COMP251: Divide-and-Conquer (2)

Assignment Project Exam Help

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Based on (Kleinberg & Tardos, 2005) & (Cormen *et al.*,2009)

How to determine the running time of a divide-and-conquer algorithm?

The Master Theorem

Recursive definition

T(n): execution time on an input of size n.

Merge Sort: $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$

Binary Search: $T(n) = T\left(\frac{n}{2}\right) + 1$

Karatsuba: $T(n) = 3 \cdot T\left(\frac{n}{2}\right) + n$

Number of recursive calls

Time to merge

Size of sub-problems

Master method

Goal. Recipe for solving common divide-and-conquer recurrences:

$$T(n) = a\,T\left(\frac{n}{b}\right) \; + \; f(n)$$

Terms.

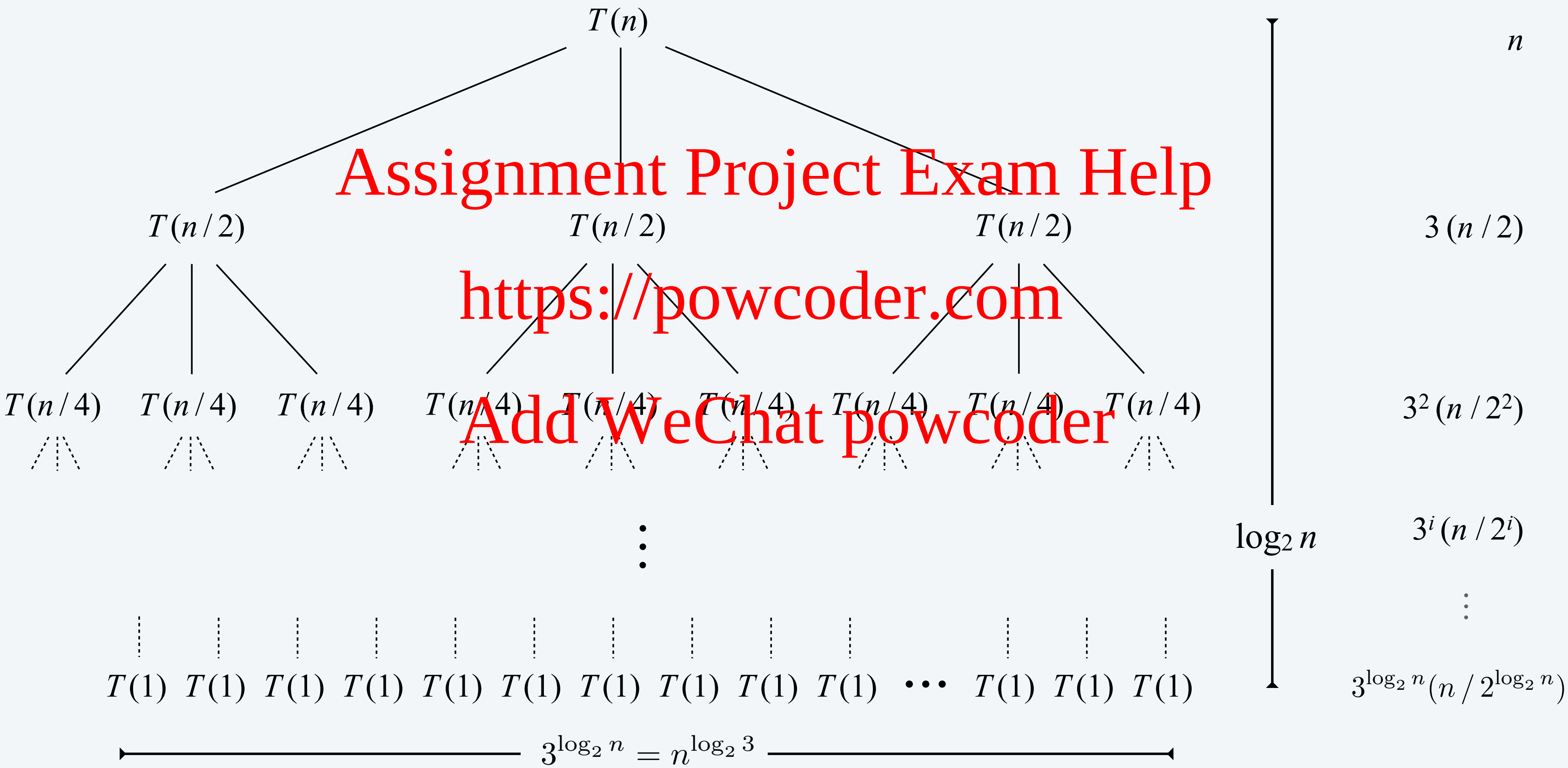
- $a \geq 1$ is the number of subproblems.
- $b > 0$ is the factor by which the subproblem size decreases.
- $f(n)$ = work to divide/merge subproblems.

