of the interviews was to reveal students' thought process and understanding of the generalizations they had developed during the classroom component of the study. The objective of the teaching experiment was to investigate students' reasoning with contextual problems about linear growth.
- Ellis describes the different types of generalization that occur when students focus on quantitative relationships as opposed to number patterns.
- The type of quantitative reasoning in which students engage is significant to promoting global generalizations about relationships.
- Engaging in "emergent-ratio quantitative reasoning" encourages global generalizations rather than numerical or recursive generalizations that can occur even in quantitatively rich situations.
- Ellis describes a generalization taxonomy that distinguishes between what students are thinking as they generalize, called generalizing actions, and what students communicate after the mental activity of generalizing, called reflection generalization. Generalizing actions is further partitioned into the following categories: relating, searching, and extending.
- Ellis found that students were more likely to engage in the generalizing activity of relating when students focused on quantities and searched for relationships.
- Contextual situations that require students to reason with two quantities may be problematic in that students fail to quantify the multiplicative relationship between the initial two quantities. Hence, there is no quantity to directly reason from. It is for this reason that the author promotes the development of an emergent-ratio that quantifies the multiplicative relationship between two initial quantities that are linearly related.

- Without the emergent-ratio quantity, students regress to generalizing the relationship between linearly related quantities by recognizing numerical patterns.
- Salient implications for classroom practice include an awareness of the type of quantitative reasoning in which students engage and focusing on contextual situations that promote the development of an emergent-ratio quantity.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

- Carlson, et al. propose a framework for studying students' mental actions when engaging in tasks involving dynamic function events.
- The theoretical framework consists of a sequence of five mental actions of covariational reasoning along with five corresponding covariational reasoning levels.
- L1 – L5 are the levels of development of images of covariation that support the mental actions (M1 – M5).
  - MA1. *Coordination*: an individual's recognition that a change in one quantity corresponds to a change in another.
  - MA2. *Direction*: not only a recognition that two quantities vary in tandem, but requires coordinating the direction of change in one variable with changes in another.
  - MA3. *Quantitative coordination*: the amount of change in one variable with respect to the amount of change in another is established.
  - MA4. *Average rate*: the average rate of change in the output variable with uniform changes in the input variable is established.
  - MA5. *Instantaneous rate*: students recognize that the instantaneous rate of change results from continually decreasing the interval of the input variable from which the average rate of change is calculated.
- The purpose of the proposed framework is to aid in the evaluation of covariational thinking to a greater extent than has been done previously.
- In light of the results of this study, the authors recommend a greater attention to students' understandings of function and covariational reasoning.
- The authors propose the development of curricular materials that encourage students' covariational reasoning abilities as well as the strategic implementation of technological resources that serve this aim.

Oehrtman, M. (2009). Collapsing dimensions, physical limitations, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

- Oehrtman investigates students' application of spontaneous metaphors when reasoning about limit concepts and examines the implications of these metaphors on students' understanding of concepts defined in terms of limits.
- Oehrtman devotes specific attention to the significance of having students spontaneously apply their conceptions to problematic situations as a means of revealing the cognitive processes on which their conceptualizations rely.
- Oehrtman defines strong metaphors as having two significant characteristics, emphasis and resonance. Emphasis refers to the metaphor's persistence across

various tasks while resonance denotes the metaphor having implications for a students' further reasoning.

- The five metaphors that were classified as strong include: a collapse in dimension; approximation and error analysis; proximity and space of point-locations; a small physical scale beyond which nothing exists; and the treatment of infinity as a number.
- Data was obtained from field notes, surveys, and a series of interviews and written assignments. The tasks for the assignments were designed to elicit the application of inquiry and sense-making.
- Weak metaphors include: motion imagery and interpretations of "approaching"; zooming imagery and interpretations of local linearity; and interpretations of arbitrarily and sufficiently.
- When applying a collapse metaphor, students visualize a physical limit attained instead of just approached, and consequently the attained limit decreases in dimension resulting specifically in physically problematic interpretations of topics associated with Riemann integration (including the fundamental theorem of calculus).
- In an approximation metaphor, students consider a limiting process terminating. Hence, it is appropriate to evaluate the accuracy of the limit approximation.
- The proximity metaphor evoked conceptualizations of closeness and was regularly applied to the meaning of continuity.
- The metaphor that treated infinity as a number was prevalent when students attempted to reason computationally with infinity, predominately in the context of evaluating infinite limits.
- Physical limitation metaphors inhibited mathematical understandings by involving physical constraints on seemingly paradoxical situations.
- Oehrtman concludes by describing the instructional facility of some of the metaphors outlined above as well as the potential origins of students' metaphors. He also cautions educators against missing opportunities to guide students to an accurate understanding of limits by ignoring metaphorical statements.

Saldanha, L., Thompson, P. W. (2002). Conceptions of sample and their relationship to statistical inference. *Educational Studies in Mathematics,* 51(3), 257-270.

- This study consisted of a teaching experiment involving 27 $11^{th}$- and $12^{th}$-grade students enrolled in a statistics course. The teaching experiment lasted 9 sessions and emphasized ideas of sample, distributions, and margins of error.
- Saldanha and Thompson investigate students thinking as they engage in a teaching experiment designed to promote students' understanding of sampling as a scheme of interrelated ideas including variability among sample statistics, distribution, and repeated random selection.
- The goal of this study was to promote the development of schematic, imagistic, and dynamic ways of thinking about sample, sampling distributions, and margins of error.
- Saldanha and Thompson found that students "had developed a multi-tiered scheme of conceptual operations centered around the images of repeatedly

sampling from a population, recording a statistic, and tracking the accumulation of statistics as they distribute themselves along a range of possibilities" (p. 261).
- Saldanha and Thompson argue that the *multiplicative conception of a sample* (MCS) promotes a deep understanding of statistical inference because MCS involves a network of interrelated images.

Tall, D., & Vinner, S. T. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

- Tall and Vinner define concept image, concept definition, and conflict factor, and use these constructs to describe how students' concept images are often problematic when students' are expected to reason with formal concept definitions. Particular attention is paid to students' concept images related to the limit of a sequence, the limit of a function, and the continuity of a function.
- In this article, the authors argue that students' concept images of limit and continuity contain features that conflict with the formal concept definitions.
- Students can maintain strong concept images that do not translate into fluency with formal definitions.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), The concept of a function: Aspects of epistemology and pedagogy (pp. 25-58). Washington, D.C.: Mathematical Association of America.

