Mathematics 3: Integral Transformations

Summary

Mathematics 3

Vorarlberg University of Applied Sciences
Bachelors's in Mechatronics

Schwartze Nicolai

loosly based on the lecture by Dipl.-Ing. Dr. Peter Pichler (2017)

Dornbirn, 25.07.2021

Contents

1 Fourier Transformation

The Fourier Transformation is a method to decompose a continuous, aperiodic signal into a continuous spectrum. This integral transformation is defined by

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt = F(\omega) \tag{1.1}$$

$$\mathcal{F}^{-1}(f(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{i\omega t} dt = f(t) \tag{1.2}$$

In the general case, $\mathcal{F}(f(t)) = F(\omega)$ is a complex function with a real and an imaginary part:

$$\begin{aligned} F(\omega) = & F_1(\omega) + iF_2(\omega) \\ & where: \\ & F_1(\omega)...\Re(F(\omega)) \\ & F_2(\omega)...\Im(F(\omega)) \end{aligned} \tag{1.3}$$

The Fourier Transform $F(\omega)$ is Hermitian that the conjugate complex $\overline{F}(\omega)$ of a Fourier Transform is equal to the Fourier Transform at the negative frequency $F(-\omega)$.

$$\overline{F}(\omega) = F(-\omega) \tag{1.4}$$

The Fourier Transformation exhibits some very useful properties that can be exploited for calculations.

1.1 Linearity

The Fourier Transformation is a linear operation which means that

$$\mathcal{F}(a \cdot f(t) \pm b \cdot g(t)) = a \cdot \mathcal{F}(t) \pm b \cdot \mathcal{F}(g(t)) \tag{1.5}$$

1.2 Differentiation

If the original function $f(t)$ converges to 0: $f(t) \to 0$ for $t \to \pm\infty$, than the Fourier Transformation of the differentiation $\mathcal{F}(f'(t))$ can be expressd as

$$\mathcal{F}(f'(t)) = i\omega\mathcal{F}(f(t)) \text{ only if: } f(t) \to 0 \text{ for } t \to \pm\infty \tag{1.6}$$

1.3 Time Shifting

If a function $f(t)$ is shifted in the time domain about a constant $f(t-a)$ the Fourier Transformation of the shifted function can be calculated by

$$\mathcal{F}(f(t-a)) = e^{-i\omega a} \cdot \mathcal{F}(f(t)) \tag{1.7}$$

1.4 Convolution

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{\tau=-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau \tag{1.8}$$

The Fourier Transformation of the convolution of two functions can also be calculated by

$$\mathcal{F}(f(t) * g(t)) = \mathcal{F}(f(t)) \cdot \mathcal{F}(g(t)) \tag{1.9}$$

2 Fourier Series

The Fourier Series is a special serious expansion for periodic, piecewise continuous functions into a function series of sine and cosine.

In the case of the complex fourier series, the trigonometric functions are further decomposed into complex Euler exponential functions.

2.1 Real Fourier Series

The real Fourier Series can be expressed by 3 parameters which are called the Euler-Fourier Parameter a_0, a_n and b_n. Depending on the symmetry of the original function $f(t)$, the calculation process can be shortened. The base frequency for all components is denoted as $\omega_0 = \frac{2\pi}{T}$.

Even Symmetric Function	Odd Symmetric Functions	Arbitrary Function
$f(t) = f(-t)$	$f(t) = -f(-t)$	no symmetry
$\int_{-a}^{a} f(t)dt = 2\int_0^a f(t)dt$	$\int_{-a}^{a} f(t)dt = 0$	no symmetry
$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos\,(n\omega_0 t)$	$f(t) = \sum_{n=1}^{\infty} b_n \cdot sin(n\omega_0 t)$	$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos\,(n\omega_0 t) + b_n \cdot sin\,(n\omega_0 t)$
$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt$	$a_0 = 0$	$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$
$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) \cdot cos(n\omega_0 t)dt$	$a_n = 0$	$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos(n\omega_0 t)dt$
$b_n = 0$	$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) sin(n\omega_0 t)dt$	$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(n\omega_0 t)dt$

2.2 Complex Fourier Series

As *sine* and *cosine* can be expressed by complex Euler-Functions. These pointers can be added together where each pointer has its own amplitude c_n called Fourier Coefficient. Again the frequency is denoted as $\omega_0 = \frac{2\pi}{T}$.

