Algorithms:
COMP3121/9101

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2. DIVIDE-AND-CONQUER

A Puzzle

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- We have already seen a prototypical "serious" algorithm designed using such a method: the MERGE-SORT.
- We split the array into two, sort the two parts recursively and then merge the two sorted arrays.
- We now look at a closely related but more interesting problem of counting inversions in an array.

Counting the number of inversions

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- Lets enumerate the movies on the ranking list of user B by assigning to the top choice of user B index 1, assign to his second choice index 2 and so on.

- For the i^{th} movie on $B's$ list we can now look at the position (i.e., index) of that movie on $A's$ list, denoted by $a(i)$.

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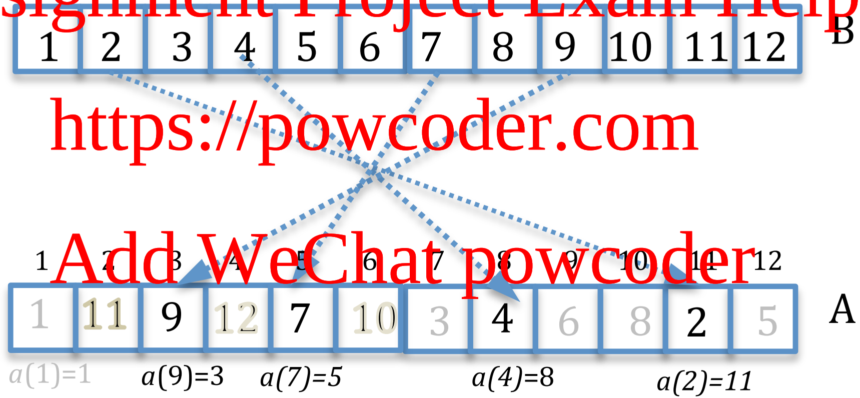

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- A good measure of how different these two users are, is the total number of *inversions*, i.e., total number of pairs of movies i, j such that movie i precedes movie j on $B's$ list but movie j is higher up on $A's$ list than the movie i.

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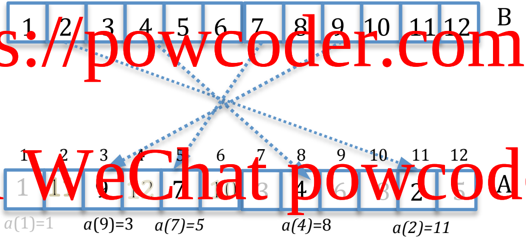
- In other words, we count the number of pairs of movies i, j such that $i < j$ (movie i precedes movie j on $B's$ list) but $a(i) > a(j)$ (movie i is in the position $a(i)$ on $A's$ list which is after the position $a(j)$ of movie j on $A's$ list.



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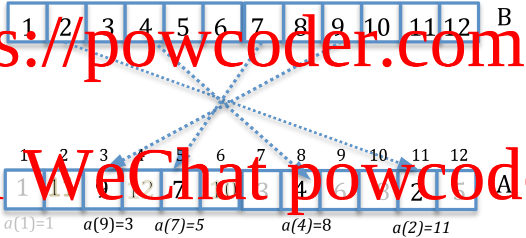

- For example 1 and 2 do not form an inversion because $a(1) < a(2)$ ($a(1) = 1$ and $a(2) = 11$ because $a(1)$ is on the first and $a(2)$ is on the 11^{th} place in A);

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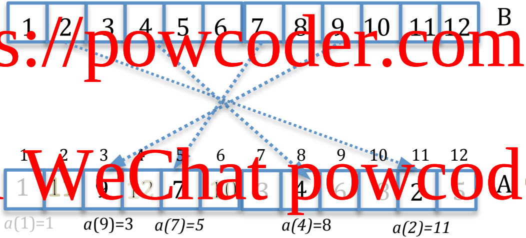

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- However, for example 4 and 7 do form an inversion because $a(7) < a(4)$ ($a(7) = 5$ because seven is on the fifth place in A and $a(4) = 8$)

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- An easy way to count the total number of inversions between two lists is by looking at all pairs $i < j$ of movies on one list and determining if they are inverted in the second list, but this would produce a quadratic time algorithm, $T(n) = \Theta(n^2)$.

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- Clearly, since the total number of pairs is quadratic in n, we cannot afford to inspect all possible pairs.
- The main idea is to tweak the MERGE-SORT algorithm, by extending it to recursively both sort an array A **and** determine the number of inversions in A.

Counting the number of inversions

- We split the array A into two (approximately) equal parts $A_{top} = A[1 \ldots \lfloor n/2 \rfloor]$ and $A_{bottom} = A[\lfloor n/2 \rfloor + 1 \ldots n]$.

