Stochastic Processes

Week 02 (Version 1.0)

Stochastic Processes
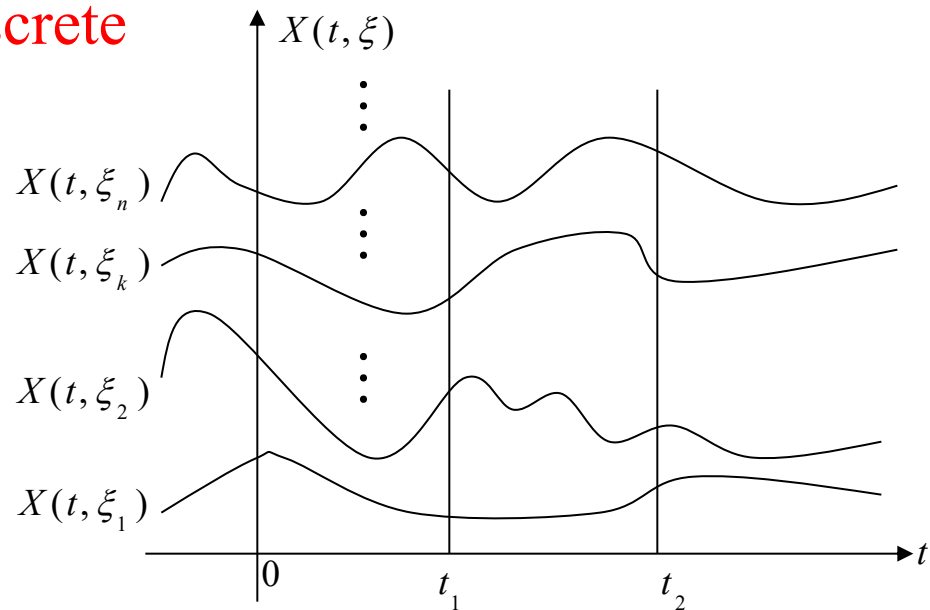
Stationary Stochastic Processes

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Outline of Week 02 Lectures

- Stochastic Processes X(t)

- First Order Statistics of X(t)

- Higher Order Statistics of X(t)

- Mean, Autocorrelation and Autocovariance of X(t)

- Stationary Stochastic Processes

- Strict-Sense Stationary (S.S.S) Processes

- Wide-Sense Stationary (W.S.S) Processes

Stochastic Processes

- Let ξ denote the random outcome of an experiment.
- To every such outcome suppose a waveform $X(t,\xi)$ is assigned.
  The collection of such waveforms form a stochastic process.

- The set of $\{\xi_k\}$ and the time
  index t can be continuous or discrete
  (countably infinite or finite)

Stochastic Processes

- For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t, \xi)$ is a specific time function. For fixed t, $X_1 = X(t_1, \xi_i)$ is a random variable

