Assignment Project Exam Help

Algorithms: COMP3121/9101

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School of Computer Science and Engineering
University of New South Wales

10. LINEAR PROGRAMMING

Problem:

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  - the price of all food per day is as low as possible.

- To obtain the corresponding constraints let us assume that we take x_i grams of each food source f_i for $1 \leq i \leq n$. Then:

Linear Programming problems - Example 1 cont.

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- Note that all constraints and the objective function are **linear**.

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- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.
- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible, i.e., that the total cost of your promises is as low as possible.

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- Such problems are MUCH harder to solve than the "plain" Linear Programming problems whose solutions can be real numbers.

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- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \ldots x_n \rangle^{\intercal}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

- Letting $\mathbf{c} = \langle c_1 \ldots c_n \rangle^{\intercal} \in \mathbb{R}^n$ and $\mathbf{b} = \langle b_1 \ldots b_m \rangle^{\intercal} \in \mathbb{R}^m$, and letting A be the matrix $A = (a_{ij})$ of size $m \times n$, we get that the above problem can be formulated simply as:
  - maximize $\mathbf{c}^{\intercal}\mathbf{x}$
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- Thus, to specify a Linear Programming optimisation problem we just have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

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- This poses no problem, because each occurrence of an unconstrained variable x_j can be replaced by the expression $x'_j - x^*_j$ where x'_j, x^*_j are new variables satisfying the constraints $x'_j \geq 0$, $x^*_j \geq 0$.

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- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, we let $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$. Some problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

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maximize $$z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{3}$$

subject to the constraints

$$x_1 + x_2 + 3x_3 \leq 30 \tag{4}$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{5}$$

$$4x_1 + x_2 + 2x_3 \leq 36 \tag{6}$$

$$x_1, x_2, x_3 \geq 0 \tag{7}$$

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- As an example, let us consider the following optimisation problem.

maximize $$z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{3}$$

subject to the constraints

$$x_1 + x_2 + 3x_3 \le 30 \tag{4}$$
$$2x_1 + 2x_2 + 5x_3 \le 24 \tag{5}$$
$$4x_1 + x_2 + 2x_3 \le 36 \tag{6}$$
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$$3x_1 + 3x_2 + 8x_3 \leq 54 \tag{8}$$

Linear Programming - Standard Form

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- Since all variables are constrained to be non-negative, we are assured that

$$3x_1 + x_2 + 2x_3 \leq 3x_1 + 3x_2 + 8x_3 \leq 54$$

Linear Programming - Standard Form

maximize: $$z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \quad (3)$$

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- Thus the objective $z(x_1, x_2, x_3)$ is bounded above by 54, i.e., $z(x_1, x_2, x_3) \leq 54$.

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- Thus the objective $z(x_1,x_2,x_3)$ is bounded above by 54, i.e., $z(x_1,x_2,x_3) \le 54$.
- Can we obtain a tighter bound? We could try to look for coefficients $y_1, y_2, y_3 \ge 0$ to be used to for a linear combination of the constraints:

$$
\begin{aligned}
y_1(x_1 + x_2 + 3x_3) &\le 30y_1\\
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- Then, summing up all these inequalities and factoring, we get

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- If we compare this with our objective (3) we see that if we choose y_1, y_2 and y_3 so that:

$$
\begin{aligned}
y_1 + 2y_2 + 4y_3 &\geq 3\\
y_1 + 2y_2 + y_3 &\geq 1\\
3y_1 + 5y_2 + 2y_3 &\geq 2
\end{aligned}
$$

then

$$3x_3 + x_2 + 2x_3 \leq x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3)$$

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Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

Linear Programming - Standard Form

- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem P:

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we have to look for y_1, y_2, y_3 which solve problem P^*:

$$
\begin{aligned}
&\text{minimise:} && z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 && (10)\\
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& && y_1 + 2y_2 + y_3 \geq 1 && (12)\\
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then $z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$ will be a tight upper bound for $z(x_1, x_2, x_3)$

- The new problem P^* is called the *dual problem* for the problem P.

- Let us now repeat the whole procedure with P^* in place of P, i.e., let us find the dual program $(P^*)^*$ of P^*.

