Mathematics 3: Integral Transformations

Summary

Mathematics 3

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Bachelors's in Mechatronics

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loosly based on the lecture by Dipl.-Ing. Dr. Peter Pichler (2017)

Dornbirn, 28.08.2021

Contents

1 Fourier Transformation

The Fourier Transformation is a method to decompose a continuous, aperiodic signal into a continuous spectrum. This integral transformation is defined by

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt = F(\omega) \tag{1.1}$$

$$\mathcal{F}^{-1}(f(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{i\omega t} dt = f(t) \tag{1.2}$$

In the general case, $\mathcal{F}(f(t)) = F(\omega)$ is a complex function with a real and an imaginary part:

$$\begin{aligned} F(\omega) = & F_1(\omega) + iF_2(\omega) \\ & where: \\ & F_1(\omega)...\Re(F(\omega)) \\ & F_2(\omega)...\Im(F(\omega)) \end{aligned} \tag{1.3}$$

The Fourier Transform $F(\omega)$ is Hermitian that the conjugate complex $\overline{F}(\omega)$ of a Fourier Transform is equal to the Fourier Transform at the negative frequency $F(-\omega)$.

$$\overline{F}(\omega) = F(-\omega) \tag{1.4}$$

The Fourier Transformation exhibits some very useful properties that can be exploited for calculations.

1.1 Linearity

The Fourier Transformation is a linear operation which means that

$$\mathcal{F}(a \cdot f(t) \pm b \cdot g(t)) = a \cdot \mathcal{F}(t) \pm b \cdot \mathcal{F}(g(t)) \tag{1.5}$$

1.2 Differentiation

If the original function $f(t)$ converges to 0: $f(t) \to 0$ for $t \to \pm\infty$, than the Fourier Transformation of the differentiation $\mathcal{F}(f'(t))$ can be expressd as

$$\mathcal{F}(f'(t)) = i\omega\mathcal{F}(f(t)) \text{ only if: } f(t) \to 0 \text{ for } t \to \pm\infty \tag{1.6}$$

1.3 Time Shifting

If a function $f(t)$ is shifted in the time domain about a constant $f(t-a)$ the Fourier Transformation of the shifted function can be calculated by

$$\mathcal{F}(f(t-a)) = e^{-i\omega a} \cdot \mathcal{F}(f(t)) \tag{1.7}$$

1.4 Convolution

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{\tau=-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau \tag{1.8}$$

The Fourier Transformation of the convolution of two functions can also be calculated by

$$\mathcal{F}(f(t) * g(t)) = \mathcal{F}(f(t)) \cdot \mathcal{F}(g(t)) \tag{1.9}$$

2 Fourier Series

The Fourier Series is a special serious expansion for periodic, piecewise continuous functions into a function series of sine and cosine.

In the case of the complex fourier series, the trigonometric functions are further decomposed into complex Euler exponential functions.

2.1 Real Fourier Series

The real Fourier Series can be expressed by 3 parameters which are called the Euler-Fourier Parameter a_0, a_n and b_n. Depending on the symmetry of the original function $f(t)$, the calculation process can be shortened. The base frequency for all components is denoted as $\omega_0 = \frac{2\pi}{T}$.

Even Symmetric Function	Odd Symmetric Functions	Arbitrary Function
$f(t) = f(-t)$	$f(t) = -f(-t)$	no symmetry
$\int_{-a}^{a} f(t)dt = 2\int_0^a f(t)dt$	$\int_{-a}^{a} f(t)dt = 0$	no symmetry
$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos\,(n\omega_0 t)$	$f(t) = \sum_{n=1}^{\infty} b_n \cdot sin(n\omega_0 t)$	$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos\,(n\omega_0 t) + b_n \cdot sin\,(n\omega_0 t)$
$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt$	$a_0 = 0$	$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$
$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) \cdot cos(n\omega_0 t)dt$	$a_n = 0$	$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos(n\omega_0 t)dt$
$b_n = 0$	$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) sin(n\omega_0 t)dt$	$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(n\omega_0 t)dt$

2.2 Complex Fourier Series

As *sine* and *cosine* can be expressed by complex Euler-Functions. These pointers can be added together where each pointer has its own amplitude c_n called Fourier Coefficient. Again the frequency is denoted as $\omega_0 = \frac{2\pi}{T}$.

