Assignment_1_ME5102

January 27, 2019

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1 Method of weighted residuals

Given the differential equation

$$\frac{d}{dt}\left[x\frac{du}{dx}\right] = \frac{2}{x^2}$$

Boundary conditions

$$u(1) = 2$$

and

$$-x\frac{du}{dx}\bigg|_{x=2} = \frac{1}{2}$$

Assuming the trial function as

$$u_h(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

Considering the first four terms only

$$u_h^{\sim}(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

Using boundary condition 1, we have

$$a_0 + a_1 + a_2 + a_3 = 2$$

Using boundary condition 2, we have

$$a_1 + 4a_2 + 12a_3 = -\frac{1}{4}$$

Substituting the above to results in the trial function

$$u_h^{\sim}(x) = 2 - \frac{1}{4}(x-1) + a_2(x-1)(x-3) + a_3(x-1)(x^2+x-11)$$

In order to get the residual, R we substitute our trial function in the given differential equation

$$R = \frac{d}{dt}\left[x\frac{du}{dx}\right] - \frac{2}{x^2}$$

$$\frac{du}{dx} = \frac{12a_3x^2 + 8a_2x - 48a_3 - 16a_2 - 1}{4}$$

$$R = \frac{36a_3x^2 + 16a_2x - 48a_3 - 16a_2 - 1}{4} - \frac{2}{x^2}$$

1.1 Exact Solution

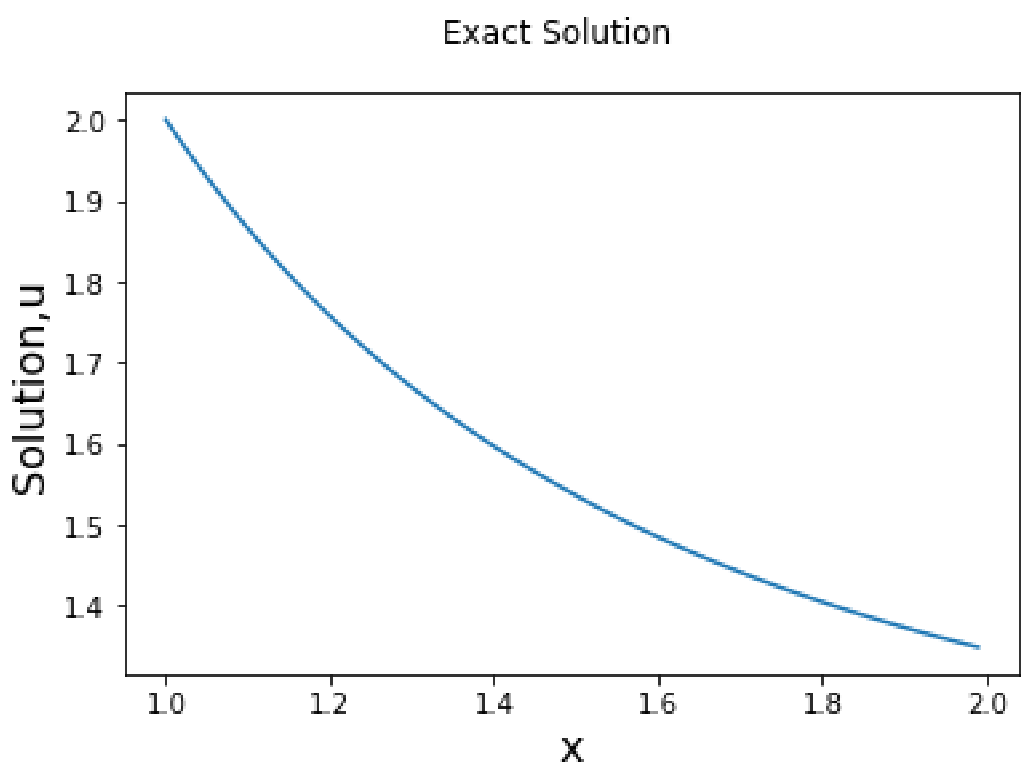
We obtain the following by integrating twice the given differential equation

$$u(x) = \frac{2}{x} + c_1 lnx + c_2$$

Finding values of the integration constants and putting them back

$$u = \frac{2}{x} + \frac{lnx}{2}$$

In [31]:
```
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(1, 2, 0.01)
#  expression for exact solution
u_exact = (2/x)+0.5*np.log(x)

plt.plot(x, u_exact)
plt.xlabel('x', fontsize=16)
plt.ylabel('Solution,u', fontsize=16)
plt.suptitle('Exact Solution')
plt.show()
```



