Tools for Message Passing

Outline

Goal: Understand interaction of "Computation/Communication".

- Shortest Paths
  - Dijkstra[a] (or BFS with weights)
- Centralized MST[b] (Minimum Spanning Tree)
  - Prim (Outline)
  - Kruskal (Outline)
- Distributed MST
  - Gallager-Humblet-Spira (SynGHS)
- Appendix

[a] This can be used as a review since the non-distributed Dijkstra algorithm may have already been covered in other courses,

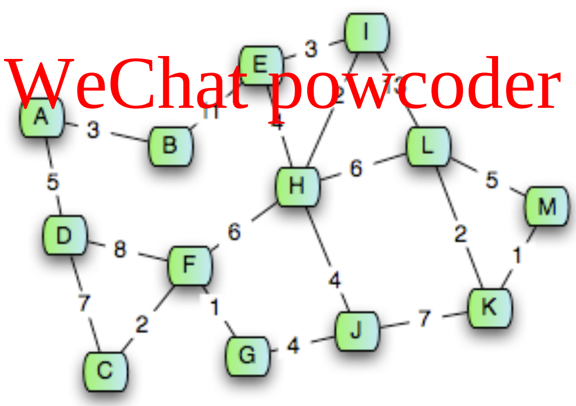
[b] This can be used as a review since the non-distributed MST material may have also been covered in other courses.

Shortest Paths

Motivation: Shortest Paths

- Consider a strongly connected (un)directed graph, with unidirectional communication between neighbors.

- BFS finds shortest path with the hop distance. How do we generalize BFS?
  - Assume that each (un)directed edge has an associated nonnegative real-valued weight.



- The weight of a path is the sum of the weights on its edges.

Shortest Path Problem

- Problem: find a shortest path from a distinguished source node to every other node, where

  - a shortest path is a path with minimum weight.
- A collection of shortest paths from the source to all the other nodes in the digraph constitutes a subtree of the digraph, all of whose edges are oriented from parent to child.

  - Does the collection of shortest paths from a given source node in an undirected graph form a tree? (E.g., consider a bidirectional ring of n nodes.) How about if the edge weights are pairwise different?[a]

[a] Why?

Applications: Shortest Paths

- Motivation for constructing such a tree comes from the desire to have a convenient structure for broadcast communication.

- Weights represent costs associated with the traversal of edges, for instance,

  - communication delay,
  - bandwidth,

  - monetary charge.

Shortest Paths' Trees (SPTs)

- A shortest paths' tree minimizes the maximum worst-case cost of communicating with any process in the network.
- We assume that every process initially knows
  1. the weight of all its incident edges.
  2. the number n of nodes in the (di)graph.
- We require that each process should determine
  1. its parent in a particular shortest paths' tree, and also
  2. its distance (i.e., the total weight of its shortest path) from the source.

Construction of BFS

- If all edges are of equal weight, then a BFS tree is also a shortest paths tree.

  - a trivial modification of the simple SynchBFS tree construction can be made to produce the distance information as well as the parent pointers.

- In synchronous systems the flooding algorithm is a simple yet efficient method to construct a BFS spanning tree.

  - in asynchronous systems the spanning tree constructed by the flooding algorithm may be far from BFS.

- In standard BFS constructions (that we have already studied) all edges have weight 1.

Construction of BFS (with weights)

- A classic BFS construction is

  - Dijkstra's Algorithm

  and can be a sychronous/asynchronous algorithm

Dijkstra's Algorithm and Relaxation

- Based on the **principle of relaxation**.

  - An approximation to correct distance gradually replaced by more accurate values until reaching the optimum solution.
  - Approximate distance to each vertex is an overestimate of true distance.

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  - Replaced by the min of its old value with the length of a newly found path.

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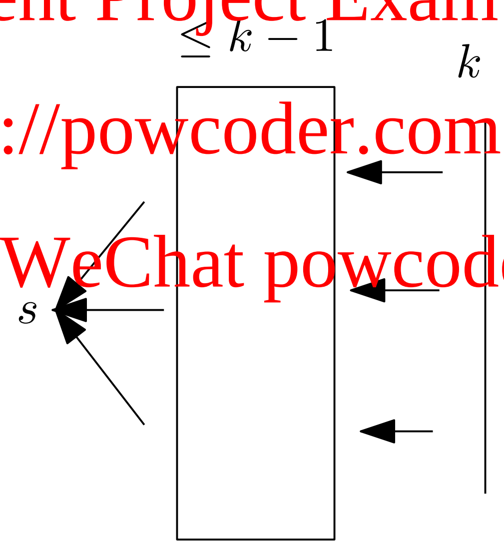
- Greedily selects a node "corresponding to a min-weight edge" that has not yet been processed, and performs this relaxation process on all of its outgoing edges.
- Processing (usually done with a heap) which also counted as part of the cost

Basic Idea of Constructing Shortest Paths

- Let S_k be the set of $k-1$ closest nodes to the destination s.

- During the kth step the kth closest node to the destination s is found by considering
  - the distance of nodes in $N \setminus S_k$ to any node in S_k.



- We will elaborate on this a bit later.[a]

[a] See discussion on *blue* edges later.

Dijkstra's Algorithm: Adding Edges (1/3)

- Finds the shortest path from the source node s to all other nodes in a weighted graph.
  - Similar to BFS, except that it keeps track of a distance $d(j)$ (length of the shortest path known so far to node from a "root node"), for each node j.
- Instead of examining all nodes in the next level, it prioritizes them by the distance d and picks just one unvisited node i with the smallest $d(i)$, whose distance $d(i)$ is now final, and updates tentative distance d for all its neighbors.
- The algorithm uses a heap to keep track of its unvisited nodes j, each with a metric $d(j)$.

Dijkstra's Algorithm: Forming a Heap (2/3)

- Removing the item with smallest metric takes $O(\log n)$ time if the heap[a] contains n items.

  - If an item's metric changes but it remains in the heap, it takes $O(\log n)$ time to adjust its position in the heap.
  - Initializing a heap of n items takes $O(n)$ time.

- Nodes no longer in the heap have been visited, and their $d(j)$ is the shortest path from s to j.

  - The shortest path can be traced backward from j to s by walking the tree from j to its parent $p(j)$, then to $p(p(j))$, and so on until reaching s.

[a] A tree-based data structure satisfying the heap property. In a max heap, for any given node C, if P is a parent node of C, then the key (the value) of P is greater than or equal to the key of C. In a min heap, the key of P is less than or equal to the key of C.

Dijkstra's Algorithm: Intuition (3/3)

- Nodes in heap have not been visited, and their $d(j)$ is tentative.
  - They split into two kinds of nodes: those with finite d (discovered in the frontier), and those with infinite d (undiscovered).
  - Each node in the frontier is incident on at least one edge from the visited nodes.
  - The node in the frontier with the smallest $d(j)$ has a very useful property on which the algorithm relies: its $d(j)$ is the true shortest distance of the path from s to j.
- The algorithm selects this node and then updates its neighbors as j moves from the frontier into the set of visited nodes.
- Algorithm finds the shortest path from s to all other nodes in $O((|V| + |E|) \log |V|)$ time; asymptotic time can be reduced with a Fibonacci heap; in practice a conventional heap is faster.

Dijkstra's Algorithm: Formally

- N set of all the nodes in (undirected) network.

- $l(i,j)$ (non-negative) cost associated with edge $\{i,j\}$.
  - $l(i,j) = +\infty$ if there is no edge between i, j.
- Let s be the node executing the algorithm in order to find shortest paths from s to all other nodes in the network.
  - s is the start or source node.
- Algorithm constructs incrementally a "current set" M of nodes:
  - M is the set of nodes incorporated so far, and
  - algorithm stops when $M = N$.
- $C(n)$ is the cost of the path from s to node n.

Dijkstra's Algorithm: Formally

- **Dijkstra's Algorithm**

1. $M = \{s\}$
2. **for each** $n \in N \setminus \{s\}$
3. $\quad C(n) = l(s, n)$
4. **while** $N \neq M$ **do**
5. $\quad M = M \cup \{w\}$ where w is chosen such that $C(w)$ is min among all $w \in N \setminus M$
6. $\quad$ **for each** $n \in N \setminus M$
7. $C(n) = \min\{C(n), C(w) + l(w, n)\}$

See example in appendix.

Discussion of Analysis

- Algorithm can be implemented so as to run in time $O(|V|^2)$, where V is the number of nodes in the graph.

- As presented, the algorithm computes weights of paths, not the paths themselves.

- Can be easily modified to compute the paths[a]
  - The last edge found in update step 7 is the first edge in a shortest path to destination,
  - can be used to compute a shortest path tree, and
  - compute routing tables.

[a] We discuss this later.

Application: Route Calculation in LSP

- Dijkstra's Algorithm used for route calculation in Link State Protocol (LSP)

- Finds shortest paths from all nodes to some fixed destination (or source)

- Requires that all edge weights are nonnegative (not a restriction for most network applications)

- Shortest paths found in order of increasing path length.

Dijkstra BFS Tree: Constructing the Routes (1/2)

- The algorithm proceeds in phases constructing trees.

  - In phase p the nodes at distance p from the root are detected.
  - Let T_p denote the tree constructed in phase p.