$$\begin{aligned} f(t) &= f(t + n \cdot T)\ n \in \mathbb{Z} \\ \text{with } \omega_0 &= \frac{2\pi}{T} \\ f(t) &= \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\omega_0 t} \end{aligned} \tag{2.1}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-in\omega_0 t} \tag{2.2}$$

Even Symmetric Function	Odd Symmetric Functions	Arbitrary Function
$f(t) = f(-t)$	$f(t) = -f(-t)$	no symmetry
$c_n = c_{-n}$	$c_n = -c_{-n}$	c_n
only real part c_n	only imaginary c_n	fully complex c_n

3 Laplace Transformation

The Laplace transformation can be used to solve differential equations. For that purpose, the Laplace transformation converts a function from the time domain into the s-domain. The notation is

$$f(t) \circ\!\!-\!\!\bullet\ \mathcal{L}(f(t)) = F(s) \tag{3.1}$$

3.1 Laplace Integral

The Laplace transformation is defined as

$$\mathcal{L}\left(f(t)\right) = \int_0^{\infty} f(t) \cdot e^{-st} dt \tag{3.2}$$

where s is a complex number $s = \delta + i\omega$. This transformation only works on functions $f(t)$ that do not grow faster than e^t. There are many standard transforamtions as seen in the following table.

3.2 Inverse Laplace Transformation

The inverse Laplace transformation brings the function brack from the s-domain into the time domain. This is noted like

$$\mathcal{L}^{-1}\left(F(s)\right) = f(t) \tag{3.3}$$

In most cases the inverse transformation can be done by a partial fraction decomposition. Therefore, the function $f(s)$ is written as a sum of fractions, where each fraction can be separately transformed.

3.3 Collection of Common Laplace-Transformation

Function $f(t)$	Transformation $F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
$e^{\pm at}$	$\frac{1}{s \mp a}$
$t \cdot e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
$t^n \cdot e^{\pm at}$	$\frac{n!}{(s \mp a)^{n+1}}$
$u(t-a)$	$\frac{1}{s}e^{-as}$
$f(t-a) \cdot u(t-a)$	$\mathcal{L}(f(t)) \cdot e^{-as}$
$\delta(t-a)$	e^{-as}
$\sqrt{t}$	$\frac{1}{2s}\sqrt{\frac{\pi}{s}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
$\sqrt{t} \cdot e^{at}$	$\frac{\sqrt{\pi}}{2(s-a)\sqrt{s-a}}$
$\frac{1}{\sqrt{t}} \cdot e^{at}$	$\frac{\sqrt{\pi}}{\sqrt{s-a}}$
$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t \cdot sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t \cdot cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$t^n \cdot sin(\omega t),\ n \in \mathbb{N}$	$\frac{i \cdot n!}{2}\left(\frac{1}{(s+i\omega)^{n+1}} - \frac{1}{(s-i\omega)^{n+1}}\right)$
$t^n \cdot cos(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s+i\omega)^{n+1}} + \frac{1}{(s-i\omega)^{n+1}}\right)$
$sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$t \cdot sinh(\omega t)$	$\frac{2\omega s}{(s^2-\omega^2)^2}$
$t \cdot cosh(\omega t)$	$\frac{s^2+\omega^2}{(s^2-\omega^2)^2}$
$t^n \cdot sinh(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s-\omega)^{n+1}} - \frac{1}{(s+\omega)^{n+1}}\right)$
$t^n \cdot cosh(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s-\omega)^{n+1}} + \frac{1}{(s+\omega)^{n+1}}\right)$
$e^{at}sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$
$e^{at}cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$
$sin(\omega t)^2$	$\frac{2\omega^2}{s(s^2+4\omega^2)}$
$cos(\omega t)^2$	$\frac{s^2+2\omega^2}{s(s^2+4\omega^2)}$
$sinh(\omega t)^2$	$\frac{2\omega^2}{s(s^2-4\omega^2)}$
$cosh(\omega t)^2$	$\frac{s^2-2\omega^2}{s(s^2-4\omega^2)}$

3.4 Properties

The Laplace transformation has some helpfule properties, that can be leveraged to solve a problem.

