one-dimensional research modelling

ELV

\subtitle{...}

Technical reference

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0.1 Introduction

This memo describes the implementation of the Preissmann scheme (**???**) in ELV for implicitly solving the **?** equations.

Liselot Arkesteijn first implemented the scheme. Performance was later improved by Victor Chavarrias.

0.2 Physical system of equations

0.2.1 Set of equations

The **?** equations in conservative form read:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \ , \tag{1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial q^2/h}{\partial x} + \frac{1}{2}g\frac{\partial h^2}{\partial x} + gh\frac{\partial \eta}{\partial x} = -C_f\frac{q^2}{h^2} \ . \tag{2}$$

The independent variables are:

- ◇ x [m] the streamwise coordinate,
- ◇ t [s] the time coordinate.

The dependent variables are:

- ◇ h [m] the flow depth,
- ◇ q [m^2/s] the specific discharge.

The constants are:

- ◇ g [m/s^2] the acceleration due to gravity,
- ◇ C_{f} [-] the non-dimensional friction coefficient.

The constant functions of x are:

- ◇ η [m] the bed elevation.

0.3 Linearization of the system of equations

We consider a reference state that is a solution to the system of equations. The reference state is a steady uniform straight flow in the x direction over an inclined plane bed. Mathematically: $h_0 =$ ct., $q_0 =$ ct., $\frac{\partial \eta}{\partial x} =$ ct. $= \frac{-C_{\mathrm{f}}{q_0}^2}{gh_0^3}$, where ct. denotes a constant different from 0 and subscript 0 indicates the reference solution.

We add a small perturbation to the reference solution denoted by $'$ and we linearise the resulting system of equations. After substituting the reference solution we obtain a system of equations of the perturbed variables:

$$\frac{\partial \mathbf{G}'}{\partial t} + \mathbf{C_0}\frac{\partial \mathbf{G}'}{\partial x} + \mathbf{B_0}\mathbf{G}' = 0 \ , \tag{3}$$

where the vector of dependent variables is:

$$\mathbf{G}' = [h', q']^{\intercal} \ , \tag{4}$$

The advective matrix in x direction is:

$$\mathbf{C_0} = \left[\begin{array}{cc} 0 & 1 \\ gh_0 - \left(\frac{q_{x0}}{h_0}\right)^2 & 2\frac{q_{x0}}{h_0} \end{array} \right] . \tag{5}$$

The matrix of linear terms is:

$$\mathbf{B_0} = \left[\begin{array}{cc} 0 & 0 \\ \frac{-3C_f q_{x0}^2}{h_0^3} & \frac{2C_f q_{x0}}{h_0^2} \end{array} \right] . \tag{6}$$

0.3.1 Boundary conditions

The equations are solved in a one-dimensional domain of length L [m] extending from $x = x_o$ until $x = x_f$ over time T [s] between $t = t_1$ and $t = t_2$.

At $t = t_1$, $h = H_1(x)$ and $q = Q_1(x)$ for $x_o \leq x \leq x_f$.

At $x = x_o$, $q = Q_o(x)$ for $t_1 \leq t \leq t_2$.

At $x = x_f$, $h = H_f(x)$ for $t_1 \leq t \leq t_2$.

0.4 Numerical discretization

0.4.1 Domain

The space domain x is discretized into N cells of equal length Δx [m]. The equations are solved at the cell centre.

0.4.2 Scheme

The non-linear set of equations (1)-(2) is discretized by means of the θ-box scheme:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\left(\frac{f_{m+1}^{n+1} - f_{m+1}^{n}}{\Delta t} + \frac{f_{m}^{n+1} - f_{m}^{n}}{\Delta t}\right) , \tag{7}$$

$$\frac{\partial f}{\partial x} = \theta\left(\frac{f_{m+1}^{n+1} - f_{m}^{n+1}}{\Delta x}\right) + (1-\theta)\left(\frac{f_{m+1}^{n} - f_{m}^{n}}{\Delta x}\right) , \tag{8}$$

where $\theta \in [0.5, 1]$ is a parameter, $m \in [1, N]$ is an index indicating cell centre number in increasing order, and $n > 1$ is an index indicating time.

For $\theta = 0.5$ one obtains the Preissmann scheme (**???**).

The first term in Equation (1) is:

The slope is approximated as:

$$\left.\frac{\partial \eta}{\partial x}\right|_m = \frac{\eta_{m+1} - \eta_m}{\Delta x} , \tag{10}$$

0.5 Solution of the algebraic system of equations

The discretized set of equations form a system of algebraic equations:

$$\mathbf{AQ} = \mathbf{B} \ . \tag{11}$$

Vector:

$$\mathbf{Q} = [h_1, h_2, \cdots, h_{N-1}, h_N, q_1, q_2, \cdots, q_{N-1}, q_N]^\intercal \ , \tag{12}$$

is the $2N$x1 vector of unknowns.

Matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \ , \tag{13}$$

is the $2N$x$2N$ matrix containing the implicit terms, which is subdivided into 4 NxN submatrices contain.

Vector:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \ , \tag{14}$$

is the $2N$x1 vector of explicit terms.

The system is solved using the Newton–Raphson method until $r < \epsilon$:

$$\mathbf{Q}^{j+1} = \mathbf{Q}^j - (\mathbf{J_0}^j)^{-1} \left(\mathbf{A}^j\mathbf{Q}^j - \mathbf{B}^j\right) \ , \tag{15}$$

where j is the iteration index. The operation to the right of equation (15) is done using function `mldivide` applied to arguments $\mathbf{J_0}^j$ and $\mathbf{A}^j\mathbf{Q}^j - \mathbf{B}^j$. The residual r is computed as:

$$r = \max\left|\left(\mathbf{A}^j\mathbf{Q}^j - \mathbf{B}^j\right)\right| \ . \tag{16}$$

References