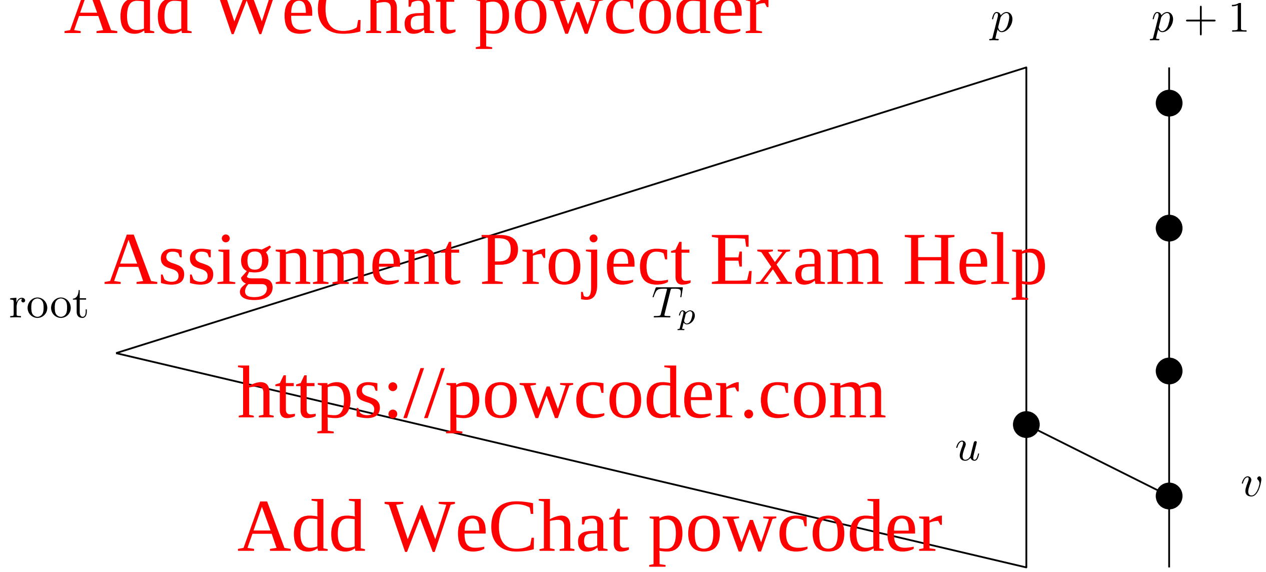
- The starting phase is $p = 1$:

  - Tree T_1 is the "star graph" consisting of the root plus all direct neighbors of the root which have min weight.

- We now determine how to update from phase p to phase $p + 1$;

Dijkstra's Algorithm Tree Construction

- Construction of tree T_p in Dijkstra's algorithm



- Broadcast from the root in phase p and echo.
- The root decides which vertex v is selected with a echo/broadcast subroutine.

Dijkstra BFS: Constructing the Routes (2/2)

repeat

1. The root starts phase p by broadcasting "start p" within T_p.
2. When receiving "start p" a leaf node u of T_p (that is, a node that was newly discovered in the last phase) sends a "join $p+1$" message to all quiet neighbors. (A neighbor v is quiet if u has not yet "talked" to v.)
3. Node v receiving first "join $p+1$" message replies with ACK and becomes a leaf of the tree T_{p+1}. /* Other nodes decline */
4. The leaves of T_p collect all the answers of their neighbors; then the leaves start an echo algorithm back to the root.
5. When the echo process terminates at the root, the root increments the phase

until there was no new node detected

Analysis of Dijkstra BFS

- **Theorem 1** *In Dijkstra's algorithm*
  - *the time complexity is* $O(D^2)$*, and*
  - *the message complexity is* $O(m + nD)$*,*

where D *is the diameter of the graph,* n *the number of nodes,*[a] *and* m *the number of edges. Adjusting the position of an element (vertex) in the heap costs a factor* $O(\log n)$*.*

[a]Note that earlier we used V for the number of nodes; here we use n.

Analysis of Dijkstra BFS

- Time Complexity
  - A broadcast/echo algorithm in T_p needs at most time $2D$.
  - Finding new neighbors at the leaves costs 2 time units.
  - Since the BFS tree height is bounded by the diameter, we have D phases, giving a total time complexity of $O(D^2)$.
- Message Complexity
  - Each node participating in broadcast/echo receives at most 1 message and sends (echoes) at most once.
  - There are D phases, so message cost is bounded by $O(nD)$.
  - On each edge there are at most 2 "join" messages.
  - Replies to "join" request are answered by 1 "ACK/NACK" (so we have at most 4 additional messages per edge).
  - Message complexity is $O(m + nD)$. Processing using the heap costs a factor $O(\log n)$.

Applications: RIP and BGP

- A distributed variant of the Bellman-Ford algorithm is used in distance-vector routing protocols, for example the Routing Information Protocol (RIP).

- The algorithm is distributed and involves a number of nodes (routers) within an Autonomous System (AS), a collection of IP networks typically owned by an ISP. It consists of the following steps:

  1. Each node calculates distances between itself and all other nodes within the AS and stores this information as a table.
  2. Each node sends its table to all neighboring nodes.
  3. When a node receives distance tables from its neighbors, it calculates the shortest routes to all other nodes and updates its own table to reflect any changes.

MST

Questions in Graph Search

- When you ask your smartphone or GPS to find the best route from Ottawa to Vancouver, how does it find a good route?

- If you ask it to help you drive from Ottawa to Paris, how does it know you cannot do that?

- If you post something on a social network for only your friends and friends of friends, how many people can see it?

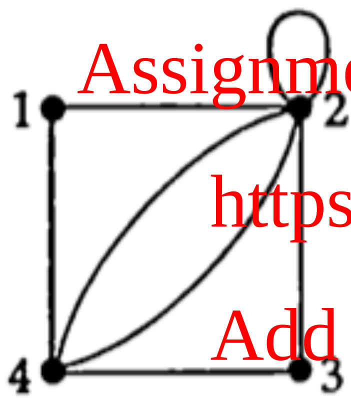
- These questions can all be posed in terms of searching a graph (also called graph traversal).

Graph Representation

- For an unweighted graph $G = (V, E)$ of n nodes, storing a binary adjacency matrix A in a conventional array using $O(n^2)$ memory works well in a graph traversal algorithm if the graph is small or if there are edges between many of the nodes.
  - In this matrix $A = (a_{ij})$, $a_{ij} = 1$ if $\{i, j\}$ is an edge, and 0 otherwise.
  - With edge weights, the value of a_{ij} in a real matrix can represent the weight of the edge $\{i, j\}$, although this assumes that in the problem at hand an edge with zero weight is the same as no edge at all.

Example Graph Representation

- A graph and its adjacency matrix.



$$
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 & \begin{array}{cccc} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\begin{array}{c} \text{row } 1 \rightarrow \\ \text{row } 2 \rightarrow \\ \text{row } 3 \rightarrow \\ \text{row } 4 \rightarrow \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}
\end{array}
$$

Graph Representation

- A road network can be represented as a weighted directed graph, with each node an intersection and each edge a road between them.

- The graph is directed because some roads are one-way, and it is unconnected because you cannot drive from Ottawa to Paris.
  - The edge weight a_{ij} represents the distance along the road from i to j;
  - all edge weights must therefore be greater than zero.[a]

- An adjacency matrix works well for a single small town, but it takes too much memory for the millions of intersections in the North American road network.

[a] There are situations where negative weights are natural to use.

Graph Representation

- Fortunately, only a handful of roads intersect at any one node, and thus the adjacency matrix is mostly zero, or sparse.

- This kind of matrix or graph is best represented with adjacency lists, where each node i has a list of nodes j that it is adjacent to, along with the weights of the corresponding edges.

Searching a Graph

- Searching a graph from a single source node, s, discovers all nodes reachable from s via $path(s)$ in the graph.

- Nodes are marked when visited, so to search the entire graph, a single-source algorithm can be repeated by starting a new search from each node in the graph, ignoring nodes that have already been visited.

- The graph traversal then performs actions on each node visited, and also records the order in which the graph was traversed.

- Often, only a small part of the graph needs to be searched.
- This can greatly speed up the solution of problems such as route planning.

Spanning Trees

- A spanning tree of a graph is a subgraph

  - that is a tree (i.e., contains no cycles), and
  - includes all of the nodes of the graph
- If the edges of the network are weighted (e.g., representing average delay expected on a given LAN[a]) a minimum weight spanning tree is one with minimum sum of edge weights
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- Two non-distributed algorithms for computing MST:
  - Prim's algorithm
  - Kruskal's algorithm
- A distributed algorithm for computing MST:
  - GHS (Gallagher, Humblet, Spira).

[a] Local Area Network.