- Sierpinska proposes a theory of how mathematics understanding is attained in general, and uses this theory to analyze the essential understandings of the notion of function and the epistemological obstacles that inhibit these understandings.
- Particular attention is paid to how the function concept evolved historically. This historical perspective motivates much of the essential understandings and prominent epistemological obstacles that encourage/inhibit understanding the notion of function.
- The aim of identifying these understandings and obstacles are to provide informed pedagogical suggestions for teaching the function concept.
- Sierpinska proposes that students make instantaneous jumps in understanding as opposed to continuous gradual progress when learning mathematics. In this paper, Sierpinska describes the understandings and epistemological obstacles that inspire/inhibit these jumps from occurring in the context of understanding the notion of a function.
- Sierpinska describes the origin of these epistemological obstacles as being one's attitudes, beliefs, and convictions of our worldview; unconscious schemes of thought; and technical knowledge and describes identification, discrimination, generalization, and synthesis as being four categories of understanding.
- Sierpinska advises educators to inspire appropriate motivation for the function concept, ensure that students maintain sufficient prerequisite knowledge, avoid prescribed contextual exercises, explicate the meaning and significance of variable, use a variety of representations, use informal definitions, and encourage

classroom discourse about the distinction between causal and functional relationships.

- Sierpinska concludes with a remark about the nature of learning mathematics in that the jumps that students must make are often not easy or painless. These jumps are accompanied by emotional tension and intellectual concentration, and unless they occur, learning is not possible.

Sfard, A. (1992). Operational origins of mathematical notions and the quandary of reification - the case of function. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy. Washington, D. C.: Mathematical Association of America.

- Sfard defines the distinction between structural and operational conceptions of function and describes the process through which one transitions from an operational to a structural conception. Moreover, Sfard makes two significant pedagogical suggestions as a result of the distinction between structural and operational conceptions and further investigates the effectiveness of these suggestions in a teaching experiment.
- A structural conception treats mathematical notions as if they referred to object-like entities while an operational conception treats mathematical notions as a computational process rather than a static construct.
- Sfard describes three steps for transitioning from operational to structural conceptions. These three steps include interiorization, condensation, and reification.
- Interiorization entails performing a process on familiar objects, condensation involves making the process more efficient by turning it into a self-contained whole, and reification results in perceiving this new entity as an object in itself.
- Sfard does not denounce the facility of spending an appropriate amount of time with an operational conception, believing that it serves as a catalyst for students' acceptance of a structural conception.
- Since reification is the last step in the transition from an operational to structural conception, Sfard devotes particular attention to sources of difficulty associated with reification.
- Sfard makes the following two pedagogical suggestions of things that should *not* be done: (1) "new conceptions should not be introduced in structural terms" and (2) "a structural conception should not be required as long as the students can do without it" (p. 69). The term *operational* is given to a method of teaching based on these two principles.
- Since the operational conception is a necessary prerequisite to developing a structural conception, Sfard petitions for an operational mode of teaching. However, without an effort at reification, students are likely to develop pseudostructural conceptions.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*(3), 247-285.

- Bridenbach et al. design and test an instructional treatment that is based on a theoretical interpretation of students' difficulties with the function concept.
- A key idea of this paper is that understanding the nature of specific processes and objects that are constructed and how they are organized when one studies mathematics serves as an essential component to applying the idea that an individual's mathematical knowledge consists of their ability to respond to problem-solving situations by constructing, reconstructing, and organizing mental processes and objects used in dealing with situations.
- Bridenbach et al. developed a well-articulated instructional treatment informed by six specific goals. In conjunction with these six goals, six activities were also developed.

Thompson, P. W. (1993). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Issues in Mathematics Education Vol. 4, (pp. 21-44). Providence, RI: American Mathematical Society.

- Thompson discusses research on students' understanding of functions and describes the importance of these understandings for the undergraduate curriculum.
- Thompson organizes his discussion of research related to function concepts around the following six themes:
    (1) Concept image and concept definition
    (2) Function as action, as process, and as object
    (3) Function as covariation of quantities and function as correspondence
    (4) Understanding phenomena and representing phenomena
    (5) Operations on numbers and operations on functions
    (6) Emergent issues
- (1) Thompson explains that mathematicians develop the aptitude to assimilate their concept images with concept definitions, which allows intuition to guide and support their reasoning. In this vein, the pedagogical suggestion is made to pay particular attention to imagery as an important aspect of pedagogy and curriculum.
- (2) Thompson describes an action conception as a recipe to apply numbers. A process conception of a function is defined as building an image of "self-evaluating" expressions.
- (2) Thompson advises an instructional emphasis on the process conception of function before developing object conceptions of function.
- (3) Thompson proposes to reflect the historical development of mathematics in the curriculum as a means of reflecting the old intuitive notion of function as covariation.

- (4) Students' struggle with applications in which covariation figures largely is a point Thompson makes and supports with prior research. This implies that there is a disparity between professors' recognition that symbolic mathematics encapsulates the dynamics of a physical situation and students' intention when they use symbolic mathematics.
- (4) Thompson cautions symbolic talk that assumes students have an appropriate image of a dynamic situation, as it will not be communicated.
- (6) Thompson argues for the need for building images of covariation in an effort to enhance students' conception of function. Thompson also proposes a "renegotiation of the didactic contract"

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

- Oehrtman et al. "provide an overview of essential processes involved in knowing and learning the function concept" (p. 150).
- Particular attention is paid to explaining why the function concept is so difficult for students to understand by describing six common obstacles that inhibit students from attaining a deep understanding of function. These six obstacles are: (1) students often view functions simply as two expressions separated by an equal sign, (2) students have difficulty distinguishing between an algebraically defined function and an equation, (3) students tend to believe that all functions should be definable by a single algebraic formula, (4) students tend to assume that functions are linear or quadratic in cases where this assumption is unwarranted, (5) students have difficulty distinguishing between visual attributes of a physical situation and similar perceptual attributes of the graph of a function that models the situation, and (6) students' commonly maintain an inability to express function relationships using function notation.
- Oehrtman et al. propose two types of dynamic reasoning abilities needed for understanding and using functions. These abilities include: (1) an ability to "develop an understanding of functions as general processes that accept input and produce output" and (2) an ability to "attend to both the changing value of the output and rate of is change as the independent variable is varied through in interval in the domain" (p. 154).
- An emphasis on the distinction between an action conception and a process conception of function is made to discuss the misconceptions that arise when students maintain an action view of function, and to recognize the facility of the process conception.
- Oehrtman et al. argue that covariational reasoning requires students to possess a process view of functions, and offer the following three recommendations for developing students' acquisition of a process conception: (1) "ask students to explain basic function facts in terms of input and output," (2) "ask about the behavior of functions on entire intervals in addition to just single points," and (3)

"ask students to make and compare judgments about functions across multiple representations" (p. 160-161).

- Oehrtman et al. conclude by solidifying the thesis that "a mature function understanding that is revealed by students' using functions fluidly, flexibly, and powerfully is typically associated with strong conceptual underpinnings" (p.167). Moreover, the researchers believe these conceptual underpinnings can be developed through strategically crafted curricula and instruction.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America.* Raleigh, NC: North Carolina State University.