3.4.1 Linearity

The Laplace transformation is a linear operation. Therefore, the linearity condition holds. The summation of multiple functions can be split and transformed seperatly.

$$\mathcal{L}\left(a \cdot f(t) \pm b \cdot g(t)\right) = a\mathcal{L}\left(f(t)\right) \pm b\mathcal{L}\left(g(t)\right) \tag{3.4}$$

3.4.2 Derivative

Differentiating a function in the time domain corresponds to a multiplication with s in the s-domain.

$$\begin{aligned}
\mathcal{L}\left(f'(t)\right) &= s \cdot \mathcal{L}\left(f(t)\right) - f(0) \\
\mathcal{L}\left(f''(t)\right) &= s^2 \cdot \mathcal{L}\left(f(t)\right) - s \cdot f(0) - f'(0) \\
\mathcal{L}\left(f^{(n)}(t)\right) &= s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)
\end{aligned} \tag{3.5}$$

3.4.3 Integration

Similary, the integration in of the function in the time domain corresponds with a multiplication of $\frac{1}{s}$ in the s-domain. This relation is especially helpfule for inverse Laplace transformations.

$$\mathcal{L}\left(\int_{\tau=0}^{t} f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}\left(f(t)\right) \tag{3.6}$$

The diagram below visualizes the correspondence in a different way.

$$\begin{tikzcd}
f(t) \arrow[r, "\mathcal{L}(...)"] \arrow[d, "\int dt"'] & F(s) \arrow[d, "\frac{1}{s}"] \\
g(t) \arrow[r, "\mathcal{L}(...)"'] & G(s)
\end{tikzcd}$$

3.4.4 Frequency - Shift

If the function $f(t)$ is multiplied with an exponential function in the time domain, than the exponent a must be subtracted or added in the transformation. The addition or subtraction must be applied to every s in the s-domain. Thus, the function gets shifted in the s-domain which is also the reason for the name of this porperty.

$$\mathcal{L}\left(f(t) \cdot e^{\pm at}\right) \rightarrow F(s \mp a) \tag{3.7}$$

3.4.5 Time - Shift

A function can also be shifted in the time domain. One way of shifting a function is to multiply it with the Heaviside function. Strictly speaking, the Heaviside "distribution" is not a function.

Heaviside Function

The hHeaviside distrubution is defined as:

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases} \tag{3.8}$$

The Heaviside function is 0 for every all time before $t < a$ and 1 for all time beyond $t \geq a$.

When a function $f(t)$ is multiplied with the Heaviside function $u(t-a)$, the result $g(t)$ is restricted to the time $t \geq a$. For $t < a$, $g(t) = 0$ and for $t \geq a$, $g(t) = f(t)$.

A function, that is multiplied with the Heaviside function, can be Laplace transformed in the following way:

$$\begin{aligned} \mathcal{L}\left(u(t-a)\right) &= \frac{e^{-a}}{s} \\ \mathcal{L}\left(f(t-a) \cdot u(t-a)\right) &= \mathcal{L}\left(f(t)\right) \cdot e^{-as} \\ \mathcal{L}^{-1}\left(F(s) \cdot e^{-as}\right) &= f(t-a) \cdot u(t-a) \end{aligned} \tag{3.9}$$

Dirac's Deltafunction

The Dirac's delta function can be thought of as an infinitly high needle at only one distinct place. It is defined as:

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases} \tag{3.10}$$

The Dirac delta impulse is the derivative of the Heaviside function, which is especially helpfule when using the derivative property from 3.4.2.

3.4.6 Convolution

The convolution of two functions $f(t)$ and $g(t)$ is often denominated by the $*$ operator. The convolution is defined as:

$$f(t) * g(t) = \int_{\tau=0}^{t} f(\tau) \cdot g(t-\tau) d\tau \tag{3.11}$$

The special property of the convolution is that the Laplace transformation of two convoluted functions in the time domain is equivalent to the muliplication in the s-domain.

$$\mathcal{L}\left(f(t) * g(t)\right) = \mathcal{L}\left(f(t)\right) \cdot \left(g(t)\right) \tag{3.12}$$

$$\mathcal{L}^{-1}\left(F(s) \cdot G(s)\right) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s)) \tag{3.13}$$