Recursion tree.

- $k = \log_b n$ levels.
- a^i = number of subproblems at level i.
- n / b^i = size of subproblem at level i.

Case 1: total cost dominated by cost of leaves

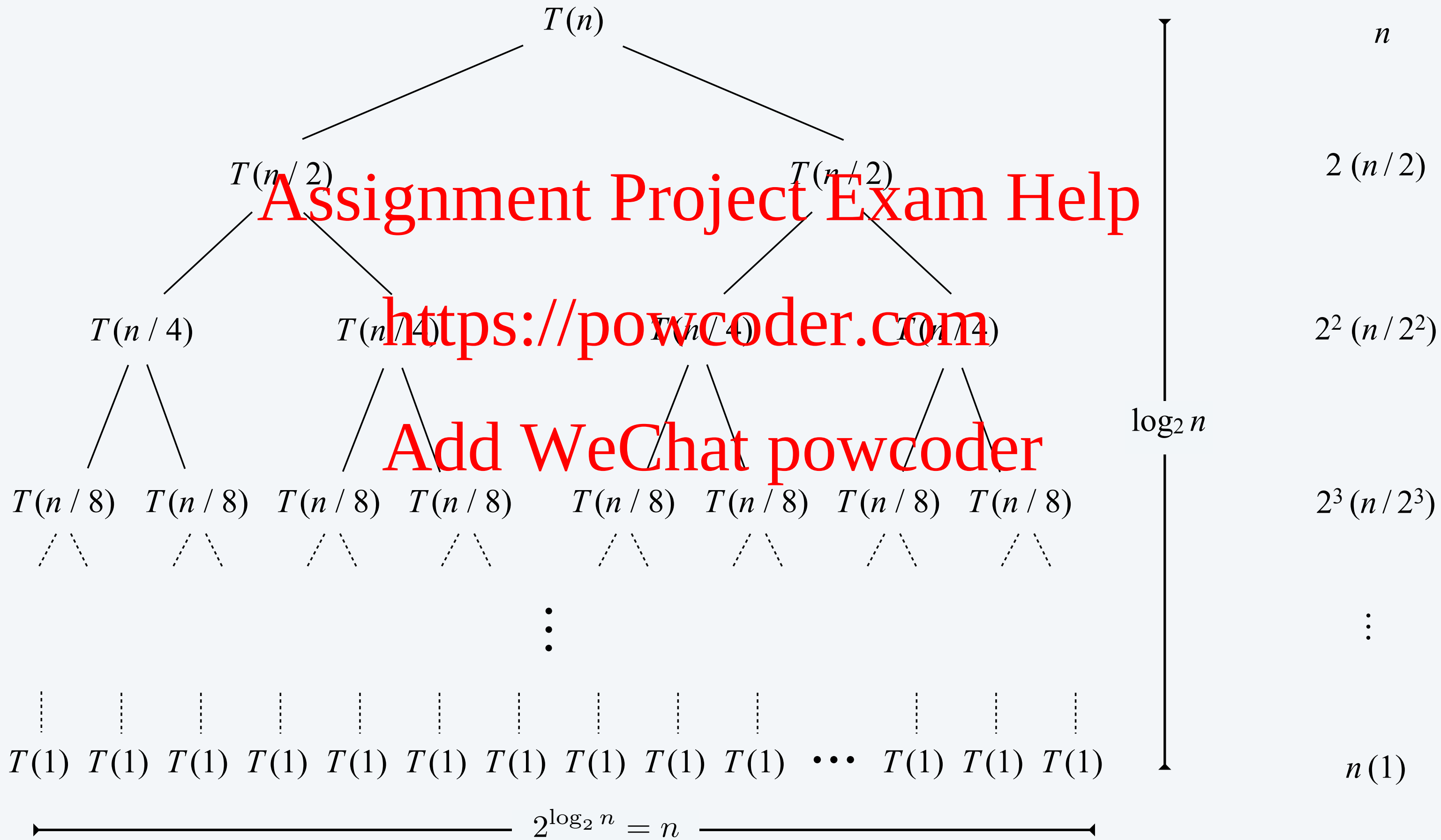
Ex 1. If $T(n)$ satisfies $T(n) = 3\,T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n^{\lg 3})$.



