COMP 250
INTRODUCTION TO COMPUTER SCIENCE

Week 10-2 : Recurrences

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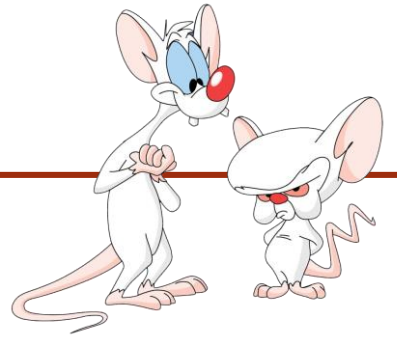
WHAT ARE WE GOING TO DO IN THIS VIDEO?



- Recurrences

ALGORITHM ANALYSIS

- We would like to find a function $T(n)$ that describes the running time of an algorithm given an input size n.
- It is relatively easy to determine $T(n)$ when our algorithms only have loops. (e.g. insertion sort from week 6)
- But how do we determine $T(n)$ for a recursive algorithm?

ALGORITHM ANALYSIS

Example: Suppose a list has n elements, what is $T(n)$ for the following algorithm?


```
reverse(list) {
   if(list.size()==1) {
      return;
   }
   firstElement = list.removeFirst();
   reverse(list); // now the list has n-1 elements
   list.addLast(firstElement);
}
```

RECURRENCES

"A recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs."

e.g. Fibonacci $F(n) = F(n-1) + F(n-2)$

We use recurrences to express the overall running time $T(n)$ of an algorithm with input size n in terms of the running time on smaller inputs.

Note that for Fibonacci number n is an input value. It is NOT the input size!

EXAMPLE 1: REVERSING A LIST

```
reverse(list) {
    if(list.size()==1) {
        return;
    }
    firstElement = list.removeFirst();
    reverse(list);
    list.addLast(firstElement);
}
```

Base case

Recursive step

$$T(1) = b, \qquad T(n) = c + T(n-1)$$

OBSERVATIONS

Q: What assumptions are we making about `removeFirst()` and `addLast()`?

A: They can be executed in constant time!

(Note that this is not true if we use an ArrayList.)

HOW TO SOLVE A RECURRENCE

There are different methods used to try to solve a recurrence:

- Forward substitution
- Back substitution
- Recursion-tree method
- Master Theorem

:

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SOLVING THE RECURRENCE USING BACK SUBSTITUTION

$$T(n) = c + T(n-1)$$

$$= c + c + T(n-2)$$

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$$T(n) = c + T(n-1)$$

$$= c + c + T(n-2)$$

$$= c + c + c + T(n-3)$$

$$= 3c + T(n-3) = \cdots$$

$$= kc + T(n-k) = \cdots$$

$$= (n-1)c + T(1)$$

$$= (n-1)c + b = c \cdot n + (b - c)$$

which is $\Theta(n)$.

EXAMPLE 2: SORTING A LIST

```
sort(list) {
    if(list.size()==1) {
        return;
    }
    minElement = list.removeMin();
    sort(list);
    list.addFirst(minElement);
}
```

Base case

Recursive step

$$T(1) = a, \quad T(n) = b + c \cdot n + T(n-1)$$

OBSERVATIONS

Q: What assumptions are we making about `addFirst()` ?

A: That can be executed in constant time.

Q: It would be ok if this step uses time proportional to n. Why??

A: Because `removeMin()` already takes time proportional to n.

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

Let's solve the following slightly simpler recurrence:

$$T(n) = cn + T(n-1)$$
$$= cn + c(n-1) + T(n-2)$$

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$$= cn + c(n-1) + T(n-2)$$
$$= cn + c(n-1) + c(n-2) + T(n-3)$$
$$\cdots$$
$$= c[n + (n-1) + (n-2) + \cdots + (n-k)] + T(n-k-1)$$
$$= c[n + (n-1) + (n-2) + \cdots + 2] + T(1), \quad when\ k = n-2$$

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$$= cn + c(n-1) + c(n-2) + T(n-3)$$
$$\cdots$$
$$= c[n + (n-1) + (n-2) + \cdots + (n-k)] + T(n-k-1)$$
$$= c[n + (n-1) + (n-2) + \cdots + 2] + T(1), \quad when\ k = n-2$$
$$= \frac{1}{2}cn^2 + \frac{1}{2}cn - c + a$$

which is $\Theta(n^2)$.

