Analysis of 2-D Lid-Driven Cavity Problem

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1 Introduction

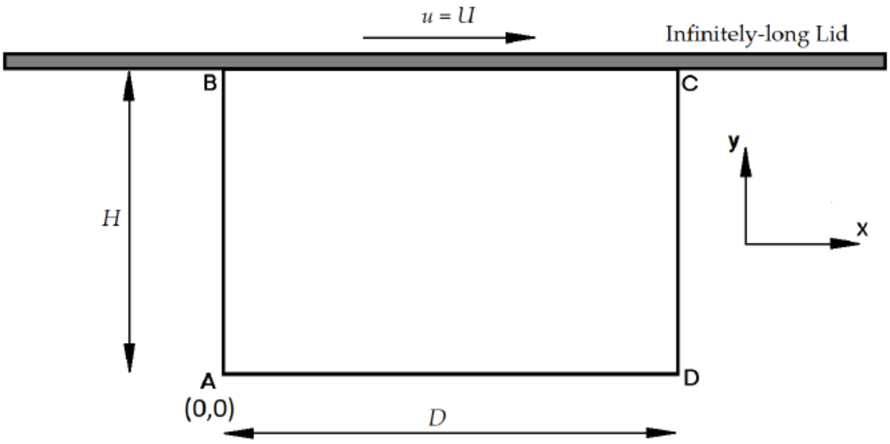

Figure 1: scaling of the WF law by $A(r, \mu^*)$

In this project, we will analyze the unsteady, viscous, incompressible,isothermal, two-dimensional, laminar flow of Newtonian fluid in a cavity covered with a lid. Consider a rectangular cavity ABCD of dimensions shown in Figure 1. AB, CD, and AD are rigid walls, whereas BC is open. The cavity is of length D and width H , in the x-y plane, as shown in Figure 1.1. The aspect ratio is defined as $R = H/D$. The Reynolds number is defined based on the velocity scale U and the length scale D. Acceleration due to gravity acts in the negative-z direction. The top of the cavity, BC, is covered with an infinitely-long rigid lid. Initially ($t \leq 0$), the fluid inside the cavity is at rest. At time $t > 0$, the lid is set in motion to the right with a constant velocity U. You are to analyze the fluid flow inside the cavity for various conditions using proper governing equations, boundary conditions, initial conditions, and numerical schemes.

2 Governing Equations

The Wiedemann Franz Law (1853) states that the ratio of the electronic contribution of the thermal conductivity κ to the electrical conductivity σ of a metal is proportional to the temperature T . [2]

Continuity equation for 2-D incompressible isothermal flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Stream function

$$\omega = -\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = -\nabla^2 \psi \tag{2}$$

Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + g_y + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{4}$$

Vorticity relation

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu\left[\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}\right] \tag{5}$$

Where, k is the Boltzmann constant, e is the electronic charge and the constant L_0 is known as the Lorenz number. For over 150 years, the Wiedemann Franz law has proven to be stable amongst the multitude of metallic systems that have been studied. But recent experiments over a couple of decades show that there are several limitations to the law, the value of Lorenz number L is not same for every materials and the law does not hold for intermediate temperatures. Experiments have shown that the value of Lorenz number, L, while roughly constant, is not exactly the same for all materials. [1, 4]

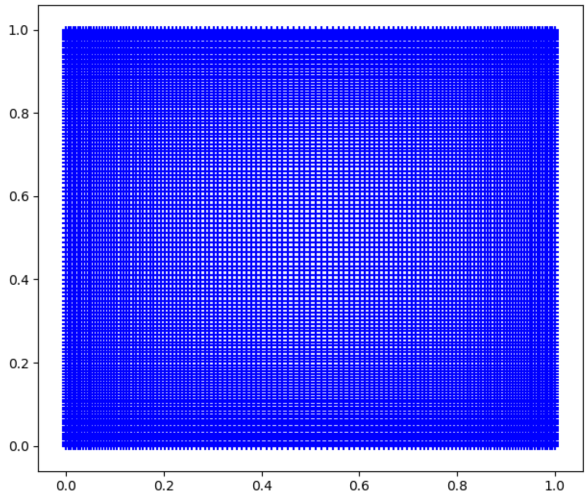

Figure 2: scaling of the WF law by $A\left(r, \mu^*\right)$

3 Initial Conditions and Boundary Conditions

At time $t = 0$ everything is at rest and hence all the values are zero. At the moment $t \geq 0$ the lid will be moving at speed $u = U$, which will try to set the fluidin motion. We have the following boundary conditions for $t \geq 0$:

No slip condition will result in zero velocity along the wall tangent

$$u(x, 0) = 0 \tag{6a}$$

$$v(D, y) = 0 \tag{6b}$$

$$u(x, H) = U \tag{6c}$$

$$u(0, y) = 0 \tag{6d}$$

The no penetration condition that the walls of the cavity are impervious results into

$$v(x, 0) = 0 \tag{7a}$$

$$u(D, y) = 0 \tag{7b}$$

$$v(x, H) = 0 \tag{7c}$$

$$u(0, y) = 0 \tag{7d}$$

4 Non-dimensionalization of the Governing Equations

It is convenient numerically to make the equations for y and w dimensionless. This means we need to introduce the appropriate scalings for the dimensionless variables.

$$\hat{x} = \frac{x}{D}, \hat{y} = \frac{y}{H}, \hat{t} = \frac{U}{D}t, \hat{u} = \frac{u}{U}, \hat{v} = \frac{v}{V_{ref}} \tag{8}$$