1.2 Collocation Method

In the collocation method, we force the residual to be zero at n points $x_1, x_2, \ldots, x_n$ within the domain.That is

$$R(x_1, a) = 0$$

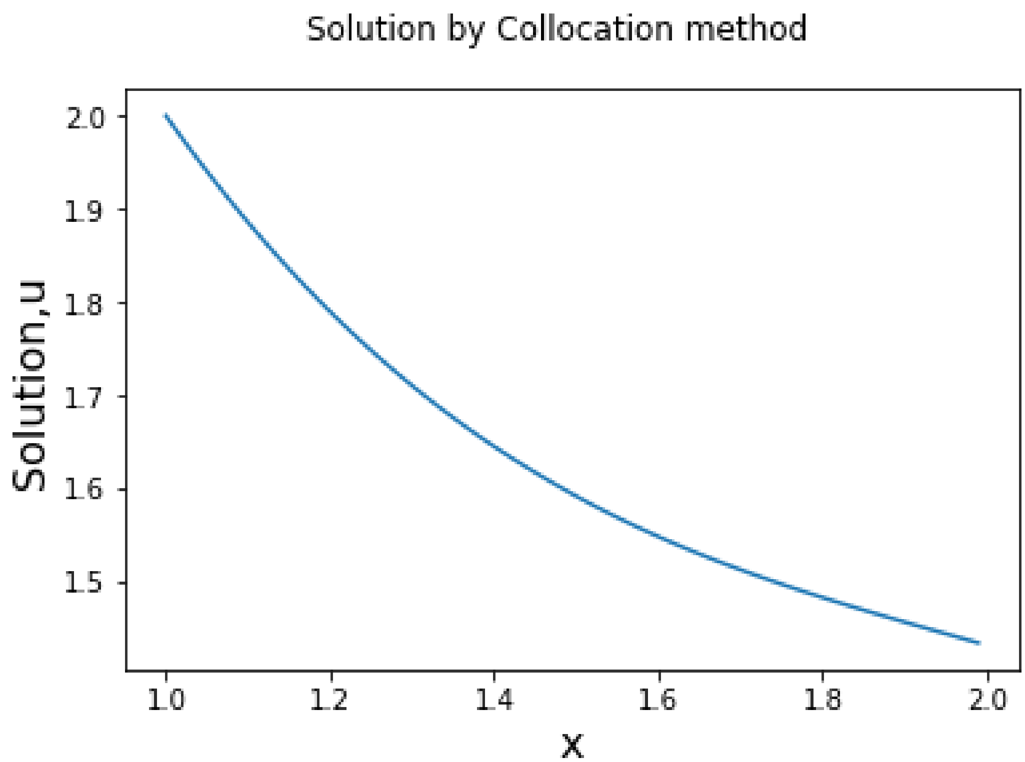
$$R(x_2, a) = 0$$

$$a = [a_1, a_2]$$

From the above equation we will get the value of a, and solution can be obtained by substituting it back in our trial function. For our problem, the domain of interest is $1 \leq x \leq 2$. Let us pick two points in this domain x_1 and x_2 such that $1 \leq x_1 < x_2 \leq 2$. In this example we choose $x_1 = 4/3$ and $x_2 = 5/3$. ,

```
In [33]: import matplotlib.pyplot as plt
         import numpy as np
         from sympy import *

         a_col = Symbol('a')
         b_col = Symbol('b')
         val_col = [4/3 ,5/3]
         eqn_col =[]

         def function(x):
                 for x in val_col:
                         # calculating residuals
                         eqn_col.append(-0.25+4*(x-1)*a_col+3*(3*(x**2)-4)*b_col-(2/(x**2)))
                 return(eqn_col)

         solve(function(val_col), [a_col,b_col])