$$\sum_{i=2}^{n} i = \frac{1}{2}n(n+1) - 1$$

EXAMPLE 3: TOWER OF HANOI

```
tower(n, start, finish, other) {
    if(n>0) {
        tower(n-1, start, other, finish)
        move from start to finish
        tower(n-1, other, finish, start)
    }
}
```

Base case is n=0

Recursive step

$$T(0) = b, \qquad T(n) = c + 2T(n-1)$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

$$T(n) = c + 2T(n-1)$$

$$= c + 2\big(c + 2T(n-2)\big)$$

$$= c(1+2) + 4T(n-2)$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

$$T(n) = c + 2T(n-1)$$

$$= c + 2\big(c + 2T(n-2)\big)$$

$$= c(1+2) + 4T(n-2)$$

$$= c(1+2) + 4[c + 2T(n-3)]$$

$$= c(1+2+4) + 8T(n-3)$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

$$T(n) = c + 2T(n-1)$$
$$= c + 2\big(c + 2T(n-2)\big)$$
$$= c(1+2) + 4T(n-2)$$
$$= c(1+2) + 4[c + 2T(n-3)]$$
$$= c(1+2+4) + 8T(n-3)$$
$$\dots$$
$$= c[1+2+4+\cdots+2^{k-1}] + 2^k T(n-k)$$
$$= c[1+2+4+\cdots+2^{n-1}] + 2^n T(0), \quad when\ k = n$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

$$T(n) = c + 2T(n-1)$$
$$= c + 2\big(c + 2T(n-2)\big)$$
$$= c(1+2) + 4T(n-2)$$
$$= c(1+2) + 4[c + 2T(n-3)]$$
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$$\cdots$$
$$= c[1+2+4+\cdots+2^{k-1}] + 2^k T(n-k)$$
$$= c[1+2+4+\cdots+2^{n-1}] + 2^n T(0), \quad when\ k = n$$
$$= c(2^n - 1) + 2^n b = (c+b)2^n - c$$

which is $\Theta(2^n)$.

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

YOU SHOULD KNOW…

$$1 + 2 + 3 + \ldots + k = ?$$

$$1 + 2 + 4 + 8 + \ldots + 2^k = ?$$

$$1 + x + x^2 + x^3 + \ldots + x^k = ?$$

EXAMPLE 4: BINARY SEARCH

```
binarySearch(list, key, left, right){
    if(left <= right){
        mid = (left + right)/2
        if(list[mid]==key)
            return mid
        else {
            if(key<list[mid])
                return binarySearch(list, key, left, mid-1)
            else
                return binarySearch(list, key, mid+1, right)
        }
    }
    return -1
}
```


$$T_w(1) = b, \qquad T_w(n) = c + T\left(\frac{n}{2}\right)$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

To simply our analysis let's assume n is a power of 2. This will not affect the order of growth of the solution.

$$T(n) = c + T(n/2)$$

$$= c + c + T(n/4)$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

To simply our analysis let's assume n is a power of 2. This will not affect the order of growth of the solution.

$$T(n) = c + T(n/2)$$
$$= c + c + T(n/4)$$
$$= c + c + c + T(n/8)$$

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To simply our analysis let's assume n is a power of 2. This will not affect the order of growth of the solution.

$$T(n) = c + T(n/2)$$
$$= c + c + T(n/4)$$
$$= c + c + c + T(n/8)$$
$$\ldots$$
$$= c \cdot k + T(n/2^k)$$
$$= \mathrm{c} \cdot \log_2 n + T(1), \quad when\ k = \log_2 n$$

