Analysis of 2-D Lid-Driven Cavity Problem

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Abstract

The lid driven cavity problem is a very standard problem in domain of Computational Fluid Dynamics and is being used as a benchmark problem in CFD. The problem although appears to be simple , holds an ample opportunity for one to explore and has endless strings attached to it. The ultimate goal of this paper is to gain insight of some of the physics that is powering the modern CFD packages. In the current work, we present our analysis for both the steady and unsteady states. We have used the derived variable approach for carrying out the analysis. For the steady state solutions, uniform grid has been used while for the unsteady state we have inplemented the non-uniform grid in order to capture the dynamics of the problem more accurately where the velocity gradients are very high. Accurate results inline with results obtained by Ghia et al [5] have been obtained. This paper also presents the methodolgy for Von-Neumann stability analysis in great depth.

1 Introduction

This problem has been solved by large numbers of people in the past by using different methodologies and schemes starting from 1966 by Burggraf [6], yet it is an active area of research even in the present. Tamer et al [8] provides a summary of all the work that has been done from 1966 till 2014 in very concise tabular format in chronological order. The problem have been simulated upto Reynolds number, $Re = 15000$ by Bruneau et al [2] in their 1990s paper by using a finite difference approach. It can be observed that Finite Difference scheme has been heavily adopted in the past for carrying out the analysis. In present work we have used the Finite Difference scheme for the unsteady state analysis while upwind scheme [4] (both first and second order) for the steady state analysis.

In this work, we have analyzed the unsteady, viscous, incompressible, isothermal, two-dimensional, laminar flow of Newtonian fluid taking place in the cavity, driven by lid of infinite length (Figure 1). This problem is solved using an in-house Python code developed by the author (https://github.com/AakashSYadav/LidDrivenCavityProblem), which solves both unsteady and steady Navier-Stokes equations using derived variable apporach, using the Finite Difference Method (FDM).

The geometery of the problem is quite simple, it consists of a rectangular cavity ABCD in the $x - y$ plane with dimensions as shown in the Figure 1, where the walls AB, CD, and AD are rigid walls, whereas BC is open. The cavity is of length, D in the x-direction and height, H in the y direction. The aspect ratio is defined as $r = H/D$. The Reynolds number is defined based on the velocity scale, U and the length scale, D. Acceleration due to gravity acts in the negative z direction. The top of the cavity, BC, is covered with an infinitely-long rigid lid. Initially ($t \leq 0$), the fluid inside the cavity is at rest. At time $t > 0$, the lid is set in motion to the right with a constant velocity U.

In section 2, we bring equtions governing the fluid flow in the appropriate form, the section that follows provides insights to boundary and initial conditions that are exisitng for this problem, and which can be used in solving the problem numerically. In section 4 and 5 we non-dimensionalise the governing equations and carry out their discretization in the later. Section 6, is the most important part of this paper which gives insight of the algorithm(s) used in this work. In the final section we present the conclusions. All the algorithms implemented in this work can be found in the appendix.

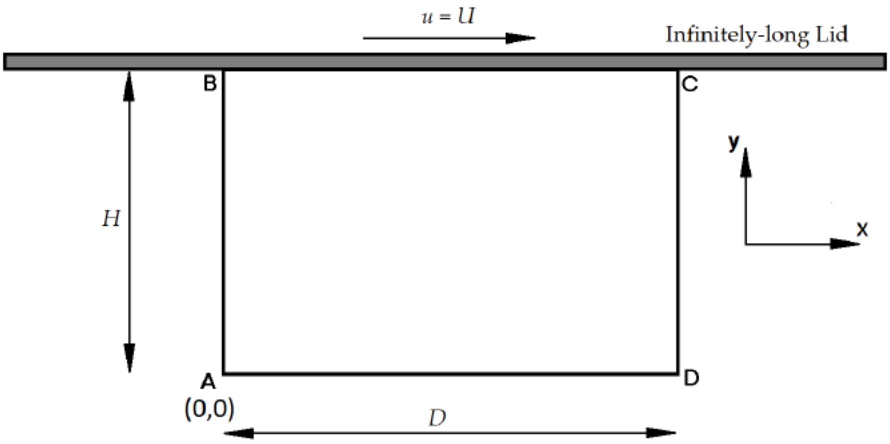

Figure 1: scaling of the WF law by $A\left(r, \mu^*\right)$

2 Governing Equations

The equation of continuity for 2-D incompressible isothermal flow [7, 1] is given by the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