         print("eqn",eqn_col)

         z_col = solve(function(val_col), [a_col,b_col])
         print("a ",z.get(a_col))
         print("b ",z.get(b_col))

         x = np.arange(1, 2, 0.01)
         u_col = 2. - (0.25)*(x-1)+(x-1)*(x-3)*z_col.get(a_col)+(x-1)*(x**2+x-11)*z_col.get(b_
         plt.plot(x, u_col)
         plt.xlabel('x', fontsize=16)
         plt.ylabel('Solution,u', fontsize=16)
         plt.suptitle('Solution by Collocation method')
         plt.show()
```

```
eqn [1.33333333333333*a + 4.0*b - 1.375, 2.66666666666667*a + 13.0*b - 0.97]
a  2.09925000000000
b  -0.356000000000000
```



1.3 Subdomain Method

The subdomain method is another way of forcing the residuals to zero. In this method, we let the "average" of the residual vanish over each domain.

$$\frac{1}{\Delta x_i}\int_{\Delta x_i} R(x)dx = 0$$

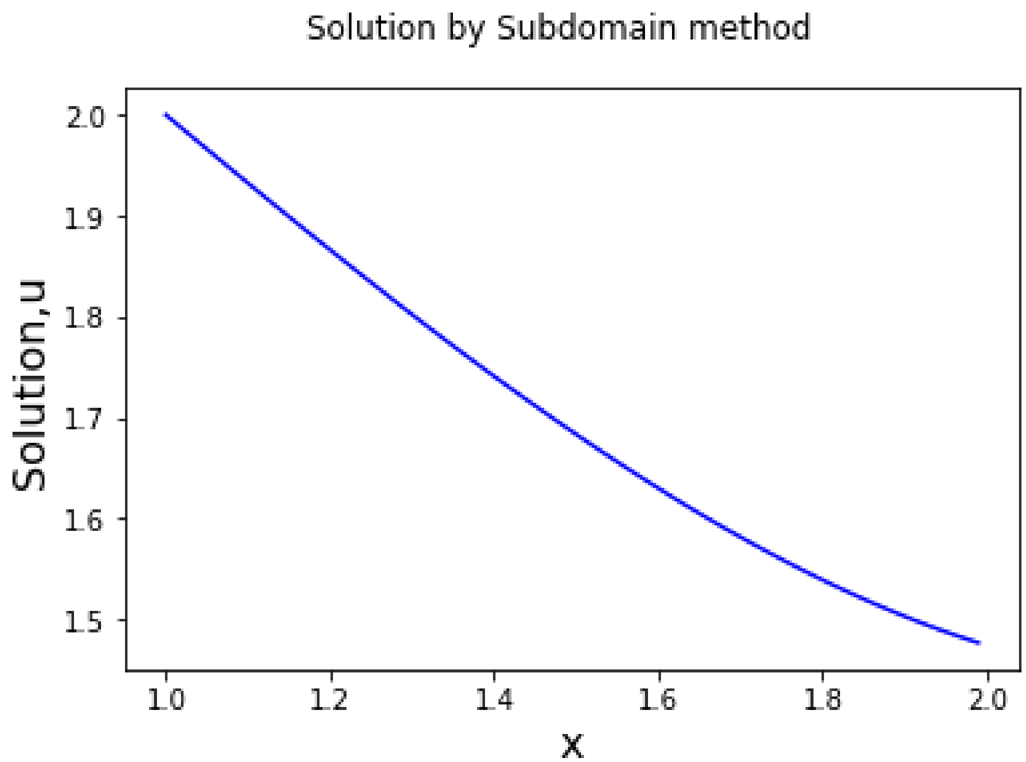
$$\frac{1}{\Delta x_1}\int_1^{\frac{3}{2}} R(x)dx = 1+\frac{5}{4}a_2+\frac{7}{12}a_3 \qquad \text{and} \qquad \frac{1}{\Delta x_2}\int_{\frac{3}{2}}^{2} R(x)dx = 1+\frac{7}{4}a_2+\frac{25}{12}a_3$$

