Universal Traversals

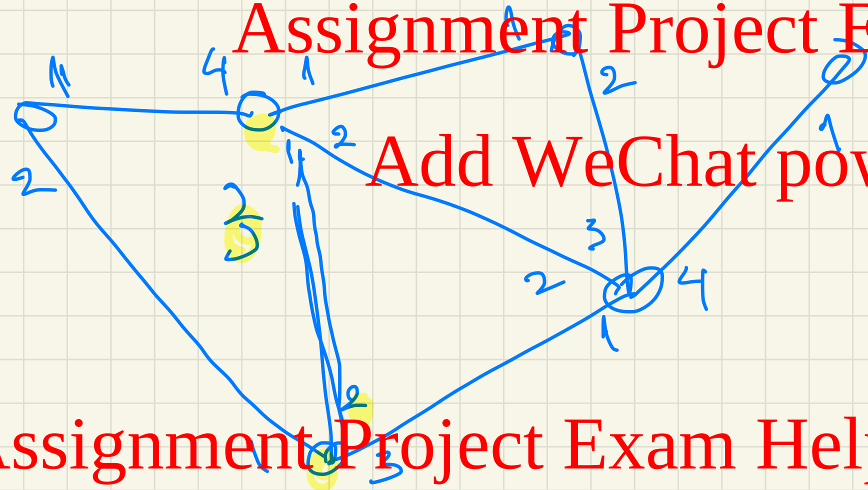
Labeling Schemes

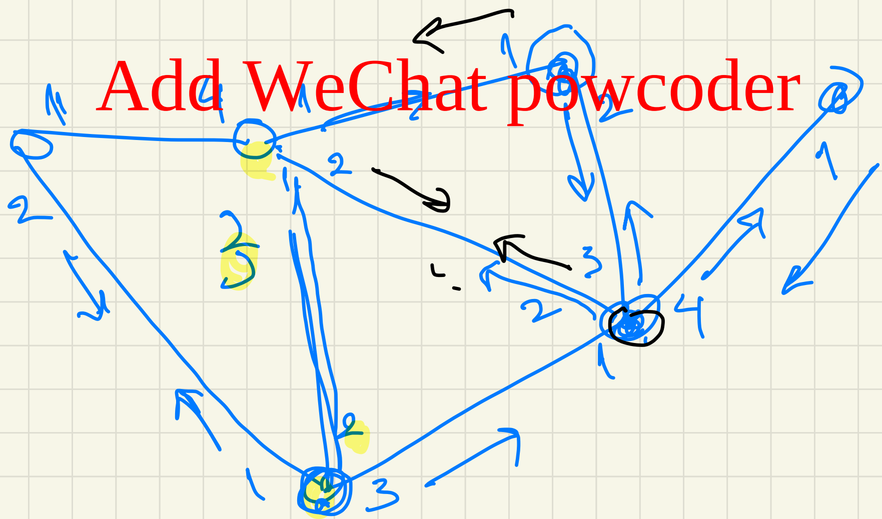
- How can we use port labeling schemes to improve communication? Give different names to the nodes

- What role do port labeling schemes play in distributed computing?

- The execution of a distributed algorithm at a node depends on the sequence of port labels that must be followed by the distributed algorithm at that node.

- Can we solve a problem in a way that all nodes follow identical sequences at each node?





At each node follow the sequence of labels

1 2 3 4 1 2 1 3

Each such sequence gives me an execution

Outline

- Probabilistic Method

- Universal traversals

Must define a sequence of port labels that will be universal!

If you think of the sequence of labels as a program

Probabilistic Method

- Using probability to prove the existence of a mathematical object is called the Probabilistic Method.

- It has many applications, especially in graph theory.
- It uses the following principle:

> If, in a given set of objects, the probability that a randomly chosen object does not have a certain property is less than 1 then then there must exist an object with this property.