- The ensemble of all such realizations $X(t, \xi)$ over time represents the stochastic process $X(t)$
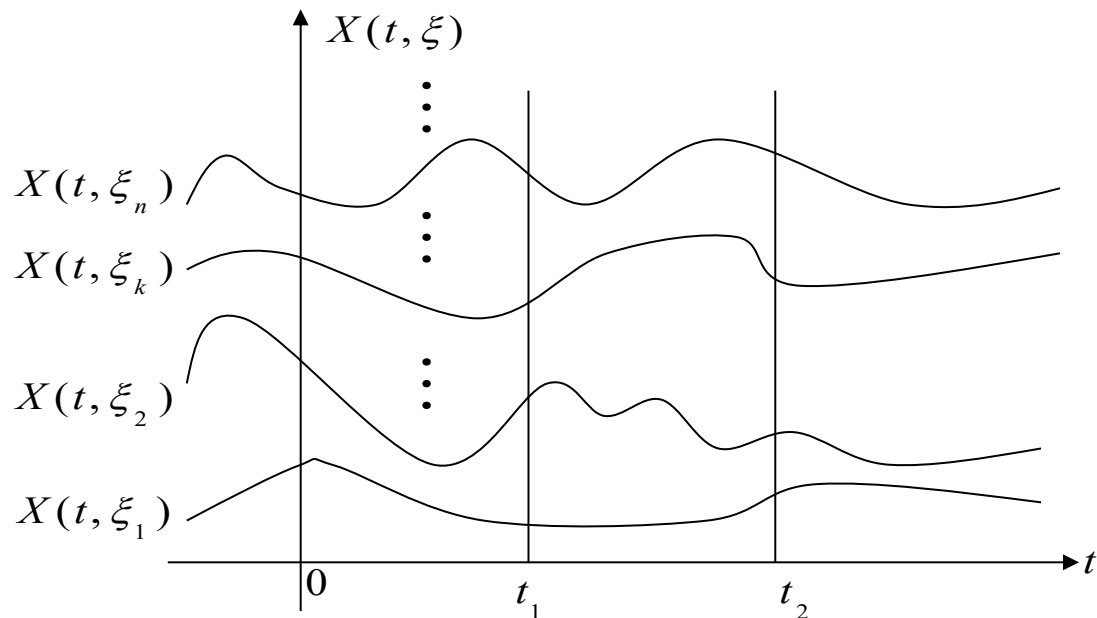
Stochastic Processes (Introduction)

Examples:

- Let $X(t) = a\cos(\omega_0 t + \varphi)$,
  where φ is a uniformly distributed random variable in $(0, 2\pi)$, represents a stochastic process.
- Stochastic processes are everywhere:
  - stock market fluctuations
  - various queuing systems
  - Earthquake Signals
  - 1-D Audios
  - 2-D Images
  - 3-D Videos

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Stochastic Processes (Density Function)

• If $X(t)$ is a stochastic process, then for fixed t, $X(t)$ represents a random variable. Its distribution function is given by:

$$F_X(x,t) = P\{X(t) \leq x\}$$

• Notice that $F_X(x,t)$ depends on t, since for a different t, we obtain a different random variable. Further:

$$f_X(x,t) \underset{\Delta}{=} \frac{dF_X(x,t)}{dx}$$

represents the first-order probability density function (pdf) of the process $X(t)$

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Stochastic Processes (Density Function)

- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively. Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

$$f_X(x_1, x_2, t_1, t_2) \overset{\Delta}{=} \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \, \partial x_2}$$

represents the second-order density function of the process $X(t)$

Stochastic Processes (Density Function of the Process)

Similarly $f_X(x_1, x_2, \cdots x_n, \; t_1, t_2 \cdots, t_n)$ represents the nth order density function of the process $X(t)$.

- Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_X(x_1, x_2, \cdots x_n, \; t_1, t_2 \cdots, t_n)$
  for all $t_i,$ $i = 1, 2, \cdots, n$ and for all n.
  (an almost impossible task in reality).

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Stochastic Processes (Mean and Autocorrelation)

Mean of a Stochastic Process:

$$\mu(t) \triangleq E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx$$

represents the mean value of a process $X(t)$. In general, the mean of a process can depend on the time index t.

Autocorrelation function of a process $X(t)$ is defined as:

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\} = \int\int x_1 x_2^* f_X(x_1,x_2,t_1,t_2)dx_1 dx_2$$

and it represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ generated from the process $X(t)$.

Stochastic Processes (Properties)

1. $R_{XX}(t_1, t_2) = R_{XX}^*(t_2, t_1) = [E\{X(t_2)X^*(t_1)\}]^*$

2. $R_{XX}(t, t) = E\{|X(t)|^2\} > 0.$ (Average instantaneous power)

3. $R_{XX}(t_1, t_2)$ represents a nonnegative definite function, i.e., for *any* set of constants $\{a_i\}_{i=1}^n$

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j^* R_{XX}(t_i, t_j) \geq 0.$$

$$E\{|Y|^2\} \geq 0 \quad \text{for} \quad Y = \sum_{i=1}^n a_i X(t_i).$$

Stochastic Processes (Autocovariance)

Autocovariance:

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

represents the **autocovariance** function of the process $X(t)$.