solving the above for a and substituiting it back in the trial function.

```
In [34]: from sympy import *
         import matplotlib.pyplot as plt
         import numpy as np
         import matplotlib.patches as mpatches

         a_sub = Symbol('a')
         b_sub = Symbol('b')
         x_sub = Symbol('x_sub')
         eqn_sub =[]
         eqn_sub.append(integrate(-0.25+4*(-1)*a_sub+3*(3*(x_sub**2)-4)*b_sub-(2/(x_sub**2)),
```

```
eqn_sub.append(integrate(-0.25+4*(x_sub-1)*a_sub+3*(3*(x_sub**2)-4)*b_sub-(2/(x_sub**
z_sub = solve(eqn_sub, [a_sub,b_sub])
print("a",z_sub.get(a_sub))
print("b",z_sub.get(b_sub))
x_sub = np.arange(1, 2., 0.01)
u_sub = 2 - 0.25*(x_sub-1)+(x_sub-1)*(x_sub-3)*z_sub.get(a_sub)+(x_sub-1)*(x_sub**2+x_

plt.plot(x_sub, u_sub , color='b', label="Subdomain")
plt.xlabel('x', fontsize=16)
plt.ylabel('Solution,u', fontsize=16)
plt.suptitle('Solution by Subdomain method')
plt.show()
```

```
a -0.327956989247312
b 0.120669056152927
```

