Stochastic Processes November 9, 2021
Sharif University of Technology CE 40-695
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Time: 20 mins

Name: Std. Number:

Quiz 1 (Solutions)

Questions

1. A pair of jointly continuous random variables, X and Y , have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \textit{in the shaded region of Figure. 1} \\ 0, & \textit{Otherwise.} \end{cases}$$

   (a) (5 points) Find c.
   (b) (5 points) Find the marginal PDFs of X and Y, i.e., $f_X(x)$ and $f_Y(y)$.
   (c) (5 points) find $E[X|Y = 1/4]$ and $Var[X|Y = 1/4]$, that is, the conditional mean and conditional variance of X given Y=1/4
   (d) (5 points) Find the conditional PDF for X given that $Y = 3/4$, i.e., $f_{X|Y}(x|3/4)$.

2. Are all memoryless systems causal? If the answer is no, give a counterexample.

3. Are all causal systems memoryless? If the answer is no, give a counterexample.

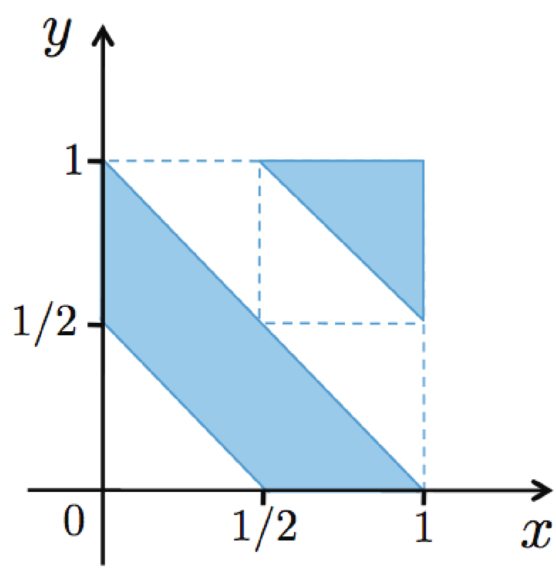

Figure 1

Solutions:
Question2: yes Question3: no

Question 1: (20 points)

(a) (5 points)
We're given that the joint PDF is constant in the shaded region, and since the PDF must integrate to 1, we know that the constant must equal 1 over the area of the region. Thus,

$$c = \frac{1}{1/2} = 2.$$

(b) (5 points)
The marginal PDFs of X and Y are found by integrating the joint PDF over all possible y's and x's, respectively. To find the marginal PDF of X, we take a particular value x and integrate over all possible y values in that vertical "slice" at $X = x$. Since the joint PDF is constant, this integral simplifies to just multiplying the joint PDF by the width of the "slice". Because the width of the slice is always $1/2$ for any $x \in [0, 1]$, we have that the marginal PDF of X is uniform over that interval:

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since the joint PDF is symmetric, the marginal PDF of Y is also uniform:

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) (5 points)
To find the conditional expectation and variance, first we need to determine what the conditional distribution is given $Y = 1/4$. At $Y = 1/4$, we take a horizontal slice of a uniform joint PDF, which gives us a uniform distribution over the interval $x \in [1/4, 3/4]$. Thus, we have

$$\mathbf{E}[X \mid Y = 1/4] = \frac{1}{2},$$

$$\text{var}(X \mid Y = 1/4) = \frac{(1/2)^2}{12} = \frac{1}{48}.$$

(d) (5 points)
At $Y = 3/4$, we have a horizontal slice of the joint PDF, which is nonzero when $x \in [0, 1/4] \cup [3/4, 1]$. Since the joint PDF is uniform, the slice will also be uniform, but only in the range of x where the joint PDF is nonzero (i.e. where (x, y) lies in the shaded region). Thus, the conditional PDF of X is

$$f_{X|Y}(x \mid 3/4) = \begin{cases} 2, & x \in [0, 1/4] \cup [3/4, 1], \\ 0, & \text{otherwise.} \end{cases}$$