Linear Programming - Standard Form

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- Combining this with (15) we get

$$3z_1 + z_2 + 2z_3 \leq 30y_1 + 24y_2 + 36y_3$$

Linear Programming - Standard Form

- Consequently, finding the dual program $(P^*)^*$ of P^* amounts to maximising the objective $3z_1 + z_2 + 2z_3$ subject to the constraints

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- So, at the first sight, looking for the multipliers y_1, y_2, y_3 did not help much, because it only reduced a maximisation problem to an equally hard minimisation problem.
- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

Linear Programming - primal/dual problem forms

- The original, *primal* Linear Program P and its *dual* Linear Program can be easily described in the most general case:

$$P: \quad \text{maximize} \qquad z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to the constraints} \qquad \sum_{j=1}^{n} a_{ij} x_j \leq b_i; \quad 1 \leq i \leq m$$

$$x_1, \ldots, x_n \geq 0,$$

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or, in matrix form,

$$P: \quad \text{maximize} \quad z(\mathbf{x}) = \mathbf{c}^{\intercal}\mathbf{x}, \ \text{subject to the constraints} \quad A\mathbf{x} \leq \mathbf{b} \ \text{ and } \ \mathbf{x} \geq 0;$$

$$P^*: \quad \text{minimize} \quad z^*(\mathbf{y}) = \mathbf{b}^{\intercal}\mathbf{y}, \ \text{subject to the constraints} \quad A^{\intercal}\mathbf{y} \geq \mathbf{c} \ \text{ and } \ \mathbf{y} \geq 0.$$

Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\intercal}\mathbf{x}$ might be.

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- **Theorem** If $x = \langle x_1 \dots x_n \rangle$ is any basic feasible solution for P and $y = \langle y_1 \dots y_m \rangle$ is any basic feasible solution for P^*, then:

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

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Proof: Since x and y are basic feasible solutions for P and P^* respectively, we can use the constraint inequalities, first from P^* and then from P to obtain

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

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$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

- Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and

Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\intercal}\mathbf{x}$ might be.
- **Theorem** If $x = \langle x_1 \ldots x_n \rangle$ is any basic feasible solution for P and $y = \langle y_1 \ldots y_m \rangle$ is any basic feasible solution for P^*, then:

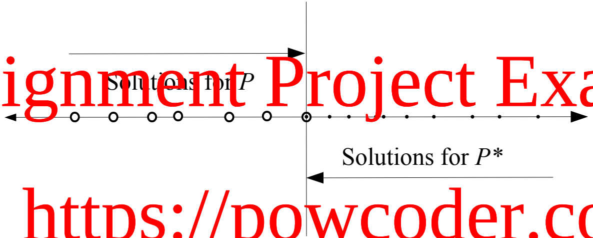
$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

Proof: Since x and y are basic feasible solutions for P and P^* respectively, we can use the constraint inequalities, first from P^* and then from P to obtain

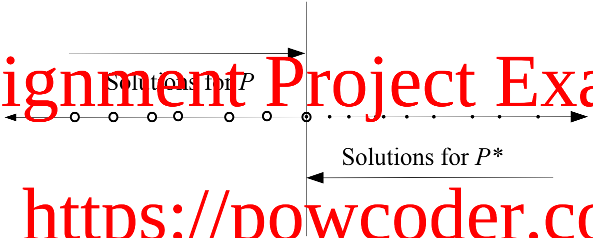
$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

- Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and
- every feasible solution of P is a lower bound for the set of feasible solutions for P^*.

Weak Duality Theorem

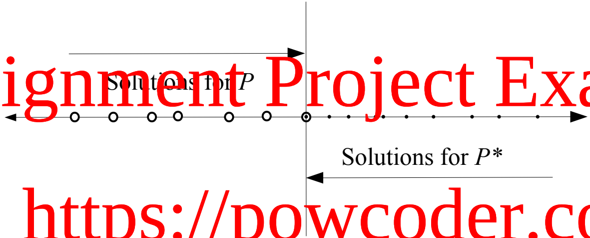


- Thus, if we find a feasible solution for P which is equal to a feasible solution to P^*, such solution must be the maximal feasible value of the objective of P and the minimal feasible value of the objective of P^*.



- Thus, if we find a feasible solution for P which is equal to a feasible solution to P^*, such solution must be the maximal feasible value of the objective of P and the minimal feasible value of the objective of P^*.
- If we use a search procedure to find an optimal solution for P we know when to stop: when such a value is also a feasible solution for P^*.

Weak Duality Theorem



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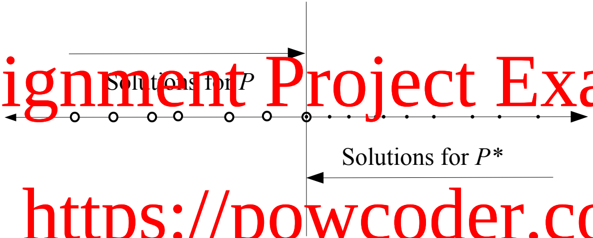
- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.

Weak Duality Theorem



- Thus, if we find a feasible solution for P which is equal to a feasible solution to P^*, such solution must be the maximal feasible value of the objective of P and the minimal feasible value of the objective of P^*.
- If we use a search procedure to find an optimal solution for P we know when to stop: when such a value is also a feasible solution for P^*.
- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.
- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

Five sisters are alone in their house. Sharon is reading a book, Jennifer is playing chess, Cathrine is cooking and Ana is doing laundry. What is Helen, the fifth sister, doing?