- Saldanha and Thompson attempt to understand the "operations involved in students coming to envision and reason about continuous covariation of quantities" (p. 2).
- The researchers use a teaching experiment with one 8$^{th}$-grade student to test the hypothesis that students' engagement with tasks requiring the coordination of two sources of information simultaneously is favorable for conceiving of a graph as composed points that record the simultaneous state of two covarying quantities.
- According to Saldanha and Thompson, covariation entails coupling two quantities so that a multiplicative object of the two quantities is formed (when forming a multiplicative object, one has the immediate, persistent, and explicit understanding that for every possible value that a given quantity can assume, the other quantity also has a value).
- Saldanha and Thompson describe the following three developmental phases for images of covariation: (1) coordination of two quantities non-simultaneously (i.e. think of one quantity, then the other, then the first, and so on), (2) time is understood as a continuous quantity which implies two quantities' values continue, and (3) both quantities being tracked for some duration is imagined and the correspondence between the two quantities is an emergent property of the image.
- The teaching experiment described in this paper entails a sequence of tasks designed in the following three phases: (1) engagement, (2) move to representation, and (3) move to reflection. The tasks were based on the activity of tracking and describing the distance from a car to each of two cities as it moved along a road.
- The student in the teaching experiment was able to reason proficiently "having developed a level of operativity at which he could intricately coordinate images of two individually varying quantities" (p. 7). Accordingly, students' developing an understanding that graphs represent a continuum of covarying quantities is nontrivial, and should not be dismissed.

Confrey, J., Smith, E. (1995). Splitting, covariation, their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

- Confrey and Smith propose a theoretical approach to exponential functions that is based on a primitive multiplicative operation called *splitting*.
- The authors advocate for uniting the constructs "splitting" and "covariation approach to functions" in an effort to construct "the exponential function based on an isomorphism between splitting and counting structures" (p. 66).
- In the first part of their paper, Confrey and Smith provide evidence "that the operation of 'splitting' arises naturally in the minds of children as a way of structuring certain problem situations" (p. 67).
- In the second part of the paper, the authors discuss and contrast the idea of a covariation approach to functions with more prominent correspondence approach.
- In the third part of their paper, Confrey and Smith present "an integrated approach to exponential functions that consists of a covariation of splitting and additive counting" (p. 67).
- The construct of splitting is introduced through the following four theoretical claims:

  > *Claim 1:* Repeated addition is inadequate for describing many multiplicative situations.
  > *Claim 2:* There is a primitive model other than repeated addition that provides an operational basis for multiplication and division. It is labeled *splitting*.
  > *Claim 3:* Splitting provides a basis for the concept of ratio. Ratio is the central concept underlying the development of a "splitting world."
  > *Claim 4:* The development of splitting and its connection to ratio creates the basis for what we call the *splitting world*, whose overall structure and developmental path differs considerably from the *counting world* that currently dominates the curriculum.

- Confrey and Smith describe the shortcomings of a correspondence approach to functions, and argue that a *covariation approach to functions* "brings the operational basis of a function to the forefront" (p. 79).
- Confrey and Smith explain how the "construction of a counting and splitting world and their juxtaposition through covariation provide the basis for the construction of an exponential function" (p. 80). In essence, coordinating the counting and splitting world is necessary when reasoning covariationally with exponential functions as an association must be made between the counting numbers ($x$-axis) and the output of a splitting process (exponential function values).
- The authors conclude with explaining the implications of a covariation approach to exponential functions. First, a covariation approach avoids ambiguity inherent to traditional exponential function presentation and second; using a covariation approach "avoids an overreliance on algebraic representation" (p. 83).
- The principal point of this paper is that combining splitting and covariation yields an effective approach to exponential functions.

## Important Figures/Tables in Literature

The following includes the most instructional figures and tables from the RUME I literature.

Thompson, P. W., & Thompson, A. G. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

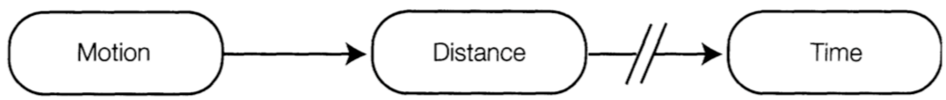



*Figure 2.* Ann's initial image: Motion entails moving a distance, but moving some distance did not automatically entail an amount of time. Rather, an amount of time was something needing to be determined after having moved an amount of distance.

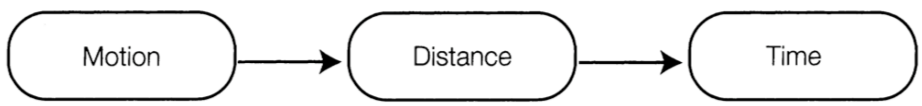



*Figure 3.* Ann's subsequent image: Motion entails moving a distance, and moving a distance entails using some time to do so.

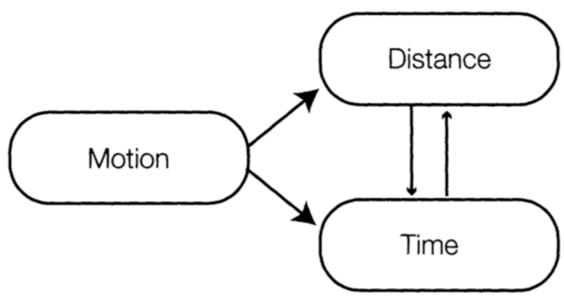



*Figure 4.* Pat's intention for Ann's understanding: Motion entails moving a distance, and it also entails some amount of time to move that distance. Distance and time become covarying quantities, as opposed to one being dependent on the other.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

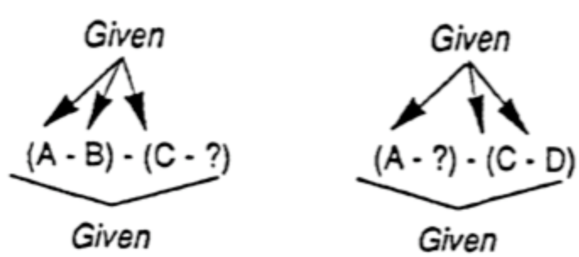



Fig. 18. Calculational structure of study's complex difference problems.

Lobato, J. & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of mathematics Behavior, 21*(1), 87-116.


Fig. 9. Terry's initial image in Interview 1: steepness can be measured by a series of height measurements. Length appears to be implicit and not clearly connected to Terry's image of steepness.

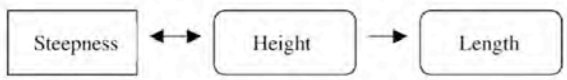



Fig. 10. Terry's subsequent image in Interview 1: length increases in importance, but changes in the length depend upon changes in the height.