3.4.7 Initial Value Theorem

With the initial value theorem one can very quickly evaluate the function at $t = 0$. For this purpose, the limit of the function multiplied with s is taken in the s-domain. For fractions it might be helpfule to apply the limit rule of de l'Hospital.

$$f(0) = \underset{s \to \infty}{lim} F(s) \cdot s \tag{3.14}$$

3.4.8 Final Value Theorem

Similar to the initial value theorem, the final value theorem can evaluate the limit of the function at $t \to \infty$.

$$\underset{t\to\infty}{lim} f(t) = \underset{s\to 0}{F(s) \cdot s} \tag{3.15}$$

3.4.9 Frequency Domain Derivative

This theorem is sometimes also called the muliplication theorem. Here $n \in \mathbb{N}$ not only denotes an natural exponent but also the number of derivatives.

$$\mathcal{L}\left(t^n \cdot f(t)\right) = (-1)^n \cdot \frac{d^{(n)}}{ds^{(n)}} F(s) \tag{3.16}$$

3.4.10 Frequency Domain Integral

Similary, this theorem is sometimes also called the division theorem. To prevent a confusion between the integration variable s and the Laplace-s, the integration variable is denoted as $\tilde{s}$.

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{\tilde{s}=s}^{\infty} F(\tilde{s}) d\tilde{s} \tag{3.17}$$

4 Z-Transformation

The Z-transformation, also called the discrete Laplace-transformation, is used for discrete sequences. The Z-transformation can be applied to solve difference equations.

$$\mathcal{Z}\left(f\left[n\right]\right) = F(z) \tag{4.1}$$

4.1 Z-Transformation Sum

The Z-transformation is calculated by evaluating the following sum:

$$Z(f[n]) = \sum_{n=0}^{\infty} f[n] z^{-n} \tag{4.2}$$

Insertion: Geometric Series

The geometric series is especially helpfule to solve many Z-transformations. It is defined as

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \tag{4.3}$$

However, this is only applicable if $|x| < 1$ holds. This is assumed for all further statements.

4.2 Inverse Z-Transformation

The inverse Z-transformation can often be solved by a partial fraction decomposition, just like the inverse Laplace-transformation. Every partial fraction is separately transformed.

$$\mathcal{Z}^{-1}\left(F(z)\right) = f[n] \tag{4.4}$$

4.3 Collection of Common Z-Transformation

Series $f[n]$	Z-Transformation $F(z)$
1 or $u[n]$	$\frac{z}{z-1}$
$\delta[n]$	1
$\delta[n-1]$	$\frac{1}{z}$
$\delta[n-2]$	$\frac{1}{z^2}$
$\delta[n-k]$	$\frac{1}{z^k}$
a^n	$\frac{z}{z-a}$
e^n	$\frac{z}{z-e}$
$(-a)^n$	$\frac{z}{z+a}$
$sin[a \cdot n]$	$\frac{z \cdot sin(a)}{z^2-2z \cdot cos(a)+1}$
$cos[a \cdot n]$	$\frac{z^2-z \cdot cos(a)}{z^2-2z \cdot cos(a)+1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z \cdot (z+1)}{(z-1)^3}$
n^3	$\frac{z \cdot (z^2+4z+1)}{(z-1)^4}$
n^4	$\frac{z \cdot (z^3+11z^2+11z+1)}{(z-1)^5}$
$sinh[a \cdot n]$	$\frac{z \cdot sinh(a)}{z^2-2z \cdot cosh(a)+1}$
$cosh[a \cdot n]$	$\frac{z^2-z \cdot cosh(a)}{z^2-2z \cdot cosh(a)+1}$
$sin\left[\frac{\pi}{2} \cdot n\right]$	$\frac{z}{z^2+1}$
$cos\left[\frac{\pi}{2} \cdot n\right]$	$\frac{z^2}{z^2+1}$

4.4 Properties

Similary to the Laplace transformation, the Z-transformation also has some important properties that can be exploited to solve most of the upcoming problems.

