Algorithms: COMP3121/9101

School of Computer Science and Engineering
University of New South Wales

3. RECURRENCES - part A

- **"Big Oh" notation:** $f(n) = O(g(n))$ is an abbreviation for:

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- Clearly, multiplying constants c of interest will be larger than 1, thus "enlarging" $g(n)$.

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- Since $c\,g(n) \leq f(n)$ if and only if $g(n) \leq \frac{1}{c} f(n)$, we have $f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$.
- **"Theta" notation:** $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$; thus, $f(n)$ and $g(n)$ have the same asymptotic growth rate.

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           Merge-Sort(A, p, q)
           Merge-Sort(A, q+1, r)
           Merge(A, p, q, r)
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Merge-Sort(A, p, r) *sorting A[p..r]*

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2. **then** $q \leftarrow \lfloor \frac{p+r}{2} \rfloor$
3. Merge-Sort(A, p, q)
4. Merge-Sort$(A, q+1, r)$
5. Merge(A, p, q, r)

- Since Merge(A, p, q, r) runs in linear time, the runtime $T(n)$ of Merge-Sort(A, p, r) satisfies

$$T(n) = 2T\left(\frac{n}{2}\right) + c\,n$$

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but it can be shown that ignoring the integer parts and additive constants is OK when it comes to obtaining the asymptotics.

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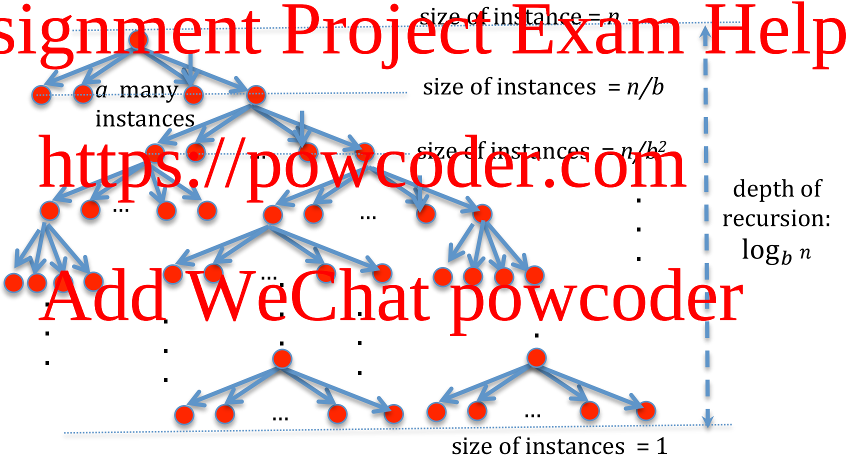

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- This is what the **Master Theorem** provides (when it is applicable).

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(But often the proof of the Master Theorem can be tweaked to obtain the asymptotic of the solution $T(n)$ in such a case when the Master Theorem does not apply; an example is $T(n) = 2T(n/2) + n \log n$).

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- So whenever we have $f = \Theta(g(n) \log n)$ we do not have to specify what base the log is - all bases produce equivalent asymptotic estimates (but we do have to specify b in expressions such as $n^{\log_b a}$).

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  Thus, condition of case 2 is satisfied; and so,

$$T(n) = \Theta(n^{\log_2 2} \log n) = \Theta(n \log n).$$

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  - Thus, in this case the Master Theorem does **not** apply!

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Continuing in this way $\log_b n - 1$ many times we get ...

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Note that so far we did not use any assumptions on $f(n)$...

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$$= O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a\, b^{\varepsilon}}{a}\right)^i\right) = O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} (b^{\varepsilon})^i\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{(b^{\varepsilon})^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right); \qquad \text{we are using } \sum_{i=0}^{m-1} q^i = \frac{q^m - 1}{q - 1}$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\varepsilon}\right)^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^\varepsilon - 1}\right)$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\varepsilon}\right)^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\varepsilon}\right)^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^{\varepsilon} - 1}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Since we had: $T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$ we get:

Case 1 - continued:

$$
\begin{aligned}
\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) &= O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{\left(b^{\lfloor \log_b n \rfloor}\right)^\varepsilon - 1}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a - \varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right) \\
&= O\left(\frac{n^{\log_b a} - n^{\log_b a - \varepsilon}}{b^\varepsilon - 1}\right) \\
&= O\left(n^{\log_b a}\right)
\end{aligned}
$$

Since we had: $T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$ we get:

$$
\begin{aligned}
T(n) &\approx n^{\log_b a} T(1) + O\left(n^{\log_b a}\right) \\
&= \Theta\left(n^{\log_b a}\right)
\end{aligned}
$$

Case 2: $f(m) = \Theta(m^{\log_b a})$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

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$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

Case 2: $f(m) = \Theta(m^{\log_b a})$

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$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right)\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \Theta\left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a^i}{(b^i)^{\log_b a}}\right) \right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i \right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1 \right)$$

$$= \Theta\left(n^{\log_b a} \lfloor \log_b n \rfloor \right)$$

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

because $\log_b n = \log_2 n \cdot \log_b 2 = \Theta(\log_2 n)$. Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = \Theta\left(n^{\log_b a} \log_b n\right) = \Theta\left(n^{\log_b a} \log_2 n\right)$$

because $\log_b n = \log_2 n \cdot \log_b 2 = \Theta(\log_2 n)$. Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

we get:

$$T(n) \approx n^{\log_b a} T(1) + \Theta\left(n^{\log_b a} \log_2 n\right)$$
$$= \Theta\left(n^{\log_b a} \log_2 n\right)$$

Case 3: $f(m) = \Omega(m^{\log_b a + \varepsilon})$ and $a\, f(n/b) \le c\, f(n)$ for some $0 < c < 1$.

