Time: 20 mins

Name: Std. Number:

Retake Quiz 1 (Solutions)

Questions

1. Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than mn is not divisible by either m or n?(9 points)

2. A simple example of a random variable is the *indicator* of an event A, which is denoted by I_A:

$$I_A(\omega) = \begin{cases} 1, & if\ \omega \in A \\ \\ 0, & Otherwise. \end{cases}$$

   (a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A and I_B are independent.(5 points)

   (b) If $X = I_A$, find $E[X]$. (3 points)

3. A system G is described by the difference equation y[n] = nu[n]. Determine whether the system is a) memoryless, b) causal, and c) time-invariant.(3 points)

Solutions:

1) We have $mn - 1$ integers less than mn. Furthermore, there are $n - 1$ multiples of m and $m - 1$ multiples of n less than mn; However, $\frac{mn}{lcm(m,n)} - 1$ of these multiples are the same. Hence by the Inclusion-Exclusion law we have $m + n - 1 - gcd(m, n)$ multiples of either m or n. By doing some algebra, the probability of not choosing a multiple of either one would be:

$$\frac{(m-1)(n-1) + gcd(m,n) - 1}{mn - 1}$$

2)

(a) We know that I_A is a random variable that maps a 1 to the real number line if ω occurs within an event A and maps a 0 to the real line if ω occurs outside of event A. A similar argument holds for event B. Thus we have,

$$I_A(\omega) = \begin{cases} 1, & \textit{with probability } P(A) \\ 0, & \textit{with probability } 1 - P(A) \end{cases}$$

If the random variables, A and B, are independent, we have $P(A \cap B) = P(A)P(B)$. The indicator random variables, I_A and I_B, are independent if, $P_{I_A,I_B}(x,y) = P_{I_A}(x)P_{I_B}(y)$. First suppose I_A and I_B are independent, we have:

$$P(A \cap B) = P_{I_A,I_B}(1,1) = P_{I_A}(1)P_{I_B}(1) = P(A)P(B)$$

now suppose A and B are independent,

$$P_{I_A,I_B}(1,1) = P(A \cap B) = P(A)P(B) = P_{I_A}(1)P_{I_B}(1)$$

$$P_{I_A,I_B}(0,1) = P(A^c \cap B) = P(A^c)P(B) = P_{I_A}(0)P_{I_B}(1)$$
$$P_{I_A,I_B}(1,0) = P(A \cap B^c) = P(A)P(B^c) = P_{I_A}(1)P_{I_B}(0)$$
$$P_{I_A,I_B}(1,1) = P(A \cap B) = P(A^c)P(B^c) = P_{I_A}(0)P_{I_B}(0)$$

(b) If $X = I_A$, we know that

$$E[X] = E[I_A] = 1.P(A) + 0.(1 - P(A)) = P(A)$$

3) a) Since the output value at time n depends only on the input value at time n, the system is memoryless.

b) Since the output does not depend on future input values, the system is causal.

c) Let $u_2[n] := u_1[n + n_0]$ with $n_0 \neq 0$ for all n, then

$$\begin{aligned} y_1[n + n_0] &= (n + n_0)u_1[n + n_0] \\ &= (n + n_0)u_2[n] \neq y_2[n] = nu_2[n]. \end{aligned}$$

Because a time-shift of n_0 in the input does not correspond to a time-shift of n_0 in the output, the system is not time-invariant.