Spanning Tree Terminology

- $G = (V, E)$ is an undirected graph
  - V is the set of nodes (vertices)
  - E is the set of edges
- $w_{i,j}$ is the weight of the edge $\{i, j\}$
- A *spanning tree* is an acyclic subgraph containing all nodes
- Weight of a tree T is the sum of its edge weights:

$$w(T) = \sum_{\{i,j\} \in E(T)} w_{i,j},$$

  where $E(T)$ is the set of edges in T.
- MST (Minimum weight Spanning Tree) is a ST (Spanning Tree) of minimum weight

Two Basic Sequential Spanning Tree Algorithms

1. Prim's Algorithm (Jarnik, 1930)

   - Start from any root node.
   - Always maintain a connected subtree (check for cycles).
   - Among possible choices add a "min weight" edge at a time

2. Kruskal's Algorithm (Boruvka, 1926)

   - Sort the edges in ascending weights
   - Always maintain a forest (check for cycles).

   - Add edges in order as long as no cycles created

Prim's Algorithm (Jarnik, 1930)[a]

- **Prim's Algorithm**

- P is current set of nodes in tree
- D_i is min weight edge from node i to a node in P
- Initially $P = \{1\}$, and $D_i = w_{i,1}$ if $\{i, 1\}$ exists, ∞ otherwise

1. Find $i \notin P$ such that D_i is minimum
2. $P = P \cup \{i\}$
3. For $j \notin P$, $D_j = \min\{D_j, w_{j,i}\}$
4. Go back to 1

Can be implemented in $O(|E| + |V| \log |V|)$ time. See example in appendix.

[a] 1) Jarnk, V. (1930), "O jistm problmu minimlnm" [About a certain minimal problem], Prce Moravsk Prodovdeck Spolenosti (in Czech), 6 (4): 57-63. 2) Prim, R. C. (November 1957), "Shortest connection networks And some generalizations", Bell System Technical Journal, 36 (6): 1389-1401,

Kruskal's Algorithm (Boruvka, 1926)[a]

- **Kruskal's Algorithm**

  1. Sort the edges of G in increasing order
  2. Consider edges in order and add edge to tree if the result does not form a cycle

- Time complexity is $O(|E| \log |E|)$. See example in appendix.

[a] 1) Kruskal, J. B. (1956). "On the shortest spanning subtree of a graph and the traveling salesman problem". Proceedings of the American Mathematical Society. 7: 48-50. 2) Borvka, Otakar. O Jistm Problmu Minimlnm. Prce Moravsk Prodovdeck Spolenosti III, no. 3 (1926): 37-58.

Distributed MST

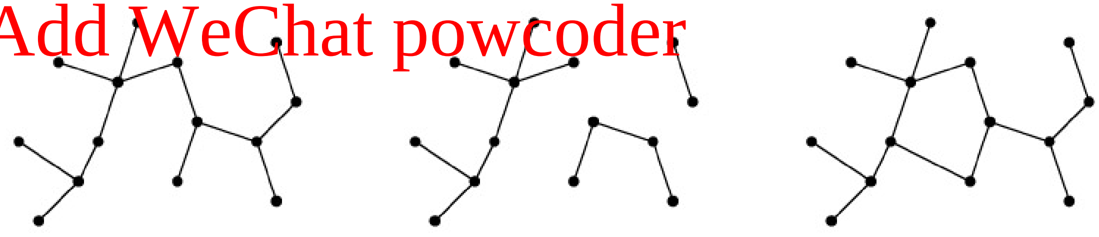
Distributed MST Algorithm

- Given a graph G, the goal is to design a distributed algorithm that always terminates, and computes an MST of G.
- At the end of an execution, each processor knows which of its incident edges belong to the MST and which do not (i.e. the processor writes in a local output register the corresponding incident edges).
- In the distributed version of the MST, a communication network is solving a problem where the input is the network itself.[a]
- This is one of the fundamental starting points of (distributed) network algorithms.

[a] A "local" version of the MST is not possible: We won't discuss this here.

Main Idea: Select

- Recall



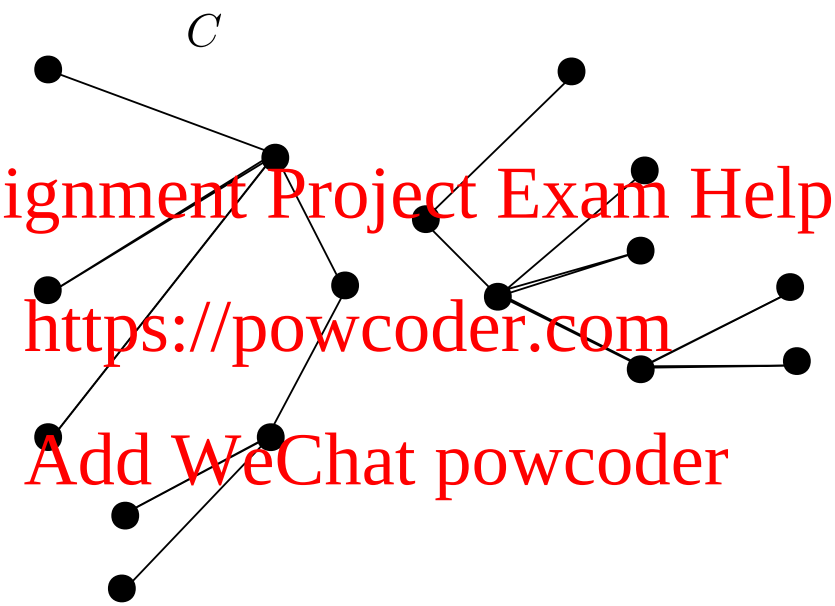
(a) is a tree, (b) is a forest, (c) is not a tree

1. Start with trivial spanning forest consisting of n individual nodes and no edges; at the start each vertex itself is a tree.

2. Repeatedly do the following: Select
   (a) an arbitrary component C (i.e., tree) in the forest, and
   (b) an arbitrary outgoing edge $e \in C$ having minimum weight among the outgoing edges of C.

Main Idea: Merge

1. For such an edge e, combine C with the component at the other end of e, including edge e in the new combined component.



2. Stop when the forest has a single component.

Assumptions

- Now we investigate how to design a Distributed Algorithm.

- We assume that no two edges of the graph have the same weight.
  - This simplifies the problem
  - simplification is not essential
  - one can always break ties by adding the IDs of adjacent vertices to the weight.

Concept of Blue Edges

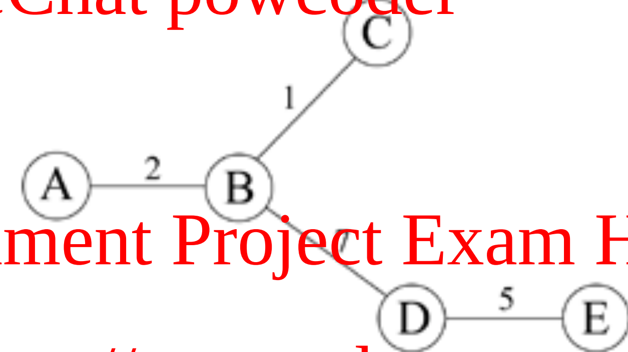
- Let T be a spanning tree of the weighted graph G
  - A subgraph $T' \subseteq T$ of T is called a fragment.

- Edge $e = \{u, v\}$ is an outgoing edge of T' if either
  - $u \in T'$ and $v \notin T'$ or,
  - $u \notin T'$ and $v \in T'$.

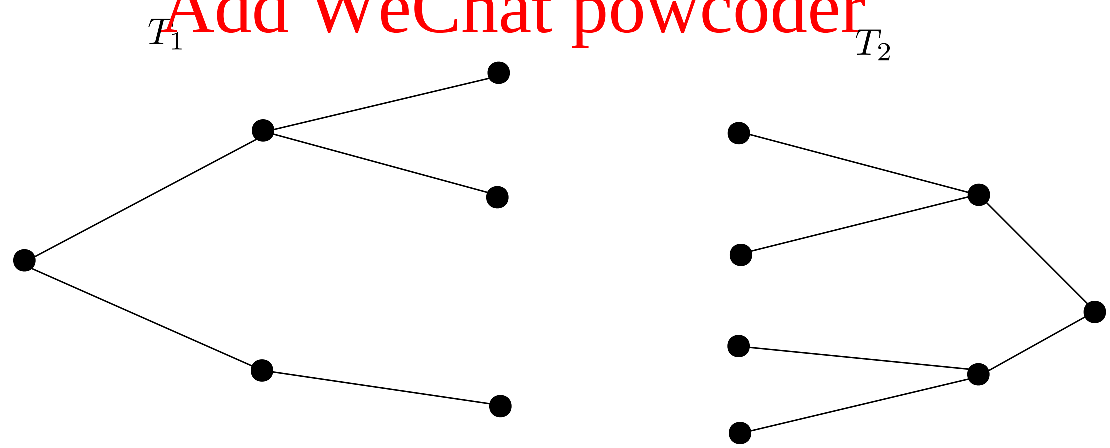
- The minimum weight outgoing edge of tree T', denoted by $b(T')$, is the so-called blue edge of T'.

