Stochastic Processes
Sharif University of Technology — CE 40-695
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Time: 20 mins

Name: Std. Number:

Quiz 9 (Markov Chains and HMMs)

1. A guy is practicing basketball and makes a shot once a minute. There is $\frac{1}{3}$ that he scores and if he does, he will gain one dollar, otherwise he loses a dollar. And if he loses all his money, he will borrow one dollar.

   (a) Formulate the model for the money the guy has so that you obtain a homogeneous Markov chain.

   (b) find the chain's transition matrix and classify its states.

   (c) What is the stationary distribution? (Extra)

2. Consider the following matrices. For the matrices that are transitition matrices draw the associated markov chain and obtain the steady state probabilities. If they do not exist, explain why.

$$a. \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ b. \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \ c. \begin{pmatrix} -a & a & 0 \\ b & -(a+b) & a \\ 0 & b & -b \end{pmatrix}$$

Solution:

1. (a) Let $Y = (Y_i, i \in \mathbb{N})$ be a sequence of independent, identically random variables with values in $-1, 1$ defined on a probability space. Let their common distribution be: $\mathbb{P}(Y_1 = -1) = \frac{2}{3}, \mathbb{P}(Y_1 = 1) = \frac{1}{3}$.
   Define $X_0 := 0$ and $X_{n+1} := \mathbf{1}_{[X_n=0]} + \mathbf{1}_{[X_n \neq 0]}(X_n + Y_{n+1}), n = 0, 1, ...$

   (b) X_n represents the money the guy has at minute n.. Then $X = (X_n, n \in \mathbb{N})$ is a homogenous discrete time Markov chain with the state space $S = \mathbb{N}_0$. Its transition matrix is as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & \dots \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \dots \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \dots \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

   (c) X admits a stationary distribution $\pi_0 = \frac{1}{4}$ and $\pi_k = \frac{3}{4}(\frac{1}{2})^k$ for $k = 1, 2, ...$ All its states are 2-periodic and positive recurrent.

2. (a) only if $-a = b \geq 0$ and $-c = d \geq 0$

(b) $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

(c) $\pi = (\pi_1, \frac{a}{b}\pi_1, (\frac{a}{b})^2\pi_1)$

$\sum \pi_i = 1$

$\pi_1 = \frac{1}{1+\frac{a}{b}+(\frac{a}{b})^2}$