$$\begin{aligned} f(t) &= f(t + n \cdot T)\ n \in \mathbb{Z} \\ \text{with } \omega_0 &= \frac{2\pi}{T} \\ f(t) &= \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\omega_0 t} \end{aligned} \tag{2.1}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-in\omega_0 t} \tag{2.2}$$

Even Symmetric Function	Odd Symmetric Functions	No Symmetry
$f(t) = f(-t)$	$f(t) = -f(-t)$	no symmetry
$c_n = c_{-n}$	$c_n = -c_{-n}$	c_n
only real part c_n	only imaginary c_n	fully complex c_n

3 Laplace Transformation

Function $f(t)$	Transformation $F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
$e^{\pm at}$	$\frac{1}{s \mp a}$
$t \cdot e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
$t^n \cdot e^{\pm at}$	$\frac{n!}{(s \mp a)^{n+1}}$
$u(t-a)$	$\frac{1}{s} e^{-as}$
$f(t-a) \cdot u(t-a)$	$\mathcal{L}(f(t)) \cdot e^{-as}$
$\delta(t-a)$	e^{-as}
$\sqrt{t}$	$\frac{1}{2s}\sqrt{\frac{\pi}{s}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
$\sqrt{t} \cdot e^{at}$	$\frac{\sqrt{\pi}}{2(s-a)\sqrt{s-a}}$
$\frac{1}{\sqrt{t}} \cdot e^{at}$	$\frac{\sqrt{\pi}}{\sqrt{s-a}}$
$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t \cdot sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t \cdot cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$t^n \cdot sin(\omega t),\ n \in \mathbb{N}$	$\frac{i \cdot n!}{2}\left(\frac{1}{(s+i\omega)^{n+1}} - \frac{1}{(s-i\omega)^{n+1}}\right)$
$t^n \cdot cos(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s+i\omega)^{n+1}} + \frac{1}{(s-i\omega)^{n+1}}\right)$
$sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$t \cdot sinh(\omega t)$	$\frac{2\omega s}{(s^2-\omega^2)^2}$
$t \cdot cosh(\omega t)$	$\frac{s^2+\omega^2}{(s^2-\omega^2)^2}$
$t^n \cdot sinh(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s-\omega)^{n+1}} - \frac{1}{(s+\omega)^{n+1}}\right)$
$t^n \cdot cosh(\omega t),\ n \in \mathbb{N}$	$\frac{n!}{2}\left(\frac{1}{(s-\omega)^{n+1}} + \frac{1}{(s+\omega)^{n+1}}\right)$
$e^{at} sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at} cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$
$e^{at} cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$
$sin(\omega t)^2$	$\frac{2\omega^2}{s(s^2+4\omega^2)}$
$cos(\omega t)^2$	$\frac{s^2+2\omega^2}{s(s^2+4\omega^2)}$
$sinh(\omega t)^2$	$\frac{2\omega^2}{s(s^2-4\omega^2)}$
$cosh(\omega t)^2$	$\frac{s^2-2\omega^2}{s(s^2-4\omega^2)}$

4 Z-Transformation

4.1 Collection of Common Z-Transformation

Series $f[n]$	Z-Transformation $F(z)$
1 or $u[n]$	$\frac{z}{z-1}$
$\delta[n]$	1
$\delta[n-1]$	$\frac{1}{z}$
$\delta[n-2]$	$\frac{1}{z^2}$
$\delta[n-k]$	$\frac{1}{z^k}$
a^n	$\frac{z}{z-a}$
e^n	$\frac{z}{z-e}$
$(-a)^n$	$\frac{z}{z+a}$
$sin[a \cdot n]$	$\frac{z \cdot sin(a)}{z^2-2z \cdot cos(a)+1}$
$cos[a \cdot n]$	$\frac{z^2-z \cdot cos(a)}{z^2-2z \cdot cos(a)+1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z \cdot (z+1)}{(z-1)^3}$
n^3	$\frac{z \cdot (z^2+4z+1)}{(z-1)^4}$
n^4	$\frac{z \cdot (z^3+11z^2+11z+1)}{(z-1)^5}$
$sinh[a \cdot n]$	$\frac{z \cdot sinh(a)}{z^2-2z \cdot cosh(a)+1}$
$cosh[a \cdot n]$	$\frac{z^2-z \cdot cosh(a)}{z^2-2z \cdot cosh(a)+1}$
$sin\left[\frac{\pi}{2} \cdot n\right]$	$\frac{z}{z^2+1}$
$cos\left[\frac{\pi}{2} \cdot n\right]$	$\frac{z^2}{z^2+1}$

