*Stochastic Processes* December 21, 2021
Sharif University of Technology CE 40-695
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Time: 20 mins

Name: Std. Number:

# Quiz 7 (Estimation, Bayes Rule)

**Questions**

1. (50%) Let $X$ be a random variable with pdf: $f(x \mid \theta) \sim$ Geometric $(\theta)$

$$f(x \mid \theta) = \theta(1-\theta)^{x-1} \quad x = 1, 2, \dots$$

Let $\Pi(\theta) = \begin{cases} 2\theta & 0 \leqslant \theta \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

Find the MAP estimate of $\theta$ given $X = 3$.

2. (50%) For the following PDF assuming that we have n i.i.d. samples from this distribution and $x_0 > 0$, find an estimation of $\theta$ using method of moments.

$$f\left(x \mid x_0, \theta\right) = \theta x_0^{\theta} x^{-\theta-1}, x \geq x_0, \theta > 1$$

**Answers**

1.

$$f(x \mid \theta) = \theta(1-\theta)^{x-1}$$
$$f(3 \mid \theta) = \theta(1-\theta)^2$$
$$f(x \mid \theta)\Pi(\theta) = \theta(1-\theta)^2 \cdot 2\theta$$
$$= 2\theta^2(1-\theta)^2$$
$$\frac{\partial}{\partial\theta}\left[\theta^2(1-\theta)^2\right] = 2\theta(1-\theta)^2 - 2(1-\theta)\theta^2 = 0$$
$$\hat{\theta}_{\text{MAP}} = \frac{1}{2}$$

2.

$$f\left(x \mid x_0, \theta\right) = \theta x_0^{\theta} x^{-\theta-1} \quad x \geq x_0, \theta > 1, \quad x_0 > 0$$

1st moment:

$$\int_{x_0}^{\infty} x f\left(x \mid x_0, \theta\right) dx = \theta x_0^{\theta} \int_{x_0}^{\infty} x^{-\theta} dx = \theta x_0^{\theta} \frac{x^{-\theta+1}}{-\theta+1}\Big|_{x_0}^{\infty} = \frac{\theta x_0^{\theta}}{-\theta+1}\left(0 - x^{-\theta+1}\right) = \frac{\theta x_0}{\theta-1}$$

Sample $1^{\text{th}}$ moment:

$$m_1 = \sum_{i=1}^{n} x_i$$

Solving for $\theta$ :

$$\frac{\theta x_0}{\theta-1} = m_1 \Longrightarrow \theta x_0 = m_1\theta - m_1 \Longrightarrow \theta = \frac{m_1}{m_1 - x_0}$$