where u and v are the components of veloity in x and y directions respectively. Furthermore we have the stream function, ψ defined as

$$u = \frac{d\psi}{dy} \tag{2a}$$

$$v = -\frac{d\psi}{dx} \tag{2b}$$

The vorticity equation of fluid dynamics describes evolution of the vorticity ω of fluid as it moves with its flow, i.e. the local rotation of the fluid. Mathematically vorticity is the curl of the flow velocity.

$$\vec{\omega} = \vec{\nabla} \times \vec{V} \tag{3a}$$

$$\vec{\omega} = (v_x - u_y)\hat{k} \tag{3b}$$

$$\vec{\omega} = -\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] = -\nabla^2\psi \tag{3c}$$

Equation 3c can be obtained by combining equations 2a, 2b and 3b.

Navier-Stokes equations which follows are perhaps the most important equations in the field of Fluid Dynamics. These equations are essentially the momentum conservation equations for isothermal, Newtonian and incopressible fluid. This can also be seen as application of Isaac Newton's second law to fluid motion.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{4a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + g_y + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{4b}$$

Equations 4a and 4b are for the fluid flow along the x and y direction respectively. One may observe that these equations also comprises of the pressure term P, which may not be required in some of the cases. We can eliminate the pressure term P in order to reduce the number of variable and simplify the anylysis by computing $\frac{\partial(\ 4a)}{\partial y} - \frac{\partial(\ 4b)}{\partial x}$, thus resulting in the below equation know as vorticity relation. This technique of approaching the problem is known as derived variable apporach.

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu\left[\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}\right] \tag{5}$$

For the sake of convienance we will drop the subscript from the ω_z hereafter.

3 Initial Conditions and Boundary Conditions

At time $t = 0$ everything is at rest and hence all the values are zero. At the moment $t \geq 0$ the lid will be moving at speed $u = U$, which will try to set the fluid in motion. We have the following boundary conditions for $t \geq 0$:
No slip condition will result in zero velocity along the wall tangent

$$u(x, 0) = 0 \tag{6a}$$

$$v(D, y) = 0 \tag{6b}$$

$$u(x, H) = U \tag{6c}$$

$$u(0, y) = 0 \tag{6d}$$

The no penetration condition that the walls of the cavity are impervious results into

$$v(x, 0) = 0 \tag{7a}$$

$$u(D, y) = 0 \tag{7b}$$

$$v(x, H) = 0 \tag{7c}$$

$$u(0, y) = 0 \tag{7d}$$

4 Non-dimensionalization of the Governing Equations

It is convenient numerically to make the equations for y and w dimensionless. This means we need to introduce the appropriate scalings for the dimensionless variables.

$$\hat{x} = \frac{x}{D}, \hat{y} = \frac{y}{H}, \hat{t} = \frac{U}{D}t, \hat{u} = \frac{u}{U}, \hat{v} = \frac{v}{V_{ref}} \tag{8}$$

Non-dimensionalising Equation 1 results into

$$\frac{U}{D}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{V_{ref}}{H}\frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{9a}$$

$$\frac{U}{D} \sim \frac{V_{ref}}{H} \tag{9b}$$

$$V_{ref} = \left(\frac{H}{D}\right) U = RU \tag{9c}$$

$$\hat{\psi} = \frac{\psi}{UD}, \hat{\omega} = \frac{\omega D}{U} \tag{10}$$

Non-dimensionalising the stream function equation (3c)

$$\frac{\hat{\omega} U}{D} = -\left[\frac{\partial^2(\hat{\psi} UD)}{\partial(\hat{x}D)^2} + \frac{\partial^2(\psi \hat{U} D)}{\partial(\hat{y}H)^2}\right] \tag{11a}$$

$$\hat{\omega} = -\left[\frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} + \frac{1}{r^2}\frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\right] \tag{11b}$$

