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Numerical Solution of 2D Flow using a Local Coordinate System

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Abstract This paper is concerned with a new numerical method of two-dimensional flow. The governing system of differential equations is transformed into an equivalent system applied over a square-grid network in order to overcome the difficulties and inaccuracies associated with the determination of characteristics near the flow boundaries. The MacCormack two-step explicit scheme with second-order accuracy is used for the solution of the transformed system of equations. Comparisons between computed and experimental data show a satisfactory agreement.

Key words: unsteady 2-D flow, MacCormack scheme, a local coordinate system, and isoperimetric element

INTRODUCTION

There are many effective methods in numerical solution of unsteady two-dimensional flow [1-5], such as Finite Difference Method (FDM), Finite Element Method (FEM), Boundary Element Method and Finite Analytic Method (FAM). The FDM has been widely used in computational fluid dynamics because of its good convergence and stability. But this method is difficult to deal with complicated boundary of computational regions. In order to avoid the disadvantage of the FDM, a new numerical method for solving two-dimensional flow is presented in this paper. The Isoparametric Element [3], which has been widely used in the FEM, is incorporated into the FDM. The flow region is subdivided into a number of small arbitrary quadrilateral elements and each element is transformed into a square element by the shape functions of Isoparametric Element. Then the water governing equations are solved by the FDM on the square element. Because this transformation is local, it is more flexible and convenient than the transformation on the whole domain. For solving the transformed water governing equations, the MacCormack two-step, predictor corrector scheme has been used. The present numerical model has been successfully used in numerical solution of a two dimensional flow.

NUMERICAL SOLUTION PROCEDURE

1. Cartesian governing equations The two-dimensional unsteady gradually varied flow equations in open channels are derived by applying the laws of conservation of mass and momentum and making the following assumptions: (1) The pressure distribution is hydrostatic; (2) The velocity distribution is uniform over the flow depth; (3) Bottom shear is large compared to other shear stresses; (4) The Channel bottom slope is small.

The resulting equations in Cartesian coordinates are [1, 2]:

$$W_t + E_x + F_y = D \tag{1}$$

in which

$$W = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}; \quad F = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix}; F = \begin{bmatrix} hu \\ huv \\ hv^2 + gh^2/2 \end{bmatrix}; \quad D = \begin{bmatrix} 0 \\ gh\left(S_{ax} - S_{fx}\right) \\ gh\left(S_{ay} - S_{fg}\right) \end{bmatrix}$$

where h is flow depth; u is depth-averaged velocity in the x-direction; v is depth-averaged velocity in the y-direction, g is the gravity acceleration; S_{0x} and S_{0y} are the channel bottom slope in the x- and y-direction defined as

$$S_{ax} = -\frac{5Z_b}{5x} \qquad S_{ay} = -\frac{5Z_b}{5y} \tag{2}$$

where Z_0 is the bottom elevation; and S_{fx} and S_{fx} are the friction slopes in the x- and y-directions, respectively, computed using the steady state friction formulas

$$S_{fx} = n^2 u\sqrt{u^2+v^2}\Big/h^{1.33} \qquad S_{fy} = n^2 v\sqrt{u^2+v^2}\Big/h^{1.33} \tag{3a, b}$$

in which n is the Manning's roughness coefficient.

2. Transformed governing equations The basic idea of the present numerical method is that the arbitrary quadrilaterals in the physical domain will be separately mapped into squares in the computational domain in dependent transformations from Cartesian x, y to the local coordinates ξ,η, as shown in Fig.1.