5 Collection of Integrals

$$\int \frac{sin(x)^2}{cos(x)}dx = \int \frac{1-cos(x)^2}{cos(x)}dx = \int \frac{1}{cos(x)}dx - sin(x)$$

$$\int \frac{1}{cos(x)}dx = \int \frac{cos(x)}{cos(x)^2}dx = \int \frac{cos(x)}{1-sin(x)^2}dx = arctanh(sin(x))$$

$$\int \frac{1}{sin(x)}dx = \int \frac{sin(x)}{sin(x)^2}dx = \int \frac{sin(x)}{1-cos(x)^2}dx = -arctanh(cos(x))$$

$$\int tan(x)dx = -ln(cos(x))$$

$$\int cot(x)dx = ln(sin(x))$$

$$\int \frac{1 - x \cdot arctan(x)}{arctan(x) \cdot (1+x^2)} = \int \left(\frac{1}{arctan(x) \cdot (1+x^2)} - \frac{x \cdot arctan(x)}{arctan(x) \cdot (1+x^2)} \right) dx$$
$$\xrightarrow{z=arctan(x)} \int \left(\frac{1}{z} - tan(z) \right) dz = ln(z) + ln(cos(z))$$

$$\int \frac{1}{x^2 \cdot \sqrt{1-x^2}} dx \xrightarrow{x=sin(u)} \int \frac{cos(u)}{sin(u)^2 \cdot cos(u)} du = \int \frac{1}{sin(x)^2} du = -cot(u)$$

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int x \cdot \frac{x}{\sqrt{1-x^2}} dx = -x\sqrt{1-x^2} + \int \sqrt{1-x^2} dx$$

$$\int sin(x)^3 \cdot e^{-cos(x)} dx \xrightarrow{u=-cos(x)} \int sin(x)^3 \cdot \frac{e^u}{sin(x)} du = \int \left(1-u^2\right) \cdot e^u du$$

$$\int \frac{1}{\sqrt{\frac{a}{x}-1}} dx = \int \frac{1}{\sqrt{\frac{a-x}{x}}} dx = \int \frac{\sqrt{x}}{\sqrt{a-x}} \xrightarrow{x=a \cdot sin(u)^2} \int \frac{\sqrt{a} \cdot sin(u) \cdot 2 \cdot a \cdot sin(u) \cdot cos(u)}{\sqrt{a - a \cdot sin(u)^2}} du = 2 \cdot a \int sin(u)^2 du$$

$$\int \frac{1}{x \cdot \sqrt{a^2x^2-1}} dx \xrightarrow{ax=cosh(u)}$$
$$\int \frac{1}{a} \cdot \frac{sinh(u)}{\frac{cosh(u)}{a} \sqrt{cosh(u)^2-1}} du = \int \frac{1}{cosh(u)} du = \int \frac{cosh(u)}{1+sinh(u)^2} du = arctan(sinh(u))$$

$$\int 2 \cdot x \cdot e^{x^2} dx = e^{x^2}$$

$$\int \frac{1}{\sqrt{x^2 \cdot a + x^2 \cdot ln(x)^2}} dx \xrightarrow{u=ln(x)} \int \frac{1}{\sqrt{a+u^2}} du = \int \frac{1}{\sqrt{a} \cdot \sqrt{1+\frac{u^2}{a}}} du = arcsinh\left(\frac{u}{\sqrt{a}}\right)$$

$$\int \frac{sin(x)}{cos(x)^3} dx = \int tan(x) \cdot \frac{1}{cos(x)^2} dx = \frac{tan(x)^2}{2}$$

$$\int \left(\frac{1}{x} e^x - \frac{1}{x^2} e^x \right) dx = \frac{1}{x} e^x - \int -x \frac{1}{x^2} e^x dx - \int \frac{1}{x^2} e^x dx = \frac{1}{x} e^x$$

$$\int \frac{2}{4x + a + \sqrt{a^2+4ax}} dx \xrightarrow{u=\sqrt{4ax+a^2}} \int \frac{u}{a(4x+a+u)} du = \int \frac{u}{4ax + a^2 + au} du = \int \frac{u}{u^2+au} du = ln(u+a)$$

$$\int \frac{1-sin(x)}{1+sin(x)} dx = \int \frac{1-sin(x)}{1+sin(x)} \cdot \frac{1-sin(x)}{1-sin(x)} dx = \int \frac{1 - 2sin(x) + sin(x)^2}{1-sin(x)^2} dx = 2 \cdot tan(x) - \frac{2}{cos(x)} - x$$