4.4.1 Linearity

The Z-transformation is a linear transformation which means that the linearity condition holds. Summations can be split and multiplicative constants can be pulled infront of the sum.

$$\mathcal{Z}\left(a \cdot f[n] \pm b \cdot g[n]\right) = aF(f[n]) \pm bF(f[n]) \tag{4.5}$$

4.4.2 Multiplication Theorem

If a sequence is multiplied with n in the time domain, the Z-transformation must be differentiated and multiplied with $-z$.

$$\mathcal{Z}(n \cdot f[n]) = -z \cdot \frac{dF}{dz} \tag{4.6}$$

4.4.3 Time Shift Right

Similarly as in continuous function, the time shift in positive time direction can be acomplished by subtraction

$$f[n] \to f[n-1] \tag{4.7}$$

The transformation of such a function can be done by:

$$
\begin{aligned}
\mathcal{Z}(f[n-1]) =& f[-1] + \frac{1}{z} \cdot \mathcal{Z}(f[n]) \\
\mathcal{Z}(f[n-2]) =& f[-2] + \frac{f[-1]}{z} + \frac{1}{z^2}\mathcal{Z}(f[n]) \\
\mathcal{Z}f([n-3]) =& f[-3] + \frac{f[-2]}{z} + \frac{f[-2]}{z^2} + \frac{1}{z^3}\mathcal{Z}(f[n])
\end{aligned}
\tag{4.8}
$$

or for any shift in positive time direction by a time steps:

$$
\mathcal{Z}(f[n-a]) \left(\sum_{i=0}^{i=(a-1)} \frac{f[-a+i]}{z^i} \right) + \frac{\mathcal{Z}(f[n])}{z^a} \tag{4.9}
$$

When considering the inverse Z-transformation, a multiplication with $\frac{1}{z}$ in the z-domain, must be shifted via a Heaviside function in the discrete time domain. An example could be:

$$
\mathcal{Z}^{-1}\left(\frac{z}{z-4} \cdot \frac{1}{z} \right) = 4^n \cdot u[n-1] \tag{4.10}
$$

4.4.4 Time Shift Left

Contrary to the time shift in positive direction, a shift into the negative time direction is done by an addition:

$$
f[n] \to f[n+1] \tag{4.11}
$$

The transformation can be expressed as

$$
\begin{aligned}
\mathcal{Z}(f[n+1]) =& z \cdot (\mathcal{Z}(f[n]) - f[0]) \\
\mathcal{Z}(f[n+2]) =& z^2 \cdot \left(\mathcal{Z}(f[n]) - f[0] - \frac{f[1]}{z} \right) \\
\mathcal{Z}(f[n+3]) =& z^3 \cdot \left(\mathcal{Z}(f[n]) - f[0] - \frac{f[1]}{z} - \frac{f[2]}{z^2} \right)
\end{aligned}
\tag{4.12}
$$

or a shift by any arbitrary constant a

$$
\mathcal{Z}(f[n+a]) = z^a \cdot \left(\mathcal{Z}(f[n]) - \sum_{j=0}^{a-1} \frac{f[j]}{z^j} \right) \tag{4.13}
$$

4.4.5 Convolution

A convolution in the discrete time domain is defined as

$$
f[n] * g[n] = \sum_{k=0}^{n} f[k] \cdot g[n-k] \tag{4.14}
$$

The Z-transformation for the convolution of two signals can be calculated by

$$
\begin{aligned}
\mathcal{Z}(f[n] * g[n]) =& \mathcal{Z}(f[n]) \cdot \mathcal{Z}(g[n]) \\
\mathcal{Z}^{-1}(F(z) \cdot G(z)) =& f[n] * g[n]
\end{aligned}
\tag{4.15}
$$

5 Collection of Integrals

$$\int \frac{sin(x)^2}{cos(x)}dx = \int \frac{1-cos(x)^2}{cos(x)}dx = \int \frac{1}{cos(x)}dx - sin(x)$$

$$\int \frac{1}{cos(x)}dx = \int \frac{cos(x)}{cos(x)^2}dx = \int \frac{cos(x)}{1-sin(x)^2}dx = arctanh(sin(x))$$

$$\int \frac{1}{sin(x)}dx = \int \frac{sin(x)}{sin(x)^2}dx = \int \frac{sin(x)}{1-cos(x)^2}dx = -arctanh(cos(x))$$