Non-dimensionalising the vorticity function equation (5)

$$\frac{\partial\left(\frac{U}{D}\hat{\omega}_z\right)}{\partial\left(\frac{D}{U}\hat{t}\right)}+\hat{u}U\frac{\partial\left(\frac{U}{D}\hat{\omega}_z\right)}{\partial(\hat{x}D)}+\frac{UH}{D}\hat{v}\frac{\partial\left(\frac{U}{D}\hat{\omega}_z\right)}{\partial(\hat{y}H)}=\nu\left[\frac{\partial^2\left(\frac{U}{D}\hat{\omega}_z\right)}{\partial(\hat{x}D)^2}+\frac{\partial^2\left(\frac{U}{D}\hat{\omega}_z\right)}{\partial(\hat{y}H)^2}\right] \tag{12a}$$

$$\frac{\partial\hat{\omega}_z}{\partial\hat{t}}+\hat{u}\frac{\partial\hat{\omega}_z}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{\omega}_z}{\partial\hat{y}}=\frac{\nu}{UD}\left[\frac{\partial^2\hat{\omega}_z}{\partial\hat{x}^2}+\frac{1}{r^2}\frac{\partial^2\hat{\omega}_z}{\partial\hat{y}^2}\right] \tag{12b}$$

Also we have

$$u=\frac{\partial\psi}{\partial y}, v=-\frac{\partial\psi}{\partial x} \tag{13}$$

Non-dimensionalisation of the above equation results into

$$\hat{u}=\frac{1}{r}\frac{\partial\hat{\psi}}{\partial\hat{y}}, \hat{v}=-\frac{1}{r}\frac{\partial\hat{\psi}}{\partial\hat{x}} \tag{14}$$

Substituting equation (13) in equation (12b) yields

$$\frac{\partial\hat{\omega}_z}{\partial\hat{t}}+\frac{1}{r}\left[\frac{\partial\hat{\psi}}{\partial\hat{y}}\frac{\partial\hat{\omega}_z}{\partial\hat{x}}-\frac{\partial\hat{\psi}}{\partial\hat{x}}\frac{\partial\hat{\omega}_z}{\partial\hat{y}}\right]=\frac{1}{Re}\left[\frac{\partial^2\hat{\omega}_z}{\partial\hat{x}^2}+\frac{1}{r^2}\frac{\partial^2\hat{\omega}_z}{\partial\hat{y}^2}\right] \tag{15}$$

where, $Re=\frac{UD}{\nu}$
Deriving the boundary conditions in the form of vorticity

$$\hat{\omega}(\hat{x},0)=-\frac{1}{r^2}\left(\frac{\partial^2\hat{\psi}}{\partial\hat{y}^2}\right)_{\hat{y}=0} \tag{16a}$$

$$\hat{\omega}(\hat{x},1)=-\frac{1}{r^2}\left(\frac{\partial^2\hat{\psi}}{\partial\hat{y}^2}\right)_{\hat{y}=1} \tag{16b}$$

$$\hat{\omega}(0,\hat{y})=-\left(\frac{\partial^2\hat{\psi}}{\partial\hat{x}^2}\right)_{\hat{x}=0} \tag{16c}$$

$$\hat{\omega}(1,\hat{y})=-\left(\frac{\partial^2\hat{\psi}}{\partial\hat{x}^2}\right)_{\hat{x}=1} \tag{16d}$$

