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$$\begin{aligned} f(x) &= \prod_{i=1}^{n}[\theta x_i^{\theta-1} I_{(0,1)}(x_i)] \\ &= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \prod_{i=1}^{n} I_{(0,1)}(x_i) \\ &= \left[I_{(0,1)}(\min\{x_1,\ldots,x_n\}) I_{(0,1)}(\max\{x_1,\ldots,x_n\})\right] \left[\left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \theta^n\right] \end{aligned}$$

By Neyman's factorization theorem, $T = \prod_{i=1}^{n} x_i$ is the sufficient statistics for θ.

ب

$$\begin{aligned} f(x) &= \prod_{i=1}^{n} \theta a x_i^{a-1} \exp\{-\theta x_i^a\} I_{(0,1)}(x_i) \\ &= (\theta a)^n \left(\prod_{i=1}^{n} x_i\right)^{a-1} \exp\{-\theta \sum_{i=1}^{n} x_i^a\} I_{(0,1)}(\min\{x_1,\ldots,x_n\}) \\ &= \left[a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1} I_{(0,1)}(\min\{x_1,\ldots,x_n\})\right] \left[\theta^n \exp\{-\theta \sum_{i=1}^{n} x_i^a\}\right] \end{aligned}$$

By Neyman's factorization theorem, $T = \sum_{i=1}^{n} x_i^a$ is the sufficient statistics for θ.

ج

Write the likelihood function using Iverson brackets to show the dependence on a:

$$L(\theta, a) = \prod_{i=1}^{n} \theta \frac{a^\theta}{x_i^{\theta+1}} [x_i \geq a] = \theta^n a^{n\theta} \prod_{i=1}^{n} \frac{1}{x_i^{\theta+1}} \prod_{i=1}^{n} [x_i \geq a].$$

But

$$\prod_{i=1}^{n} [x_i \geq a] = [(\min_i x_i) \geq a]$$

so

$$L(\theta, a) = \theta^n a^{n\theta} \left(\prod_{i=1}^{n} x_i\right)^{-(\theta+1)} [(\min_i x_i) \geq a].$$

Therefore, $(\prod_{i=1}^{n} x_i)$ and $(\min_i x_i)$ are sufficient statistics for θ and a, respectively.

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For any $\theta \in \Theta$, the posterior distribution of $\theta|x$ is:

$$\frac{f(x|\theta)\pi(\theta)}{\sum_{\theta_i\in\Theta} f(x|\theta_i)\pi(\theta_i)} \qquad \text{where } \pi(\theta) \text{ is any prior on } \theta$$

By the condition, we know it equals to some function $g(\theta, T(x))$, i.e.

$$\frac{f(x|\theta)\pi(\theta)}{\sum_{\theta_i\in\Theta} f(x|\theta_i)\pi(\theta_i)} = g(\theta, T(x))$$

where $g(x, T(x))$ is a function of θ and $T(x)$ only. Thus

$$f(x|\theta) = \frac{g(x, T(x))}{\pi(\theta)} \sum_{\theta_i\in\Theta} f(x|\theta_i)\pi(\theta_i)$$

By factorization theorem, T(x) is sufficient for θ.

ب

If $T(x)$ is sufficient, then $f(x|\theta)$ can be written as

$$f(x|\theta) = g(\theta, T(x))h(x)$$

Let $\pi(\theta)$ be an arbitrary prior distribution, then the posterior of θ is

$$\frac{f(x|\theta)\pi(\theta)}{\sum_{\theta_i\in\Theta} f(x|\theta_i)\pi(\theta_i)} = \frac{g(\theta, T(x))}{\sum_{\theta_i\in\Theta} g(\theta_i, T(x))\pi(\theta_i)}\ \pi(\theta)$$

The posterior depends on x only through $T(x)$. By factorization theorem, T(x) is sufficient for θ.

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Let $\bar{X}$ be the sample mean which is complete and sufficient for μ. Since

$$0 = E(\bar{X} - \mu)^3 = E(\bar{X}^3 - 3\mu\bar{X}^2 + 3\mu^2\bar{X} - \mu^3) = E(\bar{X}^3) - 3\mu\sigma^2/n - \mu^3$$

We obtain that

$$E\left[\bar{X}^3 - (3\sigma^2/n)\bar{X}\right] = E(\bar{X}^3) - 3\mu\sigma^2/n = \mu^3$$

for all μ. By theorem 7.3.23 the UMVUE or μ^3 is $\bar{X}^3 - (3\sigma^2/n)\bar{X}$

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مستقل از نوع تابع چگالی احتمال، درست‌نمایی با افزایش نمونه‌ها کاهش می‌یابد، چرا که تعدادی احتمال (عدد کمتر از یک) در هم ضرب می‌شوند.

در راه‌حل آلفا همان تتای صورت سوال است.