$$\int tan(x)dx = -ln(cos(x))$$

$$\int cot(x)dx = ln(sin(x))$$

$$\int \frac{1-x\cdot arctan(x)}{arctan(x)\cdot(1+x^2)} = \int \left(\frac{1}{arctan(x)\cdot(1+x^2)} - \frac{x\cdot arctan(x)}{arctan(x)\cdot(1+x^2)}\right)dx$$
$$\xrightarrow{z=arctan(x)} \int \left(\frac{1}{z} - tan(z)\right)dz = ln(z) + ln(cos(z))$$

$$\int \frac{1}{x^2\cdot\sqrt{1-x^2}}dx \xrightarrow{x=sin(u)} \int \frac{cos(u)}{sin(u)^2\cdot cos(u)}du = \int \frac{1}{sin(x)^2}du = -cot(u)$$

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int x\cdot\frac{x}{\sqrt{1-x^2}}dx = -x\sqrt{1-x^2} + \int \sqrt{1-x^2}dx$$

$$\int sin(x)^3\cdot e^{-cos(x)}dx \xrightarrow{u=-cos(x)} \int sin(x)^3\cdot\frac{e^u}{sin(x)}du = \int \left(1-u^2\right)\cdot e^u du$$

$$\int \frac{1}{\sqrt{\frac{a}{x}-1}}dx = \int \frac{1}{\sqrt{\frac{a-x}{x}}}dx = \int \frac{\sqrt{x}}{\sqrt{a-x}} \xrightarrow{x=a\cdot sin(u)^2} \int \frac{\sqrt{a}\cdot sin(u)\cdot 2\cdot a\cdot sin(u)\cdot cos(u)}{\sqrt{a-a\cdot sin(u)^2}}du = 2\cdot a\int sin(u)^2 du$$

$$\int \frac{1}{x\cdot\sqrt{a^2x^2-1}}dx \xrightarrow{ax=cosh(u)}$$
$$\int \frac{1}{a}\cdot\frac{sinh(u)}{\frac{cosh(u)}{a}\sqrt{cosh(u)^2-1}}du = \int \frac{1}{cosh(u)}du = \int \frac{cosh(u)}{1+sinh(u)^2}du = arctan(sinh(u))$$

$$\int 2 \cdot x \cdot e^{x^2} dx = e^{x^2}$$

$$\int \frac{1}{\sqrt{x^2 \cdot a + x^2 \cdot ln(x)^2}} dx \xrightarrow{u=ln(x)} \int \frac{1}{\sqrt{a + u^2}} du = \int \frac{1}{\sqrt{a} \cdot \sqrt{1 + \frac{u^2}{a}}} du = arcsinh\left(\frac{u}{\sqrt{a}}\right)$$

$$\int \frac{sin(x)}{cos(x)^3} dx = \int tan(x) \cdot \frac{1}{cos(x)^2} dx = \frac{tan(x)^2}{2}$$

$$\int \left(\frac{1}{x}e^x - \frac{1}{x^2}e^x\right) dx = \frac{1}{x}e^x - \int -x\frac{1}{x^2}e^x dx - \int \frac{1}{x^2}e^x dx = \frac{1}{x}e^x$$

$$\int \frac{2}{4x + a + \sqrt{a^2 + 4ax}} dx \xrightarrow{u=\sqrt{4ax+a^2}} \int \frac{u}{a(4x + a + u)} du = \int \frac{u}{4ax + a^2 + au} du = \int \frac{u}{u^2 + au} du = ln(u+a)$$

$$\int \frac{1 - sin(x)}{1 + sin(x)} dx = \int \frac{1 - sin(x)}{1 + sin(x)} \cdot \frac{1 - sin(x)}{1 - sin(x)} dx = \int \frac{1 - 2sin(x) + sin(x)^2}{1 - sin(x)^2} dx = 2 \cdot tan(x) - \frac{2}{cos(x)} - x$$

$$\int \frac{1}{cos(x)^3} dx = \int \frac{1}{cos(x)} \frac{1}{cos(x)^2} dx = \frac{tan(x)}{cos(x)} - \int \frac{sin(x) \cdot tan(x)}{cos(x)^2} dx =$$
$$\frac{sin(x)}{cos(x)^2} - \int \frac{1}{cos(x)^3} dx + \int \frac{cos(x)}{cos(x)^2} dx = \frac{1}{2}\left(\frac{sin(x)}{cos(x)^2} + arctanh(sin(x))\right)$$