Deriving the boundary conditions in the form of vorticity. For the bottom surface $\hat{u}=\hat{v}=0$, substituiting this equation (14) yields

$$\hat{\psi}_{(\hat{y}=0)}=c_1 \tag{17}$$

Where c_1 is the integration constant. Similarly we can obtain the equations for the other walls as

$$\hat{\psi}_{(\hat{x}=0)}=c_2, \hat{\psi}_{(\hat{x}=1)}=c_3 \tag{18}$$

For the top wall

$$\hat{u}=U=\frac{1}{r}\frac{\partial\hat{\psi}}{\partial\hat{y}}\Rightarrow\hat{\psi}=rUH+c_4=c_5 \tag{19}$$

$$\hat{\psi}_{(\hat{y}=1)}=c_5 \tag{20}$$

We have $c_1=c_2=c_3=c_5=c$ because of the continuity at the corners as shown in the Figure 2. Without any loss of generality we can set $c=0$ as ψ is a relative term. Hence

$$\hat{\psi}_{(y=0)}=\hat{\psi}_{(y=1)}=\hat{\psi}_{(x=0)}=\hat{\psi}_{(x=1)}=0 \tag{21}$$

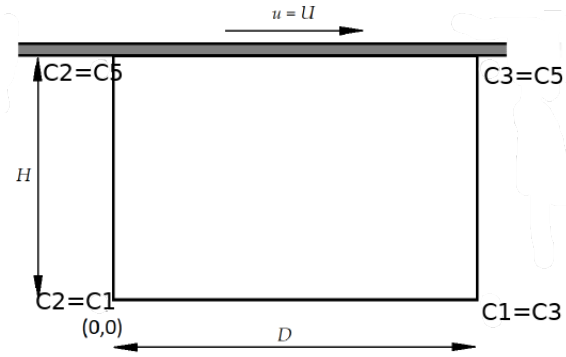

Figure 2: scaling of the WF law by $A(r, \mu^*)$

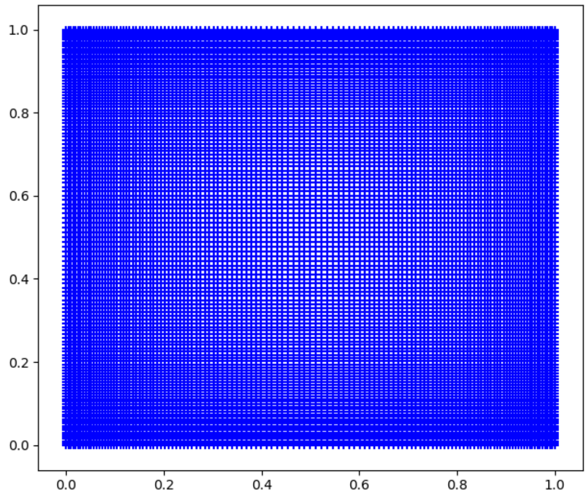

Figure 3: A 128 × 128 grid generated by the program

5 Discretization Schemes

In the analysis that follows we will drop the 'hat' notation for dimensionless variables for convienance. We will first use the usual FTCS scheme and a grid that is not equally spaced as shown in Figure 3. Derive the orders of accuracies of this scheme for both time and space. The values of Δx and Δy are not constant. You must use an appropriate stretching function so that the number of grid points vary in such a way that more grid points are concentrated closer to walls and the lid. Why is this necessary? Comment on this! Do a Von-Neumann stability analysis to find out the CFL criterion and the maximum time step to be used in your calculations. You may notice that $\Delta x, \Delta y, u, v$ are going to vary throughout the domain. So, the grid Courant number is also going to vary for every node, and every time step. So your code/program/algorithm must be adaptive; i.e, it must be able to choose the correct values of Δt so that the CFL criterion is satisfied at all times and at all locations in the domain.
Discretiztion of equation 11b using forward time and central differencing scheme. Second order accurate in space.