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\alpha) = \prod_{i=1}^{n} f(x_i;\alpha) = \prod_{i=1}^{n} \alpha^{-2} x_i e^{-x_i/\alpha} = \alpha^{-2n} \left(\prod_{i=1}^{n} x_i\right) \exp\left(\frac{-\sum_{i=i}^{n} x_i}{\alpha}\right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\alpha) = -2n \log \alpha + \sum_{i=i}^{n} \log x_i - \frac{\sum_{i=i}^{n} x_i}{\alpha}$$

To maximize this function, we take a **derivative** with respect to α.

$$\frac{d}{d\alpha} \log L(\alpha) = \frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2}$$

We set this derivative equal to **zero**, then **solve** for α.

$$\frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\boxed{\hat{\alpha} = \frac{\sum_{i=i}^{n} x_i}{2n} = \frac{\bar{x}}{2}}$$

ب

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the second moment of an exponential distribution.

$$\mathrm{E}[X] = \int_0^\infty x \cdot \alpha^{-2} x e^{-x/\alpha} dx = \frac{1}{\alpha} \int_0^\infty \frac{x^2}{\alpha} e^{-x/\alpha} dx = \frac{1}{\alpha}(2\alpha^2) = 2\alpha$$

We then set the first population moment, which is a function of α, equal to the first **sample moment**.

$$2\alpha = \frac{\sum_{i=i}^{n} x_i}{n}$$

Solving for α, we obtain the method of moments *estimator*.

$$\boxed{\tilde{\alpha} = \frac{\sum_{i=i}^{n} x_i}{2n} = \frac{\bar{x}}{2}}$$

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$$E[\hat{\theta}] = E[|x|] = \int_{-\infty}^{\infty} |x| \frac{1}{2\theta} e^{-\frac{|x|}{\theta}} dx = \frac{1}{\theta} \int_0^{\infty} x e^{-\frac{x}{\theta}} dx = \theta$$

لذا این تخمینگر unbiased است.

$$\mathrm{Var}[\hat{\theta}] = \frac{1}{n}\mathrm{Var}[|X|]$$
$$\mathrm{Var}[|X|] = E[|X|^2] - E[|X|]^2 = E[|X|^2] - \theta^2$$
$$E[X^2] = \int_{-\infty} x^2 \frac{1}{2\theta} e^{-\frac{|x|}{\theta}}\, dx = \frac{1}{\theta}\int_0 x^2 e^{-\frac{x}{\theta}}\, dx = 2\theta^2$$
$$\mathrm{Var}[|X|] = \theta^2 \to \mathrm{Var}[\hat{\theta}] = \frac{\theta^2}{n}$$
$$err_{MS}(\hat{\theta}) = \left(E[\hat{\theta}] - \theta\right)^2 + \mathrm{Var}[\hat{\theta}] = \frac{\theta^2}{n}$$

۶

اگر $P(\theta;\alpha)$ برای توزیع $P(x \mid \theta)$ یک conjugate prior باشد:

$$P(\theta \mid x) = \gamma(\alpha)P(x \mid \theta)P(\theta;\alpha)$$

که $\gamma(\alpha) = \int P(x \mid \theta')\, P(\theta';\alpha)d\theta'$

فرض کنیم $P(\theta;\alpha)$ از خانواده $D(\alpha)$ باشد. جنس توزیع $\sum_{i=1}^m \beta_i P(\theta;\alpha_i)$ را با $D'(\alpha,\beta)$ نشان می‌دهیم.

$$P(\theta \mid x) = \frac{P(x \mid \theta)P(\theta;\alpha,\beta)}{P(x)} = \frac{[\sum_i \beta_i P(\theta;\alpha_i)]P(x \mid \theta)}{\int P(x \mid \theta')P(\theta';\alpha,\beta)d\theta'} = \frac{\sum_i \beta_i P(x \mid \theta)\mathrm{P}(\theta;\alpha_i)}{\int \sum_i \beta_i P(x \mid \theta')P(\theta';\alpha_i)d\theta'}$$
$$= \frac{\sum_i \beta_i P(x \mid \theta)\mathrm{P}(\theta;\alpha_i)}{\sum_i \underbrace{\int \beta_i P(x \mid \theta')P(\theta';\alpha_i)d\theta'}_{\beta_i/\gamma_i(\alpha_i)}} = \frac{\sum_{i=1}^m \beta_i D(\alpha'_i)/\gamma_i(\alpha_i)}{\sum_{i=1}^m \beta_i/\gamma_i(\alpha_i)} = \sum_{i=1}^m \frac{\beta_i}{\left(\sum_{j=1}^m \beta_j/\gamma_j(\alpha_j)\right)\gamma_i(\alpha_i)}\mathrm{D}(\alpha'_i)$$
$$= \sum_{i=1}^m \beta'_i \mathrm{D}(\alpha'_i)$$

با توجه به اینکه عبارت آخر توزیع $D'(\alpha',\beta')$ دارد، توزیع $\sum_{i=1}^m \beta_i P(\theta;\alpha_i)$ یک *conjugate prior* برای $P(x \mid \theta)$ است.