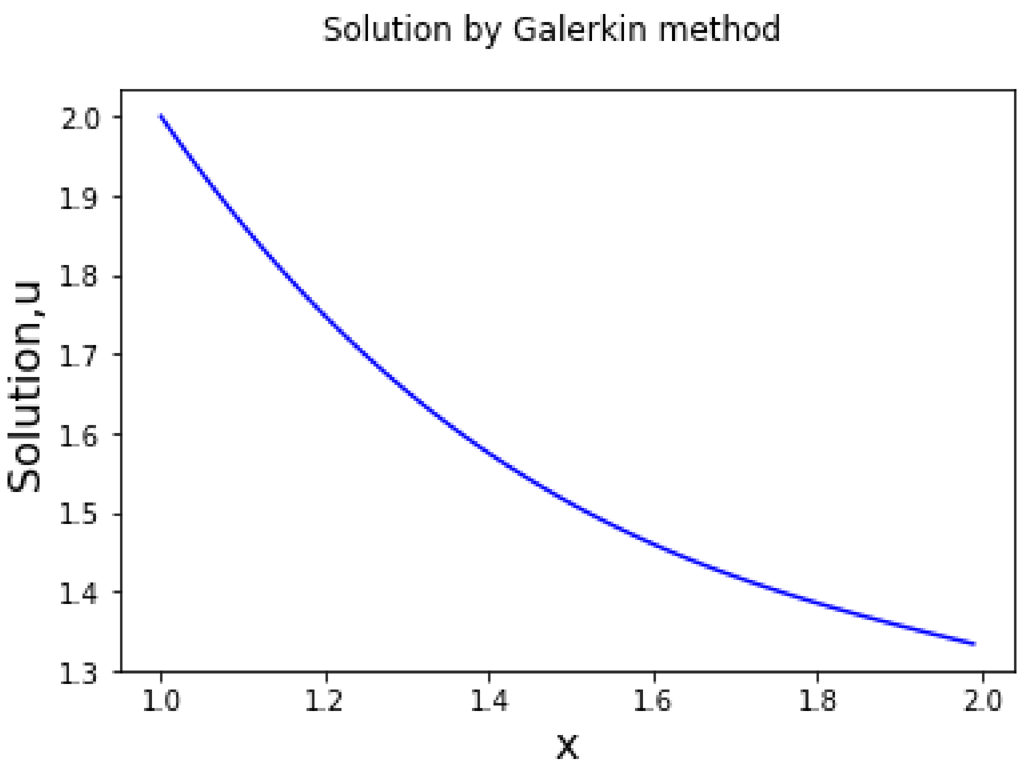


1.4 Galerkin Method

```
In [36]: from sympy import *
         import matplotlib.pyplot as plt
         import numpy as np
         import matplotlib.patches as mpatches
```

```
a_gal = Symbol('a')
b_gal = Symbol('b')
x_gal = Symbol('x')
eqn_gal =[]
r_gal = -0.25+4*(x_gal-1)*a_gal+3*(3*(x_gal**2)-4)*b_gal-(2/(x_gal**2))
print(diff(r_gal,a_gal))
eqn_gal.append(integrate(r_gal*x_gal, (x_gal, 1, 2)))
eqn_gal.append(integrate(r_gal*x_gal*x_gal, (x_gal, 1, 2)))
print(eqn_gal)
z_gal = solve(eqn_gal, [a_gal,b_gal])

x = np.arange(1, 2., 0.01)
u_gal = 2 - 0.25*(x-1)+(x-1)*(x-3)*z_gal.get(a_gal)+(x-1)*((x**2)+x-11)*z_gal.get(b_ga
plt.plot(x, u_gal , color='b')
plt.xlabel('x', fontsize=16)
plt.ylabel('Solution,u', fontsize=16)
plt.suptitle('Solution by Galerkin method')

plt.show()
```

```
4*x - 4
[3.33333333333333*a + 15.75*b - 2.0*log(2) - 0.375, 5.66666666666667*a + 27.8*b - 2.5833333333333
```



