Algorithms: COMP3121/9101

School of Computer Science and Engineering
University of New South Wales

9. STRING MATCHING ALGORITHMS

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- We now show how hashing can be combined with recursion to produce an efficient string matching algorithm.

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Rabin - Karp Algorithm

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- Thus, we can identify each string with a sequence of integers by mapping each symbol s_i into a corresponding integer i:

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- To any string $B = b_0 b_1 \dots b_{m-1}$ we can now associate an integer whose digits in base d are integers corresponding to each symbol in B:

$$h(B) = h(b_0 b_1 b_2 \dots b_m) = d^{m-1} b_0 + d^{m-2} b_1 + \dots + d \cdot b_{m-2} + b_{m-1}$$

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- This can be done efficiently using the Horner's rule:

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- Next we choose a large prime number p such that $(d+1)\,p$ still fits into a single register and define the hash value of B as $H(B) = h(B) \bmod p$.

- Recall that $A = a_0 a_1 a_2 a_3 \ldots \ldots a_s a_{s+1} \ldots a_{s+m-1} \ldots \ldots a_{N-1}$ where $N >> m$.

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- For each contiguous substring $A_s = a_sa_{s+1}\ldots a_{s+m-1}$ of string A we also compute its hash value as

$$H(A_s) = (d^{m-1}a_s + d^{m-2}a_{s+1} + \ldots + d^1a_{s+m-2} + a_{s+m-1}) \bmod p$$

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- This is where recursion comes into play: we do not have compute the hash value $H(A_{s+1})$ of $A_{s+1} = a_{s+1} a_{s+2} \ldots a_{s+m}$ "from scratch", but we can compute it efficiently from the hash value $H(A_s)$ of $A_s = a_s a_{s+1} \ldots a_{s+m-1}$ as follows.

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by multiplying both sides by d we obtain

$$d \cdot H(A_s)) \bmod p =$$
$$= (d^m a_s + d^{m-1}a_{s+1} + \dots d \cdot a_{s+m-1}) \bmod p$$
$$= (d^m a_s + (d^{m-1}a_{s+1} + \dots d^2 a_{s+m-2} + d\, a_{s+m-1} + a_{s+m}) \bmod p - a_{s+m}) \bmod p$$
$$= (d^m\, a_s + H(A_{s+1}) - a_{s+m}) \bmod p$$

- Consequently,

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- Thus, since we chose p such that $(d+1)p$ fits in a register, all the values and the intermediate results for the above expression also fit in a single register.
- Thus, for every s except $s = 0$ the value of $H(A_s)$ can be computed in constant time independent of the length of the strings A and B.

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- However, as always when we use hashing, we cannot guarantee the worst case performance.
- So we now look for algorithms whose worst case performance can be guaranteed.

- A string matching finite automaton for a string S with k symbols has $k+1$ many states $0, 1, \dots k$ which correspond to the number of characters matched thus far and a transition function $\delta(s, c)$ where s is a state and c is a character red at the moment.

String matching finite automata

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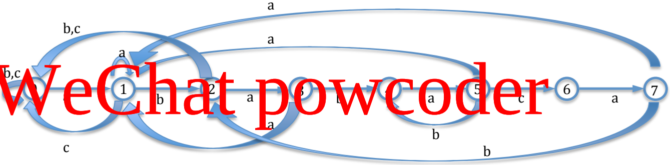
- We first look at the case when such $\delta(s, c)$ is given by a pre-constructed table.
- To make things easier to describe, we consider the string $S = ababaca$. The table defining $\delta(s, c)$ would then be

state	input a	input b	input c	
0	**1**	0	0	a
1	1	**2**	0	b
2	**3**	0	0	a
3	1	**4**	0	b
4	**5**	0	0	a
5	1	4	**6**	c
6	**7**	0	0	a
7	1	2	0	



state transition diagram for string *ababaca*

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String matching with finite automata

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- If we are at a state k this means that so far we have matched the prefix B_k; if we now see an input character a, then $\delta(k, a)$ is the largest m such that the prefix B_m of string B is the suffix of the string $B_k a$.

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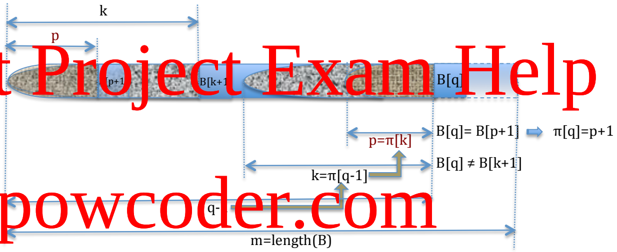
- Thus, if a happens to be $B[k+1]$, then $m = k+1$ and so $\delta(k, a) = k+1$ and $B_k a = B_{k+1}$.
- We do that by matching the string against itself: we can recursively compute a function $\pi(k)$ which for each k returns the largest integer m such that the prefix B_m of B is a proper suffix of B_k.

```
function Compute-Prefix-Function(B)
    m ← length[B]
    let π[1..m] be a new array
    π[1] = 0
    k = 0
    for q = 2 to m do
        while k > 0 and B[k+1] ≠ B[q]
            k = π[k]
        if B[k+1] == B[q]
            k = k + 1
        π[q] = k
    end for
    return π
end function
```



Assume that length of B is m and that we have already found that $\pi[q-1] = k$; to compute $\pi[q]$ we check if $B[q] = B[k+1]$; if true then $\pi[q] = k+1$; if not true then we find $\pi[k] = p$; if now $B[q] = B[p+1]$ then $\pi[q] = p+1$.

- We can now do our search for string B in a longer string A:


```
function KMP − Matcher(A, B)
    n ← length[A]
    m ← length[B]
    π = Compute − Prefix − Function(B)
    q = 0
    for i = 1 to n do
        while q > 0 and B[q + 1] ≠ A[i]
        q = π[q]
        if  B[q + 1] == A[i]
        q = q + 1
        if  q == m
            print pattern occurs with shift i − m
            q = π[q]
    end for
end function
```

- Sometimes we are not interested in finding just the prefect matches but also in matches that might have a few errors such as a few insertions, deletions and replacements.

- So assume that we have a very long string $A = a_0a_1a_2a_3\ldots\ldots a_s a_{s+1}\ldots a_{s+m-1}\ldots\ldots a_{N-1}$, a shorter string $B = b_0b_1b_2\ldots b_{m-1}$ where $m << N$ and an integer $k << m$. We are interested in finding all matches for B in A which allow up to k many errors.

- Idea: split B into $k+1$ consecutive subsequences of (approximately) equal length. Then any match in A with at most k errors must contain a subsequence which is a perfect match for a subsequence of B. Thus, we look for all perfect matches for all of $k+1$ subsequences of B and for every hit we test by brute force if the remaining parts of B have sufficient number of matches in the appropriate parts of A.

PUZZLE!!

On a rectangular table there are 25 non-overlapping round coins of equal size placed in such a way that it is not possible to add another coin without overlapping any of the existing coins and without the coin falling off the table (for a coin to stay on the table its centre must be within the table). Show that it is possible to completely cover the table with 100 coins (of course with overlapping of coins).