Non-dimensionalising Equation 1 results into

$$\frac{U}{D}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{V_{ref}}{H}\frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{9a}$$

$$\frac{U}{D} \sim \frac{V_{ref}}{H} \tag{9b}$$

$$V_{ref} = \left(\frac{H}{D}\right) U = RU \tag{9c}$$

$$\hat{\psi} = \frac{\psi}{UD}, \hat{\omega} = \frac{\omega D}{U} \tag{10}$$

Non-dimensionalising the stream function equation (2)

$$\frac{\hat{\omega} U}{D} = -\left[\frac{\partial^2(\hat{\psi} UD)}{\partial(\hat{x}D)^2} + \frac{\partial^2(\psi \hat{U} D)}{\partial(\hat{y}H)^2}\right] \tag{11a}$$

$$\hat{\omega} = -\left[\frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} + \frac{1}{r^2}\frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\right] \tag{11b}$$

Non-dimensionalising the vorticity function equation (**??**)

$$\frac{\partial\left(\frac{U}{D}\hat{\omega_z}\right)}{\partial\left(\frac{D}{U}\hat{t}\right)} + \hat{u}U\frac{\partial\left(\frac{U}{D}\hat{\omega_z}\right)}{\partial(\hat{x}D)} + \frac{UH}{D}\hat{v}\frac{\partial\left(\frac{U}{D}\hat{\omega_z}\right)}{\partial(\hat{y}H)} = \nu\left[\frac{\partial^2\left(\frac{U}{D}\hat{\omega_z}\right)}{\partial(\hat{x}D)^2} + \frac{\partial^2\left(\frac{U}{D}\hat{\omega_z}\right)}{\partial(\hat{y}H)^2}\right] \tag{12a}$$

$$\frac{\partial \hat{\omega_z}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{\omega_z}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{\omega_z}}{\partial \hat{y}} = \frac{\nu}{UD}\left[\frac{\partial^2 \hat{\omega_z}}{\partial \hat{x}^2} + \frac{1}{r^2}\frac{\partial^2 \hat{\omega_z}}{\partial \hat{y}^2}\right] \tag{12b}$$

where, $Re = \frac{UD}{\nu}$

5 Discretization Schemes

In first part of this subsection we focus on extremizing electronic thermal conductivity with respect to the chemical potential, μ^*. The analysis of equation 15 below has been done in the work of Murali et al (CJP 2011). [5]

6 Algorithm

The Fermi-Dirac integrals play very improtant role in the study of semiconductors appear frequently in semiconductor problems. It is thus a topic of special interest among physicist working in this field.

7 Results

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7.1 Grid Independence Studies

The formulation for obtaining the minimum lattice thermal conductivity is given by approach developed by Cahill [3]. To minimise the Phonon thermal conductivity, the integrand has been extremised and simplified yielding solution in the form of offset log function. The plot for the integrand essentially Planck's blackbody radiation function has also been shown in Figure 2.

7.2 Streamlines for Varying R and Re

The efficiency of the thermoelectric material is directly proportional to the electrical conductivity of the material. Maximizing the same is of the utmost importance inorder to increase its efficiency. We have the following expression for the electrical . [5]

7.3 Profiles for Steady Flow

Exact Fermi Dirac Integral expressions can be very helpful in generalizing the Wiedemann Franz Law.

8 Discussion

Exact Fermi Dirac Integral expressions can be very helpful in generalizing the Wiedemann Franz Law. Exact analytic expressions of the same will greatly assist and equip the researchers in the new material design processes. Electronic thermal conductivity, κ_e and minimum lattice thermal conductivity $\kappa_{l,min}$ have exact analytic expressions and we have obtained very interesting forms of solutions while maximising the two. More recent observations on the influence of anharmonicity on $\kappa_{l,min}$ suggest that the Polylogarithms and Lambert W can have more interesting applications.

Appendix

The below code can be used for plotting the scaling factor A as shown in Figure 1.

```
"""
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Reference(s):
http://caefn.com/cfd/hyperbolic-tangent-stretching-grid
"""

from sympy import *
import matplotlib.pyplot as plt
import numpy as np
from array import *
from scipy.sparse import *

def grid(nx,ny, gama):
    # use ty tx to change number of elements
```

```
    tx=(2)/((nx+1)+1)
    ty=(2)/((ny+1)+1)
    x=[]
    y=[]
    nx=[]
    ny=[]

    # x elements on left half
    for i in np.arange(0., 1., tx):
        nx.append(i)
    for j in nx:
        x.append(1-(np.tanh(gama*(1-(2*j)/len(nx))))/(np.tanh(gama)))

    # y elements on right half
    for i in np.arange(0., 1., ty):
        ny.append(i)
    for i in ny:
        y.append(1-(np.tanh(gama*(1-(2*i)/len(ny))))/(np.tanh(gama)))

    # mirroring x and y elements for the right half
    for i in range(len(nx)-1):
        x.append(x[len(nx)+i-1]+x[len(nx)-(i+1)]-x[len(nx)-(i+2)])
    for i in range(len(ny)-1):
        y.append(y[len(ny)+i-1]+y[len(ny)-(i+1)]-y[len(ny)-(i+2)])

    xd=[]
    yh=[]
    for i in x:
        xd.append(i/(x[len(x)-1]))
    for i in y:
        yh.append(i/(y[len(y)-1]))
    return(xd,yh)

# number of grid points in x direction
nx=128
# number of grid points un y direction
ny=128
# use gama to change the gaussian distibution
# as gama --> 0 grid becomes uniform
gama=15
c,d=grid(nx,ny,gama)
print(c)
print("\n",d)

for i in c:
    for k in d:
        plt.scatter(i,k,color='b',marker='+')
plt.show()
```

References

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