$$\int \frac{1}{cos(x)^3}dx = \int \frac{1}{cos(x)}\frac{1}{cos(x)^2}dx = \frac{tan(x)}{cos(x)} - \int \frac{sin(x)\cdot tan(x)}{cos(x)^2}dx =$$
$$\frac{sin(x)}{cos(x)^2} - \int \frac{1}{cos(x)^3}dx + \int \frac{cos(x)}{cos(x)^2}dx = \frac{1}{2}\left(\frac{sin(x)}{cos(x)^2} + arctanh(sin(x))\right)$$

$$\int e^{ax}\cdot cos(bx)dx = \frac{a}{a^2+b^2}e^{ax}cos(bx) + \frac{b}{a^2+b^2}e^{ax}sin(bx)$$

$$\int e^{ax}\cdot sin(bx)dx = \frac{a}{a^2+b^2}e^{ax}sin(bx) - \frac{b}{a^2+b^2}e^{ax}cos(bx)$$

6 Collection of Sums

$$\sum_{k=0}^{n} 1 = (n+1)$$

$$\sum_{k=0}^{n} k = \frac{n\cdot(n+1)}{2}$$

$$\sum_{k=0}^{n} a+k = \frac{(n+1)\cdot(2a+n)}{2}$$

$$\sum_{k=1}^{n} 2k+1 = n^2$$

$$\sum_{k=0}^{n} 2k = n\cdot(n+1)$$

$$\sum_{k=0}^{n} k^2 = \frac{n\cdot(n+1)^2}{6}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2\cdot(n+1)^2}{4}$$

$$\sum_{k=1}^{n} (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

$$\sum_{k=1}^{n}(2n-1)^3 = n^2(2n^2-1)$$

$$\sum_{k=0}^{n}k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{k=0}^{n}a^k = \frac{a^{n+1}-1}{a-1}$$

$$\sum_{k=0}^{n}a^{-k} = \frac{a^{-n}(a^{n+1}-1)}{a-1}$$

---

7 Collection of Trigonometric Identities

$$\begin{gathered}
sin(a) = \frac{1}{2i}\left(e^{ia}-e^{-ia}\right) \\
sin(a+b) = sin(a)\cdot cos(b) + cos(a)\cdot sin(b) \\
sin(a-b) = sin(a)\cdot cos(b) - cos(a)\cdot sin(b) \\
sin(2a) = 2\cdot sin(a)\cdot cos(a) \\
sin(a)^2 = \frac{1}{2}\cdot(1-cos(2a)) \\
sin(a)+sin(b) = 2\cdot sin\left(\frac{a+b}{2}\right)\cdot cos\left(\frac{a-b}{2}\right) \\
sin(a)-sin(b) = 2\cdot cos\left(\frac{a+b}{2}\right)\cdot sin\left(\frac{a-b}{2}\right)
\end{gathered} \tag{7.1}$$

$$\begin{gathered}
cos(a) = \frac{1}{2}\left(e^{ia}+e^{-ia}\right) \\
cos(a+b) = cos(a)\cdot cos(b) - sin(a)\cdot sin(b) \\
cos(a-b) = cos(a)\cdot cos(b) + sin(a)\cdot sin(b) \\
cos(2a) = cos(a)^2 - sin(a)^2 \\
cos(a)^2 = \frac{1}{2}\cdot(1+cos(2a))
\end{gathered}$$

$$cos(a)^2 = \frac{1}{1 + tan(a)^2}$$

$$cos(a) + cos(b) = 2 \cdot cos\left(\frac{a+b}{2}\right) \cdot cos\left(\frac{a-b}{2}\right)$$

$$cos(a) - cos(b) = -2 \cdot sin\left(\frac{a+b}{2}\right) \cdot sin\left(\frac{a-b}{2}\right)$$

$$sin(a)^2 + cos(a)^2 = 1$$

$$sin(a) \cdot cos(b) = \frac{1}{2}(sin(a+b) + sin(a-b))$$

$$cos(a) \cdot cos(b) = \frac{1}{2}(cos(a+b) + cos(a-b))$$

$$sin(a) \cdot sin(b) = \frac{1}{2}(cos(a-b) - cos(a+b))$$

$$arctan(a) + arctan\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

$$e^{ia} = cos(a) + i \cdot sin(a)$$

$$sinh(a) = \frac{1}{2} = \left(e^a - e^{-a}\right)$$

$$cosh(a) = \frac{1}{2} = \left(e + e^{-a}\right)$$

$$tanh(a) = \frac{sinh(a)}{cosh(a)}$$

$$cosh(a)^2 - sinh(a)^2 = 1$$