
Time: 20 mins

Name: Std. Number:

Quiz 5 (Gaussian processes)

Questions

1. Let $\{X(t), t \in \mathbb{R}\}$ ba a Gaussian process with covariance function $R_x(\tau)$, let's define the process $Y(t)$ as below :

$$Y(t) = X(t) - 0.4X(t-2)$$

   (a) (4 points) Compute covariance function for this process. Is it stationary?

   (b) (2 points) Is it a Gaussian process?

As usual, it is a good idea to start with the definition of the covariance function:

$$\begin{aligned} r_Y(s,t) &= \mathrm{C}[Y(s), Y(t)] = \mathrm{C}[X(s) - 0.4X(s-2), X(t) - 0.4X(t-2)] \\ &= r_X(t-s) + 0.16 r_X(t-s) - 0.4 r_X(t-s-2) - 0.4 r_X(t-s+2). \end{aligned}$$

Since this does only depend on $t-s$ (convince yourself that $r_Y(s+c, t+c) = r_Y(s,t)$ for any constant c) and since $m_Y(t)$ is constant, $\{Y(t)\}$ is a weakly stationary process. It is a Gaussian process, since every process that is a linear combination of Gaussian processes is a Gaussian process.

2. Assume that K_1 and K_2 kernels be represented as $K_i(x,y) = \Phi_i(x)^T \Phi_i(y)\ for\ i = 1,2$ where Φ_i is a mapping that maps input onto higher dimensional sapce. Then K_1 and K_2 are valid kernels.

   (a) (4 points) Find Φ_3 so that we can have $K_3(x,y) = K_1(x,y).K_2(x,y) = \Phi_3(x)^T \Phi_3(y)$(it means that multiplication of two valid kernels is a valid kernel).

   (b) (3 points) Find Φ_4 so that we can have and $K_4(x,y) = cK_1(x,y) = \Phi_4(x)^T \Phi_4(y), c > 0$

   (c) (7 points) Assume that we know $K_5(x,y) = K_1(x,y)^n$ is valid kernel. proof that $K(x,y) = e^{K_1(x,y)}$ is valid kernel.

   (Hint : use Taylor expansion)

a)

$$
\begin{aligned}
K_3(x,y) &= K_1(x,y).K_2(x,y) \\
K_1(x,y) = \Phi_1(x)^T\Phi_1(y) &= \sum_i \Phi_{1i}(x)\Phi_{1i}(y) \\
K_2(x,y) = \Phi_2(x)^T\Phi_2(y) &= \sum_j \Phi_{1j}(x)\Phi_{1j}(y) \\
K_3(x,y) &= \sum_i\sum_j \Phi_{1i}(x)\Phi_{1i}(y)\Phi_{1j}(x)\Phi_{1j}(y) \\
&lets\, define\ : \\
\Phi'_{ij}(x) &= \Phi_{1i}(x)\Phi_{2j}(x) \\
K_3(x,y) &= \sum_i\sum_j \Phi'_{ij}(x)\Phi'_{ij}(y) \\
&= \Phi'_{1i}(x)^T\Phi'_{1i}(y)
\end{aligned}
$$

b)

$$
\begin{aligned}
K_4(x,y) = cK_1(x,y) &= c\Phi_1(x)^T\Phi_1(y) \\
&= \sqrt{c}\Phi_1(x)^T\sqrt{c}\Phi_1(y) \\
\Phi_4(x) &= \sqrt{c}\Phi_1(x)
\end{aligned}
$$

c)

$$
\begin{aligned}
&Taylor\ expansion\ : \\
e^x &= \sum_{n=0}^{\infty} = \frac{x^n}{n!} \\
e^{K_1(x,y)} &= \sum_{n=0}^{\infty}\frac{K_1(x,y)^n}{n!}
\end{aligned}
$$

we know $K_1(x,y)^n$ is valid. $\frac{1}{n!}$ is positive then from part (b) we can see $\frac{K_1(x,y)^n}{n!}$ is valid and we can see easily summation of valid kernels are valid.