Examples: Concept of Blue Edges

- Example 1:



- Example 2:

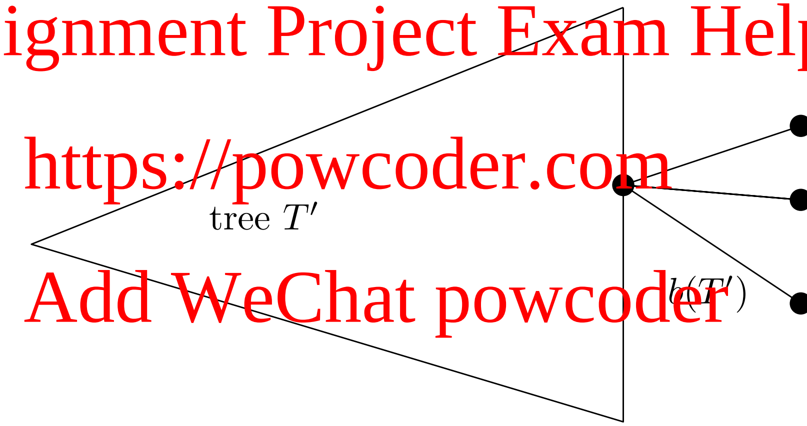


A Lemma on Adding Blue Edges

- **Lemma 1** *For a given weighted graph G (such that no two weights are the same),*

  – *let T denote the MST, and*

  – *T' be a fragment of T.*



  Then the blue edge of T' (denoted by $b(T')$) is also part of T,

  – *i.e., $T' \cup \{b(T')\} \subseteq T$.*

- So, the Lemma says that blue edges can be added to an already constructed MST fragment and maintain the MST property.

Proof of Lemma 1

- For the sake of contradiction, suppose that in the MST T there is edge $e \neq b(T')$ connecting T' with the remainder of T.

- Adding the blue edge $b(T')$ to the MST T we get a cycle including both e and $b(T')$.

- If we remove e from this cycle
  – we still have a spanning tree, and
  – since by the definition of the blue edge $w_e > w_{b(T')}$, the weight of that new spanning tree is less than the weight of T.[a]

- We have a contradiction.

[a] Here we used the fact that the edge weights are different!

Issues

- Blue edges
  - allow a fragment to grow in a greedy manner!
  - seems to be the key to a "distributed algorithm" for the "distributed MST" problem.

- Since every node itself is a fragment of the MST, every node directly has a blue edge!

- All we need to do now is to grow these fragments!
  - Essentially this is a distributed version of Kruskal's sequential algorithm.

A Distributed Algorithm

- At any given time the nodes of the graph are partitioned into fragments (rooted subtrees of the MST).

- Each fragment has a designated vertex called root (of the fragment):
  - ID of fragment is defined to be the ID of its root.

- In the course of the algorithm,
  - each node knows its parent and its children in the fragment.

- The algorithm operates in phases.
- At the beginning of a phase, nodes know the IDs of the fragments of their neighbor nodes.

SynGHS (Gallager, Humblet, Spira)

- The algorithm builds the components in "levels" (or phases).

- For each k, the level k components constitute a spanning forest, where each level k component consists of a tree that is a subgraph of the MST.

- Each level k component has at least 2^k nodes.

- Each component, at each level, has a distinguished leader node.

- The processes allow a fixed number of rounds, which is $O(n)$, to complete each level.

SynGHS: Base and Inductive Steps

- **Base Step:**

  The algorithm starts with level 0 components consisting of individual nodes and no edges.

- **Inductive Step:**

  Suppose inductively that the level k components have been determined (along with their leaders). More specifically, suppose that each process knows

  – the UID (User ID) of the leader of its component; (this UID is used as an identifier for the entire component),

  – which of its incident edges are in the component's tree.

SynGHS: Inductive Step $k \to k+1$

- To get the level $k+1$ components, each level k component conducts a search along its spanning tree edges for the MWOE (Minimum-Weight Outgoing Edge)[a] of the component.

- The leader broadcasts search requests along tree edges, using a message broadcast strategy (BFS).

- Each process finds, among its incident edges, the one of minimum weight that is outgoing from the component (if there is any such edge);
  - it does this by sending test messages along all non-tree edges, asking whether or not the other end is in the same component.

[a] Recall that we called such edges: "blue" edges of the fragment.

SynGHS: Inductive Step $k \to k+1$

- Then the processes convergecast this local min-weight edge information toward the leader (taking minima along the way).

- The minimum obtained by the leader is the MWOE[a] of the entire component.

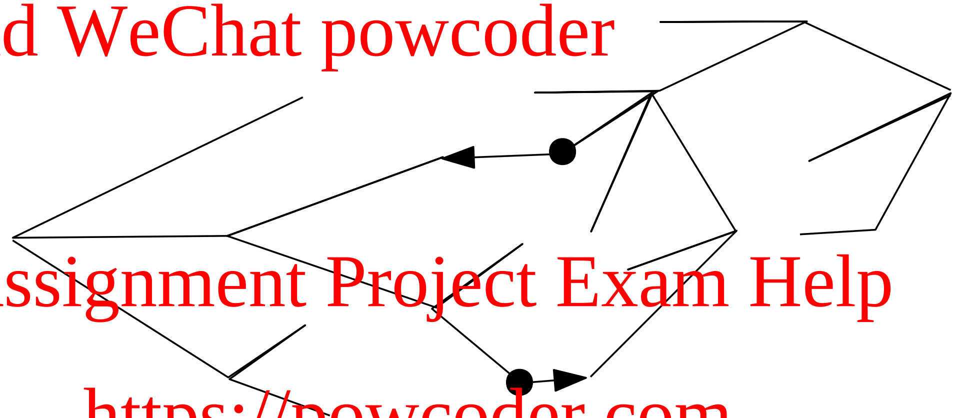
- When all level k components have found their MWOEs, the components are combined along all these MWOEs to form the level $k+1$ components.

  - This involves the leader of each level k component communicating with the component process adjacent to the MWOE, to tell it to mark the edge as being in the new tree;
  - the process at the other end of the edge is also told to do the same thing.

[a] I.e., "blue" edge

SynGHS: New Leader

- Two components at level k



- Merging two components at level k:

  - minimum weight edge will be selected.

- Moreover, a component at level k will merge only with another component at level k which corresponds to a minimum weight outgoing edge!

SynGHS: New Leader

- Then a new leader is chosen for each level $k+1$ component, as follows.

  – It can be shown that for each group of level k components that get combined into a single level $k+1$ component, there is a unique edge e that is the common MWOE of two of the level k components[a] in the group (see Lemma 2).

  – New leader: is the endpoint of e having larger UID.

  – **NB:** this new leader can identify itself using only information available locally.

- Finally, the UID of the new leader is propagated throughout the new component, using a broadcast.

[a] Recall that edge weights are pairwise distinct.

SynGHS: Conclusion

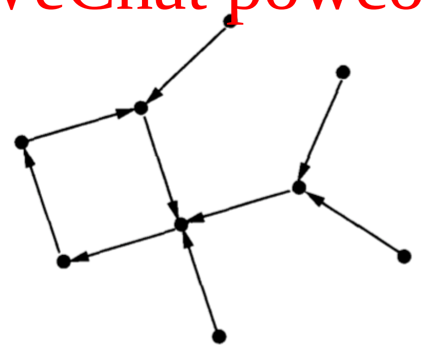
- Eventually, after some number of levels, the spanning forest consists of only a single component containing all the nodes in the network.

- Then a new attempt to find a MWOE will fail, because no process will find an outgoing edge.

- When the leader learns this, it broadcasts a message saying that the algorithm is completed.

A Key Idea of the Algorithm (1/2)

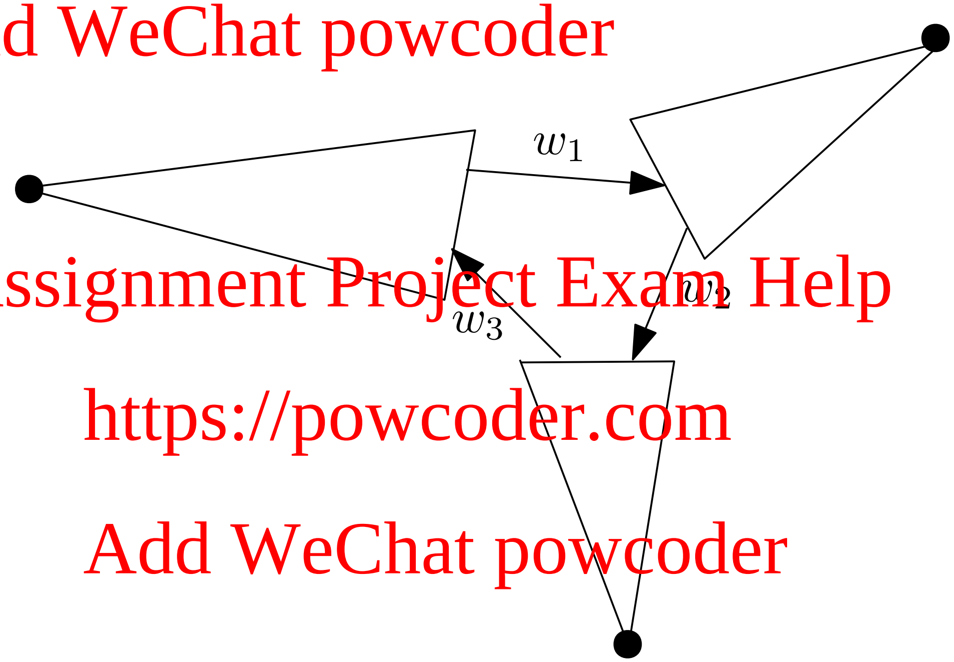
- Among each group of level k components that get combined, there is a unique (undirected) edge that is the common MWOE of both endpoint components.
  - To see this: consider the **component digraph** G', whose nodes are the level k components that combine to form one level $k+1$ component and whose edges are the MWOEs.
  - G' is a weakly connected digraph in which every node has exactly one outgoing edge.[a]



[a] A digraph is weakly connected if its undirected version, obtained by ignoring the directions of all the edges, is connected.

