Stochastic Processes Fall 2021
Sharif University of Technology CE 695-1
Dr. H.R. Rabiee

Time: 20 mins

Name: Std. Number:

Prerequisite Quiz (solutions)

Questions

1. X and Y are two independently distributed variables each having a uniform distribution on the interval [0,1]. Z being max[X,Y] and W, min[X,Y], what would E[Z-W] be?

2. Let X be a continuous random variable with PDF $f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & otherwise \end{cases}$ if $Y = \frac{2}{X} + 3$ find $E[Y]$ and $Var(Y)$

3. Let $X = \sim Uniform(\frac{-\pi}{2}, \pi)$ and $Y = \sin X$. find $f_Y(y)$.

1.

$X \to \text{uniform} \Rightarrow F_X(x) = \int_0^x \frac{1}{1}\, dx = x$

$Y \to \text{uniform} \Rightarrow F_Y(y) = \int_0^y \frac{1}{1}\, dy = y$

$P\{Z<z\} \Rightarrow P\{\max\{X, Y\} < z\} = P\{X<z, Y<z\} =$

$P\{X<z\}\, P\{Y<z\} = F_X(z)\, F_Y(z) = z^۲ \Rightarrow f_Z(z) = ۲z$

به طریق مشابه: $F_W(\omega) = 1 - (1 - F_X(\omega))(1 - F_Y(\omega)) = 1 - (1-\omega)(1-\omega) = ۲\omega - \omega^۲$

$\Rightarrow f_\omega(\omega) = ۲ - ۲\omega$

1- $E[Z] = \int_0^1 z\, f_Z(z)\, dz = \frac{۲}{۳}$

۲- $E[\omega] = \int_0^1 \omega\, f_\omega(\omega)\, d\omega = \frac{1}{۳}$

$\Rightarrow E[Z - \omega] = \frac{۲}{۳} - \frac{1}{۳} = \frac{1}{۳}$

2.

First, note that

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{2}{X}+3\right) = 4\mathrm{Var}\left(\frac{1}{X}\right),$$
$$\text{using Equation 4.4}$$

Thus, it suffices to find Var $(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2$. Using LOTUS, we have

$$E\left[\frac{1}{X}\right] = \int_0^1 x\left(2x+\frac{3}{2}\right)dx = \frac{17}{12}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \left(2x+\frac{3}{2}\right)dx = \frac{5}{2}.$$

Thus, Var$(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2 = \frac{71}{144}$. So, we obtain

$$\mathrm{Var}(Y) = 4\mathrm{Var}\left(\frac{1}{X}\right) = \frac{71}{36}.$$

3. Here $Y = g(X)$, where g is a differentiable function. Although g is not monotone, it can be divided to a finite number of regions in which it is monotone. Thus, we can use Equation 4.6. We note that since $R_X = [-\frac{\pi}{2}, \pi]$ $R_Y = [-1, 1]$. By looking at the plot of $g(x) = \sin(x)$ over $[-\frac{\pi}{2}, \pi]$, we notice that for $y \in (0, 1)$ there are two solutions to $y = g(x)$, while for $y \in (-1, 0)$, there is only one solution

In particular, if $y \in (0, 1)$, we have two solutions: $x_1 = \arcsin(y)$, and $x_2 = \pi - \arcsin(y)$. If $y \in (-1, 0)$ we have one solution, $x_1 = \arcsin(y)$. Thus, for $y \in (-1, 0)$, we have

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} \\
&= \frac{f_X(\arcsin(y))}{|\cos(\arcsin(y))|} \\
&= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}}.
\end{aligned}
$$

For $y \in (0, 1)$, we have

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\
&= \frac{f_X(\arcsin(y))}{|\cos(\arcsin(y))|} + \frac{f_X(\pi-\arcsin(y))}{|\cos(\pi-\arcsin(y))|} \\
&= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} + \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} \\
&= \frac{4}{3\pi\sqrt{1-y^2}}.
\end{aligned}
$$

To summarize, we can write

$$
f_Y(y) = \begin{cases} \frac{2}{3\pi\sqrt{1-y^2}} & -1 < y < 0 \\ \frac{4}{3\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
$$