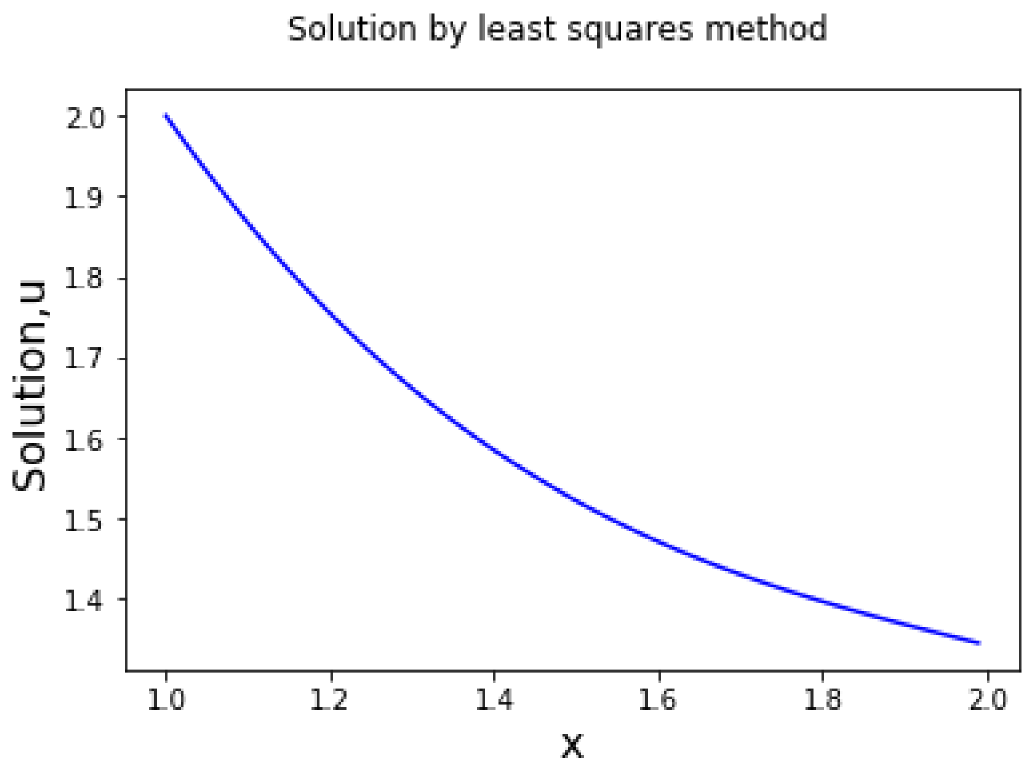
1.5 Least Squares Method

In this method we force the residuals to be zero as

$$\int_0^1 R(x)\,\frac{\partial R(x)}{\partial a_i}\,dx = 0$$

```
In [37]: from sympy import *
         import matplotlib.pyplot as plt
         import numpy as np
         import matplotlib.patches as mpatches

         a_lea = Symbol('a')
         b_lea = Symbol('b')
         x_lea = Symbol('x')
         eqn_lea =[]
         r_lea = -0.25+4*(x_lea-1)*a_lea+3*(3*(x_lea**2)-4)*b_lea-(2/(x_lea**2))
         print(diff(r_lea,a_lea))
         eqn_lea.append(integrate(r_lea*diff(r_lea,a_lea), (x_lea, 1, 2)))
         eqn_lea.append(integrate(r_lea*diff(r_lea,b_lea), (x_lea, 1, 2)))
         print(eqn_lea)
         z_lea = solve(eqn_lea, [a_lea,b_lea])
         print("a",z.get(a))
         print("b",z.get(b))
         x = np.arange(1, 2., 0.01)
         u_lea = 2 - 0.25*(x-1)+(x-1)*(x-3)*z_lea.get(a_lea)+(x-1)*((x**2)+x-11)*z_lea.get(b_le
         plt.plot(x, u_lea , color='b')
         plt.xlabel('x', fontsize=16)
         plt.ylabel('Solution,u', fontsize=16)
         plt.suptitle('Solution by least squares method')
         plt.show()
```

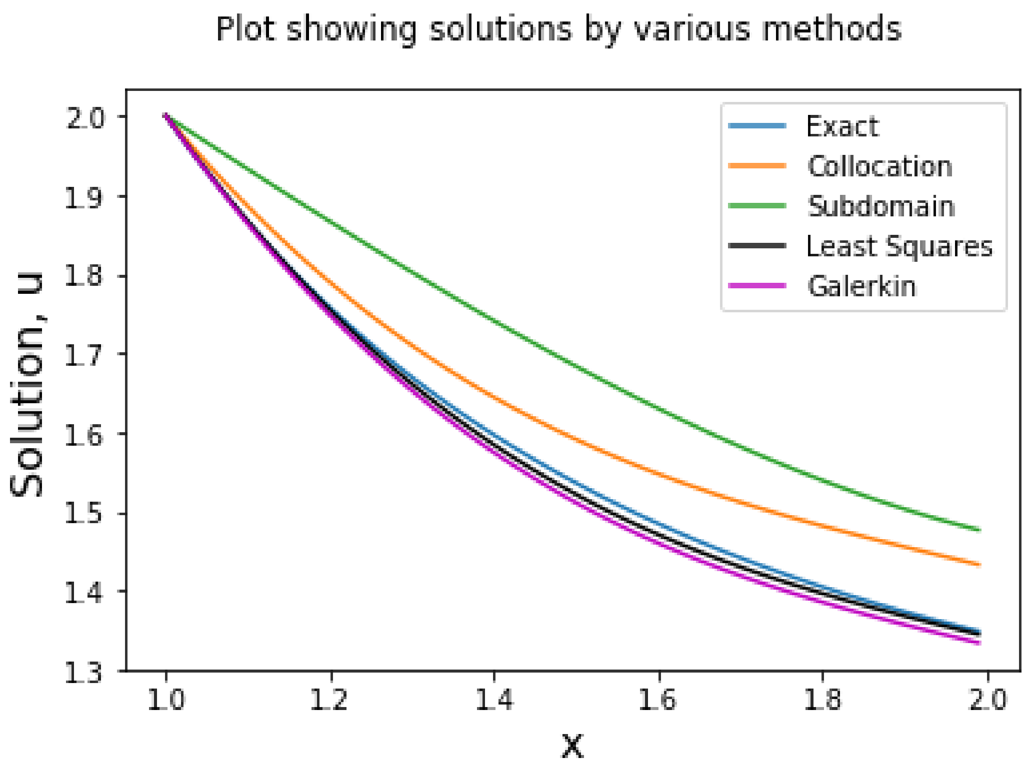
```
4*x - 4
[5.33333333333333*a + 27.0*b - 8.0*log(2) + 3.5, 27.0*a + 142.2*b - 8.25]
a 2.09925000000000
b -0.356000000000000
```