A Key Idea of the Algorithm: Example

- Can we have a cycle of length > 2 in the component graph



- If yes then, $w_1 > w_3$ and $w_2 > w_1$ and $w_3 > w_2$!

A Key Idea of the Algorithm (2/2)

- So we have the following property:

  Lemma 2 *If for a weakly connected digraph G each node has exactly one outgoing edge then G contains exactly one cycle.*

- We apply Lemma 2 to the component digraph G' to obtain the unique cycle of components.

- Because of the way G' was constructed, successive edges in the cycle must have non-increasing weights; therefore, the length of this cycle cannot be greater than 2.

- So the length of the unique cycle is exactly 2.
- But this corresponds to an edge that is the common MWOE of both adjacent components.

Importance of Synchrony in SynGHS

- Synchrony ensures when a process i tries to determine whether or not the other endpoint j of a candidate edge is in the same component, both i and j have up-to-date component UIDs.

- If the UID at j is observed to be different from that at i, we would like to be certain that i and j really are in different components, not just that they haven't yet received their component UIDs from their leaders.

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- In order to execute the levels synchronously, processes allow a predetermined number of rounds for each level.
- To be certain that all the computation for the round has completed, this number will be $O(n)$; note that $O(diameter)$ rounds are not always sufficient.
- Need to count this number of rounds is only reason that nodes need to know n.

Complexity Analysis of SynGHS

- Note first that the number of nodes in each level k component (with the possible exception of the last one) is at least 2^k.

- This can be shown by induction, using the fact that at each level, each component is combined with at least one other component at the same level.

- Therefore, the number of levels is at most $\log n$.

- Since each level takes time $O(n)$, it follows that the time complexity of SynchGHS is $O(n \log n)$.

- The communication complexity is $O((n + |E|) \cdot \log n)$, since at each level, $O(n)$ messages are sent in total along all the tree edges, and $O(|E|)$ additional messages are required for finding the local minimum-weight edges.

Issues: Asynchronous Case

- More details are needed for the asynchronous communication model.
  - It may be that some fragments (subtrees) are much larger than others, and because of that some nodes may need to wait for others, e.g., if node u needs to find out whether neighbor v also wants to merge over the blue edge $b = (u, v)$.
- These details can be solved.
  - We can bound the asynchronicity by guaranteeing that nodes only start the new phase after the last phase is done, similarly to the phase-technique of Dijkstra's Algorithm.
- This gives rise to the idea of levels which will not be discussed any further.

Issues

- The GHS algorithm is also known in the literature as BigMerge.

- The GHS algorithm can be applied in different ways.
- GHS for instance directly solves leader election in general graphs: The leader is simply the last surviving root!

- GHS is distributed but in a wireless setting there is a problem in that the number of rounds can be high.

  - In general, if you restrict the number of rounds you cannot construct a spanning tree.

  - There exist constant round algorithms on geometric graphs that construct spanners with good spanning properties.

Application

- Construct "local" planar spanners with constant stretch factor in Wireless Networks.

- Here is a simple algorithm.

1. Each node u finds its distance 2 neighborhood $N_2(u)$.
2. Each node u constructs a MST T_u (with distance wightss) of its distance 2 neighborhood $N_2(u)$.
3. $\{u, v\}$ is an edge of the spanner iff $\{u, v\}$ is an edge of both T_u and T_v.

- The resulting spanner is

1. planar,
2. has small stretch factor,
3. requires information from only two hops away.

Exercises[a]

1. Consider the following traversal algorithm. An initiator sends out a token to discover the traversal route. Define the parent of a node as one from which the token is received for the first time. All other neighboring nodes will be called neighbors. By definition, the initiator does not have a parent. The following two rules define the algorithm:

   (a) Send the token towards each neighbor exactly once.

   (b) If Rule (1a) cannot be used to send the token, then send the token to its parent.

   Show that when the token returns to the root, the entire graph has been traversed by proving the following two claims.

   (a) The token has a valid move until it returns to the root.

   (b) Eventually every node is visited by the token.

[a] No to hand in!

2. Let $G = (V, E)$ be a directed graph. A maximal strongly connected component of G is a subgraph G' such that 1) for every pair of vertices $u, v \in G'$, there is a directed path from u to v and a directed path from v to u, and 2) no other subgraph of G has G' as its subgraph. Propose a distributed algorithm to compute the maximal strongly connected component of a graph.

3. Propose an algorithm for locally repairing a spanning tree by restoring connectivity when a single node crashes. Your algorithm should complete the repair by adding the fewest number of edges. Compute the time complexity of your algorithm.

4. In a spanning tree of a graph, there is exactly one path between any pair of nodes. If a spanning tree is used for broadcasting a message, and a process crashes, some nodes will not be able to receive the broadcast. Our goal is to improve the

connectivity of the subgraph used for broadcast, so that it can tolerate the crash of one process.

Given a connected graph G, what kind of minimal subgraph would you use for broadcasting, so that messages will reach every process even if one process fails? Suggest a distributed algorithm for constructing such a subgraph. Argue why your algorithm will work, and analyze its complexity.

5. The eccentricity of a vertex v in a graph G is the maximum distance from v to any other vertex. Vertices of minimum eccentricity form the center.

   (a) Show that a tree can have at most two centers.

   (b) Design a distributed algorithm to find the center of a tree.

6. Given an undirected graph $G = (V, E)$, a matching M is a subset of E, such that no two edges are incident on the same vertex. A matching M is called maximal if there is no other

matching M' such that $M \subset M'$. Suggest a distributed algorithm for computing a maximal matching. When the algorithm terminates, each node must know its matching neighbor, if such a match exists.

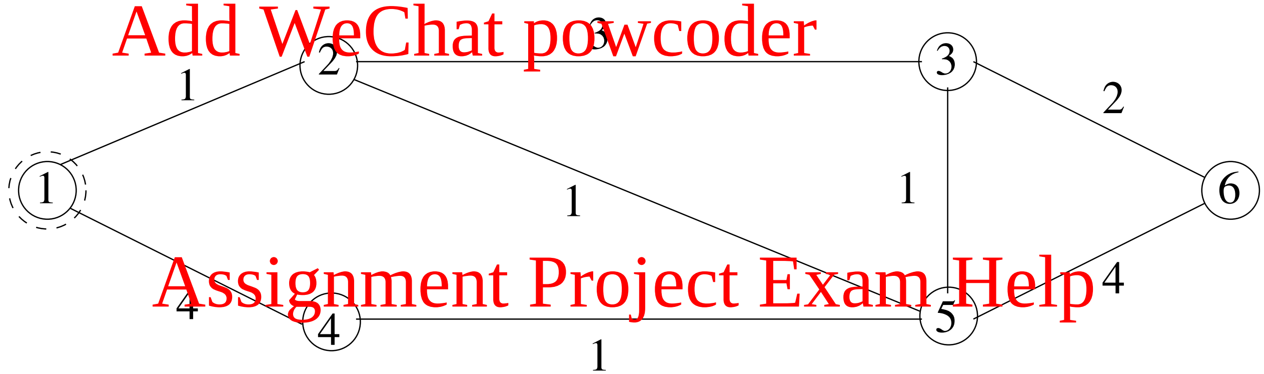
Sources

- R. Wattenhofer, Lecture Notes on Principles of Distributed Computing, ETH Spring 2012.
- N. Lynch, Distributed Algorithms, Morgan-Kaufmann, 1996.
- Gallager, R.G., Humblet, P.A., Spira, P.M.: A distributed algorithm for minimum-weight spanning trees. ACM Trans. Prog. Lang. Systems 5(1), 66-77 (1983)

Appendix

Example: Dijkstra's Algorithm

Start node $s := 1$



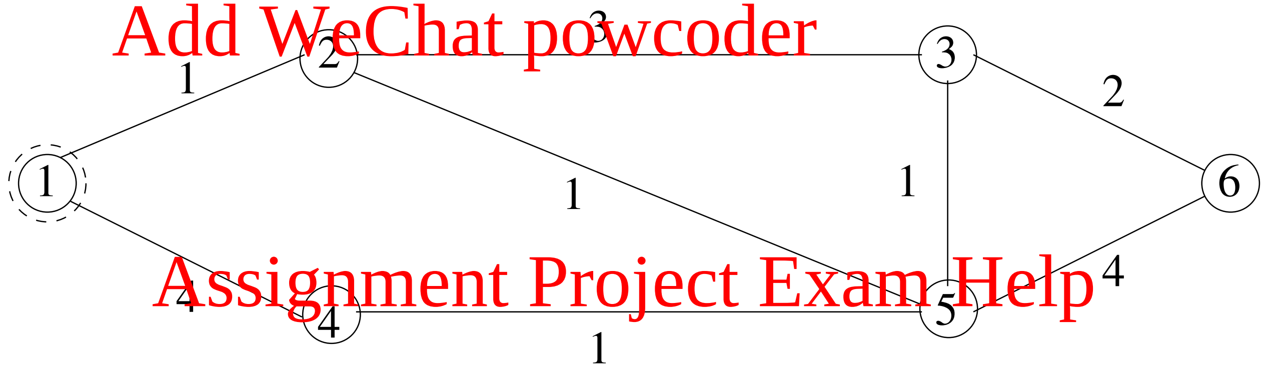
Iteration 1:

Compute all costs to 1.