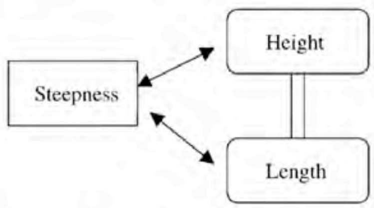



Fig. 11. Terry's emergent image in Interview 3: length and height appear to be covarying quantities of equal status.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition & Instruction, 25*(4), 439-478.

| **TYPE I: RELATING** | | **Examples** |
|---|---|---|
| *Relating Situations*: The formation of an association between two or more problems or situations. | *Connecting Back*: Connecting between a current and previously-encountered situation. | Realizing that "This gear problem is just like the swimming laps problem we did in class!" |
| | *Creating New*: Inventing a new situation viewed as similar to an existing one. | "He's walking 5 cm every 2 s. It'd be like a heart that was beating at a steady pace, 5 beats in 2 s." |
| *Relating Objects*: The formation of an association of similarity between two or more present objects. | *Property*: Associating objects by focusing on a property similar to both. | Noticing that two equations in different forms both show a multiplicative relationship between $x$ and $y$. |
| | *Form*: Associating objects by focusing on their similar form. | Noticing that "Those equations both have one thing divided by another." |
| **TYPE II: SEARCHING** | | **Examples** |
| *Same Relationship*: Performing a repeated action in order to detect a stable relationship between two or more objects. | | Dividing $y$ by $x$ for each ordered pair in a distance/time table to determine if the speed remains the same. |
| *Same Procedure*: Repeatedly performing a procedure in order to test whether it remains valid for all cases.* <br><br> *A searching action is coded as a relationship or a procedure based on the researcher's understanding of the student's understanding. One can perform the same calculational action in both cases, but the meaning of that action for the student determines whether she is searching for a relationship or performing a procedure. | | Dividing $y$ by $x$ as above without understanding what quantitative relationship is revealed by division; dividing as an arithmetic procedure to determine whether the resulting answer is the same. |
| *Same Pattern*: Checking whether a detected pattern remains stable across all cases. | | Given a series of connected matchstick triangles, noting that each extra triangle requires two more matchsticks. |
| *Same Solution or Result*: Performing a repeated action in order to determine if the outcome of the action is identical every time. | | Given an equation such as $y = 2x$, substituting multiple integers for $x$ and noticing that $y$ is always even. |

| TYPE III: EXTENDNG | Examples |
|---|---|
| *Expanding the Range of Applicability*: Applying a phenomenon to a larger range of cases than that from which it originated. | Having graphed discreet points representing constant speed, noting that it would be possible to extend the graph indefinitely for positive $x$- and $y$-values. |
| *Removing Particulars*: Removing some contextual details in order to develop a global case. | Having identified that two people walking the same speed have the same distance/time ratio at several points on the journey, generalizing that any same-speed objects will always have the same distance/time ratio for any given location. |
| *Operating*: Mathematically operating upon an object in order to generate new cases. | Knowing that $y$ increases by 6 cm for every 1 s increase for $x$, halving the (1:6) ratio to create a new ordered pair that represents the same speed. |
| *Continuing*: Repeating an existing pattern in order to generate new cases. | Knowing that $y$ increases by 6 cm for every 1 s increase for $x$, continuing the (1:6) ratio to create new ordered pairs that represent the same speed. |

FIGURE 1 Generalizing actions.

| IDENTIFICATION OR STATEMENT | | Example |
|---|---|---|
| *Continuing Phenomenon*: Identification of a dynamic property extending beyond a specific instance. | | "Every time $x$ goes up 1, $y$ goes up 5." Or, "For every second, he walks 2/3 cm." |
| *Sameness*: A statement of commonality or similarity. | *Common Property*: Identification of the property common to objects or situations. | "For each pair in the table, the ratio of centimeters to seconds is the same. So each pair must be the same speed." |
| | *Objects or Representations*: Identification of objects as similar or identical. | "Even though those equations look different, they're both relating distance and time." |
| | *Situations*: Identification of situations as similar or identical. | "This gear problem is just like the swimming laps problem we did in class!" |
| *General Principle*: A statement of a general phenomenon. | *Rule*: Description of a general formula or fact. | $s\cdot\left(\frac{2}{3}\right)=b$, or "You multiply the number the small gear turns by 2/3 to get the number the big gear turns." |
| | *Pattern*: Identification of a general pattern. | "Each triangle train is the same because you can add two matchsticks every time." |
| | *Strategy or Procedure*: Description of a method extending beyond a specific case. | "To find out if each pair represents the same speed, divide miles by hours and see if the ratio is always the same." |
| | *Global Rule*: Statement of the meaning of an object or idea. | "If the rate of change stays the same, the data are linear." |
| **DEFINITION** | | **Example** |
| *Class of Objects*: Definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon. | | "Any two gears with a 2:3 ratio of teeth will also have a 2:3 ratio of revolutions." |
| **INFLUENCE** | | **Examples** |
| *Prior Idea or Strategy*: Implementation of a previously-developed generalization. | | "You could do the same thing on this speed problem that I did with the gears. Look at the ratio each time and you see that they're the same speed." |
| *Modified Idea or Strategy*: Adaptation of an existing generalization to apply to a new problem or situation. | | "Looking at the ratio each time doesn't work on this problem, but you could divide the increase in centimeters by the increase in seconds instead, and you see that he's walking the same speed." |

FIGURE 2 Reflection generalizations.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Table 1
*Mental Actions of the Covariation Framework*

| Mental action | Description of mental action | Behaviors |
|---|---|---|
| Mental Action 1 (MA1) | Coordinating the value of one variable with changes in the other | • Labeling the axes with verbal indications of coordinating the two variables (e.g., $y$ changes with changes in $x$) |
| Mental Action 2 (MA2) | Coordinating the direction of change of one variable with changes in the other variable | • Constructing an increasing straight line<br>• Verbalizing an awareness of the direction of change of the output while considering changes in the input |
| Mental Action 3 (MA3) | Coordinating the amount of change of one variable with changes in the other variable | • Plotting points/constructing secant lines<br>• Verbalizing an awareness of the amount of change of the output while considering changes in the input |
| Mental Action 4 (MA4) | Coordinating the average rate-of-change of the function with uniform increments of change in the input variable. | • Constructing contiguous secant lines for the domain<br>• Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input |
| Mental Action 5 (MA5) | Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function | • Constructing a smooth curve with clear indications of concavity changes<br>• Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct) |

Table 2
*Levels of the Covariation Framework*

**Covariational Reasoning Levels**

The covariation framework describes five levels of development of images of covariation. These images of covariation are presented in terms of the mental actions supported by each image.

