SP_HW1_Solutions

سوال ۱.الف.

Solution: The number of emails Alice sends in the interval $[1,2]$ is a Poisson random variable with parameter λ_A. So we have:

$$\mathbf{P}(3,1) = \frac{{\lambda_A}^3 e^{-\lambda_A}}{3!}.$$

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Solution: Define T_2 as the second inter-arrival time in Alice's Poisson process. Then:

$$Y_2 = Y_1 + T_2$$

$$\mathbf{E}[Y_2 \mid Y_1] = \mathbf{E}[Y_1 + T_2 \mid Y_1] = Y_1 + \mathbf{E}[T_2] = Y_1 + 1/\lambda_A.$$

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Solution: Let $Z = Y_1^2$. Then we first find the CDF of Z and differentiate to find the PDF of Z:

$$F_Z(z) = \mathbf{P}(Y_1^2 \le z) = \mathbf{P}(-\sqrt{z} \le Y_1 \le \sqrt{z}) = \begin{cases} 1 - e^{-\lambda_A\sqrt{z}} & z \ge 0 \\ 0 & z < 0. \end{cases}$$

$$\begin{aligned} f_Z(z) = \frac{dF_Z(z)}{dz} &= \lambda_A e^{-\lambda_A\sqrt{z}}\left(\frac{1}{2}z^{-1/2}\right) \qquad (z \ge 0) \\ f_Z(z) &= \begin{cases} \frac{\lambda_A}{2\sqrt{z}} e^{-\lambda_A\sqrt{z}} & z \ge 0 \\ 0 & z < 0. \end{cases} \end{aligned}$$

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Solution:

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1}(y_1) f_{Y_2|Y_1}(y_2|y_1) \\ &= f_{Y_1}(y_1) f_{T_2}(y_2 - y_1) \\ &= \lambda_A e^{-\lambda_A y_1} \lambda_A e^{-\lambda_A(y_2 - y_1)} \qquad y_2 \ge y_1 \ge 0 \\ &= \begin{cases} \lambda_A^2 e^{-\lambda_A y_2} & y_2 \ge y_1 \ge 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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Solution: Let A be the event {exactly one arrival in the interval [0,1]}. Looking forward from time $t = 1$, the time until the next arrival is simply an exponential random variable (T). So,

$$\mathbf{E}[Y_2 \mid A] = 1 + \mathbf{E}[T] = 1 + 1/\lambda_A.$$

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Solution: Given A, the times in this interval are equally likely for the arrival Y_1. Thus,

$$\mathbf{E}[Y_1 \mid A] = 1/2.$$

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Solution: Let K be the total number of emails sent in $[0, 2]$. Let K_1 be the total number of emails sent in $[0, 1)$, and let K_2 be the total number of emails sent in $[1, 2]$. Then $K = K_1 + K_2$ where K_1 is a Poisson random variable with parameter λ_A and K_2 is a Poisson random variable with parameter $\lambda_A + \lambda_B$ (since the emails sent by both Alice and Bob after time $t = 1$ arrive according to the merged Poisson process of Alice's emails and Bob's emails). Since K is the sum of independent Poisson random variables, K is a Poisson random variable with parameter $2\lambda_A + \lambda_B$. So K has the distribution:

$$p_K(k) = \frac{(2\lambda_A + \lambda_B)^k e^{-(2\lambda_A+\lambda_B)}}{k!} \qquad k = 0, 1, \ldots.$$

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Solution: The total typing time Q associated with the email that Alice is typing at the time Bob shows up is the sum of S_0, the length of time between Alice's last email or time 0 (whichever is later) and time 1, and T_1, the length of time from 1 to the time at which Alice sends her current email. T_1 is exponential with parameter λ_A. and $S_0 = \min\{T_0, 1\}$, where T_0 is exponential with parameter λ_A.

Then,

$$Q = S_0 + T_1 = \min\{T_0, 1\} + T_1$$

and

$$\mathbf{E}[Q] = \mathbf{E}[S_0] + \mathbf{E}[T_1].$$

We have: $\mathbf{E}[T_1] = 1/\lambda_A$.

