Algorithms

COMP3121/9101

Aleks Ignjatović, ignjat@cse.unsw.edu.au
office: 504 (CSE building)
Course Admin: Anahita Namvar, cs3121@cse.unsw.edu.au

School of Computer Science and Engineering
University of New South Wales Sydney

1. INTRODUCTION

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The word "algorithm" comes by corruption of the name of *Muhammad ibn Musa al-Khwarizmi*, a Persian scientist 780-850, who wrote an important book on algebra, *"Al-kitab al-mukhtasar fi hisab al-gabr wal-muqabala"*. You are encouraged to read about him in Wikipedia.

In this course we will deal only with sequential deterministic algorithms which means that:

- they are given as sequences of steps, thus assuming that only one step can be executed at a time;
- the action of each step gives the same result whenever this step is executed for the same input.

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Course content:

- a survey of algorithm **design techniques**
- particular algorithms will be mostly used to illustrate design techniques
- emphasis on development of your algorithm design **skills**

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paperback edition available at the UNSW book store

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The hard part: how can a thief split the pile into two equal parts? Remarkably, this turns out that, most likely, there is no more efficient algorithm than the brute force: we consider all partitions of the pile and see if there is one which results in two equal parts.

Problem:

Three thieves have robbed a warehouse and have to split a pile of items without price tags on them. How do they do this in a way that ensures that each thief believes that he has got at least one third of the loot?

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- If they choose different piles, they can each take the piles they have chosen and the first thief gets the remaining pile; in this case clearly each thief thinks that he got at least one third of the loot.
- But what if the remaining two thieves choose the same pile?

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- What would be a correct algorithm?
- Let the thieves be T_1, T_2, T_3;

Algorithm:

T_1 makes a pile P_1 which he believes is 1/3 of the whole loot;

T_1 proceeds to ask T_2 if T_2 agrees that $P_1 \leq 1/3$;

If T_2 says YES, **then** T_1 asks T_3 if T_3 agrees that $P_1 \leq 1/3$;

  If T_3 says YES, **then** T_1 takes P_1;

   T_2 and T_3 split the rest as in Problem 1.

  Else if T_3 says NO, **then** T_3 takes P_1;

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Else if T_2 says NO, **then** T_2 reduces the size of P_1 to $P_2 < P_1$ such that T_2 thinks $P_2 = 1/3$;

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Hint: there is a *nested recursion* happening even with 3 thieves!

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Mathematical proofs are **NOT** academic embellishments; we use them to justify things which are not obvious to common sense!

Example: MERGESORT

Merge-Sort(A,p,r) *sorting A[p..r]*

1. **if** $p < r$
2. **then** $q \leftarrow \lfloor \frac{p+r}{2} \rfloor$
3. MERGE-SORT(A, p, q)
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4. Merging two sorted arrays always produces a sorted array, thus, the output of MERGESORT will be a sorted array.
5. The above is essentially a proof by induction, but we will never bother formalising proofs of (essentially) obvious facts.

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- **However, <u>BE VERY CAREFUL</u> what you call trivial!!**

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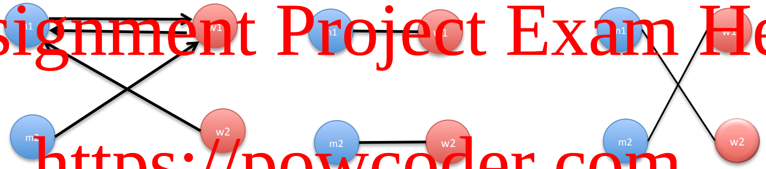
for two pairs $p = (m, w)$ and $p' = (m', w')$:

- man m prefers woman w' to woman w, **and**
- woman w' prefers man m to man m'.

Stable Matching Problem: examples

Case1:

Preferences: | stable matching | not stable



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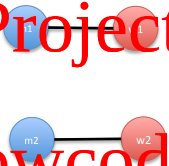
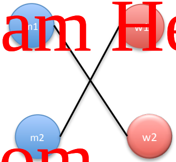
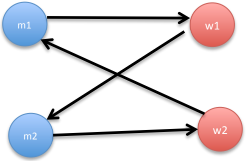


not stable

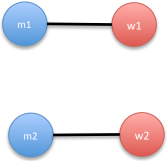


Case2:

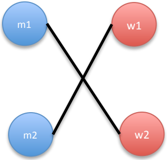
Preferences:



stable matching



also stable!



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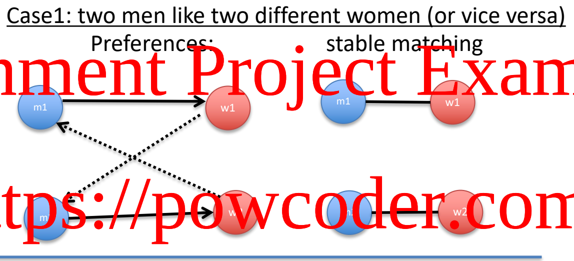

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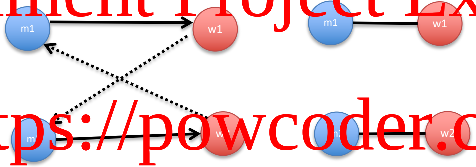
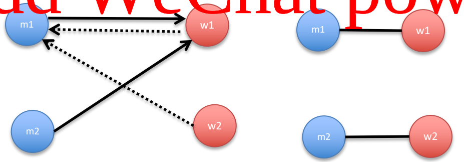
Case1: two men like two different women (or vice versa)

Preferences: stable matching



Case2: men like the same woman and women like the same man

Preferences: stable matching



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Given n men and n women, how many ways are there to match them, i.e., just to form n couples?

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    Else m is lower on her preference list than m';
      the proposal is rejected and m remains free.

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- **Contradiction!**

A Puzzle!!!

Why puzzles? It is a fun way to practice problem solving!

Problem : Tom and his wife Mary went to a party where nine more couples were present.

- Not every one knew everyone else, so people who did not know each other introduced themselves and shook hands.
- People who knew each other from before did not shake hands.
- Later that evening Tom got bored, so he walked around and asked all other guests (including his wife) how many hands they had shaken that evening, and got 19 different answers.
- How many hands did Mary shake?
- How many hands did Tom shake?

That's All, Folks!!