1.6 Comparision

In [38]:
```
import matplotlib.pyplot as plt
import numpy as np
from sympy import *
############################## col
a_col = Symbol('a')
b_col = Symbol('b')
val_col = [4/3 ,5/3]
eqn_col =[]

def function(x):
        for x in val_col:
                # calculating residuals
                eqn_col.append(-0.25+4*(x-1)*a_col+3*(3*(x**2)-4)*b_col-(2/(x**2)))
        return(eqn_col)

z_col = solve(function(val_col), [a_col,b_col])
############################## sub
a_sub = Symbol('a')
b_sub = Symbol('b')
x_sub = Symbol('x_sub')
```

```
eqn_sub =[]
eqn_sub.append(integrate(-0.25+4*(-1)*a_sub+3*(3*(x_sub**2)-4)*b_sub-(2/(x_sub**2)),
eqn_sub.append(integrate(-0.25+4*(x_sub-1)*a_sub+3*(3*(x_sub**2)-4)*b_sub-(2/(x_sub**2
z_sub = solve(eqn_sub, [a_sub,b_sub])

############################## least square
a_lea = Symbol('a')
b_lea = Symbol('b')
x_lea = Symbol('x')
eqn_lea =[]
r_lea = -0.25+4*(x_lea-1)*a_lea+3*(3*(x_lea**2)-4)*b_lea-(2/(x_lea**2))
eqn_lea.append(integrate(r_lea*diff(r_lea,a_lea), (x_lea, 1, 2)))
eqn_lea.append(integrate(r_lea*diff(r_lea,b_lea), (x_lea, 1, 2)))
z_lea = solve(eqn_lea, [a_lea,b_lea])
############################## galerian method
a_gal = Symbol('a')
b_gal = Symbol('b')
x_gal = Symbol('x')
eqn_gal =[]
r_gal = -0.25+4*(x_gal-1)*a_gal+3*(3*(x_gal**2)-4)*b_gal-(2/(x_gal**2))
eqn_gal.append(integrate(r_gal*x_gal, (x_gal, 1, 2)))
eqn_gal.append(integrate(r_gal*x_gal*x_gal, (x_gal, 1, 2)))
z_gal = solve(eqn_gal, [a_gal,b_gal])
############################## method end

x = np.arange(1, 2, 0.01)

u_col = 2. - (0.25)*(x-1)+(x-1)*(x-3)*z_col.get(a_col)+(x-1)*(x**2+x-11)*z_col.get(b_c
u_exact = (2/x)+0.5*np.log(x)
u_sub = 2 - 0.25*(x-1)+(x-1)*(x-3)*z_sub.get(a_sub)+(x-1)*(x**2+x-11)*z_sub.get(b_sub
u_lea = 2 - 0.25*(x-1)+(x-1)*(x-3)*z_lea.get(a_lea)+(x-1)*((x**2)+x-11)*z_lea.get(b_le
u_gal = 2 - 0.25*(x-1)+(x-1)*(x-3)*z_gal.get(a_gal)+(x-1)*((x**2)+x-11)*z_gal.get(b_ga

plt.plot(x, u_exact, label="Exact")
plt.plot(x, u_col, label="Collocation")
plt.plot(x, u_sub, label="Subdomain")
plt.plot(x, u_lea, label="Least Squares", color="black")
plt.plot(x, u_gal, label="Galerkin", color ="m")

plt.legend()
plt.xlabel('x', fontsize=16)
plt.ylabel('Solution, u', fontsize=16)
plt.suptitle('Plot showing solutions by various methods')
plt.show()
```


Plot showing solutions by various methods
Exact
Collocation
Subdomain
Least Squares
Galerkin
Solution, u
x
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.0
1.2
1.4
1.6
1.8
2.0

```
In [ ]:
```