Update min cost routes to node 1.

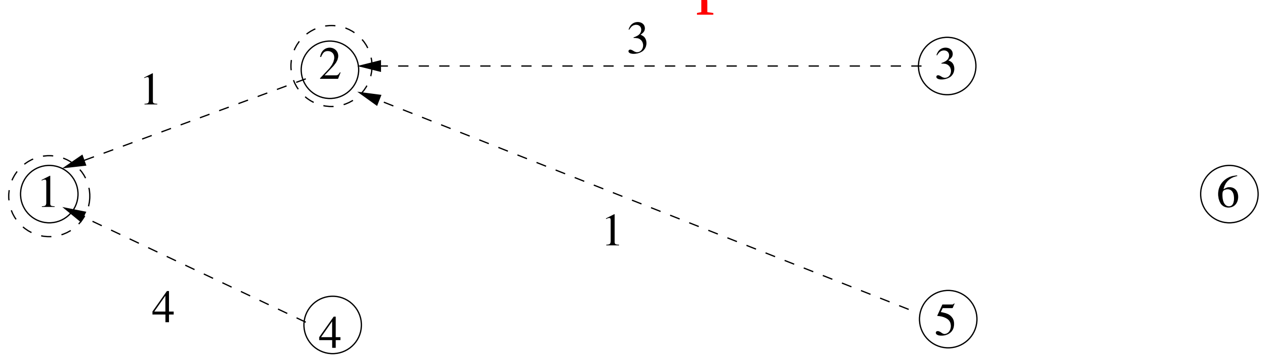
Example: Dijkstra's Algorithm

Start node is $s = 1$



Iteration 2:

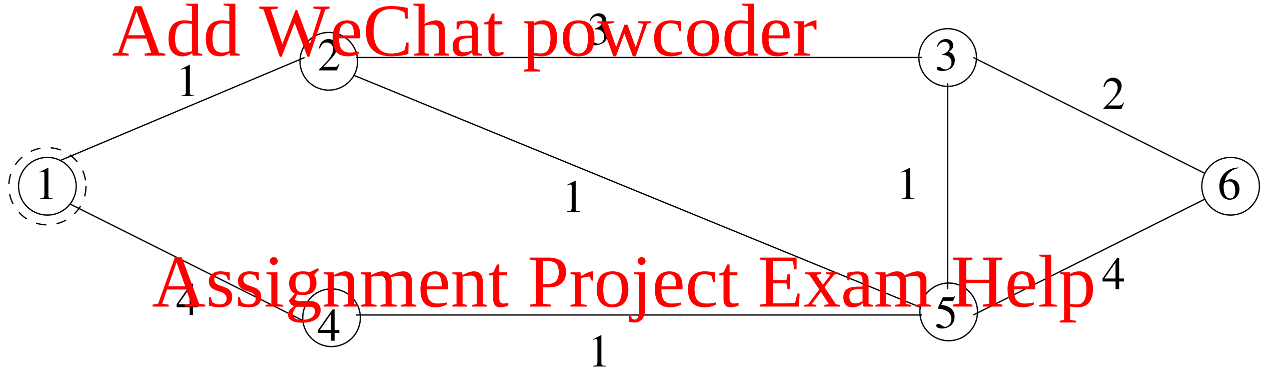
Add min cost node to M (node 2).



Update min cost routes to node 1.

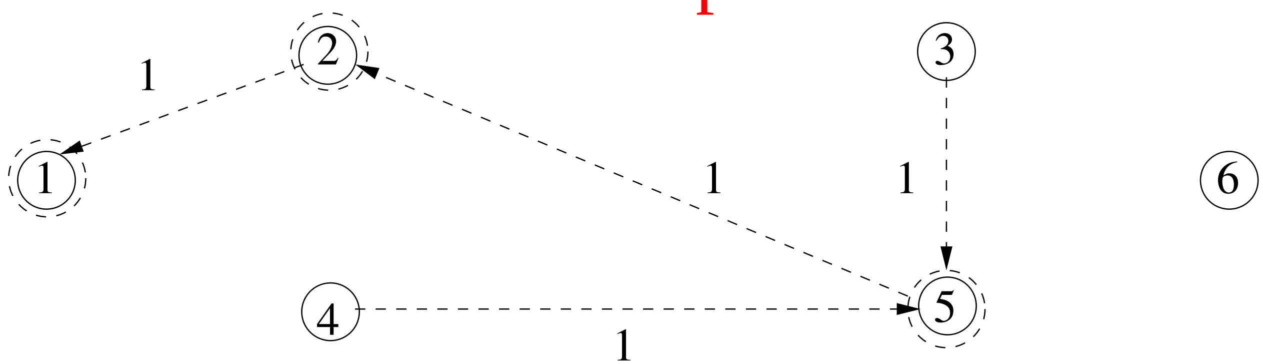
Example: Dijkstra's Algorithm

Every node executes Dijkstra's algorithm



Iteration 3:

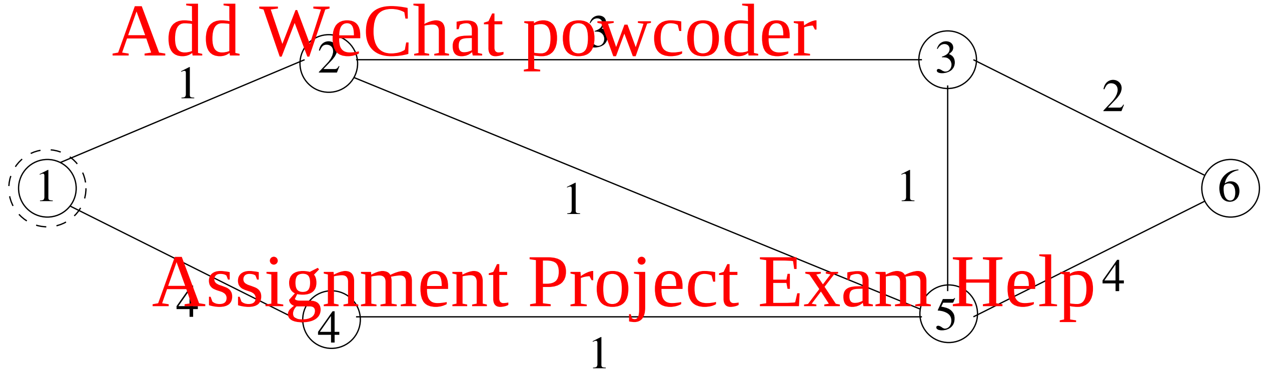
Add min cost node to M (node 5).



Update min cost routes to node 1.

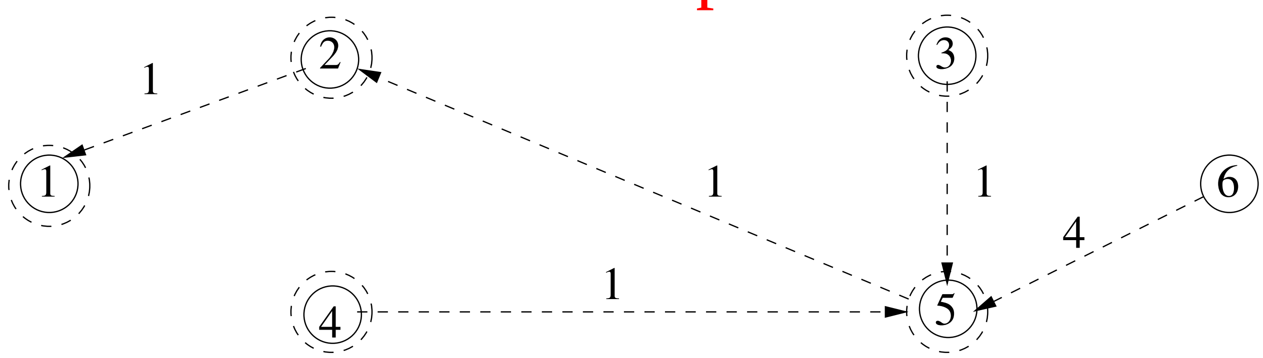
Example: Dijkstra's Algorithm

Every node executes Dijkstra's algorithm



Iterations 4:

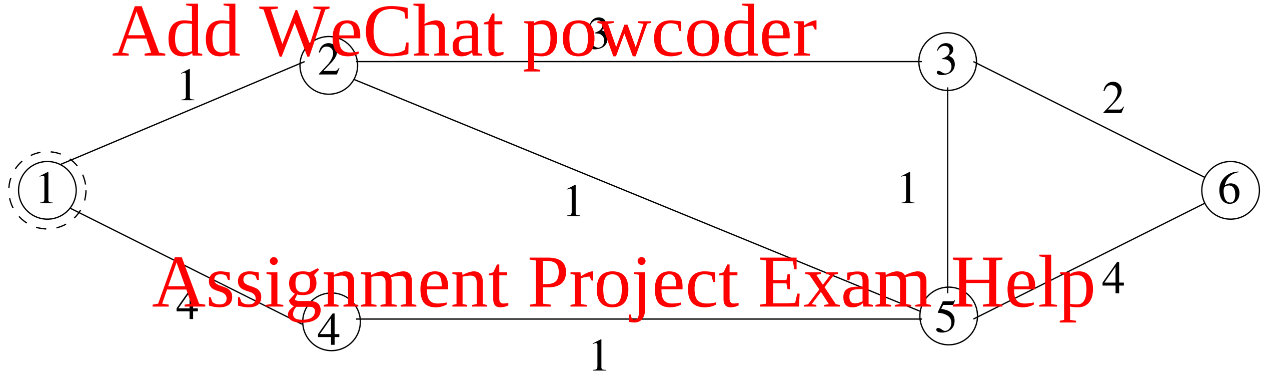
Add min cost node to M (node 3).