We can find $\mathbf{E}[S_0]$ via the law of total expectations:

$$\begin{aligned}
\mathbf{E}[S_0] = \mathbf{E}[\min\{T_0, 1\}] &= \mathbf{P}(T_0 \le 1)\mathbf{E}[T_0 \mid T_0 \le 1] + \mathbf{P}(T_0 > 1)\mathbf{E}[1|T_0 > 1] \\
&= \left(1 - e^{-\lambda_A}\right) \int_0^1 t f_{T|T_0\le 1}(t)\, dt + e^{-\lambda_A} \\
&= \left(1 - e^{-\lambda_A}\right) \int_0^1 t \frac{\lambda_A e^{-\lambda_A t}}{(1 - e^{-\lambda_A})}\, dt + e^{-\lambda_A} \\
&= \int_0^1 t\lambda_A e^{-\lambda_A t}\, dt + e^{-\lambda_A} \\
&= \frac{1}{\lambda_A} \int_0^1 t\lambda_A^2 e^{-\lambda_A t}\, dt + e^{-\lambda_A} \\
&= \frac{1}{\lambda_A}\left(1 - e^{-\lambda_A} - \lambda_A e^{-\lambda_A}\right) + e^{-\lambda_A} \\
&= \frac{1}{\lambda_A}\left(1 - e^{-\lambda_A}\right)
\end{aligned}$$

where the above integral is evaluated by manipulating the integrand into an Erlang order 2 PDF and equating the integral of this PDF from 0 to 1 to the probability that there are 2 or more arrivals in the first hour (i.e. $\mathbf{P}(Y_2 < 1) = 1 - \mathbf{P}(0, 1) - \mathbf{P}(1, 1)$). Alternatively, one can integrate by parts and arrive at the same result.

Combining the above expectations:

$$\mathbf{E}[Q] = \mathbf{E}[S_0] + \mathbf{E}[T_1] = \frac{1}{\lambda_A}\left(1 - e^{-\lambda_A}\right) + \frac{1}{\lambda_A} = \frac{1}{\lambda_A}\left(2 - e^{-\lambda_A}\right).$$

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Solution:

$$\begin{aligned}
\mathbf{P}(\text{Alice sent 4 in } [0,2] \mid \text{total 10 sent in } [0,2]) &= \frac{\mathbf{P}(\text{Alice sent 4 in } [0,2] \cap \text{total 10 sent in } [0,2])}{\mathbf{P}(\text{total 10 sent in } [0,2])} \\
&= \frac{\mathbf{P}(\text{Alice sent 4 in } [0,2] \cap \text{Bob sent 6 } [0,2])}{\mathbf{P}(\text{total 10 sent in } [0,2])} \\
&= \frac{\left(\frac{(2\lambda_A)^4 e^{-2\lambda_A}}{4!}\right)\left(\frac{(\lambda_B)^6 e^{-\lambda_B}}{6!}\right)}{\frac{(2\lambda_A+\lambda_B)^{10} e^{-2\lambda_A+\lambda_B}}{10!}} \\
&= \binom{10}{4}\left(\frac{2\lambda_A}{2\lambda_A+\lambda_B}\right)^4\left(\frac{\lambda_B}{2\lambda_A+\lambda_B}\right)^6.
\end{aligned}$$

As the form of the solution suggests, the problem can be solved alternatively by computing the probability of a single email being sent by Alice, given it was sent in the interval $[0,2]$. This can be found by viewing the number of emails sent by Alice in $[0,2]$ as the number of arrivals arising from a Poisson process with twice the rate ($2\lambda_A$) in an interval of half the duration (particularly, the interval $[1,2]$), then merging this process with Bob's process. Then the probability that an email sent in the interval $[0,2]$ was sent by Alice is the probability that an arrival in this new merged process came from the newly constructed $2\lambda_A$ rate process:

$$p = \frac{2\lambda_A}{2\lambda_A+\lambda_B}.$$

Then, out of 10 emails, the probability that 4 came from Alice is simply a binomial probability with 4 successes in 10 trials, which agrees with the solution above.

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Solution:

Let N be the number of emails Alice sent in the interval $[0,1]$. Since N is a Poisson random variable with parameter λ_A,

$$\mathbf{E}[N] = \text{var}(N) = \lambda_A = 4.$$

To apply the Chebyshev inequality, we recognize:

$$\mathbf{P}(N \geq 5) = \mathbf{P}(N-4 \geq 1) \leq \mathbf{P}(|N-4| \geq 1) \leq \frac{\text{var}(N)}{1^2} = 4.$$

In this case, the upper-bound of 4 found by application of the Chebyshev inequality is uninformative, as we already knew $\mathbf{P}(N \geq 5) \leq 1$.