Example 1

Let $z = \int_{-T}^{T} X(t) dt.$

Then

$$E[|z|^2] = \int_{-T}^{T} \int_{-T}^{T} E\{X(t_1) X^*(t_2)\} dt_1 dt_2$$

$$= \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) dt_1 dt_2$$

Example 2

$$X(t) = a \cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0, 2\pi).$$

This gives

$$\mu_X(t) = E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\}$$
$$= a \cos \omega_0 t \, E\{\cos \varphi\} - a \sin \omega_0 t \, E\{\sin \varphi\} = 0,$$

since $E\{\cos \varphi\} = \frac{1}{2\pi} \int_0^{2\pi} \cos \varphi \, d\varphi = 0 = E\{\sin \varphi\}$.

Similarly

$$R_{XX}(t_1, t_2) = a^2 E\{\cos(\omega_0 t_1 + \varphi) \cos(\omega_0 t_2 + \varphi)\}$$
$$= \frac{a^2}{2} E\{\cos \omega_0 (t_1 - t_2) + \cos(\omega_0 (t_1 + t_2) + 2\varphi)\}$$
$$= \frac{a^2}{2} \cos \omega_0 (t_1 - t_2).$$

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Stationary Stochastic Processes

- Stationary processes exhibit statistical properties that are invariant to shift in the time index.

- For example, second-order stationarity implies that the statistical properties of the pairs
  $\{X(t_1), X(t_2)\}$ and $\{X(t_1+c), X(t_2+c)\}$ are the same for *any c*.

- Similarly first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i+c)$ are the same for any c.

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Strict-Sense Stationary (S.S.S)

- In strict terms, the statistical properties are governed by the joint probability density function. Hence a process is nth-order **Strict-Sense Stationary (S.S.S)** if, for *any c:*

$$f_X(x_1, x_2, \cdots x_n, \ t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n, \ t_1 + c, t_2 + c \cdots, t_n + c)$$

where the left side represents the joint density function of the random variables $X_1 = X(t_1), \ X_2 = X(t_2), \ \cdots, \ X_n = X(t_n)$ and the right side corresponds to the joint density function of the random variables $X_1' = X(t_1 + c), \ X_2' = X(t_2 + c), \ \cdots, \ X_n' = X(t_n + c).$

- A process $X(t)$ is said to be **strict-sense stationary** if the equation is true for all $t_i, \ \ i = 1, 2, \cdots, n, \ \ n = 1, 2, \cdots$ and *any c.*

Strict-Sense Stationary (S.S.S)

For a **first-order strict sense stationary process**, from the equation we have

$$f_X(x,t) \equiv f_X(x, t+c)$$

for any c. In particular $c = -t$ gives

$$f_X(x,t) = f_X(x)$$

i.e., the first-order density of $X(t)$ is independent of t. In that case

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x) dx = \mu, \quad a\ constant.$$

Strict-Sense Stationary (S.S.S)

Similarly, for a **second-order strict-sense stationary process** we have from the equation

$$f_X(x_1, x_2, \ t_1, t_2) \equiv f_X(x_1, x_2, \ t_1 + c, t_2 + c)$$

for any c. For $c = - t_2$ we get

$$f_X(x_1, x_2, \ t_1, t_2) \equiv f_X(x_1, x_2, \ t_1 - t_2)$$

Strict-Sense Stationary (S.S.S)

- The second order density function of a strict sense stationary process depends only on the difference of the time indices $t_1 - t_2 = \tau$. In that case the autocorrelation function is given by

$$R_{XX}(t_1, t_2) \triangleq E\{X(t_1)X^*(t_2)\}$$

$$= \iint x_1 x_2^* f_X(x_1, x_2, \tau = t_1 - t_2)dx_1 dx_2$$

$$= R_{XX}(t_1 - t_2) \triangleq R_{XX}(\tau) = R_{XX}^*(-\tau),$$

- The autocorrelation function of a second order strict-sense stationary process depends only on the difference of the time indices $\tau = t_1 - t_2$.