$r = 3/2 > 1$ $\quad T(n) = (1 + r + r^2 + r^3 + \ldots + r^{\log_2 n})\, n = \dfrac{r^{1+\log_2 n} - 1}{r - 1}\, n = 3n^{\log_2 3} - 2n$

Case 2: total cost evenly distributed among levels

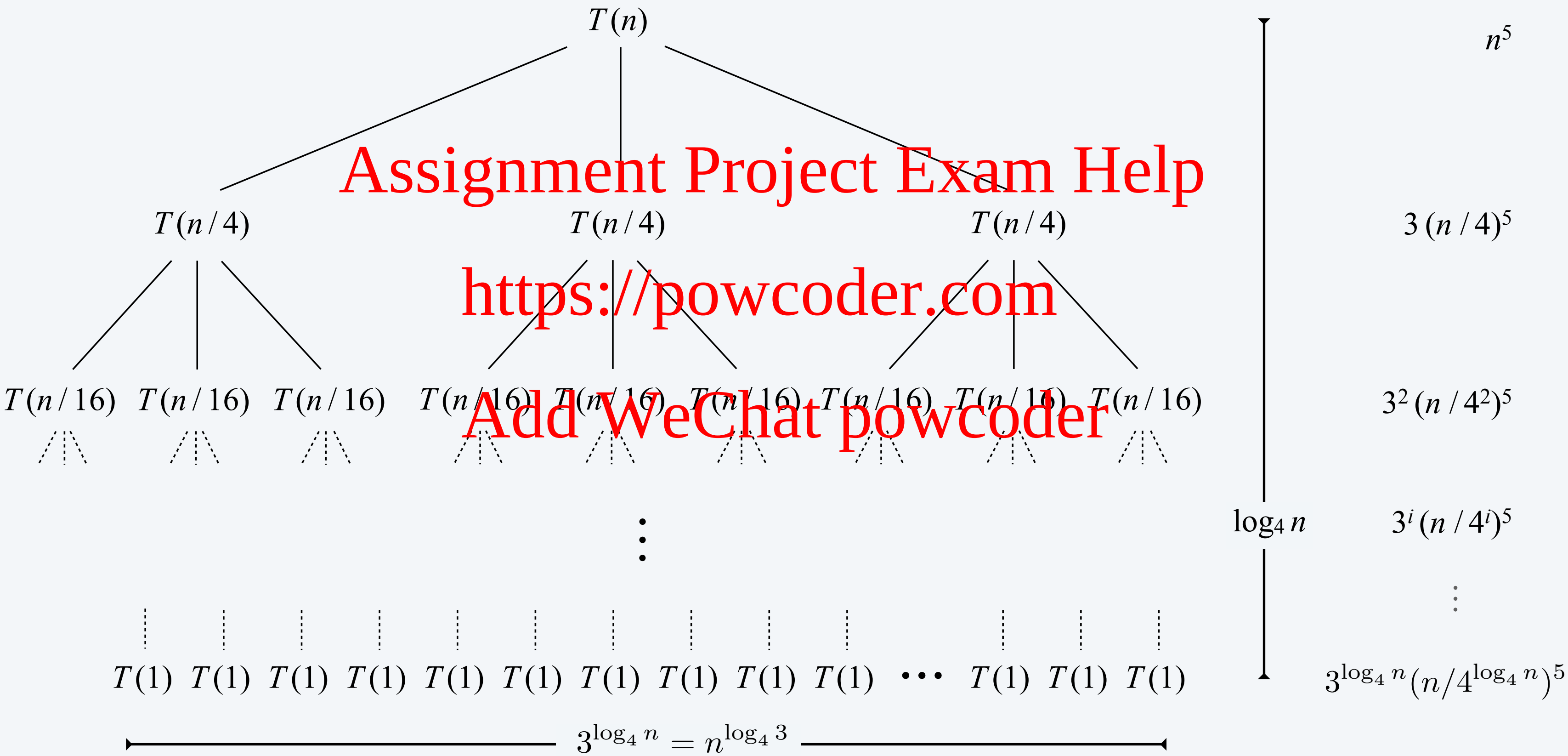
Ex 2. If $T(n)$ satisfies $T(n) = 2\,T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n \log n)$.



$r = 1$ $\qquad T(n) = (1 + r + r^2 + r^3 + \ldots + r^{\log_2 n})\, n \;=\; n\,(\log_2 n + 1)$

Case 3: total cost dominated by cost of root

Ex 3. If $T(n)$ satisfies $T(n) = 3\,T(n/4) + n^5$, with $T(1) = 1$, then $T(n) = \Theta(n^5)$.