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2} + \frac{1}{r^2}\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = -\omega_{zij} \tag{22}$$

Von-Newmann stability analysis for the above equation -
Discretiztion of equation 15 using forward time and central differencing scheme. First order time accurate

and second order space accurate.

$$\frac{\omega_{i,j}^{n+1}-\omega_{i,j}^{n}}{\Delta t}+\frac{1}{r}\left[\left(\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta y}\right)\left(\frac{\omega_{i+1,j}-\omega_{i-1,j}}{2\Delta x}\right)-\left(\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta x}\right)\left(\frac{\omega_{i,j+1}-\omega_{i,j-1}}{2\Delta y}\right)\right]$$
$$=\frac{1}{Re}\left[\left(\frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{(\Delta x)^2}\right)+\frac{1}{r^2}\left(\frac{\omega_{i,j+1}-2\omega_{i,j}+\omega_{i,j-1}}{(\Delta y)^2}\right)\right] \quad (23)$$

Von-Neutmann stability analysis for the above equation -

6 Algorithm

No.	Steps
1	Initialize u and v values
2	Define the vorticity equation
3	Solve Poisson equation for stream function
4	Solve vorticity transport equation at a forward time step $t+\Delta t$
5	Solve the Poisson equation for stream function at $t+\Delta t$
6	Find u and v using the vorticity function equation
7	Check if the CFL criteria is satisified, if yes proced to next step
8	Else change the time step size and repeat the loop

The Fermi-Dirac integrals play very improtant role in the study of semiconductors appear frequently in semiconductor problems. It is thus a topic of special interest among physicist working in this field.

7 Courant–Friedrichs–Lewy analysis

In mathematics, the Courant–Friedrichs–Lewy (CFL) condition is a necessary condition for convergence while solving certain partial differential equations (usually hyperbolic PDEs) numerically. It arises in the numerical analysis of explicit time integration schemes, when these are used for the numerical solution. As a consequence, the time step must be less than a certain time in many explicit time-marching computer simulations, otherwise the simulation produces incorrect results. The condition is named after Richard Courant, Kurt Friedrichs, and Hans Lewy who described it in their 1928 paper [3]. We have the non dimentionalized strealine vorticity formulation equation as

$$\frac{\partial\hat{\omega}}{\partial\hat{t}}+\hat{u}\frac{\partial\hat{\omega}}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{\omega}}{\partial\hat{y}}=\frac{1}{Re}\left[\frac{\partial^2\hat{\omega}}{\partial\hat{x}^2}+\frac{1}{r^2}\frac{\partial^2\hat{\omega}}{\partial\hat{y}^2}\right] \quad (24)$$

In the analysis that follows we have dropped the hat symbol i.e. ˆ for non-dimensionalised quantities. We shall now use the Fourier wave form to bring out the required conditions. The amplification is easily identified using the Fourier wave form, moreover the waveform remains preserved.

$$\omega=\sum_m A^n(t)e^{\hat{i}(k_m x+k_n y)} \quad (25)$$

Here we can assume the wavenumber of the wave in both x and y direction to be same i.e. $k_n=k_m$

$$\omega=A^n e^{\hat{i}(ik_m\Delta x+jk_n\Delta y)} \quad (26)$$

$$\omega_{i,j}^{n}=A^n e^{\hat{i}(ik_m\Delta x+jk_n\Delta y)} \quad (27a)$$

$$\omega_{i,j}^{n+1}=A^{n+1} e^{\hat{i}(ik_m\Delta x+jk_n\Delta y)} \quad (27b)$$

$$\omega^n_{i+1,j} = A^n e^{\hat{i}((i+1)k_m\Delta x + jk_n\Delta y)} \tag{27c}$$

$$\omega^n_{i-1,j} = A^n e^{\hat{i}((i-1)k_m\Delta x + jk_n\Delta y)} \tag{27d}$$

$$\omega^n_{i,j+1} = A^n e^{\hat{i}(ik_m\Delta x + (j+1)k_n\Delta y)} \tag{27e}$$

$$\omega^n_{i,j-1} = A^n e^{\hat{i}(ik_m\Delta x + (j-1)k_n\Delta y)} \tag{27f}$$