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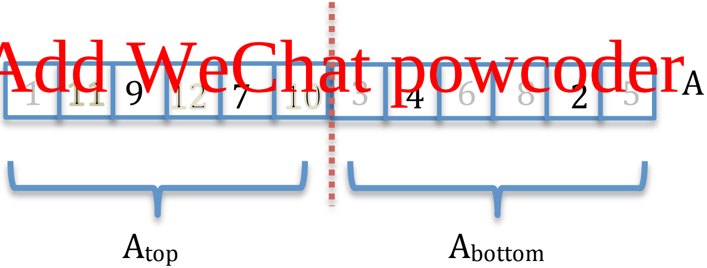
- Note that the total number of inversions in array A is equal to the sum of the number of inversions $I(A_{top})$ in A_{top} (such as 9 and 7) plus the number of inversions $I(A_{bottom})$ in A_{bottom} (such as 4 and 2) plus the number of inversions $I(A_{top}, A_{bottom})$ across the two halves (such as 7 and 4).



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- We now recursively sort arrays A_{top} and A_{bottom} also obtaining in the process the number of inversions $I(A_{top})$ in the sub-array A_{top} and the number of inversions $I(A_{bottom})$ in the sub-array A_{bottom}.

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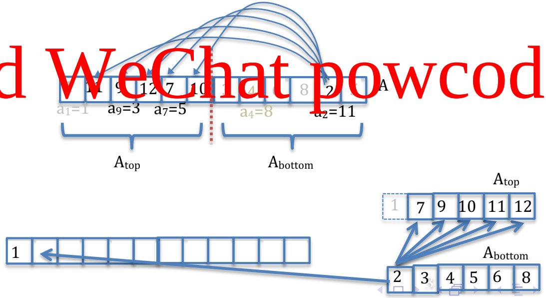
- When the next smallest element among all elements in both arrays is an element in A_{bottom}, such an element clearly is in an inversion with all the remaining elements in A_{top} and we add the total number of elements remaining in A_{top} to the current value of the number of inversions across A_{top} and A_{bottom}.



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- **Next:** we study applications of divide and conquer to arithmetic of very large integers.

Basics revisited: how do we add two numbers?

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- no asymptotically faster algorithm

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- "Simple" problems can actually turn out to be difficult!

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What we mean is that the product AB can be calculated recursively by the following program:

function $\textsc{Mult}(A, B)$

  if $|A| = |B| = 1$ **then return** AB

  else

    $A_1 \leftarrow \text{MoreSignificantPart}(A)$;

    $A_0 \leftarrow \text{LessSignificantPart}(A)$;

    $B_1 \leftarrow \text{MoreSignificantPart}(B)$;

    $B_0 \leftarrow \text{LessSignificantPart}(B)$;

    $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0)$;

    $Y \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_1)$;

    $Z \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_0)$;

    $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1)$;

    return $W\,2^n + (Y + Z)\,2^{n/2} + X$

  end if

end function

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$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \qquad (2)$$

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Claim: if $T(n)$ satisfies

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$$T\,(n) = 4\,T\left(\tfrac{n}{2}\right) + c\,n = 4\left(\left(\tfrac{n}{2}\right)^2(c+1) - \tfrac{n}{2}\,c\right) + c\,n$$

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$$\begin{aligned} T\,(n) &= 4\,T\left(\tfrac{n}{2}\right) + c\,n = 4\left(\left(\tfrac{n}{2}\right)^2(c+1) - \tfrac{n}{2}\,c\right) + c\,n \\ &= n^2(c+1) - 2c\,n + c\,n = n^2(c+1) - c\,n \end{aligned}$$

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Is there a smarter multiplication algorithm taking less than $O(n^2)$ many steps??

Remarkably, there is, but first some history:
In 1952, one of the most famous mathematicians of the 20^{th} century, Andrey Kolmogorov, conjectured that you cannot multiply in less than $\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture!! Kolmogorov was shocked!

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Take again our two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad \underbrace{XX\ldots X}_{\frac{n}{2}}\underbrace{XX\ldots X}_{\frac{n}{2}}$$

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$$\begin{aligned} AB &= A_1B_1 2^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \end{aligned}$$

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- So we have saved one multiplication at each recursion round!