Strict-Sense Stationary (S.S.S)

- Notice that the above equations are consequences of the stochastic process being first and second-order strict sense stationary.
  On the other hand, the basic conditions for the nth order stationarity are usually difficult to verify.

- In that case, we often resort to a looser definition of stationarity, known as **Wide-Sense Stationarity (W.S.S).**

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Wide-Sense Stationary (W.S.S)

A process $X(t)$ is said to be **Wide-Sense Stationary** if:

(i) $E\{X(t)\} = \mu$

and

(ii) $E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1 - t_2),$

- For wide-sense stationary processes, the mean is a constant and the autocorrelation function depends only on the difference between the time indices.

- Notice that above equations does not say anything about the nature of the probability density functions, and instead deal with the average behavior of the process.

Wide-Sense Stationary (W.S.S)

- Strict-sense stationarity always implies wide-sense stationarity.

- The converse is *not true* in general, the only exception being the Gaussian process.

This follows, since if $X(t)$ is a Gaussian process, then by definition $X_1 = X(t_1), X_2 = X(t_2), \cdots, X_n = X(t_n)$ are jointly Gaussian random variables for any $t_1, t_2 \cdots, t_n$ whose joint characteristic function is given by:

$$\phi_{\underline{X}}(\omega_1, \omega_2, \cdots, \omega_n) = e^{j\sum_{k=1}^{n} \mu(t_k)\omega_k - \sum_{l,k}^{n}\sum C_{xx}(t_i, t_k)\omega_i\omega_k / 2}$$

WSS, Gaussian Process, Cont.

If $X(t)$ is wide-sense stationary, we get

$$\phi_{\underline{X}}(\omega_1, \omega_2, \cdots, \omega_n) = e^{\, j\sum_{k=1}^{n} \mu\omega_k - \frac{1}{2}\sum_{1=1}^{n}\sum_{k=1}^{n} C_{xx}(t_i - t_k)\omega_i\omega_k}$$

and hence if the set of time indices are shifted by a constant c to generate a new set of jointly Gaussian random variables
$X'_1 = X(t_1 + c), X'_2 = X(t_2 + c), \cdots, X'_n = X(t_n + c)$ then their joint characteristic function is identical to above.

WSS, Gaussian Process, Cont.

- Thus the set of random variables $\{X_i\}_{i=1}^n$ and $\{X_i'\}_{i=1}^n$ have the same joint probability distribution for all n and all c, establishing the strict sense stationarity of Gaussian processes from its wide-sense stationarity.

To summarize if $X(t)$ is a Gaussian process, then
wide-sense stationarity (w.s.s) $\longrightarrow$ strict-sense stationarity (s.s.s)

- Notice that since the joint p.d.f of Gaussian random variables depends only on their second order statistics, which is also the basis, for wide sense stationarity, we obtain strict sense stationarity as well.

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the process

$X(t) = a\cos(\omega_0 t + \varphi),$ is wide-sense stationary, but not strict-sense stationary.

Similarly if $X(t)$ is a zero mean wide sense stationary process, then σ_z^2 reduces to:

$$\sigma_z^2 = E\{|z|^2\} = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)\,dt_1\,dt_2.$$

As t_1, t_2 varies from $-T$ to $+T$, $\tau = t_1 - t_2$ varies from $-2T$ to $+2T$. Moreover $R_{XX}(\tau)$ is a constant over the shaded region in this Fig, whose area is given by $(\tau > 0)$

$$\frac{1}{2}(2T - \tau)^2 - \frac{1}{2}(2T - \tau - d\tau)^2 = (2T - \tau)d\tau$$

and hence the above integral reduces to

$$\sigma_z^2 = \int_{-2t}^{2T} R_{XX}(\tau)(2T - |\tau|)d\tau = \tfrac{1}{2T}\int_{-2t}^{2T} R_{XX}(\tau)(1 - \tfrac{|\tau|}{2T})d\tau.$$