$$\int e^{ax} \cdot cos(bx) dx = \frac{a}{a^2 + b^2} e^{ax} cos(bx) + \frac{b}{a^2 + b^2} e^{ax} sin(bx)$$

$$\int e^{ax} \cdot sin(bx) dx = \frac{a}{a^2 + b^2} e^{ax} sin(bx) - \frac{b}{a^2 + b^2} e^{ax} cos(bx)$$

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6 Collection of Sums

$$\sum_{k=0}^{n} 1 = (n+1)$$

$$\sum_{k=0}^{n} k = \frac{n \cdot (n+1)}{2}$$

$$\sum_{k=0}^{n} a + k = \frac{(n+1) \cdot (2a+n)}{2}$$

$$\sum_{k=1}^{n} 2k + 1 = n^2$$

$$\sum_{k=0}^{n} 2k = n \cdot (n+1)$$

$$\sum_{k=0}^{n} k^2 = \frac{n \cdot (n+1)^2}{6}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2 \cdot (n+1)^2}{4}$$

$$\sum_{k=1}^{n} (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

$$\sum_{k=1}^{n} (2n-1)^3 = n^2(2n^2-1)$$

$$\sum_{k=0}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$$

$$\sum_{k=0}^{n} a^{-k} = \frac{a^{-n}(a^{n+1}-1)}{a-1}$$

7 Collection of Trigonometric Identities

$$sin(a) = \frac{1}{2i}\left(e^{ia} - e^{-ia}\right)$$

$$sin(a+b) = sin(a) \cdot cos(b) + cos(a) \cdot sin(b)$$

$$sin(a-b) = sin(a) \cdot cos(b) - cos(a) \cdot sin(b) \tag{7.1}$$

$$sin(2a) = 2 \cdot sin(a) \cdot cos(a)$$

$$sin(a)^2 = \frac{1}{2} \cdot (1 - cos(2a))$$

$$sin(a) + sin(b) = 2 \cdot sin\left(\frac{a+b}{2}\right) \cdot cos\left(\frac{a-b}{2}\right)$$

$$sin(a) - sin(b) = 2 \cdot cos\left(\frac{a+b}{2}\right) \cdot sin\left(\frac{a-b}{2}\right)$$

$$cos(a) = \frac{1}{2}\left(e^{ia} + e^{-ia}\right)$$

$$cos(a+b) = cos(a) \cdot cos(b) - sin(a) \cdot sin(b)$$

$$cos(a-b) = cos(a) \cdot cos(b) + sin(a) \cdot sin(b)$$

$$cos(2a) = cos(a)^2 - sin(a)^2$$

$$cos(a)^2 = \frac{1}{2} \cdot (1 + cos(2a))$$

$$cos(a)^2 = \frac{1}{1 + tan(a)^2}$$

$$cos(a) + cos(b) = 2 \cdot cos\left(\frac{a+b}{2}\right) \cdot cos\left(\frac{a-b}{2}\right)$$

$$cos(a) - cos(b) = -2 \cdot sin\left(\frac{a+b}{2}\right) \cdot sin\left(\frac{a-b}{2}\right)$$

$$sin(a)^2 + cos(a)^2 = 1$$

$$sin(a) \cdot cos(b) = \frac{1}{2}(sin(a+b) + sin(a-b))$$

$$cos(a) \cdot cos(b) = \frac{1}{2}(cos(a+b) + cos(a-b))$$

$$sin(a) \cdot sin(b) = \frac{1}{2}(cos(a-b) - cos(a+b))$$

$$arctan(a) + arctan\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

$$e^{ia} = cos(a) + i \cdot sin(a)$$

$$sinh(a) = \frac{1}{2} = \left(e^{a} - e^{-a}\right)$$

$$cosh(a) = \frac{1}{2} = \left(e + e^{-a}\right)$$

$$tanh(a) = \frac{sinh(a)}{cosh(a)}$$

$$cosh(a)^2 - sinh(a)^2 = 1$$