We get by substitution:

$$f(n/b) \le \frac{c}{a}\, f(n)$$

$$f(n/b^2) \le \frac{c}{a}\, f(n/b)$$

$$f(n/b^3) \le \frac{c}{a}\, f(n/b^2)$$

$$\dots$$

$$f(n/b^i) \le \frac{c}{a}\, f(n/b^{i-1})$$

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$$\dots$$

$$f(n/b^i) \leq \frac{c}{a}\,f(n/b^{i-1})$$

By chaining these inequalities we get

$$f(n/b^2) \leq \frac{c}{a}\,\underbrace{f(n/b)} \leq \frac{c}{a}\cdot\underbrace{\frac{c}{a}\,f(n)} = \frac{c^2}{a^2}\,f(n)$$

$$f(n/b^3) \leq \frac{c}{a}\,\underbrace{f(n/b^2)} \leq \frac{c}{a}\cdot\underbrace{\frac{c^2}{a^2}\,f(n)} = \frac{c^3}{a^3}\,f(n)$$

$$\dots$$

$$f(n/b^i) \leq \frac{c}{a}\,\underbrace{f(n/b^{i-1})} \leq \frac{c}{a}\cdot\underbrace{\frac{c^{i-1}}{a^{i-1}}\,f(n)} = \frac{c^i}{a^i}\,f(n)$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i}\, f(n)$$

Master Theorem Proof:

Case 3 (continued):

We got
$$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Case 3 (continued):

We got
$$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 3 (continued):

We got $$f(n/b^i) \le \frac{c^i}{a^i} f(n)$$

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Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

Case 3 (continued):

We got
$$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

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and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we had (1):

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f(n)) = O(f(n))$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

thus,

$$T(n) = \Theta(f(n))$$

Exercise 1: Show that condition

$$f(n) = \Omega(n^{\log_b a + \varepsilon})$$

follows from the condition

$$a\,f(n/b) \le c\,f(n) \quad \text{for some } 0 < c < 1.$$

Example: Let us estimate the asymptotic growth rate of $T(n)$ which satisfies

$$T(n) = 2T(n/2) + n \log n$$

Note: we have seen that the Master Theorem does **NOT** apply, but the technique used in its proof still works! So let us just unwind the recurrence and sum up the logarithmic overheads.

$$T(n) = 2\,\underbrace{T\left(\frac{n}{2}\right)} + n\log n$$

$$= 2\left(\overbrace{2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\log\frac{n}{2}}\right) + n\log n$$

$$= 2^2\,\underbrace{T\left(\frac{n}{2^2}\right)} + n\log\tfrac{n}{2} + n\log n$$

$$= 2^2\left(\overbrace{2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\log\frac{n}{2^2}}\right) + n\log\tfrac{n}{2} + n\log n$$

$$= 2^3\,\underbrace{T\left(\frac{n}{2^3}\right)} + n\log\tfrac{n}{2^2} + n\log\tfrac{n}{2} + n\log n$$

$$\ldots$$

$$= 2^{\log n}T\left(\tfrac{n}{2^{\log n}}\right) + n\log\tfrac{n}{2^{\log n-1}} + \ldots + n\log\tfrac{n}{2^2} + n\log\tfrac{n}{2} + n\log n$$

$$= nT(1) + n(\log n \times \log n - \log 2^{\log n-1} - \ldots - \log 2^2 - \log 2)$$

$$= nT(1) + n((\log n)^2 - (\log n - 1) - \ldots - 3 - 2 - 1)$$

$$= nT(1) + n((\log n)^2 - \log n(\log n - 1)/2$$

$$= nT(1) + n((\log n)^2/2 + \log n/2)$$

$$= \Theta\left(n(\log n)^2\right).$$

PUZZLE!

Five pirates have to split 100 bars of gold. They all line up and proceed as follows:

1. The first pirate in line gets to propose a way to split up the gold (for example: everyone gets 20 bars)
2. The pirates, including the one who proposed, vote on whether to accept the proposal. If the proposal is rejected, the pirate who made the proposal is killed.
3. The next pirate in line then makes his proposal, and the 4 pirates vote again. If the vote is tied (2 vs 2) then the proposing pirate is still killed. Only majority can accept a proposal. The process continues until a proposal is accepted or there is only one pirate left. Assume that every pirate :
   - above all wants to live;
   - given that he will be alive he wants to get as much gold as possible;
   - given maximal possible amount of gold, he wants to see any other pirate killed, just for fun;
   - each pirate knows his exact position in line;
   - all of the pirates are excellent puzzle solvers.

Question : What proposal should the first pirate make?

Hint: assume first that there are only two pirates, and see what happens. Then assume that there are three pirates and that they have figured out what happens if there were only two pirates and try to see what they would do. Further, assume that there are four pirates and that they have figured out what happens if there were only three pirates, try to see what they would do. Finally assume there are five pirates and that they have figured out what happens if there were only four pirates.