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

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$$T(n) = c + T(n/2)$$
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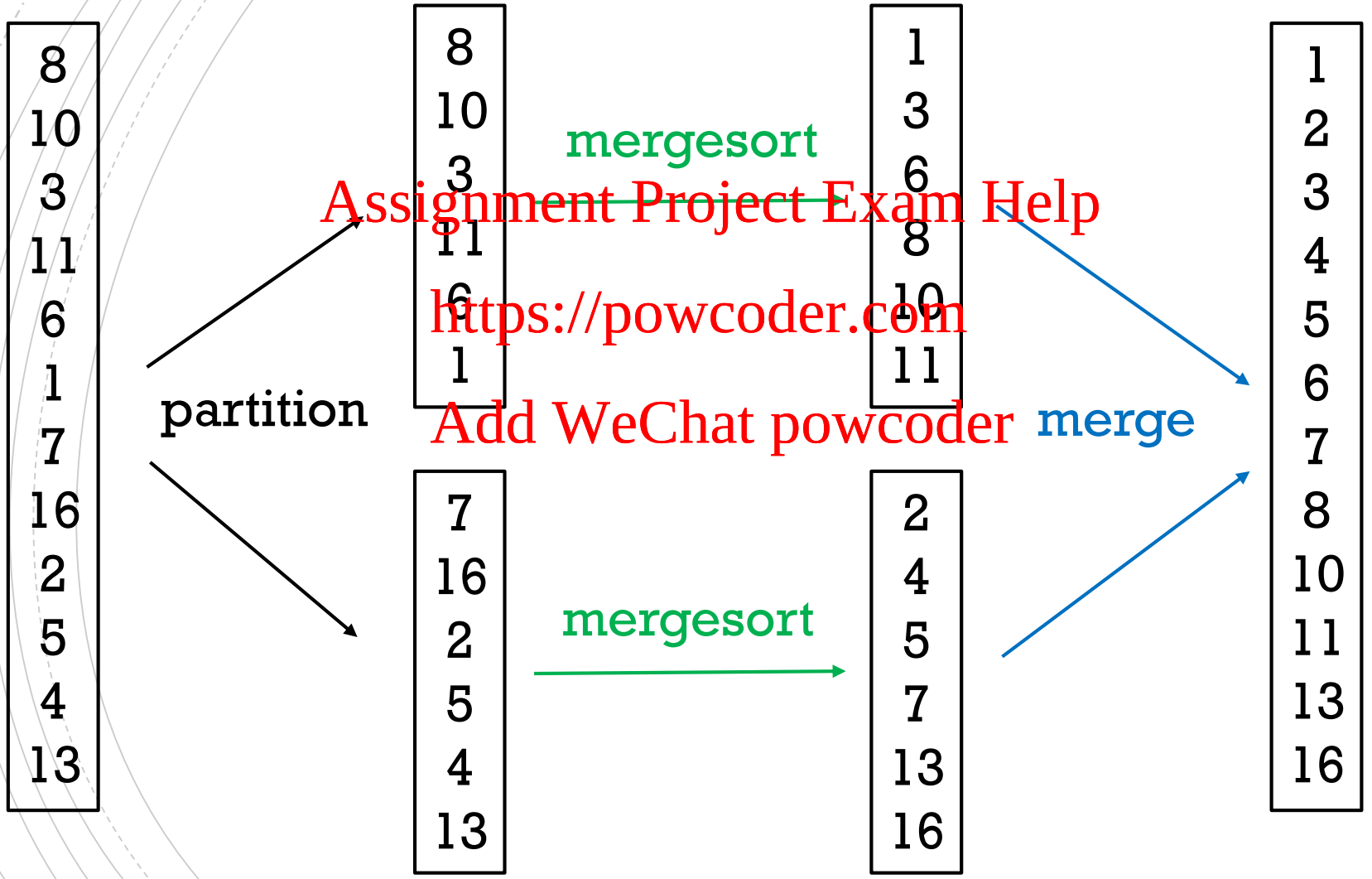
$$\ldots$$
$$= c \cdot k + T(n/2^k)$$
$$= c \cdot \log_2 n + T(1), \quad when\ k = \log_2 n$$
$$= c \cdot \log_2 n + b$$

which is $\Theta(\log_2 n)$.

RECALL MERGESORT



EXAMPLE 5: MERGESORT

```
mergesort(list){
        if (list.size() == 1)
                return list
        else {
            mid = (list.size() - 1) / 2
            list1 = list.getElements(0, mid)
            list2 = list.getElements(mid+1, list.size()-1)
            list1 = mergesort(list1)
            list2 = mergesort(list2)
            return merge(list1, list2)
        }
}
```

Base case

Recursive step

$$T(1) = a, \qquad T(n) = b + c \cdot n + 2 \cdot T\left(\frac{n}{2}\right)$$

Let's ignore the constant term for simplicity

WHAT IF n IS NOT EVEN?

Example: $t(13) = c * 13 + t(6) + t(7)$

In general, one should write the recurrence as:

$$T(n) = c\,n + T\left(floor\left(\frac{n}{2}\right)\right) + T\left(ceiling\left(\frac{n}{2}\right)\right)$$

In COMP250, one typically assumes $n = 2^k$ for recurrences that involve $T\left(\frac{n}{2}\right)$.