If $\Pr[\neg E] < 1$ then $E \neq \emptyset$

E = set of objects

Diff. Way: $\Pr[E] > 0$ then $E \neq \emptyset$

If $E = \emptyset$ then $\Pr[\emptyset] = 0$

$\Rightarrow \Pr[\neg \emptyset] = 1$

$\Rightarrow \Pr[\neg E] = 1$

i.e. showing that $\Pr[\neg E] < 1$

implies that an object in E must exist,

Union Form of the Probabilistic Method

- Consider n events $A_1, A_2, \ldots, A_n$ (not necessarily independent).

- The union (or Boole) Inequality states that

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$$

{from prob. theory}

- Therefore if we want to prove that

$$\bigcap_{i=1}^{n} (\neg A)_i \neq \emptyset$$

it is enough to show that

$$\underbrace{\Pr\left[\bigcup A_i\right]}_{<1} \leq \sum_{i=1}^{n} \Pr[A_i] < 1.$$

Expectation Form of the Probabilistic Method

- Consider an integer valued random variable X which takes only non-negative integer values.

$X = 0, 1, 2, \ldots$

- Observe that

union rule

$$
\begin{aligned}
\Pr[X > 0] &= \sum_{k>0} \Pr[X = k] \\
&\le \sum_{k>0} k \Pr[X = k] \\
&= E[X].
\end{aligned}
$$

Therefore

$$\Pr[X > 0] \le E[X] \tag{1}$$

Expectation Form of the Probabilistic Method

- Equation (1) is a special cas of Markov's inequality which states that

$$\Pr[X > kE[X]] \leq \frac{1}{k}$$

- Therefore using Equation (1) if we want to prove that

$$\Pr[X = 0] > 0$$

Exercise

it is enough to prove that

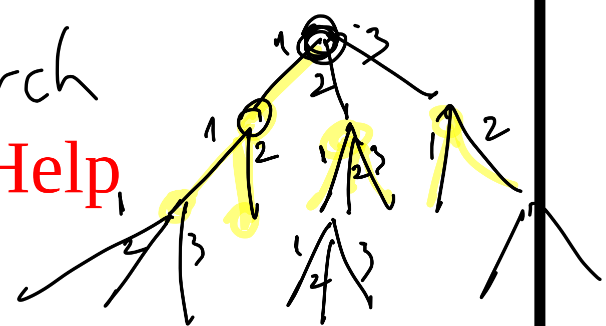
$$E[X] < 1.$$

Traversals

Explorations

- Graph traversal (also known as graph search) refers to the process of visiting (checking and/or updating) each vertex in a graph. E.g.,

  - BFS Breadth first search
  - DFS Depth



- Used in Search and Exploration.

- Each starting node is equipped with a program

  - typically a sequence of port labels that it must follow from node to node) which is used to traverse the graph.

- However, the program used may depend on the starting node.

of labels

Universal Traversals

- A sequence is universal for graphs with n vertices if for every graph and every start vertex, the walk defined by the (same) sequence will visit every vertex in the graph.

- Can you produce a universal traversal program that will work for every graph and every starting node of the graph?

- To produce a walk defined by a sequence, we need some notion of graph labeling.

  – For each vertex u, label the edges adjacent to u (ports) from 1 to $\deg(u)$ (in fact any numbering will do).
  – This is what we defined as port labelings!

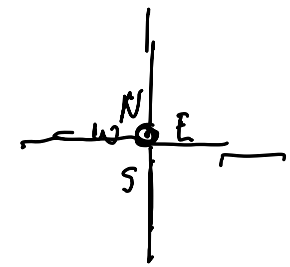
- Then a sequence is a string of edge labels which determines some walk through the graph.