**Level 1 (L1).** ***Coordination***

At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).

**Level 2 (L2).** ***Direction***

At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are *both* supported by L2 images.

**Level 3 (L3).** ***Quantitative Coordination***

At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.

**Level 4 (L4).** ***Average Rate***

At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.

**Level 5 (L5).** ***Instantaneous Rate***

At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

Saldanha, L., Thompson, P. W. (2002). Conceptions of sample and their relationship to statistical inference. *Educational Studies in Mathematics,* 51(3), 257-270.

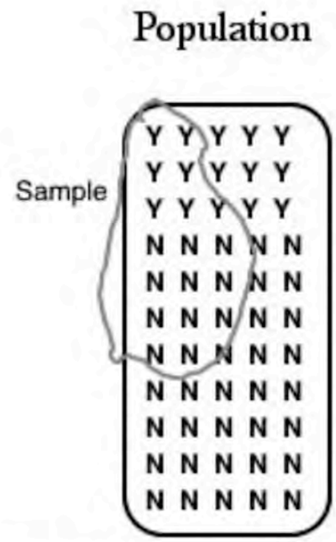



*Figure 3.* An additive image of sample entails only part-whole relationships. Resemblance between sample and population is not a salient issue. Multiple samples are seen as multiple subsets.

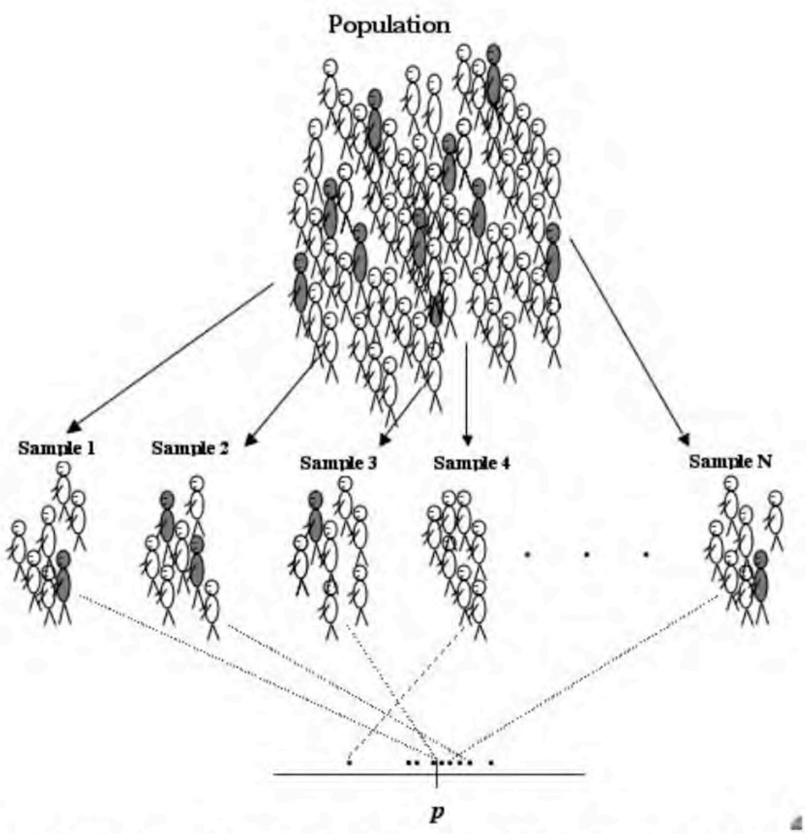



*Figure 4.* A multiplicative conception of sample entails a quasi-proportionality relationship between sample and population. Multiple samples are seen as multiple, scaled quasi mini-versions of the population.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), The concept of a function: Aspects of epistemology and pedagogy (pp. 25-58). Washington, D.C.: Mathematical Association of America.

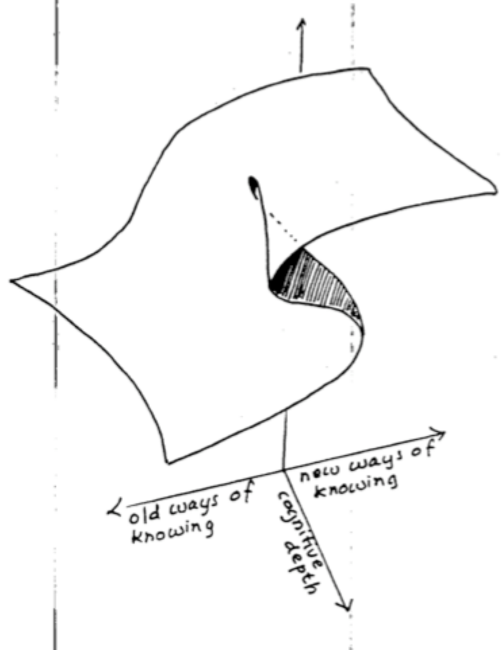



Figure 1

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*(3), 247-285.

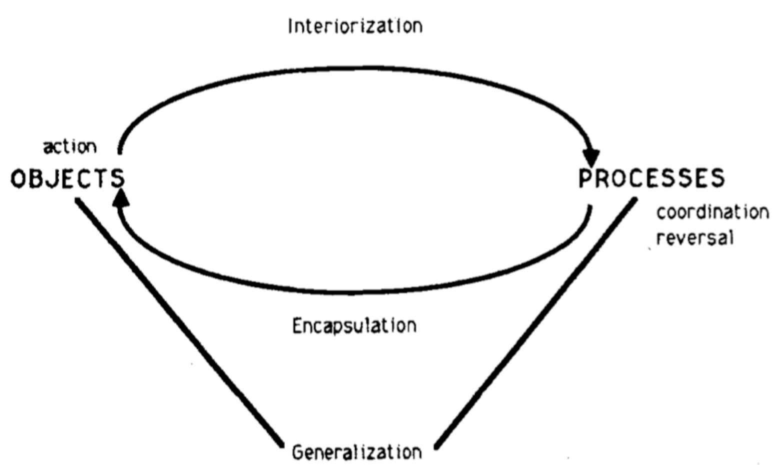



Fig. 1. Construction of objects and processes.

Thompson, P. W. (1993). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Issues in Mathematics Education Vol. 4, (pp. 21-44). Providence, RI: American Mathematical Society.