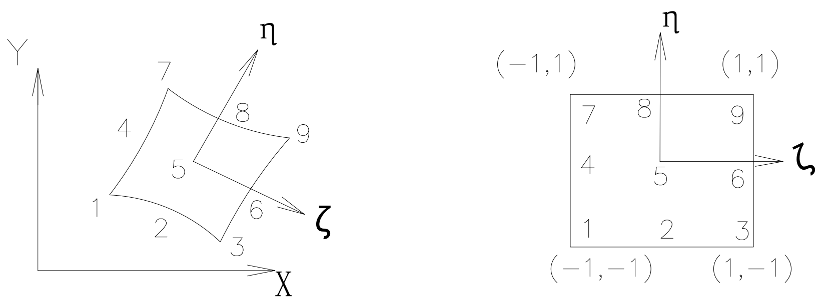


Figure 1: Transformations from Cartesian x, y to the local coordinates ξ,η

This transformation is obtained by the following shape functions

$$\phi_k = \left(\xi\xi_k + \zeta^2\right)\left(\eta\eta_k + \eta^2\right)\Big/4 \qquad k = 1,3,7,9$$

$$\phi_k = \left(\xi\xi_k + \xi^2\right)\left(1-\eta^2\right)\Big/2 \qquad k = 4,6$$

$$\phi_k = \left(1-\xi^2\right)\left(\eta\eta_k + \eta^2\right)\Big/2 \qquad k = 2,8$$

$$\phi_k = \left(1-\xi^2\right)\left(1-\eta^2\right) \qquad k = 5 \tag{4a, b, c, d}$$

where ξ_k and η_k are the coordinates of point k in the $\xi-\eta$ plane. The relations between x, y and ξ,η are

$$x = \sum_{k=1}^{9} x_k \Phi_k\left(\xi,\eta\right), \, y = \sum_{k=1}^{9} y_k \Phi_k\left(\xi,\eta\right) \tag{5a, b}$$

Where x_k and y_k are the Cartesian coordinates of the points of a cell, and x and y are the coordinates of any point of this cell, and Φ_k are the shape functions.

The derivatives of x to ξ and η, y to ξ and η, are

$$x_\xi = \sum_{k=1}^{9} x_k \frac{\partial \Phi_k}{\partial \xi}; \;\; x_\eta = \sum_{k=1}^{9} x_k \frac{\partial \Phi_k}{\partial \eta}; \;\; y_\xi = \sum_{k=1}^{9} y_k \frac{\partial \Phi_k}{\partial \xi}; \;\; y_\eta = \sum_{k=1}^{9} y_k \frac{\partial \Phi_k}{\partial \eta} \tag{6a, b, c, d}$$

The relation between x_ξ and η_y is

$$x_\xi = J\eta_y \tag{7}$$

Similarly

$$x_\eta = -J\xi_y; \;\; y_\xi = -J\eta_x; \;\; y_\eta = -J\xi_x \tag{8a, b, c}$$

If J represents the Jacobian of the transformation from the physical system to the computational local coordinate system, J is expressed as

$$J = x_\zeta y_\eta - x_\eta y_\zeta \tag{9}$$

According to the above-mentioned transformation, Eq.(1) in the local coordinate system ξ, η can be given by

$$W'_t + E'_\xi + F'_\eta = D' \tag{10}$$

in which

$$W' = J\begin{bmatrix} h \\ hu \\ hv \end{bmatrix};\ E' = J\begin{bmatrix} hU \\ hU_u + \zeta_x gh^2/2 \\ hU_v + \zeta_y gh^2/2 \end{bmatrix}; F' = \begin{bmatrix} hV \\ hV_u + \eta_x gh^2/2 \\ hV_v + \eta_y gh^2/2 \end{bmatrix}; D' = \begin{bmatrix} 0 \\ gh\left(S_{ax} - S_{fx}\right) \\ gh\left(S_{ay} - S_{fy}\right) \end{bmatrix}$$

and U, V are the velocity components in the ξ - and η -directions, respectively. They are related to u and v by

$$u = x_\xi U + x_\eta V$$

$$v = y_\xi U + y_\eta V \tag{11a, b}$$

3. Numerical discretization For the numerical solution of Eq. (10), the MacCormack scheme has been used. The MacCormack scheme was developed by MacCormack and has been widely used in computational fluid dynamics. This scheme consists of a two-step predictor corrector sequence. Flow variables are known at k time level and their values are to be determined at k+1 time level. Then for grid points i and j, the following finite difference equation may be written for Eq. (10).