$r = 3 / 4^5 < 1 \qquad n^5 \le T(n) \le (1 + r + r^2 + r^3 + \ldots)\, n^5 \le \dfrac{1}{1-r}\, n^5$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Ex. $T(n) = 3\,T(n/2) + n$.

- $a = 3,\ b = 2,\ f(n) = n,\ k = \log_2 3$.
- $T(n) = \Theta(n^{\lg 3})$.

The formula works with $\varepsilon = \log_2 3 - 1 > 0$

$f(n) = n = O\left(n^{\log_2 3 - (\log_2 3 - 1)}\right)$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) \;+\; f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Ex. $T(n) = 2\,T(n/2) + \Theta(n \log n)$.

- $a = 2,\ b = 2,\ f(n) = n \log n\ k = \log_2 2 = 1,\ p = 1$.
- $T(n) = \Theta(n \log^2 n)$.

$$f(n) = \Theta(n \log n) = \Theta(n^{\log_2 2} \log n)$$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

regularity condition holds
if $f(n) = \Theta(n^{k+\varepsilon})$

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Ex. $T(n) = 3\,T(n/4) + n^5$.

- $a = 3,\ b = 4,\ f(n) = n^5,\ k = \log_4 3$.
- $T(n) = \Theta(n^5)$.

1st property satisfied with $\varepsilon = 1 - \log_4 3$

$$f(n) = n^5 = \Omega(n^{\log_4 3 + (1 - \log_4 3)})$$

2nd property satisfied with $c = \frac{3}{4}$

$$3 \cdot \left(\frac{n}{4}\right)^5 \le c \cdot n^5$$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Pf sketch.

- Use recursion tree to sum up terms (assuming n is an exact power of b).
- Three cases for geometric series.
- Deal with floors and ceilings.

Applications

$k = \log_2 3; f(n) = n^2$
$n^2 = \Omega\left(n^{\log_2 3 + (2 - \log_2 3)}\right)$
$3 \cdot \left(\frac{n}{2}\right)^2 \leq \frac{3}{4} \cdot n^2$

T(n) = 3 * T(n/2) + n^2
⇒ T(n) = $\Theta(n^2)$ (case 3)

$k = \log_2 1 = 0; f(n) = 2^n$
$2^n = \Omega\left(n^{0+\log 2}\right)$
$1 \cdot 2^{\frac{n}{2}} \leq \frac{1}{2} \cdot 2^n$

T(n) = T(n/2) + 2^n
⇒ T(n) = $\Theta(2^n)$ (case 3)

T(n) = 16 * T(n/4) + n

⇒ T(n) = $\Theta(n^2)$ (case 1)

$k = \log_4 16 = 2; f(n) = n$
$n = O(n^{2-1})$

T(n) = 2 * T(n/2) + n log n
⇒ T(n) = n $\log^2$ n (case 2)

$k = \log_2 2 = 1; f(n) = n \log n$
$n \log n = \Theta(n^1 log^1 n)$

T(n) = 2^n * T(n/2) + n^n
⇒ Does not apply!!

Akra-Bazzi theorem

Desiderata. Generalizes master theorem to divide-and-conquer algorithms where subproblems have substantially different sizes.

Theorem. [Akra-Bazzi] Given constants $a_i > 0$ and $0 < b_i \le 1$, functions $h_i(n) = O(n / \log^2 n)$ and $g(n) = O(n^c)$, if the function $T(n)$ satisfies the recurrence:

$$T(n) = \sum_{i=1}^{k} a_i \, T\left(b_i n + h_i(n)\right) + g(n)$$

a_i subproblems of size b_i n

small perturbation to handle floors and ceilings

Then $T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}}du\right)\right)$ where p satisfies $\sum_{i=1}^{k} a_i \, b_i^p = 1$.

Ex. $T(n) = 7/4 \; T(\lfloor n/2 \rfloor) + T(\lceil 3/4 \, n \rceil) + n^2$.

- $a_1 = 7/4, \; b_1 = 1/2, \; a_2 = 1, \; b_2 = 3/4 \;\Rightarrow\; p = 2$.
- $h_1(n) = \lfloor 1/2 \, n \rfloor - 1/2 \, n, \; h_2(n) = \lceil 3/4 \, n \rceil - 3/4 \, n$.
- $g(n) = n^2 \;\Rightarrow\; T(n) = \Theta(n^2 \log n)$.

Another Divide-and-Conquer Algorithms

Dot product

Dot product. Given two length n vectors a and b, compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i \, b_i$$

$$a = \begin{bmatrix} .70 & .20 & .10 \end{bmatrix}$$

$$b = \begin{bmatrix} .30 & .40 & .30 \end{bmatrix}$$

$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Remark. Grade-school dot product algorithm is asymptotically optimal.