Update min cost routes to node 1.

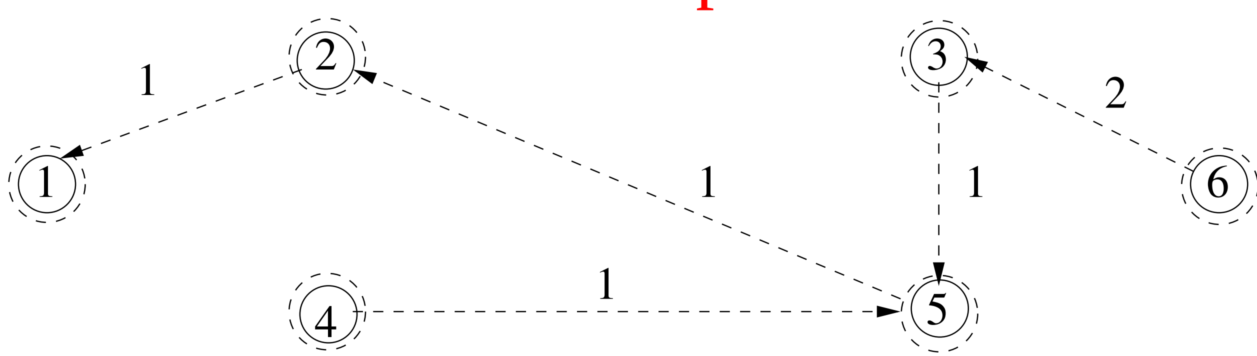
Example: Dijkstra's Algorithm

Every node executes Dijkstra's algorithm



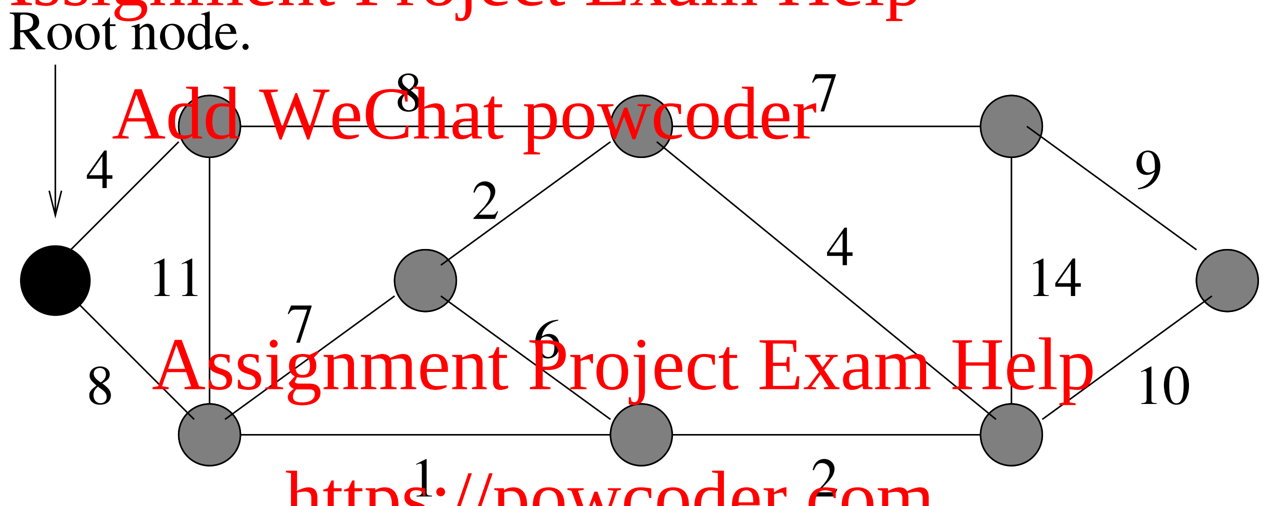
Iteration 5:

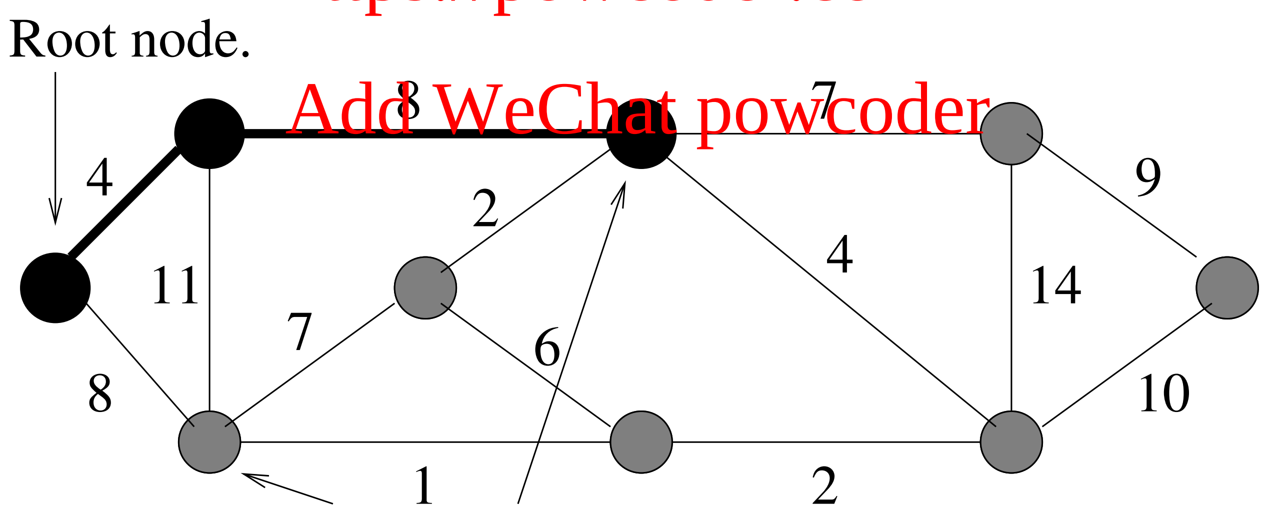
Add min cost node to M (node 6).



Update min cost routes to node 1.

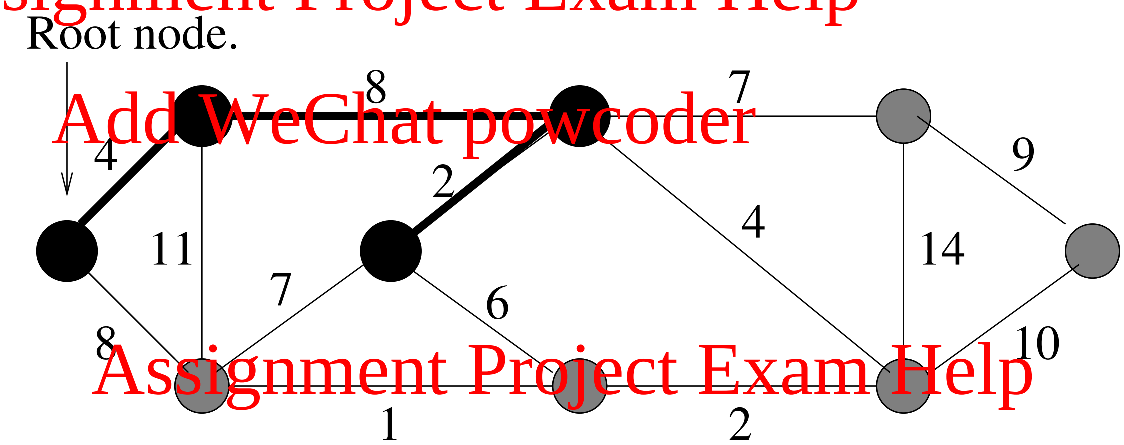
Example: Prim's Algorithm

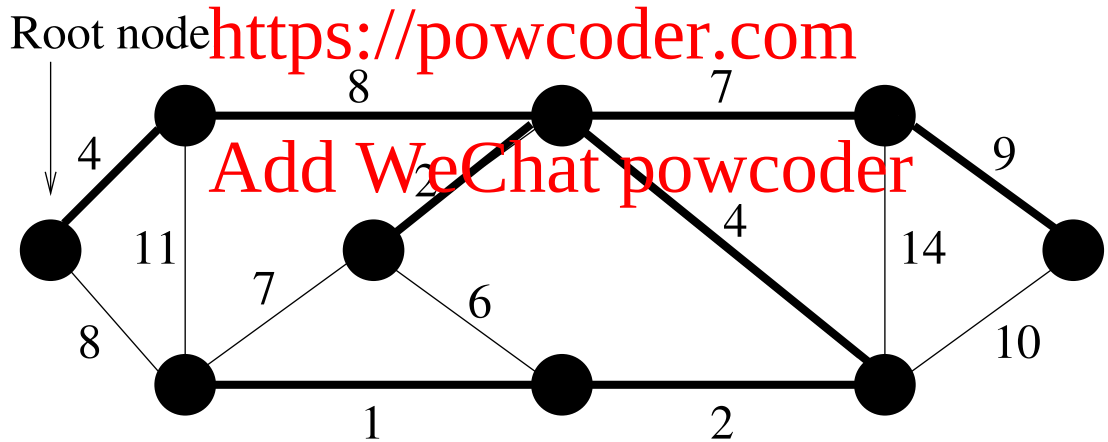




We have a choice: can add either of these two nodes.