Predictor Step

$$\overline{W}'_{i,j} = W'^k_{i,j} - \frac{\Delta t}{\Delta\xi}\nabla_\xi E'^k_{i,j} - \frac{\Delta t}{\Delta\eta}\nabla_\eta F'^k_{i,j} - \Delta t D'^k_{i,j} \tag{12}$$

Corrector Step

$$\widehat{W}'_{i,j} = \overline{W}'^k_{i,j} - \frac{\Delta t}{\Delta\xi}\Delta_\xi \overline{E}'^k_{i,j} - \frac{\Delta t}{\Delta\eta}\Delta_\eta \overline{F}'^k_{i,j} - \Delta t \overline{D}'^k_{i,j} \tag{13}$$

in which $\overline{W}'$ and $\widehat{W}$ are the intermediate values for W'.

The new values of W' are then obtained from

$$\widehat{W}'^{k+1}_{i,j} = \frac{1}{2}\left(\overline{W}'^k_{i,j} + \widehat{W}'_{i,j}\right) \tag{14}$$

The grid points are defined by subscripts i, j, and k. The scheme first uses forward space differences (∇_ξ and ∇_η) to predict an intermediate solution from known information at the k time level. Backward space differences (Δ_ξ and Δ_η) are then used in the second step to correct the predicted values. The backward and forward difference operators (∇ and Δ) are defined by $\nabla_\xi E'_{i,j} = E'_{i+1,j} - E'_{i,j}$ and $\Delta_\xi \overline{E}'_{i,j} = \overline{E}'_{i,j} - \overline{E}'_{i-1,j}$, where the subscript indicates the direction of differencing. The values of primitive variables are determined from the computed value of W' at each step as follows

$$h^{k+1} = h^{k+1} \tag{15}$$

$$u^{k+1} = \frac{(uh)^{k+1}}{h^{k+1}} \tag{16}$$

$$v^{k+1} = \frac{(vh)^{k+1}}{h^{k+1}} \tag{17}$$

where k+1 refers to an intermediate value obtained during a current predictor or corrector sequence.
In Eq. (2) and Eq. (3), U, V and $\overline{U}, \overline{V}$ are calculated by inversion of Eq. (11).

4. Boundary Condition The boundary conditions may be presented as follows:

On water section: the water depth and the velocity distribution are required;

On land boundary: the flux through the solid boundaries is zero.

5. Stability Condition The MacCormack scheme is stable if Courant-Friedrichs-Lewy (CFL) condition is satisfied. This condition for two-dimensional flow is expressed by

$$\Delta t < \min\left[\frac{\Delta x}{\sqrt{u^2+v^2}+C},\ \frac{\Delta y}{\sqrt{u^2+v^2}+C}\right] \tag{18}$$

Where c is the celerity ($c=\sqrt{gh}$). The actual time step used in the computation ($\Delta t'$) is calculated from

$$\Delta t' = C'\Delta t \tag{19}$$

where C' is a constant. Numerical experiments show that C' should be equal to or less than 0.6. For each time increment a new value for $\Delta t'$ is calculated.

RESULTS OF COMPUTATION

The numerical method has been used in the flow field of experimental open channel. The computed results are in good agreement with the experimental data.

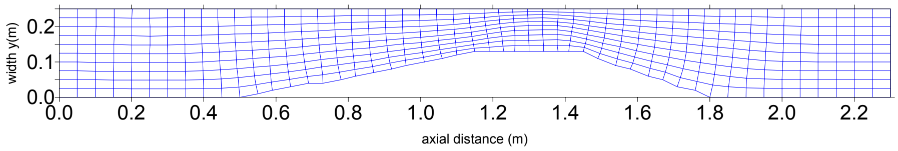


Figure 2: Orthogonal curvilinear for the physical plane

The distance between the water intake section and outflow section is about 2.3 meters; the width of the river is about 0.25 meters. The region concerned is divided into 605 elements. The physical plane is shown in Fig.2.
In the calculation, the size of space step Δs is 0.025 meters to 0.5 meters; the roughness n is about 0.023; the time step Δt is 0.2 minutes to 0.8 minutes; the viscosity coefficient ε is constant eddy viscosity; and the courant number is 2 to 5.