Matrix multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

$$\underbrace{C = AB}_{c_{ij} \,=\, \sum_{k=1}^{n} a_{ik}\, b_{kj}}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm asymptotically optimal?

Block matrix multiplication

C_{11} A_{11} A_{12} B_{11}

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

B_{11}

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix multiplication: warmup

To multiply two n-by-n matrices A and B:

- Divide: partition A and B into ½n-by-½n blocks.
- Conquer: multiply 8 pairs of ½n-by-½n matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

Running time. Apply case 1 of Master Theorem.

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Strassen's trick

Key idea. multiply 2-by-2 blocks with only 7 multiplications.
(plus 11 additions and 7 subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_1 + P_5 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &\leftarrow A_{11} \times (B_{12} - B_{22}) \\ P_2 &\leftarrow (A_{11} + A_{12}) \times B_{22} \\ P_3 &\leftarrow (A_{21} + A_{22}) \times B_{11} \\ P_4 &\leftarrow A_{22} \times (B_{21} - B_{11}) \\ P_5 &\leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &\leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &\leftarrow (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Pf.
$$\begin{aligned} C_{12} &= P_1 + P_2 \\ &= A_{11} \times (B_{12} - B_{22}) + (A_{11} + A_{12}) \times B_{22} \\ &= A_{11} \times B_{12} + A_{12} \times B_{22}. \ \checkmark \end{aligned}$$

Strassen's algorithm

STRASSEN (n, A, B)

assume n is a power of 2

IF $(n = 1)$ RETURN $A \times B$.

Partition A and B into 2-by-2 block matrices.

keep track of indices of submatrices (don't copy matrix entries)

$P_1 \leftarrow$ STRASSEN $(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN $(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN $(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN $(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN $(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

Analysis of Strassen's algorithm

Theorem. Strassen's algorithm requires $O(n^{2.81})$ arithmetic operations to multiply two n-by-n matrices.

Pf. Apply case 1 of the master theorem to the recurrence:

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Q. What if n is not a power of 2?

A. Could pad matrices with zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10 & 11 & 12 & 0 \\ 13 & 14 & 15 & 0 \\ 16 & 17 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 84 & 90 & 96 & 0 \\ 201 & 216 & 231 & 0 \\ 318 & 342 & 366 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Strassen's algorithm: practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm when n is "small".

Common misperception. *"Strassen is only a theoretical curiosity."*

- Apple reports 8x speedup on G4 Velocity Engine when $n \approx 2{,}048$.
- Range of instances where it's useful is a subject of controversy.

Linear algebra reductions

Matrix multiplication. Given two n-by-n matrices, compute their product.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	$\Theta(MM(n))$
matrix inversion	A^{-1}	$\Theta(MM(n))$
determinant	$\lvert A \rvert$	$\Theta(MM(n))$
system of linear equations	$Ax = b$	$\Theta(MM(n))$
LU decomposition	$A = L\,U$	$\Theta(MM(n))$
least squares	$\min \lVert Ax - b \rVert_2$	$\Theta(MM(n))$

numerical linear algebra problems with the same complexity as matrix multiplication

Fast matrix multiplication: theory

Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

A. Yes! [Strassen 1969] $\Theta(n^{\log_2 7}) = O(n^{2.807})$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Multiply two 3-by-3 matrices with 21 scalar multiplications?

A. Unknown. $\Theta(n^{\log_3 21}) = O(n^{2.77})$

Begun, the decimal wars have. [Pan, Bini et al, Schönhage, …]

- Two 20-by-20 matrices with 4,460 scalar multiplications. $O(n^{2.805})$
- Two 48-by-48 matrices with 47,217 scalar multiplications. $O(n^{2.7801})$
- A year later. $O(n^{2.7799})$
- December 1979. $O(n^{2.521813})$
- January 1980. $O(n^{2.521801})$

History of asymptotic complexity of matrix multiplication

year	algorithm	order of growth
?	brute force	$O(n^3)$
1969	Strassen	$O(n^{2.808})$
1978	Pan	$O(n^{2.796})$
1979	Bini	$O(n^{2.780})$
1981	Schönhage	$O(n^{2.522})$
1982	Romani	$O(n^{2.517})$
1982	Coppersmith-Winograd	$O(n^{2.496})$
1986	Strassen	$O(n^{2.479})$
1989	Coppersmith-Winograd	$O(n^{2.376})$
2010	Strother	$O(n^{2.3737})$
2011	Williams	$O(n^{2.3727})$
?	?	$O(n^{2+\varepsilon})$

number of floating-point operations to multiply two n-by-n matrices