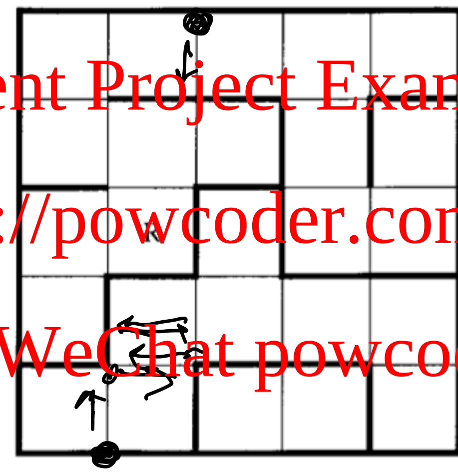
Universal Traversals on Labyrinths

- A robot is placed in a labyrinth in a $n \times n$ square grid.

- It runs a program: a sequence of commands of the form $N(orth)$, $S(outh)$, $E(ast)$, $W(est)$,

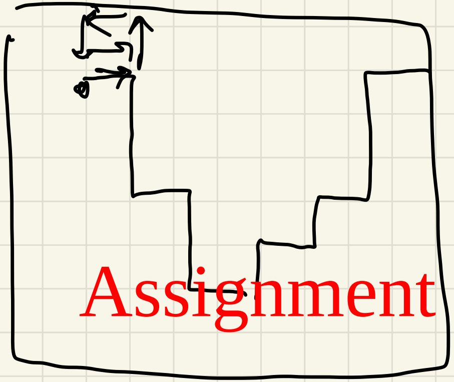
Port labels





- As an example consider the program $NESWEW$ which is given to every node.

- A robot has the sequence "$NESWEW$" and starting at a node makes movements following the sequence of labels.



fixed

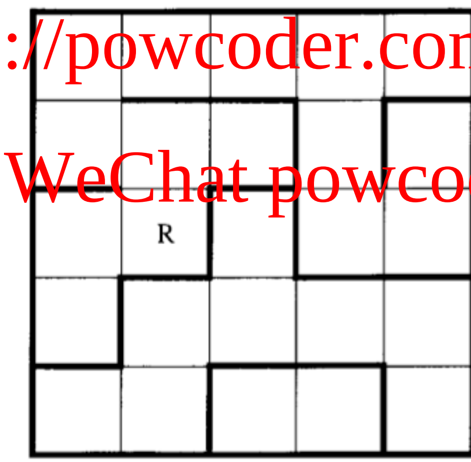
ENSWNNSW

Follow a trajectory determined by the sequence of labels given to me.

Universal Traversals

- In addition to external walls on the whole perimeter, walls are also placed between cells.

- Executing each command, the robot moves in the prescribed direction if possible (and does nothing when there is a wall in this direction).



E.g., $NESWEEEW$

Universal Traversals

n^7

- We can show that

Theorem 1 (Universal Traversals) *For any n, there exists a program that works correctly for all labyrinths of size at most $n \times n$ (independently of the positions of walls inside the square and the robot's initial position).*

- "Works correctly" means that the robot visits all reachable cells.

- To solve the traversal problem,

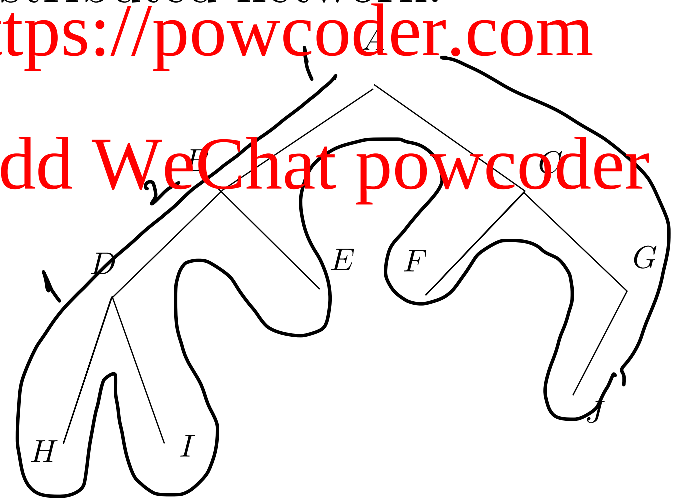
  we prove that a sufficiently long random program will work with positive probability.