To find a better bound on this probability, use the Markov inequality, which gives:

$$\mathbf{P}(N \geq 5) \leq \frac{\mathbf{E}[N]}{5} = \frac{4}{5}.$$

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Solution: With λ_A large, we assume $\lambda_A \gg 1$. For simplicity, assume λ_A is an integer. We can divide the interval $[0, 1]$ into λ_A disjoint intervals, each with duration $1/\lambda_A$, so that these intervals span the entire interval from $[0, 1]$. Let N_i be the number of arrivals in the ith such interval, so that the N_i's are independent, identically distributed Poisson random variables with parameter 1. Since N is defined as the number of arrivals in the interval $[0, 1]$, then $N = N_1 + \cdots + N_{\lambda_A}$. Since $\lambda_A \gg 1$, then N is the sum of a large number of independent and identically distributed random variables, where the distribution of N_i does not change as the number of terms in the sum increases. Hence, N is approximately normal with mean λ_A and variance λ_A.

If λ_A is not an integer, the same argument holds, except that instead of having λ_A intervals, we have an integer number of intervals equal to the integer part of λ_A ($\bar{\lambda}_A$ =floor(λ_A)) of length $1/\lambda_A$ and an extra interval of a shorter length $(\lambda_A - \bar{\lambda}_A)/\lambda_A$.

Now, N is a sum of λ_A independent, identically distributed Poisson random variables with parameter 1 added to another Poisson random variable (also independent of all the other Poisson random variables) with parameter $(\lambda_A - \bar{\lambda}_A)$. In this case, N would need a small correction to apply the central limit theorem as we are familiar with it; however, it turns out that even without this correction, adding the extra Poisson random variable does not preclude the distribution of N from being approximately normal, for large λ_A, and the central limit theorem still applies.

To arrive at a precise statement of the CLT, we must "standardize" N by subtracting its mean then dividing by its standard deviation. After having done so, the CDF of the standardized version of N should converge to the standard normal CDF as the number of terms in the sum approaches infinity (as $\lambda_A \to \infty$).

Therefore, the precise statement of the CLT when applied to N is:

$$\lim_{\lambda_A \to \infty} \mathbf{P}\left(\frac{N - \lambda_A}{\sqrt{\lambda_A}} \leq z\right) = \Phi(z)$$

where $\Phi(z)$ is the standard normal CDF.

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$$f_\Lambda(\lambda) = 2e^{-2\lambda}, \qquad \lambda \geq 0.$$

Solution:

$$\begin{aligned}
\mathbf{E}[N^2] = \mathbf{E}[\mathbf{E}[N^2 \mid \Lambda]] &= \mathbf{E}[\text{var}(N \mid \Lambda) + (\mathbf{E}[N \mid \Lambda])^2] \\
&= \mathbf{E}[\Lambda + \Lambda^2] \\
&= \mathbf{E}[\Lambda] + \text{var}(\Lambda) + (\mathbf{E}[\Lambda])^2 \\
&= \frac{1}{2} + \frac{2}{2^2} \\
&= 1.
\end{aligned}$$

Extra Question

ح. امید ریاضی زمان که هریک از آنها حداقل یک ایمیل ارسال کرده باشند، چقدر است؟ (زمان را با شروع از صفر حساب می کنیم و ایمیل های ارسال شده توسط علی در بازه (1 ,0) را هم در نظر می گیریم.)

Solution: Define U as the time from $t = 0$ until each person has sent at least one email.
Define V as the remaining time from when Bob arrives (time 1) until each person has sent at least one email (so $V = U - 1$).
Define S as the time until Bob sends his first email after time 1.
Define the event $A = \{\text{Alice sends one or more emails in the time interval } [0, 1]\} = \{Y_1 \leq 1\}$, where Y_1 is the time Alice sends her first email.
Define the event $B = \{\text{After time 1, Bob sends his next email before Alice does}\}$, which is equivalent to the event where the next arrival in the merged process from Alice and Bob's orginal processes (starting from time 1) comes from Bob's process.
We have:

$$\mathbf{P}(A) = \mathbf{P}(Y_1 \leq 1) = 1 - e^{-\lambda_A}$$

$$\mathbf{P}(B) = \frac{\lambda_B}{\lambda_A + \lambda_B}.$$

Then,

$$\begin{aligned}
\mathbf{E}[U] &= \mathbf{P}(A)\mathbf{E}[U \mid A] + \mathbf{P}(A^c)\mathbf{E}[U \mid A^c] \\
&= (1 - e^{-\lambda_A})(1 + \mathbf{E}[V \mid A]) + e^{-\lambda_A}(1 + \mathbf{E}[V \mid A^c]) \\
&= (1 - e^{-\lambda_A})(1 + \mathbf{E}[V \mid A]) + e^{-\lambda_A}(1 + \mathbf{P}(B \mid A^c)\mathbf{E}[V \mid B \cap A^c] + \mathbf{P}(B^c \mid A^c)\mathbf{E}[V \mid B^c \cap A^c]) \\
&= (1 - e^{-\lambda_A})(1 + \mathbf{E}[V \mid A]) + e^{-\lambda_A}(1 + \mathbf{P}(B)\mathbf{E}[V \mid B \cap A^c] + \mathbf{P}(B^c)\mathbf{E}[V \mid B^c \cap A^c]) \\
&= (1 - e^{-\lambda_A})(1 + \mathbf{E}[V \mid A]) + e^{-\lambda_A}\left(1 + \frac{\lambda_B}{\lambda_A + \lambda_B}\mathbf{E}[V \mid B \cap A^c] + \frac{\lambda_A}{\lambda_A + \lambda_B}\mathbf{E}[V \mid B^c \cap A^c]\right).
\end{aligned}$$

Note that $\mathbf{E}[V \mid B^c \cap A^c]$ is the expected value of the time until each of them sends one email after time 1 (since, given A^c, Alice did not send any in the interval $[0, 1]$) and given Alice sends an email before Bob. Then this is the expected time until an arrival in the merged process followed by the expected time until an arrival in Bob's process. So, $\mathbf{E}[V \mid B^c \cap A^c] = \frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_B}$.
Similarly, $\mathbf{E}[V \mid B \cap A^c]$ is the time until each sends an email after time 1, given Bob sends an email before Alice. So $\mathbf{E}[V \mid B \cap A^c] = \frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_A}$.
Also, $\mathbf{E}[V \mid A]$ is the expected time it takes for Bob to send his first email after time 1 (since, given A, Alice already sent an email in the interval $[0, 1]$). So $\mathbf{E}[V \mid A] = \mathbf{E}[S] = 1/\lambda_B$.
Combining all of this with the above, we have:

$$\begin{aligned}
\mathbf{E}[U] &= (1 - e^{-\lambda_A})(1 + 1/\lambda_B) \\
&\quad + e^{-\lambda_A}\left(1 + \frac{\lambda_B}{\lambda_A + \lambda_B}\left(\frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_A}\right) + \frac{\lambda_A}{\lambda_A + \lambda_B}\left(\frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_B}\right)\right).
\end{aligned}$$

سوال دوم

(a) Causal because $h[n] = 0$ for $n < 0$. Stable because $\sum_{n=0}^{\infty} (\frac{1}{5})^n = 5/4 < \infty$.

(b) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-2}^{\infty} (0.8)^n = 5 < \infty$.

(c) Anti-causal because $h[n] = 0$ for $n > 0$. Unstable because $\sum_{n=-\infty}^{0} (1/2)^n = \infty$

(d) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-\infty}^{3} 5^n = \frac{625}{4} < \infty$

(a) Causal because $h(t) = 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{-8}/4 < \infty$.

(b) Not causal because $h(t) \neq 0$ for $t < 0$. Unstable because $\int_{-\infty}^{\infty} |h(t)| = \infty$.

(c) Not causal because $h(t) \neq 0$ for $t < 0$. a Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{100}/2 < \infty$.

سوال سوم:

You may either prove each properties or justify answers in words.

(a) $y(t) = x(t-2) + x(2-t)$

(1) To be memoryless, a system should have an impulse response satisfying

$$h(t) = 0, \text{for all } t \neq 0$$

The impulse response of the system is

$$h(t) = \delta(t-2) + \delta(2-t)$$

The system is **not memoryless** since $h(t)$ does not satisfy the memoryless condition and the outpu depends on the past.