- Thus, the algorithm will look like this:

```
function Mult(A, B)
    if |A| = |B| = 1 then return AB
    else
        A_1 ← MoreSignificantPart(A);
        A_0 ← LessSignificantPart(A);
        B_1 ← MoreSignificantPart(B);
        B_0 ← LessSignificantPart(B);
        U ← A_0 + A_1;
        V ← B_0 + B_1;
        X ← Mult(A_0, B_0);
        W ← Mult(A_1, B_1);
        Y ← Mult(U, V);
        return W 2^n + (Y − X − W) 2^{n/2} + X
    end if
end function
```

- Thus, the algorithm will look like this:

function MULT(A, B)
  if $|A| = |B| = 1$ **then return** AB
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    $A_1 \leftarrow \text{MoreSignificantPart}(A)$;
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    $B_1 \leftarrow \text{MoreSignificantPart}(B)$;
    $B_0 \leftarrow \text{LessSignificantPart}(B)$;
    $U \leftarrow A_0 + A_1$;
    $V \leftarrow B_0 + B_1$;
    $X \leftarrow \text{MULT}(\mathrm{A_0}, \mathrm{B_0})$;
    $W \leftarrow \text{MULT}(\mathrm{A_1}, \mathrm{B_1})$;
    $Y \leftarrow \text{MULT}(\mathrm{U}, \mathrm{V})$;
    return $W\,2^n + (Y - X - W)\,2^{n/2} + X$
  end if
end function

- How fast is this algorithm?

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$$\approx 3^{\log_2 n} T(1) + c\,n\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} = 3^{\log_2 n} T(1) + 2c\,n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

So we got

$$T(n) \approx 3^{\log_2 n} T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

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$$= O(n^{\log_2 3}) = O(n^{1.58\ldots}) \ll n^2$$

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Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

- If we want to multiply two $n \times n$ matrices P and Q, the product will be a matrix R also of size $n \times n$. To obtain each of n^2 entries in R we do n multiplications, so matrix product by brute force is $\Theta(n^3)$.

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- However, we can do it faster using Divide-And-Conquer;
- We split each matrix into four blocks of (approximate) size $n/2 \times n/2$:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}; \qquad R = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

- Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{4}$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{5}$$

- We obtain:

$$\begin{aligned} a\,e + b\,g &= r & \qquad a\,f + b\,h &= s \\ c\,e + d\,g &= t & \qquad c\,f + d\,h &= u \end{aligned}$$

- Prima facie, there are 8 matrix multiplications, each running in time $T\left(\frac{n}{2}\right)$ and 4 matrix additions, each running in time $O(n^2)$, so such a direct calculation would result in time complexity governed by the recurrence

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- The first case of the Master Theorem gives $T(n) = \Theta(n^3)$, so nothing gained.

Strassen's algorithm for faster matrix multiplication

- However, we can instead evaluate:

$$A = a\,(f-h); \qquad B = (a+b)\,h; \qquad C = (c+d)\,e \qquad D = d\,(g-e);$$
$$E = (a+d)\,(e+h); \qquad F = (b-d)(g+h); \qquad H = (a-c)\,(e+f).$$

- We now obtain

$$\begin{aligned}
E+D-B+F &= (a\,e+d\,e+a\,h+d\,h)+(d\,g-d\,e)-(a\,h+b\,h)+(b\,g-d\,g+b\,h-d\,h)\\
&= a\,e+b\,g = r;\\
A+B &= (a\,f-a\,h)+(a\,h+b\,h) = a\,f+b\,h = s;\\
C+D &= (c\,e+d\,e)+(d\,g-d\,e) = c\,e+d\,g = t;\\
E+A-C-H &= (a\,e+d\,e+a\,h+d\,h)+(a\,f-a\,h)-(c\,e+d\,e)-(a\,e-c\,e+a\,f-c\,f)\\
&= c\,f+d\,h = v.
\end{aligned}$$

- We have obtained all 4 components of C using only 7 matrix multiplications and 18 matrix additions/subtractions.
- Thus, the run time of such recursive algorithm satisfies $T(n) = 7T(n/2) + O(n^2)$ and the Master Theorem yields $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.808})$.
- In practice, this algorithm beats the ordinary matrix multiplication for $n > 32$.

Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??

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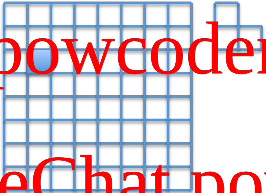
1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??
2. Can we avoid messy computations like:

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{2}\right) + cn = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn = 3^2 T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + cn \\
&= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + cn = 3^3 T\left(\frac{n}{2^3}\right) + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + cn \\
&= 3^3 T\left(\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&= 3^3\left(3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}\right) + cn\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&= 3^4 T\left(\frac{n}{2^4}\right) + cn\left(\frac{3^3}{2^3} + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) = \\
&\ldots \\
&= 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn\left(\left(\frac{3}{2}\right)^{\lfloor \log_2 n \rfloor - 1} + \ldots + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) \\
&\approx 3^{\log_2 n} T(1) + cn\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
&= 3^{\log_2 n} T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)
\end{aligned}
$$

You are given a $2^n \times 2^n$ board with one of its cells missing (i.e., the board has a hole); the position of the missing cell can be arbitrary. You are also given a supply of "dominoes" each containing 3 such squares; see the figure



Your task is to design an algorithm which covers the entire board with such "dominoes" except for the hole.

Hint: Do a divide-and-conquer recursion!

That's All, Folks!!