- We will do this using the union bound.

Preorder Traversals (1/2)

- For each $n \times n$ labyrinth, there is a program of size $4n^4$ that works for it, as each cell is reachable in at most $4n^2$ steps (round-trip) and there are at most n^2 admissible cells.

- To prove this note that for each starting cell there is a spanning tree reaching all the n^2 cells of a given labyrinth, think of it as a distributed network.



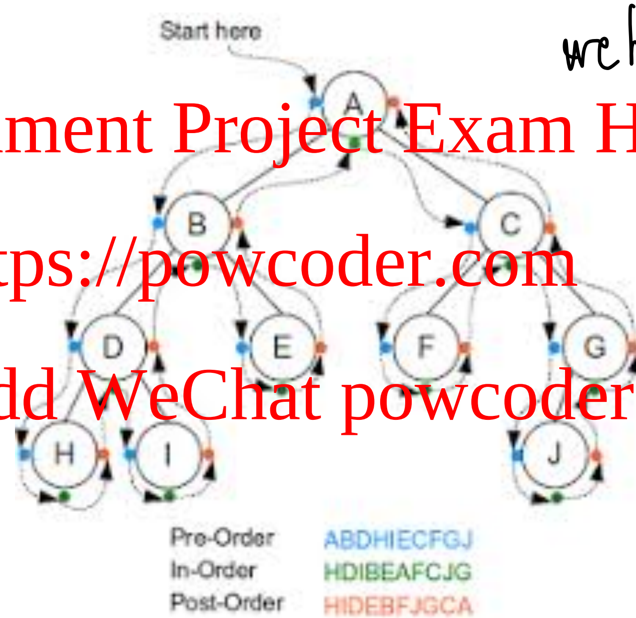
- Assign ports to each vertex (edge labels associated to the edges connected)

Preorder Traversals (2/2)

- Using a pre-order traversal of this spanning tree (which has length at most $4n^2$) every cell will be visited in at most $4n^2$ steps (round trip).

For every vertex v we have a traversal of $4n^2$ labels

$n^2 \times 4n^2$



- Therefore, a random program of size $N = 4n^4$ will work with probability at least $\epsilon = (1/4)^{4n^4}$ and fail with probability at most $1 - \epsilon$.

Universal Traversals

- Select among such programs of size $N = 4n^2$ independently and uniformly at random.

- Now for each k concatenate k such programs:

$$P_1 P_2 \cdots P_k$$

k times the size of P_i

  – By independence, a random program of size $2N$ will fail with probability at most $(1-\epsilon)^2$.

  – More generally, a random program of size kN will fail with probability at most $(1-\epsilon)^k$.

- This probability is computed for a fixed labyrinth L.
- Let F_L be the event that a program of size kN fails for the labyrinth L.
- It follows from the above that $\Pr[F_L] \leq (1-\epsilon)^k$.

Universal Traversals

- Now take the union $\bigcup_L F_L$, where L runs over all labyrinths. As a consequence,

$$\Pr\left[\bigcup_L F_L\right] \le \sum_L \Pr[F_L]$$

$$\le \ell_n (1-\epsilon)^k \quad < 1$$

where ℓ_n is the number of labyrinths in a $n \times n$ grid.

- However, we can choose k sufficiently large, so that

$$\ell_n < \frac{1}{(1-\epsilon)^k} \tag{2}$$

- So,

$$\Pr\left[\bigcup_L F_L\right] < 1$$

$$\underbrace{\Pr\left[\bigcap_L \neg F_L\right] > 0}_{\text{a program must exist.}}$$

Universal Traversals

- In particular, for a k satisfying Inequality (2), we have that

$$\Pr\left[\bigcap_L \neg F_L\right] = 1 - \Pr\left[\bigcup_L F_L\right] > 0$$

- Therefore for k sufficiently large a random program of size kN works for all labyrinths with positive probability.