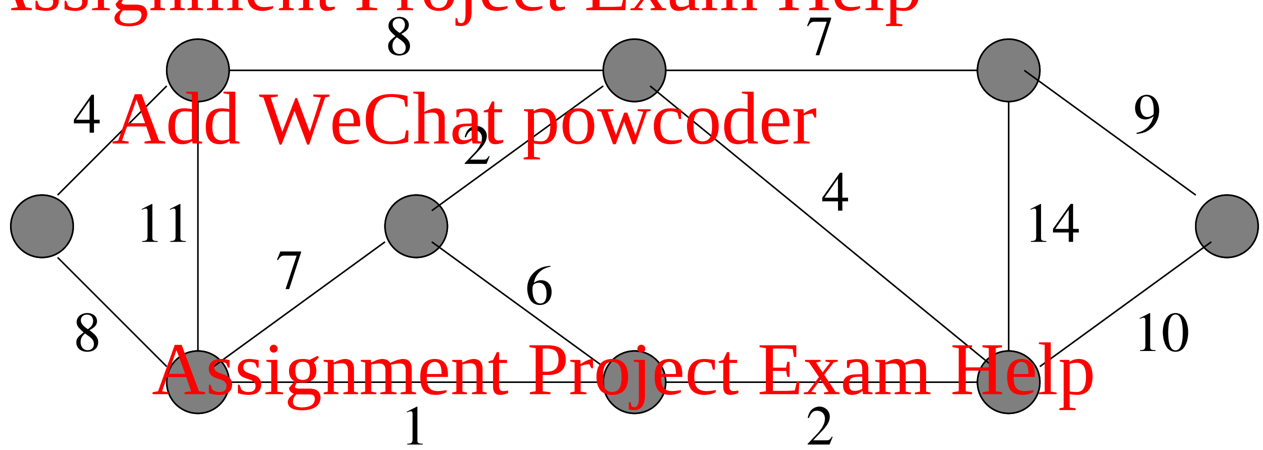
Example: Prim's Algorithm

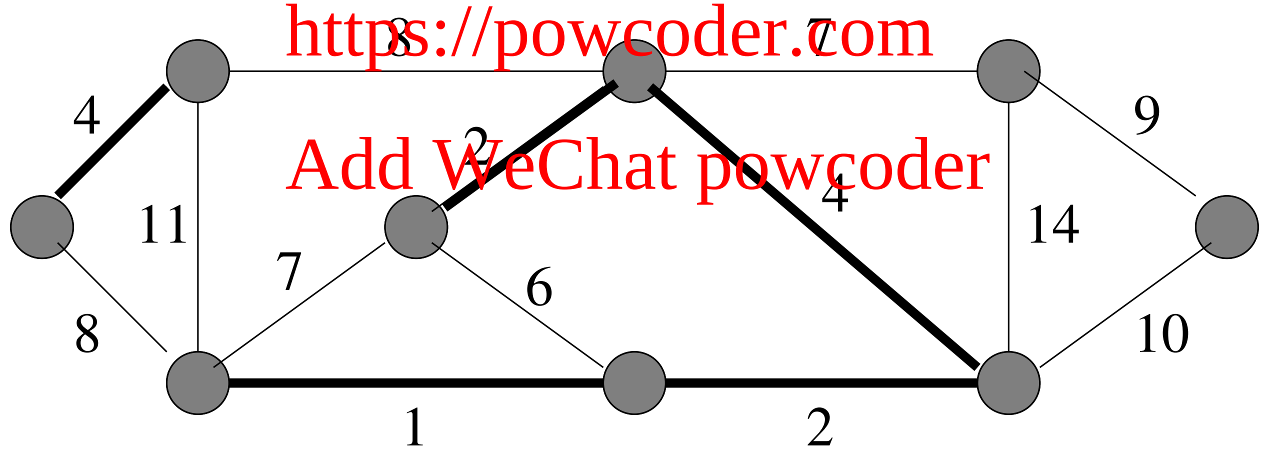




Then we add nodes adjacent to links
4. 2. 1. 7. 9 in this order

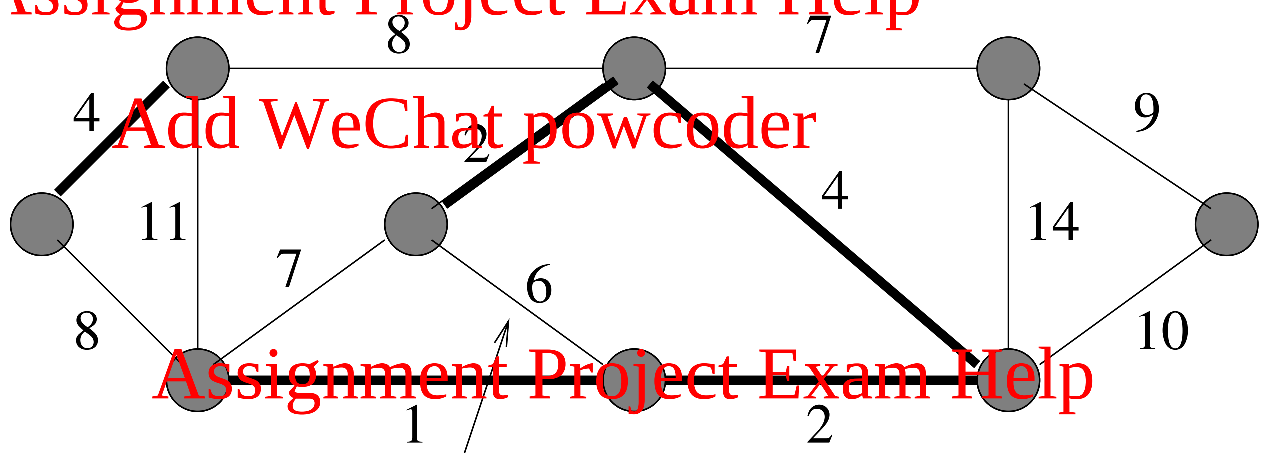
Example: Kruskal's Algorithm



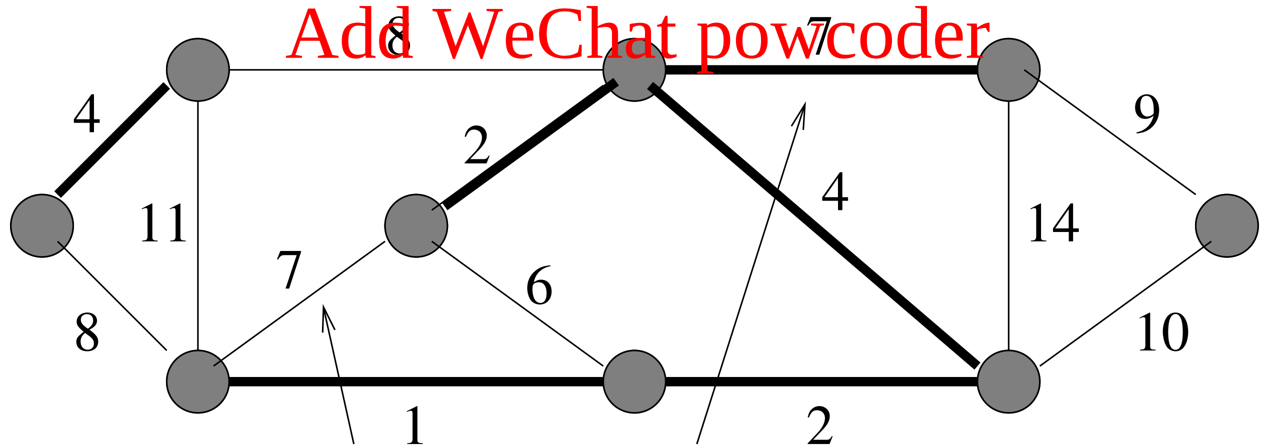


Links Added: 1. 2. 2. 4. 4

Example: Kruskal's Algorithm



We cannot add this link: creates a cycle



We can add the 7 that does not create a cycle

Example: Kruskal's Algorithm