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$D = \{x^{(1)},\dots,x^{(n)}\}$ و نمونه‌ها را بصورت یک بردار شش‌تایی که اگر Category j-ام اتفاق افتاده باشد $x_j^{(i)} = 1$ و بقیه را صفر در نظر می‌گیریم.

$$P\left(x_i^{(n+1)} = 1 \mid D\right) = \int P\left(x_i^{(n+1)} = 1 \mid \beta\right)P(\beta \mid D;\alpha_1,\alpha_2) = E_{P(\beta|D;\alpha_1,\alpha_2)}\left[P\left(x_i^{(n+1)} = 1 \mid \beta\right)\right]$$
$$= E_{P(\beta|D;\alpha_1,\alpha_2)}[\beta_i] = E[\beta_i \mid D;\alpha_1,\alpha_2]$$

تعریف می‌کنیم:

$$\sum_{n=1}^N x_i^{(n)} = m_i, \qquad \frac{1}{\beta_{(\alpha)}} = \frac{\Gamma(\sum_{=1}^6 \alpha_i)}{\prod_{i=1}^6 \Gamma_{(\alpha_i)}}$$

$$P(\beta \mid D;\alpha_1,\alpha_2) = \frac{P(D \mid \beta)P(\beta;\alpha_1,\alpha_2)}{P(D) = \int P(D \mid \beta')P(\beta';\alpha_1,\alpha_2)d\beta'}$$
$$= \frac{1/3 \times \frac{1}{\beta(\alpha_1)}\prod_{i=1}^6 \beta_i^{m_i+\alpha_{1,i}-1} + \frac{2}{3}\times\frac{1}{\beta(\alpha_2)}\prod_{i=1}^6 \beta_i^{m_i+\alpha_{2,i}-1}}{1/3 \times \frac{1}{\beta(\alpha_1)}\int \prod_{i=1}^6 {\beta'}_i^{m_i+\alpha_{1,i}-1}d\beta' + \frac{2}{3\beta_i(\alpha_2)}\int \beta_i^{m_i+\alpha_{2,i}-1}d\beta'}$$

$$\frac{1}{3}\times\frac{1}{\beta(\alpha_1)}\int \prod_{i=1}^6 \beta_i^{m_i+\alpha_{1,i}-1}d\beta' = \frac{\beta(\alpha_1+\vec{m})}{3\beta(\alpha_1)}$$

$$\frac{2}{3\beta_i(\alpha_2)}\int\prod_{i=1}^{6}{\beta'}_i^{m_i+\alpha_{2,i}-1}d\beta' = \frac{2\beta(\alpha_2+\vec{m})}{3\beta(\alpha_2)}$$

$$P(\beta \mid D;\alpha_1,\alpha_2) = \frac{1/3\times\frac{1}{\beta(\alpha_1)}\prod_{i=1}^{6}\beta_i^{m_i+\alpha_{1,i}-1}+\frac{2}{3}\times\frac{1}{\beta(\alpha_2)}\prod_{i=1}^{6}\beta_i^{m_i+\alpha_{2,i}-1}}{\frac{\beta(\alpha_1+\vec{m})\beta(\alpha_2)+2\beta(\alpha_1)\beta(\alpha_2+\vec{m})}{3\beta(\alpha_1)\beta(\alpha_2)}} =$$

$$= \frac{\beta(\alpha_2)}{\beta(\alpha_1+\vec{m})\beta(\alpha_2)+2\beta(\alpha_1)\beta(\alpha_2+\vec{m})}\beta(\alpha_1+\vec{m})\mathrm{Dir}(\alpha_1+\vec{m})$$
$$+ \frac{2\beta_1(\alpha_1)}{\beta(\alpha_1+\vec{m})\beta(\alpha_2)+2\beta(\alpha_1)\beta(\alpha_2+m)}\beta(\alpha_2+m)\mathrm{Dir}(\alpha_2+\vec{m})$$

$$E[\beta_i \mid D;\alpha_1,\alpha_2]$$
$$= \frac{\beta(\alpha_2)\beta(\alpha_1+\vec{m})}{\beta(\alpha_1+\vec{m})\beta(\alpha_2)+2\beta(\alpha_1)\beta(\alpha_2+\vec{m})}\frac{m_i+\alpha_{1,i}}{\mathrm{N}+\sum_{i=1}^{6}\alpha_{1,i}}$$
$$+ \frac{2\beta_1(\alpha_1)\beta(\alpha_2+m)}{\beta(\alpha_1+\vec{m})\beta(\alpha_2)+2\beta(\alpha_1)\beta(\alpha_2+m)}\frac{m_i+\alpha_{2,i}}{\mathrm{N}+\sum_{i=1}^{6}\alpha_{2,i}} = P\left(x_i^{(n+1)}=1 \mid D\right)$$