The more general recurrence has roughly the same solution.

SOLVING THE RECURRENCE USING BACK SUBSTITUTION

To simply our analysis let's assume n is a power of 2.

$$T(n) = cn + 2T\left(\frac{n}{2}\right)$$

$$= cn + 2\left(c\frac{n}{2} + 2T\left(\frac{n}{4}\right)\right) = cn + cn + 4T\left(\frac{n}{4}\right)$$

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$$= cn + cn + 4\left(c\frac{n}{4} + 2T\left(\frac{n}{8}\right)\right) = cn + cn + cn + 8T\left(\frac{n}{8}\right)$$

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To simply our analysis let's assume n is a power of 2.

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$$= cn + cn + 4\left(c\frac{n}{4} + 2T\left(\frac{n}{8}\right)\right) = cn + cn + cn + 8T\left(\frac{n}{8}\right)$$

...

$$= cn \cdot k + 2^k T\left(\frac{n}{2^k}\right)$$

$$= cn \cdot \log_2 n + 2^{\log_2 n} T(1), \quad when\ k = \log_2 n$$

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$$= cn + cn + 4\left(c\frac{n}{4} + 2T\left(\frac{n}{8}\right)\right) = cn + cn + cn + 8T\left(\frac{n}{8}\right)$$

...

$$= cn \cdot k + 2^k T\left(\frac{n}{2^k}\right)$$

$$= cn \cdot \log_2 n + 2^{\log_2 n} T(1), \quad when\ k = \log_2 n$$

$$= cn \log_2 n + bn$$

which is $\Theta(n \log_2 n)$.

TODAY'S RECURRENCES

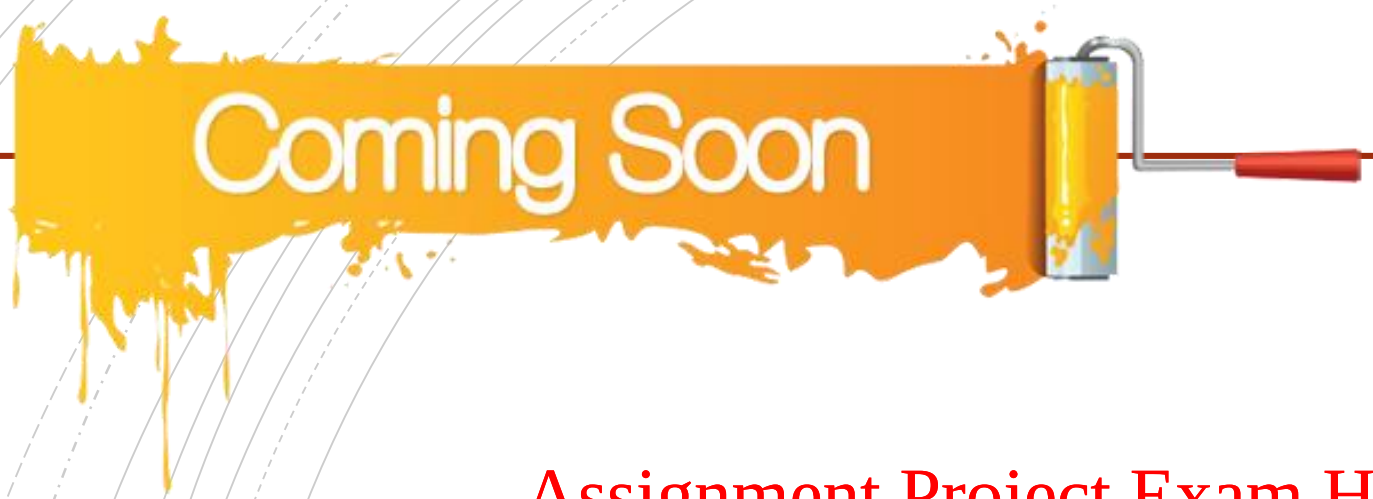
$$T(n) = c + T(n-1)$$

$$T(n) = c\,n + T(n-1)$$

$$T(n) = c + 2\,T(n-1)$$

$$T(n) = c + T\left(\frac{n}{2}\right)$$

$$T(n) = c\,n + T\left(\frac{n}{2}\right)$$



In the next videos:

- Trees