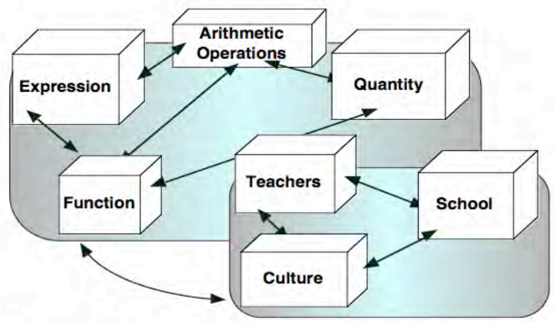
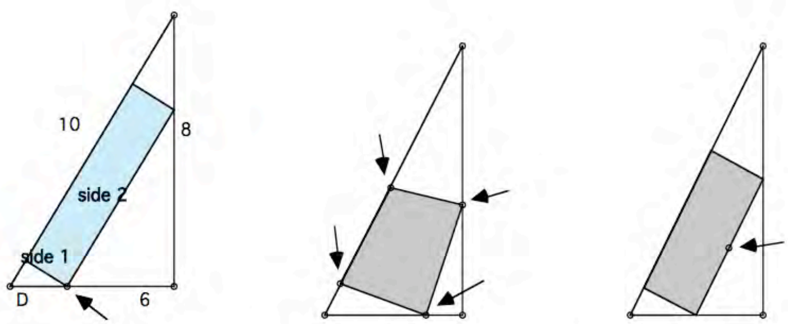
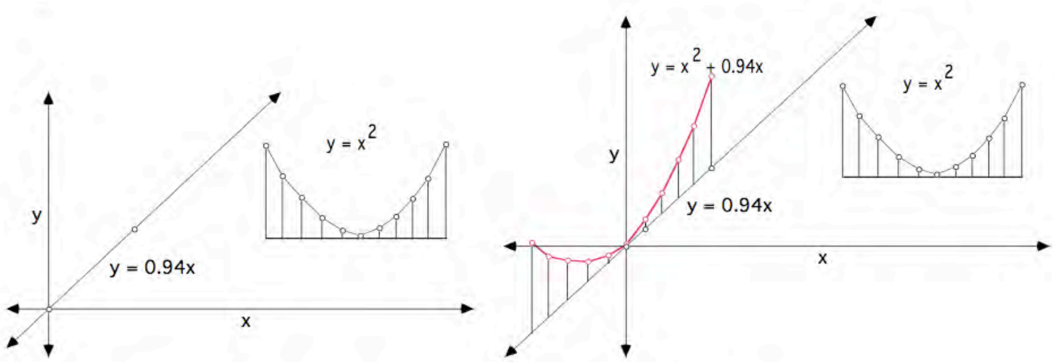



Figure 1. Students' concepts always emerge in relation to other concepts they hold, and in relation to teachers' orientatons and in relation to cultural values expressed by teachers, peers, and families.




(a) (b) (c)

Figure 5. Three categories of conceptualizing the underlying situation: (a) Move one vertex, everything moves accordingly, lengths of sides, and hence area, are a function of the distance from vertex D to corner of rectangle. (b) Move each of the rectangle's corners to get another rectangle. (c) Move a side of the rectangle; everything else moves accordingly. Area is somehow a function of "where the side is."




(a) (b)

Figure 8. (a) The functions $f(x)=0.94x$ and $g(x)=x^2$. (b) The function $h(x)=x^2+0.94x$ as the sum of $f(x)$ and $g(x)$. Varying the linear coefficient varies the slope of the graph of $f$, but the increments due to $g$ remain constant—they just get moved up or down as the linear coefficient varies.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

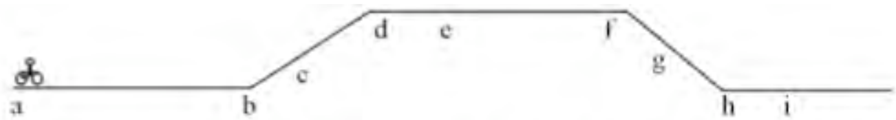



**Figure 1.** A problem in which students must distinguish between visual features of a situation and representational features of a graph. (From Monk, 1992).

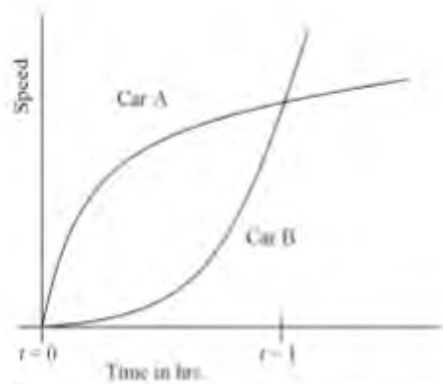



**Figure 2.** Students fail to interpret the function information conveyed by the graph.

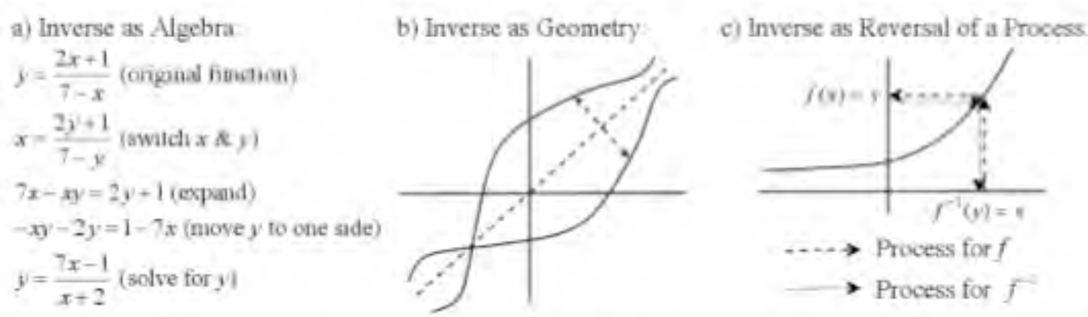



**Figure 4.** Various conceptions of the inverse of a function as a) an algebra problem, b) a geometry problem, and c) the reversal of a process. The first two of these are common among students but, in isolation, do not facilitate flexible and powerful reasoning about functional situations.

**Table 1**
*Action and Process Views of Functions*

| Action View | Process View |
|---|---|
| A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps. | A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values. |
| A student must perform or imagine *each action*. | A student can imagine the *entire process* without having to perform each action. |
| The "answer" depends on the formula. | The process is independent of the formula. |
| A student can only imagine a single value at a time as input or output (e.g., $x$ stands for a specific number). | A student can imagine all input at once or "run through" a continuum of inputs. A function is a transformation of entire spaces. |
| Composition is substituting a formula or expression for $x$. | Composition is a *coordination* of two input-output processes; input is processed by one function and its output is processed by a second function. |
| Inverse is about algebra (switch $y$ and $x$ then solve) or geometry (reflect across $y=x$). | Inverse is the *reversal of a process* that defines a mapping from a set of output values to a set of input values. |
| Finding domain and range is conceived at most as an algebra problem (e.g., the denominator cannot be zero, and the radicand cannot be negative). | Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs. |
| Functions are conceived as static. | Functions are conceived as dynamic. |
| A function's graph is a geometric figure | A function's graph defines a specific mapping of a set of input values to a set of output values. |

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America.* Raleigh, NC: North Carolina State University.

