Mathematics 3: Integral Transformations

Summary

Mathematics 3

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Bachelors's in Mechatronics

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Contents

1 Fourier Transformation

The Fourier Transformation is a method to decompose a continuous, aperiodic signal into a continuous spectrum. This integral transformation is defined by

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt = F(\omega) \tag{1.1}$$

$$\mathcal{F}^{-1}(f(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{i\omega t} dt = f(t) \tag{1.2}$$

In the general case, $\mathcal{F}(f(t)) = F(\omega)$ is a complex function with a real and an imaginary part:

$$\begin{aligned} F(\omega) = &F_1(\omega) + iF_2(\omega) \\ &where: \\ &F_1(\omega)...\Re(F(\omega)) \\ &F_2(\omega)...\Im(F(\omega)) \end{aligned} \tag{1.3}$$

The Fourier Transform $F(\omega)$ is Hermitian that the conjugate complex $\overline{F}(\omega)$ of a Fourier Transform is equal to the Fourier Transform at the negative frequency $F(-\omega)$.

$$\overline{F}(\omega) = F(-\omega) \tag{1.4}$$

The Fourier Transformation exhibits some very useful properties that can be exploited for calculations.

1.1 Linearity

The Fourier Transformation is a linear operation which means that

$$\mathcal{F}(a \cdot f(t) \pm b \cdot g(t)) = a \cdot \mathcal{F}(t) \pm b \cdot \mathcal{F}(g(t)) \tag{1.5}$$

1.2 Differentiation

If the original function $f(t)$ converges to 0: $f(t) \to 0$ for $t \to \pm\infty$, than the Fourier Transformation of the differentiation $\mathcal{F}(f'(t))$ can be expressd as

$$\mathcal{F}(f'(t)) = i\omega\mathcal{F}(f(t)) \text{ only if: } f(t) \to 0 \text{ for } t \to \pm\infty \tag{1.6}$$

1.3 Time Shifting

If a function $f(t)$ is shifted in the time domain about a constant $f(t-a)$ the Fourier Transformation of the shifted function can be calculated by

$$\mathcal{F}(f(t-a)) = e^{-i\omega a} \cdot \mathcal{F}(f(t)) \tag{1.7}$$

1.4 Convolution

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{\tau=-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau \tag{1.8}$$

The Fourier Transformation of the convolution of two functions can also be calculated by

$$\mathcal{F}(f(t) * g(t)) = \mathcal{F}(f(t)) \cdot \mathcal{F}(g(t)) \tag{1.9}$$

2 Fourier Series

The Fourier Series is a special serious expansion for periodic, piecewise continuous functions into a function series of sine and cosine.

3 Laplace Transformation

4 Z-Transformation