Substituting this in our main equation after its discretization and simplification results in

$$\begin{aligned}\frac{\frac{A^{n+1}}{A^n} - 1}{\Delta t} + u\left(\frac{e^{ik_m\Delta x} - e^{-ik_m\Delta x}}{2\Delta x}\right) + v\left(\frac{e^{ik_n\Delta y} - e^{-ik_n\Delta y}}{2\Delta y}\right) \\ = \frac{1}{Re}\left[\left(\frac{e^{ik_m\Delta x} - 2 + e^{-ik_m\Delta x}}{\Delta x^2}\right) + \frac{1}{r^2}\left(\frac{e^{ik_n\Delta y} - 2 + e^{-ik_n\Delta y}}{\Delta y^2}\right)\right]\end{aligned}$$

Let $\theta_1 = k_m\Delta x$ and $\theta_2 = k_n\Delta y$, in order to simplyfy the analysis we can choose the k_i's such that $\theta_1 = \theta_2 = \theta$. We can use the below identity on the obtained equation.

$$e^{i\theta} = cos\theta + isin\theta \tag{28}$$

Hence we can obtain the amplification factor G as

$$G = \frac{A^{n+1}}{A^n} = 1 - \frac{1}{Re}\left(\frac{4\Delta t}{\Delta x^2} + \frac{4\Delta t}{\Delta y^2}\right)\left(\frac{1 - cos\theta}{2}\right) - i\left(\frac{u\Delta t}{\Delta x} + \frac{v\Delta t}{\Delta y}\right)sin\theta \tag{29}$$

This equation can be re-written in terms of α and β as

$$G = 1 - \alpha\left(\frac{1 - cos\theta}{2}\right) + i\beta sin\theta \tag{30}$$

$$|G|^2 = 1 + \frac{\alpha^2}{4}(1-q)^2 + \beta^2(1-q^2) - \alpha(1-q) \tag{31}$$

where,

$$q = cos\theta$$

$$\alpha = \frac{1}{Re}\left[\frac{4\Delta t}{\Delta x^2} + \frac{4\Delta t}{\Delta y^2}\right]$$

$$\beta = \left[\frac{u\Delta t}{\Delta x} + \frac{v\Delta t}{\Delta y}\right]$$

For the scheme to be stable, amplification factor $|G| \leq 1$ or $|G|^2 \leq 1$. We define a polinomial in terms of q as

$$p(q) = \frac{\alpha^2}{4}(1-q)^2 + \beta^2(1-q^2) - \alpha(1-q) \leq 0 \tag{32}$$

$$p(q) = \frac{\alpha^2}{4}(1-q)^2 + \beta^2(1-q^2) - \alpha(1-q) \leq 0 \tag{33}$$

As $q = 1$ is a root of $p(q)$, we can have four types of possible parabolas as shown in Figure 4. But as $p(q) \leq 0$ the parabolas shown with dotted red line can be rejected. Hence leaving the parabolas in blue, having positive slope i.e. $p^{'}(q) > 0$ at $q = 1$. Thus at $q = 1$ we have $p^{'}(1) > 0$ or $p^{'}(1) = \alpha - 2\beta^2 > 0$ which results in the condition

$$\alpha > 2\beta^2 \tag{34}$$

Moreover, we also have the condition that $p(-1) \leq 0$ which results in $\alpha \leq 2$. Hence we have $2\beta^2 < \alpha \leq 2$. Finally, we obtain

$$\Delta t \leq \frac{Re}{2}\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2 r^2}\right]^{-1} \tag{35}$$

$$\Delta t \leq \left[\frac{u}{\Delta x} + \frac{v}{\Delta y}\right]^{-1} \tag{36}$$

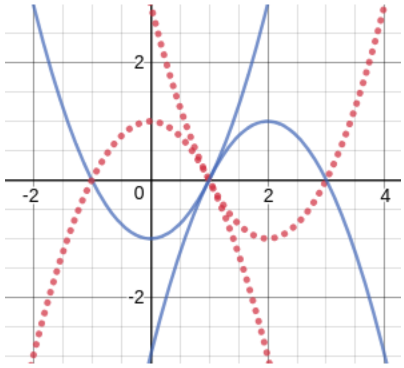

Figure 4: Possible parabolas passing through $q = 1$

8 Results

The Fermi-Dirac integrals play very improtant role in the study of semiconductors appear frequently in semiconductor problems. It is thus a topic of special interest among physicist working in this field.