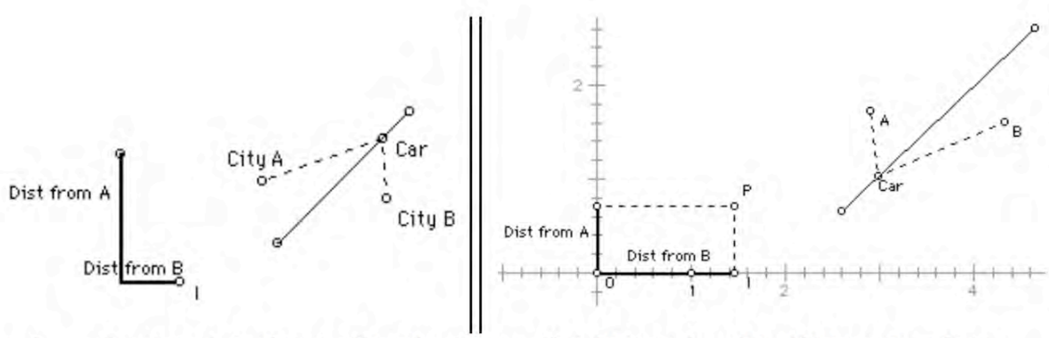



Figure 1. Two snapshots of car positions. In each snapshot, Distance from A and Distance from B are each represented by a line segment's length. In the snapshot on the right point P is displayed as the correspondence of the perpendicular segments representing AC and BC.

Confrey, J., Smith, E. (1995). Splitting, covariation, their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Table 1
*Characteristics of Counting and Splitting Worlds*

| Splitting | Counting |
|---|---|
| One is the origin | Zero is the origin |
| Splitting by *n* is the successor action | Adding one is the successor action |
| The unit (of growth) is *n* or *n*:1 | The basic unit is one |
| Multiplication and division are basic operations | Addition and subtraction are the basic operations |
| One is the identity element | Zero is the identity element |
| Reinitializing to one | Reinitializing to zero |
| Commutativity applies to multiplication | Commutativity applies to addition |
| Ratio is used to describe the interval between two successive "whole" numbers | Difference is used to describe the interval between two successive "whole" numbers |
| Composite units are made by raising "splitting units" to a higher power | Composite units are formed by aggregating counts into larger groups |
| Parts (multiplicative) are created by *n*-rooting | Parts (additive) are created by *n*-splitting |
| Exponentiation is created as repeated multiplication | Multiplication is contructed as repeated addition |
| Distributivity is applied to exponentiation over multiplication | Distributivity is applied to multiplication over addition |
| Rate is the ratio per unit time | Rate is the difference per unit time |

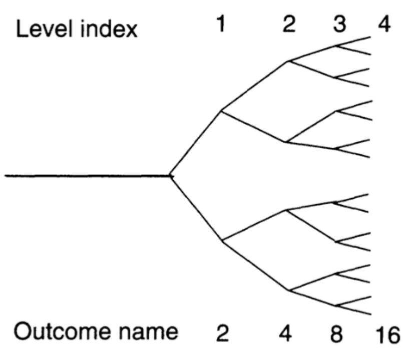



*Figure 6.* Tree diagram for 2-split.

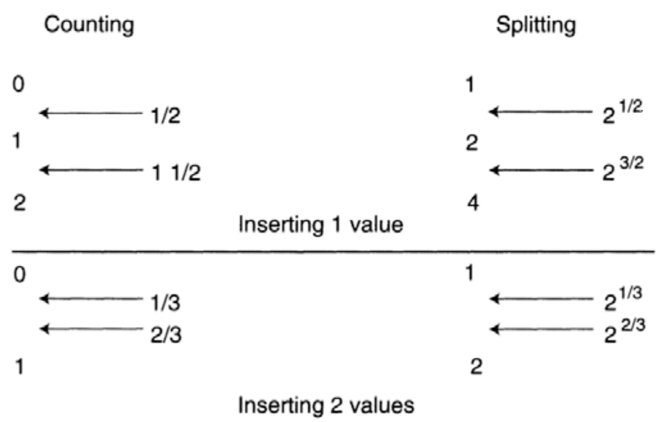



*Figure 8.* Interpolation in counting and splitting worlds contrasted.

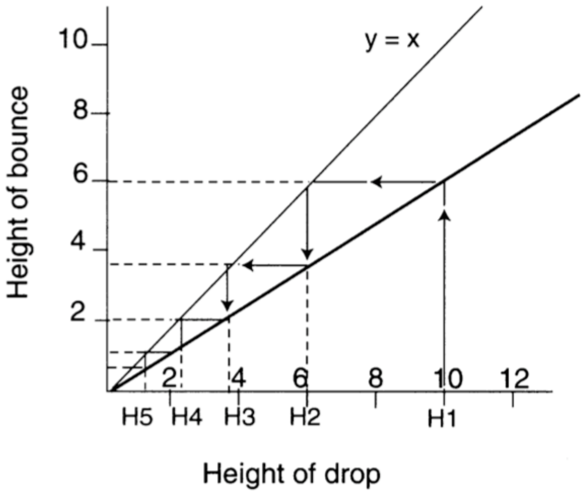



Figure 10. Interactive graph of a bouncing ball from $H_1 \ldots H_n$.

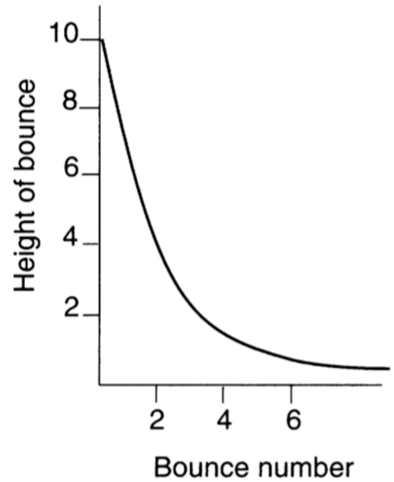



Figure 11. "Standard" graph of the height of the bouncing ball as a function of number of the bounce, drawn continously.

## Definitions of Terms

Thompson (1994b):

***Definition** (Quantity).* "A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quantity's measurability. It is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 7).

***Definition** (Internalization). "*Refers to an assimilation, an initial 'image having'" (p. 4).

***Definition** (Interiorization). "*The progressive reconstruction and organization of actions so as to enable them to be carried out in thought, as mental operations" (p. 4).

***Definition** (Scheme).* "A scheme is an organization of actions that has three characteristics: an internal state which is necessary for the activation of actions composing it, the actions themselves, and an imagistic anticipation of the result of acting" (p. 5).