(2) This system is **not time invariant**, since the output to a time shifted input $x(t-t_0)$ is

$$x(t-2-t_0) + x(2-t-t_0),$$

while the time shifted output is

$$x((t-t_0)-2) + x(2-(t-t_0)) = x(t-t_0-2) + x(2-t+t_0)$$

(3) The system is **linear** since it satisfies

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

where the system has input/output pairs,

$$x_i(t) \rightarrow y_i(t), \text{for } i = 1, 2$$

(4) **Not causal** as

$$h(t) = 0, \text{for } t < 0$$

(5) The system is **stable** since

$$\int_{-\infty}^{\infty} |h(t)| dt = 2 < \infty$$

(b) $y(t) = [\cos(3t)]x(t)$

سوال ۴.

$$h(t) = [\cos(3t)]\delta(t)$$

(1) The system is **memoryless** since

$$h(t) = 0, \text{for all } t \neq 0$$

(2) It is **not time invariant** since

$$\cos(3t)\, x(t - t_0) \neq y(t - t_0) = \cos\big(3(t - t_0)\big)\, x(t - t_0)$$

(3) **Linear** as it satisfies

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

(4) **Causal** since it's memoryless.

(5) **Stable** since

$$\int_{-\infty}^{\infty} |h(t)| dt = 1 < \infty$$

(c) $y(t) = \int_{-\infty}^{2t} x(\tau) d\tau$

(1) **Not memoryless** since it depends on the past.

(2) **Not time invariant** as

$$\int_{-\infty}^{2t} x(\tau - t_0) d\tau = \int_{-\infty}^{2t - t_0} x(\tau) d\tau \neq y(t - t_0) = \int_{-\infty}^{2(t - t_0)} x(\tau) d\tau$$

(3) **Linear** as it satisfies

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

(4) **Not causal.** For example,

$$y(1) = \int_{-\infty}^{2} x(\tau) d\tau$$

depends on the time 2 which is future input.

(5) **Not stable** since the integral takes values from $-\infty$.

(d) $y(t) = x(\frac{t}{3})$

(1) **Not memoryless** since, for example, $y(3) = x(1)$ which means that the system memorize past inputs.

(2) **Not time invariant** as

$$x\left(\frac{t}{3} - t_0\right) \neq y(t - t_0) = x(\frac{t - t_0}{3})$$

(3) **Linear** as it satisfies

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

(4) **Not causal** since, for example, $y(-3)$ depends on the value of $x(-1)$ which indexes a point in time -1 greater than -3.

(5) It is easy to see that the system is **stable** because input is mapped to output one-to-one. OR you can say it's stable since

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

(e) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$

We see that the system is **not memoryless** because output depends on the past, **linear** as it satisfies

$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$, **causal** since output only depends on the present and the past, and **stable** since $\int_{-\infty}^{\infty} |h(t)| dt < \infty$. This system is **not time invariant** as the output of $x(t - t_0)$ is different from the time shifted output $y(t - t_0)$.

Input $x(t - t_0)$ is mapped to the function $y(t) = \begin{cases} 0, & t < 0 \\ x(t - t_0) + x(t - 2 - t_0), & t \geq 0 \end{cases}$

while the time shifted output $y(t - t_0) = \begin{cases} 0, & t - t_0 < 0 \\ x(t - t_0) + x(t - t_0 - 2), & t - t_0 \geq 0 \end{cases}$

(f) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$

We see that the system is **not memoryless** because output depends on the past input, **causal** since output only depends on the present and the past, and **stable** since $\int_{-\infty}^{\infty} |h(t)| dt < \infty$. To test its linearity, see that $x_1(t) = 1 \rightarrow y_1(t) = 2$ and $x_2(t) = -2 \rightarrow y_2(t) = 0$. The input $x_1(t) + x_2(t) = -1$, however, maps to zero function, showing that the system is **not linear**. This system is **time invariant** since input $x(t - t_0)$ is mapped to the function $y(t) = \begin{cases} 0, & x(t - t_0) < 0 \\ x(t - t_0) + x(t - 2 - t_0), & x(t - t_0) \geq 0 \end{cases}$ which is equal to the time shifted output $y(t - t_0)$.

سوال ۵.

Find inverse fourier transform

$$\delta(\omega - \omega_0) + \delta(\omega + \omega_0) = \frac{1}{2\pi}[e^{-jw0T} + e^{+jw0T}] = \frac{1}{\pi}\cos[\omega_0 T]$$

$$e^{-\frac{w^2}{2}} \leftrightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{T^2}{2}}$$

$$e^{-|w|} \leftrightarrow \frac{1}{\pi}\frac{1}{1+\tau^2}$$

https://towardsdatascience.com/the-poisson-distribution-and-poisson-process-explained-4e2cb17d459