Time: 20 mins

Name: Std. Number:

Quiz 4 (Poisson Process, Point Process)

Questions

1. Let $X(t)$ be a Poisson process with rate λ.

   (a) (3 points) If we define a process $Y(t) = X(t) - \lambda t$, Is $Y(t)$ weakly stationary ? why?

   (b) (7 points) let assume that T is the time of the first event. Then, $P(T \leq s) = 1 - exp(-\lambda s)$. Show that for $0 < s < t$,

   $$P(T \leq s | X(t) = 1) = s/t$$

   Hint : $T \leq s$ is equivalent to $X(s) \geq 1$

It is not weakly stationary. The variance of $Y(t) = X(t) - \lambda t$ is the same as that of $X(t)$,

$$\mathsf{V}[Y(t)] = \mathsf{V}[X(t)] = \mathsf{E}[X(t)] = \lambda t,$$

which depends on t.

Obviously, the event that $T_1 \leq s$ is equivalent to $X(s) \geq 1$. Thus the conditional probability is

$$\begin{aligned}
\mathsf{P}(T_1 \leq s \mid X(t) = 1) &= \frac{\mathsf{P}(T_1 \leq s \text{ and } X(t) = 1)}{\mathsf{P}(X(t) = 1)} \\
&= \frac{\mathsf{P}(X(s) = 1)\mathsf{P}(X(t) - X(s) = 0)}{\mathsf{P}(X(t) = 1)} \\
&= \frac{e^{-\lambda s}(\lambda s)^1/1! \times e^{-\lambda(t-s)}(\lambda(t-s))^0/0!}{e^{-\lambda t}(\lambda t)^1/1!} = \frac{s}{t},
\end{aligned}$$

2. (10 points) Let $X(t) = N(t+1) - N(t)$ where $N(t), t \geq 0$ is a Poisson process with rate λ. Compute

   $$Cov[X(t), X(t+s)]$$

For $s < 1$

$$
\begin{aligned}
&Cov[X(t), X(t+s)] \\
&\quad = Cov[N(t+1) - N(t), N(t+s+1) - N(t+s)] \\
&\quad = Cov(N(t+1), N(t+s+1) - N(t+s)) \\
&\qquad - Cov(N(t), N(t+s+1) - N(t+s)) \\
&\quad = Cov(N(t+1), N(t+s+1) - N(t+s)) \quad (*)
\end{aligned}
$$

where the equality $(*)$ follows since $N(t)$ is independent of $N(t+s+1) - N(t+s)$. Now, for $s \leq t$,

$$
\begin{aligned}
Cov(N(s), N(t)) &= Cov(N(s), N(s) + N(t) - N(s)) \\
&= Cov(N(s), N(s)) \\
&= \lambda s
\end{aligned}
$$

Hence, from $(*)$ we obtain that, when $s < 1$,

$$
\begin{aligned}
Cov(X(t), X(t+s)) &= Cov(N(t+1), N(t+s+1)) \\
&\qquad - Cov(N(t+1), N(t+s)) \\
&= \lambda(t+1) - \lambda(t+s) \\
&= \lambda(1-s)
\end{aligned}
$$

When $s \geq 1, N(t+1) - N(t)$ and $N(t+s+1) - N(t+s)$ are, by the independent increments property, independent and so their covariance is 0.