- Therefore such a program must exist!

Efficiency of Universal Traversals

- How about the length of the sequence of port labels?
  - Can we construct a universal traversal sequence of polynomial length in polynomial time (in the size n of the graph)?

- How about efficiency?
  - Can we give an algorithm to construct a universal traversal sequence?

- Can we make the construction distributed?

It is known how to give algorithmic proofs of this but they are quite complex!

Universal Traversals on Graphs

- The Universal Traversal theorem holds on graphs of a given size n.

- Instead of N, S, E, W used in $n \times n$ grids one now uses ports and port-labelings.

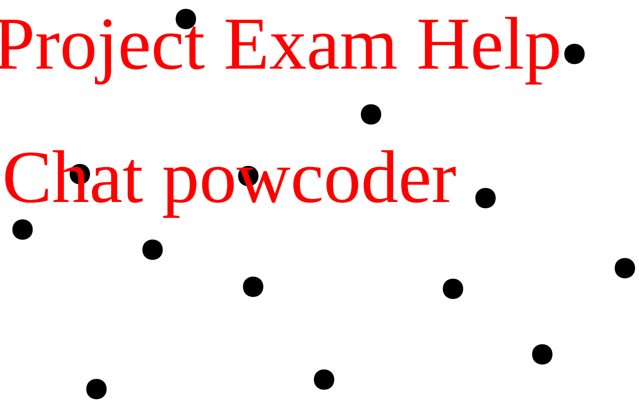
Exercises[a]

1. The traversal of the labyrinth was based on a sequence that was provided to the robot and which the robot must follow. Consider the situation where the robot constructs the sequence "on the fly": looks at the surrounding environment and based on what it sees makes its next move according to some rule. This will be a local algorithm and the moves of the robot depend on the environment. Will such an algorithm perform a successful traversal?

2. ($\star$) Use the probabilistic method to prove a Universal Traversal theorem on graphs of a given size.n. **Hint:** Instead of N, S, E, W used in $n \times n$ grids one now uses ports.

   Consider a set of points in the plane. Form n sets $A_1, A_2, \ldots, A_n$ on the plane each of which has k points.

[a] Not to submit.



The sets are arbitrary and may share points. Assume that $n < 2^k$. Show that If $n < 2^k$ then it is possible to color each point red or blue in such a way that every set A_i has both colors. **Hint:** Color each of the n points independently choosing red or blue with equal probability and use the probabilistic method.

3. A certain commodity is sold with two lottery tickets, a and b, for Prize A and Prize B, respectively. Suppose the winning probability for A and that for B are both $2/3$. Show that there

must exist a commodity with two winning tickets.[b]

4. ($\star$) The sets $S_1, S_2, \ldots, S_k$ are different subsets of a set S that has $2n$ elements. $\{S_1, S_2, \ldots, S_k\}$ is called a Sperner family[c] if $S_i \not\subseteq S_j$, for all $i \neq j$. Use the probabilistic method to prove Spener's theorem, namely "If $\{S_1, S_2, \ldots, S_k\}$ is a Sperner family then $k \leq \binom{2n}{n}$." For the proof, follow the steps below.

   (a) Consider the following process: We start with the emptyset and add random elements of S one by one until (after $2n$ steps) we get the whole set S. For a fixed subset $A \subset S$ of size a show that A will appear during this process with probability $\Pr[A] = 1/\binom{2n}{a}$.

   (b) Consider k random variables $X_1, X_2, \ldots, X_k$ so that the value of X_i is equal to 1 if the given set S_i appears during the process, otherwise, it is equal to 0. Show that the

[b] Note that the conclusion is derived without using event dependence.
[c] Sperner families have applications in Cryptography and elsewhere.

expected value of X_i is $1/\binom{2n}{s_i}$, where s_i is the number of elements in S_i,

(c) Now, consider the random variable $X = X_1 + X_2 + \cdots + X_k$. Show that this sum is less than 1 in expectation.

References

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