8.1 Grid Independence Studies

The formulation for obtaining the minimum lattice thermal conductivity is given by approach developed by Cahill [**?**]. To minimise the Phonon thermal conductivity, the integrand has been extremised and simplified yielding solution in the form of offset log function. The plot for the integrand essentially Planck's blackbody radiation function has also been shown in Figure 2.

8.2 Streamlines for Varying R and Re

The efficiency of the thermoelectric material is directly proportional to the electrical conductivity of the material. Maximizing the same is of the utmost importance inorder to increase its efficiency. We have the following expression for the electrical . [**?**]

8.3 Profiles for Steady Flow

Exact Fermi Dirac Integral expressions can be very helpful in generalizing the Wiedemann Franz Law.

9 Discussion

Exact Fermi Dirac Integral expressions can be very helpful in generalizing the Wiedemann Franz Law. Exact analytic expressions of the same will greatly assist and equip the researchers in the new material design processes. Electronic thermal conductivity, κ_e and minimum lattice thermal conductivity $\kappa_{l,min}$ have exact analytic expressions and we have obtained very interesting forms of solutions while maximising the two. More recent observations on the influence of anharmonicity on $\kappa_{l,min}$ suggest that the Polylogarithms and Lambert W can have more interesting applications.

Appendix

The below code can be used for plotting the scaling factor A as shown in Figure 1.

```
"""
Reference(s):
http://caefn.com/cfd/hyperbolic-tangent-stretching-grid
```

```
"""
from sympy import *
import matplotlib.pyplot as plt
import numpy as np
from array import *
from scipy.sparse import *

# TODO: fix the grid function for odd value of grid elements
def grid(nx,ny, gama):
    # use ty tx to change number of elements
    tx=(2)/((nx+1)+1)
    ty=(2)/((ny+1)+1)
    x=[]
    y=[]
    nx=[]
    ny=[]

    # x elements on left half
    for i in np.arange(0., 1., tx):
        nx.append(i)
    for j in nx:
        x.append(1-(np.tanh(gama*(1-(2*j)/len(nx))))/(np.tanh(gama)))

    # y elements on right half
    for i in np.arange(0., 1., ty):
        ny.append(i)
    for i in ny:
        y.append(1-(np.tanh(gama*(1-(2*i)/len(ny))))/(np.tanh(gama)))

    # mirroring x and y elements for the right half
    for i in range(len(nx)-1):
        x.append(x[len(nx)+i-1]+x[len(nx)-(i+1)]-x[len(nx)-(i+2)])
    for i in range(len(ny)-1):
        y.append(y[len(ny)+i-1]+y[len(ny)-(i+1)]-y[len(ny)-(i+2)])

    xd=[]
    yh=[]
    for i in x:
        xd.append(i/(x[len(x)-1]))
    for i in y:
        yh.append(i/(y[len(y)-1]))
    return(xd,yh)

def gridPlot(c,d):
    for i in c:
        for k in d:
            plt.scatter(i,k,color='black',marker='+')
    plt.show()

def spacing(test_list):
    res = [test_list[i + 1] - test_list[i] for i in range(len(test_list)-1)]
    return(res)

def uGrid(nx,ny):
    l1=[]
    l2=[]
    for i in range(nx+1):
        l1.append(1/(nx+1))
```

```
    for i in range(ny+1):
        l2.append(1/(ny+1))
    return(l1,l2)

## print(uGrid(10,10))
# print(grid(10,10,3))
## NOTE: call from any program using
## from gridGen import grid,gridPlot
# x,y=grid(10,10,3)   # accepts (nx, ny, game)
# gridPlot(grid(10,10,3)[0],grid(10,10,3)[1])
```

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