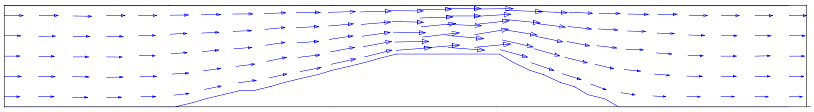
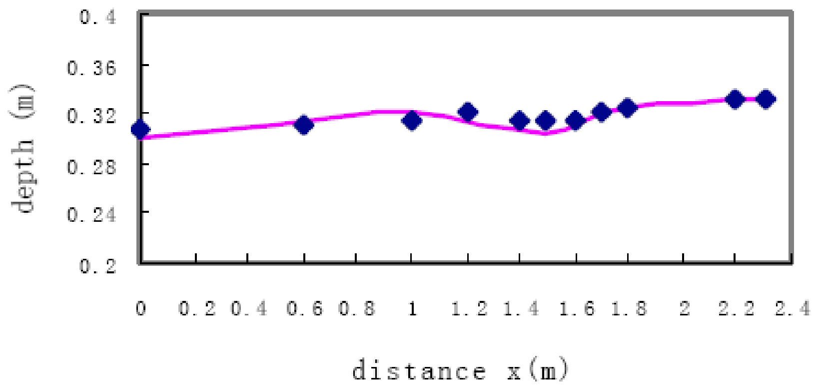


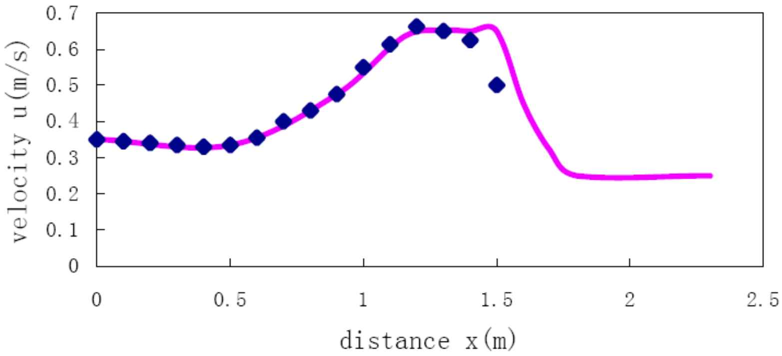
Figure 3: Computational velocity vectors

Fig. 3 shows the distribution of computational velocity vectors. Fig. 4 and Fig. 5 show the comparisons between computed results and experimental data. Fig.4 shows the comparisons of water depth between computed and experimental data on the fifth streamline from the convex riverbank. Fig. 5 shows the comparisons of velocity between computed and experimental data on the fifth streamline from the convex riverbank.



◆ measured, ———— computed

Figure 4: Comparisons of water depth between computed and experimental data on the fifth streamline from the convex riverbank



◆ measured, ———— computed

Figure 5: Comparisons of velocity between computed and experimental data on the fifth streamline from the convex riverbank

CONCLUSIONS

(1) The body-fitted non-orthogonal, local coordinate system can be easily and conveniently used to deal with the difficulties resulting from complicated shape of natural river boundaries in numerical solution procedure.

(2) The MacCormack two-step explicit scheme with second order accuracy can be employed for the solution of two-dimensional flow equations written in conservation form.

(3) The numerical method presented in this paper will be used in many aspects of practical engineering, such as environmental engineering, tidal river flow, dam-break-induced flow, spillway flow, and etc.

(4) The great advantages of this method here are based on the strong shock capturing ability of the MacCormack numerical scheme as well as on the ease of treatment of the exact solid boundary conditions.

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