Lobato & Siebert (2002):

***Definition** (Traditional transfer).* "From the traditional perspective, transfer is defined as the application of knowledge learned in one situation to another situation" (p. 89).

***Definition** (Actor-oriented transfer).* "Actor-oriented transfer is defined as the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (p. 89).

Ellis (2007):

***Definition*** *(Emergent-ratio quantitative reasoning).* "Reasoning directly with a third quantity such as endurance constitutes a different aspect of quantitative reasoning, which can be termed 'emergent ratio quantitative reasoning'" (p. 470).

***Definition*** *(Generalizing actions).* Students' mental activity as they generalize.

***Definition*** *(Reflection generalizations).* Students' final statements of generalization.

Carlson et al. (2002)

***Definition*** *(Covariational reasoning).* "We define covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354).

***Definition*** *(Pseudo-analytical thought processes/behaviors).* "The terms pseudo-analytical thought processes and pseudo-analytical behaviors identify, respectively, processes of thought and behaviors that take place without understanding, and pseudo-analytic behaviors are produced by pseudo-analytical thought processes" (p. 354).

Oehrtman (2009):

***Definition*** *(Metaphor).* Metaphor refers to an idea that students associate with the concept of limit as a means of making sense of or coping with infinity. More specifically, metaphor provides an isomorphism between formal concept definitions and students' concept images so that properties of the formal definition are assumed to be preserved and inferences transferred.

***Definition*** *(Metaphor cluster).* Due to the persistence of particular metaphors across limit concepts (i.e. definition of derivative, definition of Riemann integral, etc.), Oehrtman grouped the metaphors that emerged in a variety of contexts into five strong metaphor clusters.

***Definition*** *(Strong metaphor).* In order for a metaphor to be classified as strong it must have emphasis and resonance. Emphasis refers to the metaphor's persistence across various tasks while resonance denotes the metaphor having implications for a students' further reasoning.

Saldanha & Thompson (2002):

***Definition*** *(Multiplicative conception of a sample).* An image of sample that entails conceptual operations of multiplicative reasoning. That is, a "conception of sample as a quasi-proportional mini version of the sampled population where the 'quasi-proportionality' image emerges in anticipating a bounded variety of outcomes, were one to repeat the sampling process" (p. 266).

Tall & Vinner (1981):

***Definition*** *(Concept Image).* "The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152).

***Definition*** *(Concept definition).* A form of words that specify a particular concept. By *formal concept definition*, Tall and Vinner refer to definitions that are accepted and used by the mathematical community.

Sierpinska (1992):

***Definition*** *(Epistemological obstacles).* Epistemological obstacles "are common in the frame of some culture, whether present or past and thus seem to be the most objective obstacles to a new way of knowing" (p. 27).

Sfard (1992):

***Definition*** *(Operational conception).* An operational conception means treating mathematical notions as a computational process rather than a static construct.
***Definition*** *(Structural conception).* A structural conception means treating mathematical notions as if they referred to object-like entities.
***Definition*** *(Interiorization).* The process performed on the already familiar objects.
***Definition*** *(Condensation).* The emergence of an idea that turns this process into a more compact, self-contained whole.
***Definition*** *(Reification).* An ability to view the condensed entity as a permanent object in its own right.

Thompson (1994a)

***Definition*** *(Action conception of function).* "Students holding an action conception of function imagine that the recipe remains the same across numbers, but that they must actually apply it to some number before the recipe will produce anything. They do not necessarily view the recipe as representing a result of its application" (p. 7).
***Definition*** *(Process conception of function).* "When students build an image of 'self-evaluating' expressions they have what is called a process conception of function. They do not feel compelled to imagine actually evaluating an expression in order to think of the result of its evaluation. From the perspective of students with a process conception of function, an expression stands for what you would get by evaluating it" (p. 7).
***Definition*** *(Didactic contract).* The implicit understanding between a teacher and their students with regard to expectations for learning.

Confrey & Smith (1995)

***Definition*** *(Splitting).* A primitive multiplicative operation that provides an operational basis for multiplication and division.
***Definition*** *(Splitting world).* "The development of splitting and its connection to ratio creates the basis for what we call the splitting world, whose overall structure and developmental path differs considerably from the counting world that currently dominates the curriculum" (p. 74).

## Mathematics Education Qualifying Exam Part 1, Fall 2008

***Instructions:*** *You* <u>*must*</u> *respond to either Question 1 or Question 2. In addition, respond to 2 (two) other questions of your choosing from the remaining list. Provide responses to entire questions, not parts of multiple ones.*
*All responses should be in essay form; aim for clarity and explicitness, as well as thoroughness, concision, and coherence in your writing. The recommended length limit for each response is roughly 1000 words. If you use diagrams in your responses, it is recommended that you draw them by hand, label each with a title (e.g., "Figure x"), and insert a clear reference to each one in the appropriate place in your narrative. Turn in any diagrams with your responses. Your responses may cite literature that is not explicitly mentioned in a question. Good luck and good writing!*

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**<u>Question 2</u>**
Imagine that you have been hired by a university mathematics department as a faculty member specializing in mathematics education research. In your daily interactions with your mathematician colleagues you find yourself having to explain the raison d'être (reason for being) for mathematics education research as a scholarly field.

**<u>Question 3</u>**
a) Characterize Carlson et al.'s (2002) *covariation framework*. Your discussion should include a description of what it means to reason covariationally, what the framework is about, and it should describe the framework's various components and interconnections among them.
b) What is the ostensible usefulness of this framework? Illustrate and bolster your argument with at least one example of its usefulness drawn from the research literature (be explicit and clear as to what your example illustrates).

**<u>Question 6</u>**
In a sequence of articles—*Talking About Rates Conceptually, Parts I and II*—authors Thompson and Thompson (1994, 1996) report on a sequence of lessons involving interactions between a teacher (Bill) and a middle school student (Ann). These interactions occurred in the context of a protracted effort to help Ann develop an understanding of speed as a rate. The authors made the following statement in both articles:

> "The image of speed we intended students construct through this unit is composed of these items, which themselves are constructions:
> 1. Speed is a quantification of motion;
> 2. completed motion involves two completed quantities—distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation);
> 3. speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance;
> 4. there is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go $m$ distance units in $s$ time units at a constant speed, then at this speed you will go
>
> $a/b \times m$ distance units in $a/b \times s$." (Thompson & Thompson, 1994, p. 283)

a) Briefly interpret each of the items above (i.e., describe what they mean).
b) Describe two arguably distinct ways in which the authors used the above scheme of ideas in their research study.
c) Discuss how the authors' use of this scheme is related to what Thompson (2008) writes about in his later article *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. In particular, discuss this relationship with regard to the ideas of coherence and meaning elaborated in the later article.